

Geometric Transform and Its Applications

Instructor

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Outline

❖ Geometric Transform

❖ Its Applications

Homogenous system

Rotation Matrix

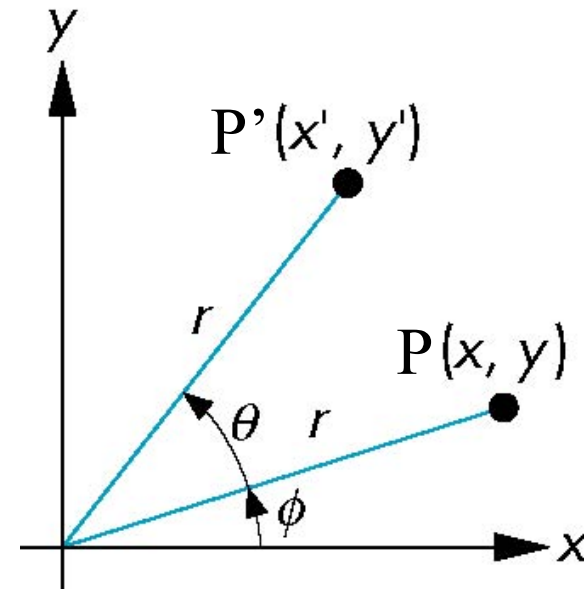
Consider Point $P(x, y)$.

In Homogenous space:

$$x = r \cos \phi$$

$$y = r \sin \phi$$

$$w = 1$$



Rotate $P(x, y)$ an positive angle θ around the origin, to Point $P'(x', y')$. In Homogenous space:

$$x' = r \cos(\theta + \phi)$$

$$y' = r \sin(\theta + \phi)$$

$$w' = 1$$

Homogenous system

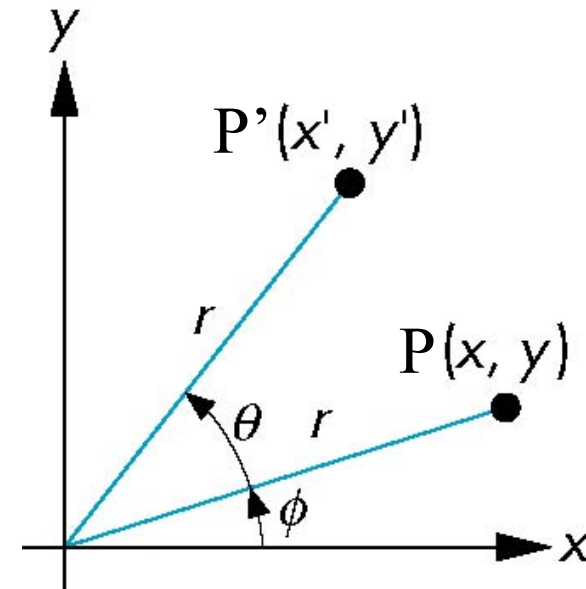
Rotation Matrix

Point $P'(x', y')$:

$$\begin{aligned}x' &= r \cos(\phi + \theta) \\&= r[\cos \phi \cos \theta - \sin \phi \sin \theta] \\&= x \cos \theta - y \sin \theta\end{aligned}$$

$$\begin{aligned}y' &= r \sin(\phi + \theta) \\&= r[\cos \phi \sin \theta + \sin \phi \cos \theta] \\&= x \sin \theta + y \cos \theta\end{aligned}$$

$$w' = 1$$



Homogenous system

Rotation Matrix

Known form of the transform matrix:

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

And, two points P and P':

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

The transformation: $P' = M.P$



$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} \\ y' &= a_{21}x + a_{22}y + a_{23} \\ 1 &= a_{31}x + a_{32}y + a_{33} \end{cases}$$

Homogenous system

Rotation Matrix

Identity:

$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= x \cos \theta - y \sin \theta \\ y' &= a_{21}x + a_{22}y + a_{23} &= x \sin \theta + y \cos \theta \\ 1 &= a_{31}x + a_{32}y + a_{33} &= 1 \end{cases}$$

Rotation matrix:

$$M_R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous system

Translation Matrix

Identity:

$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= x + dx \\ y' &= a_{21}x + a_{22}y + a_{23} &= y + dy \\ 1 &= a_{31}x + a_{32}y + a_{33} &= 1 \end{cases}$$

Translation matrix:

$$M_T = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous system

Scaling Matrix

Identity:

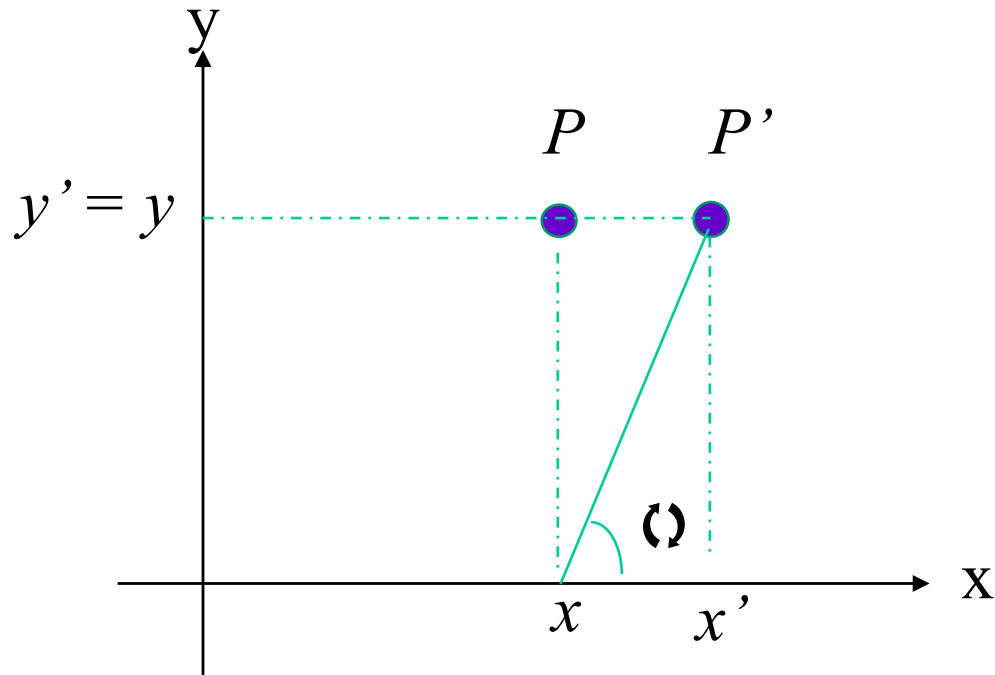
$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= S_x x \\ y' &= a_{21}x + a_{22}y + a_{23} &= S_y y \\ 1 &= a_{31}x + a_{32}y + a_{33} &= 1 \end{cases}$$

Scaling matrix:

$$M_S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous system

Shearing



$$\begin{cases} x' &= x + y \cot \theta \\ y' &= y \\ w' &= 1 \end{cases}$$

Homogenous system

Shearing

Identity:

