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Math for Machine Learning

Linear algebra - Week 2

Solving systems of equations

Matrix row reduction

Row operations that preserve singularity

Row-reduced echelon form

Row echelon form

Rank of a matrix

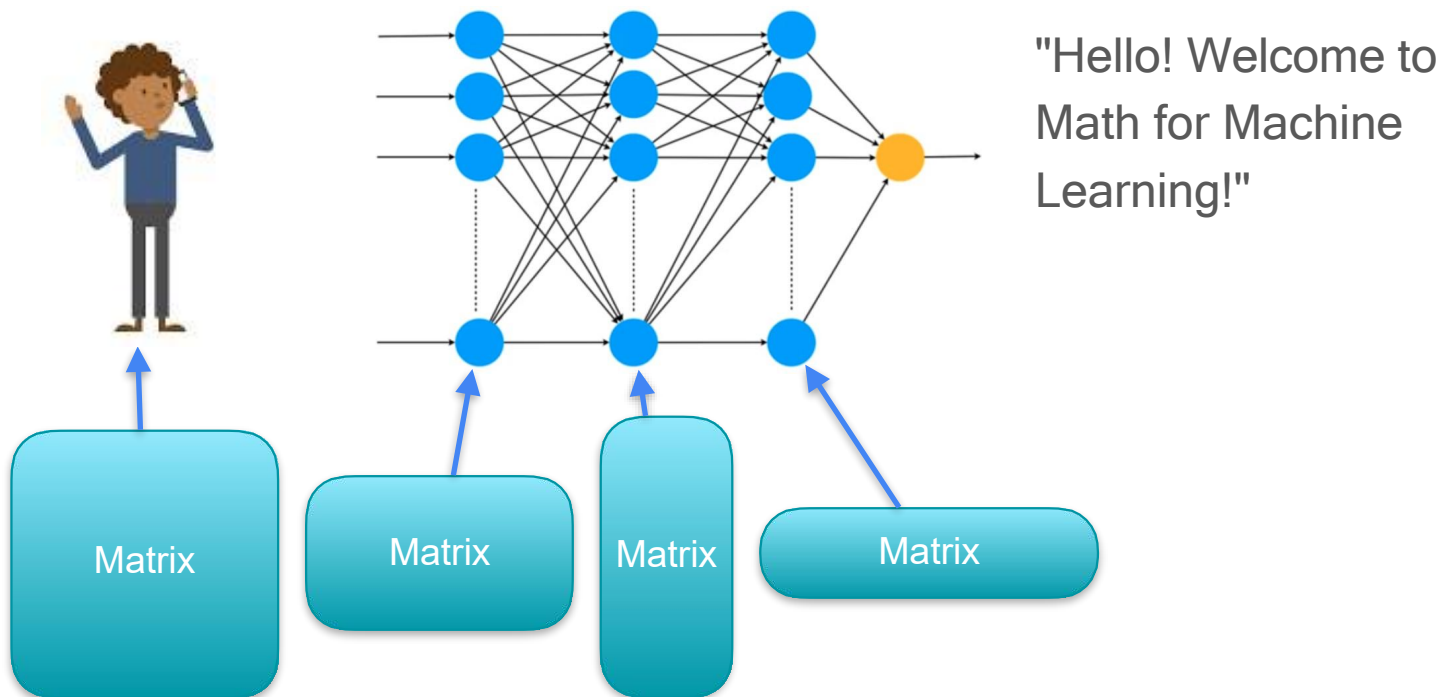


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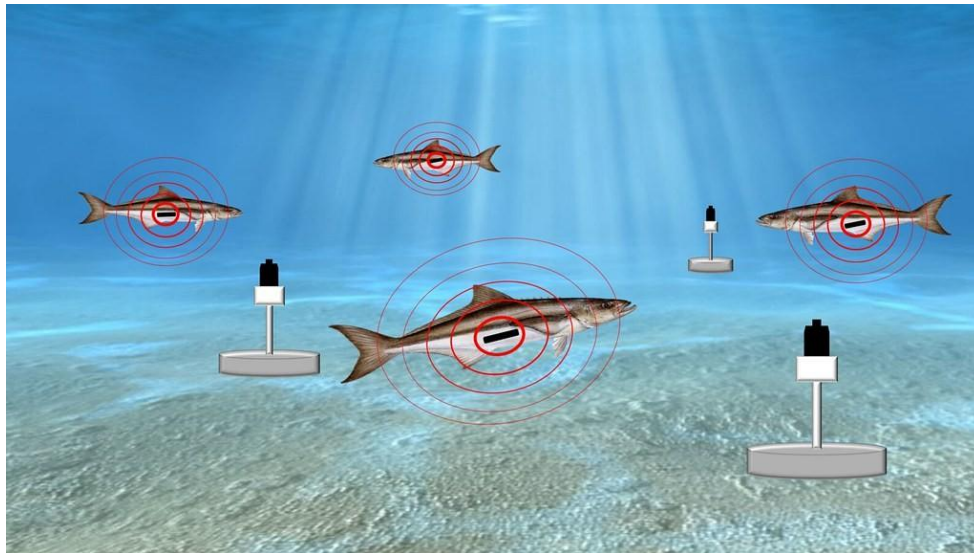
Solving System of Linear Equations

