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Math for Machine Learning

Linear algebra - Week 2

Solving systems of equations

Matrix row reduction

Row operations that preserve singularity

Row-reduced echelon form

Row echelon form

Rank of a matrix

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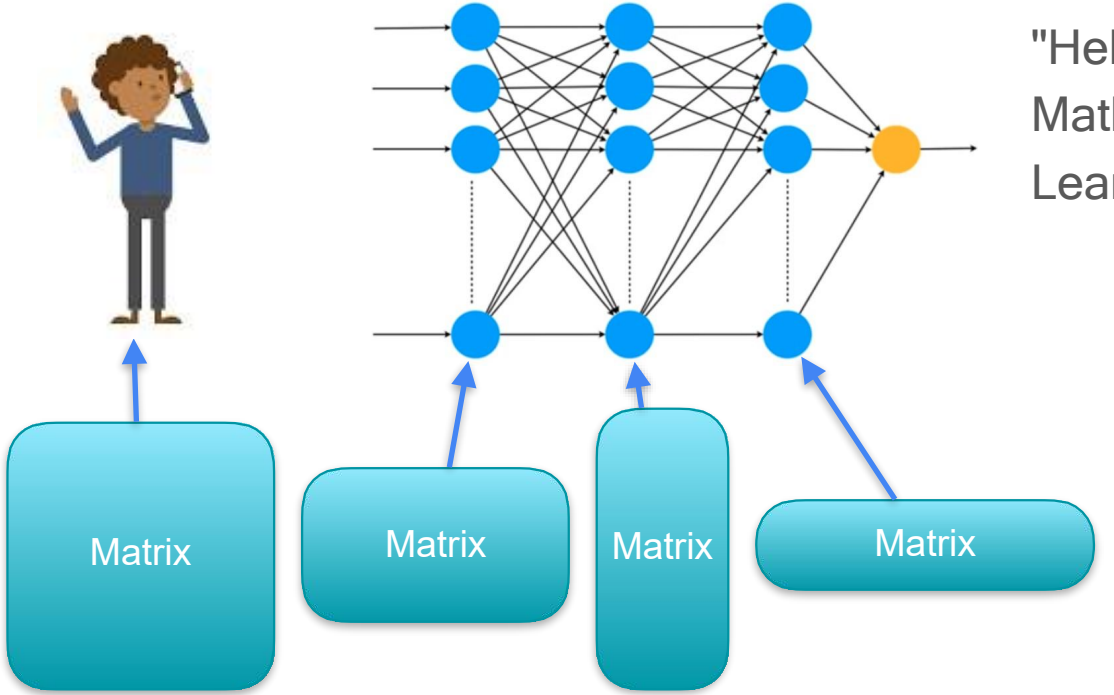
Solving System of Linear Equations

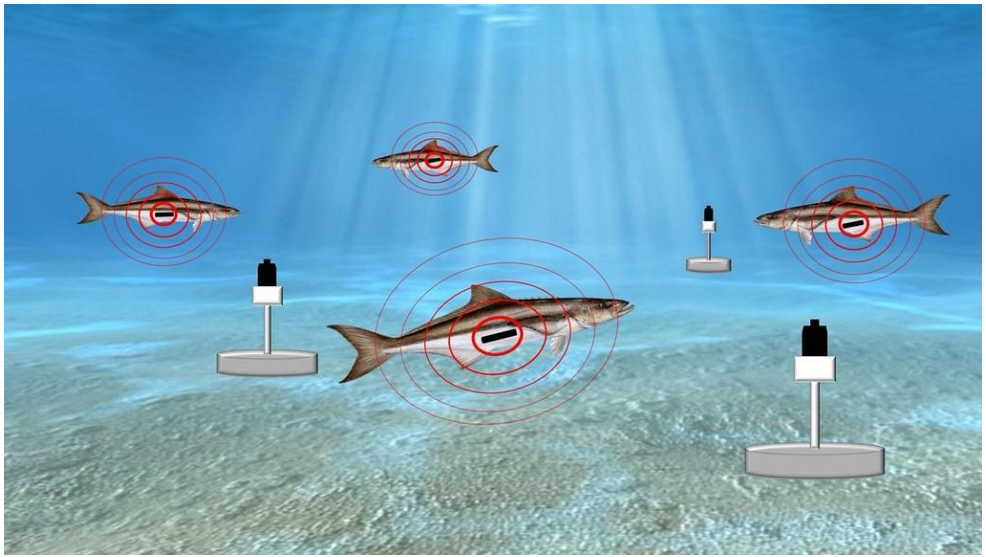
Machine learning motivation

Neural networks - Matrix operations



"Hello! Welcome to Math for Machine Learning!"

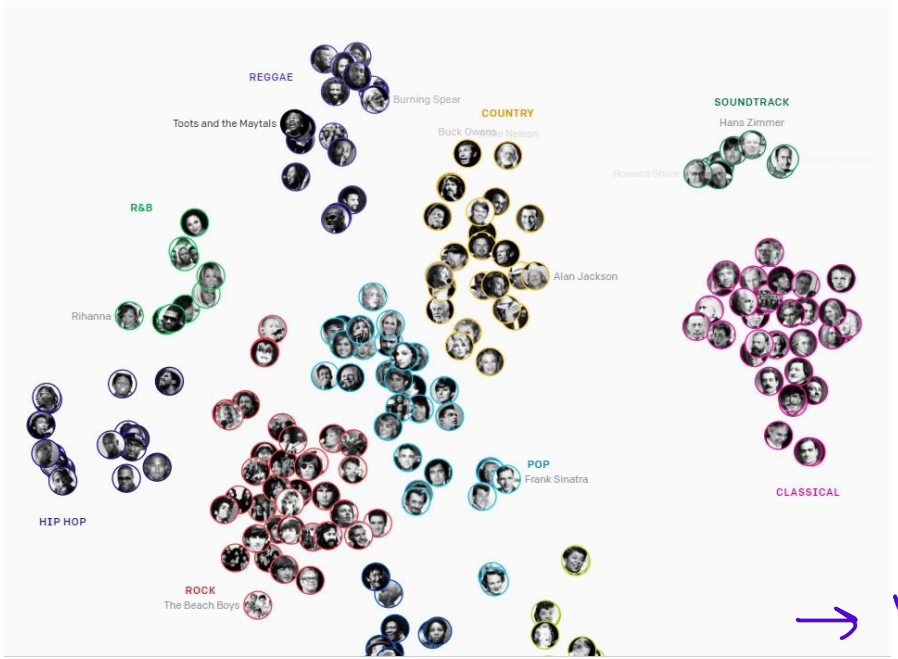
Neural networks - Sound recognition



Acoustic monitoring: Monitoring ecosystems through sounds

- Sound recognition: tracking species through sound to preserve bio-habitats.

Neural Networks - AI-generated music



Neural network generates music

- Automatic music generation: compressing music to discrete codes, then training the model on a specific genre to produce new music.

→ variational autoencoder

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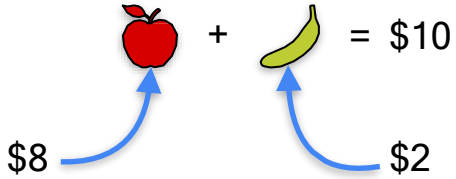
Solving System of Linear Equations

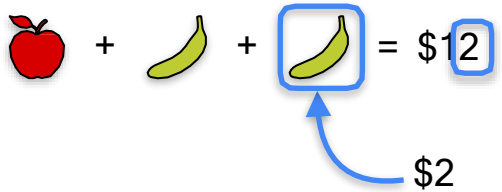
Solving non-singular system of linear equations