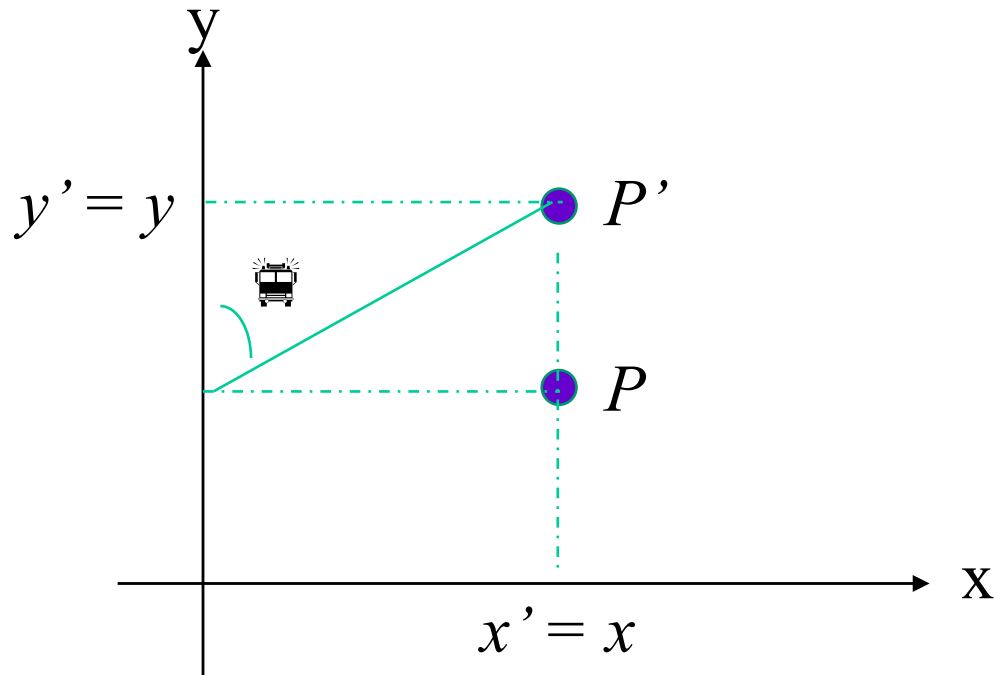
$$\begin{cases} x' &= a_{11}x + \textcolor{red}{a}_{12}y + a_{13} &= x + y \textcolor{red}{\cot \theta} \\ y' &= a_{21}x + \textcolor{green}{a}_{22}y + a_{23} &= \textcolor{green}{y} \\ 1 &= a_{31}x + a_{32}y + \textcolor{blue}{a}_{33} &= \textcolor{blue}{1} \end{cases}$$

Shearing (along x-direction) matrix:

$$M_{SHx} = \begin{bmatrix} 1 & \cot \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous system

Shearing



$$\begin{cases} x' &= x \\ y' &= x \cot \phi + y \\ w' &= 1 \end{cases}$$

Homogenous system

Shearing

Identity:

$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= x \\ y' &= \textcolor{red}{a}_{21}x + a_{22}y + a_{23} &= x \textcolor{red}{\cot \phi} + y \\ 1 &= a_{31}x + a_{32}y + \textcolor{blue}{a}_{33} &= \textcolor{blue}{1} \end{cases}$$

Shearing (along y-direction) matrix:

$$M_{SHy} = \begin{bmatrix} 1 & 1 & 0 \\ \cot \phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous system

Shearing

Shearing (along x and y-direction) matrix:

$$\begin{aligned} M_{SH} &= M_{SHy} M_{SHx} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \cot \phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \cot \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \cot \theta & 0 \\ \cot \phi & 1 + \cot \phi \cot \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Homogenous system

Affine transform

Scaling:

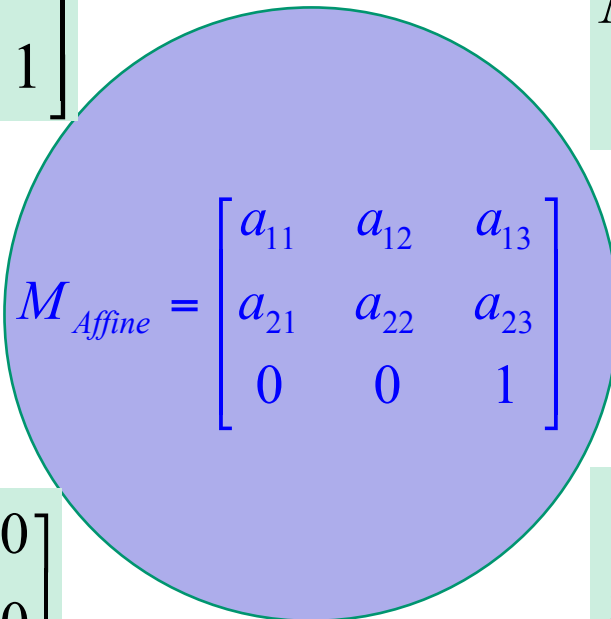
$$M_S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation:

$$M_T = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation:

$$M_R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$


$$M_{Affine} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing:

$$M_{SH} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous system

Projective transform

$$M_{\text{Projective}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

Projective transform

Estimation

$$P' = M_{\text{Projective}} P$$

$$P' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}x + a_{12}y + a_{13} \\ a_{21}x + a_{22}y + a_{23} \\ a_{31}x + a_{32}y + 1 \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + 1} \\ \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + 1} \\ 1 \end{bmatrix}$$

Projective transform

Estimation

$$\begin{aligned}x' &= \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + 1} \\y' &= \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + 1}\end{aligned}$$



$$\begin{cases} x'[a_{31}x + a_{32}y + 1] & = a_{11}x + a_{12}y + a_{13} \\ y'[a_{31}x + a_{32}y + 1] & = a_{21}x + a_{22}y + a_{23} \end{cases}$$

Projective transform

Estimation

Rearrange terms:

$$\begin{cases} a_{11}x + a_{12}y + a_{13} - a_{31}xx' - a_{32}x'y = x' \\ a_{21}x + a_{22}y + a_{23} - a_{31}xy' - a_{32}yy' = y' \end{cases}$$

Insert some dummy terms:

$$\begin{cases} a_{11}x + a_{12}y + a_{13} + a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 - a_{31}xx' - a_{32}x'y = x' \\ a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 0 + a_{21}x + a_{22}y + a_{23} - a_{31}xy' - a_{32}yy' = y' \end{cases}$$

Projective transform

Estimation

Matrix Form:

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & xx' & x'y' \\ 0 & 0 & 0 & x & y & 1 & xy' & yy' \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

a_{ij} : variables

Projective transform

Estimation

To find the value for 8 variables (a_{ij}), need to formulate at least 8 equations.

