Three-Way Decisions and Three-Way Clustering

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Abstract. A theory of three-way decisions is formulated based on the notions of three regions and associated actions for processing the three regions. Inspired by the theory of three-way decisions, some researchers have further investigated the theory of three-way decisions and applied it in different domains. After reviewing the recent studies on three-way decisions, this paper introduces the three-way cluster analysis. In order to address the problem of the uncertain relationship between an object and a cluster, a three-way clustering representation is proposed to reflect the three types of relationships between an object and a cluster, namely, belong-to definitely, uncertain and not belong-to definitely. Furthermore, this paper reviews some three-way clustering approaches and discusses some future perspectives and potential research topics based on the three-way cluster analysis.

Keywords: Three-way decisions · Three-way clustering · Uncertain Soft clustering

1 Introduction

To model a particular class of human ways of problem solving and information processing, Professor Yao [55] proposed a theory of three-way decisions. The basic ideas of three-way decisions are to divide a universal set into three pair-wise disjoint regions, or more generally a whole into three distinctive parts, and to act upon each region or part by developing an appropriate strategy [57].

The essential ideas of three-way decisions are commonly used in everyday life and widely applied in many fields and disciplines including medical decision-making, social judgement theory, hypothesis testing in statistics, management sciences and peer review process. In the last few years, we have witnessed a fast growing development and applications of three-way approaches in areas of decision making, email spam filtering, clustering analysis and so on [10, 24, 28, 64].

The term "three-way decisions" embraces all aspects of a decision-making process, including tasks such as data and evidence collection and analysis for supporting decision making, reasoning, computing in order to arrive at a particular decision, justification and explanation of a decision. The unique feature

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of three-way decisions is a type of three-way approaches (i.e., the division of a whole into three parts) to problem solving and information processing. We may replace "decisions" in "three-way decisions" by other words to have specific interpretations such as three-way computing, three-way processing, three-way classification, three-way analysis, three-way clustering, three-way recommendation, and many others [59].

2 Reviews on Three-Way Decisions

The main idea of three-way decisions is to divide a universe into three disjoint regions and to process the different regions by using different strategies. By using notations and terminologies of rough set theory [38,39,58], we give a brief description of three-way decisions as follows [64].

Suppose U is a finite nonempty set of objects or decision alternatives and D is a finite set of conditions. Each condition in D may be a criterion, an objective, or a constraint. The problem of three-way decisions is to divide, based on the set of conditions in D, U into three pair-wise disjoint regions by a mapping f:

$$f : U \longrightarrow \{\mathrm{RI}, \mathrm{RII}, \mathrm{RIII}\}. \tag{1}$$

The three regions are called Region I, Region II, and Region III, respectively.

Depending on the construction and interpretation of the mapping f, there are qualitative three-way decisions and quantitative three-way decisions. In qualitative three-way decision models, the universe is divided into three regions based on a function f that is of a qualitative nature. Quantitative three-way decision models are induced by that is of a quantitative nature. An evaluation-based three-way decision model uses an evaluation function that measures the desirability of objects with reference to the set of criteria.

It should be pointed out that we can have a more general description of three-way decisions by using more generic labels and names. For example, in an evaluation-based model of three-way decisions [55], we can use a pair of thresholds to divide a universe into three regions. If we arrange objects in an increasing order with lower values at left, then we can conveniently label the three regions as the left, middle, or right regions, respectively, or simply L, M, and R regions [64]. In a similar way, strategies for processing three regions can be described in more generic terms [5,6,57].

Originally, the concept of three-way decisions was proposed and used to interpret probabilistic rough set three regions. Further studies show that a theory of three-way decisions can be developed by moving beyond rough set theory. In fact, many recent studies go far beyond rough sets. In order to go further insights into three-way decisions and promote further research, this paper gives a brief review on the studies of three-way decisions from the following respects.

- Cost-sensitive sequential three-way decisions. Three-way decisions originate from the studies on the decision-theoretic rough set (DTRS) model. The DTRS presents a semantics explanation on how to decide a concept into positive, negative and boundary regions based on the minimization of the decision cost, rather than decision error. Li et al. [11] incorporated the three-way decisions into cost-sensitive learning and proposed a three-region cost-sensitive classification. It is evident that the boundary decision may achieve lower cost/risk than positive and negative decisions do, if available information for immediate decision is insufficient, which is consistent with human decision process [9,11]. Based on the DTRS, Ju et al. [26] constructed a generalized framework of cost-sensitive rough set with test cost and decision cost simultaneously, and further introduced multi-granulation DTRS into this field and proposed the cost-sensitive multi-granulation rough set model by considering two different costs [27], which enriches the semantics interpretation of cost-sensitive models based on the DTRS.

In real-world applications, the available information is always insufficient, or it may associate with extra costs to get available information, which leads to frequent boundary decision. However, if the available information continuously increases, the previous boundary decisions may be converted to positive or negative decisions, which forms a sequential decision process [54,56]. Li et al. proposed a cost-sensitive sequential three-way decision strategy [12,15], and introduced the method to handle the imbalance of misclassification cost and the insufficient of image information [13], and further investigated deep neural networks based on sequential granular feature extraction [14]. Considering the multilevel granular structure of real-world problems, Yang et al. proposed a unified model of sequential three-way decisions and multilevel incremental processing for complex problem solving [74].