Solving systems of equations

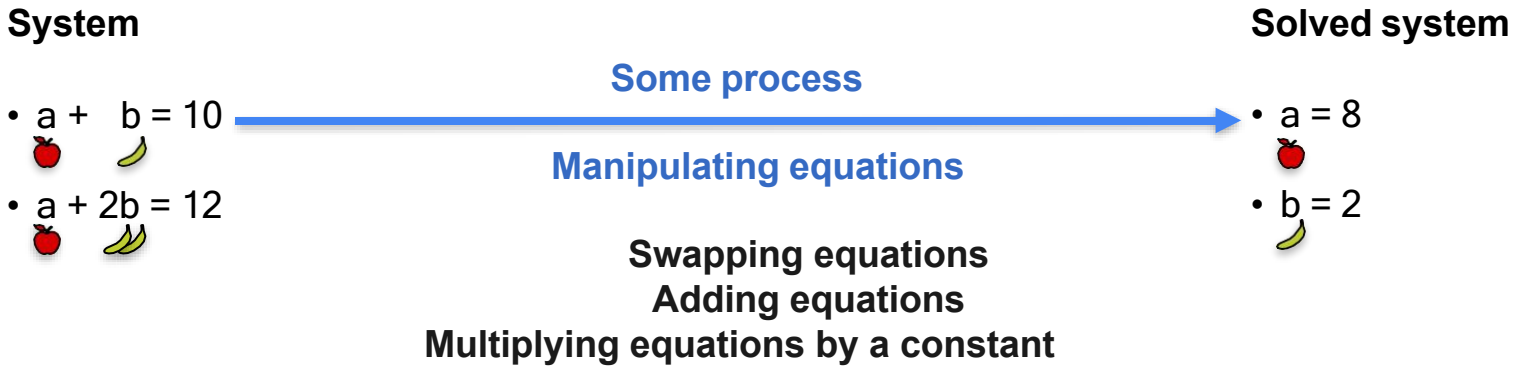
System

- a + b = 10
- a + 2b = 12

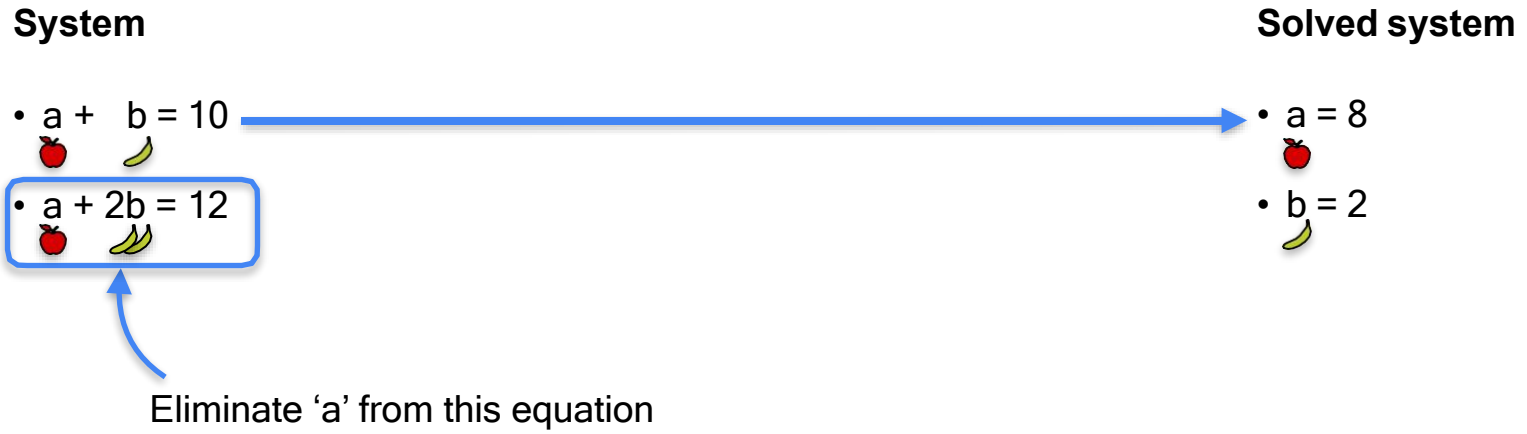




Solving systems of equations



Solving systems of equations



Manipulating equations

Multiplying by a constant

$$\begin{array}{rr} & a + b = 10 \\ * & 7 \\ \hline & 7a + 7b = 70 \end{array}$$

add 2 equations

$$\begin{array}{rr} & a + b = 10 \\ + & 2a + 3b = 22 \\ \hline & 3a + 4b = 32 \end{array}$$

Let’s do a harder example

Systems of equations

$$\begin{aligned} 5a+b&=17 \\ 4a-3b&=6 \end{aligned}$$

divide by 5 → $a + b/5 = 17/5$

divide by 4 → $a - 3/4\,b = 6/4$

$$b/5 + 3/4\,b = 17/5 - 6/4$$

$$\Rightarrow \frac{4b+15b}{20} = \frac{68-30}{20}$$

$$\Rightarrow 19b = 38$$

$$\Rightarrow b = 2$$

$$5a + 2 = 17$$

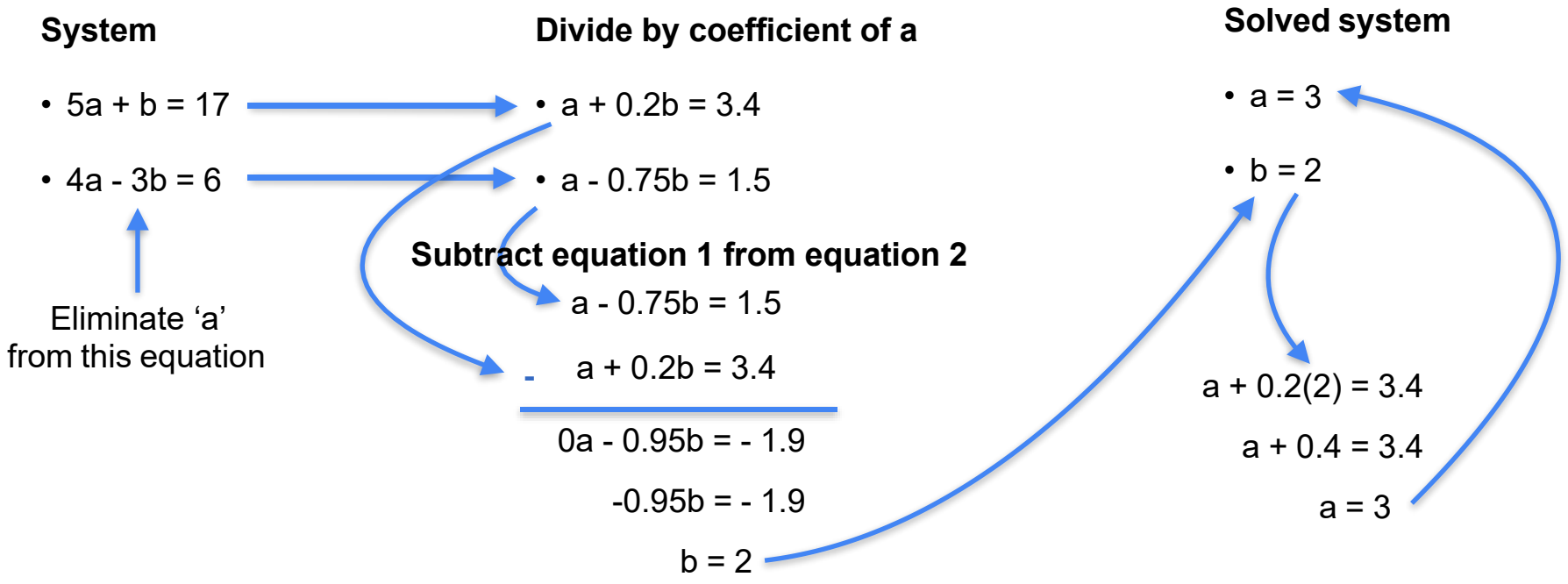
$$a = \frac{15}{5} = 3$$

$a = 3, b = 2$

solved system !!

17
× 4
68

Systems of equations



What if one of the coefficients of a is zero?

System

- $5a + b = 17$
- $3b = 6$

$b = 6/3 = 2$

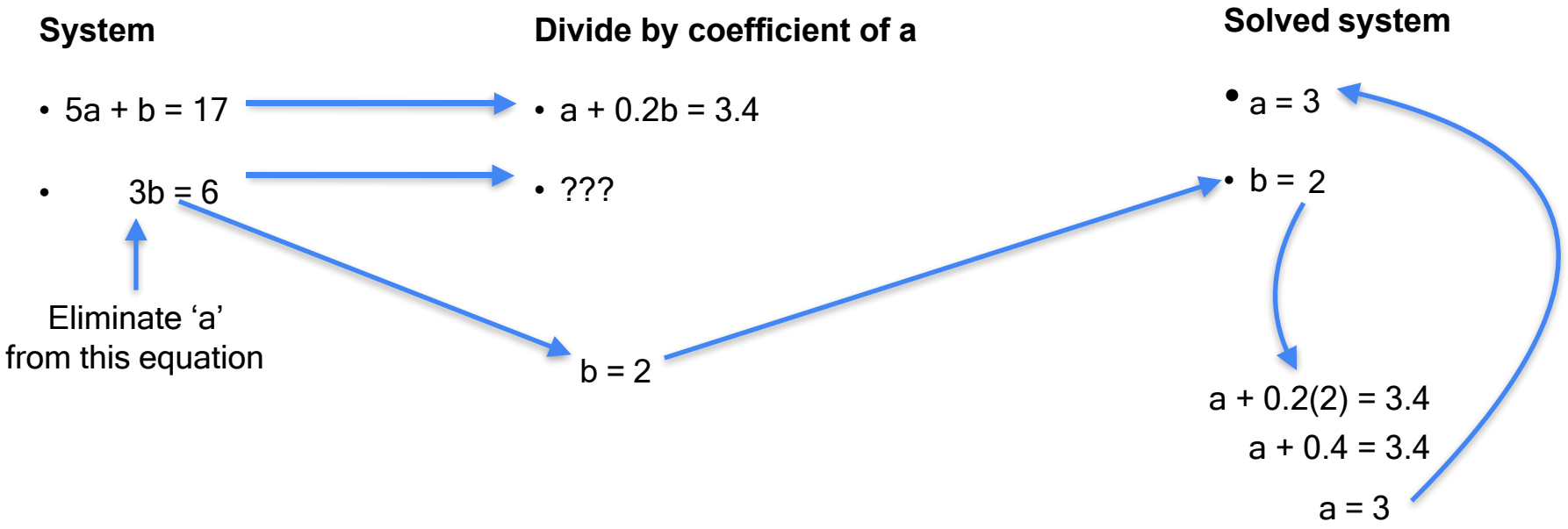
$5a + 2 = 17$

$a = 3$

Solved system

- $a = ?$
- $b = ?$

What if one of the coefficients of a is zero?



