Aerothermodynamics of High Speed Flows

Lecture 5:

Nozzle design

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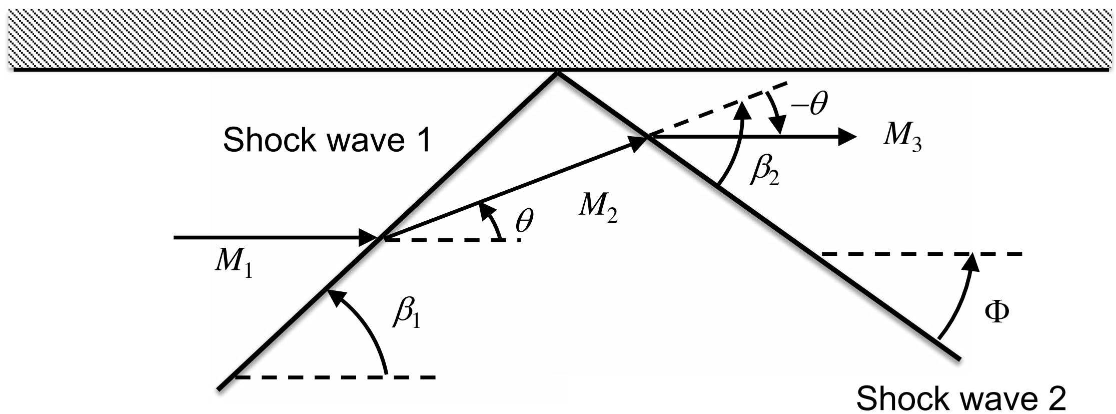
Introduction

- Before talking about nozzle design we need to address a very important issue:
  - Shock reflection
- We have already stated that shocks can exist in nozzles. As nozzles are closed spaces, the shocks will extend to the walls.
- What happens when a shock reaches a wall?
- The same question applies to expansion waves.

Shock reflection

- Consider an oblique shock with angle β_1 reaching a wall.
- The flow boundary condition is impermeability: flow cannot cross the wall.



Discussion

- The flow is deflected by an angle θ behind the shock wave.
- If the shock wave disappears at the wall, the flow will cross the solid boundary.
- Therefore the shock cannot disappear, it must be reflected.
- The reflected shock must deflect the flow by an angle -θ so that the flow remains parallel to the wall.
- Note that $M_2<M_1$. Therefore, the reflected shock is weaker than the original.

Example

- The Mach number upstream of the oblique shock is M_1=2.8 and the angle β_1 is 35°.
- Calculate the angle of the reflected shock wave to the wall Φ.
- Also calculate the Mach number behind the reflected shock M_3.

Solution

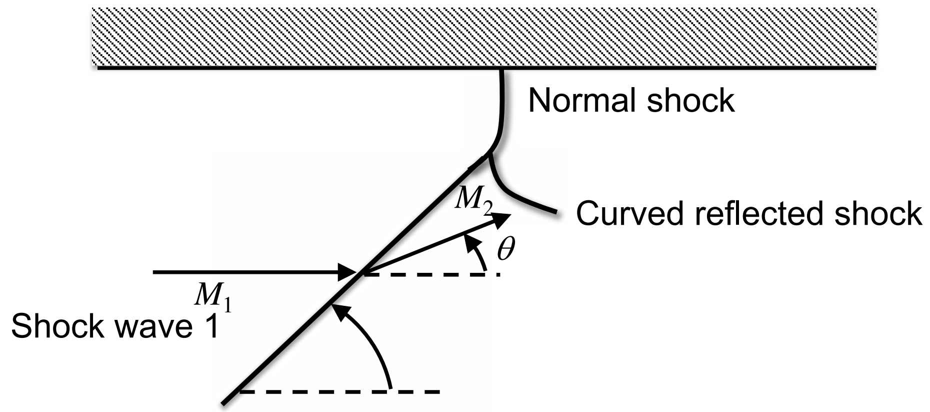
- From the oblique shock tables, for M_1=2.8, β_1=35^o:
  - The flow deflection angle is θ=16^o.
  - The Mach number behind the shock is M_2=2.06.
- The reflected shock must deflect the flow by θ=16^o and the upstream Mach number is 2.06.
- From the oblique shock tables:
  - The shock angle is β_2=45.56^o.
  - The downstream Mach number is M_3=1.45.
  - The angle Φ is $\Phi=\beta_2-\theta$=29.56^o.
- Note that the original shock wave's angle to the wall was 35^o while that of the reflected wave is lower at 29.56^o.
- Shock waves are not deflected at the same angle!

Counter-example

- The original shock wave has an angle $\beta_1=42^o$. The upstream Mach number is still $M_1=2.8$.
- The deflection angle is $\theta=22^o$. The downstream Mach number is $M_2=1.75$.
- The reflected Mach number must deflect the flow by $\theta=22^o$.
- There is no such shock wave for a Mach number of 1.75. The maximum deflection angle is 18.09^o.
- What happens now?

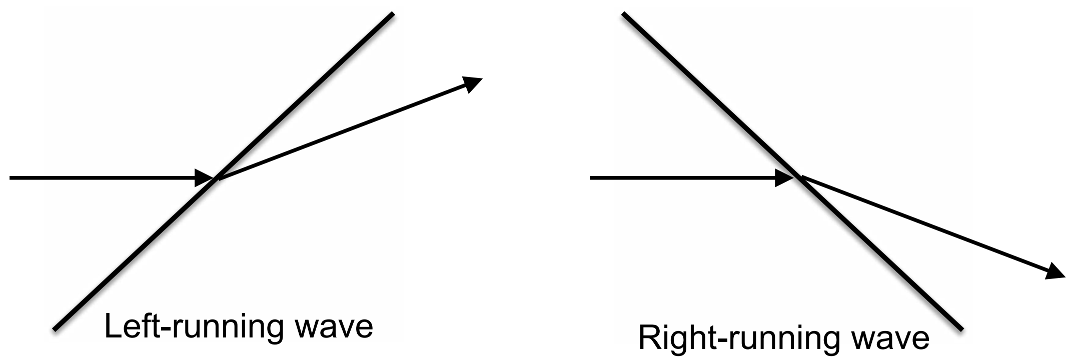
Mach reflection

- The oblique shock cannot reach the wall!
- A normal shock forms at the wall propagating downwards and curving to merge with the oblique shock.
- At the intersection a third curved reflected shock is formed.



