Computads for generalised signatures

Ioannis Markakis*

March 21, 2023

Abstract

We introduce a notion of signature whose sorts form a direct category, and study computads for such signatures. Algebras for such a signature are presheaves with an interpretation of every function symbol of the signature, and we describe how computads give rise to signatures. Generalising work of Batanin, we show that computads with certain generator-preserving morphisms form a presheaf category, and describe a forgetful functor from algebras to computads. Algebras free on a computad turn out to be the cofibrant objects for certain cofibrantly generated factorisation system, and the adjunction above induces the universal cofibrant replacement, in the sense of Garner, for this factorisation system. Finally, we conclude by explaining how many-sorted structures, weak ω-categories, and algebraic semi-simplicial Kan complexes are algebras of such signatures, and we propose a notion of weak multiple category.

1 Introduction

An important question for any kind of mathematical structure is determining the underlying data that it can be freely built from. For most algebraic structures, such as groups or rings, that data is given by a set of generators, while for *higher dimensional* structures, more structured data is needed, for example categories can be freely built from a directed graph. In all those examples, the structures of interest are algebras of some monad M on some category $\mathcal{C}$ and the free generation is expressed via the Eilenberg-Moore adjunction:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mathrm{F^M}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg_M} \arrow[l, shift left, "\mathrm{U^M}"] \end{tikzcd}$$

A similar adjunction exists for globular higher categories, where the category $\mathcal{C}$ is the category of globular sets and M some finitary monad. However, in that case, there is also a more general notion of generating datum, called *computads* or *polygraphs* [6, 12, 31], build recursively by gluing disks along their boundary spheres, similar to CW complexes.

*University of Cambridge, im496@cam.ac.uk

In this paper, we introduce and study computads for monads over more general categories $\mathcal{C}$. More precisely, we take $\mathcal{C} = [\mathcal{I}^{\mathrm{op}}, \mathrm{Set}]$ to be a category of presheaves over an arbitrary small direct category $\mathcal{I}$, the objects of which we call *sorts* and the morphisms of which we call *face maps*. Each sort i gives rise to an inclusion $\partial\mathbb{D}^i \subseteq \mathbb{D}^i$ in $\mathcal{C}$ where $\mathbb{D}^i = \mathcal{I}(-, i)$ is the representable presheaf by i and $\partial\mathbb{D}^i$ the sub-presheaf obtained by removing the top-dimensional element of $\mathbb{D}^i$. This broader framework allows us to treat computads for higher categories uniformly, regardless of the underlying geometry being globular, semi-simplicial or cubical for example.

Furthermore, we study what data can generate a monad M over such a category $\mathcal{C}$. When $\mathcal{I}$ is the terminal category, so $\mathcal{C}$ is the category of sets, an answer to this questions is the notion of *signature* from universal algebra [21]. Roughly that amounts to a set Σ of function symbols together with an *arity* function $\mathrm{ar} : \Sigma \to \mathbb{N}$ assigning to each symbol its number of inputs. A Σ-*algebra* $\mathbb{X}$ amounts to a set X together with a function

$$f^{\mathbb{X}} : X^{\mathrm{ar}(f)} \to X$$

for every function symbol $f \in \Sigma$. The monad M_Σ generated by Σ is the one induced by the free Σ-algebra adjunction $\mathrm{F}_\Sigma \dashv \mathrm{U}_\Sigma$.

A vast generalisation of signature has been proposed where $\mathcal{C}$ is a locally presentable enriched category, and the natural numbers are replaced by a small dense subcategory $\mathcal{A}$ of arities [10]. Unwrapping the definition in the case that $\mathcal{C} = [\mathcal{I}^{\mathrm{op}}, \mathrm{Set}]$ is a category of presheaves, we see that a signature Σ amounts to a presheaf of function symbols for every $A \in \mathcal{A}$. Equivalently, it is a presheaf $\Sigma : \mathcal{I}^{\mathrm{op}} \to \mathrm{Set}$ together with an arity function $\mathrm{ar} : \Sigma_i \to \mathrm{ob}\,\mathcal{A}$ for every sort i such that $\mathrm{ar}(\delta^* f) = \mathrm{ar}(f)$ for every function symbol $f \in \Sigma_i$ and face map $\delta : j \to i$. A Σ-algebra $\mathbb{X}$ is a presheaf X together with a function

$$f^{\mathbb{X}} : \mathcal{C}(\mathrm{ar}(f), X) \to X_i$$

compatible with the face maps in that

$$\delta^* f^{\mathbb{X}} = (\delta^* f)^{\mathbb{X}}.$$

The monad M_Σ generated by Σ is again the one induced by the free Σ-algebra adjunction $\mathrm{F}_\Sigma \dashv \mathrm{U}_\Sigma$. In this setting, every element $t \in (\mathrm{M}_\Sigma\, A)_i$ can be seen as a formal composite of function symbols with arity $A \in \mathcal{A}$, since it gives rise to a function

$$t^{\mathbb{X}} : \mathcal{C}(A, X) \xrightarrow{\sim} \mathrm{Alg}_\Sigma(\mathrm{F}_\Sigma\, A, X) \to X_i$$

by evaluation.

Here we propose a generalised notion of signature in the case that $\mathcal{I}$ is a small direct category. Our notion of signature generalises the previous one, in the same way that computads generalise presheaves on $\mathcal{I}$ as generating data. Our signatures consist again of a set of function symbols Σ_i for every sort i together with an arity function $\mathrm{ar} : \Sigma_i \to \mathrm{ob}\,\mathcal{C}$. However, instead of forming a presheaf, the function induced by a face map $\delta : j \to i$ sends a function symbol $f \in \Sigma_i$ of arity A to a formal composite of function symbols $t_{f,\delta} \in \mathrm{M}_{\Sigma,j} A$ of lower dimension. In order to make this definition precise, we work recursively on the dimension function $\dim : \operatorname{ob} \mathcal{I} \to \mathrm{Ord}$ of the direct category, and define signatures, Σ-computads and an adjunction inducing the monad M_Σ simultaneously by mutual induction.

A benefit of working with monads generated by a signature is that computads, their morphisms and the free algebras on them can be described as *inductively generated sets*, that is they are themselves freely generated by a set of constructors. To illustrate this concept, let Σ be a signature in the sense of universal algebra. Computads for such a signature Σ are just sets and the monad M_Σ is the one sending a set X to its set of Σ-*terms*. This is a set inductively generated by the constructors

- there exists a term $\mathrm{var}(x)$ for every $x \in C$,
- there exists a term $f[t_1, \ldots, t_{\mathrm{ar}(f)}]$ for every $f \in \Sigma$ and terms $t_1, \ldots, t_{\mathrm{ar}(f)}$.

What that means is that the set $\mathrm{M}_\Sigma X$ is an initial algebra for the polynomial endofunctor

$$G(X) = C \amalg \coprod_{f \in \Sigma} X^{\mathrm{ar}(f)}$$

specified by the constructors above. To construct the initial algebra, we let first $X_{-1} = \emptyset$ the initial set and form an increasing family of sets by letting $X_n = G(X_{n-1})$ for every natural number n. The union of this family is the initial algebra [1]. A more detailed introduction to inductively generated sets can be found in our previous work [14, Appendix A].

This presentation of computads and free algebras allows us to give simple proofs of existing results on globular computads [6, 16]. In Section 3, we show that the category of computads and certain generator-preserving maps is a presheaf topos for every generalized signature Σ. We do that by showing first that the functor of *terms*, sending a computad to the underlying presheaf of the free algebra on it, is familially representable. Following [19], we call the computads representing this functor *polyplexes* and we identify a subset of them, the *plexes*, that correspond to representable computads. Our proofs on this section are similar to the ones in our previous work [14]. This result crucially uses that the monads we consider are free on a signature, since it fails for example for the free strict ω-category monad [26].

In Section 5 we study the weak factorisation system on the category of algebra generated by the boundary inclusions $\partial\mathbb{D}^i \subseteq \mathbb{D}^i$. We call morphisms in its left class *cofibrations* and morphisms in the right class *trivial fibrations*, due to their role in the examples in Section 6. We show that free algebras on a computad and cofibrant algebras coincide, as in the case of strict ω-categories [28]. We also construct an adjunction between the category of algebras and the topos $\mathrm{Comp}_\Sigma^{\mathrm{var}}$ of computads and generator-preserving maps, extending the free algebra adjunction:

$$\begin{tikzcd}
& \mathrm{Alg}_\Sigma \arrow[dl, shift left, "\mathrm{U}_\Sigma"] \arrow[dr, shift right, "\mathrm{Und}_\Sigma"'] & \\
\mathcal{C} \arrow[ur, shift left, "\mathrm{F}_\Sigma"] \arrow[ur, phantom, "\perp"] \arrow[rr, hook] & & \mathrm{Comp}^{\mathrm{var}}_\Sigma \arrow[ul, shift right, "\mathrm{Free}_\Sigma"'] \arrow[ul, phantom, "\perp"]
\end{tikzcd}$$

We show that the induced comonad is the *universal cofibrant replacement* for this weak factorisation system [16].

Finally, in Section 6, we descirbe certain signatures of interest and their algebras. We see first that many-sorted algebraic signatures are S-sorted signatures for S a discrete category. We then show that ω-categories and algebraic semi-simplicial Kan complexes are algebras for signatures with sorts from the categories of globes and semi-simplices respectively. We conclude by proposing a new notion of fully weak unbiased multiple category as algebras of a certain signature with sorts in a cube category.

Notation

We will denote by $\mathcal{I}$ a small *direct category*, whose objects we will call *sorts* and whose morphisms we will call *face maps*. By direct category, we mean that it is equipped with a function $\dim : \mathrm{ob}\,\mathcal{I} \to \mathrm{Ord}$ to the class of ordinals satisfying that $\dim j < \dim i$ when there exists a non-identity face map $j \to i$. It is easy to see that $\mathcal{I}$ must be skeletal and have no non-identity endomorphisms. Given a presheaf $X : \mathcal{I}^{\mathrm{op}} \to \mathrm{Set}$, we will denote by X_i the value of X at a sort i and by $\delta^* : X_i \to X_j$ its value at a face map $\delta : j \to i$.

Given an ordinal α, we will let $\mathcal{I}_\alpha$ denote the full subcategory of $\mathcal{I}$ whose objects have dimension at most α. Pulling back along the subcategory inclusion induces a *truncation* functor

$$\mathrm{tr}_\alpha : [\mathcal{I}^{\mathrm{op}}, \mathrm{Set}] \to [\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}].$$

Left and right Kan extensions along the inclusion, define a left and right adjoint respectively to the truncation functor called the *skeleton* and *coskeleton* functors respectively [30, Chapter 1]. The subcategory inclusion $\mathcal{I}_\beta \subseteq \mathcal{I}_\alpha$ for $\beta \leq \alpha$ defines similarly an adjoint triple and it is easy to see that the following conditions are satisfied

$$\mathrm{tr}_\beta = \mathrm{tr}_\beta^\alpha \,\mathrm{tr}_\alpha \qquad\qquad \mathrm{tr}_\beta \,\mathrm{sk}_\beta = \mathrm{id} \qquad\qquad \mathrm{sk}_\beta = \mathrm{sk}_\alpha \,\mathrm{sk}_\beta^\alpha,$$

which we will call the *cocycle conditions*.

