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Math for Machine Learning

Linear algebra - Week 2

Solving systems of equations

Matrix row reduction

Row operations that preserve singularity

Row-reduced echelon form

Row echelon form

Rank of a matrix

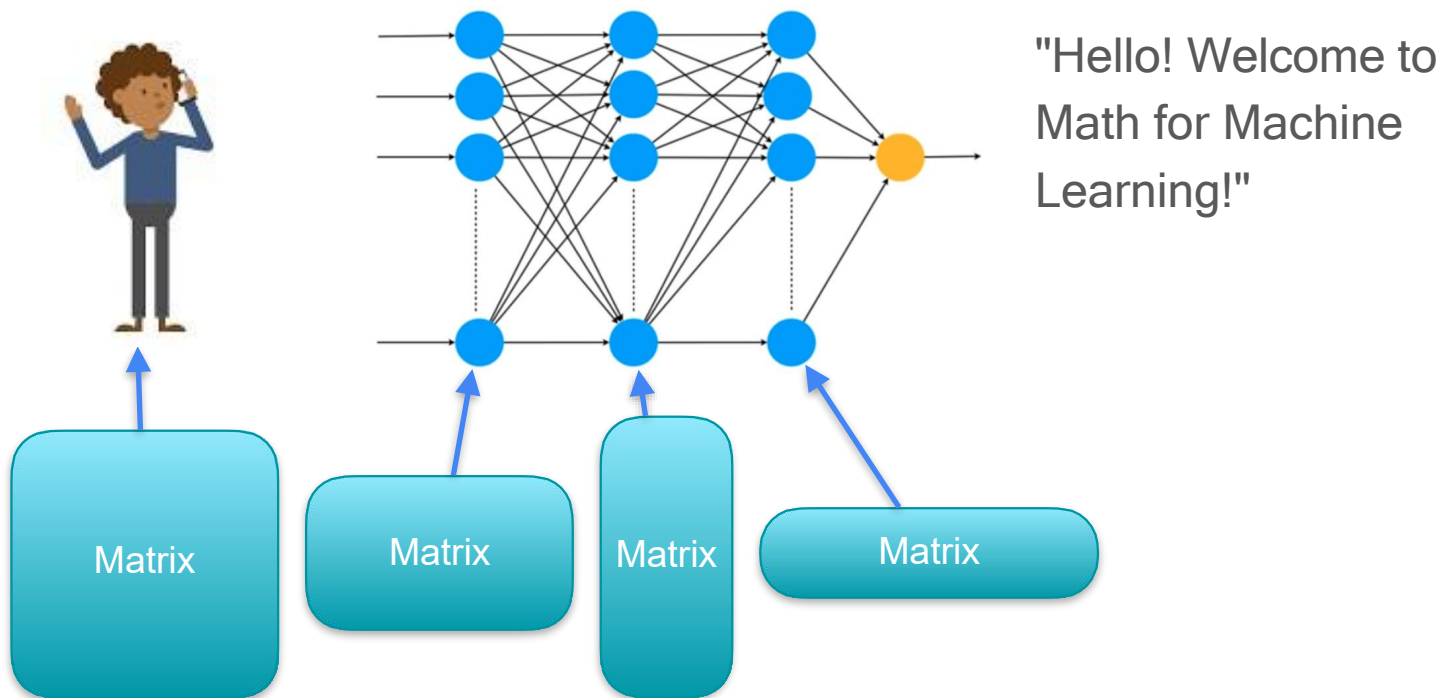


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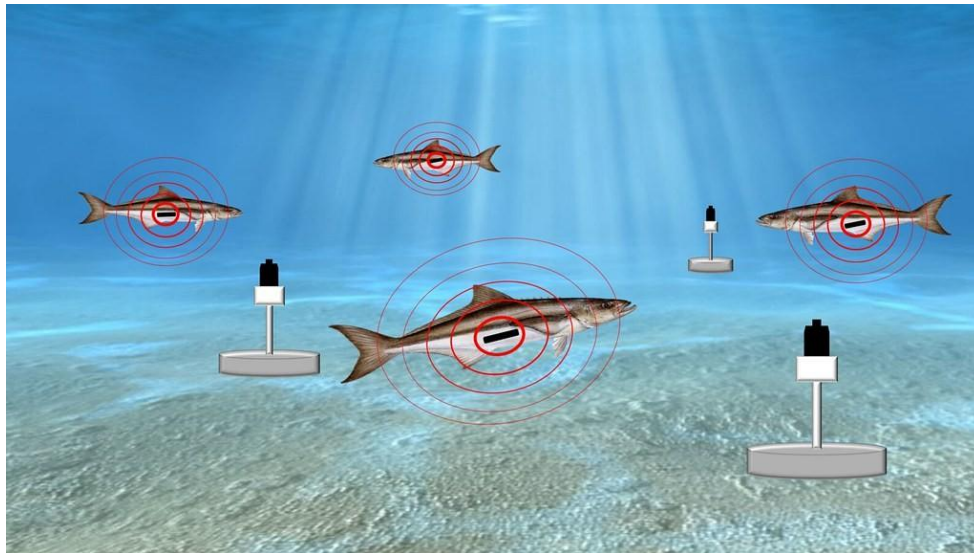
Solving System of Linear Equations