Left and right running

- In the previous examples, the initial shock is left-running:
  - An observer standing on the wave and looking downstream sees the wave running off toward the left.
- The reflected shock is right-running:
  - An observer standing on the wave and looking downstream sees the wave running off toward the right.

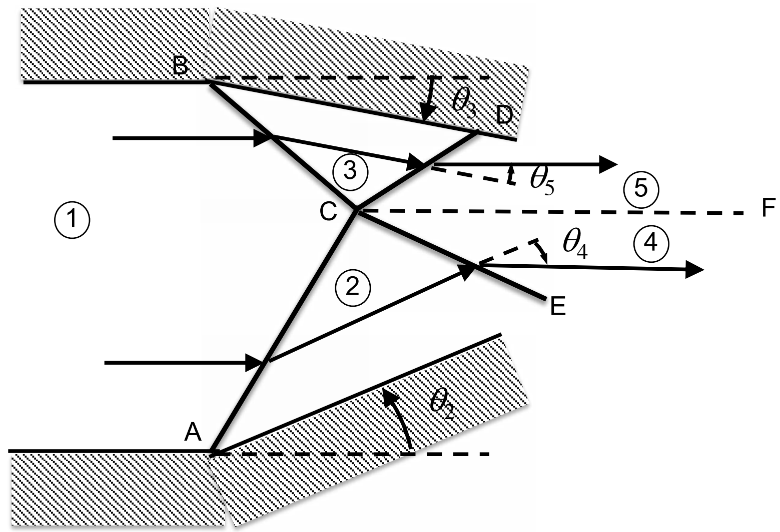


Shock intersection

- Consider the intersection between a left-running and a right-running wave:

The two shocks meet at point 2. There they are refracted into two new waves with different angles.

The line CF is a slip line: Flow cannot cross it. In other words, flow in regions 4 and 5 must be parallel.



Discussion

- The flow must be parallel in regions 4 and 5 because:
  - If the two flow directions converge, they will cross the slip line. Impossible
  - If the the two flow directions diverge, they will leave a vacuum behind them. Impossible.
- In other words, θ_3-θ_5=θ_2-θ_4.
- Furthermore, the pressure in regions 4 and 5 must be equal.
  - If it was not, the slip line would move until the two pressures became equal.
- In other words, p_4=p_5.

Example

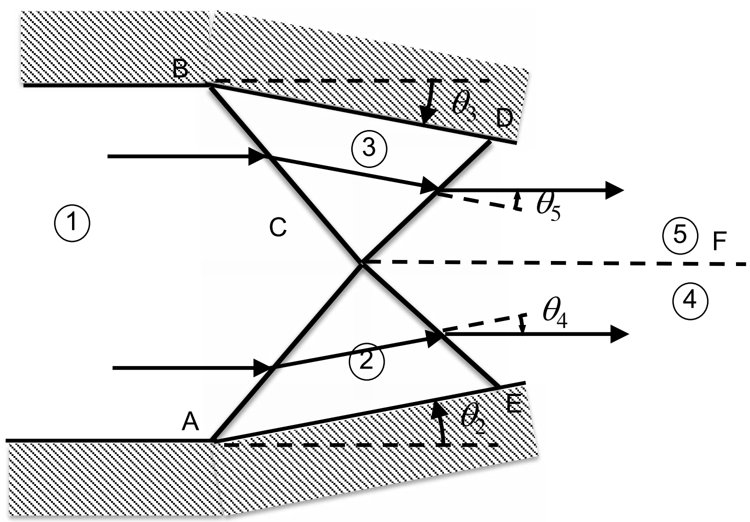
- Consider the diffuser of a supersonic wind tunnel running at $M=3$.
- The diffuser is straight and symmetric with an angle $\theta=14^{\circ}$.
- How many reflections can there be?
- What will be the Mach number at the end of the reflections?
- What is the drop in total pressure?

Solution

- Due to symmetry, the slip line CF must be horizontal.
- It follows that $\theta_2=\theta_3=\theta_4=\theta_5=\theta$.

For calculating the static pressure drop across the oblique shocks, recall that upstream of an oblique shock:

$$M_{n1} = M_1 \sin\beta$$



Solution (2)

- Calculate all the possible shock waves:

$$M_1 = 3, \theta = 14^0 \Rightarrow \beta_{2,3} = 31.24^0, M_{2,3} = 2.30, M_{n1} = 1.56, \frac{p_{0_{2,3}}}{p_{0_1}} = 0.91$$

$$M_{2,3} = 2.30, \theta = 14^0 \Rightarrow \beta_{4,5} = 38.53^0, M_{4,5} = 1.75, M_{n2,3} = 1.43, \frac{p_{0_{4,5}}}{p_{0_{2,3}}} = 0.95$$

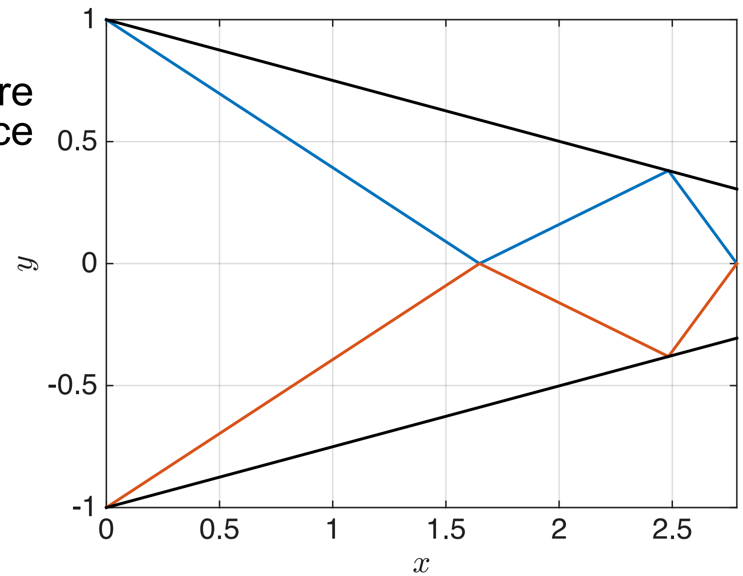
$$M_{4,5} = 1.75, \theta = 14^0 \Rightarrow \beta_{6,7} = 51.59^0, M_{6,7} = 1.23, M_{n4,5} = 1.37, \frac{p_{0_{6,7}}}{p_{0_{4,5}}} = 0.97$$

$$M_{6,7} = 1.23, \theta = 14^0 \Rightarrow \beta_{8,9} = \mathrm{NAN}$$

- One refraction and one reflection are possible.
- At the second refraction, the Mach number does not allow a 14^o flow deflection.