Machine learning motivation

Neural networks - Matrix operations



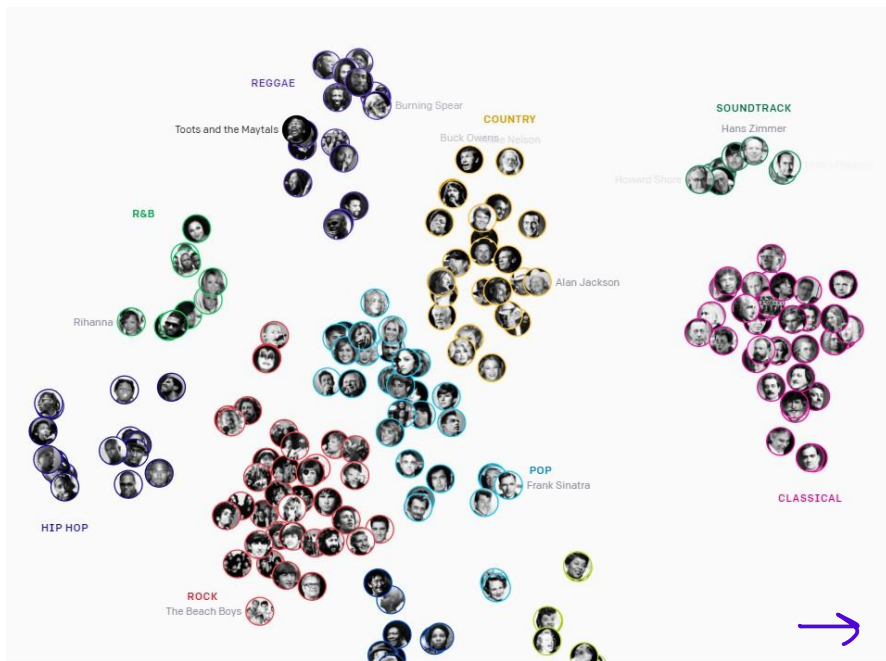
Neural networks - Sound recognition



Acoustic monitoring: Monitoring ecosystems through sounds

- Sound recognition: tracking species through sound to preserve bio-habitats.

Neural Networks - AI-generated music



Neural network generates music

- Automatic music generation: compressing music to discrete codes, then training the model on a specific genre to produce new music.

→ Variational autoencoder



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Solving System of Linear Equations

**Solving non-singular system
of linear equations**

Solving systems of equations

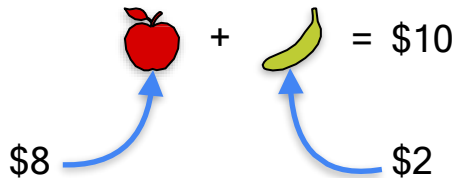
System

- $a + b = 10$

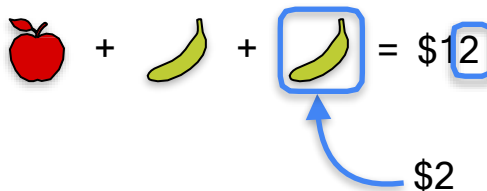


- $a + 2b = 12$









A diagram illustrating the substitution process. It shows an apple icon plus a banana icon equals \$10. Below the apple, a blue arrow points from the text '\$8' to the apple. Below the banana, a blue arrow points from the text '\$2' to the banana.



A diagram showing the final equation after substitution: an apple icon plus a banana icon plus a boxed banana icon equals \$12. A blue arrow points from the text '\$2' to the boxed banana icon. The entire equation is enclosed in a light blue rounded rectangle.

Solving systems of equations

System

- $a + b = 10$
 
- $a + 2b = 12$
 

Some process



Manipulating equations

Swapping equations

Adding equations

Multiplying equations by a constant

Solved system

- $a = 8$

- $b = 2$


Solving systems of equations

System

- $a + b = 10$



- $a + 2b = 12$



Eliminate 'a' from this equation

Solved system

- $a = 8$



- $b = 2$



Manipulating equations

Multiplying by a constant

$$\begin{array}{r} a + b = 10 \\ * \quad \quad \quad 7 \\ \hline 7a + 7b = 70 \end{array}$$

add 2 equations

$$\begin{array}{r} a + b = 10 \\ + \quad 2a + 3b = 22 \\ \hline 3a + 4b = 32 \end{array}$$