Machine learning motivation

Neural networks - Matrix operations



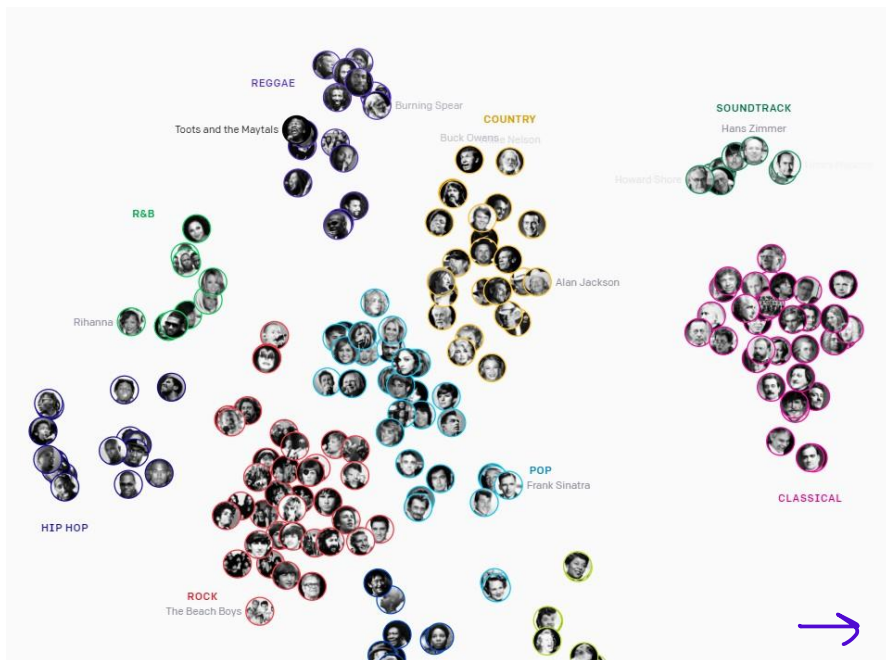
Neural networks - Sound recognition



Acoustic monitoring: Monitoring ecosystems through sounds

- Sound recognition: tracking species through sound to preserve bio-habitats.

Neural Networks - AI-generated music



Neural network generates music

- Automatic music generation: compressing music to discrete codes, then training the model on a specific genre to produce new music.

→ Variational autoencoder



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Solving System of Linear Equations

**Solving non-singular system
of linear equations**

Solving systems of equations

System

- $a + b = 10$



- $a + 2b = 12$







A diagram illustrating the substitution process for the first equation. It shows a red apple icon followed by a plus sign, then a yellow banana icon, followed by an equals sign and '\$10'. A blue arrow points from the text '\$8' to the apple icon. Another blue arrow points from the text '\$2' to the banana icon.

A diagram illustrating the substitution process for the second equation. It shows a red apple icon followed by a plus sign, then a yellow banana icon, followed by a plus sign, then a yellow banana icon inside a blue square box, followed by an equals sign and '\$12' (where the '12' is also in a blue square box). A blue arrow points from the text '\$2' to the boxed banana icon.

Solving systems of equations

System

- $a + b = 10$
 
- $a + 2b = 12$
 

Some process



Manipulating equations

Swapping equations

Adding equations

Multiplying equations by a constant

Solved system

- $a = 8$

- $b = 2$


Solving systems of equations

System

- $a + b = 10$



- $a + 2b = 12$



Eliminate 'a' from this equation

Solved system

- $a = 8$



- $b = 2$



Manipulating equations

Multiplying by a constant

$$\begin{array}{r} a + b = 10 \\ * \quad \quad \quad 7 \\ \hline 7a + 7b = 70 \end{array}$$

add 2 equations

$$\begin{array}{r} a + b = 10 \\ + \quad 2a + 3b = 22 \\ \hline 3a + 4b = 32 \end{array}$$