2 Signatures and computads

In this section, we will define the class of $\mathcal{I}$*-sorted signatures*. We will simultaneously define for every signature Σ, a category of *computads*, extending the category of presheaves on $\mathcal{I}$, and the presheaf of terms of a computad. The definition is given by transfinite recursion. More precisely, for every ordinal α, we will define a class of *$\mathcal{I}$-sorted signatures of dimension α* together with restriction functions

$$(-)_\beta : \mathrm{Sig}_{\mathcal{I}}(\alpha) \to \mathrm{Sig}_{\mathcal{I}}(\beta)$$

for $\beta \leq \alpha$. Moreover, for every such signature Σ, we will define a category of computads together with truncation functors

$$\mathrm{tr}_\beta^\Sigma : \mathrm{Comp}_\Sigma \to \mathrm{Comp}_{\Sigma_\beta}$$

for $\beta \leq \alpha$. We will also define with an adjunction

$$\begin{aligned} \mathrm{Cptd}_\Sigma &: [\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Comp}_\Sigma & \qquad \eta_\Sigma &: \mathrm{id} \Rightarrow \mathrm{Term}_\Sigma \mathrm{Cptd}_\Sigma \\ \mathrm{Term}_\Sigma &: \mathrm{Comp}_\Sigma \to [\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}] & \qquad \varepsilon_\Sigma &: \mathrm{Cptd}_\Sigma \mathrm{Term}_\Sigma \Rightarrow \mathrm{id} \end{aligned}$$

commuting with the truncation functors, in the sense that

$$\begin{aligned} \mathrm{tr}_\beta^\Sigma \mathrm{Cptd}_\Sigma &= \mathrm{Cptd}_{\Sigma_\beta} \mathrm{tr}_\beta^\alpha & \qquad \mathrm{tr}_\beta^\alpha \eta_\Sigma &= \eta_{\Sigma_\beta} \mathrm{tr}_\beta^\Sigma \\ \mathrm{tr}_\beta^\alpha \mathrm{Term}_\Sigma &= \mathrm{Term}_{\Sigma_\beta} \mathrm{tr}_\beta^\Sigma & \qquad \mathrm{tr}_\beta^\Sigma \varepsilon_\Sigma &= \varepsilon_{\Sigma_\beta} \mathrm{tr}_\beta^\Sigma . \end{aligned}$$

We will denote the monad induced by this adjunction by $(\mathrm{M}_\Sigma, \eta_\Sigma, \mu_\Sigma)$. The restriction functions and the truncation functors will be shown to satisfy the following *cocycle conditions*

$$\begin{aligned} (-)_\gamma^\alpha &= (-)_\gamma^\beta (-)_\beta^\alpha & \qquad (-)_\alpha^\alpha &= \mathrm{id} \\ \mathrm{tr}_\gamma^\Sigma &= \mathrm{tr}_\gamma^{\Sigma_\beta} \mathrm{tr}_\beta^\Sigma & \qquad \mathrm{tr}_\alpha^\Sigma &= \mathrm{id} \end{aligned}$$

for $\gamma \leq \beta \leq \alpha$.

For the rest of the section, we fix an ordinal α, and suppose that the data above has been defined for every $\beta \leq \alpha$ and it satisfies the cocycle conditions. We proceed to define that data for α as well. The apparent circularity of our definition will be explained in Remark 1.

Signatures An *$\mathcal{I}$-sorted signature* Σ of dimension α consists of

- a signature Σ_β of dimension β for every $\beta < \alpha$ satisfying for $\gamma \leq \beta$ that

$$(\Sigma_\beta)_\gamma = \Sigma_\gamma,$$

- a set Σ_i of function symbols for every sort i of dimension α, and for every function symbol $f \in \Sigma_i$,
  - a presheaf B_f on $\mathcal{I}_\alpha$, called the *arity* of the function symbol,
  - a *boundary* term $t_{f,\delta} \in \mathrm{M}_{\Sigma_\beta, j}(\mathrm{tr}_\beta B_f)$ for every non-identity face map $\delta : j \to i$, where $\beta = \dim j$ satisfying for every face map $\delta' : k \to j$ that

$$(\delta')^*(t_{f,\delta'}) = t_{f,\delta\delta'}.$$

The restriction function $(-)_\beta$ for $\beta \leq \alpha$ is the obvious projection.

Computads A Σ-*computad* C consists of

- a Σ_β-computad C_β for every $\beta < \alpha$ satisfying for $\gamma \leq \beta$ that
$$\mathrm{tr}_\gamma^{\Sigma_\beta}(C_\beta) = C_\gamma,$$
- a set V_i^C of *generators* for every sort i of dimension α,
- a *gluing* function $\phi_\delta^C : V_i^C \to \mathrm{Term}_{\Sigma_\beta,j}(C_\beta)$ for every non-identity face map $\delta : j \to i$, where $\beta = \dim j$, satisfying for every face map $\delta' : k \to j$ that
$$(\delta')^* \phi_\delta^C = \phi_{\delta\delta'}^C.$$

A *morphism of computads* $\sigma : C \to D$ consists similarly of

- morphisms $\sigma_\beta : C_\beta \to D_\beta$ for every $\beta < \alpha$ satisfying for $\gamma \leq \beta$ that
$$\mathrm{tr}_\gamma^{\Sigma_\beta}(\sigma_\beta) = \sigma_\gamma$$
- functions $\sigma_i : V_i^C \to \mathrm{Term}_{\Sigma,i}(D)$ for every sort i of dimension α satisfying for every non-identity face map $\delta : j \to i$ that
$$\delta^* \sigma_i = \mathrm{Term}_{\Sigma_\beta,j}(\sigma_\beta) \circ \phi_\delta^C$$
where $\beta = \dim j$.

The composition of a pair of morphisms $\sigma : C \to D$ and $\tau : D \to E$ is given by
$$(\tau \circ \sigma)_\beta = \tau_\beta \circ \sigma_\beta \qquad\qquad (\tau \circ \sigma)_i = \mathrm{Term}_{\Sigma,i}(\tau) \circ \sigma_i$$
for $\beta < \alpha$ and i of sort α. The identity of a computad C is given by the identities id_{C_β} and the inclusion $\mathrm{var} : V_i^C \to \mathrm{Term}_{\Sigma,i}(C)$ of generators into terms defined below. The truncation functor tr_β^Σ for $\beta \leq \alpha$ is the obvious projection.

The inclusion functor The computad $\mathrm{Cptd}_\Sigma X$ associated to a presheaf X comprises of the computads $\mathrm{Cptd}_{\Sigma_\beta} \mathrm{tr}_\beta X$ for every $\beta < \alpha$, the sets X_i for every sort i of dimension α, and the gluing functions
$$\phi_\delta^{\mathrm{Cptd}_\Sigma X} : X_i \xrightarrow{\delta^*} X_j \xrightarrow{\eta_{\Sigma_\beta, \mathrm{tr}_\beta X, j}} \mathrm{M}_{\Sigma_\beta,i}(\mathrm{tr}_\beta X)$$
for every face map $\delta : j \to i$, where $\beta = \dim j$. The morphism of computads induced by a morphism of presheaves $\sigma : X \to Y$ comprises similarly of $\mathrm{Cptd}_{\Sigma_\beta} \mathrm{tr}_\beta \sigma$ for every $\beta < \alpha$, and the composites
$$(\mathrm{Cptd}_\Sigma \sigma)_i : X_i \xrightarrow{\sigma_i} Y_i \xrightarrow{\mathrm{var}} \mathrm{M}_{\Sigma,i} Y$$
where var the inclusion of generators into terms defined below.

The presheaf of terms The presheaf $\mathrm{Term}_\Sigma(C)$ of terms of a computad C is defined as follows. For every sort j of dimension $\beta < \alpha$, we let

$$\mathrm{Term}_{\Sigma,j}(C) = \mathrm{Term}_{\Sigma_\beta,j}(C_\beta).$$

The *boundary function* δ^* induced by a face map $\delta : k \to j$ is given by

$$\mathrm{Term}_{\Sigma_\beta,j}(C_\beta) \xrightarrow{\delta^*} \mathrm{Term}_{\Sigma_\beta,k}(C_\beta) = \mathrm{Term}_{\Sigma_\gamma,k}(C_\gamma)$$

where $\gamma = \dim k$. For a sort i of dimension α, the set $\mathrm{Term}_{\Sigma,i}(C)$ is defined inductively by the constructors

- there exists a term $\mathrm{var}\, v$ for every generator $v \in V_i^C$,
- there exists a term $f[\tau]$ for every function symbol $f \in \Sigma_i$ and morphism of computads $\tau : \mathrm{Cptd}_\Sigma\, B_f \to C$.

The function δ^* induced by a non-identity face map $\delta : j \to i$ is defined recursively by

$$\delta^*(\mathrm{var}\, v) = \phi_\delta^C(v) \qquad\qquad \delta^*(f[\tau]) = \mathrm{Term}_{\Sigma_\beta,j}(\tau_\beta)(t_{f,\delta})$$

where $\beta = \dim j$. The cocycle conditions ensure that this assignment defines a presheaf on $\mathcal{I}_\alpha$. The action of a morphism of computads $\sigma : C \to D$ on a term t of sort j of dimension $\beta < \alpha$ is given by

$$\mathrm{Term}_\Sigma(\sigma)(t) = \mathrm{Term}_{\Sigma_\beta}(\sigma_\beta)(t),$$

while its action on a term of sort i of dimension α is defined recursively by

$$\mathrm{Term}_\Sigma(\sigma)(\mathrm{var}\, v) = \sigma_i(v) \qquad\qquad \mathrm{Term}_\Sigma(\sigma)(f[\tau]) = f[\sigma \circ \tau].$$

The adjunction It remains to define the unit and counit of the term adjunction. For the former, given a presheaf X on $\mathcal{I}_\alpha$, let

$$\eta_{\Sigma,X} : X \to \mathrm{Term}_\Sigma\, \mathrm{Cptd}_\Sigma\, X$$

the morphism of presheaves defined on some $x \in X_i$ by

$$\eta_{\Sigma,X,i}(x) = \begin{cases} \eta_{\Sigma_\beta,\mathrm{tr}_\beta X,i}(x), & \text{for } \dim i = \beta < \alpha \\ \mathrm{var}(x), & \text{for } \dim i = \alpha. \end{cases}$$

For the latter, given a computad C for Σ, let

$$\varepsilon_{\Sigma,C} : \mathrm{Cptd}_\Sigma\, \mathrm{Term}_\Sigma\, C \to C$$

the morphism of computads consisting of $\varepsilon_{\Sigma_\beta,C_\beta}$ for every $\beta < \alpha$ and the identity of the set

$$V_i^{\mathrm{Cptd}_\Sigma\, \mathrm{Term}_\Sigma\, C} = \mathrm{Term}_{\Sigma,i}(C)$$

for i of dimension α. Naturality of the unit and counit of the adjunction as well as the snake equations can be easily checked. This concludes the recursive definition.