Let's do a harder example

Systems of equations

$$5a + b = 17$$

$$4a - 3b = 6$$

$$\rightarrow \text{divide by } 5 \rightarrow a + b/5 = 17/5$$

$$\rightarrow \text{divide by } 4 \rightarrow a - 3/4b = 6/4$$

$$b/5 + 3/4b = 17/5 - 6/4$$

$$\Rightarrow \frac{4b + 15b}{\cancel{20}} = \frac{68 - 30}{\cancel{20}}$$

$$\Rightarrow 19b = 38$$

$$\Rightarrow b = 2$$

$$a = 3, b = 2$$

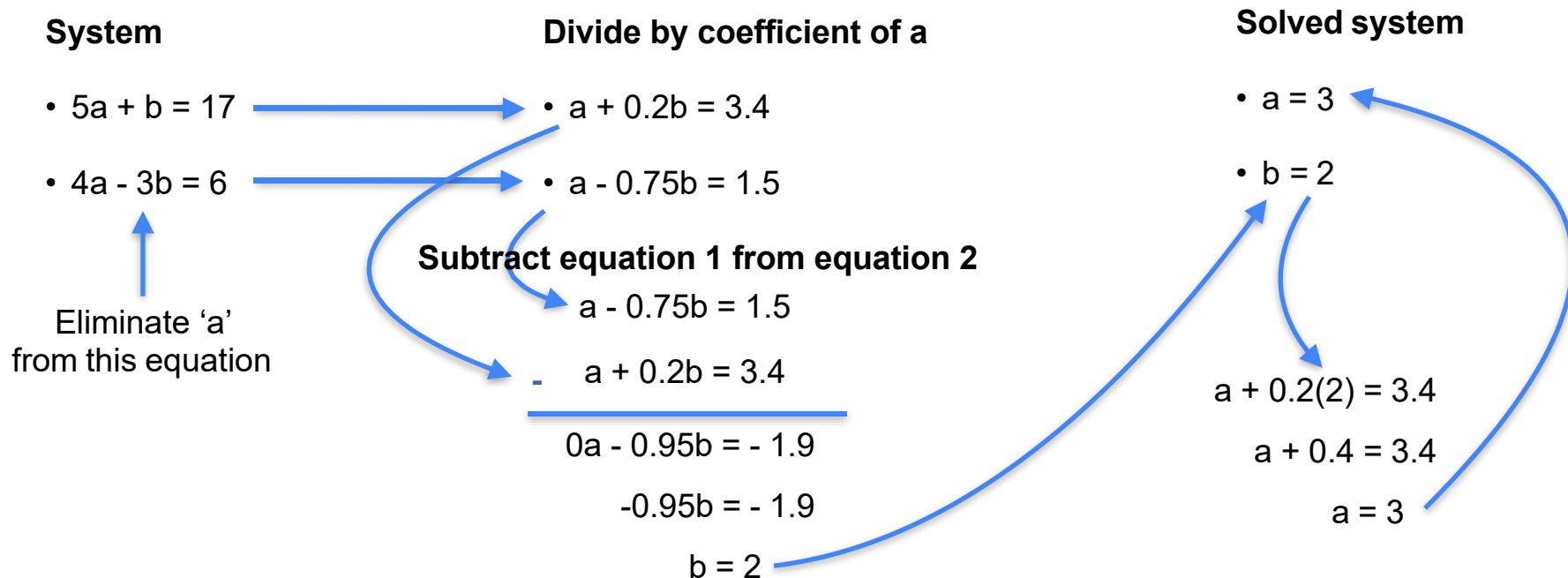
$$5a + 2 = 17$$

$$a = \frac{15}{5} = 3$$

solved system!!

$$\begin{array}{r} 17 \\ \times 4 \\ \hline 68 \end{array}$$

Systems of equations



What if one of the coefficients of a is zero?

System

- $5a + b = 17$

- $3b = 6$



$$b = 6/3 = 2$$

$$5a + 2 = 17$$

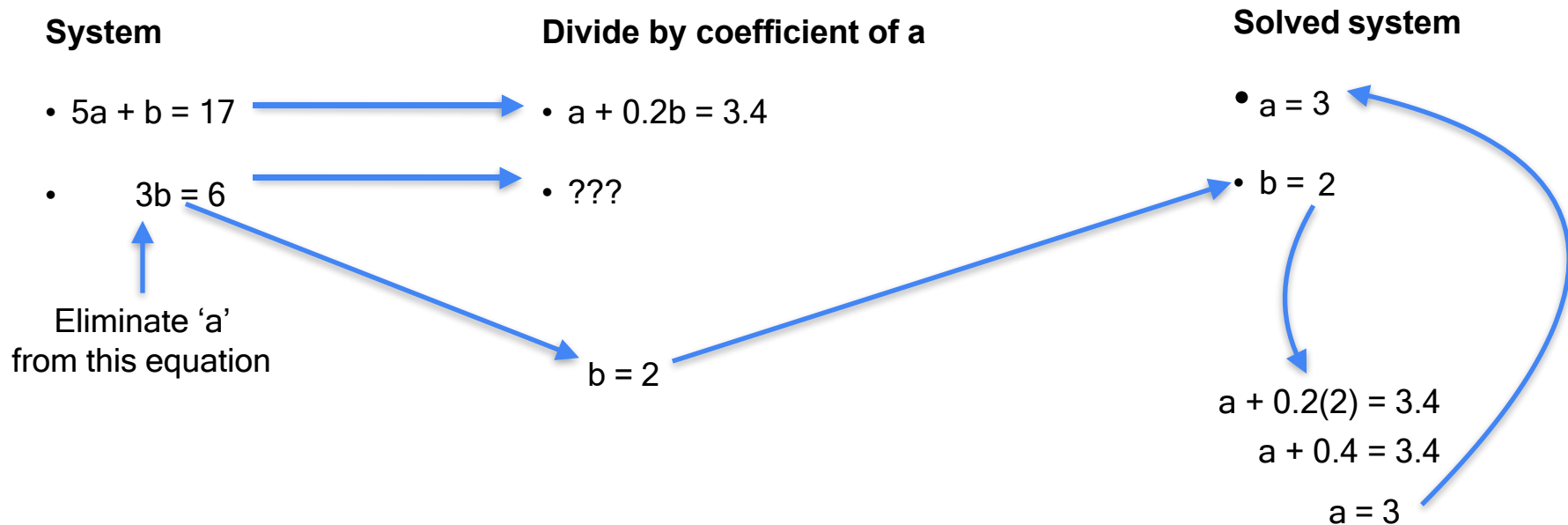
$$a = 3$$

Solved system

- $a = ?$

- $b = ?$

What if one of the coefficients of a is zero?