Let's do a harder example

Systems of equations

$$5a + b = 17$$

$$4a - 3b = 6$$

$$\rightarrow \text{divide by } 5 \rightarrow a + b/5 = 17/5$$

$$\rightarrow \text{divide by } 4 \rightarrow a - 3/4b = 6/4$$

$$b/5 + 3/4b = 17/5 - 6/4$$

$$\Rightarrow \frac{4b + 15b}{\cancel{20}} = \frac{68 - 30}{\cancel{20}}$$

$$\Rightarrow 19b = 38$$

$$\Rightarrow b = 2$$

$$a = 3, b = 2$$

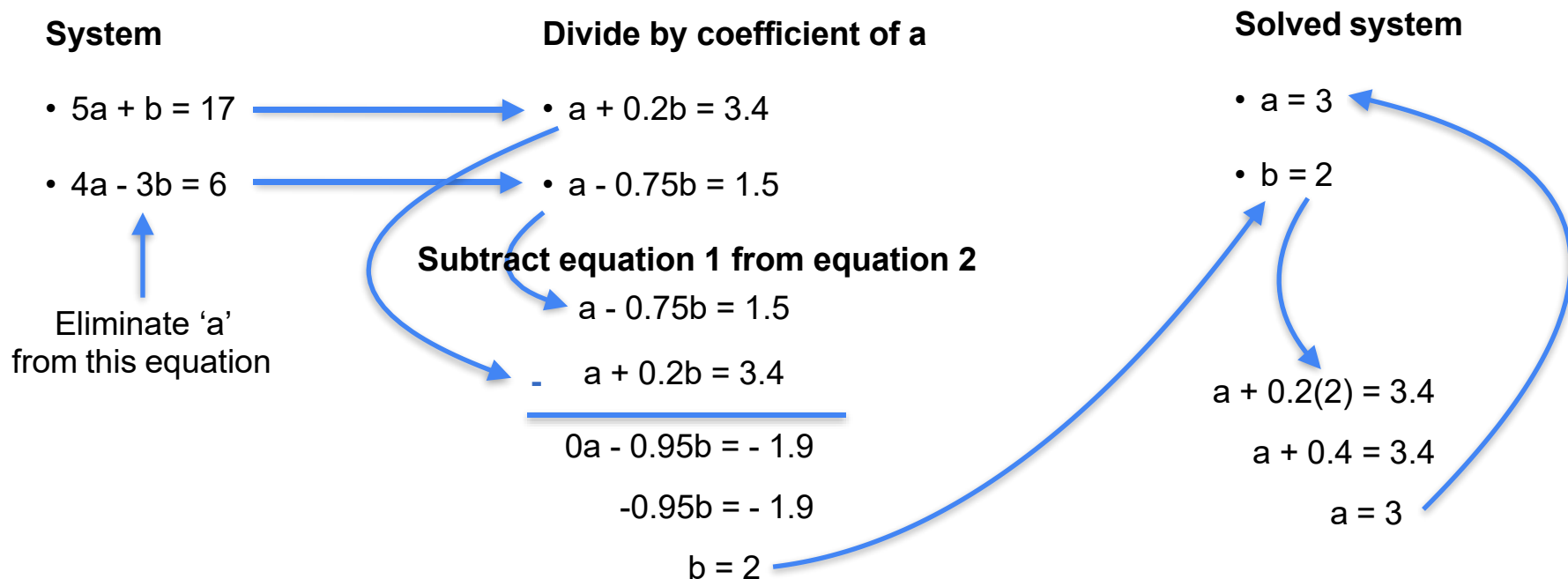
$$5a + 2 = 17$$

$$a = \frac{15}{5} = 3$$

solved system!!

$$\begin{array}{r} 17 \\ \times 4 \\ \hline 68 \end{array}$$

Systems of equations



What if one of the coefficients of a is zero?

System

- $5a + b = 17$

- $3b = 6$



$$b = 6/3 = 2$$

$$5a + 2 = 17$$

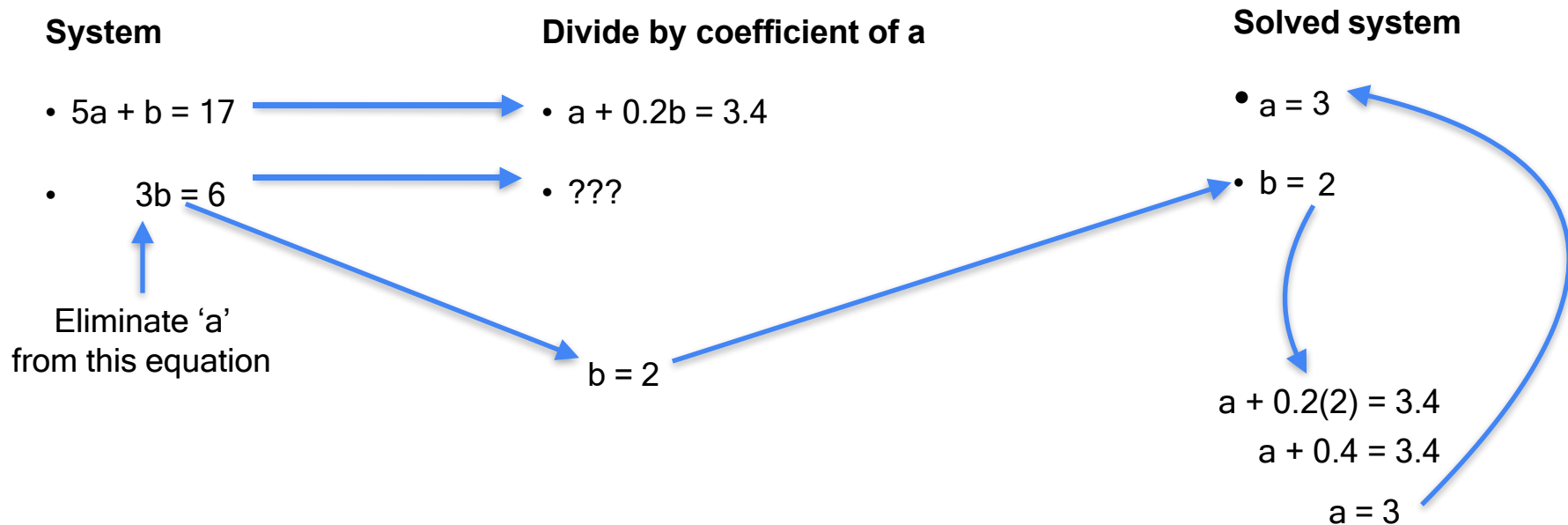
$$a = 3$$

Solved system

- $a = ?$

- $b = ?$

What if one of the coefficients of a is zero?



