Linear Data Chapter 9

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1. Is
$$\left\{ \begin{pmatrix} 1.1 \\ -3.4 \\ 0.4 \end{pmatrix}, \begin{pmatrix} 0.65 \\ 0.23 \\ -0.44 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$
a basis for $\mathbb{R}^3$? You must explain your answer to get any points.
[METHOD ONE] $\mathbb{R}^3$ is three-dimensional and the set has 4 vectors. Thus, the set cannot be a basis since the cardinality of a basis is the dimension of the space.
[METHOD TWO] A set is a basis for a vector space when it is linearly independent and spans the vector space. A set is linearly independent when you can't write any vector in the set as a linear combination of the others, but
$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1.1 \\ -3.4 \\ 0.4 \end{pmatrix} + 0 \begin{pmatrix} 0.65 \\ 0.23 \\ -0.44 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$
Thus, the set is not linearly independent and can't be a basis.

2. Let
$$\vec{x} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } W = \operatorname{span}\{\vec{x}, \vec{y}\}.$$
   (a) Show that $\{\vec{x}, \vec{y}\}$ is an orthonormal set.
   [BY HAND] We first note that
$$\langle \vec{x}, \vec{y} \rangle = (1/2)^2 - (1/2)^2 + (1/2)^2 - (1/2)^2 = 0,$$
   so they are orthogonal. We now note that
$$\|\vec{x}\| = \sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} = \sqrt{4(1/4)} = \sqrt{1} = 1$$
$$\|\vec{y}\| = \sqrt{(1/2)^2 + (-1/2)^2 + (1/2)^2 + (-1/2)^2} = \sqrt{4(1/4)} = \sqrt{1} = 1;$$
   thus, they are orthonormal.
   [MATLAB METHOD] We compute the gram matrix.
   `X=[1/2, 1/2; 1/2, -1/2; 1/2, 1/2; 1/2, -1/2]` *(with or without commas*

between entries on rows)

```
X'*X
```

This is the 2×2 identity matrix; so, the columns of X are orthonormal.

[PYTHON METHOD] We compute the gram matrix.

```
import numpy as np
X=np.array([[1/2, 1/2], [1/2, -1/2], [1/2, 1/2], [1/2, -1/2]])
X.T@X
```

This is the 2×2 identity matrix; so, the columns of X are orthonormal.

(b) Explicitly give with justification $\operatorname{proj}_W \vec{z}$.

[RIGOROUS METHOD] By (a) $\{\vec{x}, \vec{y}\}$ is an orthonormal set, and by construction, $W = \operatorname{span}\{\vec{x}, \vec{y}\}$. Thus, by Proposition 4.50,

$$
\begin{aligned}
\operatorname{proj}_W \vec{z} &= \langle \vec{z}, \vec{x} \rangle \vec{x} + \langle \vec{z}, \vec{y} \rangle \vec{y} \\
&= (3/2) \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \underbrace{((1/2) + (-1/2) + 0 + (-1/2))}_{=-1/2} \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \\
&= \begin{pmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{pmatrix} + \begin{pmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \\ 1 \end{pmatrix}.
\end{aligned}
$$

[MATLAB METHOD]

(using X from above Matlab method)

```
projW=X*X';
z=[1; 1; 0; 1]
projW*z
```

[PYTHON METHOD]

(using X from above Python method and assuming numpy already loaded above)

```
projW=X@X.T
z=np.array([1, 1, 0, 1])
projW@z
```

(c) Give an explicit element of $W^\perp$ which is not the zero vector.

There are infinitely many correct answers, but the only obvious one given the *information above is $\vec{z} - \operatorname{proj}_W \vec{z}$, since such a vector is always in $W^\perp$:*

$$\vec{z} - \operatorname{proj}_W \vec{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{pmatrix}$$

For the record, a "nice" orthonormal basis for this $W^\perp$ is:

$$\begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$$

3. Let

$$\vec{v} = \begin{pmatrix} -3 \\ 3 \\ -3 \\ \vdots \\ -3 \\ 3 \end{pmatrix} \in \mathbb{R}^{100}$$

have alternating entries -3 and 3. Further let $\mathbf{D}$ have as columns the DCT-II basis for $\mathbb{R}^{100}$.

   (a) Which coefficient of $\mathbf{D}^\top \vec{v}$, i.e., the DCT of $\vec{v}$ / change of basis of $\vec{v}$ with respect to the columns of $\mathbf{D}$ should have the largest absolute value? Explain your answer.
   This generalizes what we saw in the lab. In particular, the last vector in the DCT-II basis is the "bounciest" of the basis vectors (i.e., the coordinates of the vector are drawn from the highest frequency cosine function relative to all other DCT-II basis vectors). So the absolute values of the DCT of $\vec{v}$ should be largest in the last (100th when counting starting at 1) entry.

Bonus I Give with justification the first entry of the DCT of $\vec{v}$.
For any $N \in \mathbb{N}$, the first entry of the DCT of $\vec{x} \in \mathbb{R}^N$ is just the sum of the entries of $\vec{x}$ times $1/\sqrt{N}$. So, specifically, with $\vec{v}$, we see that there are 100 entries, half of which are 3 and half of which are -3. Thus, the sum of the entries is 0, meaning that the first entry of the DCT of $\vec{v}$ is 0.

