Linear Data Chapter 6

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1. For each of the following subsets of $\mathbb{R}^3$, explain whether or not they are a subspace of $\mathbb{R}^3$.

   (a)
   $$U = \operatorname{span}\left\{ \begin{pmatrix} 1.1 \\ -3.4 \\ 0.4 \end{pmatrix}, \begin{pmatrix} 0.65 \\ 0.23 \\ -0.44 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$
   Yes. Any span of a set of vectors in a vector space is a subspace of that vector space.

   (b)
   $$V = \left\{ \begin{pmatrix} a \\ 0 \\ a^3 \end{pmatrix} \middle|\, a \in \mathbb{R} \right\}$$
   No. The set isn't closed under vector addition or scalar multiplication. Any counter example will suffice. For example:
   $$\begin{pmatrix} 1 \\ 0 \\ 1^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1^3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 0 \\ 2^3 \end{pmatrix}.$$

   (c) $Z=$ the points in the z-axis
   Yes. Any line through the origin in $\mathbb{R}^d$ is a subspace of $\mathbb{R}^d$.

2. Assume that $f : \mathbb{R}^{100} \to \mathbb{R}^2$ is linear and that for certain $\vec{u}$, $\vec{v} \in \mathbb{R}^{100}$,
   $$f(\vec{u}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad f(\vec{v}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$
   Explicitly compute with work the following:

   (a) $f(\vec{u}+\vec{v})$
   Since f is linear,
   $$f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) $f(10\vec{v})$

Since f is linear,

$$f(10\vec{v}) = 10f(\vec{v}) = 10\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\20\end{pmatrix}$$

3. Give an example of an application of a linear transformation to audio signals.
The example in this textbook chapter is zero-padding in order to allow offset mixing. However, many of the examples of audio processing we've seen thus far are linear transformations; e.g., computing a linear combination of two digital songs of the same length is a linear transformation on the vector space of pairs of digital songs of that length, computing a moving average of a digital song is a linear transformation on the vector space of digital songs of that length, etc.

4. Assume that W is a vector space and $g, h : W \to \mathbb{R}$ are both linear maps. Show that the function

$$k : W \to \mathbb{R}^2, \quad k(w) = \begin{pmatrix}g(w)\\h(w)\end{pmatrix}$$

is linear.
Let $w, v \in W$ and $c \in \mathbb{R}$ be arbitrary vectors in W and scalar in $\mathbb{R}$, respectively. Then,

$$\begin{aligned}
k(w+v) &= \begin{pmatrix}g(w+v)\\h(w+v)\end{pmatrix} && \textit{by the definition of } k\\
&= \begin{pmatrix}g(w)+g(v)\\h(w)+h(v)\end{pmatrix} && \textit{by the linearity of } g \textit{ and } h\\
&= \begin{pmatrix}g(w)\\h(w)\end{pmatrix} + \begin{pmatrix}g(v)\\h(v)\end{pmatrix} && \textit{by the definition of vector addition in } \mathbb{R}^2\\
&= k(w) + k(v) && \textit{by the definition of } k.
\end{aligned}$$

Thus, k is additive. Similarly, we compute

$$\begin{aligned}
k(cw) &= \begin{pmatrix}g(cw)\\h(cw)\end{pmatrix} && \textit{by the definition of } k\\
&= \begin{pmatrix}cg(w)\\ch(w)\end{pmatrix} && \textit{by the linearity of } g \textit{ and } h\\
&= c\begin{pmatrix}g(w)\\h(w)\end{pmatrix} && \textit{by the definition of scalar multiplication in } \mathbb{R}^2\\
&= ck(w) && \textit{by the definition of } k.
\end{aligned}$$

Thus k also respects scaling, implying in turn that k is linear.