Remark 1. Albeit the apparent circularity of the definition of the category of computads and the functor of terms for a fixed signature Σ, this is a valid mathematical definition as explained below. Assuming that all the data has been defined for every $\beta < \alpha$, the notion of signature of dimension α can be defined. Fixing such a signature Σ, computads and their terms of sort $\beta < \alpha$ may be defined as well as the inclusion functor on objects. Then fixing a computad C, its terms of dimension α, the boundary functions, and morphisms from an arity to C are defined using induction recursion [18].

To understand the inductive recursive definition, we attach an ordinal to every term of C of dimension α and every morphism of computads with target C, its *depth*, recursively by

$$\begin{aligned} \operatorname{dpth}(\operatorname{var} v) &= 0 \\ \operatorname{dpth}(f[\tau]) &= \operatorname{dpth} \tau + 1 \\ \operatorname{dpth}(\sigma : D \to C) &= \sup\{\operatorname{dpth}(\sigma_i(v)) : \dim i = \alpha, v \in V_i^C\}. \end{aligned}$$

The definition of terms and morphisms amounts then to constructing by transfinite recursion on an ordinal γ two increasing families of sets, and forming their unions. The first family consists of the sets $\mathrm{Term}^{\gamma}_{\Sigma,i}(C;(t_\delta))$ of terms of sort i, depth at most γ and boundaries given by the terms t_δ. The second consists of the sets $\mathrm{Comp}^{\gamma}_{\Sigma}(B_f, C; (\sigma_\beta))$ of morphisms of depth at most γ from an arity $\mathrm{Cptd}_\Sigma B_f$ to C and truncations given by the morphisms σ_β. Those sets are defined by the recursive formulae

$$\mathrm{Term}^{\gamma}_{\Sigma,i}(C;(t_\delta)) = \left(\bigcap_{\delta} (\phi^C_\delta)^{-1}(t_\delta)\right) \amalg \coprod_{\substack{f\in\Sigma_i\\(\sigma_\beta)}} \bigcup_{\gamma'<\gamma} \mathrm{Comp}^{\gamma'}_{\Sigma}(B_f, C; (\sigma_\beta))$$

$$\mathrm{Comp}^{\gamma}_{\Sigma}(B_f, C; (\sigma_\beta)) = \prod_{\substack{\dim j=\alpha\\ b\in B_{f,j}}} \mathrm{Term}^{\gamma}_{\Sigma,j}(C; (\mathrm{Term}(\sigma_\beta)(t_{f,\delta})))$$

where the first coproduct is over families of morphisms $\sigma_\beta : \mathrm{Cptd}_{\Sigma_\beta} \mathrm{tr}_\beta B_f \to C_\beta$ for $\beta < \alpha$ satisfying the cocycle conditions and that $\mathrm{Term}_\Sigma(\sigma_\beta)(t_{f,\delta}) = t_\delta$ for every non-identity face map $\delta : j \to i$, where $\beta = \dim j$. The union of the first family over all γ and all families of boundary terms (t_δ) is the set of terms of sort i of C. The boundaries of those terms are given by the obvious projections. Similarly, the union of the second family gives morphisms from $\mathrm{Cptd}_\Sigma B_f$ to C and the obvious projections give the truncations of those morphisms. Once terms are defined, we may also define morphisms between arbitrary computads, and the truncation functor in general.

Observe that we defined the sets of terms and the sets of morphisms as union of increasing families over all ordinals. Such unions produce proper classes instead of sets unless they are eventually stationary. To see that this is the case in our definition, let λ a regular cardinal strictly greater than the cardinality of the set $\coprod_{\dim j=\alpha} B_{f,j}$ for every function symbol of dimension α. Existence of such cardinal follows by $\mathcal{I}$ being small and the Σ_i being sets. A simple inductive argument shows that all terms and morphisms from an arity have depth strictly less than λ, and arbitrary morphisms have depth at most λ.

Once those are defined, we may define composition of morphisms and the action of morphisms on terms mutually recursively, or equivalently by induction on depth. By induction on depth, we can then prove that composition is associative and unital, and that the action on terms is functorial. Once those properties are established, it is easy to define the action of the inclusion functor on morphisms, as well as the unit and counit of the adjunction.

Skeleton functors Given a signature Σ of dimension α and an ordinal $\beta < \alpha$, we may identify computads for Σ_β with computads for Σ that have no generators above dimension β. More precisely, we can define recursively a *skeleton functor* and a natural transformation

$$\mathrm{sk}_\beta^\Sigma : \mathrm{Comp}_{\Sigma_\beta} \hookrightarrow \mathrm{Comp}_\Sigma \qquad\qquad \kappa_\beta^\Sigma : \mathrm{sk}_\beta^\Sigma \,\mathrm{tr}_\beta^\Sigma \Rightarrow \mathrm{id}$$

satisfying the following conditions

$$\mathrm{tr}_\beta^\Sigma \,\mathrm{sk}_\beta^\Sigma = \mathrm{id} \qquad\qquad \mathrm{tr}_\beta^\Sigma \,\kappa_\beta^\Sigma = \mathrm{id} = \kappa_\beta^\Sigma \,\mathrm{tr}_\beta^\Sigma,$$

or equivalently that the skeleton functor is left adjoint to the truncation functor with unit the identity and counit κ_β^Σ. In particular, it follows that the skeleton functor is fully faithful and injective on objects.

Given a Σ_β-computad C and an ordinal $\gamma \leq \alpha$, we may define recursively

$$C^\gamma = \begin{cases} \mathrm{tr}_\gamma^{\Sigma_\beta} C, & \text{if } \gamma \leq \beta \\ ((C^{\gamma'})_{\gamma'<\gamma}, (\emptyset, \{\})_{\dim i = \gamma}) & \text{if } \beta < \gamma \leq \alpha \end{cases} \qquad\qquad \mathrm{sk}_\beta^\Sigma C = C^\alpha,$$

where $\{\}$ denotes the unique function out of the empty set. Similarly, for a morphism of Σ_β-computads $\sigma : C \to D$, we define recursively

$$\sigma^\gamma = \begin{cases} \mathrm{tr}_\gamma^{\Sigma_\beta} \sigma, & \text{if } \gamma \leq \beta \\ ((\sigma^{\gamma'})_{\gamma'<\gamma}, (\{\})_{\dim i = \gamma}) & \text{if } \beta < \gamma \leq \alpha \end{cases} \qquad\qquad \mathrm{sk}_\beta^\Sigma \sigma = \sigma^\alpha.$$

Moreover, for a Σ-computad D, we define the component of the counit at D recursively by

$$\kappa_{\beta,C,\gamma}^\Sigma = \begin{cases} \mathrm{id}_{C_\gamma}, & \text{if } \gamma \leq \beta \\ ((\kappa_{\beta,C,\gamma'}^\Sigma)_{\gamma'<\gamma}, (\{\})_{\dim i = \gamma}) & \text{if } \beta < \gamma \leq \alpha \end{cases} \qquad\qquad \kappa_{\beta,C}^\Sigma = \kappa_{\beta,C,\alpha}^\Sigma.$$

Naturality of the counit, as well as the two conditions above are easy to check.

Unbounded signatures We have assumed that the category $\mathcal{I}$ of sorts is small. A consequence of that is that there exists a least ordinal α that is greater or equal than the dimension of every sort $i \in \mathcal{I}$. The class $\mathrm{Sig}_\mathcal{I}$ of *$\mathcal{I}$-sorted signatures* is the class $\mathrm{Sig}_\mathcal{I}(\alpha)$ of signatures of dimension α, and their computads and terms are defined as above.

Observe that this definition does not really depend on α, since for any $\alpha' \geq \alpha$ the restriction function $(-)_{\alpha}^{\alpha'}$ is a bijection and the truncation functors $\mathrm{tr}_{\alpha}^{\Sigma}$ for $\Sigma \in \mathrm{Sig}_{\mathcal{I}}(\alpha')$ are isomorphisms of categories commuting with the term adjunction. Under this definition, $\mathcal{I}$-sorted signatures of arbitrary dimension β coincide with $\mathcal{I}_{\beta}$-sorted signatures.

3 Computads as presheaves

We have defined a generalised notion of morphism of computads $\sigma : C \to D$, where each generator of C is mapped to an arbitrary term of D. In contrast, the morphisms of computads usually considered are more restricted, sending instead generators to generators [6, 16] . In this section, we will identify a subcategory of computads where morphisms preserve the generators. Generalising our previous work [14], we will show that this subcategory is a presheaf topos for every signature and identify a site of definition for it. For the rest of the section, let Σ be an $\mathcal{I}$-sorted signature of some dimension α.

Definition 2. A morphism of Σ-computads $\sigma : C \to D$ is *variable-to-variable* when $\mathrm{tr}_{\beta}^{\Sigma}(\sigma)$ is variable-to-variable for every $\beta < \alpha$, and $\sigma_i(v)$ is a generator for every sort i of dimension α and generator $v \in V_i^C$.

In other words, a variable-to-variable morphism $\sigma : C \to D$ consists of variable-to-variable morphisms σ_{β} for all $\beta < \alpha$ satisfying the usual cocycle conditions and functions $\sigma_i : V_i^C \to V_i^D$ for every sort i of dimension α satisfying the gluing condition

$$\phi_{\delta}^{D} \circ \sigma_i = \mathrm{Term}_{\Sigma_{\beta}}(\sigma_{\beta}) \circ \phi_{\delta}^{C}$$

for every sort j of dimension $\beta < \alpha$, and morphism $\delta : j \to i$. variable-to-variable morphisms are closed under composition and they contain identity morphisms, so they form a subcategory that we denote by

$$\zeta_{\Sigma} : \mathrm{Comp}_{\Sigma}^{\mathrm{var}} \to \mathrm{Comp}_{\Sigma}$$

The truncation and skeleton functors preserve the class of variable-to-variable morphisms, so they restrict to an adjunction between the subcategories of variable-to-variable morphisms. Moreover, for every sort i of dimension at most α, the assignment sending a computad C to the set V_i^C of its generators of sort i can be extended to a functor

$$V_i^{\bullet} : \mathrm{Comp}_{\Sigma}^{\mathrm{var}} \to \mathrm{Set}$$

Remark 3. It is easy to see that if the composition $\sigma\tau$ of two morphisms is variable-to-variable, then the same must be true of τ. In particular, isomorphism of computads are variable-to-variable. Functoriality of $V_i^{\bullet}$ shows then that isomorphisms of computads induce bijections on the sets of generators. The converse can be shown easily by induction on the dimension of the signature.