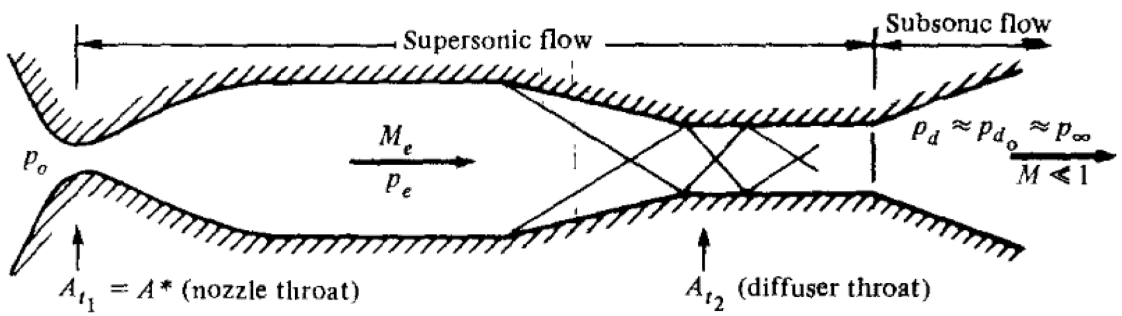
Solution (3)

- At the end of the calculation, the Mach number is 1.23.
- The diffuser total pressure ratio is 0.84 from entrance to exit.
- A Mach 3 normal shock corresponds to a total pressure ratio of 0.33.
- The diffuser has not decelerated the flow to subsonic conditions but has brought it close to sonic with much higher efficiency than a normal shock.



Practical diffusers

- In practical diffusers the last shock is a very weak Mach reflection.
- The normal shock associated with the Mach reflection leads to subsonic flow.
- The diffuser has two section:
  - One highly inclined section where the necessary flow deflections are high.
  - One flatter section where the flow deflections are much lower.
- The sonic throat lies at the end of the flat section.



From J. D. Anderson, Modern Compressible Flow

Example revisited

- Consider the previous diffuser example.
- What would happen if the diffuser angle was reduced to 2 after the first reflection?

$$M_1 = 3, \theta = 14^0 \Rightarrow \beta_{2,3} = 31.24^0, M_{2,3} = 2.30$$

First reflection

$$M_{2,3} = 2.30, \theta = 14^0 \Rightarrow \beta_{4,5} = 38.53^0, M_{4,5} = 1.75$$

$$M_{4,5} = 1.75, \theta = 2^0 \Rightarrow \beta_{6,7} = 36.69^0, M_{6,7} = 1.68$$

$$M_{6,7} = 1.68, \theta = 2^0 \Rightarrow \beta_{8,9} = 39.27^0, M_{8,9} = 1.58$$

$$M_{8,9} = 1.58, \theta = 2^0 \Rightarrow \beta_{10,11} = 40.73^0, M_{10,11} = 1.53$$

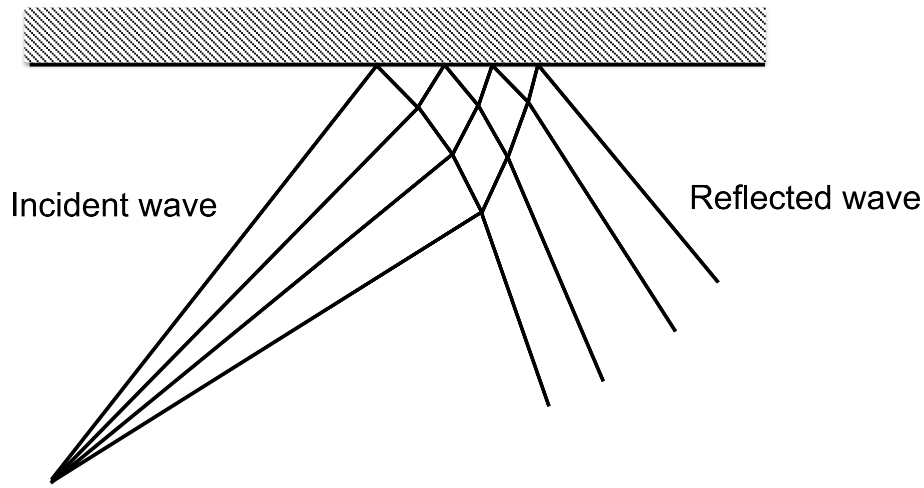
$$M_{10,11} = 1.53, \theta = 2^0 \Rightarrow \beta_{12,13} = 42.32^0, M_{12,13} = 1.48$$

$$M_{12,13} = 1.48, \theta = 2^0 \Rightarrow \beta_{14,15} = 44.07^0, M_{12,13} = 1.43$$

- etc

Expansion wave reflection

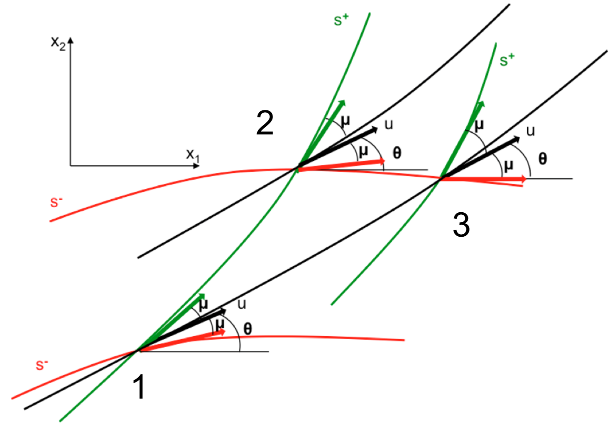
- Expansion waves are also reflected:



- They are curved in the region where the incident and reflected waves intersect:
  - Non-simple region.