Quiz

- Solve the following system of equations

System

$$\bullet \quad 2a + 5b = 46 \rightarrow a + 5/2 b = 23$$

$$\bullet \quad 8a + b = 32 \rightarrow a + b/8 = 4$$

$$a = 3, b = 8$$

$$(5/2)b - \frac{b}{8} = 19$$

$$\frac{5(4)b - b}{8} = 19$$

$$\frac{20b - b}{8} = 19$$

$$\cancel{19} b = \cancel{19}(8)$$

$$20a + 5(8) = 46$$

$$20a = 6$$

$$a = 3$$

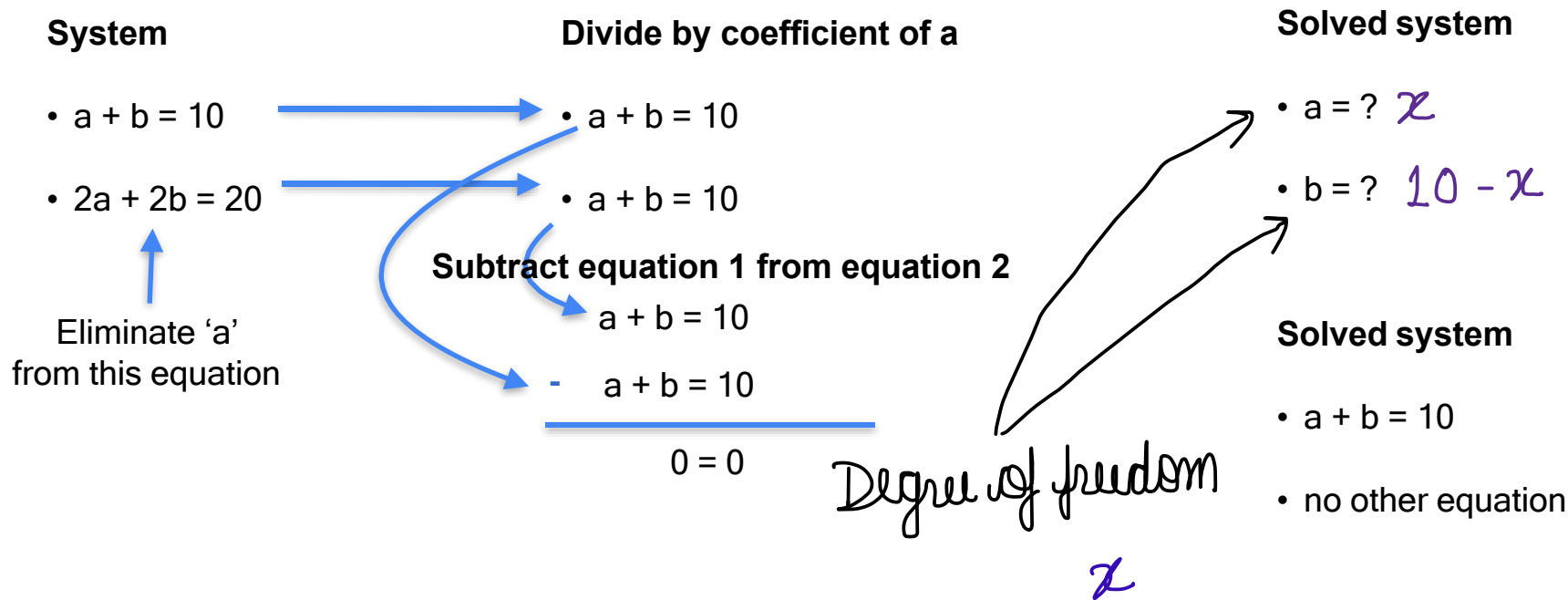


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Solving System of Linear Equations

**Solving singular system of
linear equations**

What if the system is singular (redundant)?



What if the system is singular (contradictory)?

System

- $a + b = 10 \rightarrow a + b = 10$
- $2a + 2b = 24 \rightarrow -a + b = 12$

$$\boxed{0 \neq -2}$$



No solutions

Solved system

- $a = ?$
- $b = ?$

Quiz

- Solve the following system of equations

System

- $5a + b = 11 \rightarrow a + b/5 = 11/5$

- $10a + 2b = 22 \rightarrow a + b/5 = \cancel{22/10} \quad 11/5$

$$\boxed{0 = 0}$$

^{oo}
infinite solutions

Redundant !!



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Solving System of Linear Equations

**Solving system of equations
with more variables**

Elimination method

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Elimination method

$$c/3 - 2c$$

System

- $a + b + 2c = 12 \rightarrow \textcircled{1}$
- $3a - 3b - c = 3 \rightarrow \textcircled{2}$
- $2a - b + 6c = 24 \rightarrow \textcircled{3}$

Leave 'a' by
itself

subtract eq^① from eq^②

$$-2b - \left(\frac{7}{3}\right)c = -13$$

Elimination method

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Divide each row by the coefficient of 'a'

- $a + b + 2c = 12$
- $a - b - \frac{1}{3}c = 1$
- $a - \frac{b}{2} + 3c = 12$

Use the first equation to remove 'a' from the others

- Isolated 'a'
- $a + b + 2c = 12$
 - $-2b - \frac{7}{3}c = -11$
 - $-\frac{3}{2}b + c = 0$
- Solve this new system of 2 equations

Elimination method

System

- $a + b + 2c = 12$
- $-2b - 7/3 c = -11$
- $-3/2 b + c = 0$

Divide last two rows by the coefficient of b

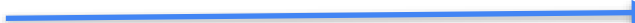

- $a + b + 2c = 12$
- $b + 7/6 c = 11/2$
- $b - 2/3 c = 0$

Use the second equation to remove 'b' from the third

- $a + b + 2c = 12$
 - $b + 7/6 c = 11/2$
 - $-11/6 c = -11/2$
- Isolated 'b'
- $c = 3$

Elimination method

System

- $a + b + 2c = 12$  $a + 2 + 6 = 12$
 $a = 4$
- $b + 7/6 c = 11/2$  $b + 7/2 = 11/2$
 $b = 2$
- $c = 3$