It is easy to see that in general the category of computads is neither complete or cocomplete. For example, it has no terminal object, so long as the signature contains at least one function symbol. On the contrary, the subcategory of variable-to-variable morphisms is both complete and cocomplete, and the inclusion functor preserves colimits and connected limits.

Proposition 4. *The category of computads and variable-to-variable morphisms is cocomplete and the inclusion ζ_Σ is cocontinuous.*

Proof. Let $F : \mathcal{D} \to \mathrm{Comp}^{\mathrm{var}}_\Sigma$ a small diagram of computads and variable-to-variable morphisms. By induction on the dimension α of the signature, we may assume that colimit cocones $(\sigma_{\beta,d} : (Fd)_\beta \to C_\beta)_{d\in\mathcal{D}}$ have been constructed for all $\beta < \alpha$ and that they are preserved strictly by the truncation functors. We may then form the colimit of sets of generators $(\sigma_{i,d} : V_i^{Fd} \to V_i^C)_{d\in\mathcal{D}}$ for every sort i of dimension α.

Let C the computad consisting of C_β for every $\beta < \alpha$, the sets of generators V_i^C for i of dimension α and the gluing functions $\phi_\delta^C : V_i^C \to \mathrm{Term}_{\Sigma_\beta,j}(C_\beta)$ for every non-identity face map $\delta : j \to i$, defined by the universal property of the colimit on the morphisms

$$V_i^{Fd} \xrightarrow{\phi_\delta^{Fd}} \mathrm{Term}_{\Sigma_\beta,j}((Fd)_\beta) \xrightarrow{\mathrm{Term}(\sigma_{\beta,d})} \mathrm{Term}_{\Sigma_\beta,j}(C_\beta).$$

where $\beta = \dim j$. The morphisms $\sigma_{\beta,d}$ and the functions $\sigma_{i,d}$ assemble to a cocone of variable-to-variable morphisms $(\sigma_d : Fd \to C)_{d\in\mathcal{D}}$, whose universal property in the category of computads and the subcategory of variable-to-variable morphisms can be easily verified. Moreover, it is strictly preserved by the truncation functors, which concludes the induction. $\square$

Proposition 5. *The category of computads and variable-to-variable morphisms has a terminal object $\mathbb{1}_\Sigma$.*

Proof. Suppose that terminal computads $\mathbb{1}_{\Sigma_\beta}$ are given for every $\beta < \alpha$, strictly preserved by the truncation functors. For every sort i of dimension α, we may then form the limit

$$V_i^{\mathbb{1}_\Sigma} = \lim_{\substack{\dim j = \beta < \alpha \\ \delta : j \to i}} \mathrm{Term}_{\Sigma_\beta,j}(\mathbb{1}_{\Sigma_\beta})$$

The terminal computad $\mathbb{1}_\Sigma$ consists of the computads $\mathbb{1}_{\Sigma_\beta}$ and those sets for every sort i of dimension α. Its gluing functions are the obvious projections out of the limit. $\square$

The existence of a terminal computad implies that the restricted functor of terms $\mathrm{Term}^{\mathrm{var}}_\Sigma = \mathrm{Term}_\Sigma \circ \zeta_\Sigma$ can be refined to a functor with target the slice category over $\mathrm{Term}_\Sigma(\mathbb{1}_\Sigma)$. This slice category is equivalent to the category of presheaves on the category of elements of $\mathrm{Term}_\Sigma(\mathbb{1}_\Sigma)$, whose objects we will call *polyplexes* following [19].

Definition 6. A *polyplex* is a term of the terminal computad.

We will denote the sort of a polyplex p by $\mathrm{sort}(p)$. Polyplexes form a category Pplex_Σ, where morphisms $\delta : p \to p'$ are face maps $\mathrm{sort}(p) \to \mathrm{sort}(p')$ satisfying that $\delta^*(p') = p$. The refinement of the functor of terms described in the previous paragraph is given by the functor

$$\begin{aligned}\mathcal{T}_\Sigma &: \mathrm{Comp}^{\mathrm{var}}_\Sigma \times \mathrm{Pplex}^{\mathrm{op}}_\Sigma \to \mathrm{Set}\\ \mathcal{T}_{\Sigma,p} C &= \{t \in \mathrm{Term}_{\Sigma,\mathrm{sort}(p)}(C) \;:\; \mathrm{Term}_\Sigma(!)(t) = p\}\end{aligned}$$

where ! is the unique variable-to-variable morphism to the terminal computad.

Proposition 7. *The functor $\mathcal{T}_{\Sigma,p}$ is representable for every polyplex p.*

Proof. By induction on α, we may first assume that $\mathcal{T}_{\Sigma_\beta,p}$ is representable for every polyplex p of dimension $\beta < \alpha$. If $|p|_\beta$ is a Σ_β-computad representing it, then for every Σ-computad C, there exists a natural isomorphism

$$\mathcal{T}_{\Sigma,p} C = \mathcal{T}_{\Sigma_\beta,p} C_\beta \cong \mathrm{Comp}^{\mathrm{var}}_{\Sigma_\beta}(|p|_\beta, C_\beta) \cong \mathrm{Comp}^{\mathrm{var}}_\Sigma(\mathrm{sk}^\Sigma_\beta |p|_\beta, C),$$

so $\mathcal{T}_{\Sigma,p}$ is represented by $|p| = \mathrm{sk}_\beta |p|_\beta$. It remains to show that $\mathcal{T}_{\Sigma,p}$ is representable for polyplexes p of dimension α as well. We will construct such a representation recursively on the depth of p.

Given a polyplex p' of dimension less than α, or depth less than that of p, we will denote by $|p'|$ the computad representing $\mathcal{T}_{\Sigma,p'}$. We will also denote by $t_{p'}$ the universal term in $\mathcal{T}_{\Sigma,p'}(|p'|)$ inducing the representation. Finally, given a morphism of such polyplexes $\delta : p'' \to p'$, we will denote by $|\delta| : |p''| \to |p'|$ the morphism corresponding to the natural transformation $\mathcal{T}_\delta$.

Suppose first that p is a polyplex of dimension α and depth 0, and let i its sort. Then there exists a family of polyplexes $p_\delta \in \mathrm{Term}_{\Sigma,j}(\mathbb{1}_\Sigma)$ indexed by non-identity morphisms $\delta : j \to i$, satisfying the usual cocycle condition, and that $p = \mathrm{var}(p_\delta)$. Then the colimit

$$D = \operatorname*{colim}_{\substack{\dim j < \alpha\\ \delta : j \to i}} |p_\delta|$$

is a computad with no generators of dimension α from the description of colimits in Proposition 4. Let inc_δ the canonical inclusion to the colimit, and let $|p|$ the computad consisting of D_β for all $\beta < \alpha$, has unique generator $*$ of sort i with gluing functions

$$\phi^{|p|}_\delta(*) = \mathcal{T}_{\Sigma,p_\delta}(\mathrm{inc}_\delta)(t_{p_\delta}),$$

and no other generator of dimension α. Let also $t_p = \mathrm{var}(*)$. Using the universal property of the colimit defining D and that D has no top-dimensional generators, it is easy to see that evaluation at t_p induces a natural isomorphism

$$\mathrm{Comp}^{\mathrm{var}}_\Sigma(|p|, C) \cong \{v \in V^C_i \;:\; \phi^C_\delta(v) \in \mathcal{T}_{\Sigma,p_\delta} C \text{ for all } \delta\} \cong \mathcal{T}_p C$$

for every computad C.

Suppose finally that p has positive depth, so that it is of the form $p = f[\tau]$ for some function symbol $f \in \Sigma_i$ and $\tau : \mathrm{Cptd}_\Sigma B_f \to \mathbb{1}_\Sigma$. We may then form the transpose $\tau^\dagger : B_f \to \mathrm{Term}_\Sigma(\mathbb{1}_\Sigma)$ of τ under the term adjunction, which sends $b \in B_{f,j}$ to $\tau_j(b)$. By the inductive hypothesis, we may form the colimit

$$|p| = \operatorname*{colim}_{b \in B_{f,j}} |\tau^\dagger|(b)$$

over the category of elements of B_f. variable-to-variable morphisms $\rho : |p| \to C$ are in natural bijection to families of terms $\hat{\rho}(b) \in \mathcal{T}_{\Sigma,\tau^\dagger(b)}C$ compatible with the boundary maps, or equivalently to morphisms $\hat{\rho} : B_f \to \mathrm{Term}_\Sigma(C)$ such that $\mathrm{Term}(!)\hat{\rho} = \tau^\dagger$. Those correspond in turn to morphisms $\hat{\rho}^\dagger : \mathrm{Cptd}_\Sigma B_f \to C$ such that $\tau =! \circ \hat{\rho}^\dagger$, or equivalently to terms $f[\hat{\rho}^\dagger] \in \mathcal{T}_p C$. □

Corollary 8. *The restricted functor of terms* $\mathrm{Term}_\Sigma^{var}$ *preserves connected colimits.*

Proof. For every sort i, the functor $\mathrm{Term}_{\Sigma,i}^{\mathrm{var}}$ is the coproduct of the representable $\mathcal{T}_{\Sigma,p}$ over all polyplexes p of sort i. The result follows by continuity of representable functors, and commutativity of connected limits of sets over arbitrary coproducts. □

Corollary 9. *The category of computads and variable-to-variable morphisms is complete and the inclusion ζ_Σ preserves connected limits.*

Proof. We have already shown that $\mathrm{Comp}_\Sigma^{\mathrm{var}}$ has a terminal object, so it suffices to show that it has connected limits preserved by the inclusion into Comp_Σ. For that, let $F : \mathcal{D} \to \mathrm{Comp}_\Sigma^{\mathrm{var}}$ a small, connected diagram of computads and variable-to-variable morphisms. By induction on the dimension of the signature, we may assume that limit cones $(\sigma_{\beta,d} : C_\beta \to (Fd)_\beta)_{d\in\mathcal{D}}$ have been constructed for every $\beta < \alpha$ and that they are strictly preserved by the inclusion functors. We may then form the limits of the sets of generators $(\sigma_i : V_i^C \to V_i^{Fd})_{d\in\mathcal{D}}$ for every sort i of dimension α. For every non-identity face map $\delta : j \to i$, we have from Corollary 8 that the morphisms

$$\mathrm{Term}_{\Sigma_\beta,j}(\sigma_{\beta,d}) : \mathrm{Term}_{\Sigma_\beta,j}(C_\beta) \to \mathrm{Term}_{\Sigma_\beta,j}((Fd)_\beta),$$

where $\beta = \dim j$, form a limit cone. We let $\phi_\delta^C : V_i^C \to \mathrm{Term}_{\Sigma_\beta,j}(C_\beta)$ the function induced by the universal property of the limit on the functions

$$V_i^C \xrightarrow{\sigma_{i,d}} V_i^{Fd} \xrightarrow{\phi_\delta^{Fd}} \mathrm{Term}_{\Sigma_\beta,j}((Fd)_\beta).$$

