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Math for Machine Learning

Linear algebra - Week 1

Systems of linear equations
Singular and non-singular matrices
Determinants
Rank of a matrix
Row reduction
Null space

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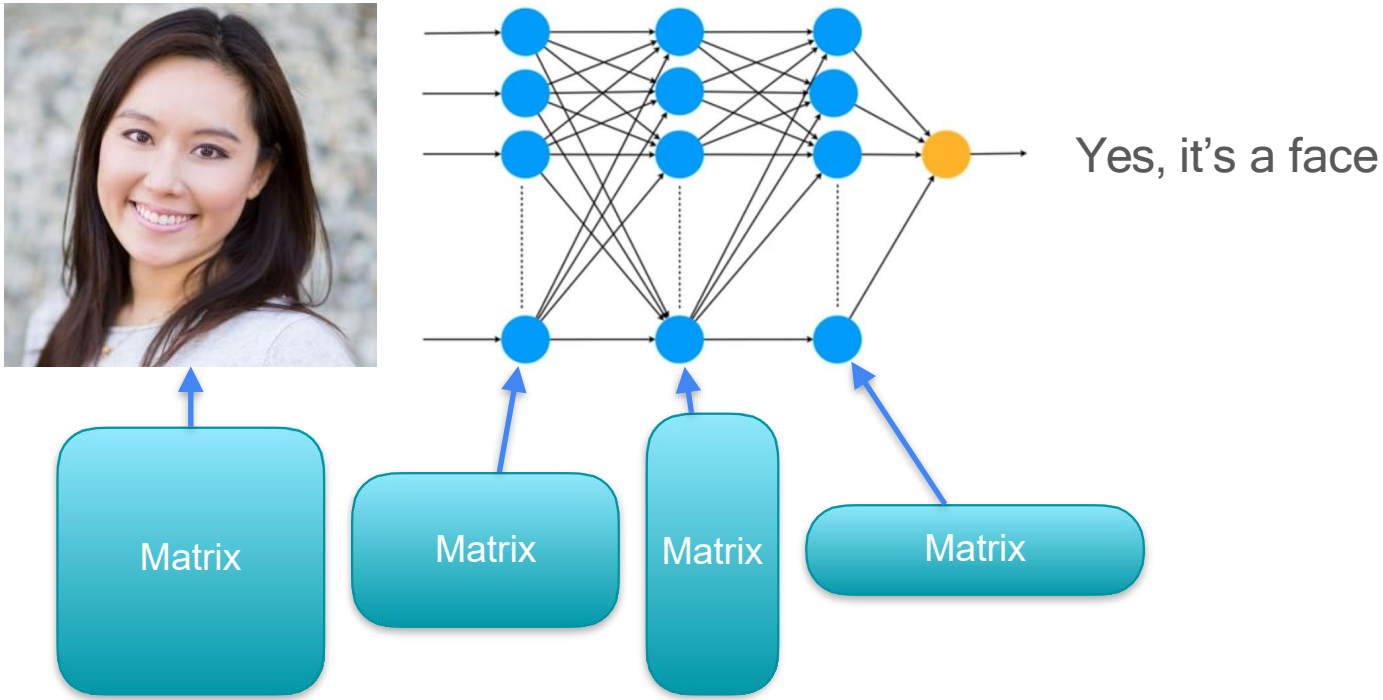
System of Linear Equations

Machine learning motivation

Neural networks - Matrix operations



Neural networks - image recognition

Image recognition in a busy street in New York.

- Image recognition: Getting the computer to see images and recognize what is on them.

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System of Linear Equations

System of sentences

Systems of sentences

System 1


The dog is **black**


The cat is **orange**

Complete

Non-singular

more informative

System 2


The dog is **black**


The dog is **black**

Redundant

Singular

less informative

System 3


The dog is **black**


The dog is white

Contradictory

Singular

less informative

Systems of sentences

1 dog
1 cat
1 Bird

System 1

The dog is **black**
The cat is **orange**
The bird is **red**

complete
(non-singular)

System 2

The dog is **black**
The dog is **black**
The bird is **red**

Redundant
(singular)

System 3

The dog is **black**
The dog is **black**
The dog is **black**

Redundant
(singular)

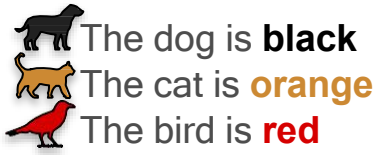
System 4

The dog is **black**
The dog is white
The bird is **red**

contradictory
(singular)

Systems of sentences

System 1

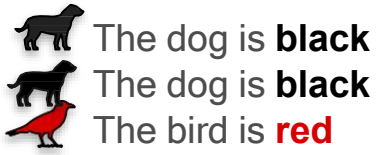


The dog is **black**
The cat is **orange**
The bird is **red**

Complete

Non-singular

System 2

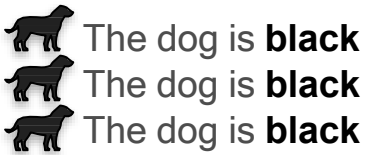


The dog is **black**
The dog is **black**
The bird is **red**

Redundant

Singular

System 3

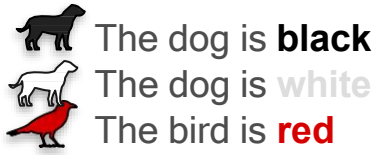


The dog is **black**
The dog is **black**
The dog is **black**

Redundant

Singular

System 4



The dog is **black**
The dog is white
The bird is **red**

Contradictory

Singular

Quiz: Systems of sentences

dog ⇒ Black
cat ⇒ Orange
bird ⇒ Red

Given this system:

- Between the dog, the cat, and the bird, one is red.
- Between the dog and the cat, one is orange.
- The dog is black.

Problem 1:

What color is the bird? → Red

informative

Problem 2:

Is this system singular or non-singular? → non singular (complete system)

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System of Linear Equations

System of equations

Sentences → Equations

Sentences

Between the dog and the cat, one is black.

Sentences with numbers

The price of an apple and a banana is $10.

Equations

a + b = 10

Quiz: Systems of equations 1

You go two days in a row and collect this information:

- **Day 1:** You bought an apple (x) and a banana (y) and they cost $10.
- **Day 2:** You bought an apple and two bananas and they cost $12.

Question: How much does each fruit cost?

$$x + y = 10$$
$$x + 2y = 12$$

$y = 2 \Rightarrow$ Banana

$x = 8 \Rightarrow$ Apple

Quiz: Systems of equations 2

You go two days in a row and collect this information:

- **Day 1:** You bought an apple (x) and a banana (y) and they cost $10.
- **Day 2:** You bought two apples and two bananas and they cost $20.

Question: How much does each fruit cost?

Not enough info!

$x + y = 10 \quad / \quad x = 10 - y$

$2x + 2y = 20$

$2(10 - y) + 2y = 20 \Rightarrow 20 - 2y + 2y = 20$

Solution: Systems of equations 2

- **Day 1:** You bought an apple and a banana and they cost $10.

- **Day 2:** You bought two apples and two bananas and they cost $20.

Same thing!!!

8	2
5	5
8.3	1.7
0	10

Infinitely many solutions!

Quiz: Systems of equations 3

You go two days in a row and collect this information:

- **Day 1:** You bought an apple (x) and a banana (y) and they cost $10.
- **Day 2:** You bought two apples and two bananas and they cost $24.

contradictory

Question: How much does each fruit cost?

$x + y = 10$

$2x + 2y = 24$

$2(x+y) = 24 \Rightarrow x+y = 12$

mistake in solution!

ye kya chai?

Solution: Systems of equations 3

- **Day 1:** You bought an apple and a banana and they cost $10.

- **Day 2:** You bought two apples and two bananas and they cost $24.

Contradiction!

No solutions!

Systems of equations

System 1

- a + b = 10
- a + 2b = 12

Unique solution:

a = 8

b = 2

Complete

Non-singular

System 2

- a + b = 10
- 2a + 2b = 20

Infinite solutions

a = 8 , 7 , 6 …

b = 2 3 4

Redundant

Singular

System 3

- a + b = 10
- 2a + 2b = 24

No solution

Contradictory

Singular

What is a linear equation?

Linear

$a + b = 10$

$2a + 3b = 15$

$3.4a - 48.99b + 2c = 122.5$

Non-linear

$a^2 + b^2 = 10$

$\sin(a) + b^2 = 15$

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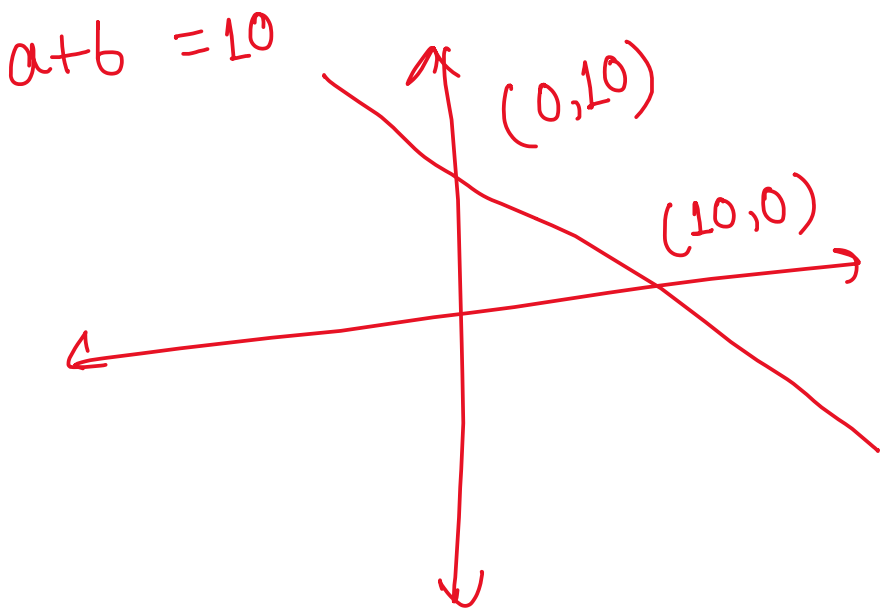
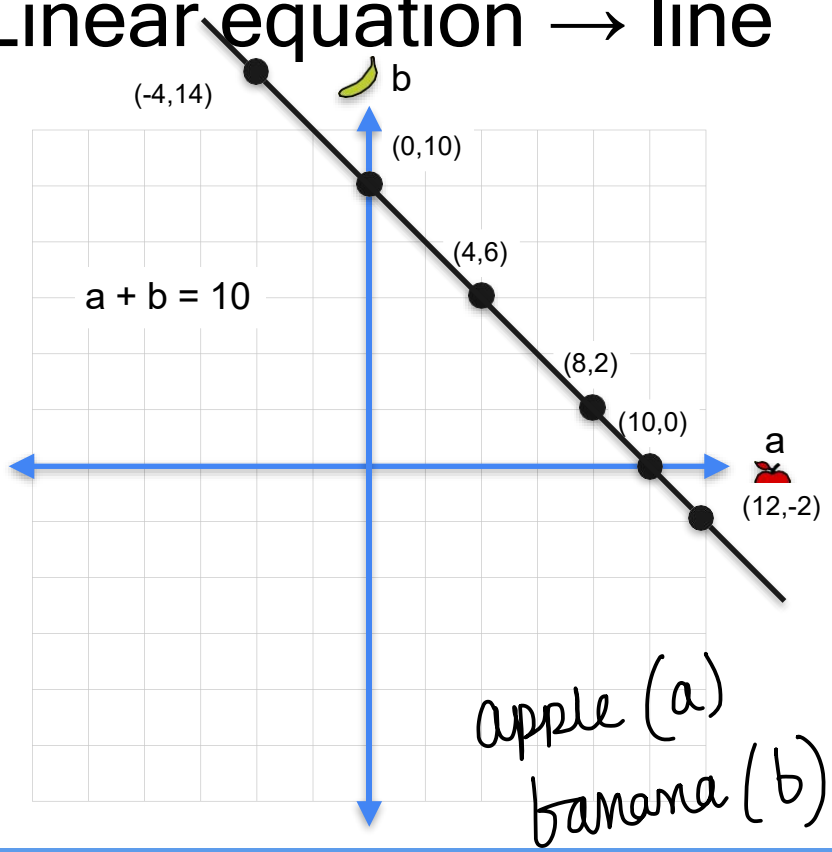
System of Linear Equations