i.e., need at least four pairs of mapping points:

$$\begin{aligned} &\langle P_1(x_1, y_1), P'_1(x'_1, y'_1) \rangle, \\ &\langle P_2(x_2, y_2), P'_2(x'_2, y'_2) \rangle, \\ &\langle P_3(x_3, y_3), P'_3(x'_3, y'_3) \rangle, \\ &\langle P_4(x_4, y_4), P'_4(x'_4, y'_4) \rangle, \\ &\dots \text{ maybe many more } \dots \end{aligned}$$

Projective transform

Estimation

for N pairs:

$$\begin{array}{c}
 \text{Pair 1:} \\
 \text{Pair 2:} \\
 \text{Pair 3:} \\
 \text{Pair 4:} \\
 \text{Additional pair:}
 \end{array}
 \begin{array}{c}
 \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 x'_1 & -x'_1 y_1 \\
 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1 y'_1 & -y_1 y'_1 \\
 x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2 x'_2 & -x'_2 y_2 \\
 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2 y'_2 & -y_2 y'_2 \\
 x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3 x'_3 & -x'_3 y_3 \\
 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3 y'_3 & -y_3 y'_3 \\
 x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4 x'_4 & -x'_4 y_4 \\
 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4 y'_4 & -y_4 y'_4 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{bmatrix}
 \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ \vdots
 \end{bmatrix}
 \end{array}
 =
 \begin{array}{c}
 \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \\ \vdots
 \end{bmatrix}
 \end{array}$$

Projective transform Estimation



Matrix form:

$$\begin{matrix} 2N \times 8 \\ A \end{matrix} \times \begin{matrix} 8 \times 1 \\ h \end{matrix} = \begin{matrix} 2N \times 1 \\ b \end{matrix}$$

$$\begin{matrix} 8 \times 2N & 2N \times 8 \\ A^T & A \end{matrix} \times \begin{matrix} 8 \times 1 \\ h \end{matrix} = \begin{matrix} 8 \times 2N & 2N \times 1 \\ A^T & b \end{matrix}$$

$$\begin{matrix} 8 \times 8 \\ (A^T A) \end{matrix} \times \begin{matrix} 8 \times 1 \\ h \end{matrix} = \begin{matrix} 8 \times 1 \\ (A^T b) \end{matrix}$$