Quiz

- Solve the following system of equations

System

$$\bullet 2a + 5b = 46 \rightarrow a + 5/2 b = 23$$

$$\bullet 8a + b = 32 \rightarrow a + b/8 = 4$$

$$a = 3, b = 8$$

$$(5/2)b - \frac{b}{8} = 19$$

$$\frac{5(4)b - b}{8} = 19$$

$$\frac{20b - b}{8} = 19$$

$$\cancel{19} b = \cancel{19}(8)$$

$$20a + 5(8) = 46$$

$$20a = 6$$

$$a = 3$$

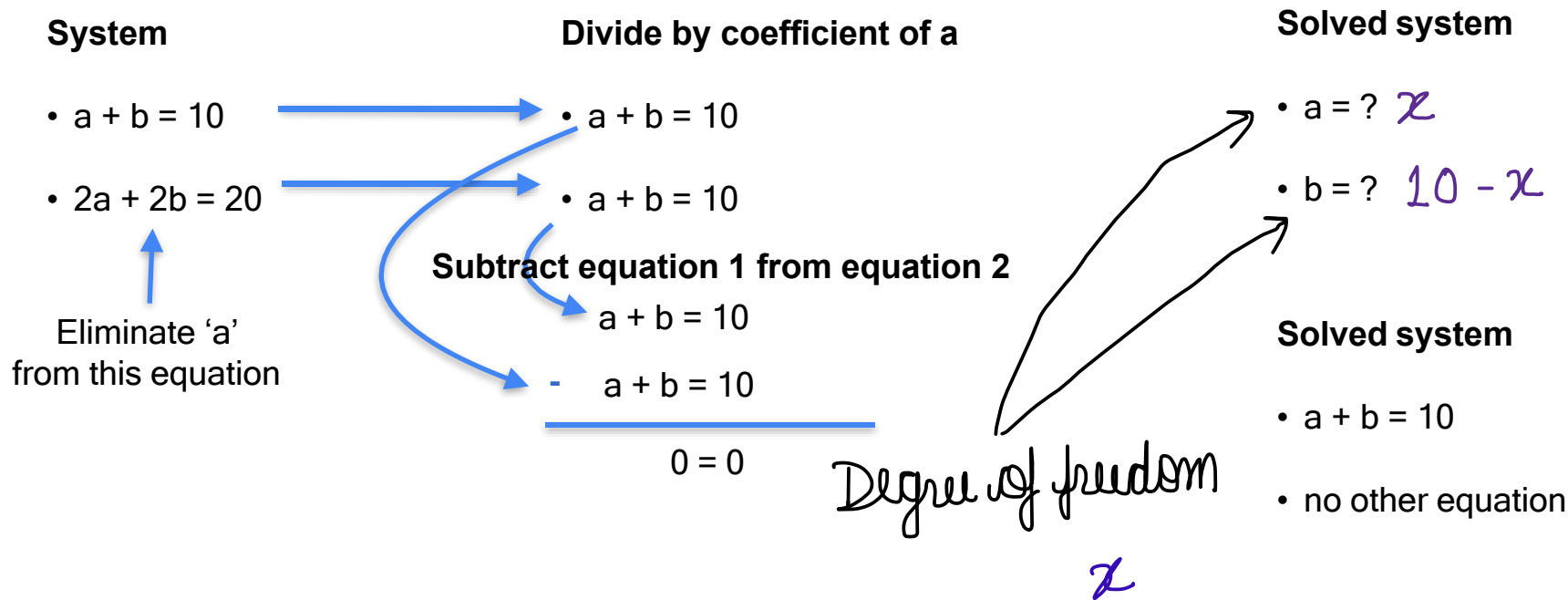


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Solving System of Linear Equations

**Solving singular system of
linear equations**

What if the system is singular (redundant)?



What if the system is singular (contradictory)?

System

- $a + b = 10 \rightarrow a + b = 10$
- $2a + 2b = 24 \rightarrow -a + b = 12$

$$\boxed{0 \neq -2}$$



No solutions

Solved system

- $a = ?$
- $b = ?$

Quiz

- Solve the following system of equations

System

- $5a + b = 11 \rightarrow a + b/5 = 11/5$

- $10a + 2b = 22 \rightarrow a + b/5 = \cancel{22/10} \quad 11/5$

$$\boxed{0 = 0}$$

^{oo}
infinite solutions

Redundant !!



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Solving System of Linear Equations

**Solving system of equations
with more variables**

Elimination method

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Elimination method

$$c/3 - 2c$$

System

- $a + b + 2c = 12 \longrightarrow \textcircled{1}$
- $3a - 3b - c = 3 \longrightarrow \textcircled{2}$
- $2a - b + 6c = 24 \longrightarrow \textcircled{3}$

Leave 'a' by
itself

subtract eq^① from eq^②

$$-2b - \left(\frac{7}{3}\right)c = -13$$

Elimination method

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Divide each row by the coefficient of 'a'

- $a + b + 2c = 12$
- $a - b - \frac{1}{3}c = 1$
- $a - \frac{b}{2} + 3c = 12$

Use the first equation to remove 'a' from the others

- Isolated 'a'
- $a + b + 2c = 12$
 - $-2b - \frac{7}{3}c = -11$
 - $-\frac{3}{2}b + c = 0$
- Solve this new system of 2 equations

Elimination method

System

- $a + b + 2c = 12$
- $-2b - 7/3 c = -11$
- $-3/2 b + c = 0$

Divide last two rows by the coefficient of b

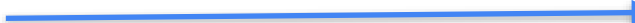

- $a + b + 2c = 12$
- $b + 7/6 c = 11/2$
- $b - 2/3 c = 0$

Use the second equation to remove 'b' from the third

- $a + b + 2c = 12$
 - $b + 7/6 c = 11/2$
 - $-11/6 c = -11/2$
- Isolated 'b'
- $c = 3$

Elimination method

System

- $a + b + 2c = 12$  $a + 2 + 6 = 12$
 $a = 4$
- $b + 7/6 c = 11/2$  $b + 7/2 = 11/2$
 $b = 2$
- $c = 3$

Replace $c = 3$
in the second
equation, get
 $b = 2$

Replace $c = 3$
and $b = 2$ in the
first equation,
get $a = 4$

The solution is
 $a = 4$
 $b = 2$
 $c = 3$



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Solving System of Linear Equations