System of equations as lines

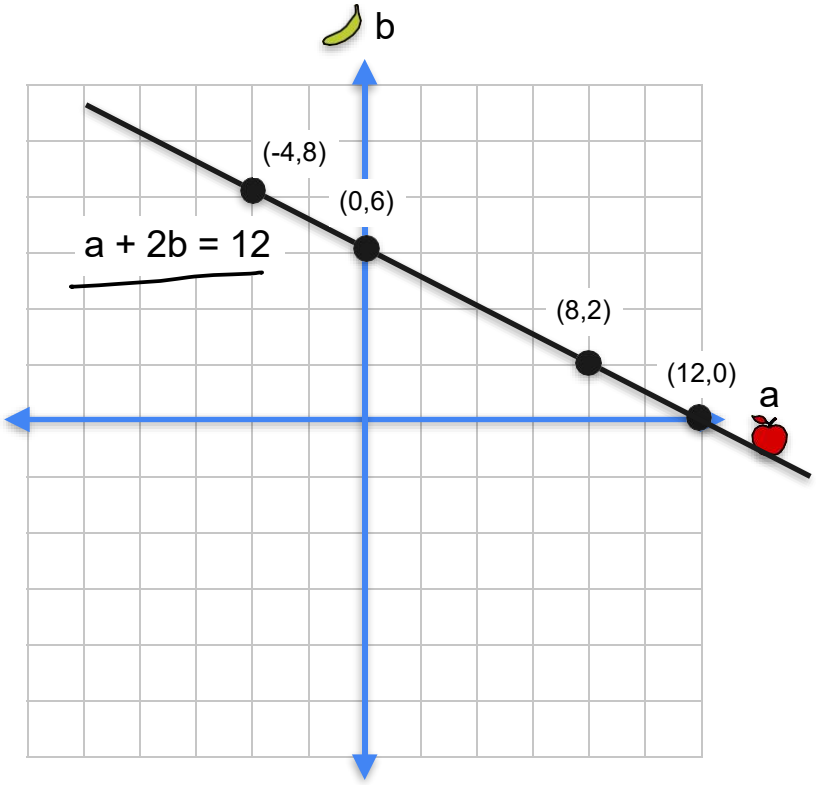
Linear equation → line



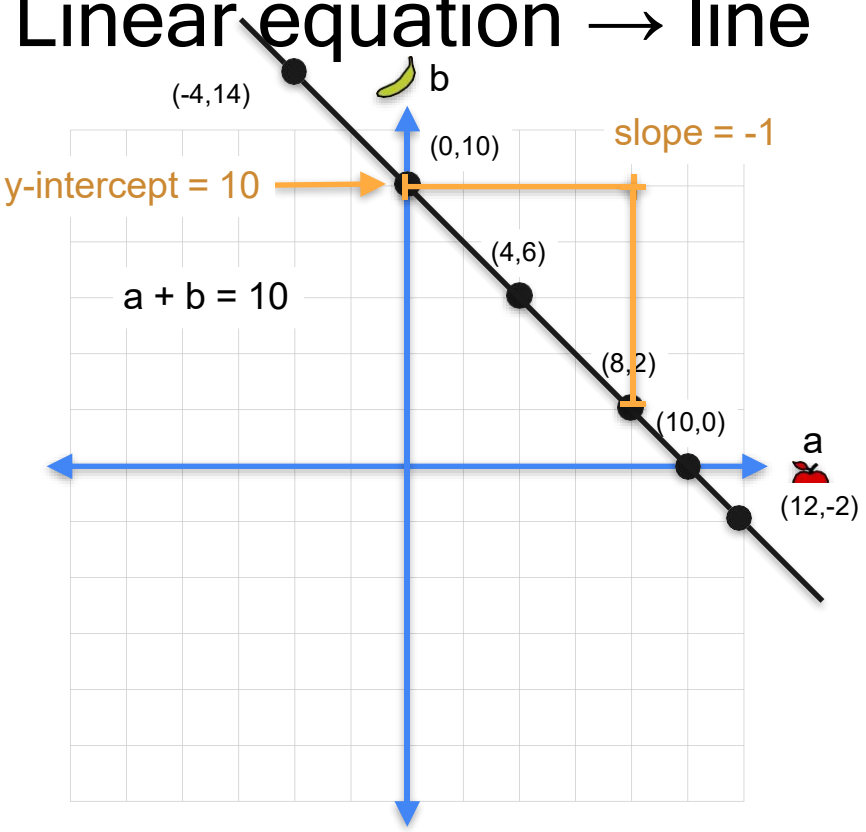
Linear equation → line

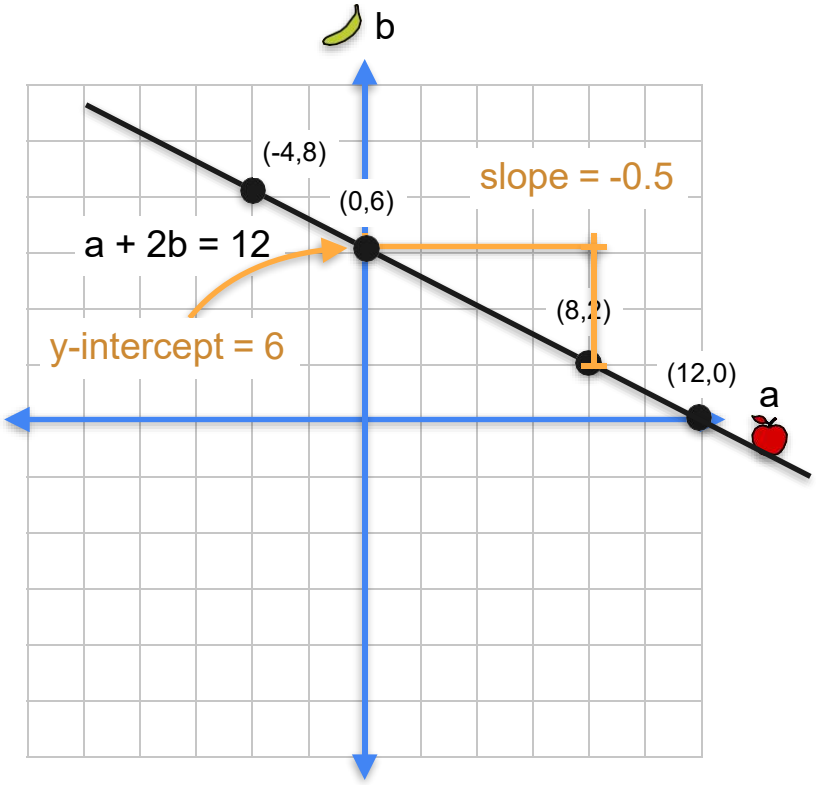


apple (a)
banana (b)



Linear equation → line



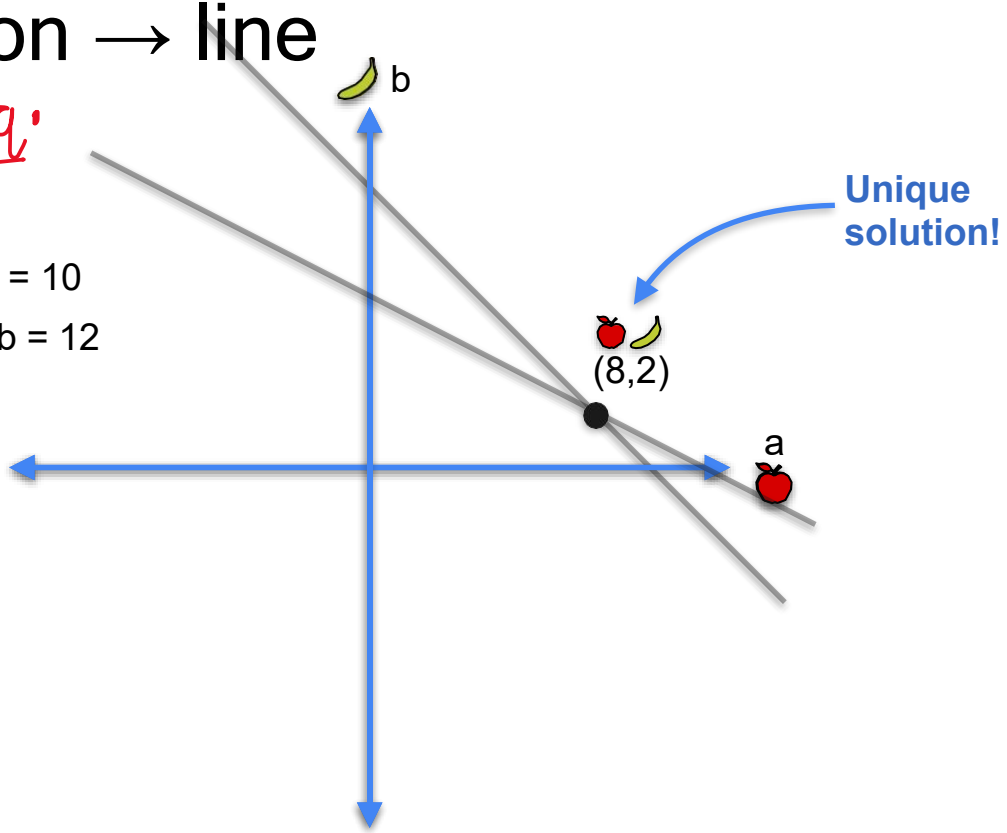


Linear equation → line

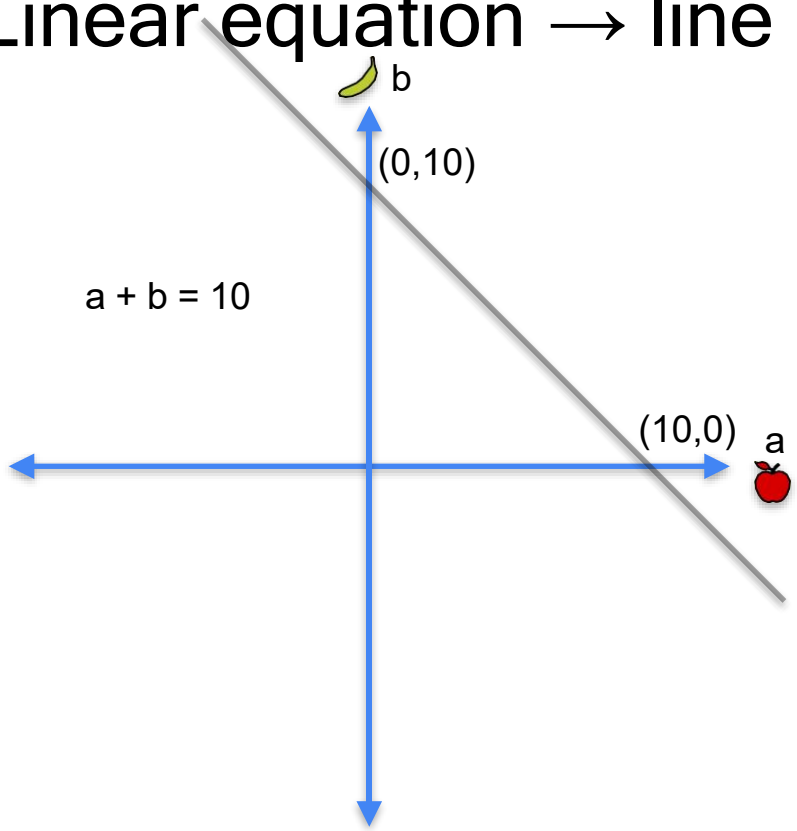
system of linear eq.

