Producer Heterogeneity in Macroeconomics
Part I: Firm Heterogeneity, Lectures 2-3: Hopenhayn-style models

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Introduction: 3 workhorse models

→ Aiyagari Model → Went through by yourself.

- **Lucas (1978)**: combination of diminishing returns in production & heterogeneity in productivity yields a nondegenerate firm-size distribution
- **Hopenhayn (1992)** workhorse model of industry *dynamics*
  - likewise emphasizes heterogeneity in productivity + diminishing returns in production
  - steady state model: firms enter, grow and exit, but overall distribution of firms is unchanged
  - Perfect competition
- Will also (briefly) consider **Melitz (2003)** model
  - heterogeneity in productivity + diminishing returns in *preferences*
  - monopolistic competition

Ex-Ante vs Ex-Post Heterogeneity

- Two different ways that we could introduce idiosyncratic heterogeneity in productivity, z_i
    1. ex-*ante* heterogeneity: z_i observed before plant is created
    2. ex-*post* heterogeneity: z_i observed after plant is created Hopenhayn Model 1992
- No inherent right or wrong; both are plausible, but the choice does matter
    - Sterk, Sedlacek, Pugsley (2021, AER): "The Nature of Firm Growth" look for reference
- Lucas span-of-control model traditionally assumes ex ante heterogeneity, calling it "heterogeneous manager ability" $\Rightarrow$ high z_i become managers (selection)
    - recall Lecture 1...
- Hopenhayn (and Melitz) models traditionally assume ex-post heterogeneity: pay a fixed cost to get a random draw z_i

Plan for today & next lecture

- Simple static models: Hopenhayn vs. Melitz setup
- Hopenhayn (1992): partial-equilibrium model
- Hopenhayn & Rogerson (1993): GE & adjustment costs PSet

Simple static models

Static Hopenhayn: environment

- Constant measure M of firms, indexed by i
  - no entry/exit yet
- Constant number of workers N
- Firms (establishments) are heterogeneous in productivity, once-and-for-all draw $z \sim G(z)$
  - no time-varying productivity yet
- Production function

$$y = zn^{\alpha}, \; 0 < \alpha < 1$$

- Perfect competition in product and labor markets
- Equilibrium wage w equalizes agg. demand for labor with (fixed) supply of labor

Static Hopenhayn: firm problem

- Profit maximization (normalizing price of good $p = 1$):

$$\pi(z) = \max_n zn^\alpha - wn$$

- Optimal labor demand for firm i:

$$\alpha z_i n_i^{\alpha-1} = w$$

- Since w is the same for all firms, **Marginal Product of Labor equalized across firms**:

$$\boxed{\frac{z_j}{z_i} = \left(\frac{n_j}{n_i}\right)^{1-\alpha}}$$

↑ higher productivity
↓ less labour to keep the
MPL the same.

for any two firms i and j

Static Hopenhayn: efficiency

- Consider a social planner who wants to maximize production

$$\max_{n_i} Y = \int y_i di = \int z_i n_i^\alpha di \quad \text{s.t.} \quad N = \int n_i di$$

- FOC implies for any firm i, with λ the multiplier on the constraint

$$\alpha z_i n_i^{\alpha-1} = \lambda$$

- Hence, **efficient allocation also requires MPL equalized across producers**, with higher-z firms hiring more labor

$$\frac{z_j}{z_i} = \left(\frac{n_j}{n_i}\right)^{1-\alpha}$$

for any two firms i and j

Static Melitz: environment

- **Monopolistic competition** instead of perfect competition
- Intermediates production **linear** in labor:

$$y_i = \tilde{z}_i n_i,$$

  where $\tilde{z}_i = z_i^{1/\eta}$, so $\text{MPL}_i = \tilde{z}_i$
- Final good is produced by combining a continuum of intermediates/varieties:

$$Y = \left(\int_0^M y_i^\eta di \right)^{\frac{1}{\eta}}, \quad 0 < \eta < 1$$

Concavity in preferences makes the diminishing returns as the $y_i \uparrow$

  – demand elasticity $\sigma \equiv \frac{1}{1-\eta} > 1 \rightarrow$ **diminishing returns**

- Standard optimization yields usual isoelastic demand & optimal price index

I need to think about the CES

$$y_i = \left(\frac{p_i}{P} \right)^{\frac{1}{\eta-1}} Y \qquad P = \left(\int p_i^{\frac{\eta}{\eta-1}} di \right)^{\frac{\eta-1}{\eta}}$$

Correct Derivation for CES

Starting with CES:

$$Y = \left(\sum_i y_i^\rho \right)^{\frac{1}{\rho}}$$

Consumer's Problem:

Maximize Y subject to budget constraint: $\sum p_i y_i = I$

First-order conditions:

$$\frac{\partial Y}{\partial y_i} = \lambda p_i$$

Where λ is the Lagrange multiplier.

