An Updated Set of Nonlinear Eigenvalue Problems Version 4.1

Nicholas J. Higham* Gian Maria Negri Porzio* Françoise Tisseur*

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Abstract

This is the description of the new changes made in version 4.1 of the NLEVP MATLAB toolbox with respect to version 4.0. The collection and its organization are described in a separate paper. A user's guide describes how to install and use the NLEVP MATLAB toolbox.

1 Introduction

NLEVP is a MATLAB toolbox, which provides a collection of nonlinear eigenvalue problems. For details of the design and organization of the collection, and a description of the problems in version 3.0, see [7]. The NLEVP toolbox is available on the GITHUB repository at `https://github.com/ftisseur/nlevp`. The MATLAB codes are also available at MATLAB Central File Exchange. For details of how to install and use the toolbox see [6].

This document describes the changes made in version 4.1 of NLEVP, which consist in minor bugs fixings and the addition of 6 new problems with respect to version 4.0 (and 28 with respect to version 3.0). See Table 3 for their names.

A convenient general form for expressing the matrix-valued function $F : \mathbb{C} \to \mathbb{C}^{m\times n}$ defining a nonlinear eigenvalue problem (NEP) $F(\lambda)x = 0$, is

$$F(\lambda) = \sum_{i=0}^{k} f_i(\lambda)A_i, \tag{1}$$

where the $f_i : \mathbb{C} \to \mathbb{C}$ are nonlinear functions and $A_i \in \mathbb{C}^{m\times n}$. Particular cases include

- quadratic eigenvalue problems (QEPs) corresponding to $k = 2$ and $f_i(\lambda) = \lambda^i$, $i = 0, 1, 2$,
- polynomial eigenvalue problems (PEPs) corresponding to $f_i(\lambda) = \lambda^i$, $i = 0, \dots, k$,
- rational eigenvalue problems (REPs), where the $f_i(\lambda)$ are rational functions of λ.

We give in Table 1 a list of identifiers for the types of problems available in the collection and in Table 2 a list of identifiers that specify the properties of problems in the collection. These properties can be used to extract specialized subsets of the collection for use in numerical experiments. We refer to [7, Sec. 2] for their definition. The identifiers `banded`, `tridiagonal`, and `low-rank` had been added to version 4.0 of NLEVP. We recall that the property `banded` is given to problems with coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ having bandwidth less than $n/5$.

2 Collection of new problems

This section contains a brief description of the new problems added to version 4.1 of the collection. The identifiers for the problem properties are listed inside curly brackets after the name of each problem. The new problems are listed in Table 3. All the problems are summarized in Table 4 and Table 5.

*School of Mathematics, The University of Manchester, Manchester, M13 9PL, UK ({`nick.higham`, `francoise.tisseur`, `gianmaria.negriporzio`}`@manchester.ac.uk`).

Table 1: Problems available in the collection and their identifiers.

qep	quadratic eigenvalue problem
pep	polynomial eigenvalue problem
rep	rational eigenvalue problem
nep	other nonlinear eigenvalue problem

Table 2: List of identifiers for the problem properties.

nonregular	symmetric	hyperbolic
real	hermitian	elliptic
nonsquare	T-even	overdamped
sparse	*-even	proportionally-damped
scalable	T-odd	gyroscopic
parameter-dependent	*-odd	low-rank
solution	T-palindromic	
random	*-palindromic	
tridiagonal	T-anti-palindromic	
banded	*-anti-palindromic	

Table 3: New problems in NLEVP version 4.1.

Nonlinear	clamped_beam_1d	nep3	neuron_dde
	schrodinger_abc	square_root	time_delay3

Table 4: The first half of the problems in NLEVP.