Matrix row reduction

Systems of equations to matrices

Original system

- $5a + b = 17$
- $4a - 3b = 6$

Intermediate System

- $a + 0.2b = 3.4$
- $b = 2$

Solved system

- $1a + 0b = 3$
- $0a + 1b = 2$

Original matrix

5	1
4	-3

Upper diagonal matrix

1	0.2
0	1

Row echelon form

Diagonal matrix

1	0
0	1

Reduced row echelon form

Systems of equations to matrices

Original system

- $a + b = 10$
- $2a + 2b = 20$

Intermediate System

- $a + b = 10$
- $0a + 0b = 0$

Original matrix

1	1
2	2

Upper diagonal matrix

1	1
0	0

Row echelon form

Systems of equations to matrices

Original system

- $5a + b = 11$
- $10a + 2b = 22$



$$a + 0.2b = 2.2$$

$$0a + 0b = 0$$

original matrix

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$$



Row echelon form

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

Systems of equations to matrices

Original system

- $5a + b = 11$
- $10a + 2b = 22$

Intermediate System

- $a + 0.2b = 2.2$
- $0a + 0b = 0$

Original matrix

5	1
10	2

Upper diagonal matrix

1	0.2
0	0

Row echelon form

Systems of equations to matrices

Original system

- $0a + 0b = 0$
- $0a + 0b = 0$

Intermediate System

- $0a + 0b = 0$
- $0a + 0b = 0$

Original matrix

0	0
0	0

Upper diagonal matrix

0	0
0	0

Row echelon form

Row echelon form

1	•	•	•	•
0	1	•	•	•
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



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Solving System of Linear Equations

**Row operations that
preserve singularity**

Switching rows

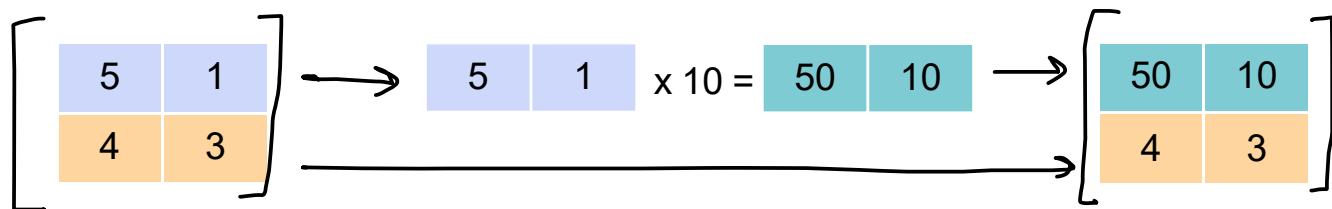
5	1
4	3

Determinant $5 \cdot 3 - 1 \cdot 4 = 11$

4	3
5	1

Determinant $4 \cdot 1 - 3 \cdot 5 = -11$

Multiplying a row by a (non-zero) scalar

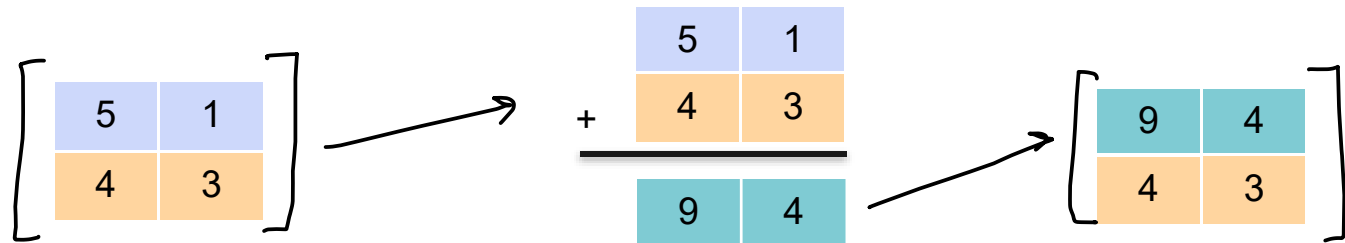


Determinant

$$\begin{aligned} & 5(3) - 4(1) \\ & = 11 \end{aligned}$$

$$\begin{aligned} \text{Determinant} &= 5 \cdot (10 \cdot 3) - 1 \cdot (10 \cdot 4) \\ &= 10(11) \end{aligned}$$

Adding a row to another row



Determinant

$$\begin{aligned} & 5(3) - 4(1) \\ &= 11 \end{aligned}$$

Determinant

$$\begin{aligned} & 9(3) - 4(4) \\ &= 11 \end{aligned}$$

$$\begin{cases} x+y=4 \\ -6x+2y=16 \end{cases} \rightarrow \begin{array}{r} y+x=4 \\ y-3x=8 \\ \hline 4x=-4 \\ x=-1 \end{array}$$

$$y=5$$

$$\begin{bmatrix} -3 & 8 & 1 \\ 2 & 2 & -1 \\ -5 & 6 & 2 \end{bmatrix}$$

$$-3(4+6) - 8(4-5) + 1(12+10)$$

$$\Rightarrow -30 + 8 + 22 = 0$$

(Singular)

$$\begin{bmatrix} 4 & -3 \\ 7 & -8 \end{bmatrix} \Rightarrow \begin{array}{l} -32 + 21 \\ = -11 \text{ (Non singular)} \end{array}$$

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix} \Rightarrow ac - ab = a(c-b) \neq 0$$

$$\begin{array}{l} 2a \quad s = 2\% \\ a \quad cd = 3\% \\ z \quad b = 4\% \end{array}$$

\$10,000

$$3a + z = 10,000 \longrightarrow \textcircled{1}$$

$$\underline{0.04a + 0.03a + 0.04z = 260}$$

$$7a + 4z = 26000$$

$$1.75a + z = 6500 \longrightarrow \textcircled{2}$$

$$1.25a = 3500$$

$$a = 2800$$

$$z = 10000 - 3(2800) = 1600$$



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Rank \rightarrow how much information is the matrix carrying / or the corresponding system of linear eqns

Solving System of Linear Equations

Rank of a matrix

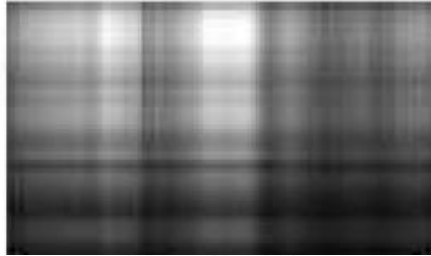
singular value decomposition (SVD)

Compressing Images - Reducing rank

Original (Rank 200)