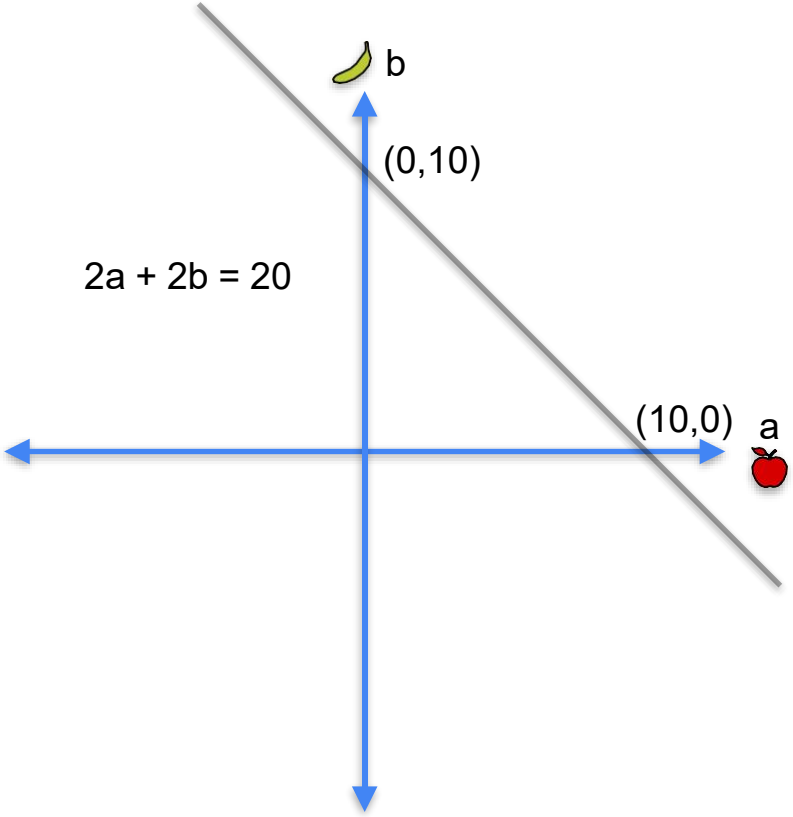
$a + b = 10$

$a + 2b = 12$

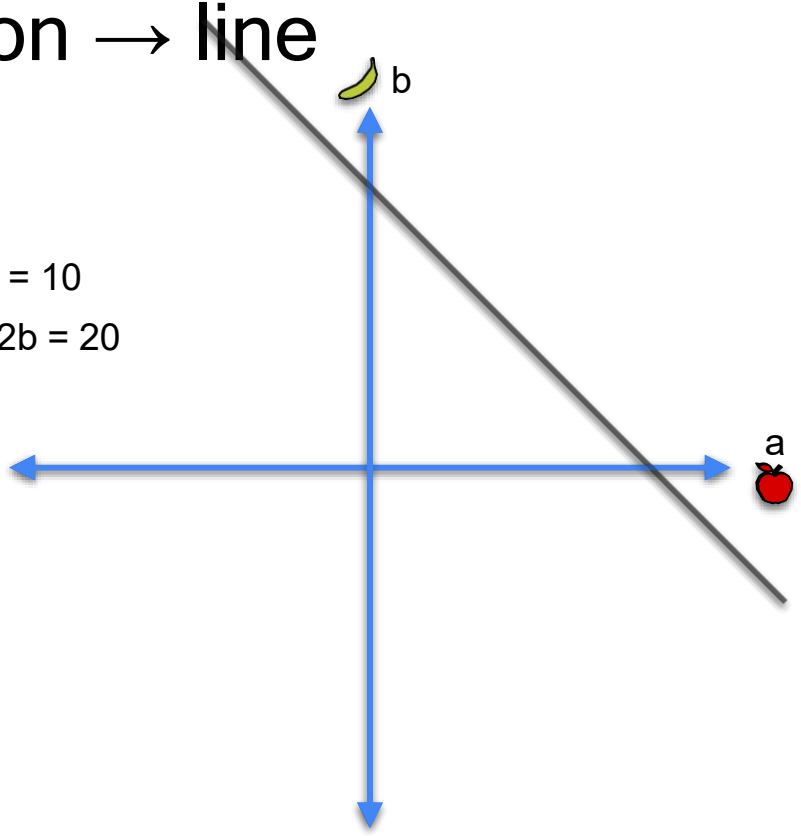


Linear equation → line





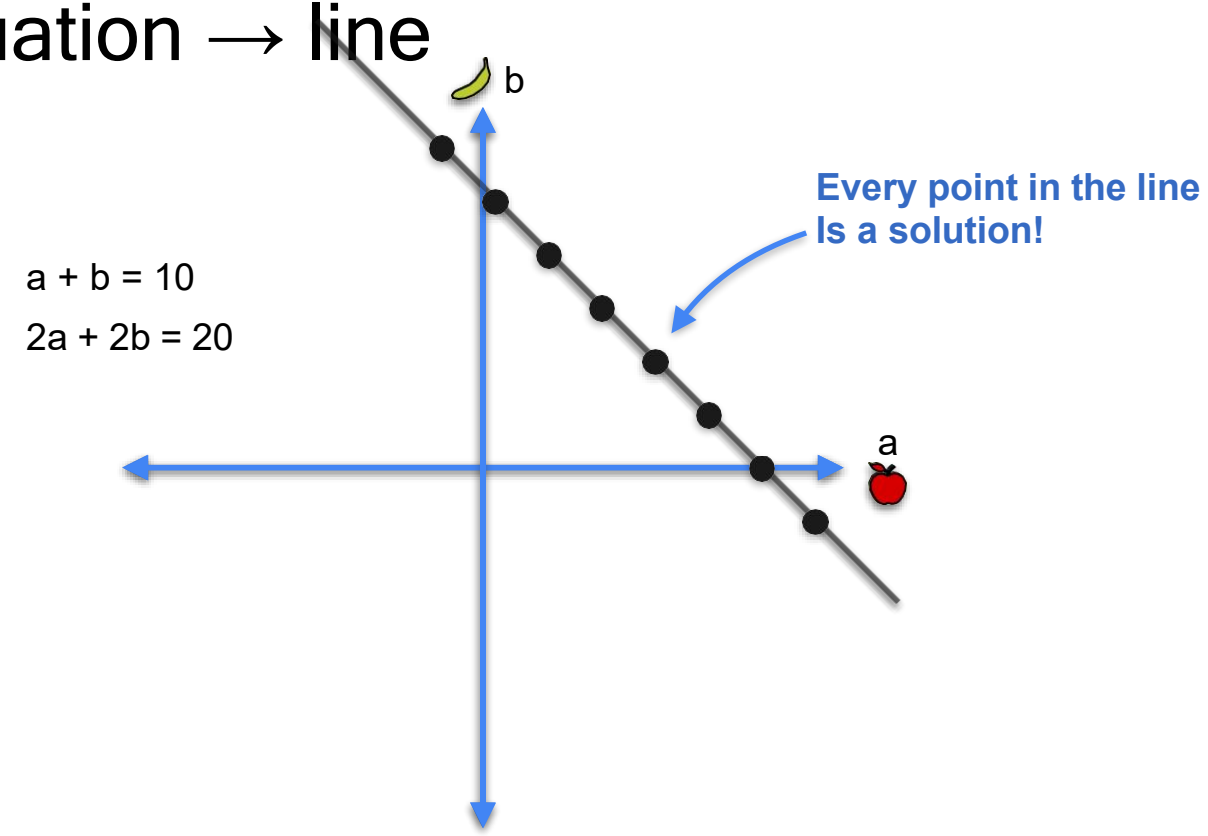
Linear equation → line



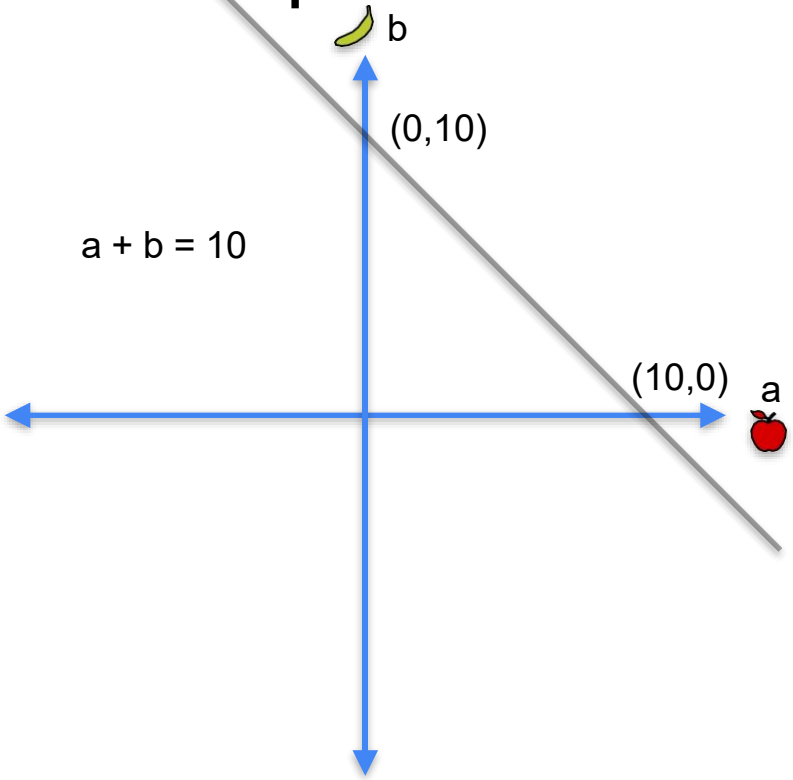
$a + b = 10$

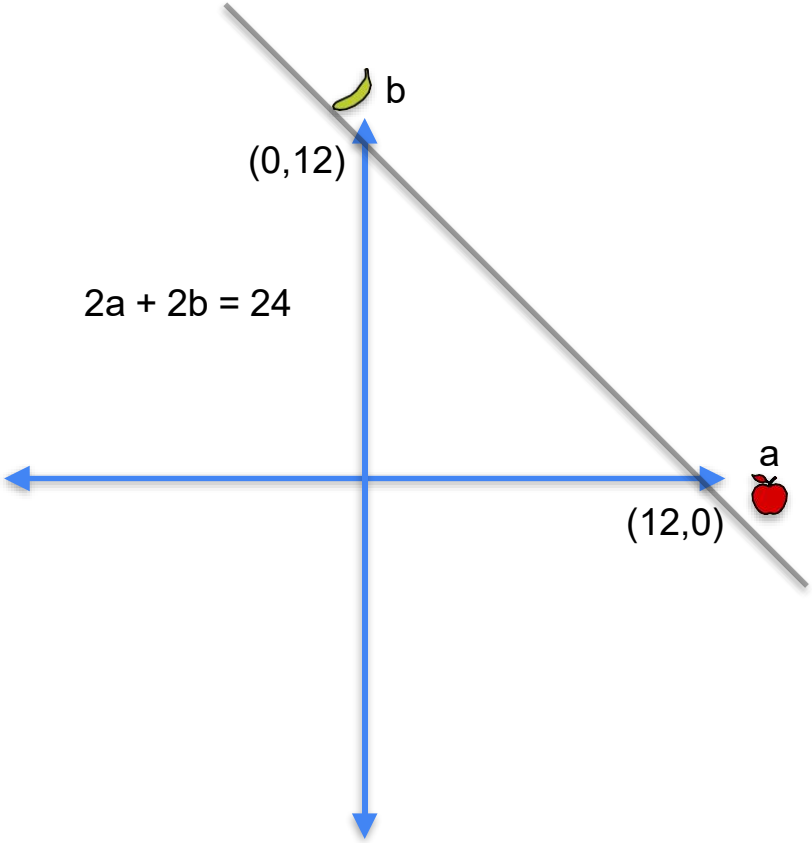
$2a + 2b = 20$

Linear equation → line

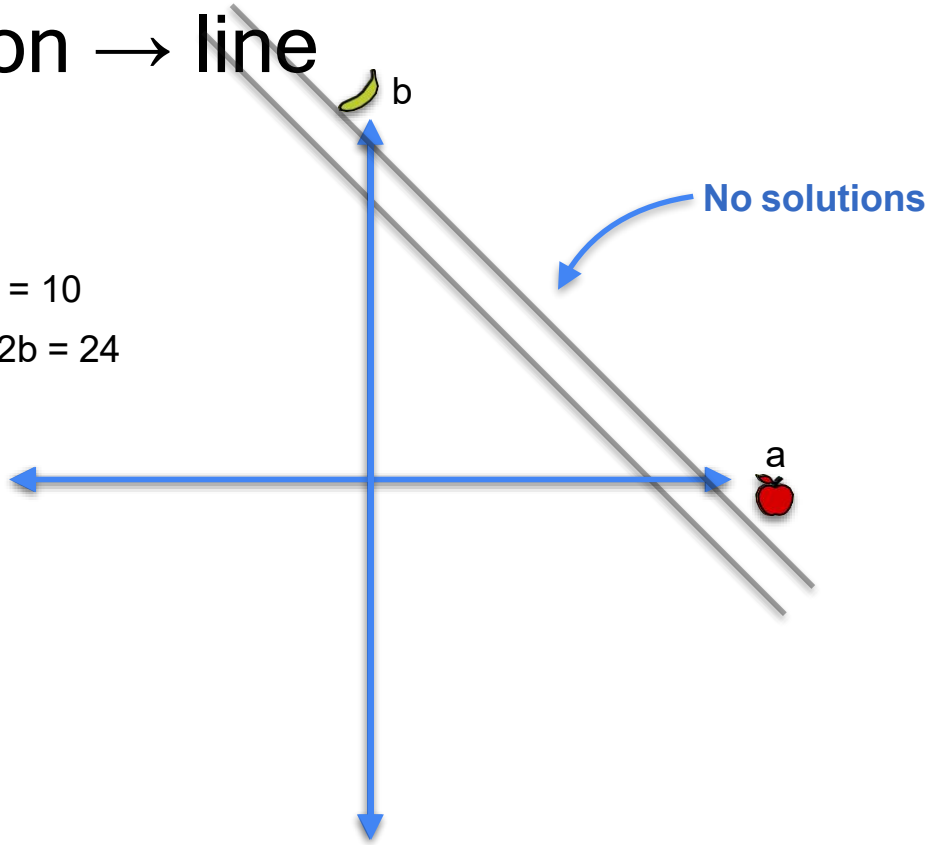


Linear equation → line





Linear equation → line

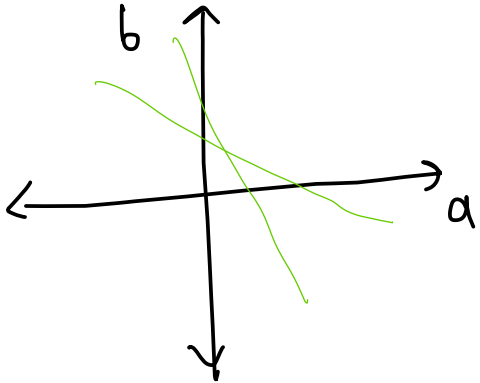


Systems of equations as lines

System 1

$a+b=10$

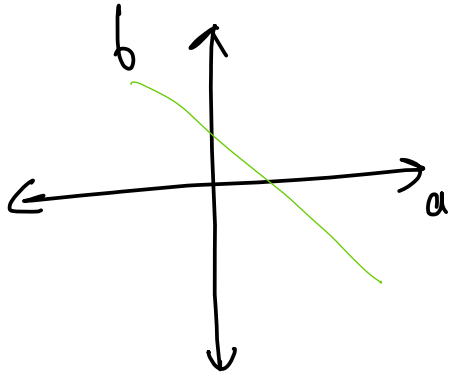
$a+2b=12$



System 2

$a+b=10$

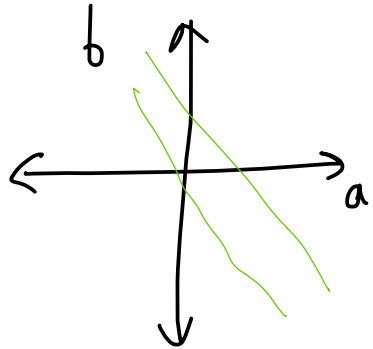
$2a+2b=20$



System 3

$a+b=10$

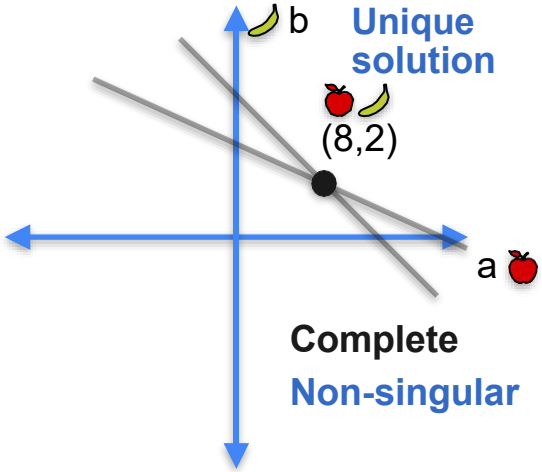
$2a+2b=24$



Systems of equations as lines

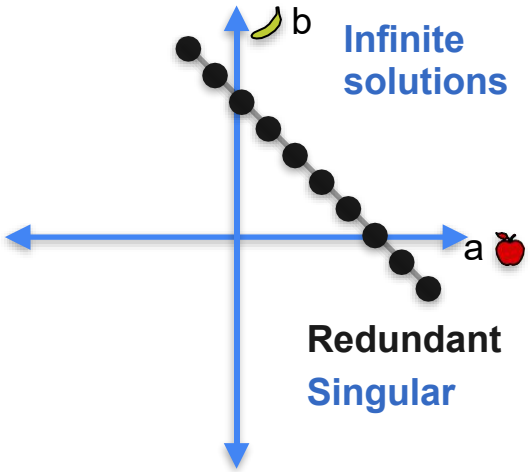
System 1

- a + b = 10
- a + 2b = 12



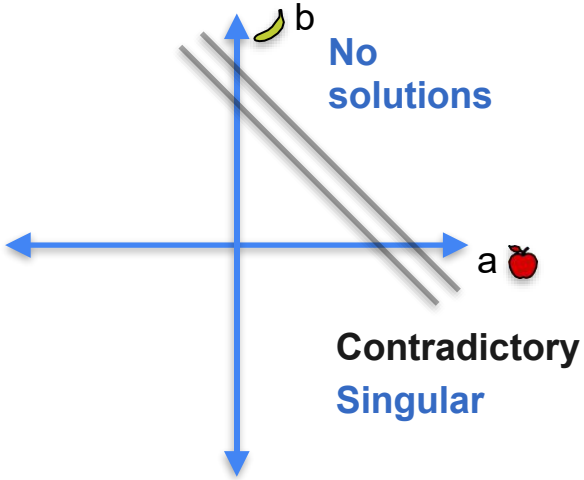
System 2

- a + b = 10
- 2a + 2b = 20



System 3

- a + b = 10
- 2a + 2b = 24



Quiz

$3a + 2(2a-3) = 8$

$3a + 4a - 6 = 8 \Rightarrow 7a - 6 = 8$

$7a = 14, \; a = 2$

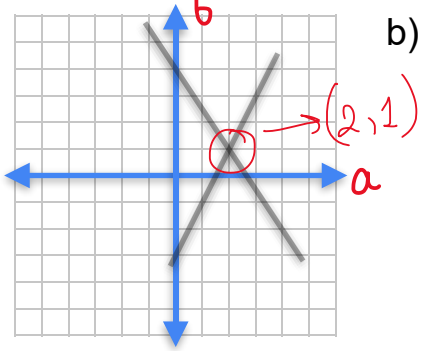
$b = 2(2) - 3 = 1$

Problem 1

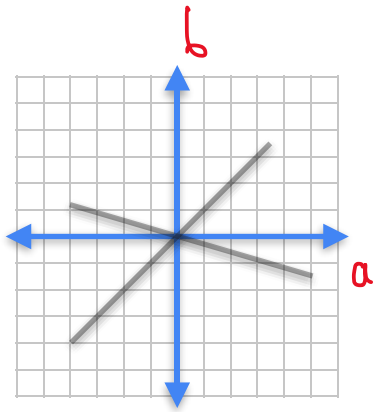
Which of the following plots corresponds to the system of equations:

- 3a + 2b = 8
- 2a - b = 3

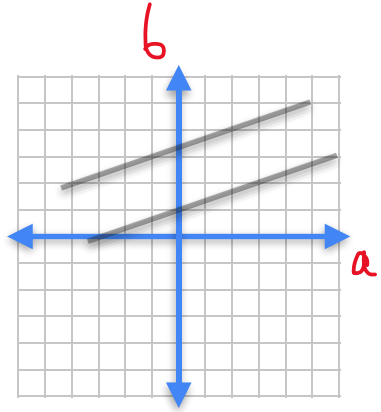
a)



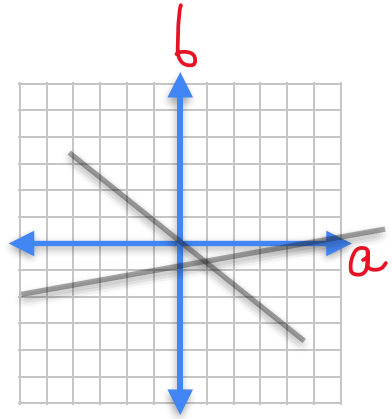
b)



c)



d)

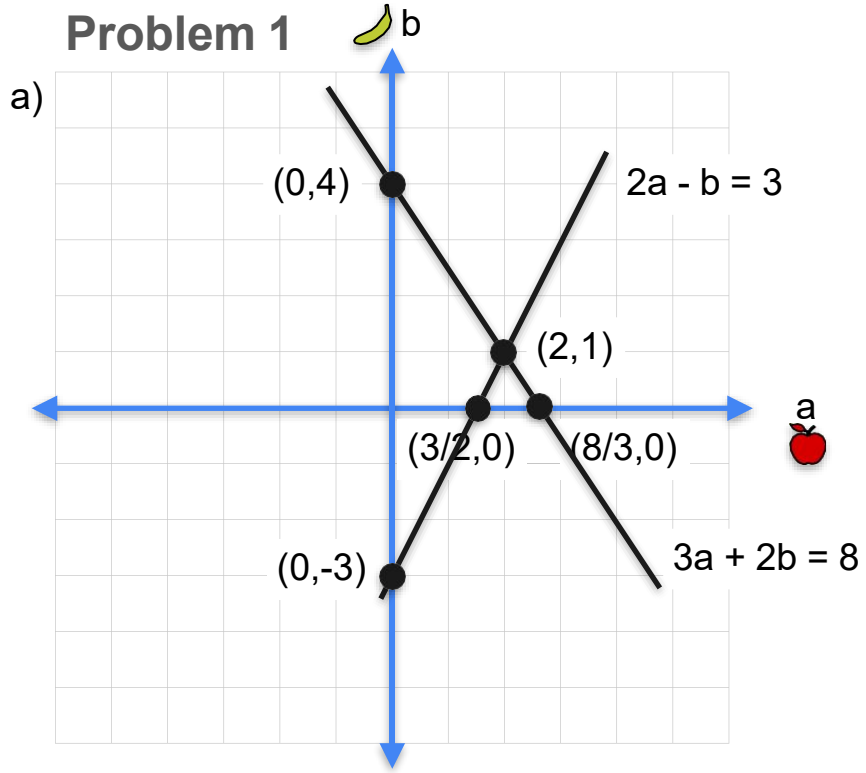


Problem 2

Is this system singular or non-singular? → non singular (unique solution)

Solution

Problem 1

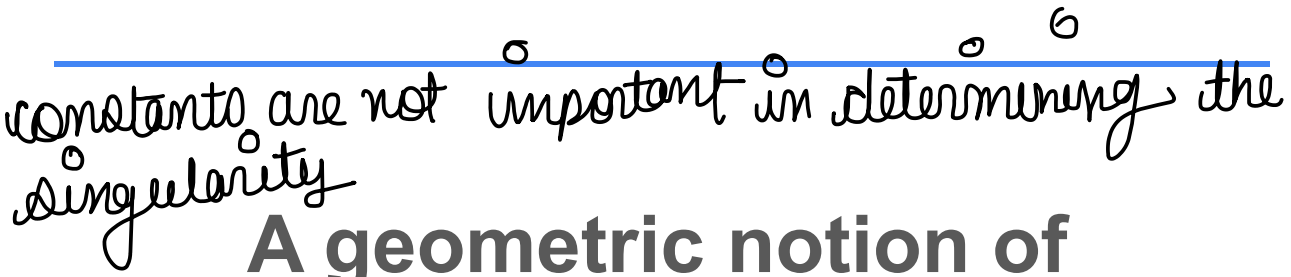