- Determining the thresholds. Compared to two-way decisions approaches, three-way decisions approaches introduce deferment decision through a pair of thresholds (α, β). Therefore, for the three-way decision models, a great challenge is acquirement of a set of pairs of thresholds (α, β). Thus, Shang and Jia [23,25] studied this problem from an optimization viewpoint, in which the thresholds and corresponding cost functions for making three-way decisions can be learned from given data without any preliminary knowledge [17]. Zhang and Zou et al. [82] proposed a cost-sensitive three-way decisions model based on constructive covering algorithm (CCA); Zhang and Xing et al. [83] introduced CCA to the three-way decisions procedure and proposed a new three-way decisions model based on CCA to obtain POS, NEG and BND automatically.

Yao and his group explored the use of game-theoretic rough set (GTRS) model to handle thresholds determination issue. Afridi et al. [1] constructed a three-way clustering approach for handling missing data by introducing a method of thresholds determination based on a tradeoff game between the properties of accuracy and generality of clusters. Besides, Zhang and Yao applied GTRS in multi-criteria based three-way classification problem [76]. By considering probabilistic rough sets based models of game-theoretic rough sets for inducing three-way decisions, Rehman et al. [44] proposed an architecture of protein functions classification with probabilistic rough sets based three-way decisions.

- Three-way decisions with DTRS. Considering that incomplete data with missing values are very common in many data-intensive applications. Luo et al. [32] proposed an incremental approach for updating probabilistic rough approximations, with the variation of objects in an incomplete information system. Yang et al. [71] proposed the notions of weighted mean multi-granulation decision-theoretic rough set, optimistic multi-granulation decision-theoretic rough set, and pessimistic multi-granulation decision-theoretic rough set in an incomplete information system. Based on the DTRS, Liu et al. [30] proposed a novel three-way decision model by defining a new relation to describe the similarity degree of incomplete information.

Recently, Yao [60] have extended the theory of three-way decisions to the framework of interval sets and the corresponding three-way concept analysis in incomplete contexts. Li et al. [16] studied three-way cognitive concept learning via multi-granularity, and designed a three-way cognitive computing system which is in fact a dynamic process to update three-way granular concepts. Li et al. [13] simulated the human decision-making process, and proposed a dynamic sequential three-way decision method for cost-sensitive face recognition, by considering available information increases continuously. To deal with the problem of incremental overlapping clustering, Yu et al. [65] designed a dynamic three-way decision strategy to update the clustering when the data increase. Liu et al. [29] considered the dynamic change of loss functions in the DTRS with the time, and further proposed the dynamic three-way decision model. Zhang et al. [80] introduced a new three-way decision model based on dynamic decision making with the updating of attribute values.

- Three-way attribute reduction. The combination of three-way decisions and attribute reducts has theoretical significance and applicable prospects. In this regard, Chen et al. [3] discussed reduction issue based on three-way decisions in neighborhood rough sets. By utilizing double-quantitative measure, Zhang et al. [81] established a hierarchical reduct system, including qualitative/quantitative reducts, tolerant/approximate reducts. Furthermore, Zhang et al. [79] introduced three-way decisions into attribute reducts, and constructed a novel framework of three-way attribute reducts, aiming to directly quantify the final reduction action. Ren and Wei [45] studied three-way concept analysis, and proposed an approach for attribute reductions of three-way concept lattices. Ma and Yao [36] gived a general definition of class-specific attribute reducts, and thus, introduced the class-specific attribute reducts framework on the perspective of three-way decision.
- Three-way decisions and other theories. There are lots of excellent results on the combination of three-way decisions and other theories such as Dempster-Shafer theory, fuzzy sets, formal concept analysis and so on.

Wang et al. [51] proposed a Dempster-Shafer theory based intelligent three-way group sorting method. Zhao and Hu [75]investigated fuzzy and interval-valued fuzzy probabilistic rough sets and proposed their corresponding three-way decisions models, which are appropriate for fuzzy events. Hu [21] established the framework of three-way decisions spaces based on partially order sets and studied three-way decisions based on hesitant fuzzy sets [21,22]. In order to generate decision rules in incomplete information systems, Yang and Tan [72] constructed the evaluation function by combining the intuitionistic fuzzy set and the three-way decisions. To overcome the limitation of the existing three-way decisions models in uncertainty environment, Zhai et al. [78] extended the rough fuzzy set to tolerance rough set, thus, proposed the three-way decisions model based on tolerance rough fuzzy sets. Based on linguistic information-based decision-theoretic rough fuzzy sets, Sun et al. [49] established the corresponding three-way decisions approach to solve multiple attribute group decision problem. To mine three-way concepts to support three-way decisions in formal context, Li et al. [16] studied three-way cognitive concept learning via multi-granularity. Qi et al. [42] proposed the three-way concept analysis based on combining three-way decisions [55] and formal concept analysis [7]. Besides, Ren and Wei investigated the attribute reductions method over three-way concept lattices [45]. Aimed at analyzing the uncertainty and incompleteness in single-valued neutrosophic set, Singh [47] proposed three-way formal fuzzy concept lattice representation. With the issue of three-way concept lattices construction, Qian et al. [43] proposed approaches to create the three-way concept lattices based on the concept lattices of Type I-combinatorial context and Type II-combinatorial context. Yu et al. [73] made efforts on characterizing three-way concept lattices and three-way rough concept lattices, which enriched the theory of three-way concept lattices.