Rank 1



Rank 2



Rank 5



Rank 15





Rank 50



Systems of information

goal: "determine the color"

System 1



 The dog is **black**
 The cat is **orange**

Two sentences

Two pieces of information

Rank = 2

System 2

 The dog is **black**
 The dog is **black**

Two sentences

One piece of information

Rank = 1

System 3

 The dog
 The dog

Two sentences

Zero pieces of information

Rank = 0

Systems of equations



System 1

$$a + b = 0$$



$$a + 2b = 0$$



 	
1	1
1	2

Rank = 2

Two equations

Two pieces of information

Rank = 2



System 2

$$a + b = 0$$



$$2a + 2b = 0$$



 	
1	1
2	2

Rank = 1



Two equations

One piece of information

Rank = 1

System 3

$$0a + 0b = 0$$

 	
0	0
0	0



Rank = 0

Two equations

Zero pieces of information

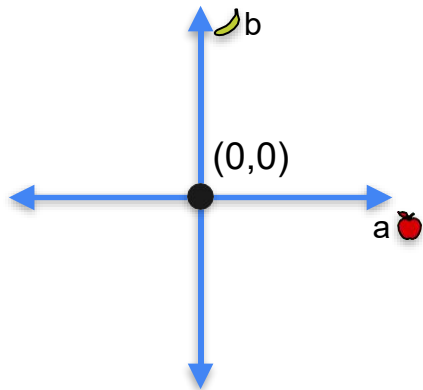
Rank = 0



Rank and solutions to the system

	
1	1
1	2

Rank = 2

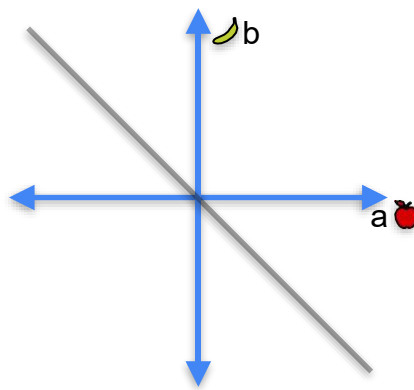
Dimension of solution space = 0





	
1	1
2	2

Rank = 1

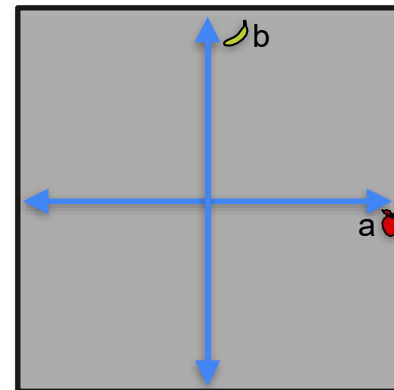
Dimension of solution space = 1





	
0	0
0	0

Rank = 0

Dimension of solution space = 2





Rank of a matrix





1	1
1	2

Rank = 2



1	1
2	2

Rank = 1





0	0
0	0

Rank = 0

Dimension of solution space = 0 Dimension of solution space = 1 Dimension of solution space = 2

$$\text{Rank} = 2 - (\text{Dimension of solution space})$$



Rank and singularity



1	1
1	2

Rank = 2



Non-singular



1	1
2	2

Rank = 1

Singular



0	0
0	0

Rank = 0

Singular

Quiz: Rank of a matrix

Determine the rank of the following two matrices

Matrix 1

5	1
-1	3

Rank = 2 # solution space of dimension 0
→ $15 + 1 = 16$, Non singular

Matrix 2

2	-1
-6	3

solution space of dimension 1
→ $6 - 6 = 0$, singular
Rank = 1



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Solving System of Linear Equations

**Rank of a matrix:
General case**

Rank for matrices

System 1

$$\begin{aligned}a + b + c &= 0 \\a + 2b + c &= 0 \\a + b + 2c &= 0\end{aligned}$$



3 Equations
3 Pieces of information

Rank 3

1	1	1
1	2	1
1	1	2

System 2

avg →

$$\begin{aligned}a + b + c &= 0 \\a + b + 2c &= 0 \\a + b + 3c &= 0\end{aligned}$$



3 Equations
2 Pieces of information

Rank 2

1	1	1
1	1	2
1	1	3

System 3

x₂
x₃

$$\begin{aligned}a + b + c &= 0 \\2a + 2b + 2c &= 0 \\3a + 3b + 3c &= 0\end{aligned}$$



3 Equations
1 Piece of information

Rank 1

1	1	1
2	2	2
3	3	3

System 4

$$\begin{aligned}0a + 0b + 0c &= 0 \\0a + 0b + 0c &= 0 \\0a + 0b + 0c &= 0\end{aligned}$$



3 Equations
0 Pieces of information

Rank 0

0	0	0
0	0	0
0	0	0

Question

- Is there an easier way to calculate the rank?
- Answer: Yes! As before, it is the number of ones in the diagonal of the reduced row echelon form of the matrix.



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Solving System of Linear Equations

Row echelon form

Row echelon form of a matrix

Original matrix

Row echelon form

5	1
4	-3



1	0.2
0	1

5	1
10	2



1	1
0	0

0	0
0	0



0	0
0	0

Row echelon form

Original matrix

5	1
4	-3

Divide each row by
the leftmost coefficient

1	0.2
1	-0.75

$R_2 - R_1$

1	-0.75
-	
1	0.2
<hr/>	
0	-0.95

Divide the second row by
the leftmost non-zero coefficient

1	0.2
0	-0.95

Row echelon form

1	0.2
0	1

Row echelon form for singular matrices

Original matrix

5	1
10	2

Divide each row by
the leftmost coefficient

1	0.2
1	0.2

$$\begin{array}{r} R_2 - R_1 \\ \begin{array}{|c|c|} \hline 1 & 0.2 \\ \hline 1 & 0.2 \\ \hline \end{array} \\ - \\ \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \end{array} \end{array}$$

Row echelon form

1	0.2
0	0

Divide the second row by
the leftmost non-zero coefficient

1	0.2
?	?