Quiz

- Solve the following system of equations

System

- $2a + 5b = 46$ $\longrightarrow$ $a + 5/2\, b = 23$
- $8a + b = 32$ $\longrightarrow$ $a + b/8 = 4$

$$(5/2)b - \frac{b}{8} = 19$$

$$\frac{5(4)b - b}{8} = 19$$

$$\frac{20b - b}{8} = 19$$

$$19b = 19(8)$$

$$2a + 5(8) = 46$$

$$2a = 6$$

$$a = 3$$

$a = 3, b = 8$

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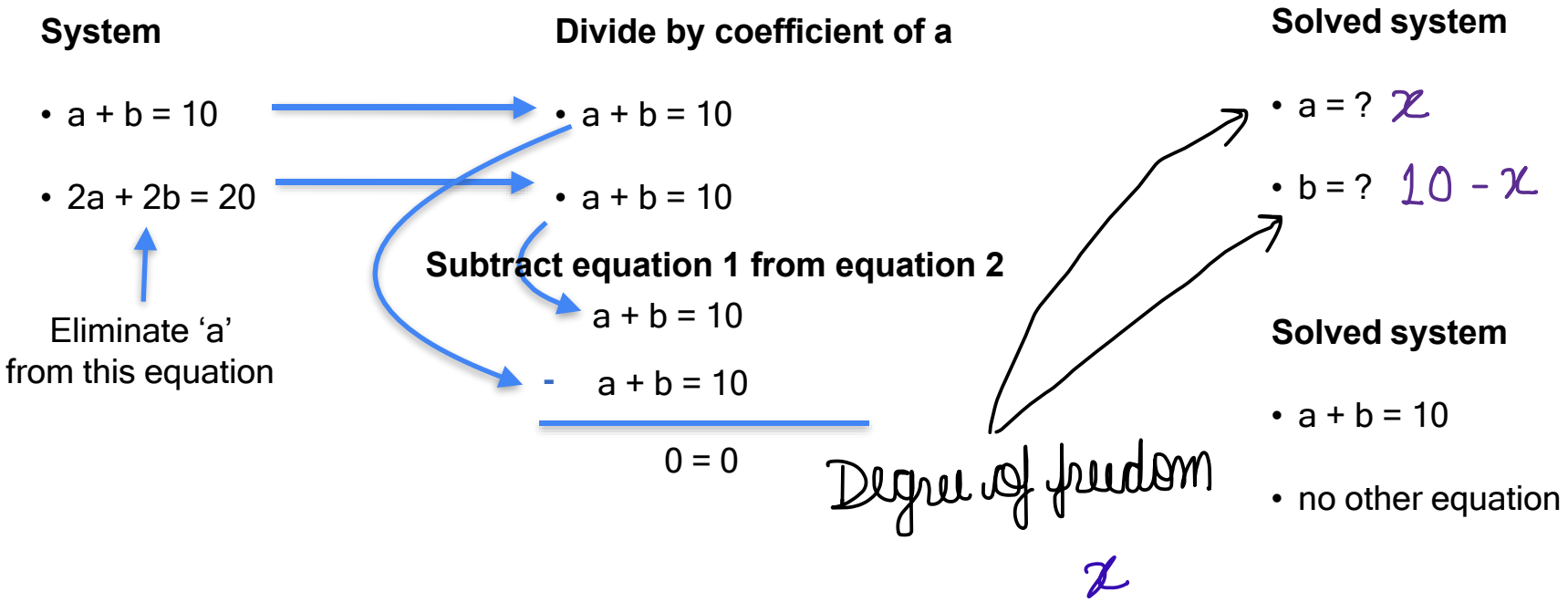
Solving System of Linear Equations

Solving singular system of linear equations

What if the system is singular (redundant)?



What if the system is singular (contradictory)?

System

- a + b = 10 → $a + b = 10$
- 2a + 2b = 24 → $-a + b = 12$

$0 \neq -2$

No solutions

Solved system

- a = ?
- b = ?

Quiz

- Solve the following system of equations

System

- $5a + b = 11$ → $a + b/5 = 11/5$
- $10a + 2b = 22$ → $a + b/5 = \cancel{22/10}\ 11/5$

$0 = 0$

Redundant !!

infinite solutions

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Solving System of Linear Equations

Solving system of equations with more variables

Elimination method

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Elimination method

$c/3 - 2c$

System

- $a + b + 2c = 12$ ⟶ $a + b + 2c = 12$ ⟶ ①
- $3a - 3b - c = 3$ ⟶ $a - b - c/3 = 1$ ⟶ ②
- $2a - b + 6c = 24$ ⟶ $a - b/2 + 3c = 12$ ⟶ ③

Leave 'a' by itself

subtract eq① from eq②

$$-2b - (7/3)c = -13$$

Elimination method

System

- a + b + 2c = **12**
- 3a - 3b - c = **3**
- 2a - b + 6c = **24**

Divide each row by the coefficient of ‘a’

- a + b + 2c = **12**
- a - b - 1/3 c = **1**
- a - b/2 + 3c = **12**

Use the first equation to remove ‘a’ from the others

- a + b + 2c = **12**
- -2b - 7/3 c = **-11**
- - 3/2 b + c = **0**

Isolated ‘a’

Solve this new system of 2 equations

Elimination method

System

- a + b + 2c = **12**
- -2b - 7/3 c = **-11**
- - 3/2 b + c = **0**

Divide last two rows by the coefficient of b

- a + b + 2c = **12**
- b + 7/6 c = **11/2**
- b - 2/3 c = **0**

Use the second equation to remove ‘b’ from the third

- a + b + 2c = **12**
- b + 7/6 c = **11/2**
- -11/6 c = **-11/2**

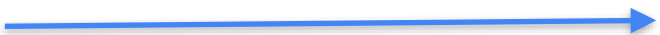
Isolated ‘b’