Characteristic lines

- Two characteristic lines pass from every point in a supersonic flow, a left running (S^+) and a right running (S^-) characteristic.
- Characteristic lines are Mach lines:
  - The S^+ characteristic has an angle $\theta+\mu$, where μ is the local Mach angle.
  - The S^- characteristic has an angle $\theta-\mu$.
- Characteristics are normally curved. The slope of the characteristic is known at a point where all the flow parameters are known.
- All points on a characteristic line have the same value of $s_{+,-}$.
  - The quantity $s_-=\theta+\nu(M)$ is constant on a left running characteristic.
  - The quantity $s_+=\theta-\nu(M)$ is constant on a right running characteristic.
  - $\nu(M)$ is the Prandtl-Meyer function and θ is the local flow angle.
- In the diagram on the right, points 1 and 2 lie on the same S^+ and points 2 and 3 lie on the same S^- characteristic:
  - $\theta_1-\nu(M_1)=\theta_2-\nu(M_2)$
  - $\theta_2+\nu(M_2)=\theta_3+\nu(M_3)$



Equations

- The important equations to remember are:
  - Prandtl-Meyer function:

$$\nu(M)=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}-\tan^{-1}\sqrt{M^2-1},\ \text{ with } \nu(1)=0$$

  - Mach angle: $\mu=\sin^{-1}\frac{1}{M}$
  - Characteristics: $\begin{aligned}\theta+\nu(M)&=s_-\\ \theta-\nu(M)&=s_+\end{aligned}$ or $\begin{aligned}\theta&=\frac{1}{2}(s_-+s_+)\\ \nu(M)&=\frac{1}{2}(s_--s_+)\end{aligned}$
  - Slope of characteristics:

$$\lambda_{\pm}=\tan(\theta\pm\mu)$$

Linearized characteristics

- Curved characteristics cannot be easily handled.
  - We don't know the shape of the curve unless we solve numerical the flow equation.
- It is possible to simplify the analysis by assuming that characteristic lines are straight line segments.
- This is a very powerful procedure:
  - If we know the flow parameters at two points in the flow, we can calculate the equations of the characteristics that pass through them.
  - We can then calculate the intersection of the S^+ characteristic from the first point with the S^- characteristic from the second.
  - This intersection defines a third point in the flow, at which we can calculate all the flow parameters.
  - We can also calculate the intersection of the S^- characteristic from the first point with the S^+ characteristic from the second.
  - We have now calculated a fourth point in the flow.

Linearized characteristics (2)

- The coordinates, flow angle and Mach number are known at points 1 and 2.

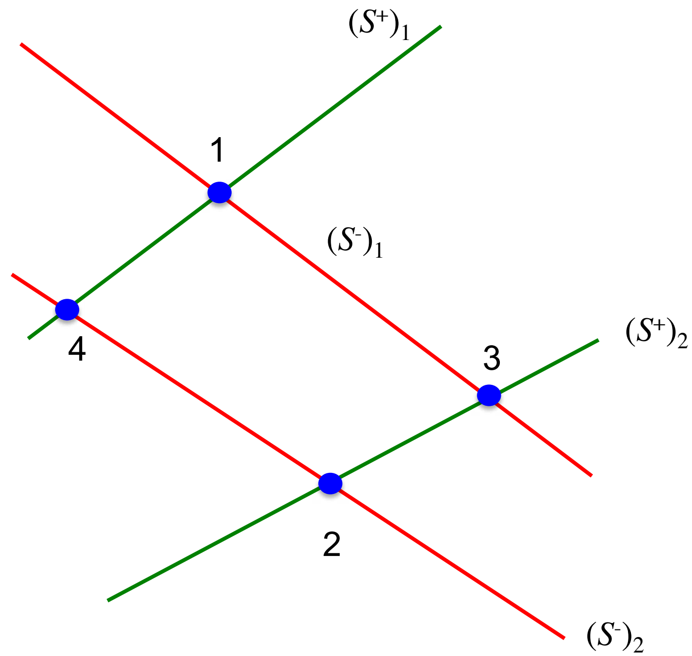
$$x_{1,2}, y_{1,2}, \theta_{1,2}, M_{1,2}$$

- We can calculate the equations of the characteristic lines, i.e.

$$\left(S^{+}\right)_{1,2}:\ y = \tan\left(\theta_{1,2} + \mu_{1,2}\right)\left(x - x_{1,2}\right) + y_{1,2}$$

$$\left(S^{-}\right)_{1,2}:\ y = \tan\left(\theta_{1,2} - \mu_{1,2}\right)\left(x - x_{1,2}\right) + y_{1,2}$$

- The coordinates of point 3 are the intersection of lines $(S^{+})_2$ and $(S^{-})_1$.
- The coordinates of point 4 are the intersection of lines $(S^{+})_1$ and $(S^{-})_2$.