- Applications on three-way decisions. Since the theory of three-way decisions has been proposed, scholars have applied the idea to different applications. Yu and her group studied overlapping clustering [61], determining the number of clusters [62], incremental clustering [65] and so on, based on the three-way decision theory. They also applied the idea to refine and detect social community [66]. Min and his group applied three-way decisions to the incremental mining of frequent itemsets [18,37]. Shang and Jia combined the three-way decisions solution with text sentiment analysis to improve the performance of sentiment classification [85]. Miao and his group applied three-way decision into Chinese emotion recognition [50], and achieved an excellent result. Zhang and Wang studied the issue of sentiment uncertainty analysis, and applied three-way decisions to sentiment classification with sentiment uncertainty [77], with considering the scenarios of context dependent sentiment classification and topic-dependent sentiment classification. In order to solve multi-label sentiment classification, Ren and Wang [46] proposed the method of three-way decisions to recognize the multi-label sentiment orientation of Chinese text. Li and his group utilized cost-sensitive sequential three-way decision to face recognition [13]. Miao and his group proposed a novel algorithm for image segmentation with noise in the framework of decision-theoretic rough set model [8]. Three-way decisions also have been adapted to solve group decision making problem, by combining with theories of decision-theoretic rough sets [35], two universes fuzzy decision-theoretic rough set [48], cloud model [20] and prospect theory [31]. Moreover, the theory of three-way decisions has also been used in other fields such as email spam filtering [84] and recommender system [19].

3 Clustering Approaches for Uncertain Relationships Between Objects and Clusters

The task of cluster analysis or clustering is to group similar objects into the same cluster and dissimilar objects into different clusters. Obviously, there are three relationships between an object and a cluster: (1) the object certainly belongs to the cluster, (2) the object certainly does not belong to the cluster, and (3) the object might or might not belong to the cluster. It is a typical three-way decision processing to decide the relationship between an object and a cluster. Such relationships will inspire us to introduce the three-way decisions into the cluster analysis problem.

In the existing clustering approaches, some approaches such as fuzzy clustering, rough clustering and interval clustering, have been proposed to deal with this kind of uncertain relationship between objects and clusters. Sometimes, we also say that these approaches are soft clustering or overlapping clustering based on the meaning that an object can belong to more than one cluster. In other words, soft clustering technologies aim to relax the hard boundary of clusters by soft constraints, so that it can deal with problems such as overlapping clusters, outliers and uncertain objects [41].

Fuzzy c-means (FCM) is a method of clustering which allows an object to belong to more than one cluster. In the FCM, similarities between objects and each cluster are described by membership degrees based on the fuzzy sets theory, and all objects are assigned to k fuzzy clusters. However, it cannot get an exact representation of clusters by fuzzy sets. To solve this issue, Lingras and Peters [34] applied the rough sets theory to clustering, they presented a new cluster representation that an object can belong to multiple clusters with the concepts of lower and upper approximations. In rough clustering, every cluster might have the fringe region (boundary region) to decrease cluster errors. Objects in fringe regions need more information so that they can be assigned to certain clusters eventually. Next, they combined rough sets to k-means and proposed the rough k-means clustering which each cluster is described by a lower and upper approximation. Since changes in general lead to uncertainty, the appropriate methods for uncertainty modeling are needed in order to capture, model, and predict the respective phenomena considered in dynamic environments, Peters et al. [40] proposed the dynamic rough clustering to detect changing data structures. In addition, Lingras and Yan [33] developed fuzzy clustering by combining rough clustering, in which a cluster is represented by a lower and upper approximation and two thresholds α and β are used to divide the two approximations.

Considering clusters presented as interval sets with lower and upper approximations in rough k-means clustering are not adequate to describe clusters, Chen and Miao [2] proposed an interval set clustering based on decision theory.

The rough sets theory has played an important role in dealing with uncertainty. Yao introduced the Bayes risk decision-making into rough sets and proposed the decision-theoretic rough set model, then proposed the concept of three-way decisions [53]. The theory of three-way decisions extends binary-decisions in order to overcome some drawbacks of binary-decisions. Inspired by the three-way decisions, Yu [68] proposed a framework of three-way cluster analysis. The three-way clustering redefines the clustering representation and has been applied to dealing with some problems such as overlapping incremental clustering [65], community detection [66] and high-dimensional data clustering [67]. Similar to rough clustering using a pair of lower and upper approximations to represent a cluster, three-way clustering describes a cluster by a pair of sets. Generally speaking, rough clustering usually restricts to the rough k-means and its extension algorithms. The intersections between any two core regions do not have to be empty in the three-way clustering, it is different to that the intersection between any two lower approximations is empty in rough clustering. For example, we have shown some real-world cases in the reference [66], in which some objects are core elements of two communities. Usually, uncertain objects in fringe regions need further treatment in three-way clustering when further information can be obtained.

In the above, we have discussed the existing approaches for dealing with uncertain relationships. Rough clustering and interval clustering can also be regarded as the approaches of three-way decisions in some sense, in which the fringe objects are described well.

4 Three-Way Cluster Analysis

In cluster analysis, we need to solve two essential problems. One is how to represent a cluster. Another one is how to obtain the clusters, namely, how to develop clustering algorithms. In this section, this paper will introduce a novel framework of three-way cluster analysis. The basic idea of three-way clustering concludes two aspects: (1) the result of clustering is three-way, and (2) the three-way decision strategy is used during the process of clustering.

4.1 Representation of Three-Way Clustering

Let $U = \{\mathbf{x}_1, \cdots, \mathbf{x}_n, \cdots, \mathbf{x}_N\}$ be a finite set, called the universe or the reference set. $\mathbf{x}_n$ is an object which has D attributes, namely, $\mathbf{x}_n = (x_n^1, \cdots, x_n^d, \cdots, x_n^D)$. x_n^d denotes the value of the d-th attribute of the object $\mathbf{x}_n$, where $n \in \{1, \cdots, N\}$, and $d \in \{1, \cdots, D\}$.