For any two varieties i and j:

$$\frac{\partial Y / \partial y_i}{\partial Y / \partial y_j} = \frac{p_i}{p_j}$$

This gives us:

$$\frac{y_i^{\rho-1}}{y_j^{\rho-1}} = \frac{p_i}{p_j}$$

Solving for the quantity ratio:

$$\left(\frac{y_i}{y_j}\right)^{\rho-1} = \frac{p_i}{p_j}$$

$$\frac{y_i}{y_j} = \left(\frac{p_i}{p_j}\right)^{\frac{1}{\rho-1}}$$

Taking logs:

$$\ln\left(\frac{y_i}{y_j}\right) = \frac{1}{\rho-1}\ln\left(\frac{p_i}{p_j}\right)$$

Computing elasticity:

$$\sigma = \frac{d\ln(y_i/y_j)}{d\ln(p_i/p_j)} = \frac{1}{\rho - 1}$$

The Key Insight

The elasticity of substitution tells us: **"If good i becomes 1% more expensive relative to good j, by what percentage do consumers reduce their consumption of i relative to j?"**

For CES, this elasticity is **constant** - it's always $\sigma = \frac{1}{\rho-1}$ regardless of:

- The current quantities (y_i, y_j)
- The current prices (p_i, p_j)
- How rich or poor the consumer is

Static Melitz: firm problem

→ Price dispersion question by Luiza
→ Midrigan-Xu Edmond

- Firm optimization under monopolistic competition:

$$\max_{y_i, p_i} \; p_i y_i - w\frac{y_i}{\tilde{z}_i} \quad \text{s.t.} \quad y_i = \left(\frac{p_i}{P}\right)^{\frac{1}{\eta-1}} Y, \quad 0 < \eta < 1$$

- Pricing (CES): with markup $\mu = \frac{\sigma}{\sigma-1} = \frac{1}{\eta}$,

$$p_i = \mu\, MC_i = \frac{1}{\eta}\frac{w}{\tilde{z}_i}.$$

  - More productive firms ($\tilde{z}_i \uparrow$) set lower p_i, sell more, earn higher revenue/profits
  - $MR_i = (1 - \frac{1}{\sigma})p_i = \frac{p_i}{\mu}$
- Equivalent labor-choice formulation: $\max_{n_i} \; R_i(\tilde{z}_i n_i) - w n_i$

$$\Rightarrow \frac{\partial R_i}{\partial n_i} = \underbrace{\frac{\partial R_i}{\partial y_i}}_{MR_i} \underbrace{\frac{\partial y_i}{\partial n_i}}_{MPL_i = \tilde{z}_i} - w = 0 \;\Rightarrow\; MR_i \cdot MPL_i = w$$

Comparison

- **Both approaches yield a non-degenerate firm-size distribution due to the combination of productivity heterogeneity & diminishing returns**
- *But* differences in market structure are important
    - Hopenhayn: $MR = p$ common $\Rightarrow\ p \cdot MPL_i = w \Rightarrow MPL$ equalizes
    - Melitz: *Marginal Revenue Product of Labor* is equalized across firms, but MPL *isn't*
- This contrast will be important when going to the data
    - e.g. thinking about misallocation (Lecture 4)

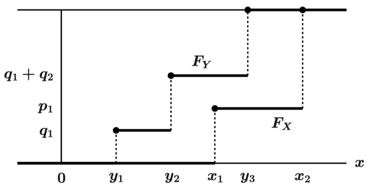
Hopenhayn (1992)

Setup

- Discrete time $t = 0, 1, 2, \ldots$
- Output and input prices, p and w, taken as given
- Household demand: exogenous demand function $D(p)$ where $D'(p) < 0$
- Continuum of measure-zero **firms**, produce with labor as only input and DRS: $y = zf(n)$, where f is s. concave
- Static profits $\pi(z, p, w) \equiv \max_n (pzf(z)) - wn - c_f$, where $c_f > 0$ is per-period fixed cost of operating
- Timing within a period
  1. incumbents decide to stay or exit, entrants decide to enter or not
  2. incumbents that stay pay c_f, entrants pay c_e (in labor units)
  3. after paying c_f or c_e, operating firms learn their productivity draws

Setup: stochastic processes

Definition: X dominates Y by First Order Stochastic Dominance (FOSD) if for all x, $F_X(x) \leq F_Y(x)$. Q. What is happening?



- Individual productivity draws follow 1st-order **Markov** process with distribution function $F(z'|z)$
  - $F(\cdot|z)$ is strictly decreasing in z, i.e. if $z_1 > z_2$ then $F(\cdot|z_1)$ first-order stochastically dominates $F(\cdot|z_2)$
- Persistence implies if your profits are high (low) today, they are expected to stay high (low) in the near future $\rightarrowtail$ important for exit decision
- Entrants draw initial productivity z_0 from separate distribution
  - Having entrants and incumbents draw productivity from different distributions allows non-trivial firm size distribution → Young firms are different from old/incumbant firms.

Overview of our next steps

- Incumbent firms' problem
- Entrant firms' problem & free-entry condition
- Distribution of firms
- Market clearing
- Equilibrium: stationary recursive competitive equilibrium
- Solution algorithm
- Comparative statics

Incumbents

- Solving the incumbent firm's usual static problem yields profits $\pi(z; p, w)$
- Let $V(z)$ denote the value of incumbency to a firm with current productivity draw z
  - implicitly also conditioning on sequence of prices $\{p_t, w_t\}_{t=0}^{\infty}$ a firm takes as given
- Bellman equation

$$V(z) = \pi(z; p, w) + \beta \max\{\int V(z')dF(z'|z), 0\} \tag{1}$$

  - implicit assumption: scrap value is zero
- Since profits are increasing in z and F is monotone, V is also increasing in z
- There exists an exit threshold z^* s.t. firm exits if $z < z^*$, solves (for interior cases)

$$\int V(z')dF(z'|z^*) = 0$$

  - threshold equalizes expected value of the firm with its scrap value
  - does this mean firms never incur negative profits?