Linearized characteristics (3)

- As an example, the coordinates of point 3 are given by:

$$x_3 = \frac{y_1 - \tan(\theta_1 - \mu_1)x_1 - y_2 + \tan(\theta_2 + \mu_2)x_2}{\tan(\theta_2 + \mu_2) - \tan(\theta_1 - \mu_1)}$$

$$y_3 = \tan(\theta_2 + \mu_2)(x_3 - x_2) + y_2$$

- At point 3:

$$\theta_3 + \nu(M_3) = (s_-)_1 = \theta_1 + \nu(M_1)$$

$$\theta_3 - \nu(M_3) = (s_+)_2 = \theta_2 - \nu(M_2)$$

- These are two equations with two unknowns, θ_3 and M_3:

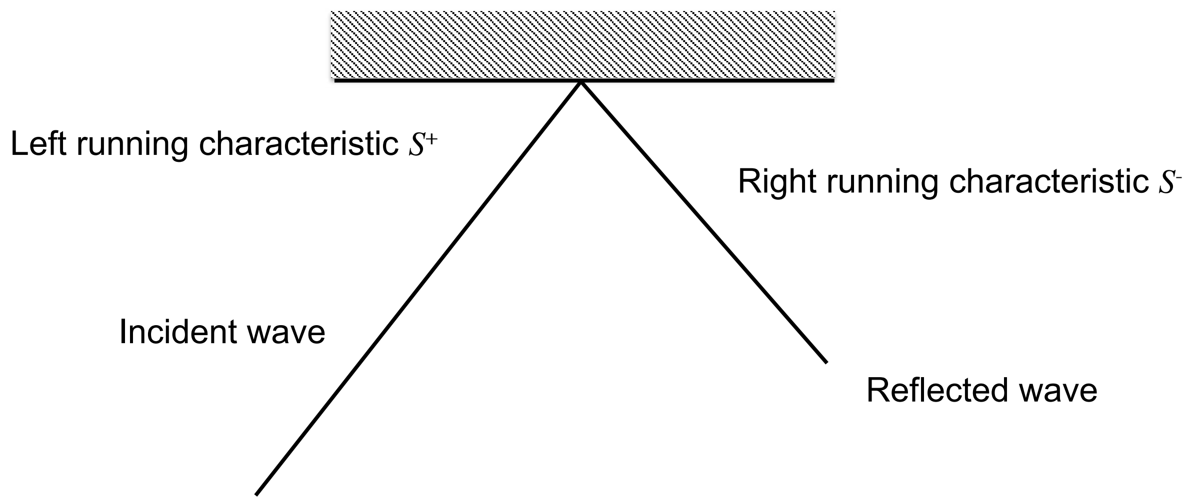
$$\theta_3 = \frac{1}{2}\left((s_-)_1 + (s_+)_2\right)$$

$$\nu(M_3) = \frac{1}{2}\left((s_-)_1 - (s_+)_2\right) \qquad (1)$$

- We have now fully characterized point 3.

Expansion wave reflection

- The reflection of a single expansion wave can be analyzed using the method of characteristics.

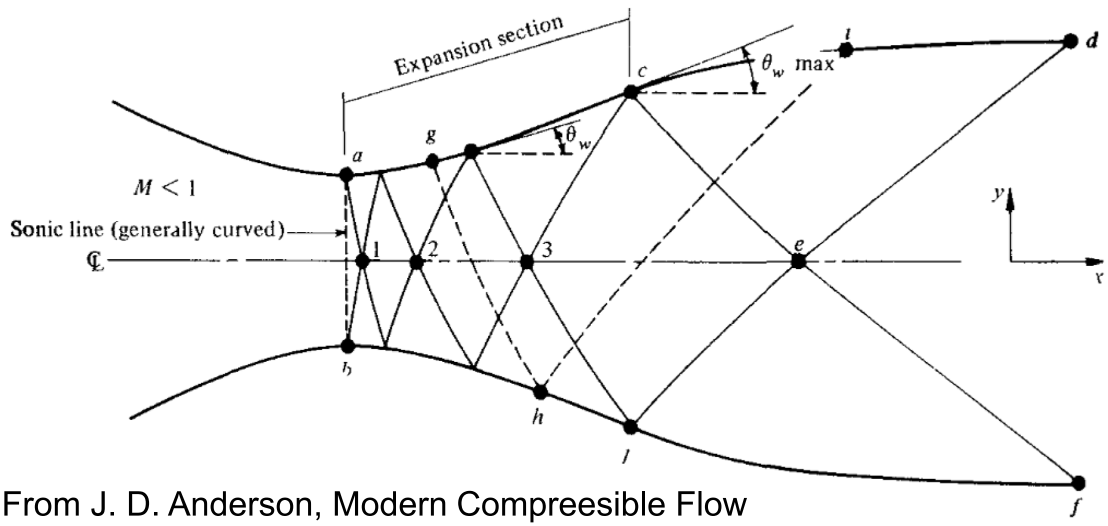


Nozzle design

- Quasi-1D flow can give us a lot of information on flow conditions inside a nozzle.
- However, it cannot be used to design the shape of the nozzle.
- The flow is in reality 2D and must be treated accordingly.
- If the shape of the nozzle is not appropriate, shocks can occur inside it.
- We can use the method of characteristics to design the shape of a nozzle.

Nozzle geometry

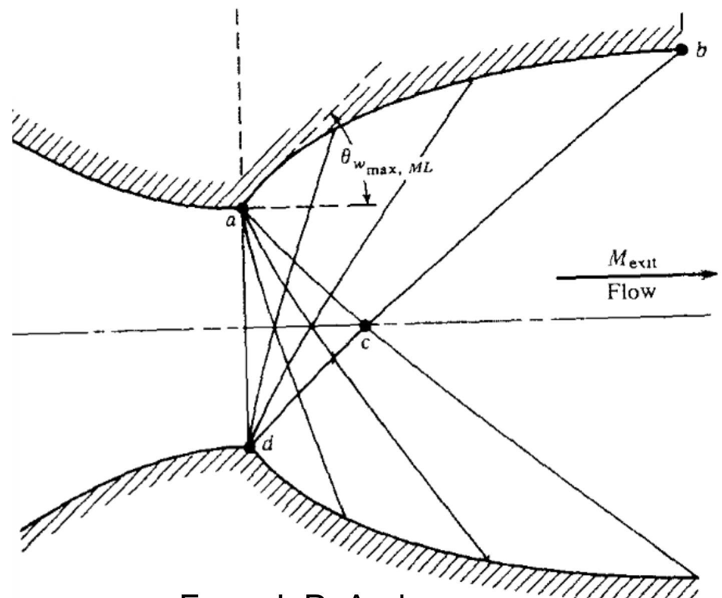
- Nozzles can contain two sections:
  - An expansion section
  - A straightening section



