Linear Data Chapter 5

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1. For the matrix
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
explicitly compute by hand (with work shown) the following.

   (a) $\mathbf{I}_2\mathbf{A}$, where $\mathbf{I}_2$ is the 2×2 identity matrix.
   Since multiplying by the (appropriately sized) identity matrix doesn't change a matrix,
$$\mathbf{I}_2\mathbf{A} = \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

   (b) $\mathbf{A}^{-1}$
   Using the formula for inverses of 2×2 matrices, we first test
$$1(1) - 0(1) = 1 \neq 0,$$
   which tells us $\mathbf{A}$ is invertible with inverse
$$\mathbf{A}^{-1} = \frac{1}{1(1) - 0(1)} \begin{pmatrix} 1 & -1 \\ -0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

   (c) $\mathbf{A}^2$
   We compute
$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1(1) + 1(0) & 1(1) + 1(1) \\ 0(1) + 1(0) & 0(1) + 1(1) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

   (d) $\mathbf{A}^{-2}$
   [OPTION ONE] We can square $\mathbf{A}^{-1}$:
$$\mathbf{A}^{-2} = \left(\mathbf{A}^{-1}\right)^2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1(1) + -1(0) & 1(-1) - 1(1) \\ 0(1) + 1(0) & 0(-1) + 1(1) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$
   [OPTION TWO] We can invert $\mathbf{A}^2$:
$$\mathbf{A}^{-2} = \left(\mathbf{A}^2\right)^{-1} = \frac{1}{1(1) - 0(2)} \begin{pmatrix} 1 & -2 \\ -0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

(e) the diagonal of $\mathbf{A}$

It is simply $1, 1$.

2. Let

$$\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.8 \end{pmatrix}$$

Either explicitly compute $\mathbf{D}^{-1}$ or explain why it doesn't exist.

[METHOD ONE] $\mathbf{D}$ *is not invertible because multiplying vectors in* $\mathbb{R}^6$ *on the left by* $\mathbf{D}$ *zeros out the fourth entry. Thus, the matrix multiplication cannot be undone.* *[METHOD TWO]* $\mathbf{D}$ *is a diagonal matrix, and diagonal matrices are invertible if and only if each diagonal element is non-zero.*

3. Assume that $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E}$ are all 3×3 matrices such that

$$\mathbf{BC} = \begin{pmatrix} -6 & 4 & 4 \\ -1 & 0 & 3 \\ 3 & 2 & -7 \end{pmatrix}, \quad \mathbf{CB} = \begin{pmatrix} -5 & 4 & 4 \\ -1 & -2 & 4 \\ 1 & 6 & -6 \end{pmatrix}, \quad \mathbf{BE} = \begin{pmatrix} -2 & 2 & 4 \\ -2 & 1 & 3 \\ 2 & -5 & -7 \end{pmatrix}, \quad \mathbf{EC} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -2 & 0 & 4 \end{pmatrix}$$

Explicitly compute the following by hand. (I.e., write out the entries of the 3×3 matrix.)

(a) [3 points] $\mathbf{B}(\mathbf{E} + \mathbf{C})$

By the distributivity of matrix multiplication,

$$\mathbf{B}(\mathbf{E} + \mathbf{C}) = \mathbf{BE} + \mathbf{BC} = \begin{pmatrix} -2 & 2 & 4 \\ -2 & 1 & 3 \\ 2 & -5 & -7 \end{pmatrix} + \begin{pmatrix} -6 & 4 & 4 \\ -1 & 0 & 3 \\ 3 & 2 & -7 \end{pmatrix}$$
$$= \begin{pmatrix} -2-6 & 2+4 & 4+4 \\ -2-1 & 1+0 & 3+3 \\ 2+3 & -5+2 & -7-7 \end{pmatrix} = \begin{pmatrix} -8 & 6 & 8 \\ -3 & 1 & 6 \\ 5 & -3 & -14 \end{pmatrix}.$$

(b) $(\mathbf{E} + \mathbf{B})\mathbf{C}$

By the distributivity of matrix multiplication,

$$(\mathbf{E} + \mathbf{B})\mathbf{C} = \mathbf{EC} + \mathbf{BC} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -2 & 0 & 4 \end{pmatrix} + \begin{pmatrix} -6 & 4 & 4 \\ -1 & 0 & 3 \\ 3 & 2 & -7 \end{pmatrix}$$
$$= \begin{pmatrix} 1-6 & -1+4 & -1+4 \\ 1-1 & -1+0 & -1+3 \\ -2+3 & 0+2 & 4-7 \end{pmatrix} = \begin{pmatrix} -5 & 3 & 3 \\ 0 & -1 & 2 \\ 1 & 2 & -3 \end{pmatrix}$$