Replace $c = 3$
in the second
equation, get
 $b = 2$

Replace $c = 3$
and $b = 2$ in the
first equation,
get $a = 4$

The solution is
 $a = 4$
 $b = 2$
 $c = 3$



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Solving System of Linear Equations

Matrix row reduction

Systems of equations to matrices

Original system

- $5a + b = 17$
- $4a - 3b = 6$

Intermediate System

- $a + 0.2b = 3.4$
- $b = 2$

Solved system

- $1a + 0b = 3$
- $0a + 1b = 2$

Original matrix

5	1
4	-3

Upper diagonal matrix

1	0.2
0	1

Row echelon form

Diagonal matrix

1	0
0	1

Reduced row echelon form

Systems of equations to matrices

Original system

- $a + b = 10$
- $2a + 2b = 20$

Intermediate System

- $a + b = 10$
- $0a + 0b = 0$

Original matrix

1	1
2	2

Upper diagonal matrix

1	1
0	0

Row echelon form

Systems of equations to matrices

Original system

- $5a + b = 11$
- $10a + 2b = 22$



$$a + 0.2b = 2.2$$

$$0a + 0b = 0$$

original matrix

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$$



Row echelon form

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

Systems of equations to matrices

Original system

- $5a + b = 11$
- $10a + 2b = 22$

Intermediate System

- $a + 0.2b = 2.2$
- $0a + 0b = 0$

Original matrix

5	1
10	2

Upper diagonal matrix

1	0.2
0	0

Row echelon form

Systems of equations to matrices

Original system

- $0a + 0b = 0$
- $0a + 0b = 0$

Intermediate System

- $0a + 0b = 0$
- $0a + 0b = 0$

Original matrix

0	0
0	0

Upper diagonal matrix

0	0
0	0

Row echelon form

Row echelon form

1	•	•	•	•
0	1	•	•	•
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



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Solving System of Linear Equations

**Row operations that
preserve singularity**

Switching rows

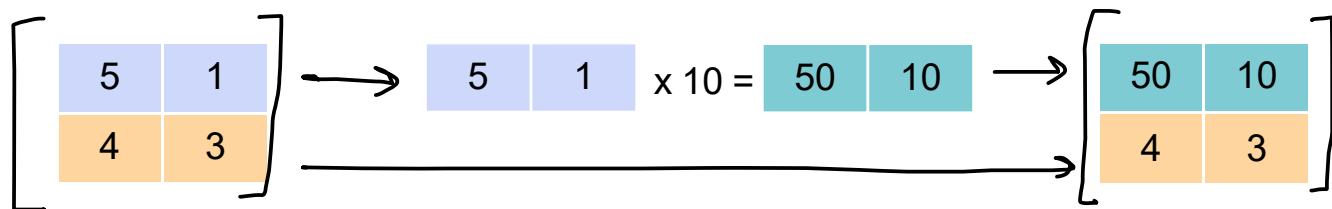
5	1
4	3

Determinant $5 \cdot 3 - 1 \cdot 4 = 11$

4	3
5	1

Determinant $4 \cdot 1 - 3 \cdot 5 = -11$

Multiplying a row by a (non-zero) scalar

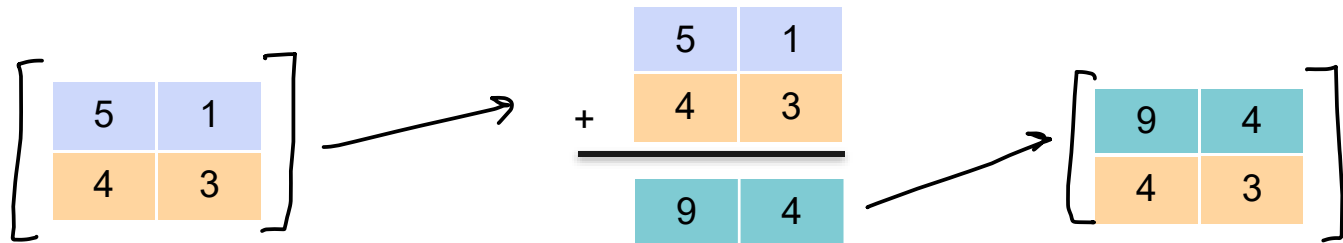


Determinant

$$\begin{aligned} & 5(3) - 4(1) \\ & = 11 \end{aligned}$$

$$\begin{aligned} \text{Determinant} &= 5 \cdot (10 \cdot 3) - 1 \cdot (10 \cdot 4) \\ &= 10(11) \end{aligned}$$

Adding a row to another row



Determinant

$$\begin{aligned} & 5(3) - 4(1) \\ &= 11 \end{aligned}$$

Determinant

$$\begin{aligned} & 9(3) - 4(4) \\ &= 11 \end{aligned}$$

$$\begin{cases} x+y=4 \\ -6x+2y=16 \end{cases} \rightarrow \begin{array}{r} y+x=4 \\ y-3x=8 \\ \hline 4x=-4 \\ x=-1 \end{array}$$

$y=5$

$$\begin{bmatrix} -3 & 8 & 1 \\ 2 & 2 & -1 \\ -5 & 6 & 2 \end{bmatrix}$$

$$-3(4+6) - 8(4-5) + 1(12+10)$$

$$\Rightarrow -30 + 8 + 22 = 0$$

(Singular)

$$\begin{bmatrix} 4 & -3 \\ 7 & -8 \end{bmatrix} \Rightarrow -32 + 21 = -11 \text{ (Non singular)}$$