4. Using Python/Jupyter or Matlab/Matlab Live Script, compute the following.

   (a) Determine the dimension of the span of the following set of vectors:

$$\begin{pmatrix} 3 \\ 4 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ -12 \\ 12 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 1 \\ -2 \end{pmatrix},$$

The dimension of the span is 2.
In Matlab:

```
X=[3, 8, 1, -1; 4, 0, -4, -4; -4, -12, -2, 1; 5, 12, 1, -2]
rank(X)
```

(with or without commas between elements in rows, may put the matrix directly into rank, may define each vector separately and then stack the columns)
In Python:

```
import numpy as np
from numpy.linalg import matrix_rank as rank
X=np.array([[3, 8, 1, -1],[4, 0, -4, -4],[-4, -12, -2, 1],[5, 12, 1,
-2]])
rank(X)
```

(may put the matrix directly into rank, may define each vector separately and then stack the columns using `np.column_stack`*)*

(b) Generate an orthonormal basis for the span of the vectors above.
In Matlab:

```
[U, , ]=svd(X);
U(:,1:2)
```

(may also put the matrix directly into svd)
In Python:

```
U = np.linalg.svd(X)[0]
U[:,0:2]
```

(may also put the matrix directly into svd)

5. Let

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 0 \\ 3/5 \\ -4/5 \end{pmatrix}, \quad \text{and} \quad \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Define $V = \text{span}\{\vec{v}, \vec{u}\}$.

(a) Show that $\vec{v}$ and $\vec{u}$ are orthonormal.
We compute

$$\begin{aligned}
\langle \vec{v}, \vec{u} \rangle &= 1(0) + 0(3/5) + 0(-4/5) = 0 \\
\|\vec{v}\| &= \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1 \\
\|\vec{u}\| &= \sqrt{0^2 + (3/5)^2 + (-4/5)^2} = \sqrt{9/25 + 16/25} = \sqrt{25/25} = 1.
\end{aligned}$$

Thus, the vectors are orthonormal.

(b) Do $\{\vec{u}, \vec{v}\}$ form a basis for V?
[METHOD ONE] Orthogonal vectors are always bases for their spans.
[METHOD TWO] By definition $\{\vec{u}, \vec{v}\}$ span their span. Two vectors are linearly independent when neither is a scalar multiple of the other. Given the location of

zeros in the vectors, they must be linearly independent. Thus, they are a linearly independent spanning set, i.e., a basis.

(c) Explicitly compute $\operatorname{proj}_V \vec{w}$.
By parts (a) and (b), $\vec{v}, \vec{u}$ are an orthonormal basis for V. Thus,

$$\begin{aligned}
\operatorname{proj}_V \vec{w} = \langle \vec{w}, \vec{v} \rangle \vec{v} + \langle \vec{w}, \vec{u} \rangle \vec{u} &= \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3/5 \\ -4/5 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 3/5 \\ -4/5 \end{pmatrix} \\
&= (1(1) + 1(0) + 1(0)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (1(0) + 1(3/5) + 1(-4/5)) \begin{pmatrix} 0 \\ 3/5 \\ -4/5 \end{pmatrix} \\
&= 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1/5) \begin{pmatrix} 0 \\ 3/5 \\ -4/5 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -3/25 \\ 4/25 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -3/25 \\ 4/25 \end{pmatrix}
\end{aligned}$$

(d) Explicitly give a non-zero vector in $V^\perp$.
[METHOD ONE] We know $\operatorname{proj}_{V^\perp} \vec{w} \in V^\perp$; thus, we compute

$$\operatorname{proj}_{V^\perp} \vec{w} = \vec{w} - \operatorname{proj}_V \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -3/25 \\ 4/25 \end{pmatrix} = \begin{pmatrix} 0 \\ (25+3)/25 \\ (25-4)/25 \end{pmatrix} = \begin{pmatrix} 0 \\ 28/25 \\ 21/25 \end{pmatrix}$$

[METHOD TWO (overkill but correct)] Any element in $V^\perp$ must be orthogonal to $\vec{v}$ and $\vec{u}$ since they are in V. Further, by linearity of the inner product, any vector orthogonal to both vectors will be orthogonal to everything in their span. To that end, we compute for arbitrary $a, b, c \in \mathbb{R}$:

$$\begin{aligned}
\left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \vec{v} \right\rangle &= a(1) + b(0) + c(0) = a \\
\left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \vec{u} \right\rangle &= a(0) + b(3/5) + c(-4/5).
\end{aligned}$$

If both inner products are 0, we have $a = 0$ and

$$0 = 3b/5 - 4c/5 \quad \Rightarrow \quad 0 = 3b - 4c \quad \Rightarrow \quad 4c = 3b \quad \Rightarrow \quad c = 3b/4.$$

Thus, the elements of $V^\perp$ are

$$\left\{ \begin{pmatrix} 0 \\ b \\ 3b/4 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 3/4 \end{pmatrix} \right\}$$

Any non-zero element would then work.