Problem 2

Since the lines cross at a unique point, the system is non-singular.

separate singular from non singular

System of Linear Equations

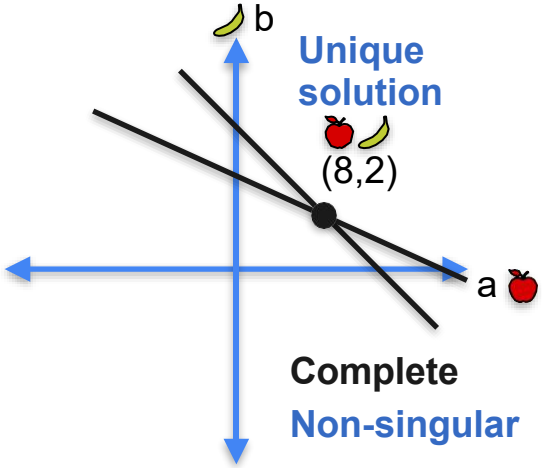


A geometric notion of singularity

Systems of equations as lines

System 1

- a + b = 10
- a + 2b = 12

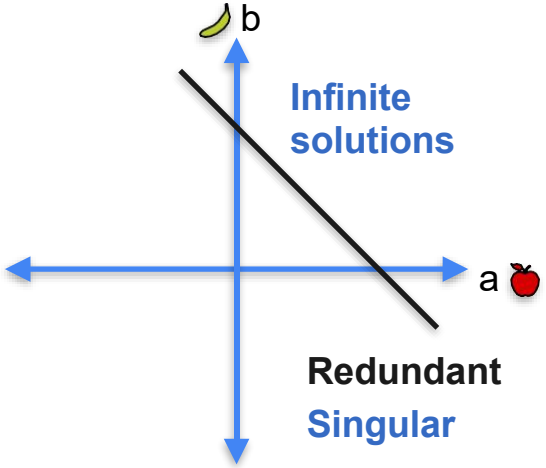


Complete

Non-singular

System 2

- a + b = 10
- 2a + 2b = 20

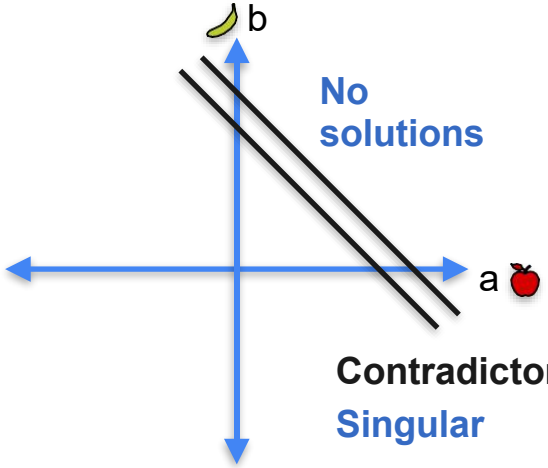


Redundant

Singular

System 3

- a + b = 10
- 2a + 2b = 24



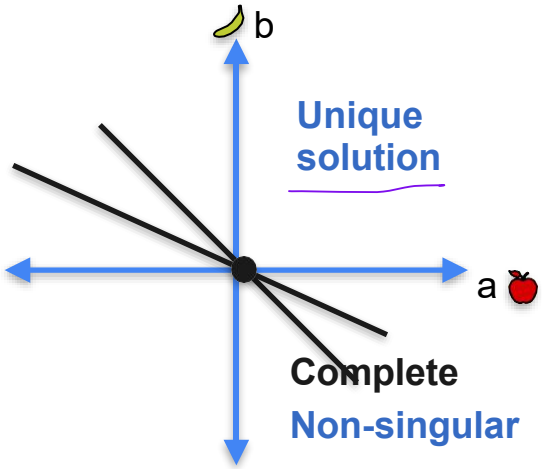
Contradictory

Singular

Systems of equations as lines

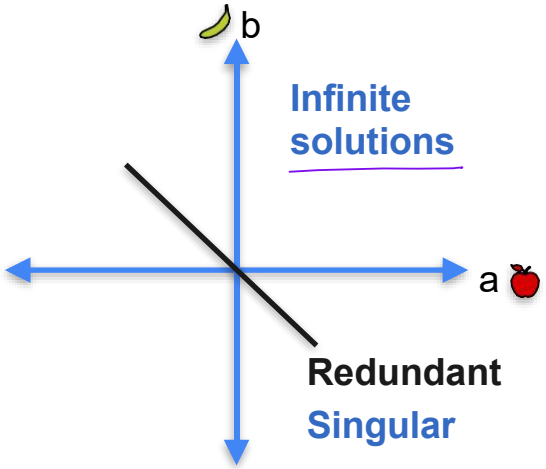
System 1

- a + b = 0
- a + 2b = 0



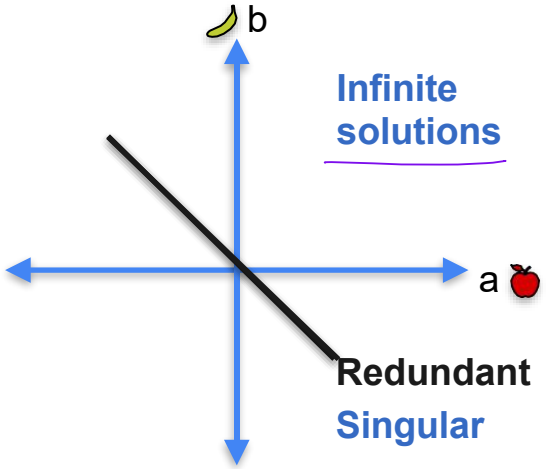
System 2

- a + b = 0
- 2a + 2b = 0



System 3

- a + b = 0
- 2a + 2b = 0



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System of Linear Equations

Singular vs nonsingular matrices

Systems of equations as matrices

System 1

- a + b = 0
- a + 2b = 0

Non singular system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Non singular matrix

unique solution

System 2

- a + b = 0
- 2a + 2b = 0

singular system

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

singular Matrix

infinitely many solutions

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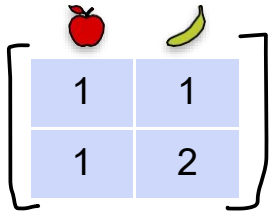
System of Linear Equations

Linear dependence and independence

Linear dependence between rows

Non-singular

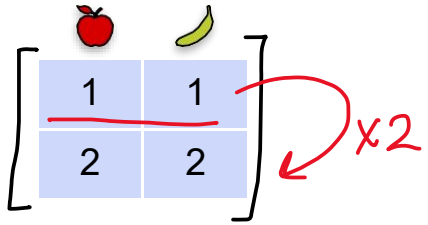
- a + b = 0
- a + 2b = 0



Rows are linearly "independent"