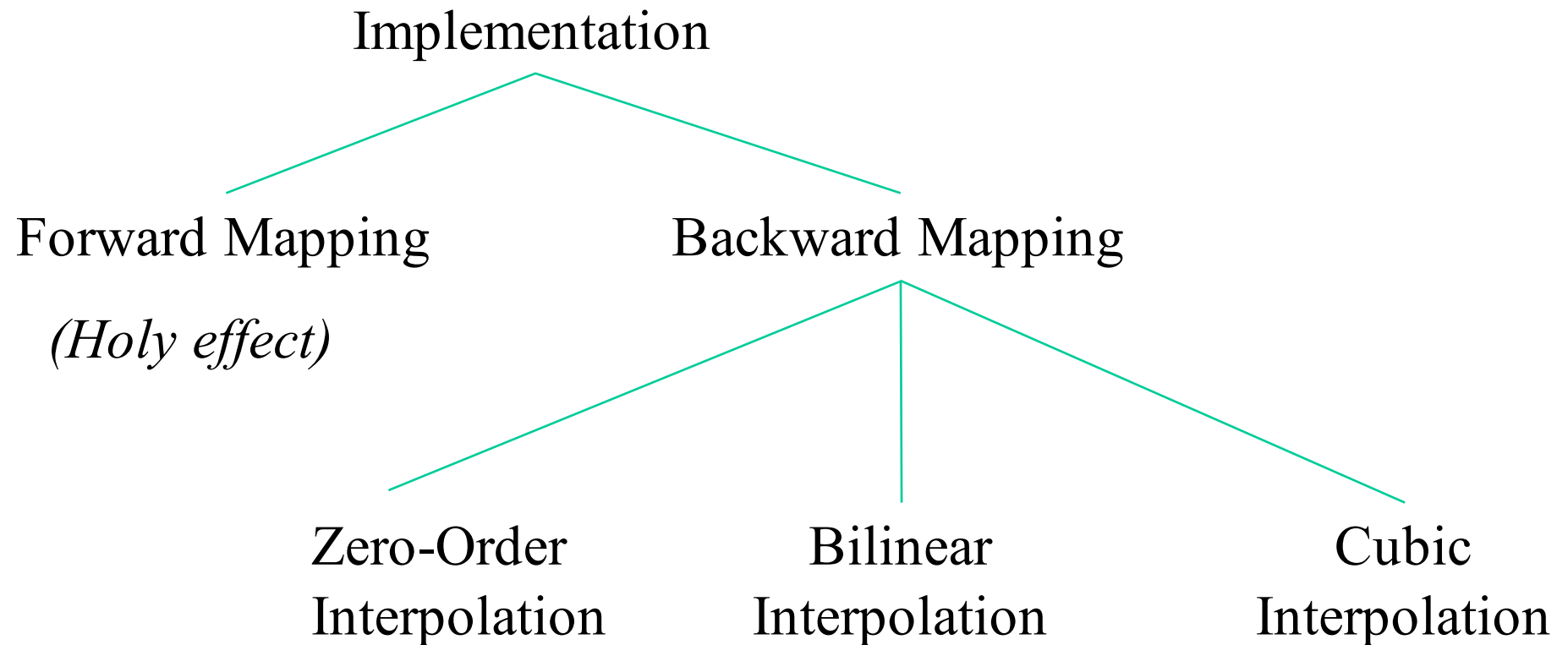
Projective transform Estimation

Solution: $h = (A^T A)^{-1} A^T b$

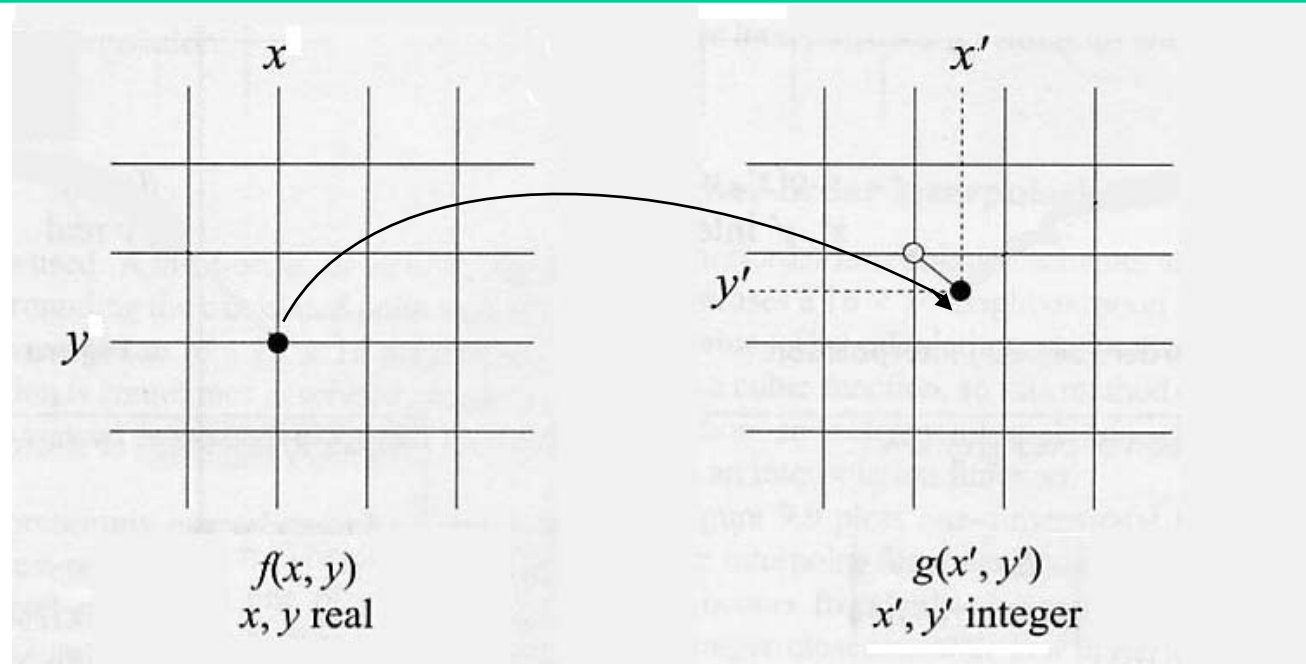
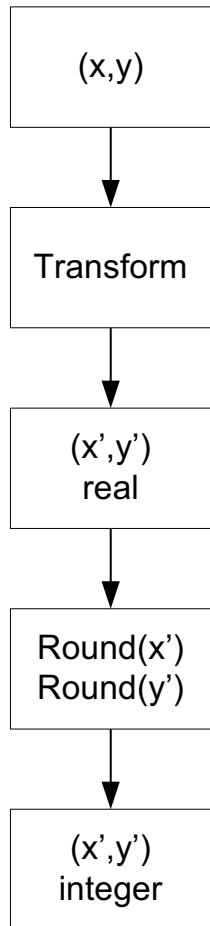
$$h = (A)^{-1} b$$

Matlab: $h = A \backslash b$

Implementation of a transformation



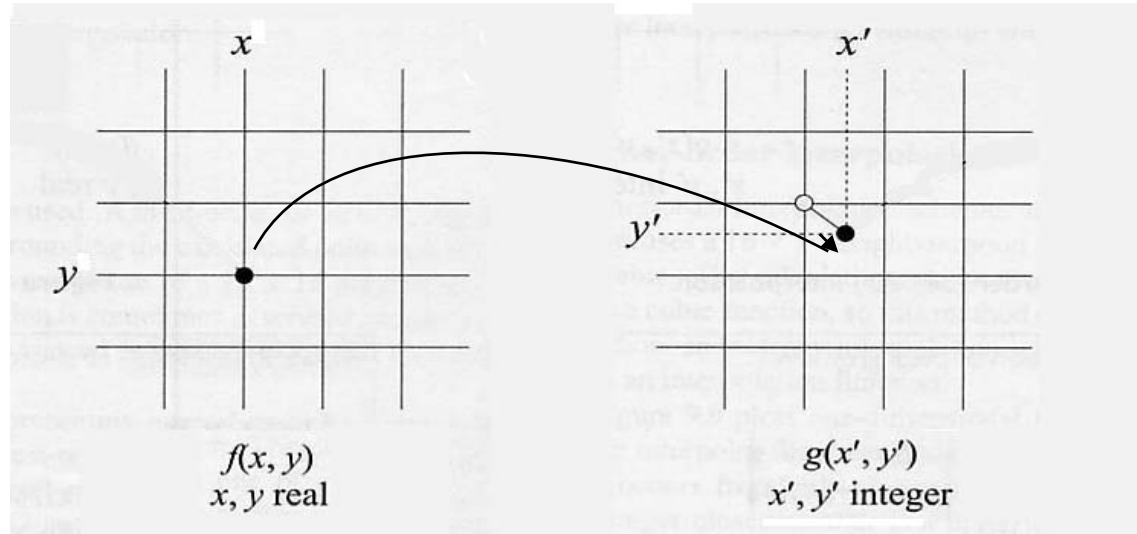
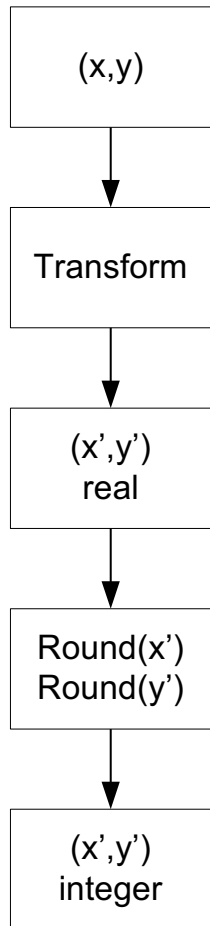
Forward mapping