Row echelon form for singular matrices

Row echelon form

Original matrix

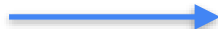
0	0	→	?	?
0	0		?	?

Divide each row by
the leftmost coefficient

Row echelon form, singularity, and rank

Non-singular matrix

5	1
4	-3



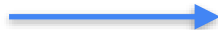
1	0.2
0	1

Rank 2

2 ones in the diagonal

Singular matrix

5	1
10	2



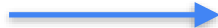
1	0.2
0	0

Rank 1

1 one in the diagonal

Singular matrix

0	0
0	0



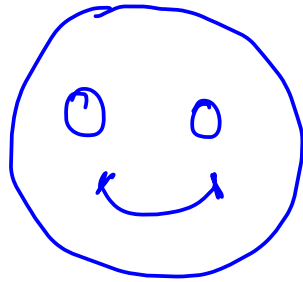
0	0
0	0

Rank 0

0 ones in the diagonal

A matrix is **non singular** if and only if its

Row echelon form has **only ones and
no zeros**





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Solving System of Linear Equations

**Row echelon form:
General case**

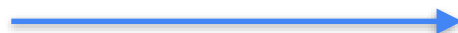
Row echelon form

System

- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

Matrix

1	1	2
3	-3	-1
2	-1	6



System

- $a + b + 2c = 12$
- $-6b - 7c = -33$
- $6c = 18$

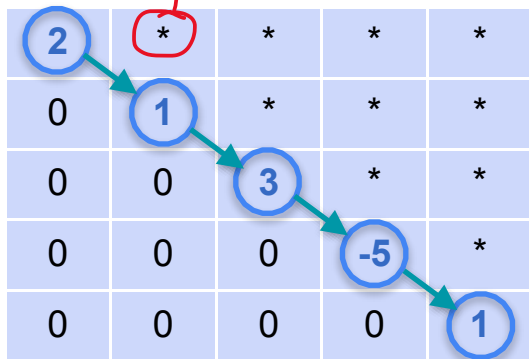
Row echelon form matrix

1	1	2
0	-6	7
0	0	6



Row echelon form

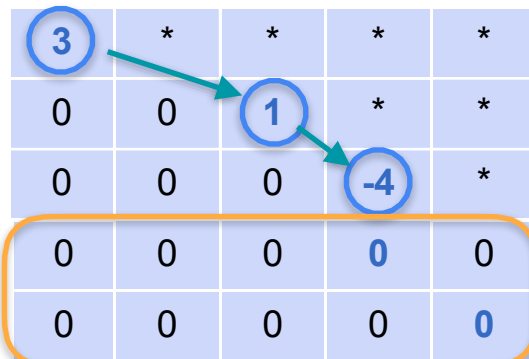
can be zero / non-zero "doesn't matter"



A 5x5 matrix in row echelon form. The pivots are circled in blue and connected by green arrows from top-left to bottom-right. The pivot values are 2, 1, 3, -5, and 1. The first pivot (2) is in the first column. The second pivot (1) is in the second column, and the entry above it is marked with a red circle and a red arrow pointing to the handwritten text "can be zero / non-zero". The third pivot (3) is in the third column. The fourth pivot (-5) is in the fourth column. The fifth pivot (1) is in the fifth column. All entries below the pivots are zero. All other entries are marked with an asterisk (*).

2	*	*	*	*
0	1	*	*	*
0	0	3	*	*
0	0	0	-5	*
0	0	0	0	1

Rank 5

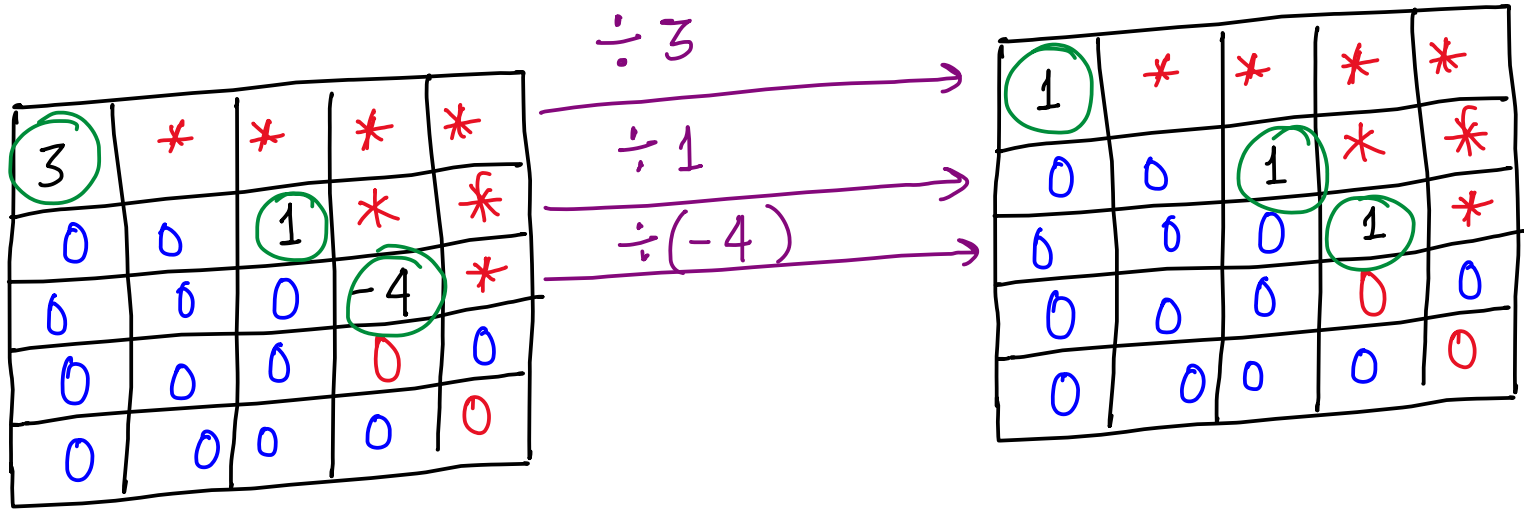


A 5x5 matrix in row echelon form. The pivots are circled in blue and connected by green arrows from top-left to bottom-right. The pivot values are 3, 1, and -4. The first pivot (3) is in the first column. The second pivot (1) is in the third column. The third pivot (-4) is in the fourth column. The last two rows are entirely zero and are highlighted with an orange border. All entries below the pivots are zero. All other entries are marked with an asterisk (*).