Singular system

- a + b = 0
- 2a + 2b = 0



2nd row is linearly dependent upon 1st row

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$|M| = 0$ # singular

$|M| \neq 0$ # non singular

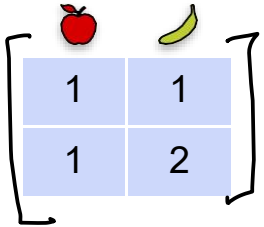
System of Linear Equations

The determinant

↳ fatajat sa batayega if matrix is singular

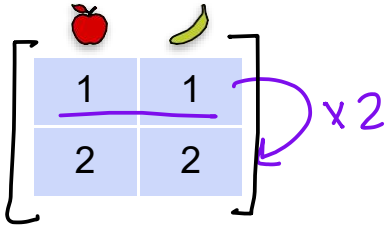
Linear dependence between rows

Non-singular matrix



linearly independent

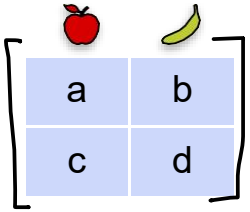
Singular matrix



$\begin{bmatrix} 1 & 1 \end{bmatrix} * 2 = \begin{bmatrix} 2 & 2 \end{bmatrix}$

linearly dependent

Determinant



Determinant

$ad - bc = 0$

Matrix is singular if

$$\begin{bmatrix} a & b \end{bmatrix} * k = \begin{bmatrix} c & d \end{bmatrix}$$

$ak = c$

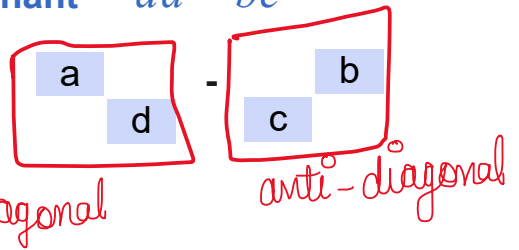
$bk = d$

$\Rightarrow \quad c/a = d/b = k \quad \Rightarrow \quad ad = bc$

Determinant

a	b
c	d

Determinant ad bc



$$\begin{matrix} ak & c \\ k & d \end{matrix}$$

$$\frac{c}{a} = \frac{d}{b} = k$$

Matrix is singular if

a	b	* k	=	c	d

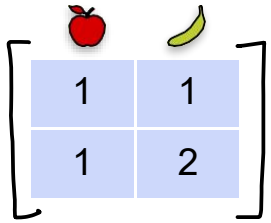
$$\begin{matrix} ad & bc \end{matrix}$$

Determinant

$$\begin{matrix} ad & bc \end{matrix}$$

Determinant

Non-singular matrix



1	1
1	2

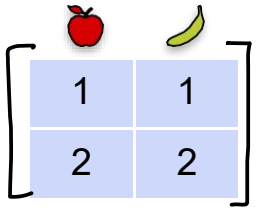
Determinant

$1(2) - 1(1)$

$\Rightarrow 2 - 1 = 1$

Non singular

Singular matrix



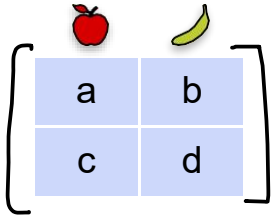
1	1
2	2

Determinant

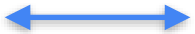
$1(2) - 2(1)$

$\Rightarrow 2 - 2 = 0$ singular

Determinant and singularity



Matrix is singular



$$ad - bc$$

Determinant is zero

Quiz: Determinant

Problem 1: Find the determinant of the following matrices

Matrix 1

$$\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \rightarrow 15 + 1 = 16 \text{ (non-singular)}$$

Matrix 2

$$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \rightarrow 6 - 6 = 0 \text{ (singular)}$$

Problem 2: Are these matrices singular or non-singular?

Quiz

1. $2b + 3m = 15$
   $2b + 4m = 16$

2. $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

3. $8 - 6 = 2$

4. Linearly independent

5. $m = 1$
   $b = 6$

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System of Linear Equations

System of equations (3x3)

Quiz: Systems of equations

Problem 1: You're trying to figure out the price of apples, bananas, and cherries at the store. You go three days in a row, and bring this information.

- **Day 1:** You bought an apple, a banana, and a cherry, and paid $10.
- **Day 2:** You bought an apple, two bananas, and a cherry, and paid $15.
- **Day 3:** You bought an apple, a banana, and two cherries, and paid $12.

How much does each fruit cost?

$$a + b + c = 10$$
$$a + 2b + c = 15$$
$$a + b + c = 12$$

Solution: Systems of equations

$a+b+c=10$

$a+2b+c=15$

$a+b+2c=12$

$\rightarrow \quad c = 10-(a+b)$

$a+2b+(10-a-b)=15$

$\Rightarrow \quad a+2b+10-a-b=15$

$b=5$

$a+b+2(10-a-b)=12$

$\Rightarrow a+b+20-2a-2b=12$

$\Rightarrow \quad a+b=8$

$\Rightarrow \quad a+5=8$

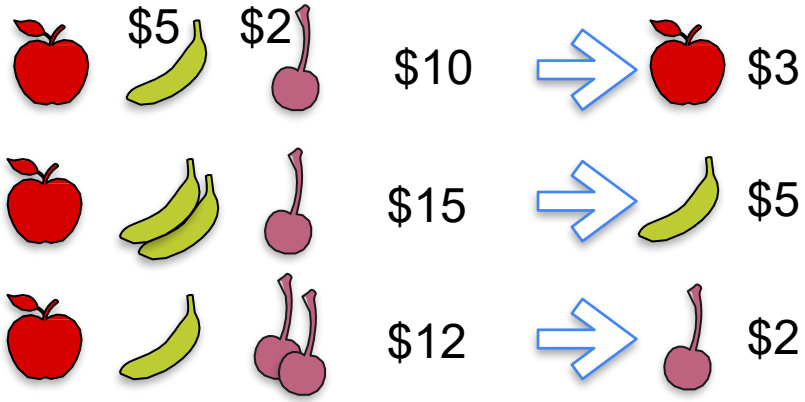
$\Rightarrow \quad a=3$

$c=10-(3+5)$

$c=10-8=2$

$a=3, b=5, c=2$

Solution: Systems of equations



System of equations 1

$$a + b + c = 10$$
$$a + 2b + c = 15$$
$$a + b + 2c = 12$$

10 5

$(a+b+c) + b$

$(a+b+c) + c$

10 2

Solution

$a = 3$

$b = 5$

$c = 2$

Quiz: More systems of equations

System 2

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 20$

$10 + c = 15, c = 5$
$10 + 2c = 20, c = 5$

infinitely many sol.

Redundant

System 3

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 18$

$10 + c = 15, c = 5$
$10 + 2c = 18, c = 4$

No solution

contradictory

System 4

$a + b + c = 10$
$2a + 2b + 2c = 20$
$3a + 3b + 3c = 30$

$2(a + b + c) = 20$
$3(a + b + c) = 30$

infinitely many sol.

Redundant

Solutions: More systems of equations

System 2

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 20$

Infinitely many sols.

$c = 5$

$a + b = 5$

(0,5,5), (1,4,5), (2,3,5), …

System 3

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 18$

No solutions

From 1st and 2nd:
$c = 5$
From 2nd and 3rd:
$c = 3$

System 4

$a + b + c = 10$
$2a + 2b + 2c = 20$
$3a + 3b + 3c = 30$

Infinitely many solutions

Any 3 numbers that add to 10 work.
(0,0,10), (2,7,1), …

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System of Linear Equations

Singular vs non-singular matrices

Constants don't matter for singularity

System 1

$a + b + c = 10$
$a + 2b + c = 15$
$a + b + 2c = 12$

unique solution
Complete
Non-singular

System 2

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 20$

∞ many sol.
Redundant
Singular

System 3

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 18$

no solution
Contradictory
Singular

System 4

$a + b + c = 10$
$2a + 2b + 2c = 15$
$3a + 3b + 3c = 20$

∞ many sols.
Redundant
Singular

Constants don’t matter for singularity

System 1

$a + b + c = 10$
$a + 2b + c = 15$
$a + b + 2c = 12$

$a + b + c = 0$
$a + 2b + c = 0$
$a + b + 2c = 0$

System 2

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 20$

$a + b + c = 0$
$a + b + 2c = 0$
$a + b + 3c = 0$

System 3

$a + b + c = 10$
$a + b + 2c = 15$
$a + b + 3c = 18$

$a + b + c = 0$
$a + b + 2c = 0$
$a + b + 3c = 0$

System 4

$a + b + c = 10$
$2a + 2b + 2c = 20$
$3a + 3b + 3c = 30$

$a + b + c = 0$
$2a + 2b + 2c = 0$
$3a + 3b + 3c = 0$

Constants don’t matter for singularity

System 1

$a + b + c = 0$
$a + 2b + c = 0$
$a + b + 2c = 0$

System 2

$a + b + c = 0$
$a + b + 2c = 0$
$a + b + 3c = 0$

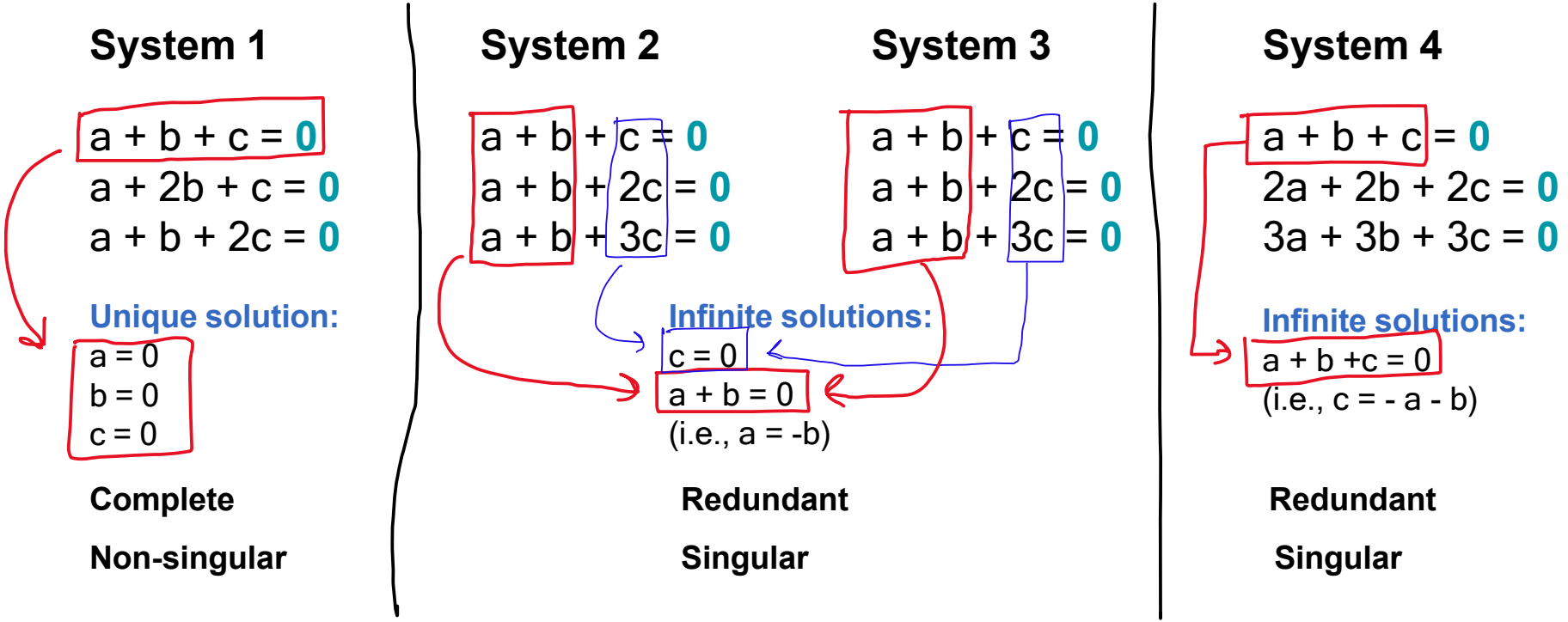
System 3

$a + b + c = 0$
$a + b + 2c = 0$
$a + b + 3c = 0$

System 4

$a + b + c = 0$
$2a + 2b + 2c = 0$
$3a + 3b + 3c = 0$

Constants don't matter for singularity



Constants don’t matter for singularity

System 1

$a + b + c = 0$
$a + 2b + c = 0$
$a + b + 2c = 0$

1	1	1
1	2	1
1	1	2

Non-singular

System 2

$a + b + c = 0$
$a + b + 2c = 0$
$a + b + 3c = 0$

1	1	1
1	1	2
1	1	3

Singular

System 3

$a + b + c = 0$
$a + b + 2c = 0$
$a + b + 3c = 0$

System 4

$a + b + c = 0$
$2a + 2b + 2c = 0$
$3a + 3b + 3c = 0$

1	1	1
2	2	2
3	3	3

Singular

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System of Linear Equations

System of equations as planes (3x3)