c = 3

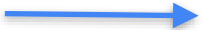
Elimination method

System

- $a + b + 2c = 12$
- $b + 7/6\ c = 11/2$
- $c = 3$



$a + 2 + 6 = 12$
$a = 4$



$b + 7/2 = 11/2$
$b = 2$

Replace c = 3 in the second equation, get b = 2

Replace c = 3 and b = 2 in the first equation, get a = 4

The solution is
a = 4
b = 2
c = 3

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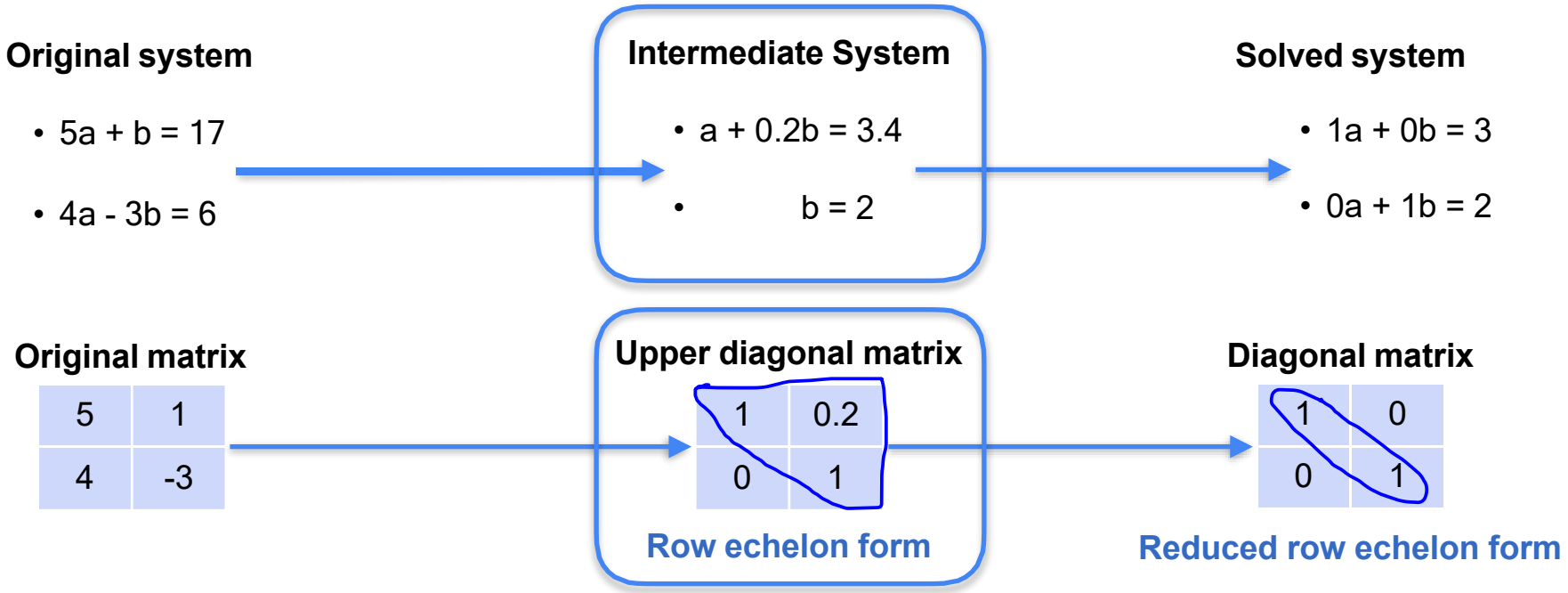
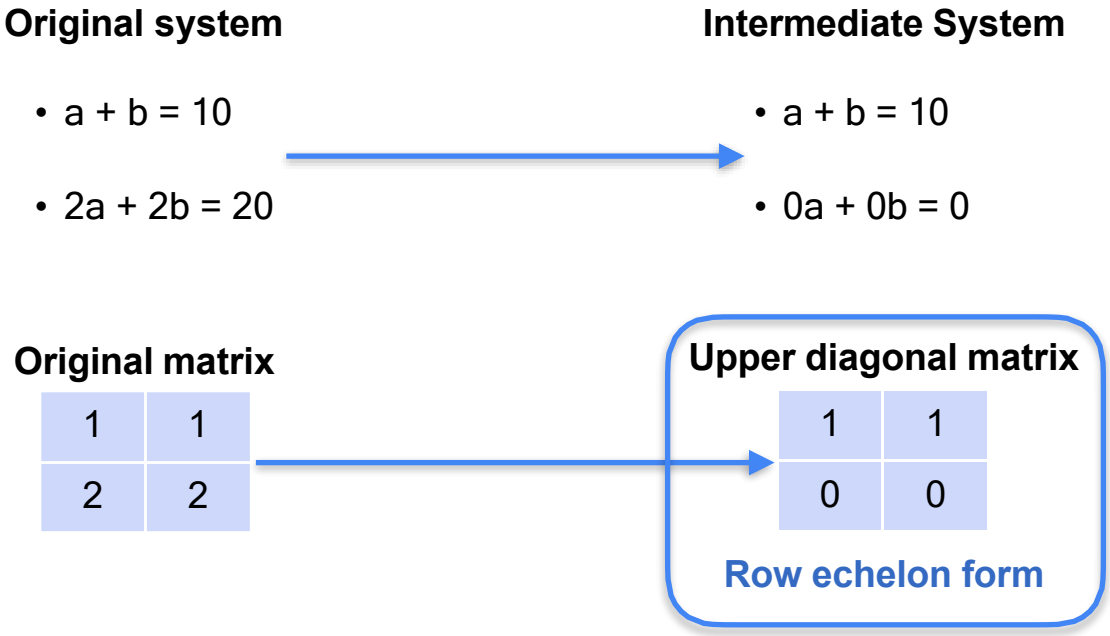
Solving System of Linear Equations

Matrix row reduction

Systems of equations to matrices



Systems of equations to matrices



Systems of equations to matrices

Original system

- 5a + b = 11
- 10a + 2b = 22

$a + 0.2b = 2.2$

$0a + 0b = 0$

original matrix

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$$

Row echelon form

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

Systems of equations to matrices

Original system

- $5a + b = 11$
- $10a + 2b = 22$

Intermediate System

- $a + 0.2b = 2.2$
- $0a + 0b = 0$

Original matrix

5	1
10	2

Upper diagonal matrix

1	0.2
0	0

Row echelon form

Systems of equations to matrices

Original system

- $0a + 0b = 0$
- $0a + 0b = 0$

Intermediate System

- $0a + 0b = 0$
- $0a + 0b = 0$

Original matrix

0	0
0	0

Upper diagonal matrix

0	0
0	0

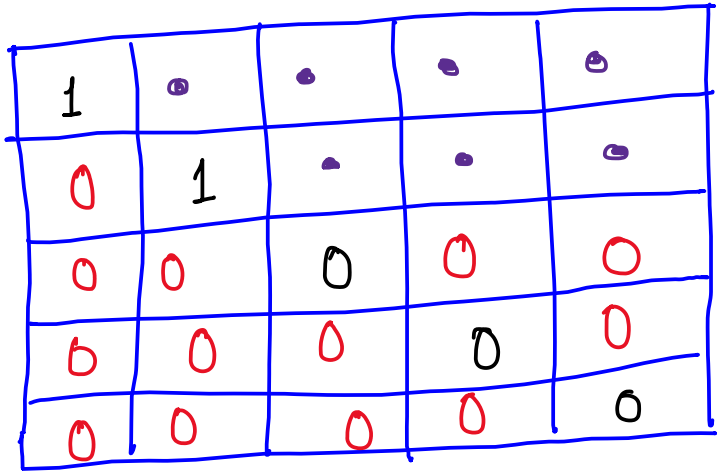
Row echelon form

Row echelon form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



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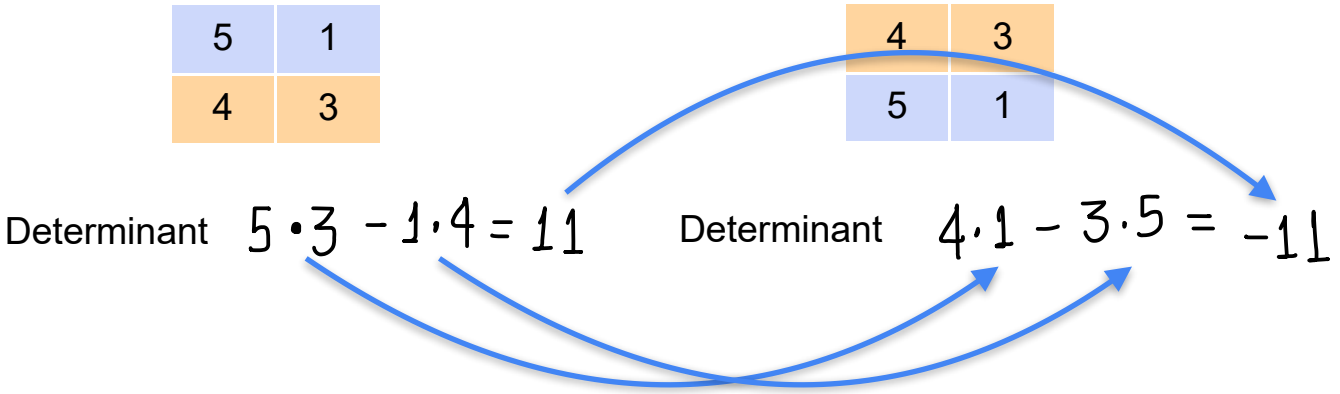
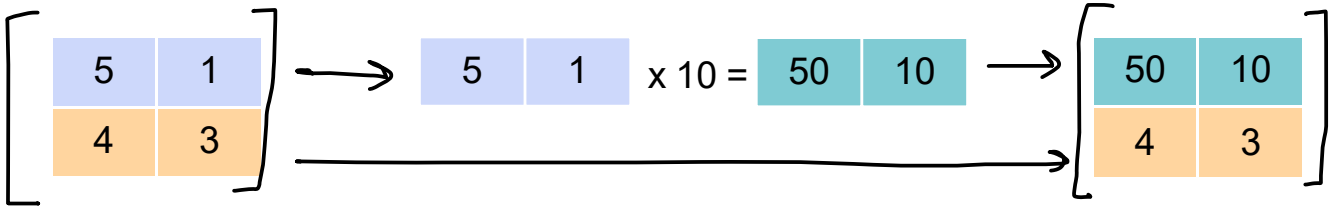
Solving System of Linear Equations