The result of clustering scheme $\mathbf{C} = \{C^1, \cdots, C^k, \cdots, C^K\}$ is a family of clusters of the universe, in which K means this universe is composed of K clusters. According to Vladimir Estivill-Castro, the notion of a "cluster" cannot be precisely defined, which is one of the reasons why there are so many clustering algorithms [4]. There is a common denominator: a group of data objects. In the existing works, a cluster is usually represented by a single set, namely, $C^k = \{\mathbf{x}_1^k, \cdots, \mathbf{x}_i^k, \cdots, \mathbf{x}_{|C^k|}^k\}$, abbreviated as C without ambiguous.

From the view of making decisions, the representation of a single set means, that the objects in the set belong to this cluster definitely and the objects not in the set do not belong to this cluster definitely. This is a typical result of two-way decisions. For hard clustering, one object just belongs to one cluster; for soft clustering, one object might belong to more than one cluster. However, this representation cannot show which objects might belong to this cluster, and it cannot intuitively show the influence degree of the object during the processing of forming the cluster. Obviously, the use of three regions to represent a cluster is more appropriate than the use of a crisp set, which also directly leads to three-way decisions based interpretation of clustering.

In contrast to the general crisp representation of a cluster, we represent a three-way cluster C as a pair of sets:

$$C = (Co(C), Fr(C)). \tag{2}$$

Here, $Co(C) \subseteq U$ and $Fr(C) \subseteq U$. Let $Tr(C) = U - Co(C) - Fr(C)$. Then, $Co(C)$, $Fr(C)$ and $Tr(C)$ naturally form the three regions of a cluster as Core Region, Fringe Region and Trivial Region respectively. If $\mathbf{x} \in Co(C)$, the object $\mathbf{x}$ belongs to the cluster C definitely; if $\mathbf{x} \in Fr(C)$, the object $\mathbf{x}$ might belong to C; if $\mathbf{x} \in Tr(C)$, the object $\mathbf{x}$ does not belong to C definitely. These subsets have the following properties.

$$\begin{aligned} &U = Co(C) \cup Fr(C) \cup Tr(C), \\ &Co(C) \cap Fr(C) = \emptyset, \\ &Fr(C) \cap Tr(C) = \emptyset, \\ &Tr(C) \cap Co(C) = \emptyset. \end{aligned} \tag{3}$$

If $Fr(C) = \emptyset$, the representation of C in Eq. (2) turns into $C = Co(C)$; it is a single set and $Tr(C) = U - Co(C)$. This is a representation of two-way decisions. In other words, the representation of a single set is a special case of the representation of three-way cluster.

Furthermore, according to Formula (3), we know that it is enough to represent expediently a cluster by the core region and the fringe region.

In another way, for $1 \leq k \leq K$, we can define a cluster scheme by the following properties:

$$\begin{aligned} &(i) \text{ for } \forall k,\ Co(C^k) \neq \emptyset; \\ &(ii)\ \bigcup_{k=1}^{K} (Co(C^k) \cup Fr(C^k)) = U. \end{aligned} \tag{4}$$

Property (i) implies that a cluster cannot be empty. This makes sure that a cluster is physically meaningful. Property (ii) states that any object of U must definitely belong to or might belong to a cluster, which ensures that every object is properly clustered.

With respect to the family of clusters, $\mathbf{C}$, we have the following family of clusters formulated by three-way representation as:

$$\mathbf{C} = (\{Co(C^1), Fr(C^1)), \cdots, (Co(C^k), Fr(C^k)), \cdots, (Co(C^K), Fr(C^K))\}. \tag{5}$$

Obviously, we have the following family of clusters formulated by two-way decisions as:

$$\mathbf{C} = \{Co(C^1), \cdots, Co(C^k), \cdots, Co(C^K)\}. \tag{6}$$

Under the representation, we can formulate the soft clustering and hard clustering as follows. For a clustering, if there exists $k \neq t$, such that

$$\begin{array}{l} (1)\ Co(C^k) \cap Co(C^t) \neq \emptyset,\ \textit{or} \\ (2)\ Fr(C^k) \cap Fr(C^t) \neq \emptyset,\ \textit{or} \\ (3)\ Co(C^k) \cap Fr(C^t) \neq \emptyset,\ \textit{or} \\ (4)\ Fr(C^k) \cap Co(C^t) \neq \emptyset, \end{array} \tag{7}$$

we call it is a soft clustering; otherwise, it is a hard clustering.

As long as one condition of Eq. (7) is satisfied, there must exist at least one object belonging to more than one cluster.

Obviously, the representation of three-way brings the following advantages: the representation of a single set is a special case of the representation of three-way cluster; it intuitively shows that which objects are core of the cluster, and which ones are fringe of the cluster; it diversifies the type of overlapping; and it reduces the searching space when focusing on the overlapping/fringe objects.

4.2 An Evaluation-Based Three-Way Cluster Model

In this subsection, we will introduce an evaluation-based three-way cluster model, which produces three regions by using an evaluation function and a pair of thresholds on the values of the evaluation function. The model partially addresses the issue of trisecting a universal set into three regions.

Suppose there are a pair of thresholds (α, β) and $\alpha \geq \beta$. Although evaluations based on a total order are restrictive, they have a computational advantage. One can obtain the three regions by simply comparing the evaluation value with a pair of thresholds. Based on the evaluation function $v(\mathbf{x})$, we get the following three-way decision rules:

$$\begin{array}{l} Co(C^k) = \{x \in U | v(\mathbf{x}) > \alpha\}, \\ Fr(C^k) = \{x \in U | \beta \leq v(\mathbf{x}) \leq \alpha\}, \\ Tr(C^k) = \{x \in U | v(\mathbf{x}) < \beta\}. \end{array} \tag{8}$$

Yao proposed an evaluation-based three-way decisions model in the reference [57]. Naturally, an similar evaluation-based three-way cluster model is depicted in Fig. 1. We can divide the universe U according to Eq. 8 and design different strategies to process the three regions.