(c) $\mathbf{E}^\top \mathbf{B}^\top$

By algebraic properties of matrix multiplication:

$$\mathbf{E}^\top \mathbf{B}^\top = (\mathbf{BE})^\top = \begin{pmatrix} -2 & 2 & 4 \\ -2 & 1 & 3 \\ 2 & -5 & -7 \end{pmatrix}^\top = \begin{pmatrix} -2 & -2 & 2 \\ 2 & 1 & -5 \\ 4 & 3 & -7 \end{pmatrix}$$

4. Consider the matrices

$$\mathbf{G} = \begin{pmatrix} 1 & -3 \\ -1 & 2 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} 1 & -3 & -1 \\ -1 & 2 & 0 \\ 0 & 5 & 5 \end{pmatrix},$$

noting that $H(:,3) = 2H(:,1) + H(:,2)$.

- Is **G** invertible? Explain your answer.
  Only square matrices can be invertible; thus, **G** *is not invertible. In other words, multiplying vectors on the left by* **G** *is a function from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$*, which can't be surjective / onto ("nice enough" functions can't move from smaller to cover larger dimensions) and thus can't be bijective / invertible.*
- Is **H** invertible? Explain your answer.
  There is a result in the book that any matrix which has a column which is a linear combination of other columns can't be invertible. More explicitly, since matrix-vector multiplication can be expressed as a linear combination of the columns, we have that

$$\mathbf{H}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = H(:,3) = 2H(:,1) + H(:,2) = \mathbf{H}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix};$$

  i.e., multiplying by **H** *isn't injective / one-to-one and hence not invertible.*

5. Assume there is the following simplified grade book:

Name	Homework	Labs	Final Exam	Project
Avery	95	98	90	100
Blake	90	96	92	95
Carlos	83	79	79	90
Dax	55	30	65	60

Assume that the weights used to compute the final grades are homework 0.3, labs 0.2, the final 0.35, and the project 0.15.

- Write an explicit formula to compute Avery's final grade using a single inner product.

Avery's final grade is

$$0.3(95) + 0.2(98) + 0.35(90) + 0.15(100) = \left\langle \begin{pmatrix} 95 \\ 98 \\ 90 \\ 100 \end{pmatrix}, \begin{pmatrix} 0.3 \\ 0.2 \\ 0.35 \\ 0.15 \end{pmatrix} \right\rangle$$

It wasn't necessary to compute Avery's grade to get credit, but it is 94.6*.*

- Write an explicit formula to compute everyone's final grade simultaneously using a single matrix-vector product.
  We need to multiply each student's grades times the weights to compute the final grade. Thus, the formula is

$$\begin{pmatrix} 95 & 98 & 90 & 100 \\ 90 & 96 & 92 & 95 \\ 83 & 79 & 79 & 90 \\ 55 & 30 & 65 & 60 \end{pmatrix} \begin{pmatrix} 0.3000 \\ 0.2000 \\ 0.3500 \\ 0.1500 \end{pmatrix}$$

  It wasn't necessary to compute all of the grades to get credit, but they are

Avery	*94.6*
Blake	*92.65*
Carlos	*81.85*
Dax	*54.25*

- Give an example of a linear algebra operation performed on the above data that is not informative to a professor who is trying to understand the students' performance in the course.
  There are many correct answers. They just must be correctly justified or more generally forming linear combinations of the rows or columns with negative scalars. E.g., multiplying the homework grades by -2.35 *isn't informative.*

6. [3 points] Using Python/Jupyter or Matlab/Matlab Live Script, perform the following:

- Define $\mathbf{M}$ be to a random 3×3 matrix.
  In Matlab:
  `M=rand(3)` *(with or without semicolon)*
  In Python:

```
import numpy as np
from numpy import linalg as LA
M=np.random.rand(3,3)
```

- Test if $\left(\mathbf{M}^\top\right)^{-1}$ and $\left(\mathbf{M}^{-1}\right)^\top$ are equal (up to floating point arithmetic).
  In Matlab:

```
norm(inv(M)'-inv(M'))<10e-14
```

  In Python:

```
np.allclose(LA.inv(M.T),LA.inv(M).T) or
np.allclose(LA.inv(np.transpose(M)),np.transpose(LA.inv(M)))
```