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I don't understand it

  - threshold equalizes expected value of the firm with its scrap value
  - does this mean firms never incur negative profits? no! may expect sufficient mean-reversion in z

Entrants & free-entry condition

- Potential entrants are ex ante identical
- Recall: potential entrant must pay $c_e > 0$ to set up, draw $z \sim G(z)$, start producing next period

- Value of entrant

$$V_e(\cancel{z}) = -c_e + \beta \int V(z) dG(z)$$

- **Free-entry** condition: letting m be the mass of entrants, in equilibrium

$$\beta \int V(z) dG(z) \leq c_e,$$

V(z) is dependent upon the number of firms

with strict ~~in~~equality whenever $M > 0$

- NB: could be that for some parameters the equilibrium features no entry, i.e. $M = 0$ with $V_e(z) < 0$

Distribution of firms

- Distribution of firms over productivity space is an **aggregate state** for the economy
- Let $\mu_t(\mathcal{Z})$ be the measure of incumbents with productivity $z \in \mathcal{Z}$
  - endogenous and, in general, evolves over time
  - we will look for *stationary* equilibrium, so $\mu_{t+1} = \mu_t = \mu$
- **LoM:** the measure of incumbents with productivity $z \in [0, z')$ at $t+1$ is

$$\mu_{t+1}([0, z')) = \int F(z'|z)\mathbf{1}\left[z \geq z_t^*\right]\mu_t(dz) + m_{t+1}G(z'), \quad \forall z'$$

  - suppressing dependence on price path again

Goods market clearing

- Assumed exogenous industry demand curve $D(p)$
- Supply curve is endogenous:

$$Y = \int y(z, p_t, w_t)\mu_t(dz)$$

  - Why do (c_e, c_f) not show up? $\to$ paid in labor units
- Market clears when $Y = D(p)$
  - $D(p)$ decreasing, Y increasing in price
- Numeraire: can choose either p_t or w_t as numeraire. We'll set $w = 1$.

Stationary equilibrium

- A **stationary recursive competitive equilibrium** (SRCE) consists of a pricing function $p*$, mass of entrants m^*, cutoff productivity z^*, and distribution μ, such that
  1. goods market clears,
  2. incumbents make optimal exit decisions,
  3. there is no further incentives to enter,
  4. μ is consistent with individual decisions.

- Q: what's the difference with respect to an Aiyagari-type model?
  $\Rightarrow$ we need to also determine the endogenous number of firms

- Q: What would the equilibrium definition be if we *didn't* impose stationarity?
  $\Rightarrow$ it would consist of sequences $\{p_t, m_t, z_t^*, \mu_t\}_{t=0}^{\infty}$ such that (1)-(3) hold and μ_t is defined cursively be the law of motion

Solving for equilibrium: sketch (1)

- Competitive equilibrium of this model has a structure that is often called "**block recursive**": the equilibrium price can be computed without the information on the distribution of state variables across incumbent firms
  - simplifies life (computation speed) by a *lot*

- **Step 1:** Solving for the optimal price p^* given a positive mass of entrants $m > 0$:
  1. guess price p_0. Compute $\pi(z; p_0), n(z; p_0); y(z; p_0)$ for all grid points
  2. iterate on the Bellman equation, e.g. using VFI, yielding $V(z)$
     – the solution of this problem also implies the optimal exit rule z^*
  3. given $V(z)$, check free entry condition: if not satisfied, update guess and return to 1.1; if satisfied, move to Step 2

Solving for equilibrium: sketch (2)

- Competitive equilibrium of this model has a structure that is often called "block recursive": the equilibrium price can be computed without the information on the **distribution of state variables across incumbent firms**

- **Step 2:** use the law of motion and goods market clearing condition to find m^*
  1. (i) guess a measure of entrants, m_0; given this calculate the stationary distribution μ_0
     – the RHS depends on the price found in Part 1 via the exit threshold $z^*(p^*)$
  2. (ii) given this μ_0, calculate the total industry supply and check the market clearing condition : if not satisfied, go back to 2.1 and guess new entrant measure
     – e.g. if supply too low, guess higher entrant mass

Computing the stationary distribution

- Recall the law of motion:

$$\mu_{t+1}([0, z')) = \int F(z'|z)\mathbf{1}\left[z \geq z^*\right]\mu_t(dz) + m_{t+1}G(z'), \quad \forall z'$$

- At stationary equilibrium: $\mu_{t+1} = \mu_t = \mu^*$ and $m_{t+1} = m_t = m^*$. very simple on the computer.
- To compute this, discretize z on a grid $\{z_1, z_2, \ldots, z_{N_z}\}$
- Now μ becomes a vector and we can write:

$$\mu^* = \Phi^T \mu^* + m \cdot g$$

where:

  - Φ is the $N_z \times N_z$ transition matrix: $\Phi_{ij} = F(z_j|z_i) \cdot \mathbf{1}[z_i \geq z^*]$
  - g is the entry distribution vector: $g_i = G(z_i)$
  - m is the mass of entrants

