清華大學 数学作业纸

班级：2+31 姓名：韩佑硕 编号：2021?80?0 科目：信号处理 第 1 页

1. 证明欧拉公式

证明：$e^z = \lim_{z\to\infty}\left(1+\frac{z}{n}\right)^z$

$e^{ix} = \lim_{n\to\infty}\left(1+\frac{ix}{n}\right)^n$

令 $r = \left|1+\frac{ix}{n}\right| = \sqrt{1+\left(\frac{x}{n}\right)^2}$,

$\tan(\theta_n) = \frac{x/n}{1} = \frac{x}{n} \Rightarrow \theta_n = \arctan\left(\frac{x}{n}\right)$

$\Rightarrow 1+\frac{ix}{n} = \sqrt{1+\left(\frac{x}{n}\right)^2}\left(\cos(\theta_n)+i\sin(\theta_n)\right)$

$\left(1+\frac{ix}{n}\right)^n = \left(1+\frac{x^2}{n^2}\right)^{\frac{n}{2}}\left(\cos(n\theta_n)+i\sin(n\theta_n)\right)$

模：$\lim_{n\to\infty}\left(1+\frac{x^2}{n^2}\right)^{\frac{n}{2}} = \lim_{n\to\infty}\left(\left(1+\frac{x^2}{n^2}\right)^{\frac{n^2}{x^2}}\right)^{\frac{x^2}{2n}} = \lim_{n\to\infty} e^{\frac{x^2}{2n}} = 1$

θ: $\lim_{n\to\infty} n\cdot\arctan\left(\frac{x}{n}\right) \xrightarrow{\text{L'Hôpital}} \lim_{u\to 0}\frac{xu}{u} = x$ $\left(u=\frac{1}{n},\ \arctan(xu)\approx xu\right)$

$\Rightarrow e^{ix} = 1(\cos(x)+i\sin(x))$ □

2. 证明三角函数序列 $\{1, \cos(\omega_1 t+\varphi_1), \cos(2\omega_1 t+\varphi_2), \cdots, \cos(n\omega_1 t+\varphi_n)\}$ 是在 $[0, \frac{2\pi}{\omega_1}]$ 区间正交函数集

证明：令 $\varphi_0 = 0$，$\forall i, j \in \{0,1,2,\cdots,n\}$

当 $i \neq j$ 时，$\int_0^{\frac{2\pi}{\omega_1}} \cos(i\omega_1 t+\varphi_i)\cos(j\omega_1 t+\varphi_j)dt \overset{def}{=} \langle i,j\rangle$

$= \frac{1}{2}\int_0^{\frac{2\pi}{\omega_1}} \underbrace{\cos((i+j)\omega_1 t+\varphi_i+\varphi_j)}_{I_1} + \underbrace{\cos((i-j)\omega_1 t+\varphi_i-\varphi_j)}_{I_2}dt$

$\because$ $i \pm j$ 是非零整数 $\therefore \int_0^{\frac{2\pi}{\omega_1}} I_1 dt = 0$, $\int_0^{\frac{2\pi}{\omega_1}} I_2 dt = 0$

$\Rightarrow \int_0^{\frac{2\pi}{\omega_1}} \cos(i\omega_1 t+\varphi_i)\cos(j\omega_1 t+\varphi_j)dt = 0$

当 $i=j$ 时，$\langle i,j\rangle = \int_0^{\frac{2\pi}{\omega_1}} \cos^2(i\omega_1 t+\varphi_i)dt = \int_0^{\frac{2\pi}{\omega_1}} \frac{1}{2}[1+\cos(2i\omega_1 t+2\varphi_i)]dt$

$= \frac{\pi}{\omega_1} \neq 0$ □

3. $f(t) = \sin t\cos(2t) + 5\cos(3t)\sin(4t)$

$= \frac{1}{2}(\sin(3t)-\sin t) + \frac{5}{2}(\sin 7t + \sin t) = 2\sin t + \frac{1}{2}\sin(3t) + \frac{5}{2}\sin(7t)$ □