$$P' = M.P$$

$$g(\text{round}(x'), \text{round}(y')) = f(x, y)$$

Forward mapping



For each pixel P

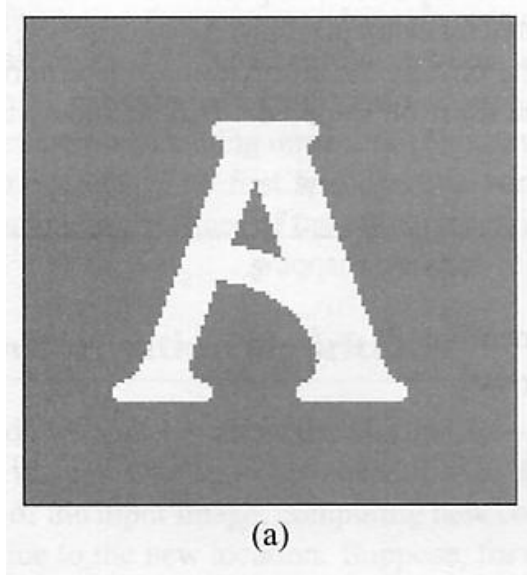
Compute $P' = M.P$

$g(\text{round}(x'), \text{round}(y'))$
 $= f(x, y)$

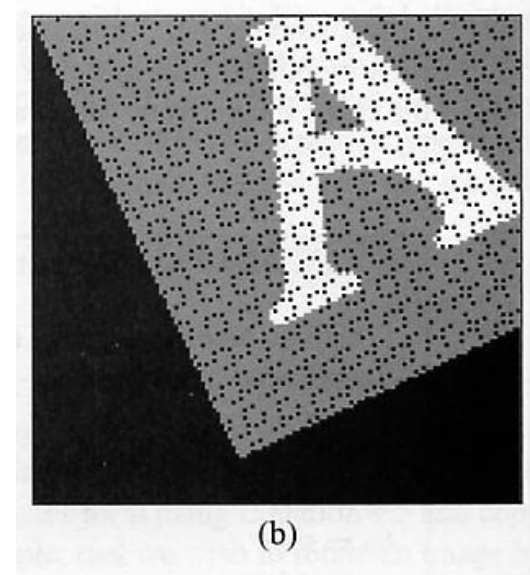
End for

Forward mapping

Example



$f(x, y)$

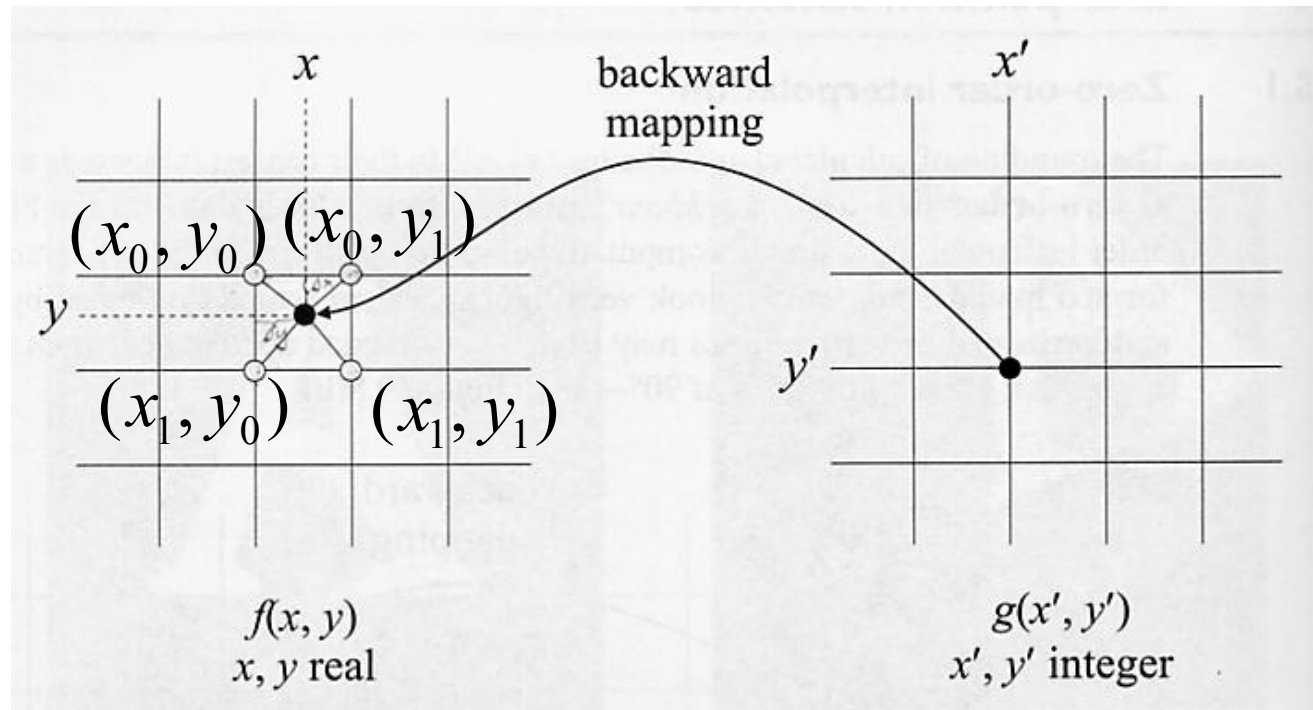


$g(x', y')$

Rotate $f(x, y)$ an positive angle 25°

$$M_R = \begin{bmatrix} \cos 25^\circ & -\sin 25^\circ & 0 \\ \sin 25^\circ & \cos 25^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

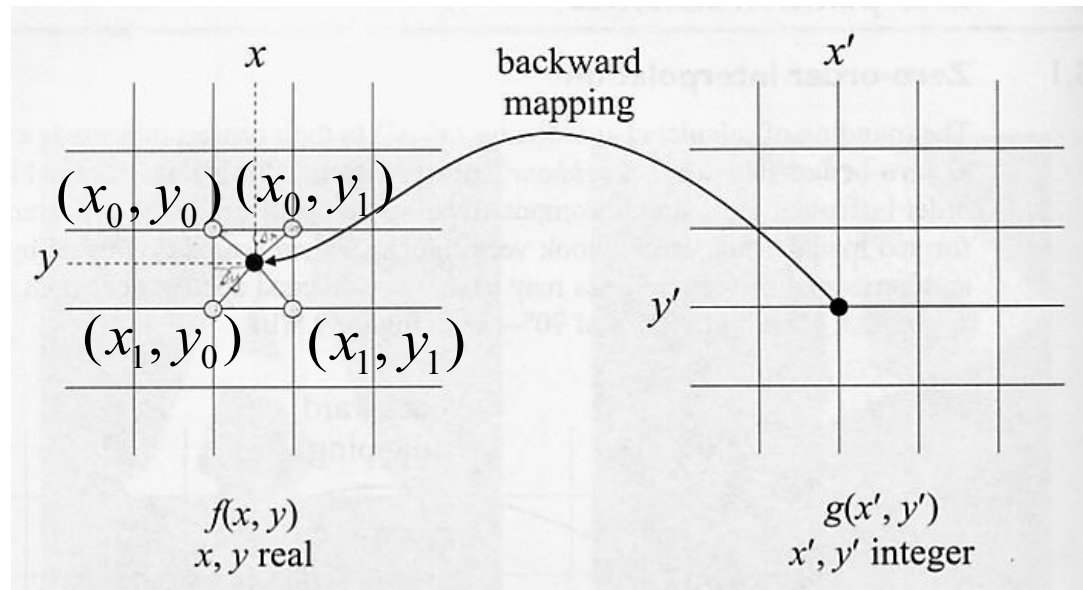
Backward mapping



$$P = M^{-1} \cdot P'$$

$$g(x', y') = \text{interpolation}[f(x_0, y_0)]$$

Backward mapping



For each pixel P'

Compute $P = M^{-1} \cdot P'$

$g(x', y') =$
 $interpolation[f(x_0, y_0)]$

End for

Backward mapping

Inverse transform

Scaling:

$$M_S = \begin{bmatrix} \frac{1}{S_x} & 0 & 0 \\ 0 & \frac{1}{S_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation:

