Linear Data Chapter 10

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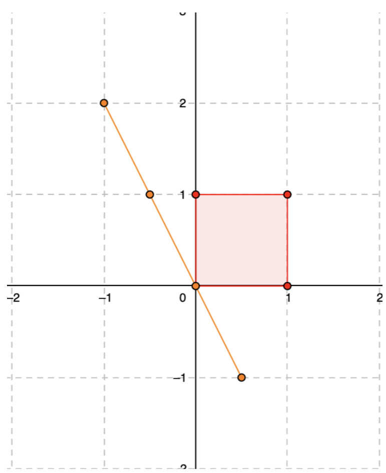
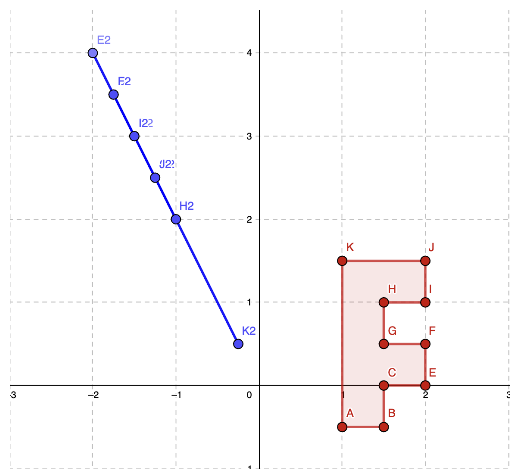
1. (a) Give an example of an element of the null space / kernel of the matrix $\mathbf{A}$ from the coding demonstration for Section 10.1.
   Any vector in $\mathbb{R}^{1{,}000{,}001}$ which only has bass/low frequencies.

   (b) [5 points] Give an example of an element of the image / column space of the matrix $\mathbf{A}$ from thhe coding demonstration for Section 10.1.
   Any vector in $\mathbb{R}^{1{,}000{,}001}$ which only has treble/high frequencies.

2. Consider the Geogebra screenshots below showing the effects of multiplication by a certain 2×2 matrix $\mathbf{B}$.





   (a) What must the rank of $\mathbf{B}$ be? Explain your answer.
   Since $\mathbf{B}$ is not invertible, we know that it has a non-trivial null space / kernel and the rank must be less than 2. However, the only matrices with rank 0 are those which map everything to the zero vector, which is clearly not the case here. Thus, $\operatorname{rank}(\mathbf{B}) = 1$.

   (b) Give at least one element of the image / column space of $\mathbf{C}$ which is not the zero vector. Explain your answer.
   We can see in the images some of the points output by multiplication by $\mathbf{C}$. The easiest two to parse are $(-1, 2)$ and $(-2, 4)$. However, any point on either of the two line segments would work (i.e., a point of the form $(t, -2t)$ for $-2 \leq t \leq 1/2$).

Also, since we know that the image of $\mathbf{C}$ is a vector space, one further knows that the entire line $(t, -2t)$ for $t \in \mathbb{R}$ is the image of $\mathbf{C}$; so, any point on it would be an acceptable answer.

(c) Describe what elements of the cokernel of $\mathbf{C}$ look like.
Since the image of $\mathbf{C}$ is the line $y = -2x$, the elements of the cokernel are just all of the parallel lines, i.e., $y = -2x + b$ for each choice of $b \in \mathbb{R}$.

3. Using Python/Jupyter or Matlab/Matlab Live Script, perform the following.

(a) Set

$$\mathbf{M} = \begin{pmatrix} -2 & 2 & -2 \\ 0 & -2 & 0 \\ 3 & 2 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Matlab:
```
M=[-2, 2, -2; 0, -2, 0; 3, 2, 3]
b=[1;5;5]
z=[1;1;1]
```
Python:
```
import numpy as np
M=np.array([[-2, 2, -2],[0, -2, 0],[3, 2, 3]])
b=np.array([1,5,5])
z=np.array([1,1,1])
```

(b) Determine if $\mathbf{M}\vec{x} = \vec{b}$ has a solution $\vec{x}$. If it does have a solution, give the solution.

The equation does not have a solution. Code to show that:
Matlab:
```
x=lsqr(M,b)
```
`norm(M*x-b)` *is not within floating point arithmetic of zero*
Python:
```
from numpy.linalg import lstsq as lsqr
x=lsqr(M,b,rcond=None)[0]
np.allclose(M@x,b)
```

(c) Set $\vec{c} = \mathbf{M}\vec{z}$.
Matlab:
```
c=M*z
```
Python:
```
c=M@z
```

(d) Generate another solution to $\mathbf{M}\vec{x} = \vec{c}$ than $\vec{x} = \vec{z}$.
We can add any element of the null space (which the calculations below show is one-dimensional) of $\mathbf{M}$ to $\vec{z}$ to get another solution. E.g.:
Matlab:
```
N=null(M)
```

```
norm(M*(z+N)-c,Inf)<1e-15
```

Python:

```
from scipy.linalg import null_space as null
N=null(M)
np.allclose(M*(z+N),c)
```

(e) Explain how the answer to part (d) is related to the coimage of $\mathbf{M}$.

Each element of the image of $\mathbf{M}$ *has an associated set of points that map to it under multiplication by* $\mathbf{M}$. *Each set of such points is an element of the coimage.*