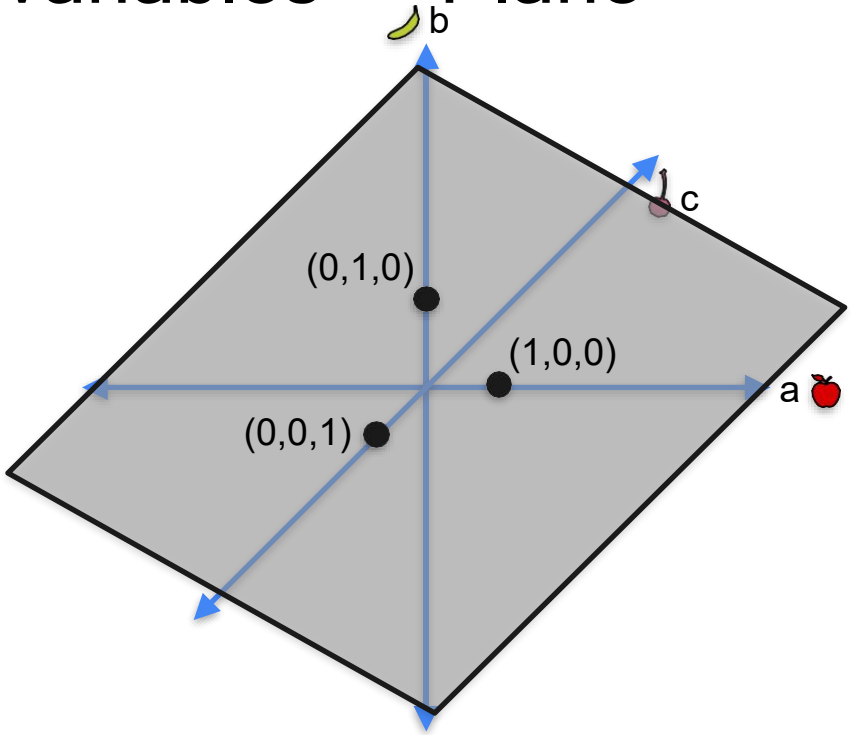
Linear equation in 3 variables -> Plane

$a + b + c = 1$

$1 + 0 + 0 = 1$

$0 + 1 + 0 = 1$

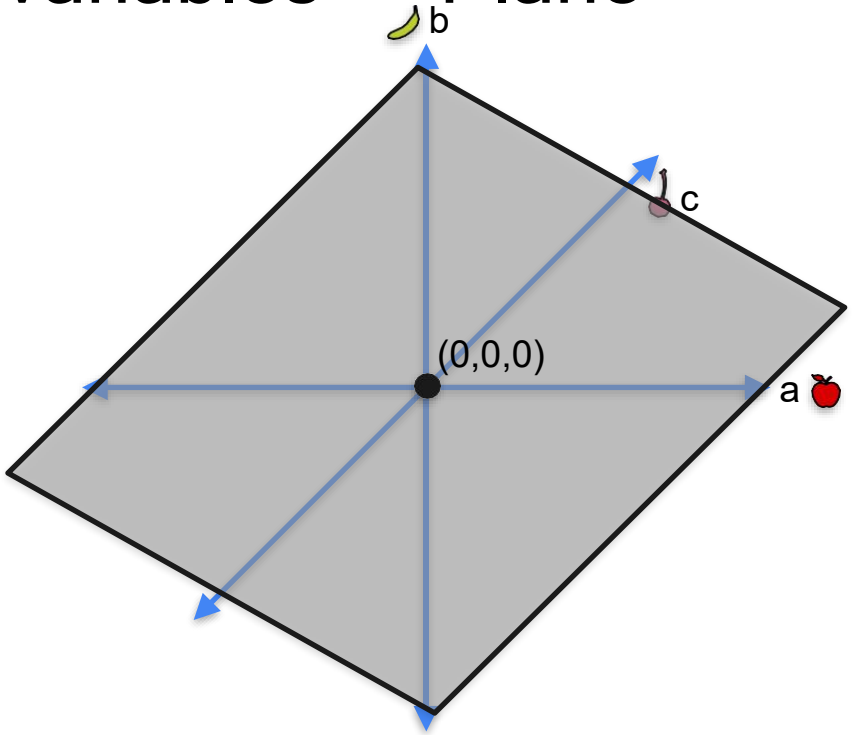
$0 + 0 + 1 = 1$



Linear equation in 3 variables -> Plane

$3a - 5b + 2c = 0$

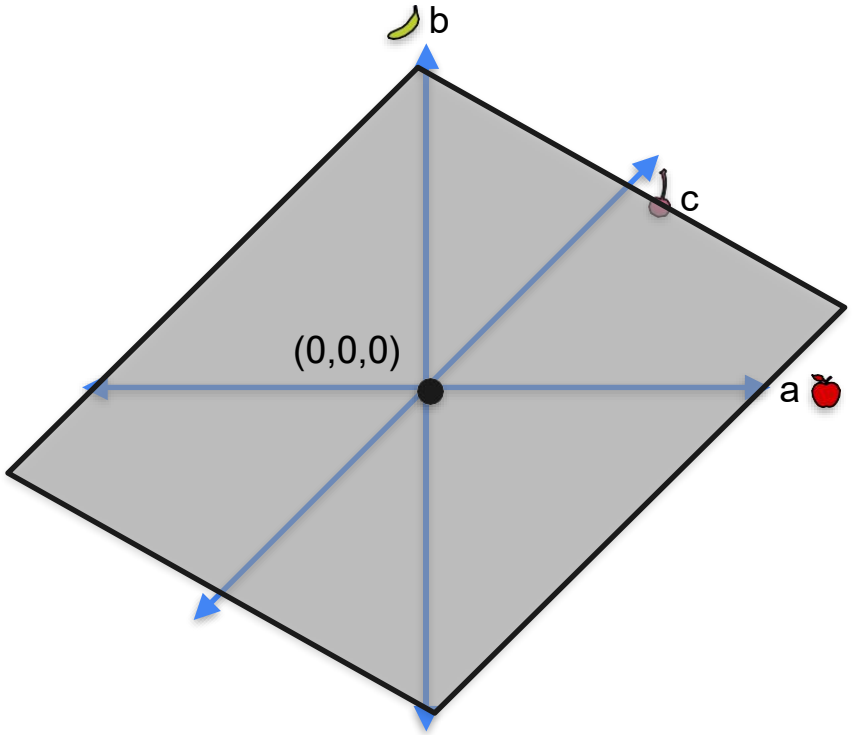
$3(0) + 5(0) + 2(0) = 0$



System 1

System 1

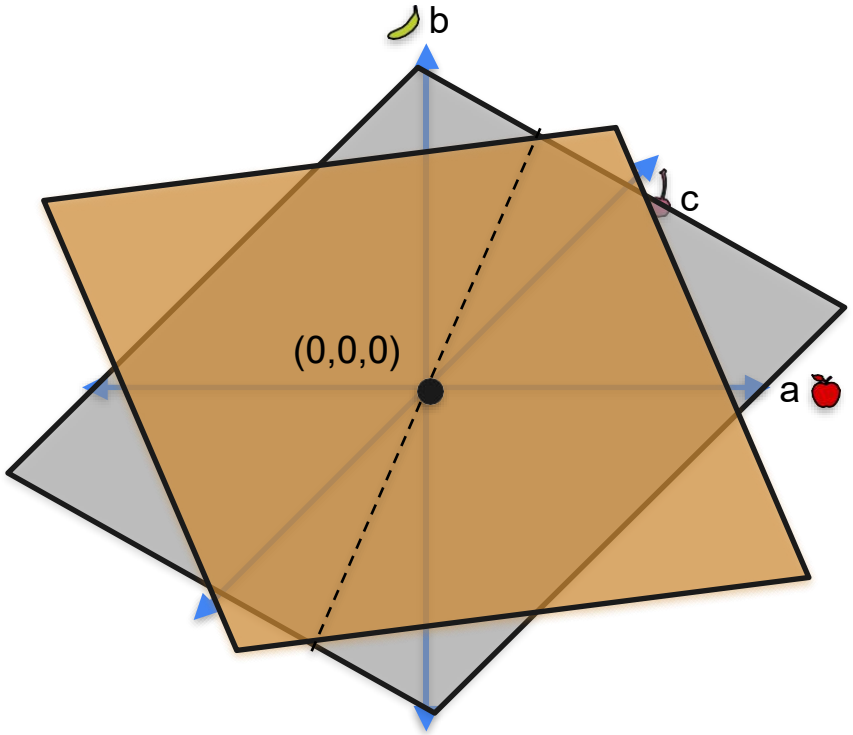
- a + b + c = **0**
- a + 2b + c = **0**
- a + b + 2c = **0**



System 1

System 1

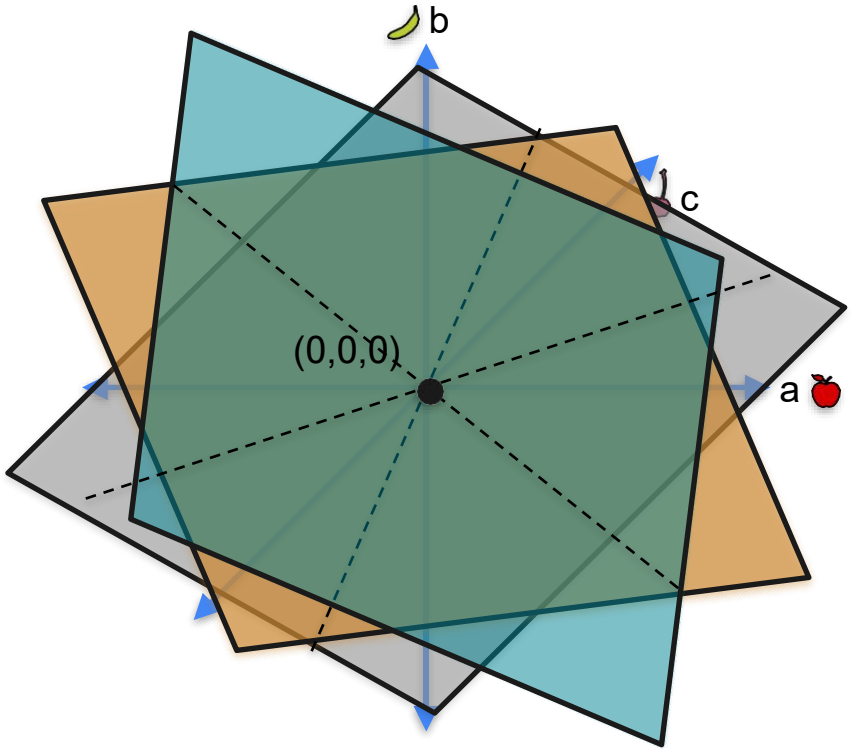
- a + b + c = **0**
- a + 2b + c = **0**
- a + b + 2c = **0**



System 1

System 1

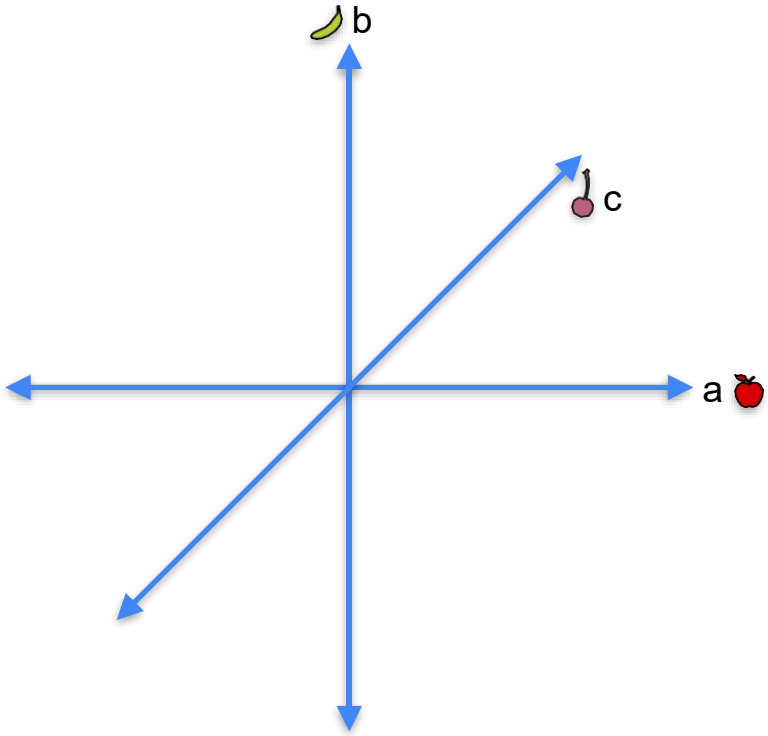
- a + b + c = **0**
- a + 2b + c = **0**
- a + b + 2c = **0**



System 2

System 2

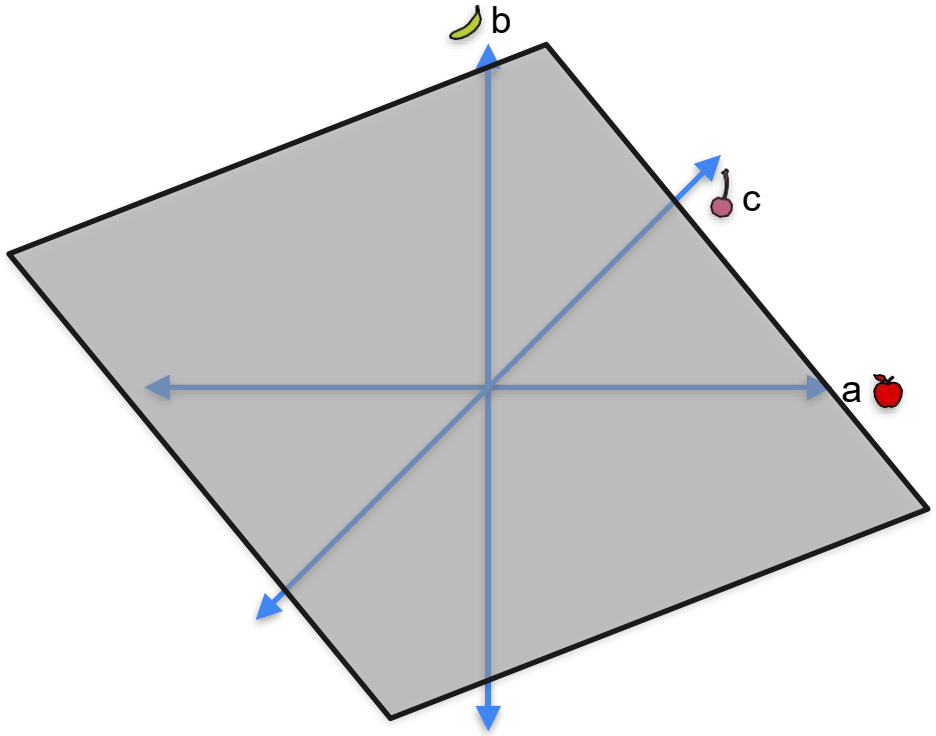
- a + b + c = **0**
- a + b + 2c = **0**
- a + b + 3c = **0**