Let C the computad consisting of C_β for all $\beta < \alpha$, the set V_i^C for every sort i of dimension α and the gluing functions above. The morphisms $\sigma_{\beta,d}$ and the functions $\sigma_{i,d}$ give rise to a cone $(\sigma : C \to Fd)_{d\in\mathcal{D}}$, whose universal property in the subcategory of variable-to-variable morphisms can be verified immediately. Its universal property in the category of computads and all morphisms follows then easily by preservation of connected limits by $\mathrm{Term}_\Sigma^{\mathrm{var}}$, Finally, this limit cone is strictly preserved by the truncation functors, which concludes the induction. □

At this point, we have all the ingredients needed to show that $\mathrm{Comp}_\Sigma^{\mathrm{var}}$ is a presheaf topos. We define a *plex* to be a generator of the terminal computad $\mathbb{1}_\Sigma$. Plexes form a direct subcategory Plex_Σ of $\mathrm{Comp}_\Sigma^{\mathrm{var}}$, where

$$\mathrm{Plex}_\Sigma(p, p') = \mathrm{Comp}_\Sigma^{\mathrm{var}}(|p|, |p'|) \cong \mathcal{T}_{\Sigma,p}(|p'|)$$

and $\dim p = \dim(\mathrm{sort}(p))$. Plexes familially represent the functors sending a computads to its generators, in the sense that there exist natural isomorphisms

$$V_i^C \cong \coprod_{\substack{p \in \mathrm{Plex}_\Sigma \\ \mathrm{sort}(p) = i}} \mathrm{Comp}_\Sigma^{\mathrm{var}}(|p|, C).$$

From the construction of colimits of variable-to-variable morphisms in Proposition 4, it easy to see that the functors represented by the plexes are cocontinuous. Moreover, they jointly reflect isomorphisms by Remark 3. The following theorem is then an immediate consequence of [14, Proposition 5.14].

Theorem 10. *The nerve functor* $N : \mathrm{Comp}_\Sigma^{var} \to [\mathrm{Plex}_\Sigma^{op}, \mathrm{Set}]$ *defined by*

$$(NC)(p) = \mathrm{Comp}_\Sigma^{var}(|p|, C) \cong \{v \in V_{\mathrm{sort}(p)}^C \ : \ \mathrm{Term}_\Sigma(!)(\mathrm{var}\, v) = p\},$$

is an equivalence of categories.

4 Algebras over a signature

In this section, we introduce the semantics of $\mathcal{I}$-sorted signatures. We define algebras for Σ to be algebras for the term monad M_Σ, and give a simpler description of them in terms of presheaves equipped with a function for every function symbol of Σ, satisfying certain boundary axioms dictated by the boundary terms. By definition, computads gives rise to free algebras, and we will show that our generalised morphisms of computads are precisely the morphisms between the algebras they generate. As before, in this section, Σ denotes an $\mathcal{I}$-sorted signature of some dimension α.

Definition 11. A Σ-*algebra* is an algebra for the monad $(\mathrm{M}_\Sigma, \eta_\Sigma, \mu_\Sigma)$ induced by the term adjunction $\mathrm{Cptd}_\Sigma \dashv \mathrm{Term}_\Sigma$. We will denote their category by $\mathrm{Alg}_\Sigma = \mathrm{Alg}_{\mathrm{M}_\Sigma}$.

An algebra $\mathbb{X} = (X, u_\mathbb{X})$ consists therefore of a *carrier* presheaf X on $\mathcal{I}_\alpha$, and a morphism $u^\mathbb{X} : \mathrm{M}_\Sigma(X) \to X$, the M_Σ-*action*, satisfying the following unit and associativity axioms

$$u^\mathbb{X} \circ \eta_\Sigma = \mathrm{id} \qquad\qquad u^\mathbb{X} \circ \mu_\Sigma = u^\mathbb{X} \circ \mathrm{M}_\Sigma(u^\mathbb{X}).$$

The unit axiom prescribes the value of the action on generators by

$$u^\mathbb{X}(\mathrm{var}\, x) = x$$

for every $i \in \mathcal{I}_\alpha$ and $x \in X_i$. The associativity axiom, on the other hand, gives rise to a recursive formula for composite terms: Given a composite term $t = f[\tau]$, we may form the transpose $\tau^\dagger : B_f \to \mathrm{M}_\Sigma(X)$ of τ under the term adjunction. By definition, $\tau = \varepsilon_{\Sigma,X} \circ \mathrm{Cptd}_\Sigma(\tau^\dagger)$, so

$$u^{\mathbb{X}}(f[\tau]) = u^{\mathbb{X}}\mu_\Sigma(f[\mathrm{Cptd}_\Sigma(\tau^\dagger)]) = u^{\mathbb{X}}(f[\mathrm{Cptd}_\Sigma(u^{\mathbb{X}}\tau^\dagger)]). \tag{1}$$

We define the *interpretation* of a function symbol $f \in \Sigma_i$ of dimension $\beta \leq \alpha$ in $\mathbb{X}$ to be the function

$$f^{\mathbb{X}} : [\mathcal{I}_\beta^{\mathrm{op}}, \mathrm{Set}](B_f, \mathrm{tr}_\beta X) \to X_i$$

given on a morphism $\sigma : B_f \to \mathrm{tr}_\beta X$ by

$$f^{\mathbb{X}}(\sigma) = u^{\mathbb{X}}(f[\mathrm{Cptd}_\Sigma(\sigma)]).$$

A recursive formula shows that an algebra $\mathbb{X}$ is uniquely determined by its carrier presheaf and the interpretations of the function symbols. Moreover, those interpretation functions can be freely chosen, so long as certain boundary condition is satisfied.

Proposition 12. *Let X a presheaf on $\mathcal{I}_\alpha$. Actions $u : \mathrm{M}_\Sigma(X) \to X$ are in bijection to families of*

- *actions $u_\beta : \mathrm{M}_{\Sigma_\beta}(\mathrm{tr}_\beta X) \to \mathrm{tr}_\beta X$ for $\beta < \alpha$*
- *functions $\hat{f} : [\mathcal{I}^{op}, \mathrm{Set}](B_f, X) \to X_i$ for i of dimension α and $f \in \Sigma_i$,*

satisfying the usual cocycle conditions, and the following boundary condition

$$\delta^* \hat{f} = u_\beta(\mathrm{M}_{\Sigma_\beta}(-)(t_{f,\delta}))$$

for non-identity face maps $\delta : j \to i$, where $\beta = \dim j$.

Proof. As explained above, the morphisms u_β and the functions $\hat{f}$ determine u uniquely. Conversely, given such morphisms and functions, we can build such u recursively. We first define u on terms of dimension $\beta < \alpha$ to coincide with u_β and define it on generators $x \in X_i$ of dimension α by

$$u(\mathrm{var}\, x) = x.$$

This assignment is compatible with the boundary maps by the cocycle conditions and the fact that each u_β satisfies the unit axiom. We define then u on composite term $t = f[\tau]$ of dimension α by the recursive formula (1):

$$u(f[\tau]) = \hat{f}(u \circ \tau^\dagger).$$

This assignment is compatible with face maps by the boundary condition, hence it defines a morphism $u : \mathrm{M}_\Sigma(X) \to X$.

This morphism satisfies the associativity axiom on terms of dimension less than α, and the unit axiom on all terms. The unit axiom also implies the associativity axiom for generators of dimension α, so let $t = f[\rho]$ a composite term of $\mathrm{Cptd}_\Sigma \mathrm{M}_\Sigma(X)$ of dimension α. We may assume that the associativity axiom holds in the image of the transpose $\rho^\dagger : B_f \to \mathrm{M}_\Sigma \mathrm{M}_\Sigma X$ of ρ by induction on depth, and compute that

$$
\begin{aligned}
u \circ \mathrm{M}_\Sigma(u)(t) &= u(f[(\mathrm{Cptd}_\Sigma u) \circ \rho]) \\
&= \hat{f}(u \circ ((\mathrm{Cptd}_\Sigma u) \circ \rho)^\dagger) \\
&= \hat{f}(u \circ \mathrm{M}_\Sigma(u) \circ \rho^\dagger) \\
&= \hat{f}(u \circ \mu_\Sigma \circ \rho^\dagger) \\
&= \hat{f}(u \circ (\varepsilon_\Sigma \rho)^\dagger) \\
&= u(f[\varepsilon_\Sigma \rho]) \\
&= u \circ \mu_\Sigma(t)
\end{aligned}
$$

Therefore, the associativity axiom holds for all terms, and $\mathbb{X} = (X, u)$ is an algebra.

It remains to show that this algebra gives rise to the data that we started from. The cocycle conditions imply that $\mathrm{tr}_\beta\, u = u$ for every $\beta < \alpha$. Let therefore $f \in \Sigma_i$ of dimension α and $\tau : B_f \to X$. By definition of transposition and the unit axiom, we have that

$$u \circ \mathrm{Cptd}_\Sigma(\tau)^\dagger = u \circ \mathrm{M}_\Sigma(\tau) \circ \eta_\Sigma = u \circ \eta_\Sigma \circ \tau = \tau$$

and hence that

$$f^{\mathbb{X}}(\tau) = u(f[\mathrm{Cptd}_\Sigma(\tau)]) = \hat{f}(u \circ \mathrm{Cptd}_\Sigma(\tau)^\dagger) = \hat{f}(\tau).$$

We see that the interpretation of f in this algebra coincides with the function $\hat{f}$, proving the bijection of the proposition. □

Morphisms of algebras $f : \mathbb{X} \to \mathbb{Y}$ are the morphisms $\sigma : X \to Y$ satisfying that

$$u^{\mathbb{Y}} \circ (\mathrm{M}_\Sigma\, \sigma) = \sigma \circ u^{\mathbb{X}}.$$

This equation easily implies that morphisms of algebras must preserve the interpretation functions

$$\sigma \circ f^{\mathbb{X}} = f^{\mathbb{Y}}(\mathrm{tr}_\beta\, \sigma \circ -) \tag{2}$$

for every $f \in \Sigma_i$ of dimension $\beta \leq \alpha$. The recursive formula (1) immediately implies that the converse also holds.

Proposition 13. *A morphism of algebras $\sigma : \mathbb{X} \to \mathbb{Y}$ is a morphism between their underlying presheaves $\sigma : X \to Y$ satisfying equation (2) for every function symbol of Σ.*

In the rest of the section, we will study the connection between computads and algebras. The category of algebras is equipped by definition with an adjunction $\mathrm{F}_\Sigma \dashv \mathrm{U}_\Sigma$ to the category of presheaves on $\mathcal{I}_\alpha$ and with a functor K_Σ from the category of computads, making the following triangle commute in both directions.

$$\begin{tikzcd}
 & {[\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}]} \arrow[dl, shift left, "\mathrm{Cptd}_\Sigma"] \arrow[dr, shift left, "\mathrm{F}_\Sigma"'] & \\
\mathrm{Comp}_\Sigma \arrow[ur, shift left, "\mathrm{Term}_\Sigma"] \arrow[rr, "\mathrm{K}_\Sigma"'] & & \mathrm{Alg}_\Sigma \arrow[ul, shift left, "\mathrm{U}_\Sigma"']
\end{tikzcd}$$

The forgetful functor U_Σ sends an algebra to its carrier presheaf, while the comparison functor K_Σ sends a computad C to its presheaf of terms $\mathrm{Term}_\Sigma(C)$ equipped with the action $u^C = \mathrm{Term}_\Sigma(\varepsilon_{\Sigma,C})$. Henceforth, we will treat K_Σ as a subcategory inclusion and suppress it in the notation. This abuse of notation partially justified by Proposition 15.