From J. D. Anderson, Modern Compreesible Flow

Minimum length nozzles

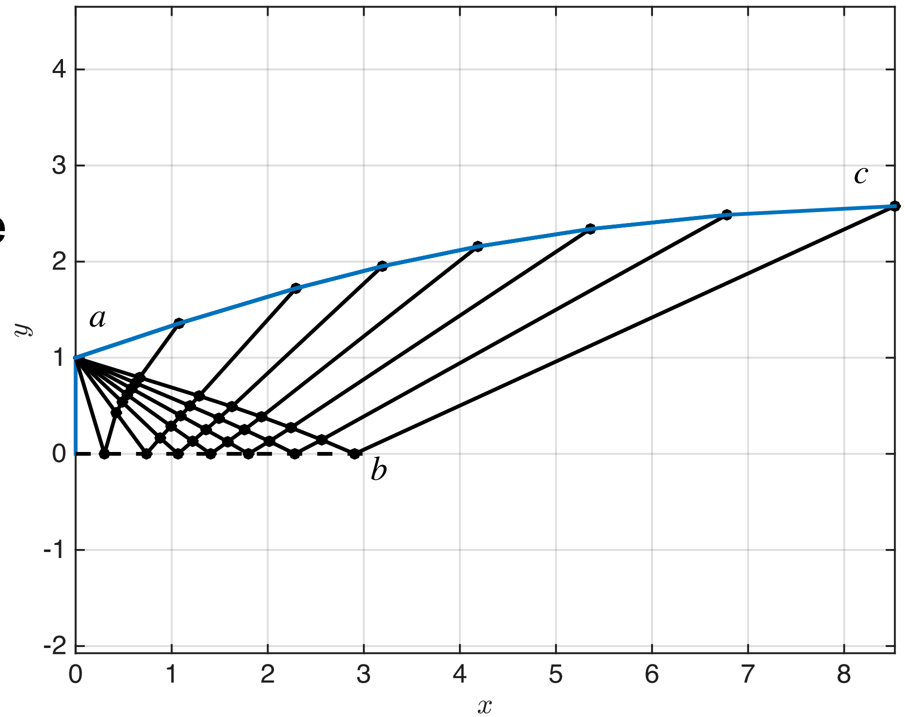
- Nozzles with an expansion section create better quality flow but are long.
  - They are preferred for wind tunnels.
- Minimum length nozzles have no expansion section.
  - They are preferred for rocket engines.



From J. D. Anderson,
Modern Compreesible Flow

Symmetry

- As with diffusers, we can analyze only half of the nozzle if it is symmetric.
- Note that the maximum wall angle, $\theta_{w,\max}$, occurs at the throat.



Mach number and maximum angle

- At point c the Mach number is the design Mach number, M_e, and the flow direction is θ=0°.
- At point b the flow direction is θ=0° because the point lies on the centerline.
- Points b and c lie on the same left running characteristic, i.e.:

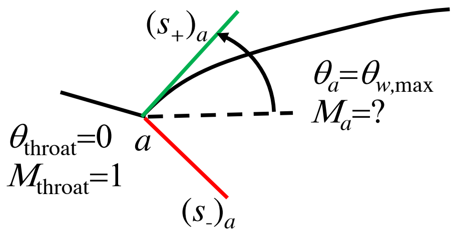
$$\theta_c - \nu(M_c) = \theta_b - \nu(M_b) \Rightarrow \nu(M_b) = \nu(M_e)$$

- Points a and b lie on the same right running characteristic, i.e:

$$\theta_{w,\max} - \nu(M_a) = \theta_b - \nu(M_b) \Rightarrow \theta_{w,\max} - \nu(M_a) = \nu(M_e)$$

Point a

- Consider point a:
  - It is a Prandtl-Meyer expansion.



  - On the s_+ characteristics passing through a:

$$\theta_{throat} - \nu\left(M_{throat}\right) = \theta_a - \nu\left(M_a\right)$$

  - Substituting: $\theta_a = \nu\left(M_a\right)$ or $\nu\left(M_a\right) = \theta_{w,\max}$
  - We can find the Mach number behind the expansion.
  - Note also that $(s_-)_a = \theta_a + \nu\left(M_a\right) = 2\theta_{w,\max}$, $(s_+)_a = \theta_a - \nu\left(M_a\right) = 0$

Maximum nozzle angle

- Consider the right running characteristics passing through points a and b

$$(s_-)_a = 2\theta_{w,\max}$$
$$(s_-)_b = \nu(M_e)$$

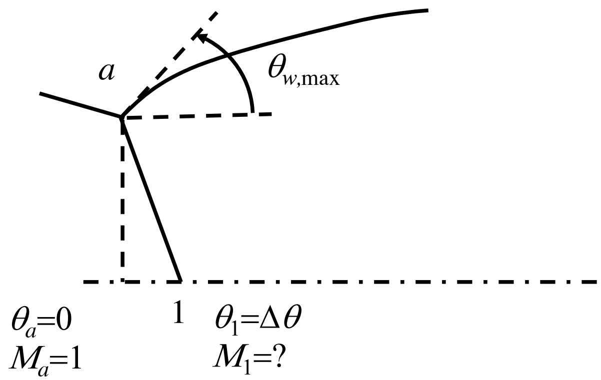
- But the two points lie on the same right running characteristics, so that $(s_-)_a=(s_-)_b$.
- Therefore:

$$\theta_{w,\max} = \frac{\nu(M_e)}{2}$$

- Which means that the maximum nozzle angle is uniquely defined by the design exit Mach number.
- Note that this is only the case for minimum length nozzles.

Point 1

- Now consider point 1, a point on the centerline and close to the throat.



