Linear Data Chapter 11

Written by: Emily J. King

1. Assume that $\mathbf{C}$ is a 4×4 matrix, where

$$\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \in \mathcal{E}_{-2}(\mathbf{C})$$

   Explicitly compute $\mathbf{C}^3\vec{y}$.

2. Consider the following matrices

$$\mathbf{M} = \begin{pmatrix} 4 & 4 & 0 & 6 \\ 4 & -6 & 1 & 0 \\ 0 & 1 & -4 & 5 \\ 6 & 0 & 5 & -6 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   where $\mathbf{M}$ is invertible. Define $\mathbf{A} = \mathbf{M}\mathbf{D}\mathbf{M}^{-1}$. You must work all of the following problems by hand with justification to receive credit.

   (a) Explicitly list the eigenvalues of $\mathbf{A}$.

   (b) Explicitly give all elements of the null space of $\mathbf{A}$.

   (c) Explicitly compute the rank of $\mathbf{A}$ by hand.

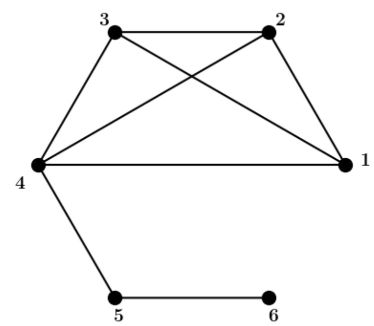
   (d) Explicitly compute the determinant of $\mathbf{A}$ by hand.

   (e) Let $\mathbf{I}$ be the 4×4 identity matrix. Explicitly compute $(\mathbf{A} + 2\mathbf{I})(\mathbf{A} + \mathbf{I})(\mathbf{A} - 3\mathbf{I})\mathbf{A}$.

   (f) Let $\vec{y} \in \mathbb{R}^4$ be a random vector and let $\mathbf{U}$ be a random 4×4 orthogonal matrix. If you were to perform power iteration on the symmetric matrix $\mathbf{U}\mathbf{D}\mathbf{U}^\top$ (that is, multiply the vector by $\mathbf{U}\mathbf{D}\mathbf{U}^\top$, normalize, then take that vector multiply by $\mathbf{U}\mathbf{D}\mathbf{U}^\top$ and normalize, and repeat this process) for 1000 iterations, what is most likely to be the output? Fully justify your answer without any reference to Matlab/Python.

3. Are all square matrices diagonalizable? If so, explain why. If not, give an example of a non-diagonalizable matrix.

4. Consider the graph/network below.



(a) Explicitly give the graph Laplacian $\mathbf{L}$ of the graph/network.

(b) Explicitly give (the entries of) a non-zero vector in the nullspace of this graph Laplacian $\mathbf{L}$. Fully justify your answer without any reference to Matlab/Python.

(c) Which of the following is most likely to be a Fielder eigenvector of the graph above? Fully justify your answer without any reference to Matlab/Python.

$$\text{A.}\begin{pmatrix} -0.3151 \\ -0.3151 \\ -0.3151 \\ -0.1620 \\ 0.3759 \\ 0.7312 \end{pmatrix} \quad \text{B.}\begin{pmatrix} -0.3151 \\ 0.3151 \\ 0.3151 \\ 0.1620 \\ 0.3759 \\ 0.7312 \end{pmatrix} \quad \text{C.}\begin{pmatrix} -0.3151 \\ -0.3151 \\ -0.3151 \\ -0.1620 \\ -0.3759 \\ -0.7312 \end{pmatrix} \quad \text{D.}\begin{pmatrix} -0.3151 \\ -0.3151 \\ 0.3151 \\ 0.1620 \\ -0.3759 \\ -0.7312 \end{pmatrix}$$

5. Consider the following (left) stochastic matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1/3 & 1/5 \\ 0 & 2/3 & 2/5 \\ 0 & 0 & 2/5 \end{pmatrix}$$

(a) Recall that if $\mathbf{Q}$ is a $d \times d$ (left) stochastic matrix and $\vec{e}_i$ is the ith standard basis vector in $\mathbb{R}^d$, then $\mathbf{Q}\vec{e}_i$ is a probability vector where the jth coordinate is the probability of the system transitioning from state i to state j.
If a system explained by $\mathbf{P}$ is in state 3, what are the probabilities of it transitioning to state 1 or state 2 or remaining in state 3?

(b) Give and run the Matlab/Python commands to compute the eigenvectors and eigenvalues of $\mathbf{P}$.

(c) Given your answer to (b), give the steady state vector of $\mathbf{P}$.

(d) Explain why intuitively your answer to (c) makes sense.