System 2

System 2

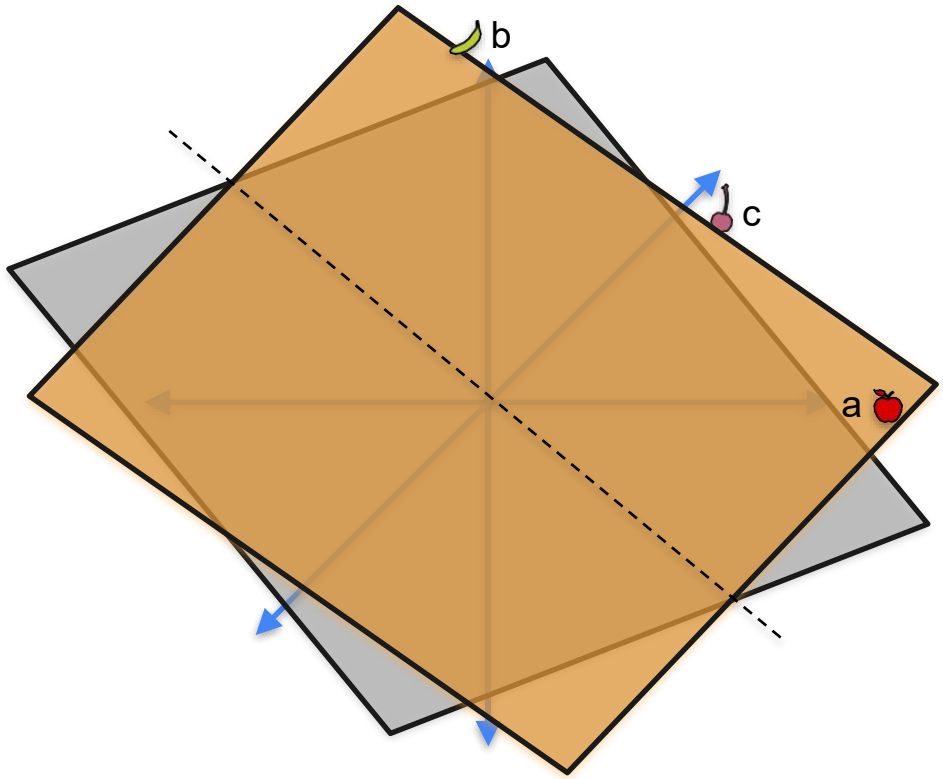
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



System 2

System 2

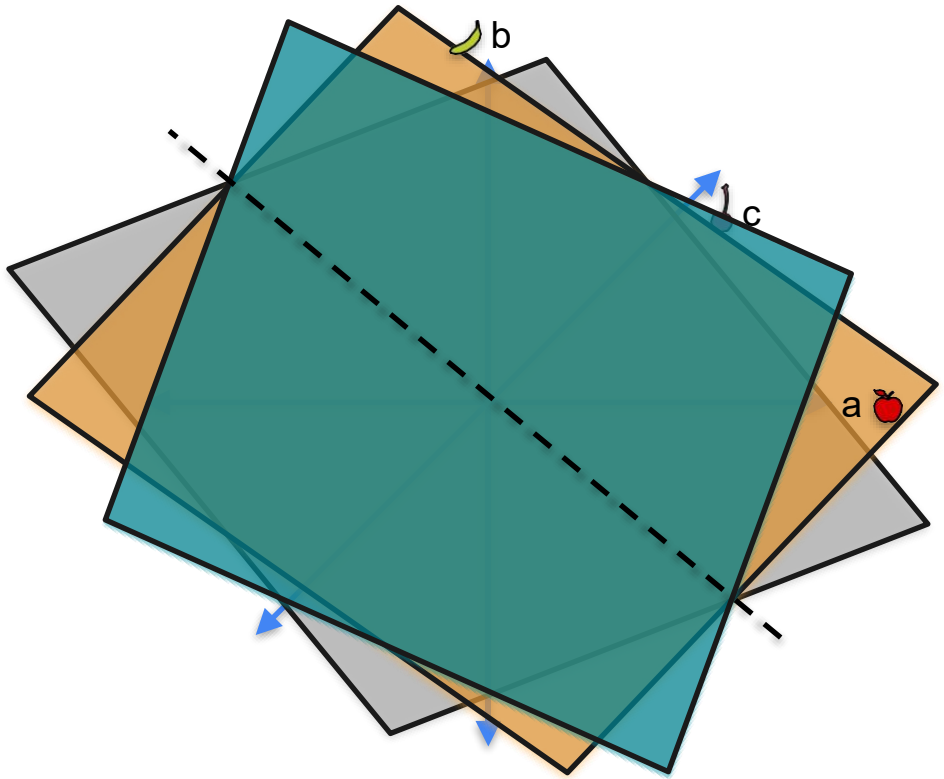
- a + b + c = **0**
- a + b + 2c = **0**
- a + b + 3c = **0**



System 2

System 2

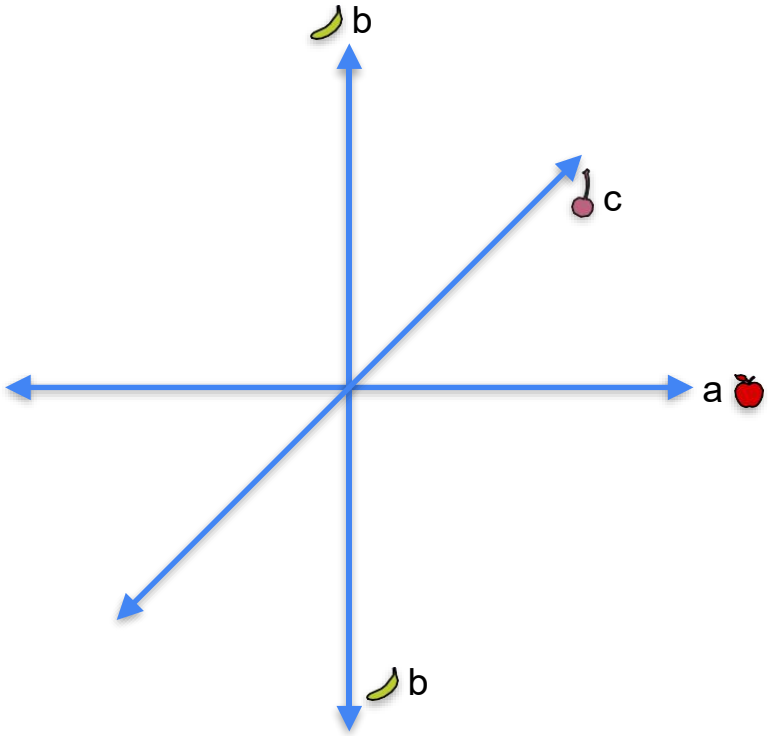
- a + b + c = **0**
- a + b + 2c = **0**
- a + b + 3c = **0**



System 3

System 3

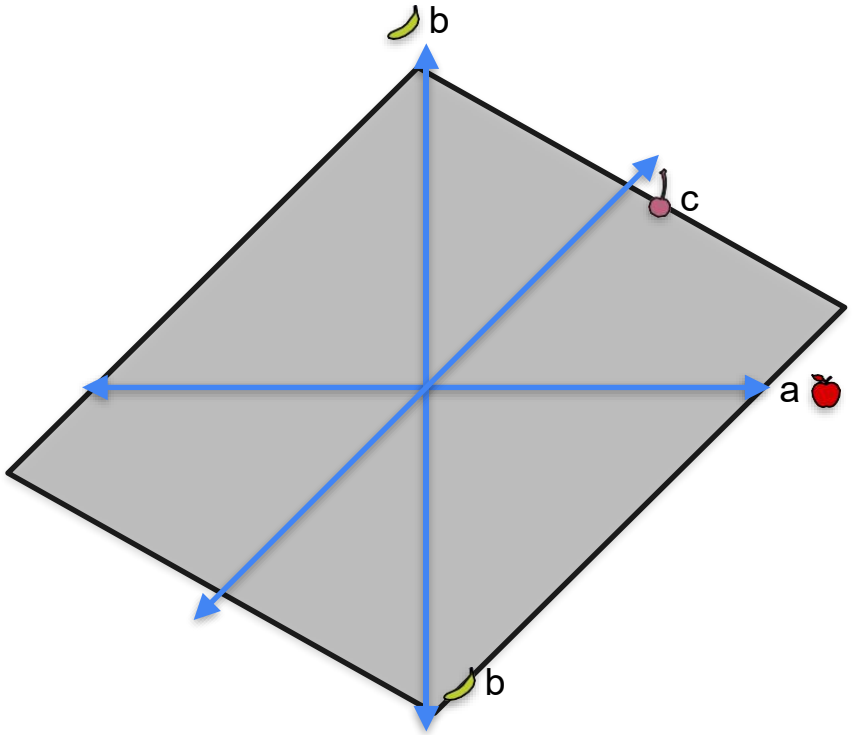
- $a + b + c = 0$ (same)
- $2a + 2b + 2c = 0$ (same plane)
- $3a + 3b + 3c = 0$ (same plane)



System 3

System 3

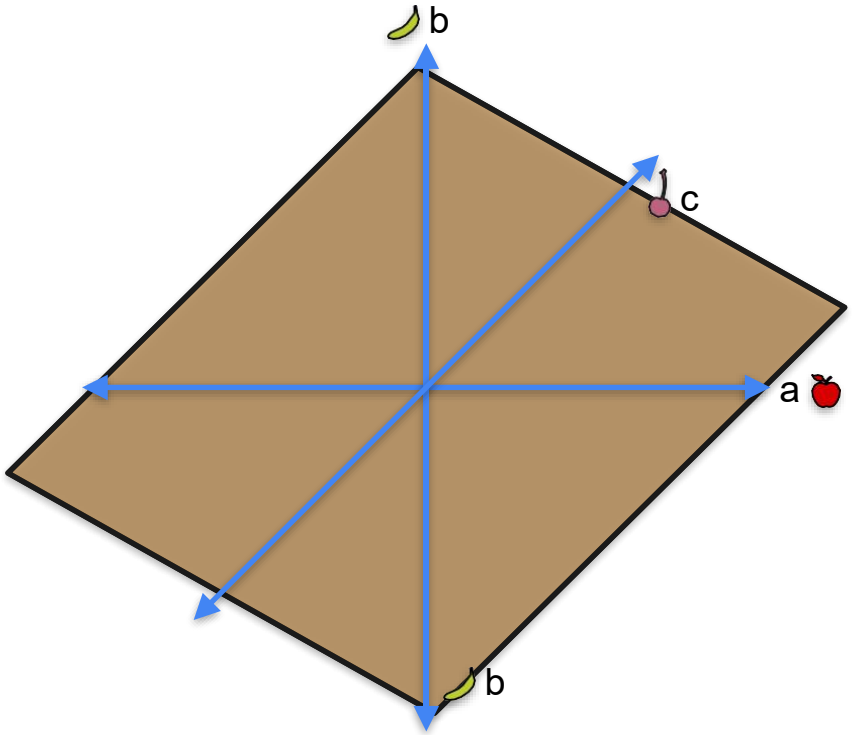
- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$



System 3

System 3

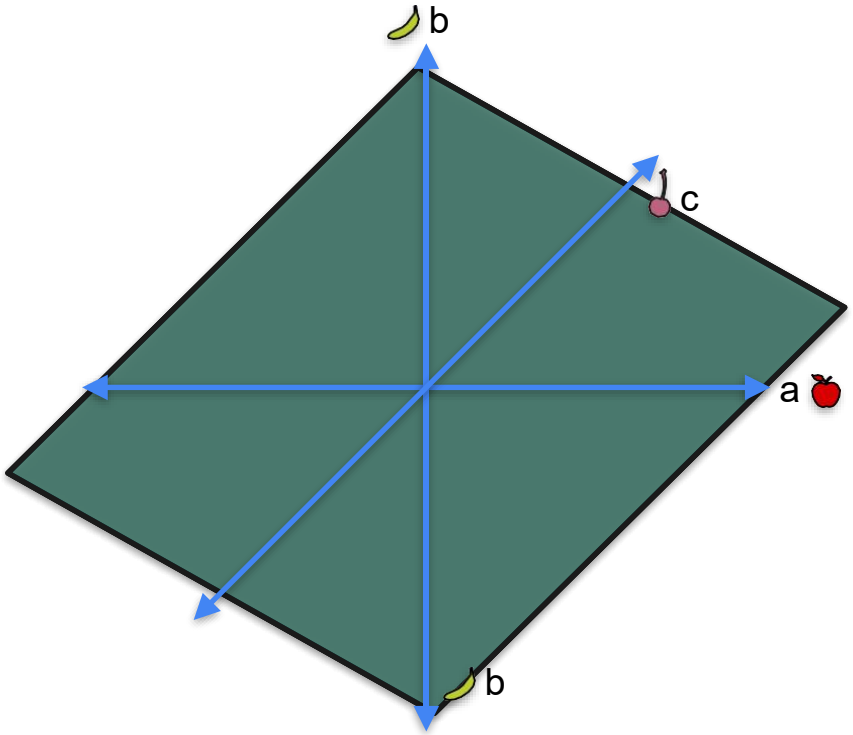
- $a + b + c = 0$
- $2a + 2b + 2c = 0$ ⇐
- $3a + 3b + 3c = 0$