- As the point lies on the centerline, the flow deflection must be zero.
- However, we cannot start the scheme if we set the flow deflection to zero.
- Instead, we choose a small flow deflection $\Delta\theta$.
- Then, $\theta_a - \nu(M_a) = \theta_1 - \nu(M_1) \quad \text{or} \quad \nu(M_1) = \Delta\theta$
- And, hence $(s_-)_1 = \theta_1 + \nu(M_1) = 2\Delta\theta, \ (s_+)_1 = \theta_1 - \nu(M_1) = 0$

Expansion fan

- Now we can draw the entire expansion fan as n lines.
- The flow deflections caused by the expansion linear are equally spaced from $\Delta\theta$ to $\theta_{w,\max}$.
- As they are all characteristic lines passing by point a, they will be characterized by:

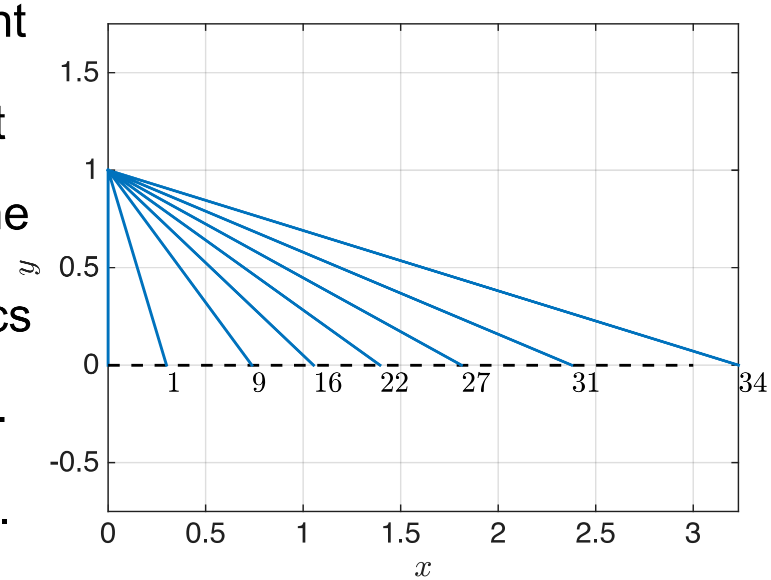
$$\left(s_-\right)_j = 2\theta_j,\ \left(s_+\right)_j = \theta_j - \nu\left(M_j\right) = 0 \qquad (2)$$

- As they are all right-running, the slope of each line is given by:

$$\lambda_{-_j} = \tan\left(\theta_j - \mu_j\right) = \tan\left(\theta_j - \sin^{-1}\frac{1}{M_j}\right)$$

Expansion fan (2)

- The current form of the expansion fan is straight lines.
- However, we know that each characteristic line will be reflected from the centerline.
- Reflected characteristics will interact with the incident characteristics.
- This phenomenon has not been computed yet.

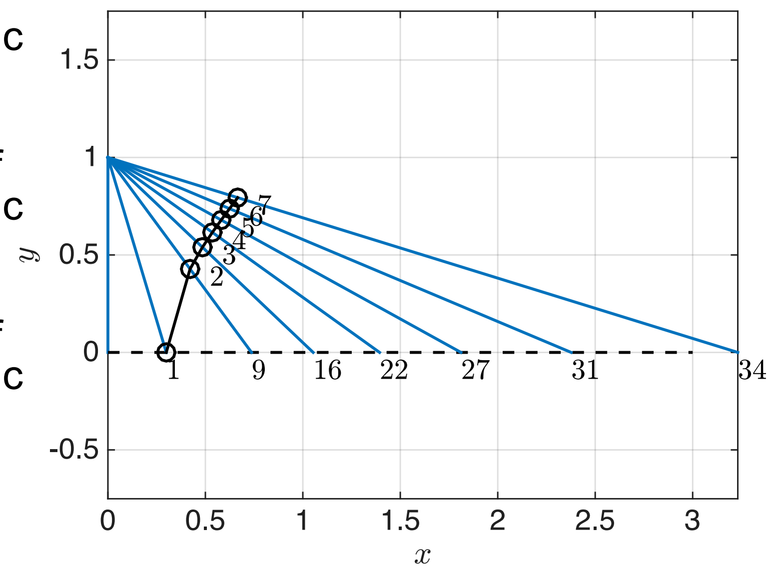


Point 1 (again)

- Two characteristics pass by point 1:
  - The right running with slope $\lambda_{-_1} = \tan(\theta_1 - \mu_1)$
  - The left running with slope $\lambda_{+_1} = \tan(\theta_1 + \mu_1)$
- As point 1 lies on the centerline, the two characteristics describe fully the reflection of the expansion wave:
  - The right running characteristic is the incident wave
  - The left running characteristic is the reflected wave.
- Recall that we decided that $\theta_1 = \Delta\theta$. Therefore the two waves are not reflected at the same angle.

First reflection

- Point 2 is the intersection of the left running characteristic from point 1 with the right running characteristic $(s_{-})_9$.
- Point 3 is the intersection of the left running characteristic from point 2 with the right running characteristic $(s_{-})_{16}$.
- Point 4 is the intersection of the left running characteristic from point 3 with the right running characteristic $(s_{-})_{22}$.
- Etc.



Flow deflections

- Points 1-7 lie on the same left-running characteristic with $(s_+)_1$=0 (see equation 2).
- At point 2 the right-running characteristic is $(s_-)_9 = 2\Delta\theta$ (equation 2 again). Then, from equation 1:

$$\theta_2 = \frac{1}{2}\big((s_-)_9 + (s_+)_1\big) = \Delta\theta$$

$$\nu(M_3) = \frac{1}{2}\big((s_-)_9 - (s_+)_1\big) = \Delta\theta$$

- From equation 2, at point 7, $(s_-)_{34} = 2\theta_{w,max}$ i.e.

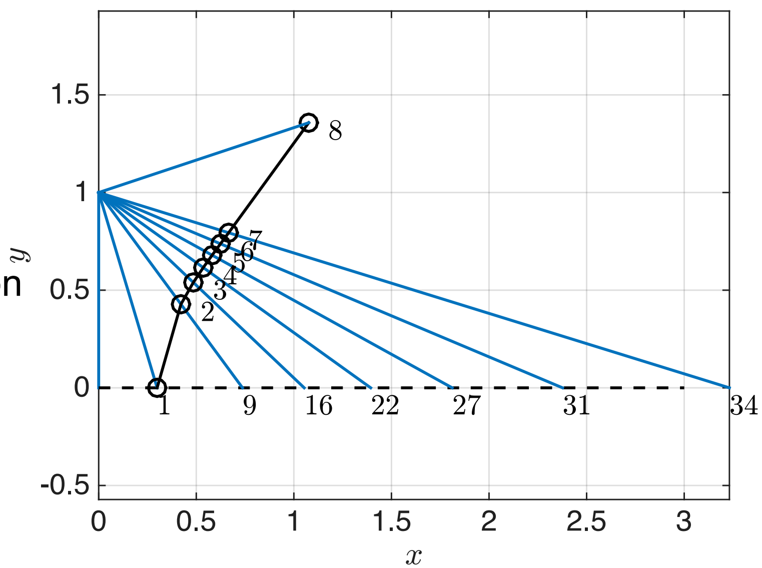
$$\theta_7 = \frac{1}{2}\big((s_-)_{34} + (s_+)_1\big) = \theta_{w,max}$$

$$\nu(M_7) = \frac{1}{2}\big((s_-)_{34} - (s_+)_1\big) = \theta_{w,max}$$

- which means that after the last intersection the flow is already parallel to the wall.