Row operations that preserve singularity

Switching rows



Multiplying a row by a (non-zero) scalar



Determinant

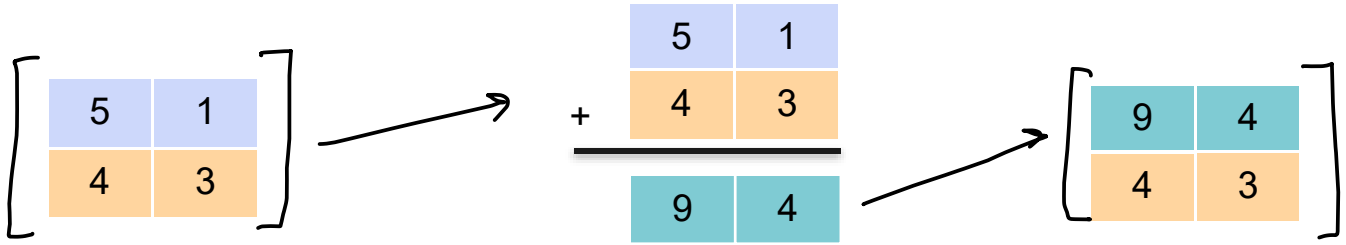
$5(3) - 4(1)$

$= 11$

Determinant $= 5 \cdot (10 \cdot 3) - 1 \cdot (10 \cdot 4)$

$= 10\,(11)$

Adding a row to another row



Determinant

$5(3) - 4(1)$

$= 11$

Determinant

$9(3) - 4(4)$

$= 11$

$$\begin{cases} x+y=4 \\ -6x+2y=16 \end{cases} \rightarrow \begin{array}{r} y+x=4 \\ y-3x=8 \\ \hline 4x=-4 \end{array}$$

$y=5$ $\quad x=-1$

$$\begin{bmatrix} -3 & 8 & 1 \\ 2 & 2 & -1 \\ -5 & 6 & 2 \end{bmatrix}$$

$$-3(4+6)-8(4-5)+1(12+16)$$

$$\Rightarrow -30+8+22=0$$

(Singular)

$$\begin{bmatrix} 4 & -3 \\ 7 & -8 \end{bmatrix} \Rightarrow -32+21$$

$$= -11 \text{ (Non singular)}$$

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix} \Rightarrow ac-ab = a(c-b) \neq 0$$

2a $s = 2\%$
a $cd = 3\%$
z $b = 4\%$

$\$10,000$

$3a + z = 10,000 \longrightarrow (1)$

$0.04a + 0.03a + 0.04z = 260$

$7a + 4z = 26000$

$1.75a + z = 6500 \longrightarrow (2)$

$1.25a = 3500$

$a = 2800$

$z = 10000 - 3(2800) = 1600$

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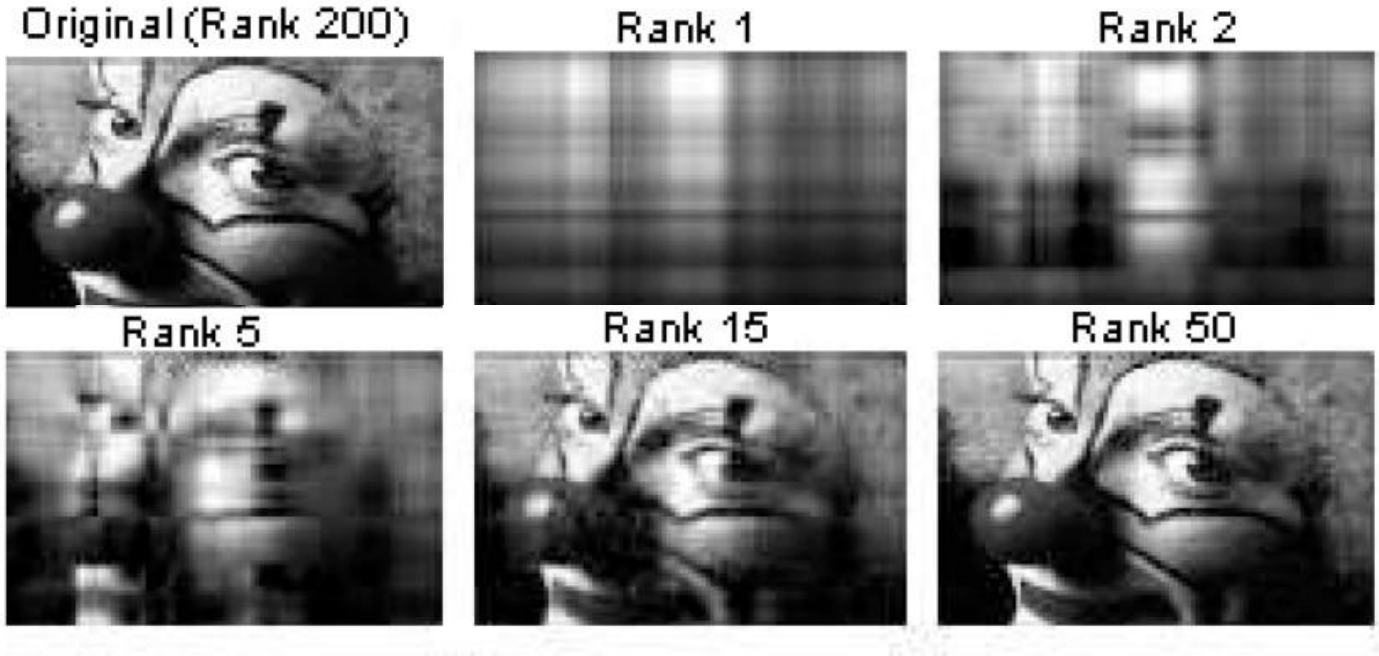
Rank → how much information is the matrix carrying / or the corresponding system of linear eqns

Solving System of Linear Equations

Rank of a matrix

singular value decomposition (SVD)

Compressing Images - Reducing rank



Systems of information

goal: "determine the color"

System 1


The dog is **black**


The cat is **orange**

Two sentences

Two pieces of information

Rank = 2

System 2


The dog is **black**


The dog is **black**

Two sentences

One piece of information

Rank = 1

System 3


The dog


The dog

Two sentences

Zero pieces of information

Rank = 0

Systems of equations

System 1

a + b = 0

a + 2b = 0

1	1
1	2

Rank = 2

Two equations

Two pieces of information

Rank = 2

System 2

a + b = 0

2a + 2b = 0

1	1
2	2

Rank = 1

Two equations

One piece of information

Rank = 1

System 3

0a + 0b = 0

0a + 0b = 0

0	0
0	0

Rank = 0

Two equations

Zero pieces of information

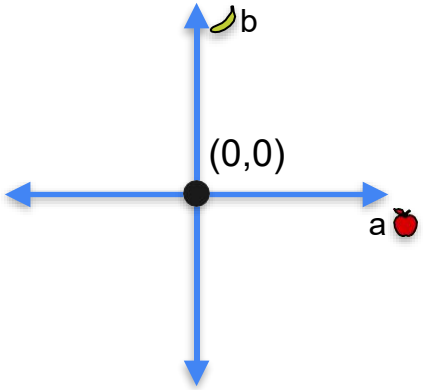
Rank = 0

Rank and solutions to the system

1	1
1	2

Rank = 2

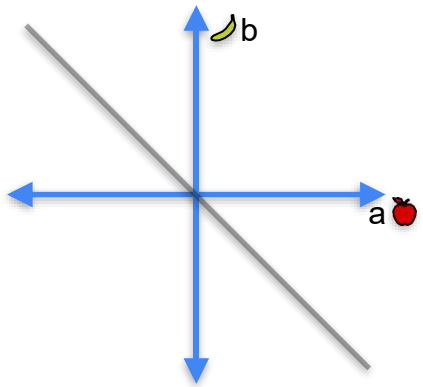
Dimension of solution space = 0



1	1
2	2

Rank = 1

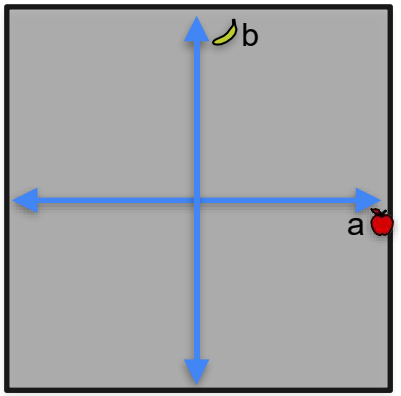
Dimension of solution space = 1



0	0
0	0

Rank = 0

Dimension of solution space = 2



Rank of a matrix

1	1
1	2

Rank = 2

Dimension of solution space = 0

1	1
2	2

Rank = 1

Dimension of solution space = 1

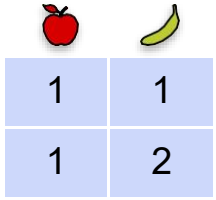
0	0
0	0

Rank = 0

Dimension of solution space = 2

Rank = 2 - (Dimension of solution space)

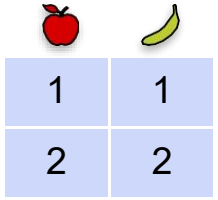
Rank and singularity



1	1
1	2

Rank = 2

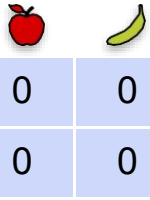
Non-singular



1	1
2	2

Rank = 1

Singular



0	0
0	0

Rank = 0

Singular

Quiz: Rank of a matrix

Determine the rank of the following two matrices

Matrix 1

5	1
-1	3

→ $15+1=16$, Non Singular

Rank = 2 # solution space of dimension 0

Matrix 2

2	-1
-6	3

→ $6-6=0$, Singular

Rank = 1

solution space of dimension 1

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Solving System of Linear Equations

Rank of a matrix: General case

Rank for matrices

System 1

$a + b + c = 0$ ✓
$a + 2b + c = 0$ ✓
$a + b + 2c = 0$ ✓

3 Equations
3 Pieces of information

Rank 3

1	1	1
1	2	1
1	1	2

System 2

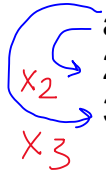
$a + b + c = 0$ ✓
$a + b + 2c = 0$ ✗
$a + b + 3c = 0$ ✓

3 Equations
2 Pieces of information

Rank 2

1	1	1
1	1	2
1	1	3

System 3



$a + b + c = 0$ ✓
$2a + 2b + 2c = 0$ ✗
$3a + 3b + 3c = 0$ ✗

3 Equations
1 Piece of information

Rank 1

1	1	1
2	2	2
3	3	3

System 4

$0a + 0b + 0c = 0$ ✗
$0a + 0b + 0c = 0$ ✗
$0a + 0b + 0c = 0$ ✗

3 Equations
0 Pieces of information

Rank 0

0	0	0
0	0	0
0	0	0

Question

- Is there an easier way to calculate the rank?
- Answer: Yes! As before, it is the number of ones in the diagonal of the reduced row echelon form of the matrix.