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix} \Rightarrow ac - ab = a(c-b) \neq 0$$

$$\begin{array}{l} 2a \quad s = 2\% \\ a \quad cd = 3\% \\ z \quad b = 4\% \end{array}$$

\$10,000

$$3a + z = 10,000 \longrightarrow \textcircled{1}$$

$$\underline{0.04a + 0.03a + 0.04z = 260}$$

$$7a + 4z = 26000$$

$$1.75a + z = 6500 \longrightarrow \textcircled{2}$$

$$1.25a = 3500$$

$$a = 2800$$

$$z = 10000 - 3(2800) = 1600$$



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Rank \rightarrow how much information is the matrix carrying / or the corresponding system of linear eqns

Solving System of Linear Equations

Rank of a matrix

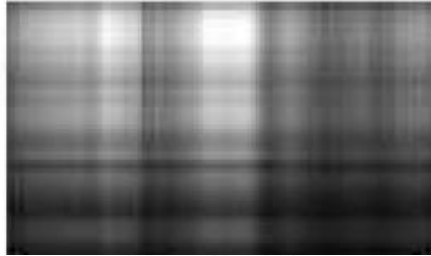
singular value decomposition (SVD)

Compressing Images - Reducing rank

Original (Rank 200)



Rank 1



Rank 2



Rank 5



Rank 15



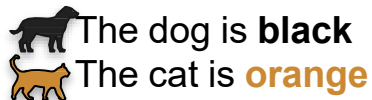
Rank 50



Systems of information

goal: "determine the color"

System 1

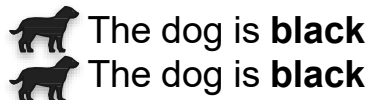


Two sentences

Two pieces of information

Rank = 2

System 2



Two sentences

One piece of information

Rank = 1

System 3



Two sentences

Zero pieces of information

Rank = 0

Systems of equations



System 1

$$a + b = 0$$



$$a + 2b = 0$$



 	
1	1
1	2

Rank = 2

Two equations

Two pieces of information

Rank = 2



System 2

$$a + b = 0$$



$$2a + 2b = 0$$



 	
1	1
2	2

Rank = 1



Two equations

One piece of information

Rank = 1

System 3

$$0a + 0b = 0$$

 	
0	0
0	0



Rank = 0

Two equations

Zero pieces of information

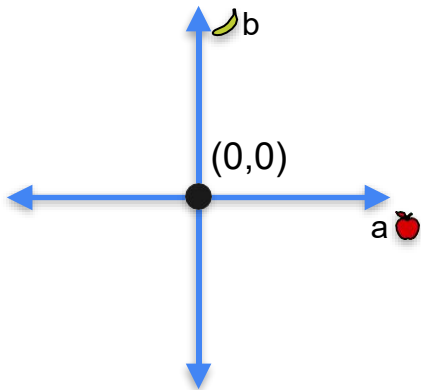
Rank = 0



Rank and solutions to the system

	
1	1
1	2

Rank = 2

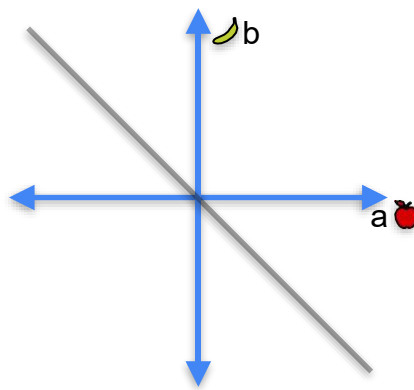
Dimension of solution space = 0





	
1	1
2	2

Rank = 1

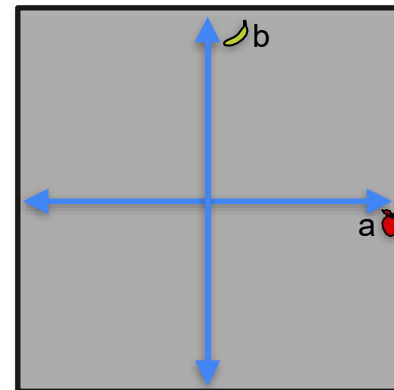
Dimension of solution space = 1





	
0	0
0	0

Rank = 0

Dimension of solution space = 2





Rank of a matrix





1	1
1	2

Rank = 2



1	1
2	2

Rank = 1





0	0
0	0

Rank = 0

Dimension of solution space = 0 Dimension of solution space = 1 Dimension of solution space = 2

$$\text{Rank} = 2 - (\text{Dimension of solution space})$$



Rank and singularity



1	1
1	2

Rank = 2



Non-singular



1	1
2	2

Rank = 1

Singular



0	0
0	0

Rank = 0

Singular

Quiz: Rank of a matrix

Determine the rank of the following two matrices

Matrix 1

5	1
-1	3

Rank = 2 # solution space of dimension 0
→ $15 + 1 = 16$, Non singular

Matrix 2

2	-1
-6	3

solution space of dimension 1
→ $6 - 6 = 0$, singular
Rank = 1



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Solving System of Linear Equations

**Rank of a matrix:
General case**

Rank for matrices

System 1

$$\begin{aligned}a + b + c &= 0 \\a + 2b + c &= 0 \\a + b + 2c &= 0\end{aligned}$$



3 Equations
3 Pieces of information

Rank 3

1	1	1
1	2	1
1	1	2

System 2

avg →

$$\begin{aligned}a + b + c &= 0 \\a + b + 2c &= 0 \\a + b + 3c &= 0\end{aligned}$$



3 Equations
2 Pieces of information

Rank 2

1	1	1
1	1	2
1	1	3

System 3

x₂ *x₃*

$$\begin{aligned}a + b + c &= 0 \\2a + 2b + 2c &= 0 \\3a + 3b + 3c &= 0\end{aligned}$$



3 Equations
1 Piece of information

Rank 1

1	1	1
2	2	2
3	3	3

System 4

$$\begin{aligned}0a + 0b + 0c &= 0 \\0a + 0b + 0c &= 0 \\0a + 0b + 0c &= 0\end{aligned}$$



3 Equations
0 Pieces of information

Rank 0

0	0	0
0	0	0
0	0	0

Question

- Is there an easier way to calculate the rank?
- Answer: Yes! As before, it is the number of ones in the diagonal of the reduced row echelon form of the matrix.