Intersection with the wall

- The point where the reflected characteristic reaches the wall is the intersection between the left running characteristic from point 7 and the wall.
- The wall is modeled as a straight line with slope $\tan\theta_{w,\max}$ starting at point a.
- Point 8 lies at the intersection between a line with slope $\tan\theta_{w,\max}$ starting at point a and the left-running characteristic coming from point 7.
- The flow direction and Mach number at point 8 are identical to those at point 7.



Point 9

- Point 9 has not been properly calculated yet.
- It lies on the same right running characteristic as point 2. Therefore

  $(s_{-})_9=(s_{-})_2$.
- As point 9 lies on the centerline, $\theta=0$. This means that: $(s_{-})_9 = \theta_9 + \nu(M_9) \Rightarrow \nu(M_9) = (s_{-})_2$
- Its left running characteristic is given by

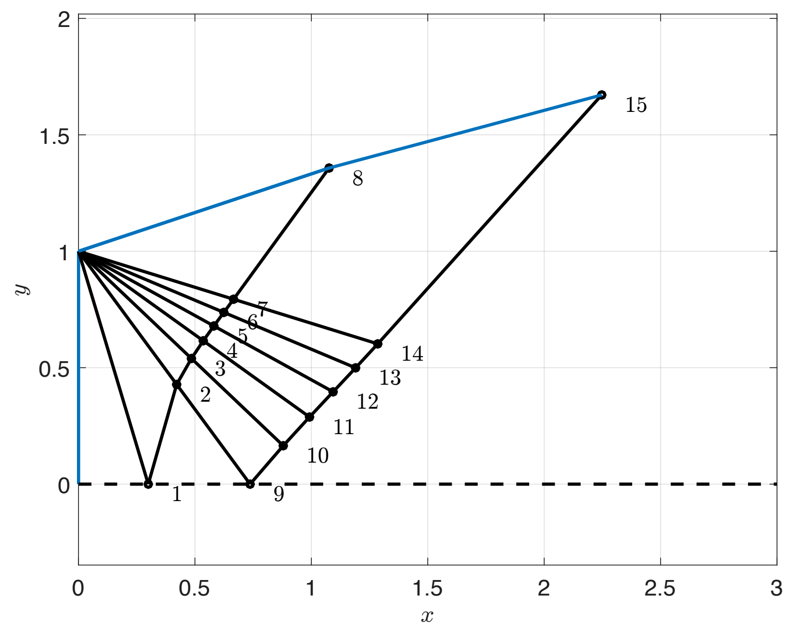
$$(s_{+})_9 = \theta_9 - \nu(M_9) \Rightarrow (s_{+})_9 = -\nu(M_9)$$

Pinpointing points 9-15

- Point 9 is the intersection of the right running characteristic from point 2 and the centerline.
- We can generalize by saying that any point lying on the centerline is the intersection between the right running characteristic from the second point on the previous reflection and the centerline.
- Point 10 is the intersection between the right running characteristic from point 3 and the left running characteristic from point 9.
- And so on until we get to point 14.
- Point 15 lies on the wall. All its values (Mach number, deflection angle, characteristics etc) are equal to those at point 14.
- Point 15 is the intersection of a line with slope $\tan\theta_{14}$ starting at point 8 and the left-running characteristic from point 14.

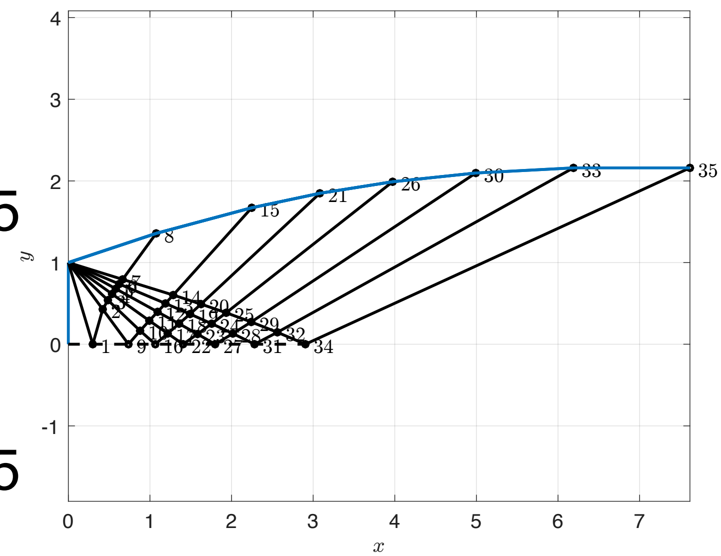
Two reflections

- The result after the reflections of two expansion waves is:
- The wall should not deflect the flow; its slope is always given by the flow direction after the last refraction:
  - Point 8 has slope $\tan\theta_7$
  - Point 15 has sloped $\tan\theta_{14}$
  - Etc.



Complete nozzle

- After n reflections we had the complete design.
- The Mach number at points 34 and 35 is the desired Mach number.
- The flow direction at points 34 and 35 is $\theta_{34}=\theta_{35}=0$.



Discussion

- The method is approximate:
  - Characteristic lines are not truly straight.
  - They are approximated as straight.
- Even the sonic line is curved in reality.
- Better approximations can be obtained using finite differences.