Connection to Markov chain theory

- The stationary condition $\mu^* = \Phi^T \mu^* + m \cdot g$ is just finding the **stationary distribution of a Markov chain**
- The transition matrix Φ captures:
  - Productivity dynamics: $F(z'|z)$
  - Exit decisions: firms with $z < z^*$ exit (corresponding rows of Φ are zeros)
- The term $m \cdot g$ represents the constant inflow of entrants each period
- At steady state: exit flow = entry flow
- Standard Markov chain theory tells us this has a unique solution if exit occurs with positive probability

Method 1: Iteration

- Start with an initial guess μ_0 (e.g., $\mu_0 = m \cdot g$)
- Iterate until convergence:

$$\mu_{k+1} = \Phi^T \mu_k + m \cdot g$$

- Stop when $\|\mu_{k+1} - \mu_k\| <$ tolerance
- **Interpretation:** Each iteration adds one more generation of firms
    - $\mu_0 = m \cdot g$ (just current entrants)
    - $\mu_1 = \Phi^T(m \cdot g) + m \cdot g = m(g + \Phi^T g)$ (current + last period's survivors)
    - $\mu_2 = m(g + \Phi^T g + (\Phi^T)^2 g)$ (current + 1-2 period survivors)
    - ...
    - $\mu_\infty = m \sum_{k=0}^{\infty} (\Phi^T)^k g$ (all cohorts)
- This is a **contraction mapping**: converges exponentially fast

Method 2: Direct inversion

- Start from the stationary condition:

$$\mu^* = \Phi^T \mu^* + m \cdot g$$

- Rearrange as a linear system:

$$\mu^* - \Phi^T \mu^* = m \cdot g$$
$$(I - \Phi^T)\mu^* = m \cdot g$$

- Solve directly:

$$\boxed{\mu^* = m \cdot (I - \Phi^T)^{-1} g}$$

- This works as the infinite sum from iteration has a closed form

$$(I - \Phi^T)^{-1} = \sum_{k=0}^{\infty} (\Phi^T)^k$$

  This geometric series converges because $\rho(\Phi^T) < 1$

The two methods are equivalent

- **Iteration** explicitly computes the partial sums:

$$\mu_k = m \cdot \sum_{j=0}^{k} (\Phi^T)^j g$$

As $k \to \infty$, this converges to μ^*

- **Direct inversion** analytically evaluates the infinite sum:

$$\mu^* = m \cdot \sum_{j=0}^{\infty} (\Phi^T)^j g = m \cdot (I - \Phi^T)^{-1} g$$

- Same answer, different computational approaches! When to use which?
  - Small systems ($N_z < 1000$): direct inversion is typically faster
  - Large systems: iteration may be preferable (can exploit sparsity)
  - Iteration is often more intuitive

Exploiting linearity in entry mass

- Key insight: $\mu^*(p, m)$ is **linear** in the entry mass m

$$\boxed{\mu^*(p,m) = m \cdot \underbrace{(I - \Phi^T)^{-1} g}_{\text{distribution per unit entry}}}$$

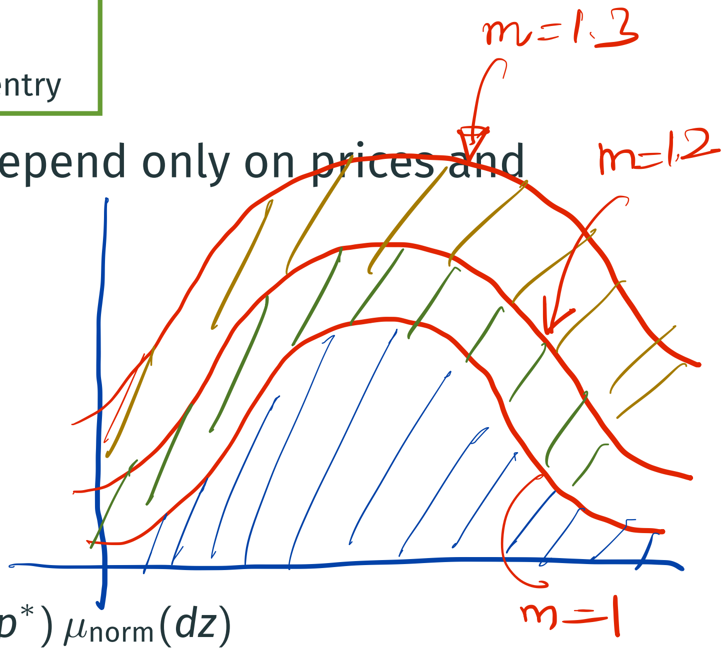
- Why? The transition matrix Φ and entry distribution g depend only on prices and policies, not on m
- This allows us to separate the problem into two steps:
  1. **Compute normalized distribution** (for $m = 1$):

$$\mu_{\text{norm}} = (I - \Phi^T)^{-1} g$$

  2. **Find entry mass m^*** that clears the goods market:

$$m^* = \frac{D(p^*)}{Y_{\text{norm}}} \quad \text{where} \quad Y_{\text{norm}} = \int y(z, p^*)\, \mu_{\text{norm}}(dz)$$

  3. **Scale:** $\mu^* = m^* \cdot \mu_{\text{norm}}$
- Computationally efficient: only need to invert $(I - \Phi^T)$ once!