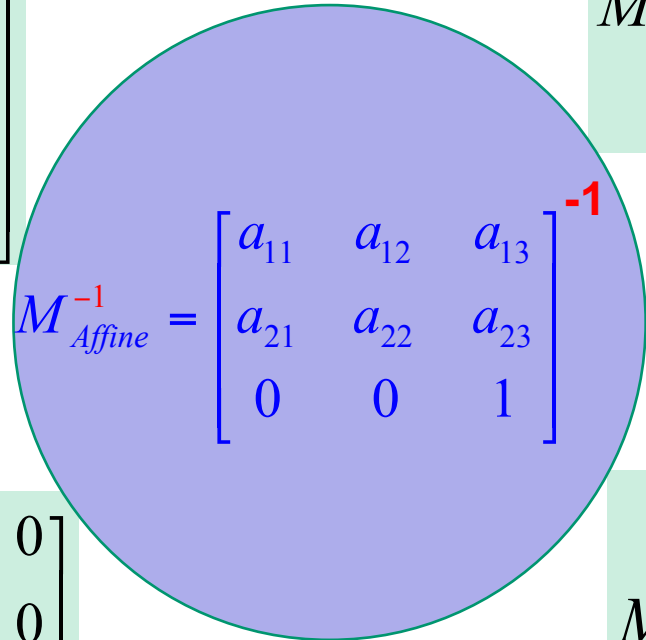
$$M_T = \begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation:

$$M_R = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

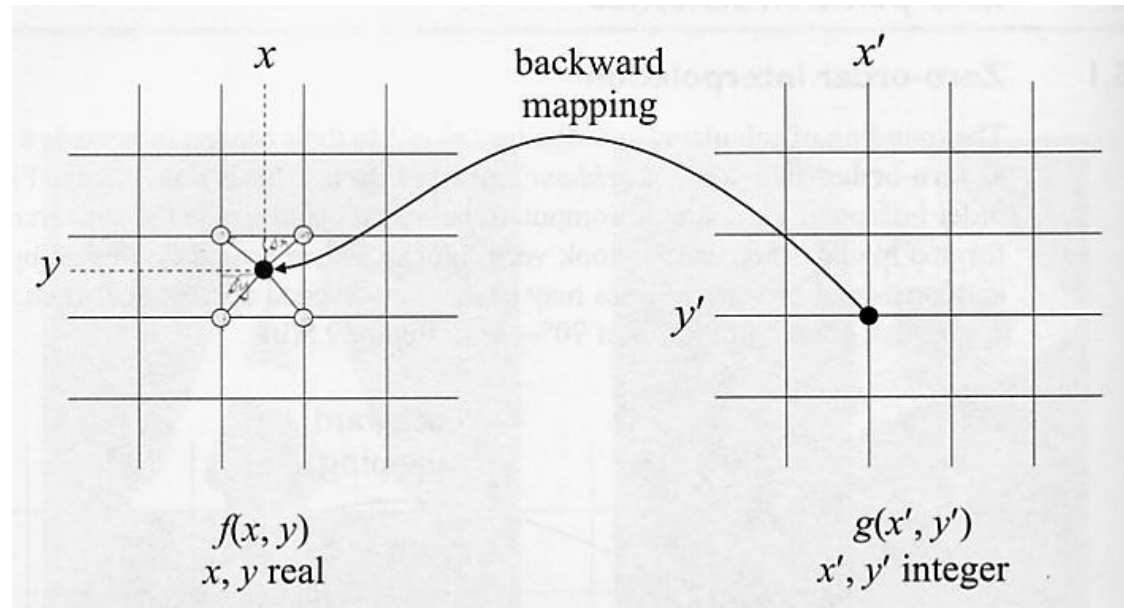
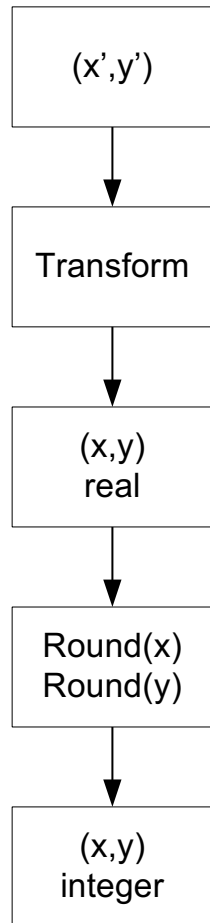
Shearing:

$$M_{SH} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$


$$M_{Affine}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Backward mapping

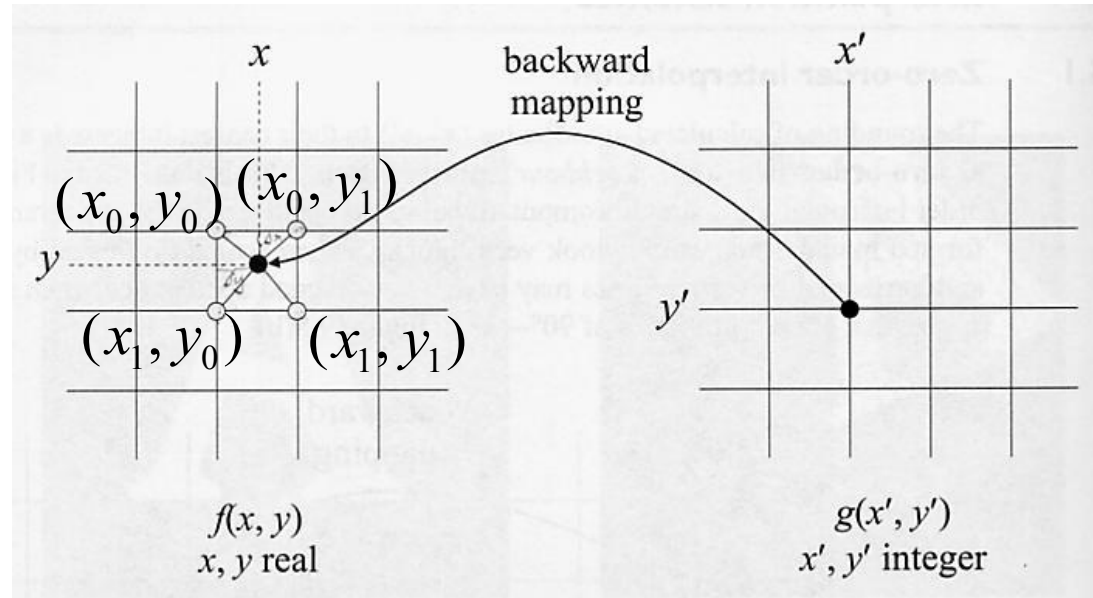
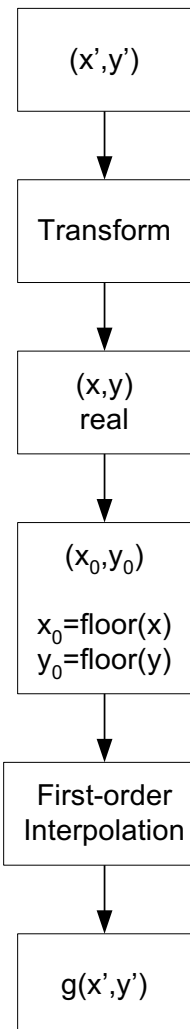
Zero-order interpolation