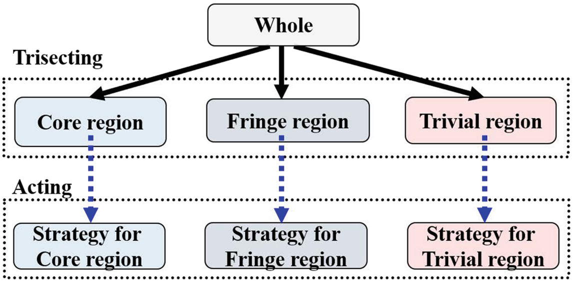

Fig. 1. An Evaluation-based Three-way Cluster Model

Based on the model, we have to pay attention to the following three points.

– About the evaluation function $v(\mathbf{x})$. It will be specified accordingly when an algorithm is devised. In fact, in order to devise the evaluation function, we can refer to the similarity measures or distance measures, probability, possibility functions, fuzzy membership functions, Bayesian confirmation measures, subsethood measures and so on.
– About the three-way thresholds α and β. For an evaluation-based model, we need to investigate ways to compute and to interpret a pair of thresholds. An optimization framework can be designed to achieve such a goal. That is, a pair of thresholds should induce a trisection that optimizes a given objective function. By designing different objective functions for different applications, we gain a great flexibility.
– The three-way decision strategy used during the process of clustering. Shortly, it concludes two aspects such as how to get the three-regions of a cluster and how to act on the three regions.

Of course, the previous two items serve to the third item. In other words, the basic research issues of three-way clustering are about how to obtain the three regions and how to act on the three regions, which is similar to the researches on three-way decisions.

4.3 Some Researches on Three-Way Clustering

In this subsection, I will summarize and discuss some issues and research points about the three-way clustering.

- Representation of three-way clustering. As discussed in Sect. 4.1, we can use a pair of sets to represent a cluster in three-way representation. Some works have been proposed in view of rough sets [34], interval sets [2], decision-theoretic rough sets [62] and mathematical morphology [52]. We can also represent the model of three-way clustering by using fuzzy set, shadow sets and other models. Different interpretations of three-way clustering could give different solutions to different kinds of clustering problems.

- How to get the three-way clustering. It is a good way to extend from the classical two-way decisions clustering approaches. The following properties are important to the efficiency and effectiveness of a novel algorithm: how to decide the thresholds, how to know the truth number of clusters. Yu et al. [69] proposed a method to determine the thresholds automatically based on gravitational search during the processing of clustering.
- Developing new clustering approaches for more uncertainty situations such as dynamic, incomplete data or multi-source data. For example, we had proposed a tree-based three-way clustering method for incremental overlapping clustering [65], a three-way decisions clustering algorithm for incomplete data based on attribute significance and miss rate [63], a semi-supervised three-way clustering framework for multi-view data [70], a three-way decision clustering approach for high dimensional data [67], and so on [68].
- Application of three regions. We can put forward the three-way clustering strategy to the application fields such as social network services, cyber marketing, E-commerce, recommendation service and other fields. Through the further work on the fringe region, we can know the influence degree of the object during the processing of forming the cluster, which is very helpful in some practical applications. For example, Yu et al. [66] have presented a method to detect and refine overlapping regions in complex networks by three-way clustering.

5 Conclusions

The notion of three-way decisions was introduced for meeting the needs to properly explain three regions of probabilistic rough sets. The theory of three-way decisions moves far beyond this original goal. We have seen a more general theory that embraces ideas from many fields and disciplines. This paper introduces most of recent studies on three-way decisions, in order to demonstrate the value and power as well as the great potentials of three-way decisions. For purpose of giving an example of researches related to three-way decisions, a three-way cluster analysis approach is introduced in this paper, which mainly addresses the problem that the uncertain relationship between an object and a cluster.

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References

1. Afridi, M.K., Azam, N., Yao, J.T., Alanazi, E.: A three-way clustering approach for handling missing data using GTRS. Int. J. Approximate Reasoning **98**, 11–24 (2018)
2. Chen, M., Miao, D.Q.: Interval set clustering. Expert Syst. Appl. **38**(4), 2923–2932 (2011)