Numerical example: parametrization

- Preferences and technology:

$$y = zn^{\alpha}, \quad D(p) = \bar{D}/p$$

- Firm productivity follows an AR(1) in logs:

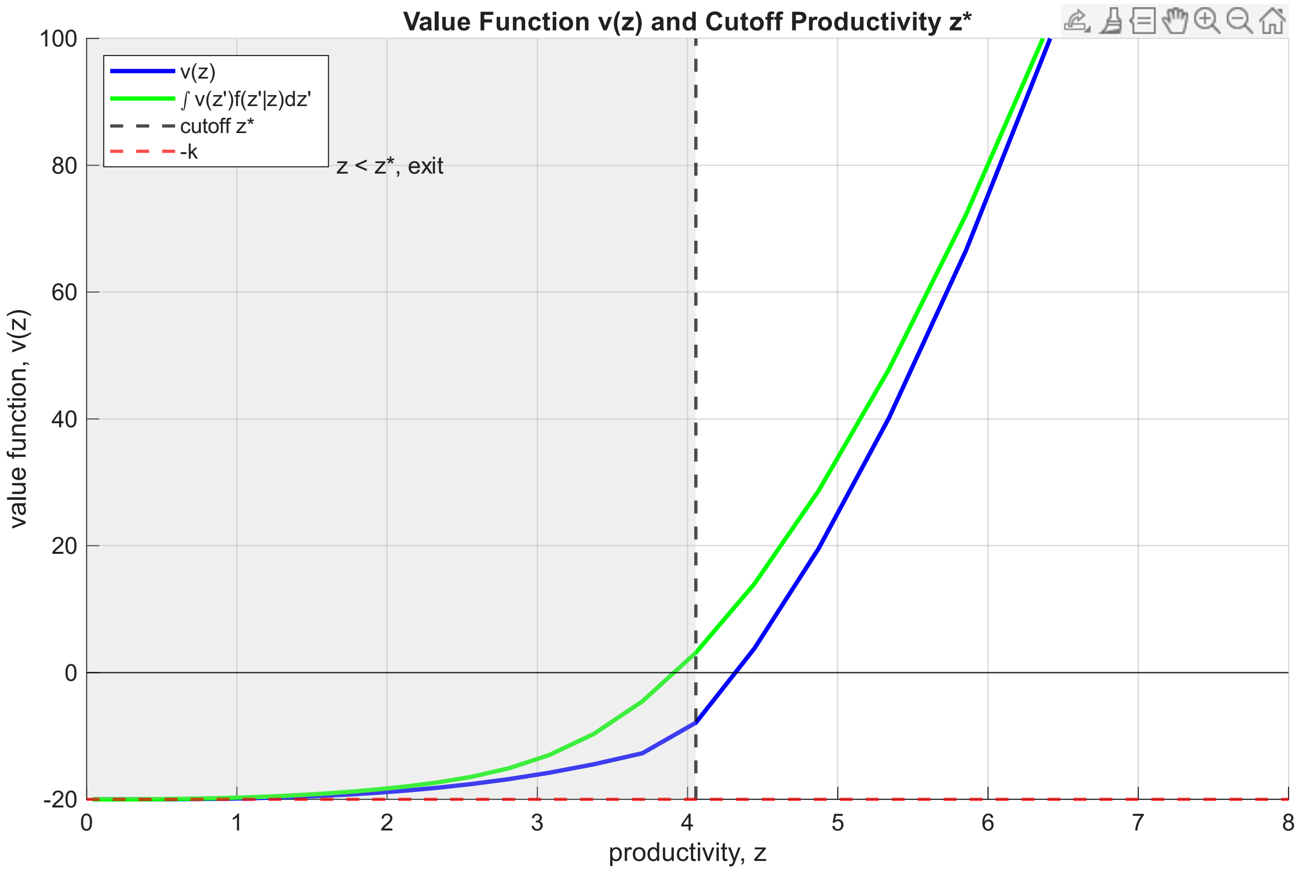
$$\log z_{t+1} = (1-\rho)\log \bar{z} + \rho \log z_t + \sigma\varepsilon_{t+1}$$

- Parameter values (where the period length is 5 years):