Lemma 14. *If morphisms $\sigma, \tau : C \to \mathbb{X}$ from a computad to an algebra agree on all generators of C, then they are equal.*

Proof. By induction on the dimension α of the signature, we may assume that $\mathrm{tr}_\beta\, \sigma = \mathrm{tr}_\beta\, \tau$ for all $\beta < \alpha$, so it remains to show that they agree on composite terms of dimension α as well. Given a term $t = f[\rho]$, we may further assume by structural induction that the morphisms agree on terms in the image of the transpose $\rho^\dagger$ as well. Using that, we see that

$$\begin{aligned}
\sigma(f[\rho]) &= \sigma \circ u^C(f[\mathrm{Cptd}_\Sigma\, \rho^\dagger]) \\
&= u^{\mathbb{X}} \circ \mathrm{M}_\Sigma(\sigma)(f[\mathrm{Cptd}_\Sigma\, \rho^\dagger]) \\
&= u^{\mathbb{X}}(f[\mathrm{Cptd}_\Sigma(\sigma\rho^\dagger)]) \\
&= u^{\mathbb{X}}(f[\mathrm{Cptd}_\Sigma(\sigma'\rho^\dagger)]) \\
&= \cdots = \tau(f[\rho]),
\end{aligned}$$

so the two morphisms are equal. □

Proposition 15. *The functor K_Σ is fully faithful.*

Proof. The composite $\mathrm{Term}_\Sigma = \mathrm{U}_\Sigma\, \mathrm{K}_\Sigma$ is clearly faithful, so the same must hold of K_Σ. To show that it is full, let C, D be computads and $\sigma : \mathrm{K}_\Sigma\, C \to \mathrm{K}_\Sigma\, D$ a morphism of algebras. By induction on dimension, we may assume that there exists a unique morphisms of computads $\rho_\beta : C_\beta \to D_\beta$ for every $\beta < \alpha$, such that

$$\mathrm{K}_{\Sigma_\beta}(\rho_\beta) = \mathrm{tr}_\beta\, \sigma. \tag{3}$$

Uniqueness of those morphisms implies that the usual cocycle conditions are satisfied. Let $\rho : C \to D$ consist of ρ_β for all $\beta < \alpha$ and the functions

$$\rho_i = \sigma_i \circ \mathrm{var} : V_i^C \to \mathrm{Term}_{\Sigma,i}(D)$$

for every sort i of dimension α. Using equation (3), it is easy to see that ρ is a well-defined morphism of computads. The morphisms of algebras σ and $\mathrm{K}_\Sigma(\rho)$ agree on generators, so they must be equal by Lemma 14. □

Proposition 16. *Morphisms $\sigma : C \to \mathbb{X}$ from a computad to an algebra $\mathbb{X}$ are in bijection to*

- *morphisms $\rho_\beta : \operatorname{tr}_\beta C \to \operatorname{tr}_\beta \mathbb{X}$ for every $\beta < \alpha$,*
- *functions $\rho_i : V_i^C \to X_i$ for every sort i of dimension α,*

satisfying the usual cocycle conditions, and the boundary condition

$$\delta^* \rho_i = \rho_{\beta,j} \phi_\delta^C$$

for non-identity face maps $\delta : j \to i$, where $\beta = \dim j$.

Proof. A morphism σ gives rise to such data by letting $\rho_\beta = \operatorname{tr}_\beta \sigma$ for $\beta < \alpha$ and letting $\rho_i = \sigma_i \circ \operatorname{var}$ for i of dimension α. Moreover, this assignment is injective by Lemma 14, so it remains to construct a morphism $\sigma : C \to \mathbb{X}$ out of that data.

On terms of dimension $\beta < \alpha$, we let σ coincide with ρ_β, while on generators of sort i of dimension α, we let σ coincide with ρ_i. this assignment is compatible with the face maps by the cocycle and boundary conditions. Given a composite term $t = f[\tau]$ of sort i, we may assume recursively on depth that σ has been defined on the image of the transpose $\tau^\dagger$ and let

$$\sigma(f[\tau]) = f^{\mathbb{X}}(\sigma\tau^\dagger).$$

Given a non-identity face map $\delta : j \to i$, we let $\beta = \dim j$ and calculate that

$$\begin{aligned}
\delta^*\sigma(f[\tau]) &= \delta^* f^{\mathbb{X}}(\sigma\tau^\dagger) \\
&= u^{\mathbb{X}} \circ (\mathrm{M}_\sigma\, \sigma) \circ (\mathrm{M}_\Sigma\, \tau^\dagger)(t_{f,\delta}) \\
&= \sigma \circ \mathrm{Term}_\Sigma(\varepsilon_{\Sigma,C}) \circ (\mathrm{M}_\Sigma\, \tau^\dagger)(t_{f,\delta}) \\
&= \sigma \circ \mathrm{Term}_\Sigma(\tau)(t_{f,\delta}) \\
&= \sigma(\delta^*(f[\tau])),
\end{aligned}$$

so this assignment defines a morphism of presheaves $\sigma : \mathrm{Term}_\Sigma C \to X$, which restricts to the morphisms ρ_β and the functions ρ_i. It remains to show that σ is a morphism of algebras. By Proposition 13, it suffices to show that it preserves the interpretations of every function symbol of Σ. This is the case for function symbols of dimension $\beta < \alpha$, since ρ_β is a morphism of algebras. On the other hand, let $f \in \Sigma_i$ of dimension α and $\hat{\tau} : B_f \to \mathrm{Term}_\Sigma(C)$ a morphism. Then

$$\begin{aligned}
\sigma(f^C(\hat{\tau})) &= \sigma(f[\varepsilon_{\Sigma,C}\, \mathrm{Cptd}_\Sigma(\hat{\tau})]) \\
&= f^{\mathbb{X}}(\sigma(\varepsilon_{\Sigma,C}\, \mathrm{Cptd}_\Sigma(\hat{\tau}))^\dagger) \\
&= f^{\mathbb{X}}(\sigma\hat{\tau}^{\dagger\dagger}) = f^{\mathbb{X}}(\sigma\hat{\tau})
\end{aligned}$$

so σ is a morphism of algebras. □

Corollary 17. *Morphisms $\sigma : C \to \mathbb{X}$ from a computad with no generators of dimension at least β to an arbitrary algebra $\mathbb{X}$ are in bijection to families of morphisms $\sigma_\gamma : \operatorname{tr}_\gamma^\Sigma C \to \operatorname{tr}_\gamma^\Sigma \mathbb{X}$ for every $\gamma < \beta$ satisfying the usual cocycle conditions.*

5 Cofibrancy of computads

Computads have recently proven useful in the study of the homotopy theory of ω-categories. The inclusion of the free ω-category on a sphere into the free ω-category on a disk cofibrantly generate a weak factorisation system in the category of ω-categories and strict morphisms for which computads are cofibrant. Moreover, there exists an adjunction between computads and ω-categories [6] generating the *universal cofibrant replacement* comonad for this factorisation system [16]. For strict ω-categories, this weak factorisation system is part of a model structure [24], and it has been shown that conversely every cofibrant strict ω-category is free on a computad [28].

Most of those results hold for our computads for $\mathcal{I}$-sorted theories verbatim. Proposition 16 describes a universal property of algebras free on a computad, analogous to the universal property of free ω-categories [32], which can be used to construct a right adjoint to the free algebra functor

$$\mathrm{Free}_\Sigma : \mathrm{Comp}^{\mathrm{var}}_\Sigma \xhookrightarrow{\zeta_\Sigma} \mathrm{Comp}_\Sigma \xhookrightarrow{\mathrm{K}_\Sigma} \mathrm{Alg}_\Sigma\,.$$

We will show that the comonad induced by this adjunction is the universal cofibrant replacement for certain weak factorisation system, and that algebras free on a computad are the cofibrant objects.

To set the notation, fix an $\mathcal{I}$-sorted signature Σ of some dimension α. The *representable computad* on a sort $i \in \mathcal{I}_\alpha$ is the computad

$$\mathbb{D}^i_\Sigma = \mathrm{Cptd}_\Sigma(\mathcal{I}_\alpha(-,i)).$$

Its *boundary* is the computad obtained by removing its top-dimensional generators, or equivalently by

$$\partial\mathbb{D}^i_\Sigma = \mathrm{Cptd}_\Sigma\left(\operatorname*{colim}_{\delta:j\to i}\mathcal{I}_\alpha(-,j)\right)$$

where the colimit is over all non-identity morphisms $\delta : j \to i$. The *boundary inclusion*

$$\iota_{\Sigma,i} : \partial\mathbb{D}^i_\Sigma \hookrightarrow \mathbb{D}^i_\Sigma$$

is the morphism of computads induced by the morphism of presheaves whose component at δ is $\delta_* = (\delta \circ -)$. The following proposition is an immediate consequence of the Yoneda lemma and the term adjunction.

Proposition 18. *The computad $\mathbb{D}^i$ represents the functor $\mathrm{Term}_{\Sigma,i}$ of terms of sort i. Its boundary $\partial\mathbb{D}^i_\Sigma$ represents the functor of types of sort i*

$$\mathrm{Type}_{\Sigma,i} = \lim_{\delta:j\to i} \mathrm{Term}_{\Sigma,j}$$

and the boundary inclusion $\iota_{\Sigma,i}$ induces the natural transformation sending a term t to the family of terms $(\delta^* t)$.

Using instead the adjunction $\mathrm{F}_\Sigma \dashv \mathrm{U}_\Sigma$, it is easy to see that the functor sending an algebra $\mathbb{X}$ to the set X_i is represented by the free algebra on $\mathbb{D}^i_\Sigma$, and that the functor represented by the free algebra on $\partial\mathbb{D}^i_\Sigma$ admits a similar description.

The set of boundary inclusions $I = \{\iota_{\Sigma,i} \ : \ i \in \mathcal{I}_\alpha\}$ cofibrantly generates a weak factorisation system $(\mathcal{L}, \mathcal{R})$ in the category of algebras via the small object argument, in light of Corollary 35. We will call morphisms of $\mathcal{L}$ *cofibrations* and morphism of $\mathcal{R}$ *trivial fibrations*. We will call an algebra $\mathbb{X}$ *cofibrant* when the unique morphism $\emptyset \to \mathbb{X}$ from the initial algebra is a cofibration. We will also say that a morphism $\mathbb{X} \to \mathbb{Y}$ is a *cofibrant replacement* when it is a trivial fibration and $\mathbb{X}$ is cofibrant.