Name	Description
acoustic_wave_1d	Acoustic wave problem in 1 dimension.
acoustic_wave_2d	Acoustic wave problem in 2 dimensions.
bcc_traffic	QEP from stability analysis of chain of non-identical cars.
bent_beam	6-by-6 NEP from a bent beam model.
bicycle	2-by-2 QEP from the Whipple bicycle model.
bilby	5-by-5 QEP from bilby population model.
buckling_plate	3-by-3 NEP from a buckling plate model.
butterfly	Quartic matrix polynomial with T-even structure.
canyon_particle	NEP from the Schroedinger equation on a canyon-like shape.
cd_player	QEP from model of CD player.
clamped_beam_1d	NEP from 1D clamped beam model with delayed feedback control.
circular_piston	Sparse QEP from model of circular piston.
closed_loop	2-by-2 QEP associated with closed-loop control system.
concrete	Sparse QEP from model of a concrete structure.
damped_beam	QEP from simply supported beam damped in the middle.
damped_gyro	QEP from a damped gyroscopic system.
deformed_consensus	n-by-n QEP from multi-agent systems theory.
dirac	QEP from Dirac operator.
disk_brake100	100-by-100 QEP from a disk brake model.
disk_brake4669	4669-by-4669 QEP from a disk brake model.
distributed_delay1	3-by-3 NEP from distributed delay system.
elastic_deform	QEP from elastic deformation of anisotropic material.
fiber	NEP from fiber optic design.
foundation	Sparse QEP from model of machine foundations.
gen_hyper2	Hyperbolic QEP constructed from prescribed eigenpairs.
gen_tantipal2	T-anti-palindromic QEP with eigenvalues on the unit circle.
gen_tpal2	T-palindromic QEP with prescribed eigenvalues on the unit circle.
gun	NEP from model of a radio-frequency gun cavity.
hadeler	NEP due to Hadeler.
hospital	QEP from model of Los Angeles Hospital building.
intersection	10-by-10 QEP from intersection of three surfaces.
loaded_string	REP from finite element model of a loaded vibrating string.
metal_strip	QEP related to stability of electronic model of metal strip.
mirror	Quartic PEP from calibration of cadioptric vision system.
mobile_manipulator	QEP from model of 2-dimensional 3-link mobile manipulator.
nep1	2-by-2 basic NEP example.
nep2	3-by-3 basic NEP example.
nep3	NEP with weighted norm coefficients.
neuron_dde	2-by-2 NEP from a neural-network DDE.
omnicam1	9-by-9 QEP from model of omnidirectional camera.

Table 5: The second half of the problems in NLEVP.

Name	Description
omnicam2	15-by-15 QEP from model of omnidirectional camera.
orr_sommerfeld	Quartic PEP arising from Orr-Sommerfeld equation.
pdde_stability	QEP from stability analysis of discretized PDDE.
pdde_symmetric	n-by-n NEP from a partial delay differential equation.
photonic_crystal	REP from dG-FEM of wave propagation in a periodic medium.
pillbox_cavity	170562-by-170562 NEP from a RF pillbox cavity.
pillbox_small	20-by-20 NEP from a RF pillbox cavity.
planar_waveguide	Quartic PEP from planar waveguide.
plasma_drift	Cubic PEP arising in Tokamak reactor design.
power_plant	8-by-8 QEP from simplified nuclear power plant problem.
qep1	3-by-3 QEP with known eigensystem.
qep2	3-by-3 QEP with known, nontrivial Jordan structure.
qep3	3-by-3 parametrized QEP with known eigensystem.
qep4	3-by-4 QEP with known, nontrivial Jordan structure.
qep5	3-by-3 nonregular QEP with known Smith form.
railtrack	QEP from study of vibration of rail tracks.
railtrack_rep	REP from study of vibration of rail tracks.
railtrack2	Palindromic QEP from model of rail tracks.
railtrack2_rep	REP from model of rail tracks.
relative_pose_5pt	Cubic PEP from relative pose problem in computer vision.
relative_pose_6pt	QEP from relative pose problem in computer vision.
sandwich_beam	NEP from model of a clamped sandwich beam.
schrodinger	QEP from Schrodinger operator.
schrodinger_abc	NEP from Schrodinger equation with absorbing boundary condition.
shaft	QEP from model of a shaft on bearing supports with a damper.
sign1	QEP from rank-1 perturbation of sign operator.
sign2	QEP from rank-1 perturbation of 2*sin(x) + sign(x) operator.
sleeper	QEP modelling a railtrack resting on sleepers.
speaker_box	QEP from model of a speaker box.
spring	QEP from finite element model of damped mass-spring system.
spring_dashpot	QEP from model of spring/dashpot configuration.
square_root	Square root of a skew-symmetric matrix.
surveillance	21-by-16 QEP from surveillance camera callibration.
time_delay	3-by-3 NEP from time-delay system.
time_delay2	2-by-2 NEP from a time-delay system.
time_delay3	NEP with high-variance-norm coefficients.
utrecht1331	1331-by-1331 QEP from propagation of sound waves.
wing	3-by-3 QEP from analysis of oscillations of a wing in an airstream.
wiresaw1	Gyroscopic QEP from vibration analysis of a wiresaw.
wiresaw2	QEP from vibration analysis of wiresaw with viscous damping.

Clamped_beam1d {`nep, real, sparse, parameter_dependent, scalable, tridiagonal, banded, low-rank`}. This nonlinear eigenvalue problem has the form

$$F(\lambda)x = (\lambda I + A_0 + \mathrm{e}^{-\lambda\tau} A_1)x,$$

where I, A_0 and $A_1 \in \mathbb{R}^{n\times n}$ and I is the identity matrix. More precisely,

$$A_0 = \begin{bmatrix} -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -n & n \end{bmatrix}, \quad A_1 = e_n e_n^T,$$

where e_n is the nth column of the $n \times n$ identity matrix. The parameter τ is a positive real number that can be chosen by the user. The default is $\tau = 1$. The problem arises from a 1D clamped beam model with delayed feed control and can be found in Section 5.2 of [32].