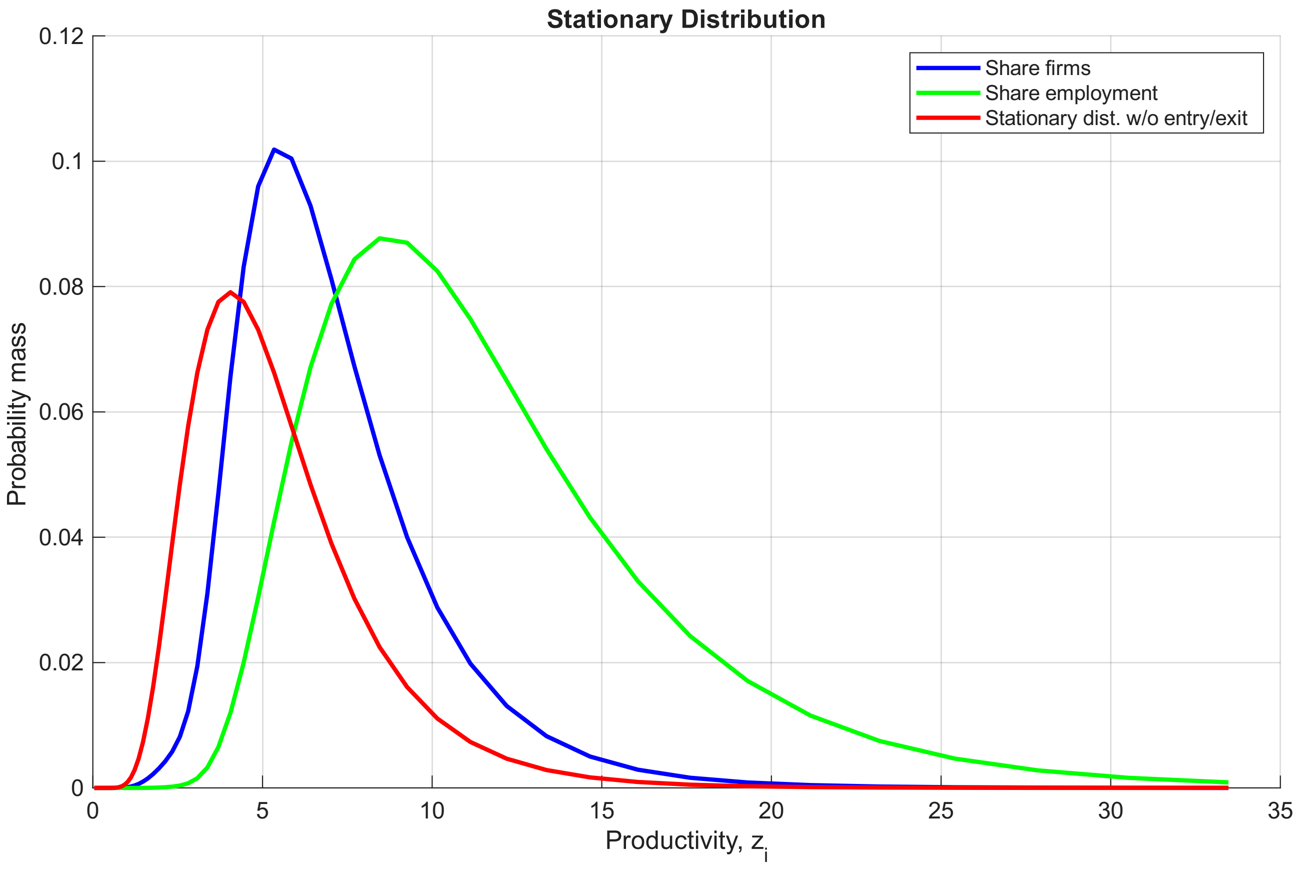
$$\begin{array}{cccc} \alpha = 2/3, & \beta = 0.80, & c_f = 20, & c_e = 40 \\ \log \bar{z} = 1.40, & \sigma = 0.20, & \rho = 0.9, & \bar{D} = 100 \end{array}$$

- Approximate the AR(1) process with a Markov chain on 101 nodes
  - standard discretization methods for stationary AR(1) processes
  - e.g. in this example I used the Tauchen method

Numerical example: value function, cutoff



Numerical example: distribution



Comparative statics: increased entry barriers (1)

- What happens as entry cost c_e increases?
  - think of total entry cost as $c_e + \kappa$, where c_e is technological and κ is an additional, wasteful cost we'll interpret as "entry barriers"

not entirely clear on it

- Unambiguous implications:
  - $\uparrow$ expected discounted profits
  - $\downarrow$ exit threshold z^* $\Rightarrow$ less selection, incumbents make more profits, more continue
  - $\downarrow$ entry rate $\Rightarrow$ higher avg. age of firms (we'll come back to this)
  - $\downarrow$ entry/ exit rate $m^*/\mu^*(\mathbb{R})$
  - $\uparrow$ price p^*

Comparative statics: increased entry barriers (2)

- Ambiguous implications for firm-size distribution
  - price effect ($p^* \uparrow$) $\Rightarrow$ increase output and employment
  - selection effect ($z^* \downarrow$) $\Rightarrow$ more incumbent firms are relatively-low productivity firms

- What does this exercise tell us about heterogeneity in policy “treatment” effects across firms?

Comparative statics: increased entry barriers (2)

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$\Rightarrow$ conflict of interest between incumbent firms and potential entrants

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  $\Rightarrow$ conflict of interest between incumbent firms and potential entrants
- What's your conjecture: how would $\kappa \uparrow$ affect selection in Melitz?

Hopenhayn & Rogerson (1993)

Overview

→ Original Hopenhayn mainly a theoretical Model

- **Motivation**: we learned in Lecture 1 that there are large labor market flows at individual firm level (job creation & destruction)
  - also, changes in employment at the firm level tend to be lumpy →
- What are the consequences of **policies** that make it costly for firms to adjust employment levels (e.g., taxes on job destruction)?
- **Hopenhayn-Rogerson (1993)**: quantitative application of Hopenhayn model
  - introduce (non-convex) adjustment costs $\Rightarrow$ a firm's lagged employment is an endogenous state variable
  - also introduce GE, instead of assuming exogenous demand
- We'll later *apply* this model to study misallocation

Setup

- Time $t = 0, 1, 2, \dots$
- Will focus on stationary equilibrium, so suppress time subscripts where no loss of clarity
- Output prices p_t and input prices taken as given; we use labor as numeraire ($w_t = 1\ \forall t$)
- Agents:
    - Heterogeneous firms: produce with labor given idiosyncratic productivity; entry and exit decisions; face tax on job destruction
    - Representative household: supplies labor elastically, chooses consumption; receives firm profits and government transfer
    - Government: collects tax from firms, rebates to household