Proposition 19. *Free algebras on a computad are cofibrant.*

Proof. We will show that the unique morphism $\emptyset \to C$ is a transfinite composition of pushouts of coproducts of boundary inclusions in the category of algebras for every computad C. Closure of cofibrations under those operations is a standard result [22]. In order to show that, we will introduce a variant of the skeleton functors.

Let $\operatorname{sk}^\Sigma_{\partial\beta} C$ the computad obtained by removing all generators of C of dimension at least β for every $\beta \le \alpha + 1$

$$\operatorname{sk}^\Sigma_{\partial\beta} C = \operatorname{sk}^\Sigma_\beta((\operatorname{tr}^\Sigma_\gamma C)_{\gamma<\beta}, (\emptyset, \{\})_{\dim j=\beta}).$$

Inclusions of the sets of generators give rise to variable-to-variable morphisms

$$\kappa^\beta_\gamma : \operatorname{sk}^\Sigma_{\partial\gamma} C \to \operatorname{sk}^\Sigma_{\partial\beta}$$

for every $\gamma \le \beta$ satisfying the obvious cocycle conditions. That data defines a chain of variable-to-variable morphisms which is cocontinuous by Proposition 4 and has transfinite composite $\emptyset \to C$. It remains to show that this chain remains cocontinuous when viewed as a chain of algebras and that the morphisms $\kappa^{\beta+1}_\beta$ are pushouts of coproducts of boundary inclusions for every $\beta \le \alpha$.

For the former, let $\beta \le \alpha$ a limit ordinal and let

$$\tau^\gamma : \operatorname{sk}^\Sigma_{\partial\gamma} C \to \mathbb{X}$$

for $\gamma < \alpha$ a cocone under the restriction of this chain to β. Then the morphisms

$$\tau_\gamma = \operatorname{tr}^\Sigma_\gamma(\tau^{\gamma+1}) : \operatorname{tr}^\Sigma_\gamma \operatorname{sk}^\Sigma_{\partial\beta} C \to \mathbb{X}$$

satisfy the cocycle conditions and give rise to a morphism $\tau : \operatorname{sk}^\Sigma_{\partial\beta} C \to \mathbb{X}$ by Corollary 17. By Lemma 14 it follows immediately that τ is the unique morphism such that $\tau^\gamma = \tau\kappa^\beta_\gamma$ for every $\gamma < \beta$. Therefore, this chain is cocontinuous in when viewed as a chain of algebras.

For the latter, given an ordinal $\beta \le \alpha$, we may form the commutative square

$$\begin{tikzcd}
\coprod\limits_{\substack{\dim i=\alpha \\ v\in V_i^C}} \partial\mathbb{D}^i_\Sigma \arrow[r, "(\iota_{\Sigma,i})"] \arrow[d, "\phi"'] & \coprod\limits_{\substack{\dim i=\alpha \\ v\in V_i^C}} \mathbb{D}^i_\Sigma \arrow[d, "\psi"] \\
\mathrm{sk}^\Sigma_{\partial\beta} C \arrow[r, "\kappa^{\beta+1}_\beta"'] & \mathrm{sk}^\Sigma_{\partial(\beta+1)} C
\end{tikzcd}$$

where ψ classifies the generators of C of dimension β under the bijection of Proposition 18, and ϕ classifies their boundary types. The left adjoints F_Σ preserves colimits, while Cptd_Σ reflects them, so both coproducts of the square are also coproducts of algebras. Using Corollary 17 for the computads on the bottom row and Proposition 16, we see that this square is a pushout of algebras. □

In the proof of the proposition above, we described each computad as a transfinite composite of pushouts of coproducts of boundary inclusions. This descriptions allows us to define a right adjoint

$$\mathrm{Und}_\Sigma : \mathrm{Alg}_\Sigma \to \mathrm{Comp}^{\mathrm{var}}_\Sigma$$

to the free functor Free_Σ sending a computad to the algebra it generates. This adjunctions defines a comonad on the category of algebras, whose underlying pointed endofunctor we will denote by

$$\mathrm{Cof}_\Sigma : \mathrm{Alg}_\Sigma \to \mathrm{Alg}_\Sigma$$
$$r_\Sigma : \mathrm{Cof}_\Sigma \Rightarrow \mathrm{id}\,.$$

The definition of Und_Σ is recursive on the dimension α of the signature, so suppose that the adjunctions $\mathrm{Free}_{\Sigma_\beta} \dashv \mathrm{Und}_{\Sigma_\beta}$ have been defined for all $\beta < \alpha$ and that they are compatible with the truncation functors.

The underlying computad $\mathrm{Und}_\Sigma \mathbb{X}$ of an algebra $\mathbb{X}$ consists of the computads $\mathrm{Und}_{\Sigma_\beta} \mathrm{tr}^\Sigma_\beta \mathbb{X}$ for $\beta < \alpha$ and the sets of generators and gluing functions defined by the following pullback square

$$\begin{tikzcd}
V_i^{\mathrm{Und}_\Sigma \mathbb{X}} \arrow[rr, dashed] \arrow[d, dashed, "(\phi_\delta^{\mathrm{Und}_\Sigma \mathbb{X}})"'] & & X_i \arrow[d, "(\delta^*)"] \\
\lim\limits_{\substack{\dim j=\beta<\alpha \\ \delta:j\to i}} (\mathrm{Cof}_{\Sigma_\beta} \mathrm{tr}^\Sigma_\beta \mathbb{X})_j \arrow[r, "(r_{\Sigma_\beta})"'] & \lim\limits_{\substack{\dim j=\beta<\alpha \\ \delta:j\to i}} (\mathrm{tr}^\Sigma_\beta \mathbb{X})_j \arrow[r, equal] & \lim\limits_{\substack{\dim j=\beta<\alpha \\ \delta:j\to i}} X_j.
\end{tikzcd}$$

for every sort i of dimension α. The variable-to-variable morphism $\mathrm{Und}_\Sigma \sigma$ induced by a morphism $\sigma : \mathbb{X} \to \mathbb{Y}$ consists similarly of the morphisms $\mathrm{Und}_{\Sigma_\beta} \mathrm{tr}^\Sigma_\beta \sigma$ for $\beta < \alpha$ and the functions $V_i^{\mathrm{Und}_\Sigma \mathbb{X}} \to V_i^{\mathrm{Und}_\Sigma \mathbb{Y}}$ for i of dimension α with components

$$(\mathrm{Und}_\Sigma \sigma)_i((t_\delta), x) = (((\mathrm{Cof}_{\Sigma_\beta} \mathrm{tr}^\Sigma_\beta \sigma)(t_\delta)), \sigma(x)).$$

This assignment is functorial and commutes clearly with the truncation functors. The counit of the adjunction $r_{\Sigma,\mathbb{X}} : \operatorname{Cof}_\Sigma \mathbb{X} \to \mathbb{X}$ is the morphism corresponding to the morphisms $r_{\Sigma_\beta, \operatorname{tr}^\Sigma_\beta \mathbb{X}}$ and the projection functions $V_i^{\operatorname{Und}_\Sigma \mathbb{X}} \to X_i$ under the bijection of Proposition 16.

Proposition 20. *The functor* $\operatorname{Und}_\Sigma$ *is right adjoint to* $\operatorname{Free}_\Sigma$.

Proof. Variable-to-variable morphisms $\sigma : C \to \operatorname{Und}_\Sigma \mathbb{X}$ consist of variable-to-variables morphisms $\sigma_\beta : C_\beta \to \operatorname{Und}_{\Sigma_\beta} \operatorname{tr}^\Sigma_\beta \mathbb{X}$ satisfying the usual cocycle conditions and a pair of functions

$$\sigma_{i,1} : V_i^C \to \lim_{\substack{\dim j = \beta < \alpha \\ \delta : j \to i}} (\operatorname{tr}_{\Sigma_\beta} \mathbb{X})_j$$

$$\sigma_{i,2} : V_i^C \to X_i$$

satisfying gluing conditions and that $(\sigma_{i,1}, \sigma_{i,2})$ defines a function into the pullback $V_i^{\operatorname{Und}_\Sigma \mathbb{X}}$. The gluing condition is equivalent to

$$\sigma_{i,1} = (\operatorname{Term}_{\Sigma_\beta}(\sigma_\beta) \circ \phi^C_\delta)_{\delta : j \to i},$$

in the presence of which the other condition becomes

$$\delta^* \sigma_{i,2} = r_{\Sigma_\beta} \operatorname{Term}_{\Sigma_\beta, j}(\sigma_\beta) \phi^C_\delta$$

for non-identity face map $\delta : j \to i$ where $\beta = \dim j$. By the inductive hypothesis, the morphisms σ_β are in bijection to morphisms $\sigma_\beta^\dagger : C_\beta \to \operatorname{tr}^\Sigma_\beta \mathbb{X}$ satisfying the same cocycle conditions. Under this bijection, the condition above becomes

$$\delta^* \sigma_{i,2} = \sigma^\dagger_{\beta,j} \phi^C_\delta,$$

so the morphisms $\sigma^\dagger_\beta$ and the functions $\sigma_{i,2}$ determine uniquely a morphism $\sigma^\dagger : C \to \mathbb{X}$ by Proposition 16. Using Lemma 14, it is easy to see that this bijection is given by $\sigma^\dagger = r_{\Sigma,\mathbb{X}} \circ \operatorname{Free}_\Sigma \sigma$. Therefore, the bijection is natural and r_Σ is the counit of the adjunction. $\square$

Recall that trivial fibrations are the morphisms in the right class of the factorisation system generated by the boundary inclusions, so they are morphisms of algebras $\sigma : \mathbb{X} \to \mathbb{Y}$ such that every commutative square of the form

$$\begin{tikzcd} \partial \mathbb{D}^i_\Sigma \arrow[r, "T"] \arrow[d, hook] & \mathbb{X} \arrow[d, "\sigma"] \\ \mathbb{D}^i_\Sigma \arrow[r, "t"] \arrow[ur, dashed, "v(i,T,t)"] & \mathbb{Y} \end{tikzcd}$$

admits a diagonal lift $v(i,T,t)$. An *algebraic trivial fibration* is instead a trivial fibration with a choice of lifts for every such square. Morphisms of algebraic trivial fibrations with common target $\mathbb{Y}$ are morphisms in the slice category $\operatorname{Alg}_\Sigma / \mathbb{Y}$ preserving the lifts.