System 3

System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$ ⇦



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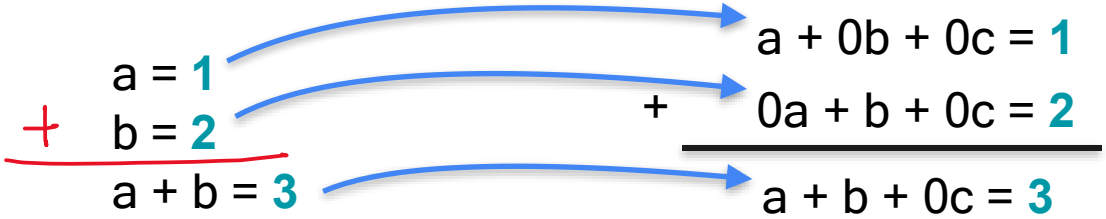
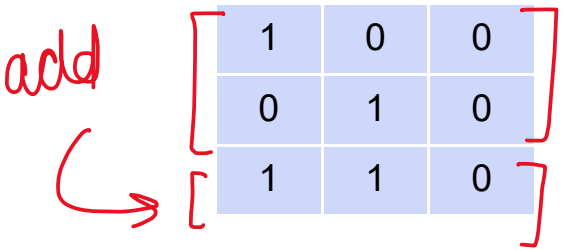
System of Linear Equations

Linear dependence and independence (3x3)

Linear dependence and independence





Row 1 + Row 2 = Row 3

Row 3 **depends** on rows 1 and 2

Rows are **linearly dependent**

Linear dependence and independence

$$
\begin{array}{rl}
 & a + b + c = 0 \\
+ & 2a + 2b + 2c = 0 \\
\hline
 & 3a + 3b + 3c = 0
\end{array}
$$

add

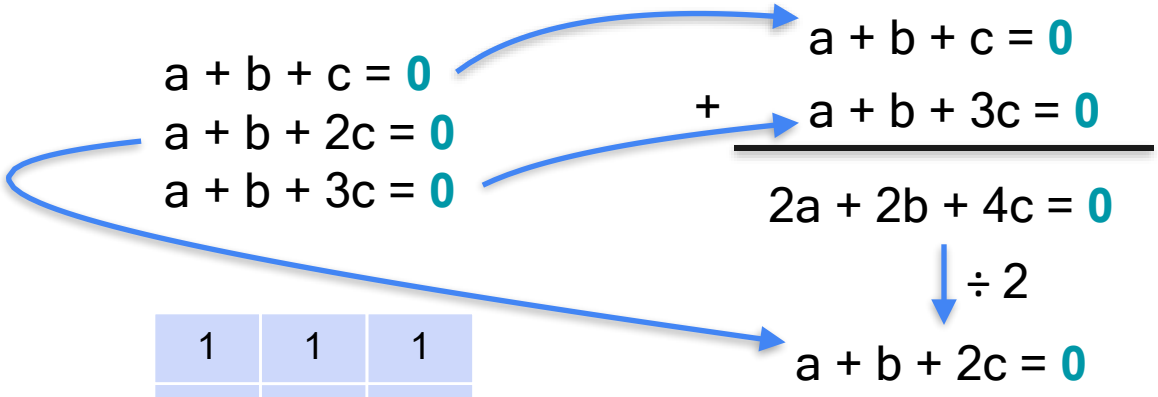
1	1	1
2	2	2
3	3	3

Row 1 + Row 2 = Row 3

Row 3 is dependent on Row 1 & 2

Rows are linearly dependent

Linear dependence and independence



1	1	1
1	1	2
1	1	3

Average of Row 1 and Row 3 is Row 2

Row 2 **depends** on rows 1 and 3

Rows are **linearly dependent**

Linear dependence and independence

Non singular

$a + b + c = 0$
$a + 2b + c = 0$ → No relations between equations
$a + b + 2c = 0$

unique solution

1	1	1
1	2	1
1	1	2

No relations between rows

Rows are **linearly independent**

Quiz: Linear dependence and independence

Problem: Determine if the following matrices have linearly dependent or independent rows

singular

1	0	1
0	1	0
3	2	3

Linearly Independent ✗

$3R_1 + 2R_2 = R_3$

singular

1	1	1
1	1	2
0	0	-1

Linearly Dependent

$R_1 - R_2 = R_3$ ✓

non-singular

1	1	1
0	2	2
0	0	3

Linearly Independent ✓

singular

1	2	5
0	3	-2
2	4	10

Linearly Dependent

$R_3 = 2R_1$ ✓

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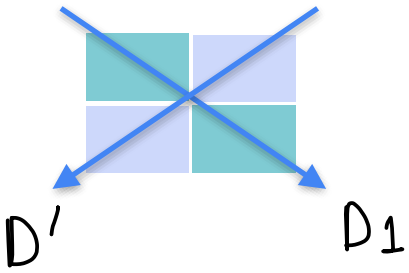
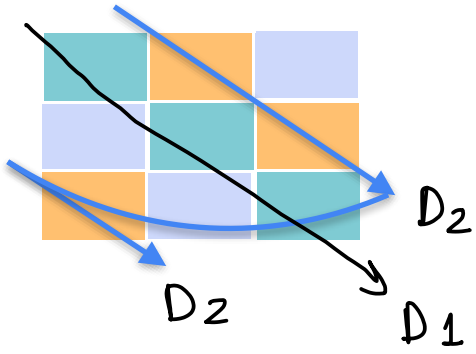
System of Linear Equations

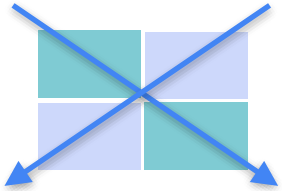
The determinant (3x3)

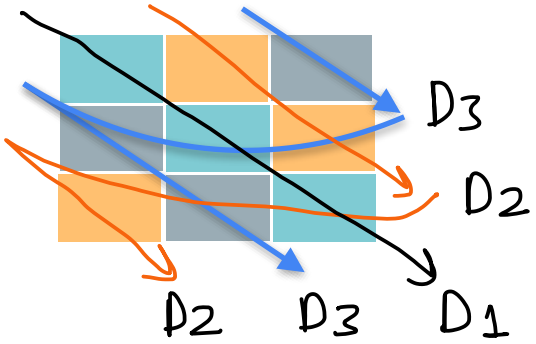
Diagonals in a 3x3 matrix



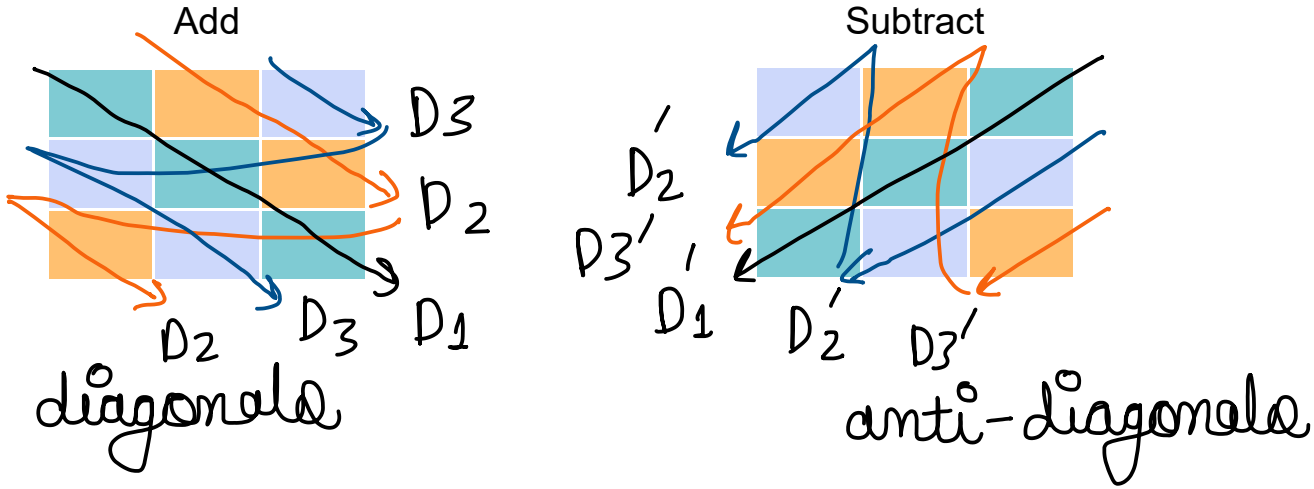


Diagonals in a 3x3 matrix





Determinant



The determinant

1	1	1
1	2	1
1	1	2

sum(diagonals) − sum(anti-diagonals)

$[(1)(2)(2) + (1)(1)(1) + (1)(1)(1)] = 4+1+1 = 6$

$[(1)(2)(1) + (1)(1)(1) + (2)(1)(1)] = 2+1+2 = 5$

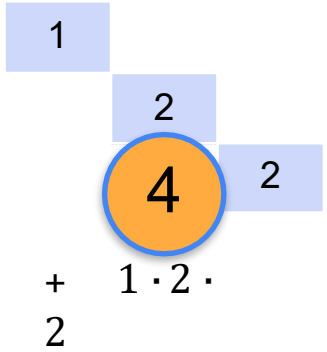
$6-5 = 1$

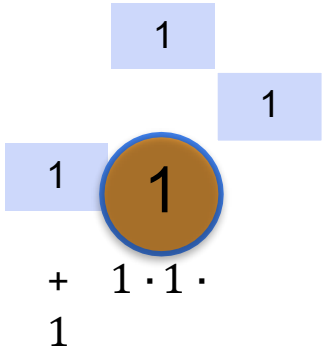
The determinant

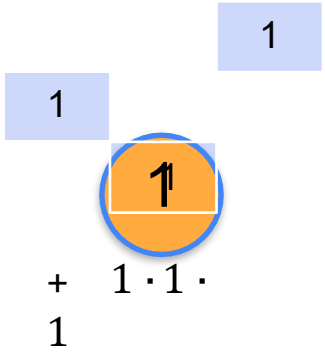
1	1	1
1	2	1
1	1	2

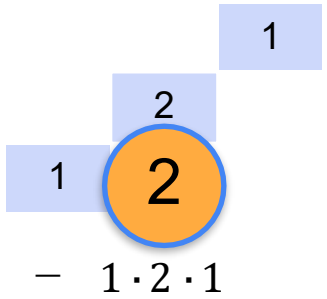
Det = 4+1+1
-2-1-2

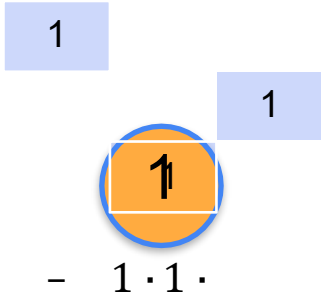
= 1

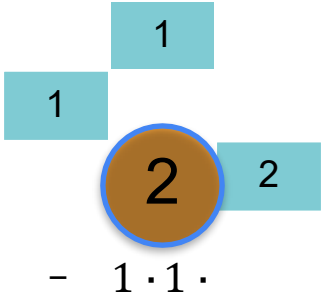












Quiz: Determinants

Problem: Find the determinant of the following matrices (from the previous quiz). Verify that those with determinant 0 are precisely the singular matrices.

1	0	1
0	1	0
3	3	3

$3+0+0-3-0-0=0$

singular

1	1	1
1	1	2
0	0	-1

$-1+0+0-0-0+1=0$

singular

1	1	1
0	2	2
0	0	3

$6+0+0-0-0-0=6$

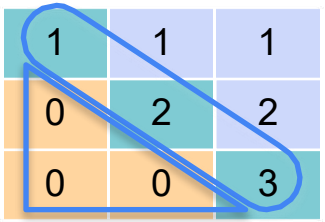
non-singular

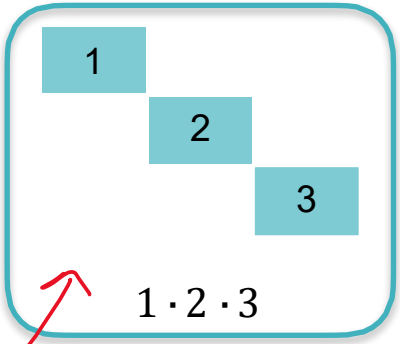
1	2	5
0	3	-2
2	4	10

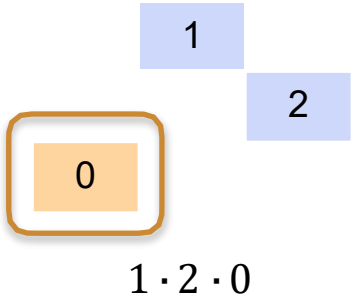
$30-8+0-30+8-0=0$

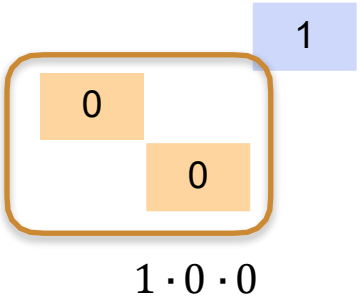
singular

The determinant (upper triangular matrix)







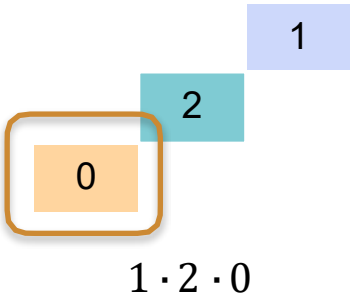


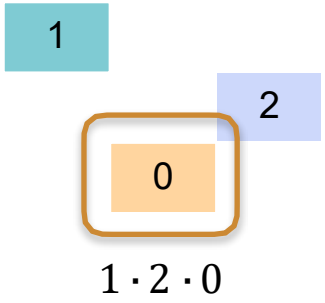
Det = 6+0+0-0-0-0

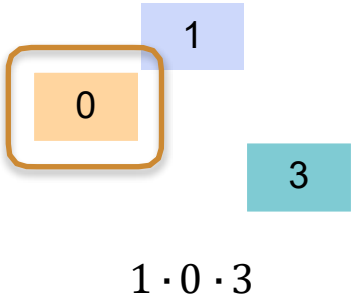
= 6

Determinant

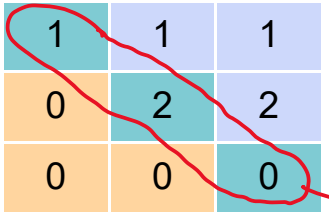
"product of diagonal"







The determinant (upper triangular matrix)



product

$(1)(2)(0) = 0$

Determinant

DeepLearning.AI

System of Linear Equations

Conclusion