Household

- Representative household's problem:

$$\max_{C_t, N_t} \sum_{t=0}^{\infty} \beta^t \theta \ln C_t - N_t$$

$$\text{s.t. } p_t C_t = N_t + \Pi_t - T_t$$

where $\theta > 0$ and $\beta = \frac{1}{1+r}$ → Endogeneous labour supply

- Yields consumption policy function $C(p) = \frac{\theta}{p}$ and labor supply function $N = N(p) = \theta - \Pi - T$
  - using budget constraint

Firms

- Produce final good with $y = zf(n)$, with f strictly increasing and strictly concave (DRS)
- Static profits :

$$pzf(n_t) - n_t - g(n_t, n_{t-1}) - c_f,$$

where c_f is per-period fixed cost of operating and $g(n_t, n_{t-1})$ captures **labor adjustment cost**, both in units of labor

Two state variables
n_{t-1}, z_{t-1}

- Tax in form of **firing costs** implies adjustment cost function

$$g(n_t, n_{t-1}) = \tau \times \max\{0, n_{t-1} - n_t\}, \ \tau \geq 0$$

– other specifications straightforward

- Past employment n_{t-1} is now a state variable (!)

Timing within a period

- Incumbent begins period with (z_{-1}, n_{-1})
- Decides to exit or not
- If exit, receive $-g(0, n_{-1})$ this period, zero forever after
- If stay, draw new productivity $z \sim F(z|z_{-1})$, make employment decision, receive profits, start next period

Incumbents

- Let $V(z, n)$ denote **value function** for firm that *had* employment n last period, has decided to operate, and just drawn z

- **Bellman equation:**

$$V(z, n) = \max_{n' \geq 0}\{pzf(n') - n' - g(n', n) - c_f + \beta \max[-g(0, n'), , \int V(z', n')dF(z', z)]\}$$

- Define **policy functions** for optimal **employment** $n' = n^d(z, n; p)$ and **exit** $o(z, n; p) \in \{0, 1\}$, where $o(\cdot) = 1$ is exit

Entry

- Potential entrants ex-ante identical, as before
- An entrant firm must pay the entry cost $c_e > 0$ to enter and, if they do, draw $z \sim G(z)$; start producing next period with $n_{t-1} = 0$
  - original H-R paper assumes entrants produce in same period
- Denoting by $m \geq 0$ the mass of entrants, free entry condition:

$$\beta \int V(z, 0; p) dG(z) \leq c_e,$$

with strict inequality whenever $m > 0$

Stationary distribution

- Let $\mu(z, n)$ denote the distribution of firms across states z, n
- Let $\psi(z', n'|z, n)$ denote the **transition** from (z, n) to (z', n'):

$$\psi(z', n'|z, n) = F(z'|z) \times \mathbf{1}\{n' = n^d(z, n)\} \times (1 - o(z, n))$$

  depends on the *exogenous* Markov process and the *endogenous* labor and exit policy functions

- **Law of motion:**

$$\mu_{t+1}(z', n') = \int \psi(z', n'|z, n) d\mu_t(z, n) + m_{t+1} G(z') 1\{n' = 0\}$$

- Stationary distribution has $\mu_{t+1} = \mu_t = \mu$ and, so, solves a linear system
- Stationary distribution depends on two equilibrium objects, i.e. p and m, and satisfies linearity, so $\mu(p, m) = m \times \mu(p, 1)$

Aggregation

- Aggregate production

$$Y = \int \int z f(n^d(z, n; p)) d\mu(z, n)$$

- Aggregate labor demand

$$N^d(p, m) = \int \int \left(n^d(z, n; p) + c_f \right) d\mu(z, n)$$

Solving for equilibrium: sketch

- We can solve for the equilibrium using the same approach as in the original Hopenhayn model, except both the dynamic programming problem and solving for the invariant distribution is more involved

1. Step 1a: guess price p_0 and solve the dynamic problem of incumbents
2. Step 1b: Check if p_0 satisfies the free-entry condition

$$\beta \int V(z, 0; p_0) dG(z) = c_e$$

→ You only starts producing in the next period

   - if no: return to 1a with new guess p_1
   - if yes: proceed to step 2, denote price p^*
3. Step 2a: given p^* and the associated policy functions for incumbents, consider $m = 1$ and solve for the stationary distribution $\mu(p^*, 1)$
4. Step 2b: find the scale factor m^* for the distribution $\mu(z, n)$ that ensures the goods market clears

Calibration approach in Hopenhayn-Rogerson (1993)

- One period is set at five years (because they want to have a meaningfull entry & exit of the firms)
- Assume log utility & $f(\cdot) = zn^{\alpha}$ & AR-1 for z
- Externally calibrated: $\beta = 0.8$ (4% annual real rate); $\alpha = 0.64$ (why?) Labor share
- Remaining parameters: assume that the frictionless case corresponds to the U.S. economy, find the parameter values so that various statistics from the model-generated data match the corresponding data moments
    - average size/employment & auto-correlation of employment ($\to$ stochastic process)
    - exit rate ($\to c_f$)
    - size distribution of young firms ($\to G$)
    - free-entry condition holds with $p = 1$ ($\to c_e$)
    - steady-state labor supply ($\to \theta$)