3. Chen, Y.M., Zeng, Z.Q., Zhu, Q.X., Tang, C.H.: Three-way decision reduction in neighborhood systems. Appl. Soft Comput. **38**, 942–954 (2016)
4. Estivill-Castro, V.: Why so many clustering algorithms: a position paper. ACM SIGKDD Explor. Newslett. **4**(1), 65–75 (2002)
5. Gao, C., Yao, Y.Y.: Actionable strategies in three-way decisions. Knowl.-Based Syst. **133**, 141–155 (2017)
6. Gao, C., Yao, Y.: Actionable strategies in three-way decisions with rough sets. In: Polkowski, L., et al. (eds.) IJCRS 2017. LNCS (LNAI), vol. 10314, pp. 183–199. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-60840-2_13
7. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer, Heidelberg (1999). https://doi.org/10.1007/978-3-642-59830-2
8. Li, F., Miao, D.Q., Liu, C.H., Yang, W.: Image segmentation algorithm based on the decision-theoretic rough set model. CAAI Trans. Intell. Syst. **9**(2), 143–147 (2014)
9. Li, H.X., Zhou, X.Z.: Risk decision making based on decision-theoretic rough set: a three-way view decision model. Int. J. Comput. Intell. Syst. **4**(1), 1–11 (2011)
10. Li, H.X., Zhou, X.Z., Li, T.R., Wang, G.Y., Miao, D.Q., Yao, Y.Y.: Decision-Theoretic Rough Set Theory and Recent Progress. Science Press, Beijing (2011). (In Chinese)
11. Li, H., Zhou, X., Zhao, J., Huang, B.: Cost-sensitive classification based on decision-theoretic rough set model. In: Li, T., et al. (eds.) RSKT 2012. LNCS (LNAI), vol. 7414, pp. 379–388. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-31900-6_47
12. Li, H., Zhou, X., Huang, B., Liu, D.: Cost-sensitive three-way decision: a sequential strategy. In: Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013. LNCS (LNAI), vol. 8171, pp. 325–337. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-41299-8_31
13. Li, H.X., Zhang, L.B., Huang, B., Zhou, X.Z.: Sequential three-way decision and granulation for cost-sensitive face recognition. Knowl.-Based Syst. **91**, 241–251 (2016)
14. Li, H.X., Zhang, L.B., Zhou, X.Z., Huang, B.: Cost-sensitive sequential three-way decision modeling using a deep neural network. Int. J. Approximate Reasoning **85**, 68–78 (2017)
15. Li, H.X., Zhou, X.Z., Huang, B.: Cost-sensitive sequential three-way decisions. In: Liu, D., Li, T.R., Miao, D.Q., Wang, G.Y., Liang, J.Y. (eds.): Three-Way Decisions and Granular Computing, pp. 42–59. Science Press, Beijing (2013). (In Chinese)
16. Li, J.H., Huang, C.C., Qi, J.J., Qian, Y.H., Liu, W.Q.: Three-way cognitive concept learning via multi-granularity. Inf. Sci. **378**, 244–263 (2016)
17. Li, W., Huang, Z., Jia, X.: Two-phase classification based on three-way decisions. In: Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013. LNCS (LNAI), vol. 8171, pp. 338–345. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-41299-8_32
18. Li, Y., Zhang, Z.H., Chen, W.B., Min, F.: TDUP: an approach to incremental mining of frequent itemsets with three-way-decision pattern updating. Int. J. Mach. Learn. Cybernet. **8**(2), 441–453 (2017)
19. Huang, J.J., Wang, J., Yao. Y.Y., Zhong, N.: Cost-sensitive three-way recommendations by learning pair-wise preferences. Int. J. Approximate Reasoning **86**, 28–40 (2017)
20. Hu, J.H., Yang, Y., Chen, X.H.: Three-way linguistic group decisions model based on cloud for medical care product investment. J. Intell. Fuzzy Syst. **33**(6), 3405–3417 (2017)

21. Hu, B.Q.: Three-way decision spaces based on partially ordered sets and three-way decisions based on hesitant fuzzy sets. Knowl.-Based Syst. **91**, 16–31 (2016)
22. Hu, B.Q.: Three-way decisions based on semi-three-way decision spaces. Inf. Sci. **382**, 415–440 (2017)
23. Jia, X., Li, W., Shang, L., Chen, J.: An optimization viewpoint of decision-theoretic rough set model. In: Yao, J.T., Ramanna, S., Wang, G., Suraj, Z. (eds.) RSKT 2011. LNCS (LNAI), vol. 6954, pp. 457–465. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-24425-4_60
24. Jia, X.Y., et al.: Theory of Three-Way Decisions and Application. Nanjing University Press, Nanjing (2012). (In Chinese)
25. Jia, X.Y., Tang, Z.M., Liao, W.H., Shang, L.: On an optimization representation of decision-theoretic rough set model. Int. J. Approximate Reasoning **55**, 156–166 (2014)
26. Ju, H.R., Yang, X.B., Yu, H.L., Li, T.J., Yu, D.J., Yang, J.Y.: Cost-sensitive rough set approach. Inf. Sci. **355**, 282–298 (2016)
27. Ju, H.R., Li, H.X., Yang, X.B., Zhou, X.Z., Huang, B.: Cost-sensitive rough set: a multi-granulation approach. Knowl.-Based Syst. **123**, 137–153 (2017)
28. Liu, D., Li, T.R., Miao, D.Q., Wang, G.Y., Liang, J.Y.: Three-Way Decisions and Granular Computing. Science Press, Beijing (2013). (In Chinese)
29. Liu, D., Li, T., Liang, D.: Three-way decisions in dynamic decision-theoretic rough sets. In: Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013. LNCS (LNAI), vol. 8171, pp. 288–299. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-41299-8_28
30. Liu, D., Liang, D.C., Wang, C.C.: A novel three-way decision model based on incomplete information system. Knowl.-Based Syst. **91**, 32–45 (2016)
31. Liu, S.L., Liu, X.W., Qin, J.D.: Three-way group decisions based on prospect theory. J. Oper. Res. Soc. **69**(1), 25–35 (2018)
32. Luo, C., Li, T., Chen, H.: Dynamic maintenance of three-way decision rules. In: Miao, D., Pedrycz, W., Ślęzak, D., Peters, G., Hu, Q., Wang, R. (eds.) RSKT 2014. LNCS (LNAI), vol. 8818, pp. 801–811. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-11740-9_73
33. Lingras, P., Yan, R.: Interval clustering using fuzzy and rough set theory. In: Annual Meeting of the North American Fuzzy Information Processing Society, vol. 2, pp. 780–784. IEEE (2004)
34. Lingras, P., Peters, G.: Applying rough set concepts to clustering. In: International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing, pp. 23–37. Springer, London (2012). https://doi.org/10.1007/978-1-4471-2760-4_2
35. Liang, D.C., Liu, D., Kobina, A.: Three-way group decisions with decision-theoretic rough sets. Inf. Sci. **345**, 46–64 (2016)
36. Ma, X.A., Yao, Y.Y.: Three-way decision perspectives on class-specific attribute reducts. Inf. Sci. **450**, 227–245 (2018)
37. Min, F., Zhang, Z.H., Zhai, W.J., Shen, R.P.: Frequent pattern discovery with tri-partition alphabets. Inf. Sci. **000**, 1–18 (2018)
38. Pawlak, Z.: Rough sets. Int. J. Comput. Inform. Sci. **11**, 341–356 (1982)
39. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning About Data. Kluwer Academic Publishers, Dordrecht (1991)
40. Peters, G., Weber, R., Nowatzke, R.: Dynamic rough clustering and its applications. Appl. Soft Comput. **12**(10), 3193–3207 (2012)
41. Peters, G., Crespo, F., Lingras, P., Weber, R.: Soft clustering-fuzzy and rough approaches and their extensions and derivatives. Int. J. Approximate Reasoning **54**(2), 307–322 (2013)