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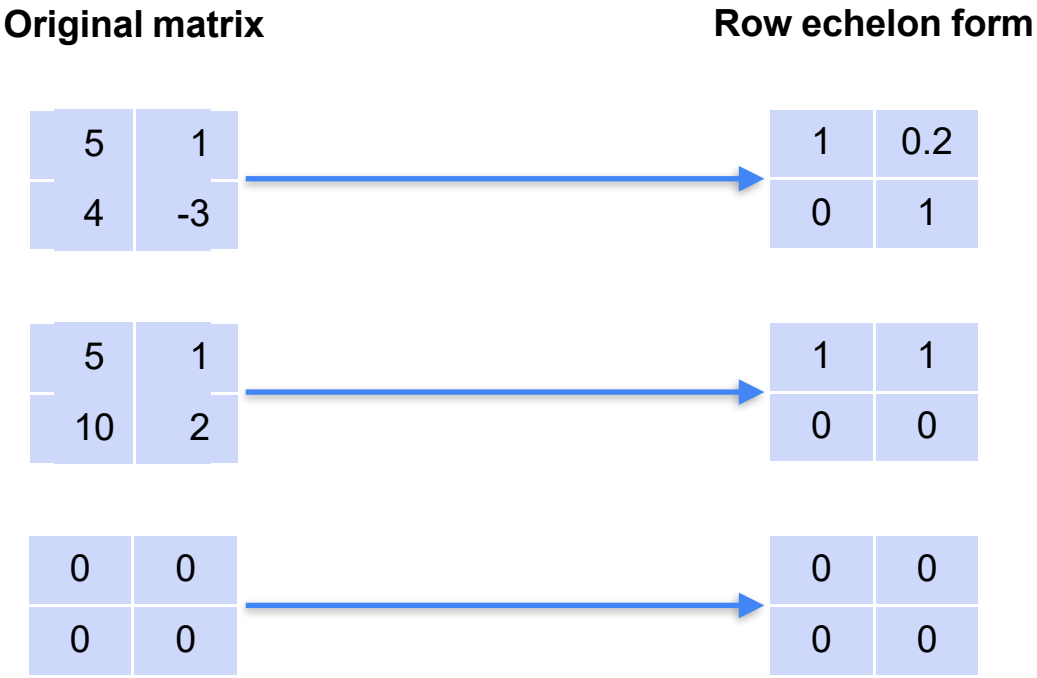
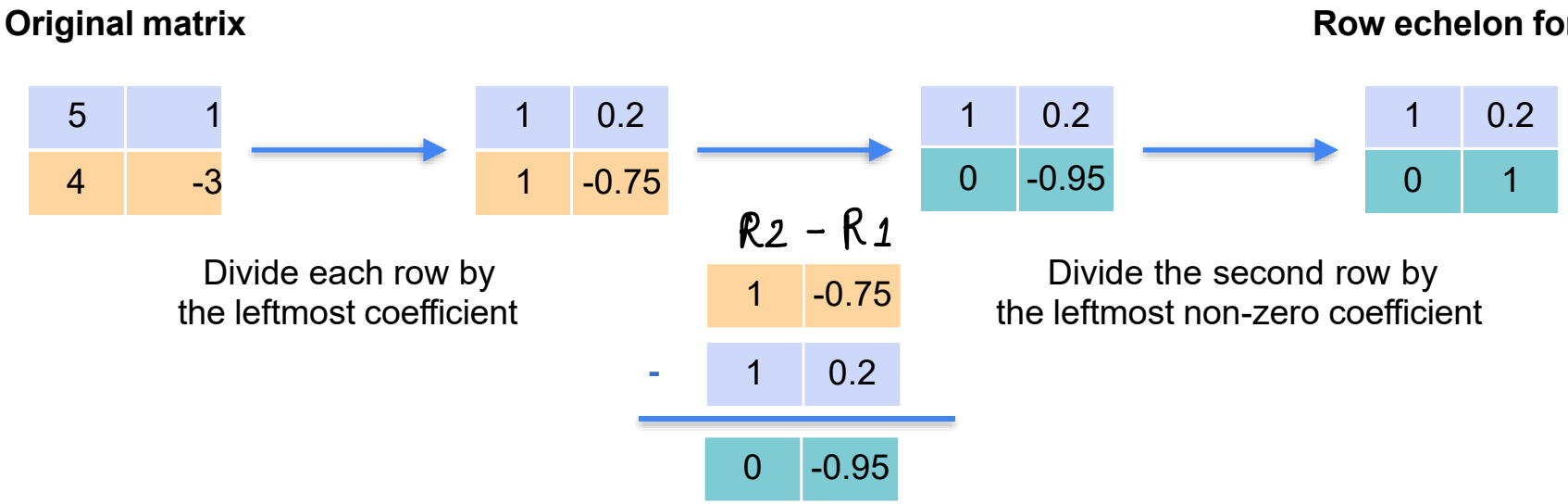
Solving System of Linear Equations

Row echelon form

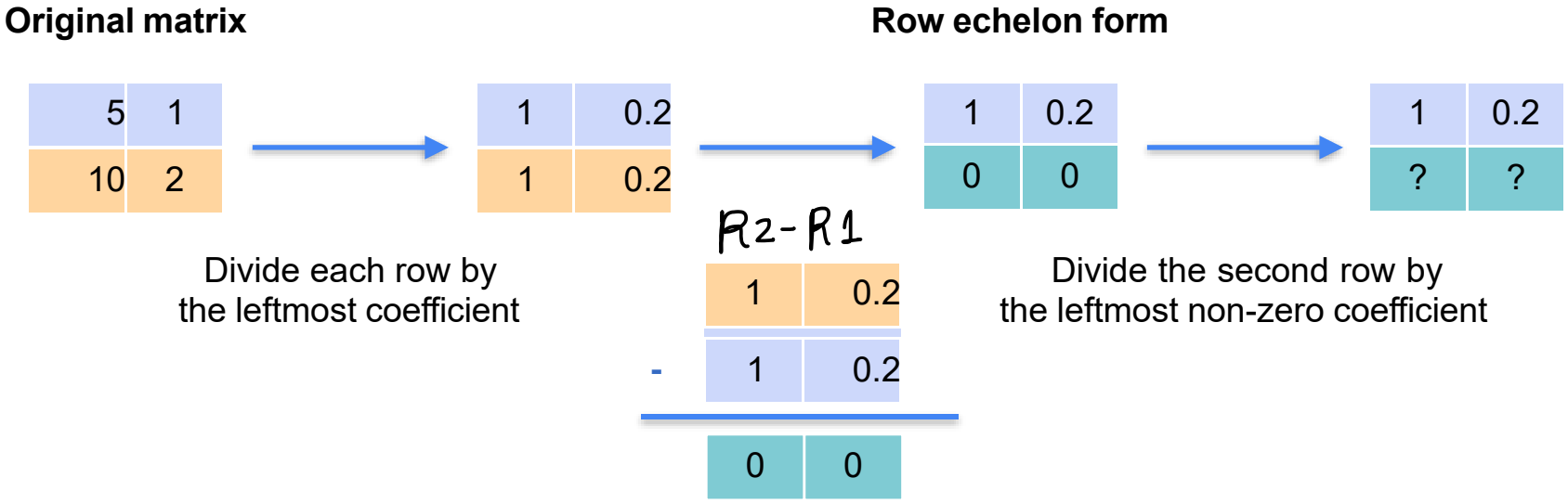
Row echelon form of a matrix



Row echelon form



Row echelon form for singular matrices



Row echelon form for singular matrices

Row echelon form

Original matrix

Divide each row by
the leftmost coefficient

Row echelon form, singularity, and rank

Non-singular matrix

5	1
4	-3

1	0.2
0	1

Rank 2

2 ones in the diagonal

Singular matrix

5	1
10	2

1	0.2
0	0

Rank 1

1 one in the diagonal

Singular matrix

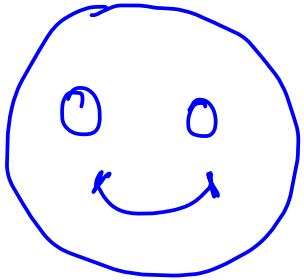
0	0
0	0

0	0
0	0

Rank 0

0 ones in the diagonal

A matrix is non singular if and only if its Row echelon form has only ones and no zeros