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Solving System of Linear Equations

Row echelon form

Row echelon form of a matrix

Original matrix

Row echelon form

5	1
4	-3



1	0.2
0	1

5	1
10	2



1	1
0	0

0	0
0	0



0	0
0	0

Row echelon form

Original matrix

5	1
4	-3

Divide each row by
the leftmost coefficient

1	0.2
1	-0.75

$R_2 - R_1$

1	-0.75
-	
1	0.2
<hr/>	
0	-0.95

Divide the second row by
the leftmost non-zero coefficient

Row echelon form

1	0.2
0	-0.95

1	0.2
0	1

Row echelon form for singular matrices

Original matrix

5	1
10	2

Divide each row by
the leftmost coefficient

1	0.2
1	0.2

$R_2 - R_1$

1	0.2
-	
1	0.2
<hr/>	
0	0

Row echelon form

1	0.2
0	0

Divide the second row by
the leftmost non-zero coefficient

1	0.2
?	?

Row echelon form for singular matrices

Row echelon form

Original matrix

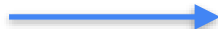


Divide each row by
the leftmost coefficient

Row echelon form, singularity, and rank

Non-singular matrix

5	1
4	-3



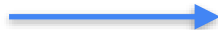
1	0.2
0	1

Rank 2

2 ones in the diagonal

Singular matrix

5	1
10	2



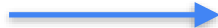
1	0.2
0	0

Rank 1

1 one in the diagonal

Singular matrix

0	0
0	0



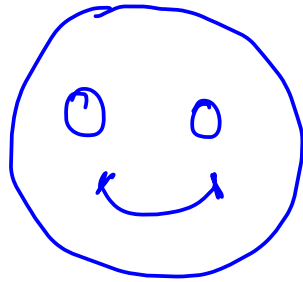
0	0
0	0

Rank 0

0 ones in the diagonal

A matrix is **non singular** if and only if its

Row echelon form has **only ones and
no zeros**





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Solving System of Linear Equations

**Row echelon form:
General case**

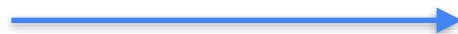
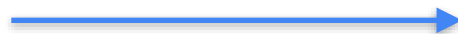
Row echelon form

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Matrix

1	1	2
3	-3	-1
2	-1	6



System

- $a + b + 2c = 12$
- $-6b - 7c = -33$
- $6c = 18$

Row echelon form matrix

1	1	2
0	-6	7
0	0	6

Row echelon form

can be zero / non-zero "doesn't matter"

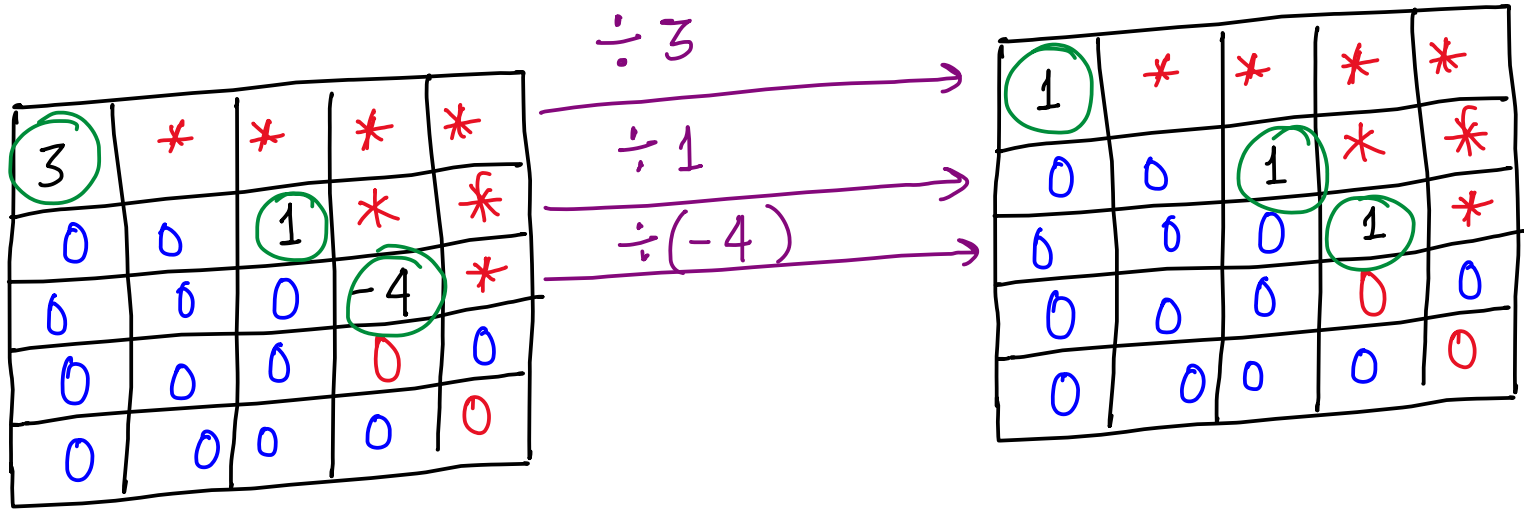
2	*	*	*	*
0	1	*	*	*
0	0	3	*	*
0	0	0	-5	*
0	0	0	0	1