Nep3 {`nep, scalable, parameter-dependent, random`}. This nonlinear matrix valued function has the form

$$F(\lambda) = \lambda I + A_0 + e^{2\mathrm{i}\lambda} A_1 + \sqrt{\lambda + 4} A_2,$$

where I is the identity matrix and $A_j \in \mathbb{R}^{n\times n}$ are uniformly distributed with Frobenius norm equal to 1 except $\|A_1\| = \alpha$. The default parameters are $n = 10$ and $\alpha = 10^6$. Users may set the random seed; the default is equal to 42.

Neuron_dde {`nep, real, parameter-dependent`}. This nonlinear eigenvalue problem has the form

$$F(\lambda)x = \begin{bmatrix} \lambda + k - \beta\mathrm{e}^{-\lambda\tau_s} & -a_{12}\mathrm{e}^{-\lambda\tau_2} \\ -a_{21}\mathrm{e}^{-\lambda\tau_1} & \lambda + k - \beta\mathrm{e}^{-\lambda\tau_s} \end{bmatrix} x = 0$$

and comes from a two-neurons neural network model with feedback, where multiple time delays are included. The parameter τ_s represents the delay of the feedback, while τ_1 and τ_2 are the delays caused by the propagation time of the signal, as described in [30, Sec. 1&2]. Finally, β represents the feedback strength, while a_{ij} are the connections strengths. The user can define each of these values, thanks to the optional parameter `param`, which has the form $[k, \beta, a_{12}, a_{21}, \tau_s, \tau_1, \tau_2]$. The default values are $[0.5, -1, 1, 2.5, 0.01, 1, 1]$, as in Experiment 3 from the same paper.

Schrodinger_abc {`nep, real, sparse, low-rank, banded, scalable`}. This problem can be found in the Julia package NEP-PACK by Jarlebring et al. [24] under the name `schrodinger_movebc`, where they consider the Schrödinger eigenvalue problem on the interval $[0, L_1]$

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x^2} - V(x) - \lambda\right)\psi(x) = 0, & x \in [0, L_1] \\ \psi(0) = 0, & \\ \psi(L_1) = 0, & \end{cases}$$

with potential function $V(x)$ defined as

$$V(x) = \begin{cases} 1 + \sin(\alpha x), & x \in [0, L_0], \\ V_0, & x \in \,]L_0, L_1[. \end{cases}$$

Avoiding the discretization of the segment $[L_0, L_1]$ and restricting to $[0, L_0]$ yields the nonlinear eigenvalue problem

$$F(\lambda)x = (A - \lambda I_n + g(\lambda)B_1 + f(\lambda)B_2)x,$$

where

$$f(\lambda) = \frac{\sinh((L_1 - L_0)\sqrt{\lambda + V_0})}{\sqrt{\lambda + V_0}}, \qquad g(\lambda) = \cosh((L_1 - L_0)\sqrt{\lambda + V_0}).$$

The matrices are

$$A = n^2 \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & 1 & -2 & 1 \\ & & 0 & 0 \end{bmatrix} - \begin{bmatrix} V(L_0/(n)) & & & \\ & \ddots & & \\ & & V((n-1)L_0/n) & \\ & & & 0 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix},$$

and

$$B_1 = e_n e_n^T, \qquad B_2 = \frac{n}{2} e_n e_{n-2}^T - 2n e_n e_{n-1}^T + 3\frac{n}{2} e_n e_n^T,$$

where e_j is the j-th column of the identity matrix. The user may set the size of the problem n: the default value is $n = 1000$, while $V_0 = 10$, $L_0 = 1$ and $L_1 = 8$.

Square_root {`nep, real, sparse`}. This nonlinear eigenvalue problem has the form

$$F(\lambda) = A - \sqrt{\lambda} I_{20},$$

where $I_k \in \mathbb{R}^{k \times k}$ is the identity matrix and$A \in \mathbb{R}^{20 \times 20}$ is the skew-symmetric matrix

$$A = \begin{bmatrix} 4I_{10} & B \\ -B & 4I_{10} \end{bmatrix}, \qquad B = \begin{bmatrix} 20 & -10 & & \\ -10 & \ddots & \ddots & \\ & \ddots & \ddots & -10 \\ & & -10 & 20 \end{bmatrix}.$$

The eigenvalues of A lie on the imaginary segment $[4 - 40\mathrm{i}, 4 + 40\mathrm{i}]$, and they are the square of the eigenvalues of $F(z)$. This problem first appeared in [11].

Time_delay3 {`nep, real, scalable, parameter-dependent, random`}. This matrix valued function has the form

$$F(\lambda) = \lambda I + \sum_{j=1}^{k} \mathrm{e}^{-\lambda j} A_j,$$

where $A_j \in \mathbb{R}^{n \times n}$ are uniformly distributed random matrices with $\|A_j\|_\infty = 10^{2j}$ and I is the identity matrix. The value of k is chosen by the user (the default is $k = 2$), who may also set the random seed; the default is equal to 0.

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