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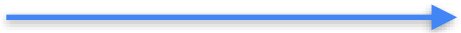
Solving System of Linear Equations

Row echelon form: General case

Row echelon form

System

- a + b + 2c = **12**
- 3a - 3b - c = **3**
- 2a - b + 6c = **24**

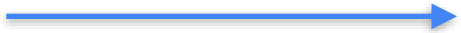


System

- a + b + 2c = **12**
- -6b - 7c = **-33**
- 6c = **18**

Matrix

1	1	2
3	-3	-1
2	-1	6

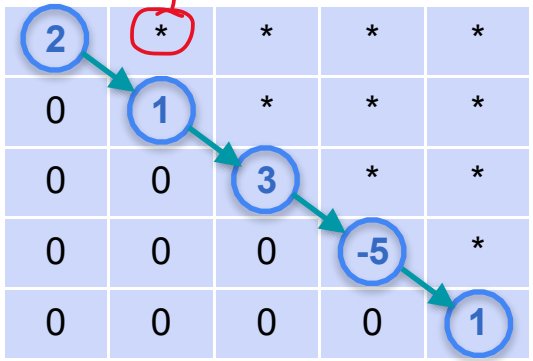


Row echelon form matrix

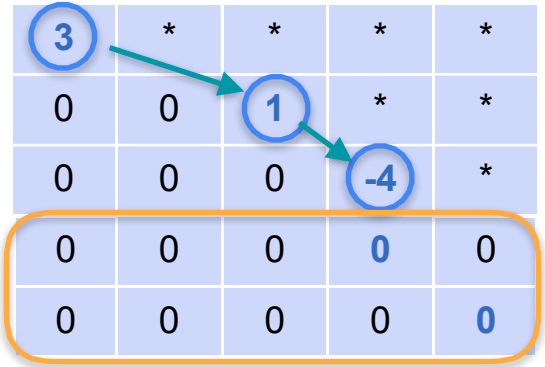
1	1	2
0	-6	7
0	0	6

Row echelon form

can be zero / non-zero "doesn't matter"

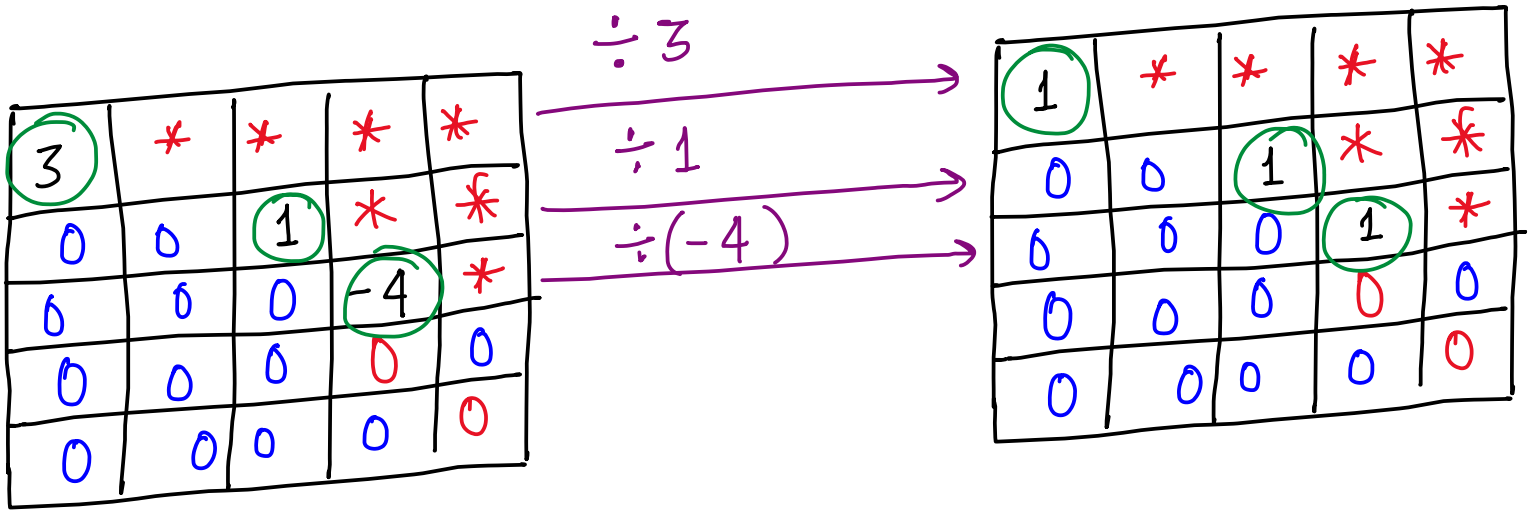


Rank 5



Rank 3

- Zero rows at the bottom
- Each row has a pivot (leftmost non-zero entry)
- Every pivot is to the right of the pivots on the rows above
- Rank of the matrix is the number of pivots



for this class, pivots are 1

Another example

Matrix

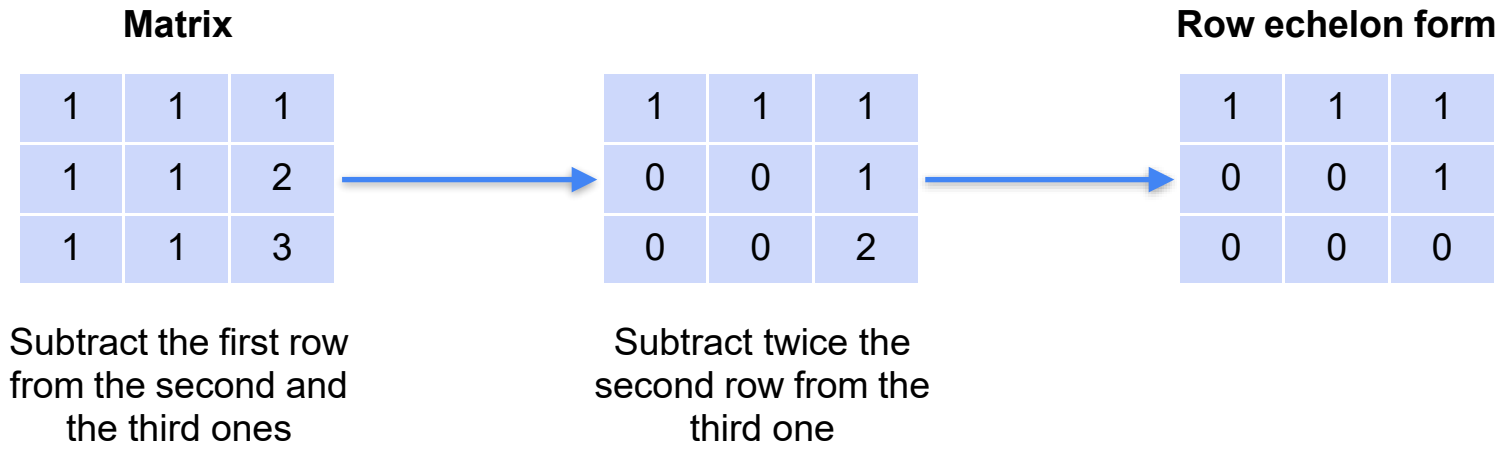
1	1	1
1	2	1
1	1	2

Row echelon form

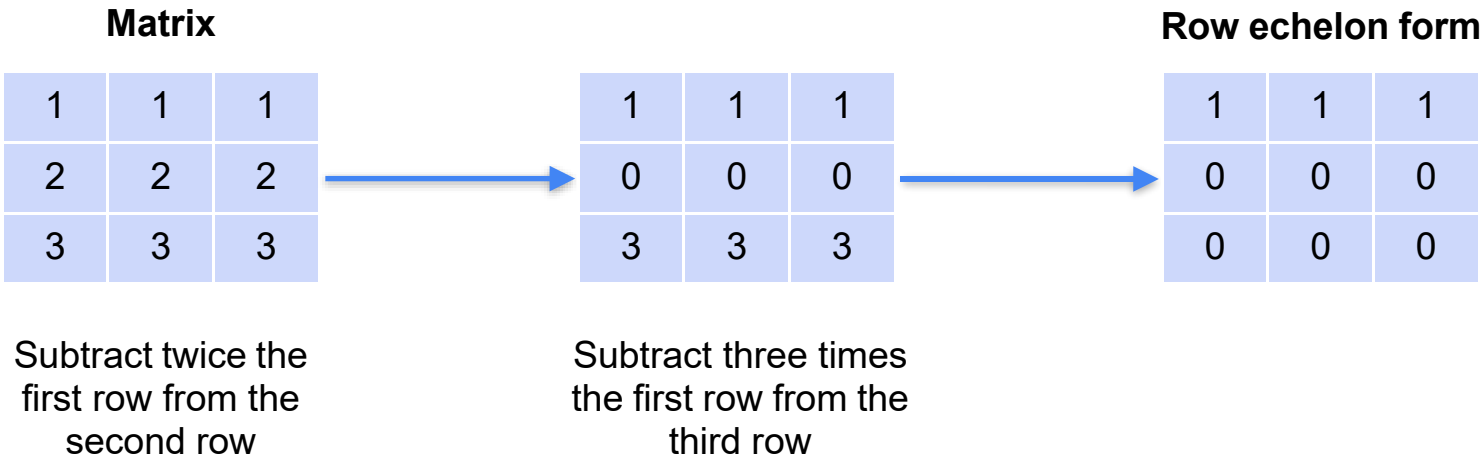
1	1	1
0	1	0
0	0	1

Subtract the first row from the second and the third ones

What if the matrix is singular?