42. Qi, J., Wei, L., Yao, Y.: Three-way formal concept analysis. In: Miao, D., Pedrycz, W., Ślęzak, D., Peters, G., Hu, Q., Wang, R. (eds.) RSKT 2014. LNCS (LNAI), vol. 8818, pp. 732–741. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-11740-9_67
43. Qian, T., Wei, L., Qi, J.J.: Constructing three-way concept lattices based on apposition and subposition of formal contexts. Knowl.-Based Syst. **116**, 39–48 (2017)
44. Rehman, H.U., Azam, N., Yao, J.T., Benso, A.: A three-way approach for protein function classification. PLoS ONE **12**(2), e0171702 (2017)
45. Ren, R.S., Wei, L.: The attribute reductions of three-way concept lattices. Knowl.-Based Syst. **99**, 92–102 (2016)
46. Ren, F.J., Wang, L.: Sentiment analysis of text based on three-way decisions. J. Intell. Fuzzy Syst. **33**(1), 245–254 (2017)
47. Singh, P.K.: Three-way fuzzy concept lattice representation using neutrosophic set. Int. J. Mach. Learn. Cybernet. **8**(1), 69–79 (2017)
48. Sun, B.Z., Ma, W.M., Xiao, X.: Three-way group decision making based on multigranulation fuzzy decision-theoretic rough set over two universes. Int. J. Approximate Reasoning **81**, 87–102 (2017)
49. Sun, B.Z., Ma, W.M., Li, B.J., Li, X.N.: Three-way decisions approach to multiple attribute group decision making with linguistic information-based decision-theoretic rough fuzzy set. Int. J. Approximate Reasoning **93**, 424–442 (2018)
50. Wang, L., Miao, D., Zhao, C.: Chinese emotion recognition based on three-way decisions. In: Ciucci, D., Wang, G., Mitra, S., Wu, W.-Z. (eds.) RSKT 2015. LNCS (LNAI), vol. 9436, pp. 299–308. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-25754-9_27
51. Wang, B., Liang, J.: A novel intelligent multi-attribute three-way group sorting method based on dempster-shafer theory. In: Miao, D., Pedrycz, W., Ślęzak, D., Peters, G., Hu, Q., Wang, R. (eds.) RSKT 2014. LNCS (LNAI), vol. 8818, pp. 789–800. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-11740-9_72
52. Wang, P.X., Yao, Y.Y.: CE3: a three-way clustering method based on mathematical morphology. Knowl.-Based Syst. **155**, 54–65 (2018). https://doi.org/10.1016/j.knosys.2018.04.029
53. Yao, Y.: Three-way decision: an interpretation of rules in rough set theory. In: Wen, P., Li, Y., Polkowski, L., Yao, Y., Tsumoto, S., Wang, G. (eds.) RSKT 2009. LNCS (LNAI), vol. 5589, pp. 642–649. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-02962-2_81
54. Yao, Y.Y., Deng, X.F.: Sequential three-way decisions with probabilistic rough sets. In: Proceedings of 10th IEEE International Conference on Cognitive Informatics & Cognitive Computing, pp. 120–125. IEEE (2011)
55. Yao, Y.: An outline of a theory of three-way decisions. In: Yao, Y.T., et al. (eds.) RSCTC 2012. LNCS (LNAI), vol. 7413, pp. 1–17. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-32115-3_1
56. Yao, Y.: Granular computing and sequential three-way decisions. In: Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013. LNCS (LNAI), vol. 8171, pp. 16–27. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-41299-8_3
57. Yao, Y.: Rough sets and three-way decisions. In: Ciucci, D., Wang, G., Mitra, S., Wu, W.-Z. (eds.) RSKT 2015. LNCS (LNAI), vol. 9436, pp. 62–73. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-25754-9_6
58. Yao, Y.Y.: The two sides of the theory of rough sets. Knowl.-Based Syst. **80**, 67–77 (2015)