Numerical example

Firing Regulation
→ does it increase job security
→ GE effects of firing regulations

- Suppose production function $y = z n^{\alpha}$
- Adjustment cost function $\tau \times \max[0, n - n']$
- AR-1 in logs for z

$$\log z' = (1 - \rho) \log \bar{z} + \rho \log z + \sigma \epsilon'$$

- Parameter values similar to before, for illustration

$$\alpha = 2/3, \;\; \beta = 0.8, \;\; c_f = 20, \;\; c_e = 40, \log \bar{z} = 1.40, \sigma = 0.20, \rho = 0.9, \theta = 100$$

- Assume, for simplicity, the entrants draw their productivity from the invariant distribution of z

Analysis of firing taxes: inaction region

- If no adjustment costs ($\tau = 0$), marginal product of labor is equalized across firms, as before and n' is independent of n

$$zf'(n') = \frac{1}{p}$$

This will not be equalized across the firms.

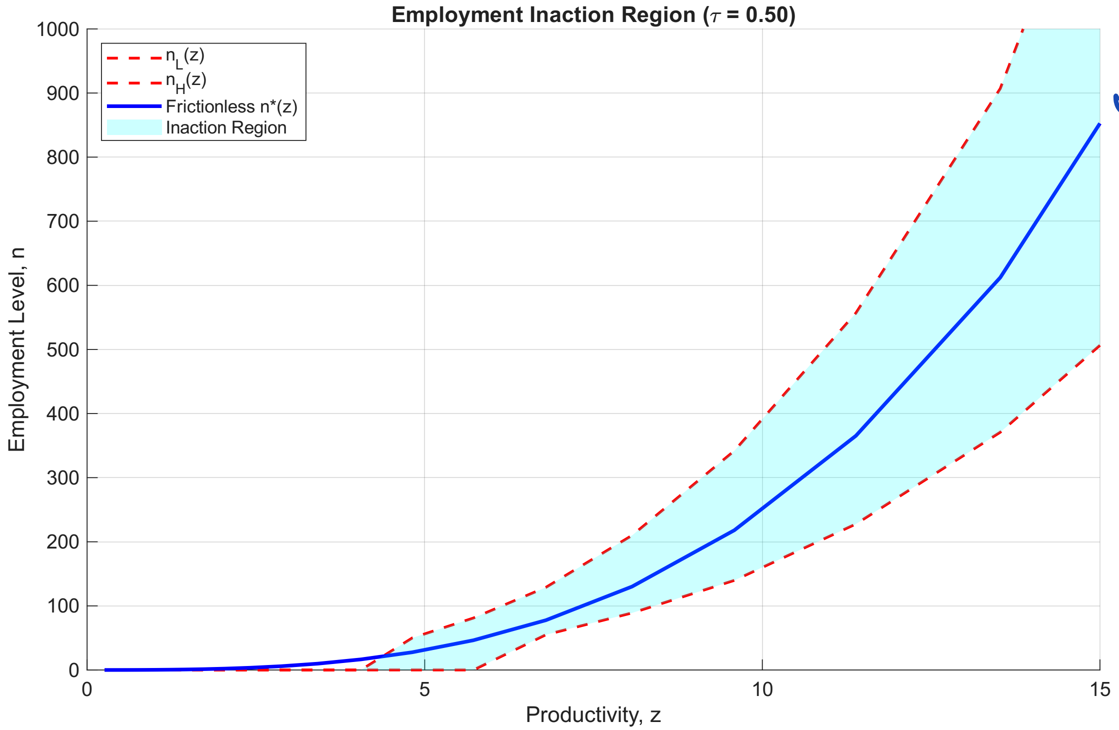
- When there are adjustment costs ($\tau > 0$), there is an **inaction region:** the firm may not find optimal to re-adjust labor - even if z has changed

$$n^d(z, n) = n' = n \quad \text{if} \quad n \in (n_L(z), n_H(z))$$

  - higher τ widens the inaction region for each z

- Implication 1: non-convex adjustment costs generate **lumpy adjustment**
  - alternative: convex (e.g., quadratic) adjustment costs implies firms will adjust slowly but no inaction region

Inaction region



Models wil uncertainity region

- Note that “frictionless” ignores fixed costs

Aggregate statistics

- Effect of higher τ on **employment** ex-ante ambiguous
  - direct effect: firms *fire* less because of the taxes
  - equilibrium effect on hiring: firms do not *hire* much even with a positive *z* shock, given that, in the future, the firm may have to fire these extra workers

τ	Price	Output	Productivity	Employment (productive)	JD (=JC) Rate
0.000	1.000	100.00	100.00	100.00	0.2877
0.200	1.026	97.49	98.89	98.58	0.2047
0.500	1.051	95.16	95.70	99.44	0.1370

$\tau \uparrow$ $P \uparrow$ | $z \downarrow$ | Employment is non-monotone | More job security

Wrap-up

Where are we in (Part I) of the course?

- Lecture 1: facts about (US) firm heterogeneity & why it might matter for macro
- Lectures 2-3: Hopenhayn & Hopenhayn-Rogerson workhorse models
- Next: To what extent does **(mis-) allocation** of resources across businesses matter for aggregate productivity?