What if the matrix is singular?



Rank for matrices

Matrix 1

1	1	1
1	2	1
1	1	2

Rank = 3

Matrix 2

1	1	1
1	1	2
1	1	3

Rank = 2

Matrix 3

1	1	1
2	2	2
3	3	3

Rank = 1

Matrix 4

0	0	0
0	0	0
0	0	0

Rank = 0

Row echelon forms

1	1	1
0	1	0
0	0	1

Number of pivots = 3

1	1	1
0	0	1
0	0	0

Number of pivots = 2

1	1	1
0	0	0
0	0	0

Number of pivots = 1

0	0	0
0	0	0
0	0	0

Number of pivots = 0

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Solving System of Linear Equations

Reduced row echelon form

"solved system"

Systems of equations to matrices

Original system

- $5a + b = 17$
- $4a - 3b = 6$

Intermediate System

- $a + 0.2b = 3.4$
- $b = 2$

Solved system

- $1a + 0b = 3$
- $0a + 1b = 2$

Original matrix

5	1
4	-3

Upper diagonal matrix

1	0.2
0	1

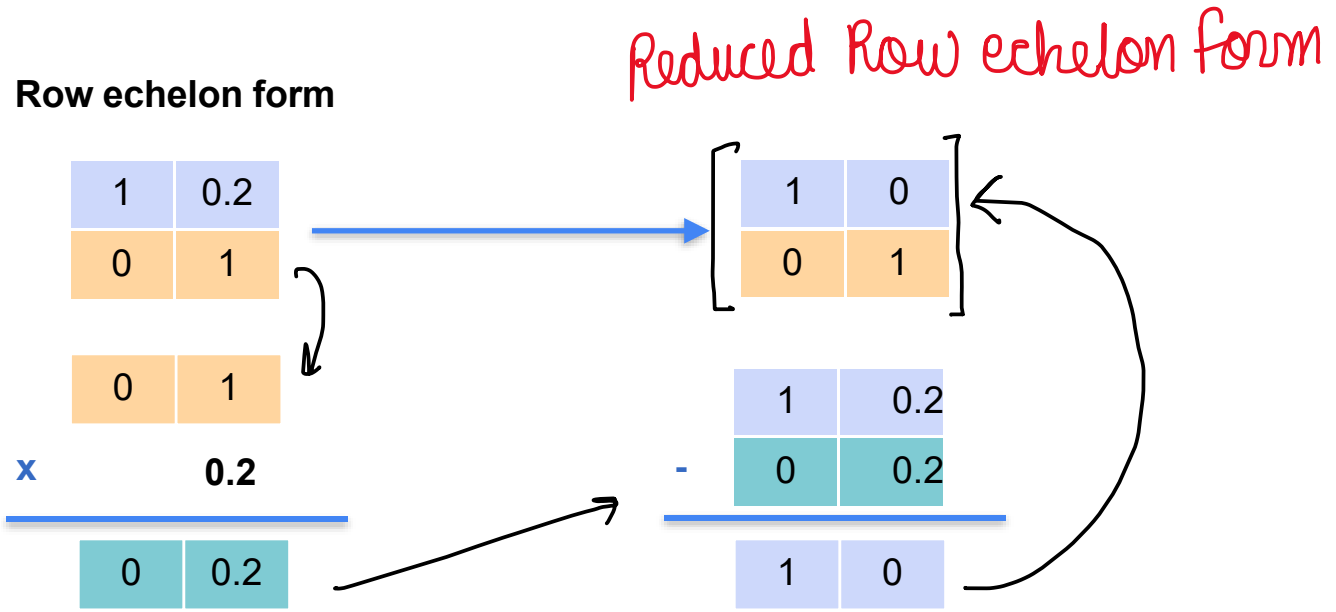
Row echelon form

Diagonal matrix

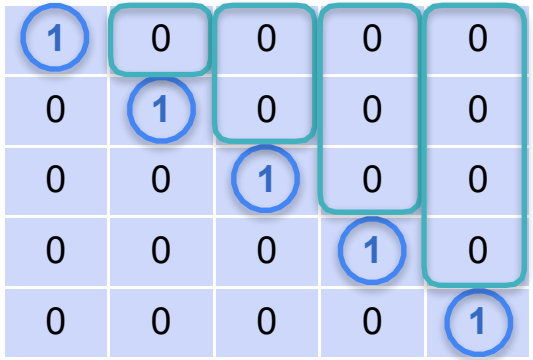
1	0
0	1

Reduced row echelon form

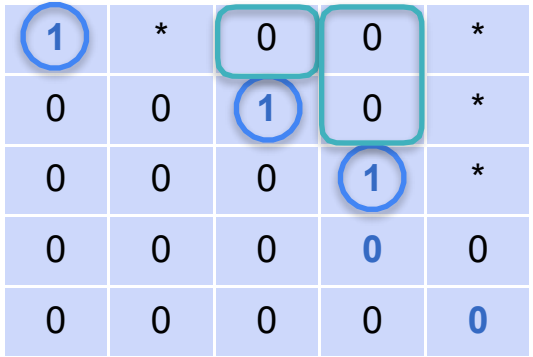
Reduced row echelon form



Reduced row echelon form



Rank 5



Rank 3

- Is in row echelon form
- Each pivot is a 1
- Any number above a pivot is 0
- Rank of the matrix is the number of pivots

Reduced row echelon form

Row echelon form

3	*	*	*	*
0	0	2	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Divide each row by the value of the pivot

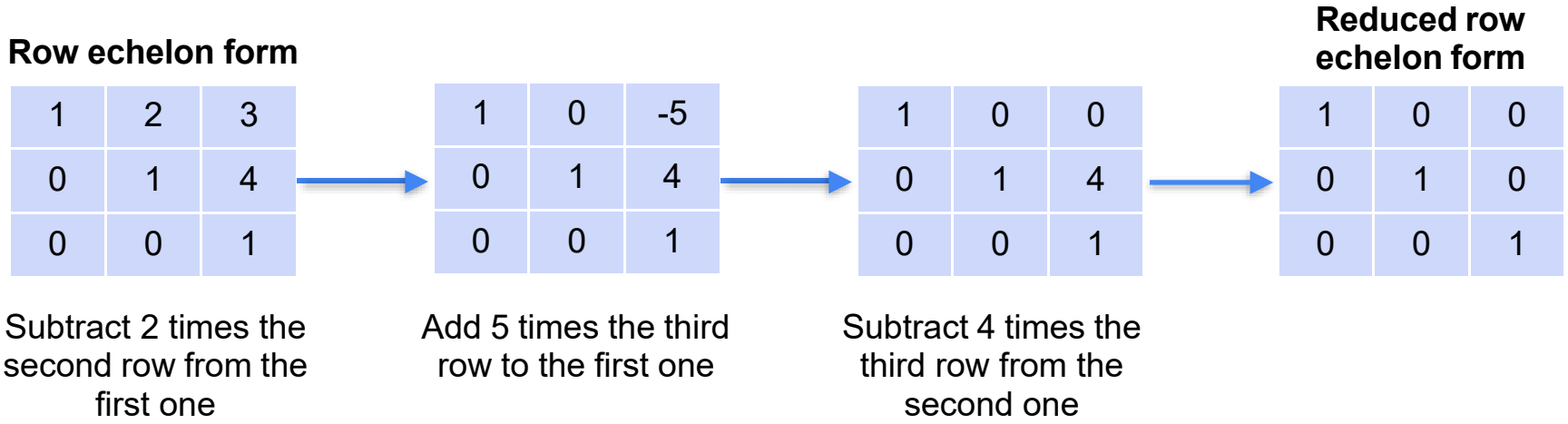
(leading coefficients)

Reduced row echelon form

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Turn anything above a pivot to 0

Reduced row echelon form



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Solving System of Linear Equations

Conclusion