$$g(x', y') = f[\text{round}(x), \text{round}(y)]$$

Backward mapping

Bilinear interpolation



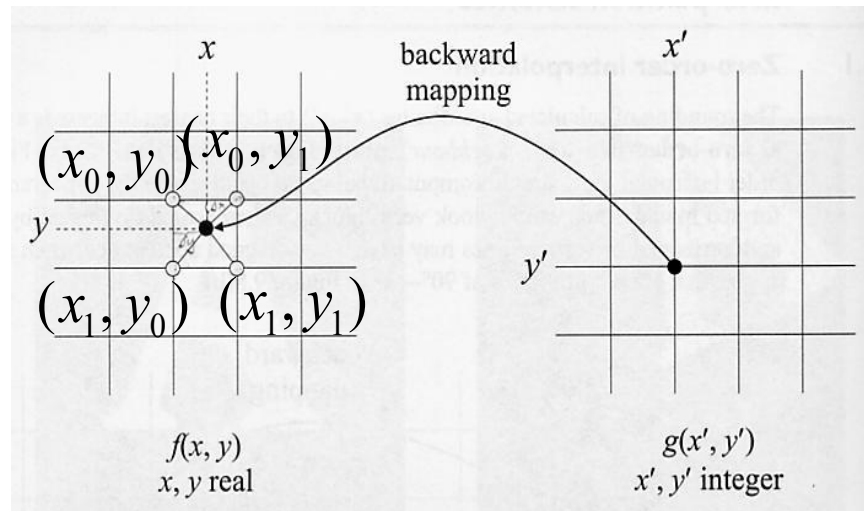
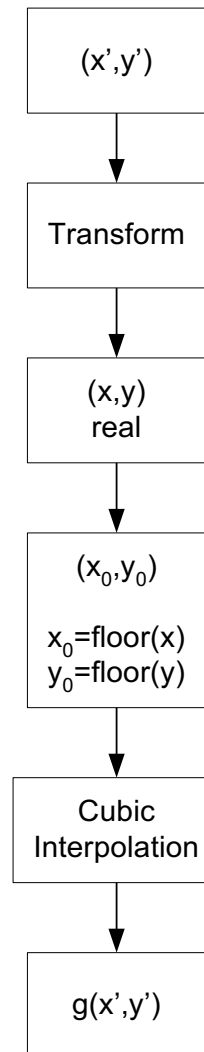
$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

$$\begin{aligned}
 g(x', y') = & f(x_0, y_0) + [f(x_1, y_0) - f(x_0, y_0)]\Delta x \\
 & + [f(x_0, y_1) - f(x_0, y_0)]\Delta y \\
 & + [f(x_1, y_1) + f(x_0, y_0) - f(x_0, y_1) - f(x_1, y_0)]\Delta x\Delta y
 \end{aligned}$$

Backward mapping

Cubic interpolation



$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

$$g(x', y') = \sum_{m=-1}^2 \sum_{n=-1}^2 f(x_0 + m, y_0 + n) R(m - \Delta x) R(\Delta y - n)$$