Rank 5

3	*	*	*	*
0	0	1	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

Rank 3

- Zero rows at the bottom
- Each row has a pivot (leftmost non-zero entry)
- Every pivot is to the right of the pivots on the rows above
- Rank of the matrix is the number of pivots



for this class, pivots are 1

Another example

Matrix

1	1	1
1	2	1
1	1	2



Row echelon form

1	1	1
0	1	0
0	0	1

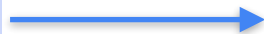
Subtract the first row
from the second and
the third ones

What if the matrix is singular?

Matrix

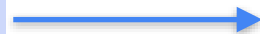
1	1	1
1	1	2
1	1	3

Subtract the first row
from the second and
the third ones



1	1	1
0	0	1
0	0	2

Subtract twice the
second row from the
third one



Row echelon form

1	1	1
0	0	1
0	0	0

What if the matrix is singular?

Matrix

1	1	1
2	2	2
3	3	3

Subtract twice the
first row from the
second row



1	1	1
0	0	0
3	3	3

Subtract three times
the first row from the
third row



Row echelon form

1	1	1
0	0	0
0	0	0

Rank for matrices

Matrix 1

1	1	1
1	2	1
1	1	2

Rank = 3

Matrix 2

1	1	1
1	1	2
1	1	3

Rank = 2

Matrix 3

1	1	1
2	2	2
3	3	3

Rank = 1

Matrix 4

0	0	0
0	0	0
0	0	0

Rank = 0

Row echelon forms

1	1	1
0	1	0
0	0	1

Number of pivots = 3

1	1	1
0	0	1
0	0	0

Number of pivots = 2

1	1	1
0	0	0
0	0	0

Number of pivots = 1

0	0	0
0	0	0
0	0	0

Number of pivots = 0



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Solving System of Linear Equations

Reduced row echelon form

"solved system"

Systems of equations to matrices

Original system

- $5a + b = 17$
- $4a - 3b = 6$

Intermediate System

- $a + 0.2b = 3.4$
- $b = 2$

Solved system

- $1a + 0b = 3$
- $0a + 1b = 2$

Original matrix

5	1
4	-3

Upper diagonal matrix

1	0.2
0	1

Row echelon form

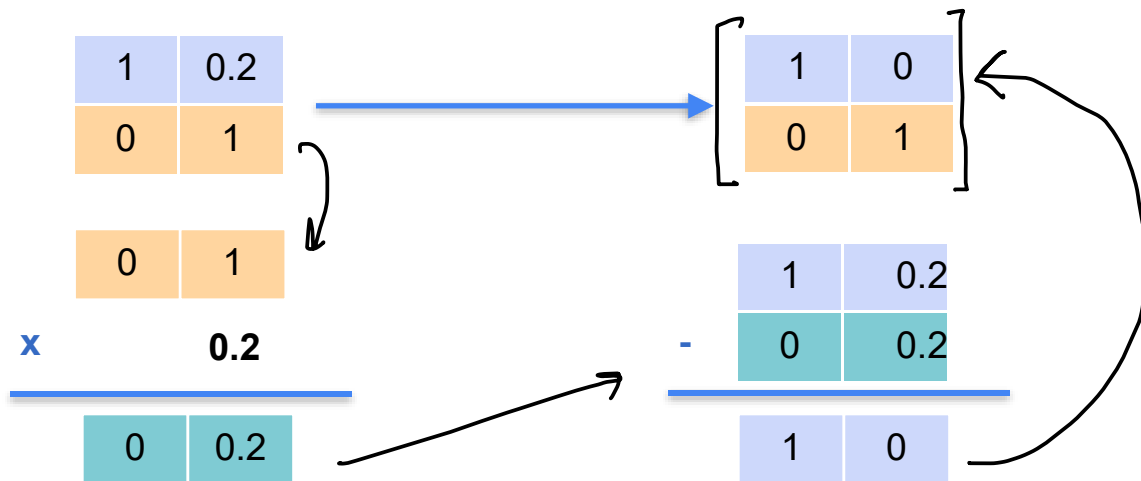
Diagonal matrix

1	0
0	1

Reduced row echelon form

Reduced row echelon form

Row echelon form



Reduced Row echelon form

Reduced row echelon form

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

Rank 5

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Rank 3

- Is in row echelon form
- Each pivot is a 1
- Any number above a pivot is 0
- Rank of the matrix is the number of pivots

Reduced row echelon form

Row echelon form

3	*	*	*	*
0	0	2	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Divide each row by
the value of the pivot

(leading coefficients)

Reduced row
echelon form

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Turn anything above a
pivot to 0

Reduced row echelon form

Row echelon form

1	2	3
0	1	4
0	0	1

Subtract 2 times the second row from the first one

1	0	-5
0	1	4
0	0	1

Add 5 times the third row to the first one

1	0	0
0	1	4
0	0	1

Subtract 4 times the third row from the second one

Reduced row echelon form

1	0	0
0	1	0
0	0	1



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Solving System of Linear Equations

Conclusion