59. Yao, Y.Y.: Three-way decisions and cognitive computing. Cogn. Comput. **8**(4), 543–554 (2016)
60. Yao, Y.Y.: Interval sets and three-way concept analysis in incomplete contexts. Int. J. Mach. Learn. Cybernet. **8**(1), 3–20 (2017)
61. Yu, H., Wang, Y.: Three-way decisions method for overlapping clustering. In: Yao, J.T., et al. (eds.) RSCTC 2012. LNCS (LNAI), vol. 7413, pp. 277–286. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-32115-3_33
62. Yu, H., Liu, Z.G., Wang, G.Y.: An automatic method to determine the number of clusters using decision-theoretic rough set. Int. J. Approximate Reasoning **55**(1), 101–115 (2014)
63. Yu, H., Su, T., Zeng, X.: A three-way decisions clustering algorithm for incomplete data. In: Miao, D., Pedrycz, W., Ślęzak, D., Peters, G., Hu, Q., Wang, R. (eds.) RSKT 2014. LNCS (LNAI), vol. 8818, pp. 765–776. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-11740-9_70
64. Yu, H., Wang, G.Y., Li, T.R., Liang, J.Y., Miao, D.Q., Yao, Y.Y.: Three-Way Decisions: Methods and Practices for Complex Problem Solving. Science Press, Beijing (2015). (In Chinese)
65. Yu, H., Zhang, C., Wang, G.Y.: A tree-based incremental overlapping clustering method using the three-way decision theory. Knowl.-Based Syst. **91**, 189–203 (2016)
66. Yu, H., Jiao, P., Yao, Y.Y., Wang, G.Y.: Detecting and refining overlapping regions in complex networks with three-way decisions. Inf. Sci. **373**, 21–41 (2016)
67. Yu, H., Zhang, H.: A three-way decision clustering approach for high dimensional data. In: Flores, V., et al. (eds.) IJCRS 2016. LNCS (LNAI), vol. 9920, pp. 229–239. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-47160-0_21
68. Yu, H.: A framework of three-way cluster analysis. In: Polkowski, L., et al. (eds.) IJCRS 2017. LNCS (LNAI), vol. 10314, pp. 300–312. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-60840-2_22
69. Yu, H., Chang, Z.H., Li, Z.X., Wang, G.Y.: An efficient three-way clustering algorithm based on gravitational search. In: ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. ACM (2018)
70. Yu, H., Wang, X.C., Wang, G.Y., Zeng, X.H.: An active three-way clustering method via low-rank matrices for multi-view data. Inf. Sci. (2018). https://doi.org/10.1016/j.ins.2018.03.009
71. Yang, H.L., Guo, Z.L.: Multigranulation decision-theoretic rough sets in incomplete information systems. Int. J. Mach. Learn. Cybernet. **6**(6), 1005–1018 (2015)
72. Yang, X., Tan, A.: Three-way decisions based on intuitionistic fuzzy sets. In: Polkowski, L., et al. (eds.) IJCRS 2017. LNCS (LNAI), vol. 10314, pp. 290–299. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-60840-2_21
73. Yu, H.Y., Li, Q.G., Cai, M.J.: Characteristics of three-way concept lattices and three-way rough concept lattices. Knowl.-Based Syst. **146**, 181–189 (2018)
74. Yang, X., Li, T.R., Fujita, H., Liu, D., Yao, Y.Y.: A unified model of sequential three-way decisions and multilevel incremental processing. Knowl.-Based Syst. **134**, 172–188 (2017)
75. Zhao, X.R., Hu, B.Q.: Fuzzy probabilistic rough sets and their corresponding three-way decisions. Knowl.-Based Syst. **91**, 126–142 (2016)
76. Zhang, Y., Yao, J.T.: Multi-criteria based three-way classifications with game-theoretic rough sets. In: Kryszkiewicz, M., Appice, A., Rybinski, H., Skowron, A., Ślęzak, D., Raś, Z.W. (eds.) ISMIS 2017. LNCS (LNAI), vol. 10352, pp. 550–559. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-60438-1_54

77. Zhang, Z., Wang, R.: Applying three-way decisions to sentiment classification with sentiment uncertainty. In: Miao, D., Pedrycz, W., Ślęzak, D., Peters, G., Hu, Q., Wang, R. (eds.) RSKT 2014. LNCS (LNAI), vol. 8818, pp. 720–731. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-11740-9_66
78. Zhai, J.H., Zhang, Y., Zhu, H.Y.: Three-way decisions model based on tolerance rough fuzzy set. Int. J. Mach. Learn. Cybernet. **8**(1), 35–43 (2017)
79. Zhang, X.Y., Miao, D.Q.: Three-way attribute reducts. Int. J. Approximate Reasoning **88**, 401–434 (2017)
80. Zhang, Q.H., Lv, G.X., Chen, Y.H., Wang, G.Y.: A dynamic three-way decision model based on the updating of attribute values. Knowl.-Based Syst. **142**, 71–84 (2018)
81. Zhang, X.Y., Miao, D.Q.: Double-quantitative fusion of accuracy and importance: systematic measure mining, benign integration construction, hierarchical attribute reduction. Knowl.-Based Syst. **91**, 219–240 (2016)
82. Zhang, Y., Zou, H., Chen, X., Wang, X., Tang, X., Zhao, S.: Cost-sensitive three-way decisions model based on CCA. In: Cornelis, C., Kryszkiewicz, M., Ruiz, E.M., Bello, R., Ślęzak, D., Shang, L. (eds.) RSCTC 2014. LNCS (LNAI), vol. 8536, pp. 172–180. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-08644-6_18
83. Zhang, Y., Xing, H., Zou, H., Zhao, S., Wang, X.: A three-way decisions model based on constructive covering algorithm. In: Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013. LNCS (LNAI), vol. 8171, pp. 346–353. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-41299-8_33
84. Zhou, B., Yao, Y.Y., Luo, J.G.: Cost-sensitive three-way email spam filtering. J. Intell. Inf. Syst. **42**, 19–45 (2014)
85. Zhou, Z., Zhao, W., Shang, L.: Sentiment analysis with automatically constructed lexicon and three-way decision. In: Miao, D., Pedrycz, W., Ślęzak, D., Peters, G., Hu, Q., Wang, R. (eds.) RSKT 2014. LNCS (LNAI), vol. 8818, pp. 777–788. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-11740-9_71