3	*	*	*	*
0	0	1	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

Rank 3

- Zero rows at the bottom
- Each row has a pivot (leftmost non-zero entry)
- Every pivot is to the right of the pivots on the rows above
- Rank of the matrix is the number of pivots



for this class, pivots are 1

Another example

Matrix

1	1	1
1	2	1
1	1	2



Row echelon form

1	1	1
0	1	0
0	0	1

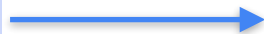
Subtract the first row
from the second and
the third ones

What if the matrix is singular?

Matrix

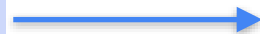
1	1	1
1	1	2
1	1	3

Subtract the first row
from the second and
the third ones



1	1	1
0	0	1
0	0	2

Subtract twice the
second row from the
third one



Row echelon form

1	1	1
0	0	1
0	0	0

What if the matrix is singular?

Matrix

1	1	1
2	2	2
3	3	3

Subtract twice the
first row from the
second row



1	1	1
0	0	0
3	3	3

Subtract three times
the first row from the
third row



Row echelon form

1	1	1
0	0	0
0	0	0

Rank for matrices

Matrix 1

1	1	1
1	2	1
1	1	2

Rank = 3

Matrix 2

1	1	1
1	1	2
1	1	3

Rank = 2

Matrix 3

1	1	1
2	2	2
3	3	3

Rank = 1

Matrix 4

0	0	0
0	0	0
0	0	0

Rank = 0

Row echelon forms

1	1	1
0	1	0
0	0	1

Number of pivots = 3

1	1	1
0	0	1
0	0	0

Number of pivots = 2

1	1	1
0	0	0
0	0	0

Number of pivots = 1

0	0	0
0	0	0
0	0	0

Number of pivots = 0



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Solving System of Linear Equations

Reduced row echelon form

"solved system"

Systems of equations to matrices

Original system

- $5a + b = 17$
- $4a - 3b = 6$

Intermediate System

- $a + 0.2b = 3.4$
- $b = 2$

Solved system

- $1a + 0b = 3$
- $0a + 1b = 2$

Original matrix

5	1
4	-3

Upper diagonal matrix

1	0.2
0	1

Row echelon form

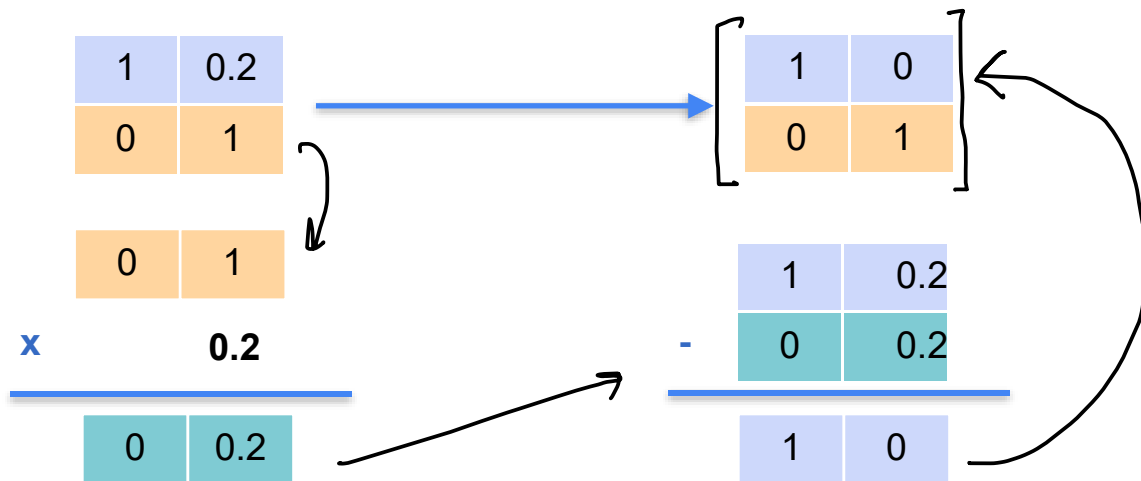
Diagonal matrix

1	0
0	1

Reduced row echelon form

Reduced row echelon form

Row echelon form



Reduced Row echelon form

Reduced row echelon form

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

Rank 5

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Rank 3

- Is in row echelon form
- Each pivot is a 1
- Any number above a pivot is 0
- Rank of the matrix is the number of pivots

Reduced row echelon form

Row echelon form

3	*	*	*	*
0	0	2	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Divide each row by
the value of the pivot

(leading coefficients)

Reduced row
echelon form

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

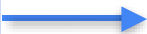
Turn anything above a
pivot to 0

Reduced row echelon form

Row echelon form

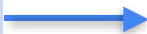
1	2	3
0	1	4
0	0	1

Subtract 2 times the second row from the first one



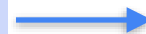
1	0	-5
0	1	4
0	0	1

Add 5 times the third row to the first one



1	0	0
0	1	4
0	0	1

Subtract 4 times the third row from the second one



Reduced row echelon form

1	0	0
0	1	0
0	0	1



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Solving System of Linear Equations

Conclusion