$$R(k) = \frac{1}{6} [P(k+2)^3 - 4P(k+1)^3 - 4P(k-1)^3 + 6P(k)^3]$$

$$P(z) = \begin{cases} z & z > 0 \\ 0 & z \leq 0 \end{cases}$$

Backward mapping

Example: zero-order interpolation



Scaling



Backward mapping

Example: bilinear interpolation



Scaling



Backward mapping

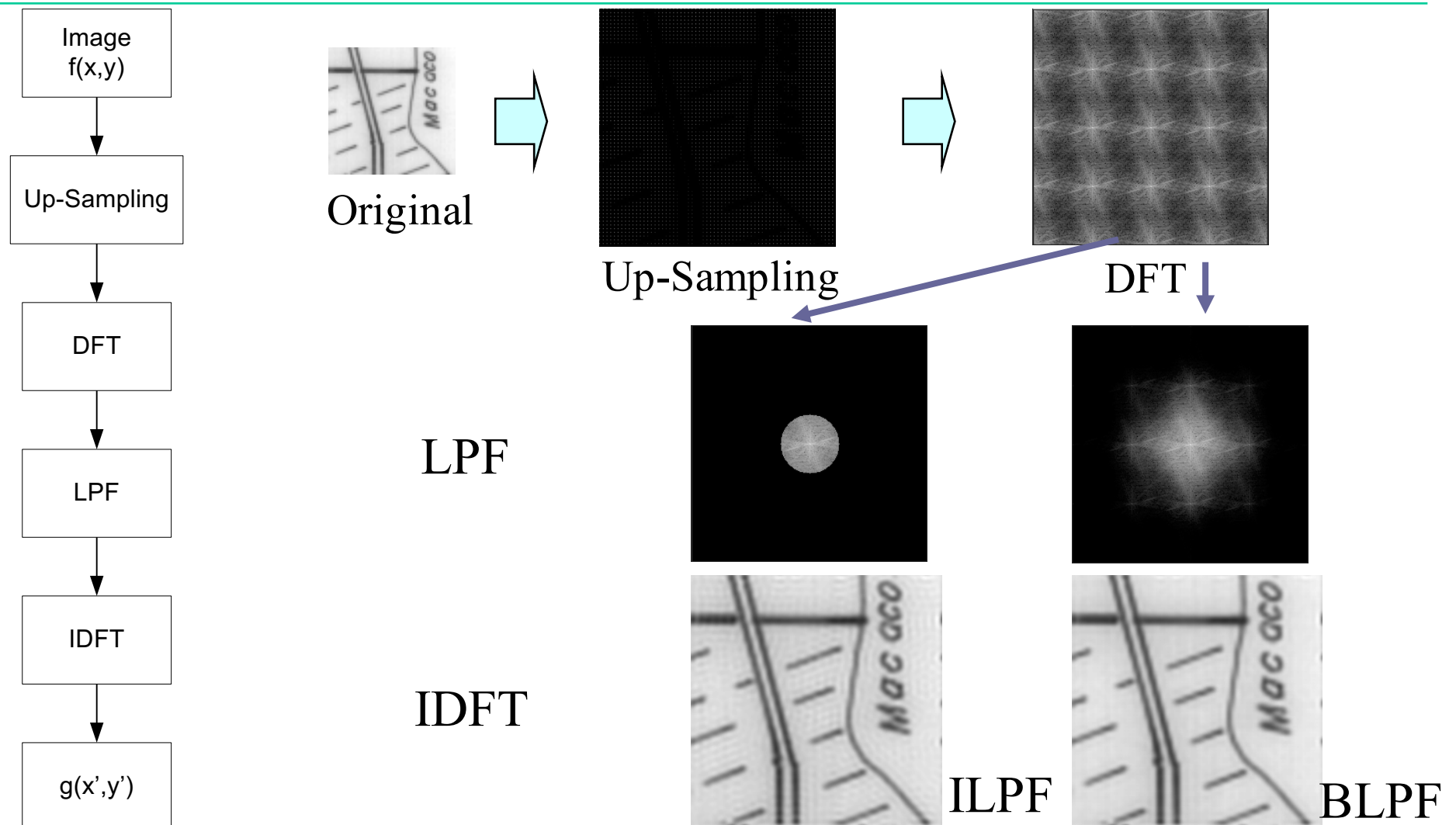
Example: cubic interpolation



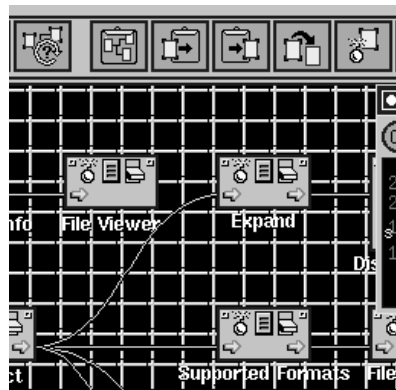
Scaling



Processing in frequency domain

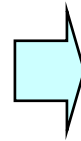


Processing in frequency domain



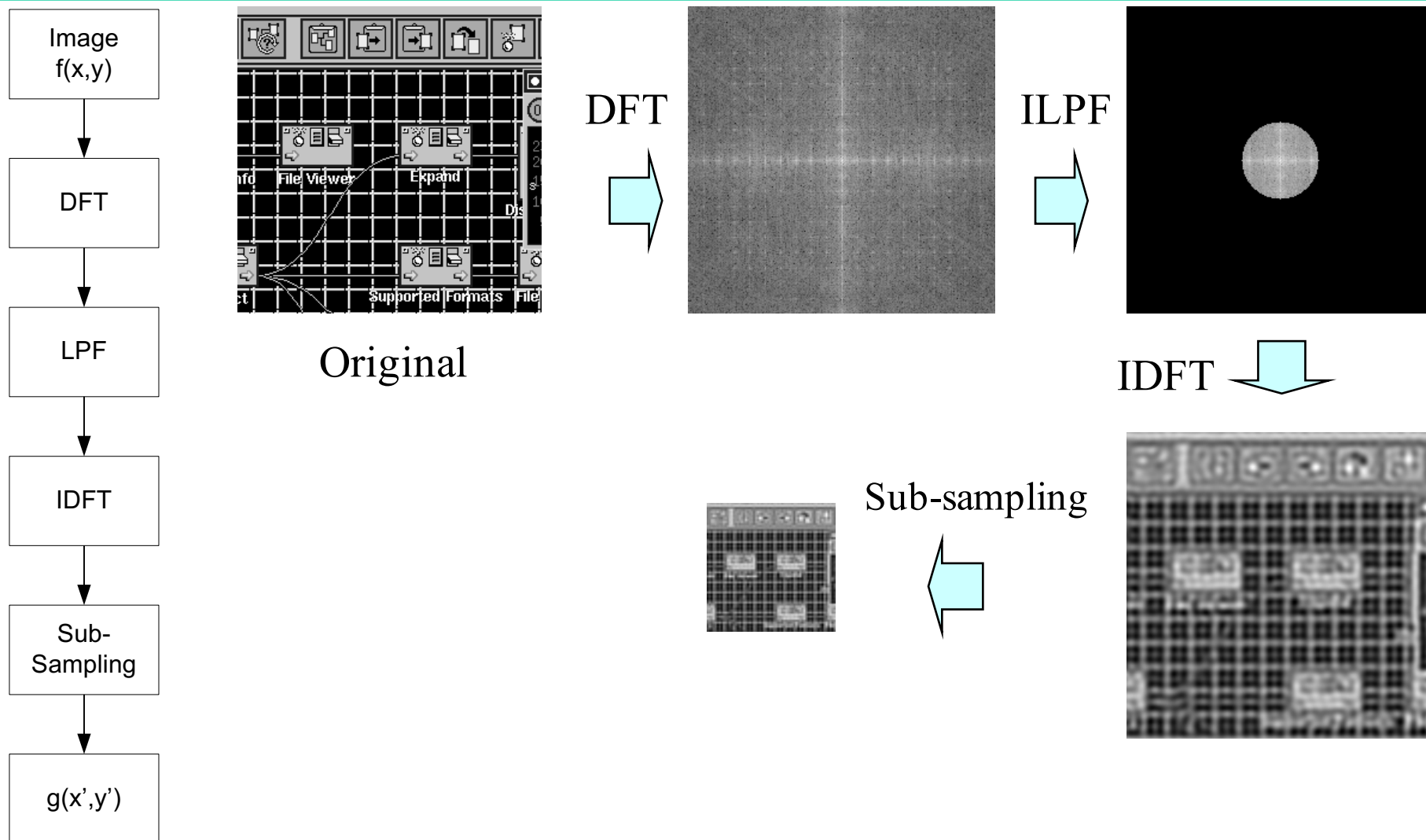
Original

Sub-sampling



Reduced Image

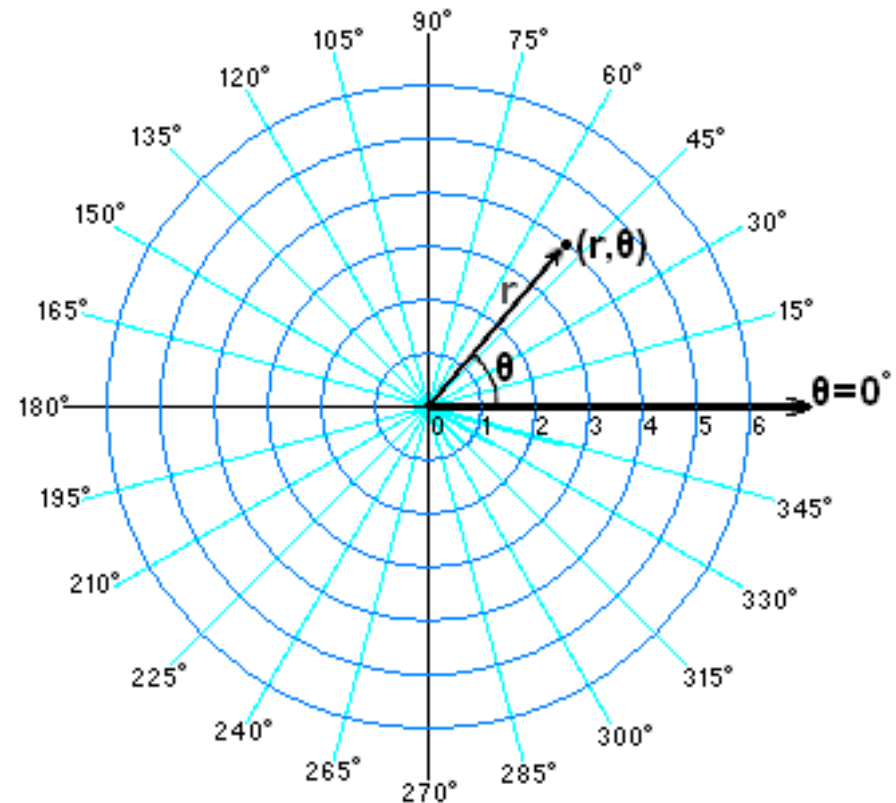
Processing in frequency domain



Polar Transform

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$



Polar Transform

Example