The counit $r_{\Sigma,\mathbb{X}} : \mathrm{Cof}_\Sigma \mathbb{X} \to \mathbb{X}$ can be equipped with the structure of an algebraic trivial fibration as follows: A commutative square of the form

$$\begin{tikzcd} \partial\mathbb{D}^i_\Sigma \arrow[r, "T"] \arrow[d, hook] & \mathrm{Cof}_\Sigma \mathbb{X} \arrow[d, "r_{\Sigma,\mathbb{X}}"] \\ \mathbb{D}^i_\Sigma \arrow[r, "t"] & \mathbb{X} \end{tikzcd}$$

amounts to the choice of compatible terms $T = (t_\delta)_{\delta:j\to i}$ of $\mathrm{Cof}_\Sigma \mathbb{X}$ together with an element $t \in X_i$ satisfying the compatibility condition

$$r_{\Sigma,\mathbb{X}}(t_\delta) = \delta^* t,$$

or equivalently a generator (T, t) of $\mathrm{Cof}_\Sigma \mathbb{X}$ of sort i. A lift for this square is then given by the morphism corresponding to the term

$$v(i, T, t) = \mathrm{var}(T, t).$$

It is not hard to see that $(r_{\Sigma,\mathbb{X}}, v)$ is initial among algebraic trivial fibrations with target $\mathbb{X}$. Given any algebraic trivial fibration $(\sigma : \mathbb{Y} \to \mathbb{X}, v')$, we may define a morphism $\tau : \mathrm{Cof}_\Sigma \mathbb{X} \to \mathbb{Y}$ of algebraic trivial fibrations recursively by letting for every $\beta \leq \alpha$, the morphism $\tau_\beta : \mathrm{tr}^\Sigma_\beta \mathrm{Cof}_\Sigma \mathbb{X} \to \mathrm{tr}^\Sigma_\beta \mathbb{Y}$ correspond under the bijection of Proposition 16 to the morphisms τ_γ for $\gamma < \beta$ and the functions $\tau_i : V_i^{\mathrm{Und}_\Sigma \mathbb{X}} \to Y_i$ given by

$$\tau_i((t_\delta)_{\delta:j\to i}, t) = v'(i, (\tau_\beta(t_\delta))_{\delta:j\to i}, t).$$

The morphism $\tau = \tau_\alpha : \mathrm{Cof}_\Sigma \mathbb{X} \to \mathbb{Y}$ is a morphism of algebraic trivial fibrations, since $r_{\Sigma,\mathbb{X}} = \sigma\tau$ and

$$\tau \circ v(i, T, t) = v'(i, \tau \circ T, t)$$

for every commutative square as above. Moreover, τ is unique by Lemma 14. This observation combined with the recognition criterion [16, Proposition 2.6] show the following corollary.

Corollary 21. *The pointed endofunctor* $(\mathrm{Cof}_\Sigma, r_\Sigma)$ *underlies the universal cofibrant comonad for the weak factorisation system cofibrantly generated by the boundary inclusions.*

The existence of a cofibrant replacement functor taking values in free algebras together with Cauchy completeness of the category of computads, shown in Corollary 32 allow us to prove the converse of Proposition 19 by the same argument used for strict ω-categories [28].

Corollary 22. *Cofibrant algebras are free on a computad.*

Proof. Let $\mathbb{X}$ a cofibrant algebra. By Corollary 21, there exists a computad C and a trivial fibration $r : C \to \mathbb{X}$. Since $\mathbb{X}$ is cofibrant, the map r admits a section $s : \mathbb{X} \to C$. Being a section, the endomorphism $sr : C \to C$ is idempotent, so by Proposition 19, there exists a computad D and morphisms $\iota : D \to C$ and $\pi : C \to D$ such that

$$\pi\iota = \mathrm{id} \qquad\qquad \iota\pi = sr$$

It follows that $r\iota : D \to \mathbb{X}$ is an isomorphism with inverse $\pi s : \mathbb{X} \to D$, so $\mathbb{X}$ is free on a computad. □

6 Higher categories

Over the past decades, many definitions of higher categorical structures have been proposed, a number of which can be described as presheaves on some category of shapes $\mathcal{I}$ that are either equipped with extra operations (algebraic models) or satisfying certain conditions (geometric models). It is often possible in some occasions to replace the latter with the former [11, 29], which can more easily described in our setting. In this section, we will explain how Leinster's weak ω-categories [25], and algebraic semi-simplicial Kan complexes, a model of ∞-groupoids, are algebras for certain signatures. We will also propose a signature for fully weak multiple categories, inspired by the signature for ω-categories. Given that opetopic higher categories [5] are presheaves on the direct category of opetopes [13], and fair categories are presheaves on the direct category 'fat Delta' [23], we believe that some variant of them should also be describable as algebras of some signature.

Universal algebra The discrete category on any set S can be made into a direct category by equipping it with a constant dimension function. Then S-sorted signatures are given by an S-indexed family of sets $\Sigma = (\Sigma_s)_{s\in S}$ together with an S-indexed family of sets B_f for every $s \in S$ and $f \in \Sigma_s$. Algebras for such a signature are again S-indexed families of sets $X = (X_s)_{s\in S}$ equipped with a function

$$f^{\mathbb{X}} : \mathrm{Set}^S(B_f, X) \to X_s$$

for every sort $s \in S$ and function symbol $f \in \Sigma_s$.

Letting S be a singleton, one recovers the usual notion of (infinitary) signature of universal algebra. Letting, for example, $\Sigma = \{+, 0, -\}$ consist of three elements with arities a set with two elements, no elements and a unique element respectively, one recovers the language of group theory. Groups can be described as algebras satisfying certain equational axioms.

Modules over a ring can similarly be expressed as algebras for some signature satisfying certain equations. The set S of sorts in this case consists of two elements R and V, representing the ring elements and the vectors. The set Σ_R contains two function symbols $+^R, \cdot^R$ of arity $y_R \amalg y_R$, for $y_\bullet$ the Yoneda embedding, two function symbols $0^R, 1^R$ with arity the empty family, and one function symbol $-^R$ with arity y_R. The set Σ_V contains similarly function symbols $+^V, 0^V, -^V$ with the obvious arities and a function symbol $\cdot^V$ with arity $y_R \amalg y_V$. We hope that out of those examples, it is clear how to incorporate arbitrary many-sorted signatures in our framework.

Kan complexes A classical result in algebraic topology shows that the homotopy theory of spaces and simplicial sets are equivalent. More recently, it was shown that one may define a weak model structure on a semi-simplicial set, presenting the same theory. The cofibrations of this weak model structure are monomorphisms, generating acyclic cofibrations are the horn inclusions, and weak equivalences are the morphisms that become homotopy equivalences of spaces after realisation [20, Theorem 5.5.6]. This structure can be transferred to an actual model structure on the category of algebraically fibrant objects that we discuss below [11, Example 33].

To set the notation, let Δ_+ the category of non-empty finite ordinals and strictly monotone functions. This category has as objects natural numbers $[n] = \{0, \ldots, n\}$ and morphisms generated by the *face maps*

$$\delta_i^n : [n-1] \to [n]$$
$$\delta_i^n(j) = \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } i \leq j \end{cases}$$

for $n > 0$ and $0 \leq i \leq n$ under the simplicial identity

$$\delta_i^{n+1}\delta_j^n = \delta_{j+1}^{n+1}\delta_i^n$$

for $n > 0$ and $0 \leq i \leq j \leq n$. It is clear that Δ_+ is a direct category with dimension function $\dim([n]) = n$. Presheaves on it are called *semi-simplicial sets*.

We will denote by $\Delta[n]$ the semi-simplicial set represented by $[n]$, and define its *boundary* $\partial\Delta[n]$ to be its semi-simplicial subset obtained by removing the top-dimensional element $\mathrm{id}_{[n]}$. For every $n > 0$ and $0 \leq k \leq n$, we define the semi-simplicial horn $\Lambda^k[n]$ to be the subset of $\partial\Delta[n]$ obtained by removing the face δ_k^n. An *algebraic semi-simplicial Kan complex* $\mathbb{X}$ is a semi-simplicial set X equipped with a choice of lifts for every diagram of the form

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "\sigma"] \arrow[d, hook] & X \\ \Delta[n]. \arrow[ur, dashed] & \end{tikzcd}$$

The choices of lifts clearly amount to two operations

$$\mathrm{face}^{\mathbb{X}}_{k,n} : [\Delta_+^{\mathrm{op}}, \mathrm{Set}](\Lambda^k[n], X) \to X_{n-1}$$
$$\mathrm{fill}^{\mathbb{X}}_{k,n} : [\Delta_+^{\mathrm{op}}, \mathrm{Set}](\Lambda^k[n], X) \to X_n$$

satisfying certain boundary conditions, that provide the image of the missing face and of the interior of $\Delta[n]$ respectively. Morphisms of algebraic semi-simplicial Kan complexes are morphisms of presheaves preserving the chosen lifts.

From the description above, it is easy to extract a Δ_+-sorted signature $\Sigma_{\mathrm{Kan}} = (\Sigma_{\mathrm{Kan},n})_{n\in\mathbb{N}}$ whose algebras are algebraic semi-simplicial Kan complexes. The signature $\Sigma_{\mathrm{Kan},n}$ has two families of function symbols of sort $[n]$. The first one consists of the symbols $\mathrm{face}_{k,n+1}$ for $0 \le k \le n+1$ with arity $\Lambda^k[n+1]$ and with boundary terms

$$t_{\mathrm{face}_{k,n+1},\delta} = \mathrm{var}(\delta_k^{n+1} \circ \delta)$$

for every non-identity morphism $\delta : [m] \to [n]$. The second one consists of the symbols $\mathrm{fill}_{k,n}$ for $n > 0$ and $0 \le k \le n$ with arity $\Lambda^k[n]$ and with boundary terms

$$t_{\mathrm{fill}_{k,n},\delta} = \begin{cases} \mathrm{face}_{k,n}[\mathrm{id}], & \text{if } \delta = \delta_k^n, \\ \mathrm{var}(\delta), & \text{otherwise.} \end{cases}$$

It is easy to see that those families of boundary terms satisfy the cocycle conditions for giving a type, and that they correspond to the boundary conditions that the operations $\mathrm{face}_{k,n}$ and $\mathrm{fill}_{k,n}$ must satisfy.

As explained above, the category of algebraic semi-simplicial Kan complexes admits a model structure equivalent to spaces. Weak equivalences of algebraic Kan complexes are morphisms that become homotopy equivalences after geometrically realising the underlying semi-simplicial set, while the two weak factorisation systems are cofibrantly generated by the set of boundary inclusions $\partial\Delta[n] \subseteq \Delta[n]$, and by the set of horn inclusions $\Lambda^k[n] \subseteq \Delta[n]$ respectively, seen as morphisms of free algebraic semi-simplicial sets. In particular, the (cofibrations, trivial fibrations) weak factorisation system coincides with the one discussed in Section 5, so computads are the cofibrant complexes.

Globular categories The motivating example for this work is the theory of globular weak ω-categories [7, 25]. This example was studied extensively in our previous work [14], where a $\mathbb{G}$-sorted signature was provided whose algebras coincide with Leinster's ω-categories. The underlying category $\mathbb{G}$ of sorts is the category of globes with objects natural numbers and morphisms generated by

$$s_n, t_n : (n) \to (n+1)$$

under the globularity conditions

$$s_{n+1}s_n = t_{n+1}s_n \qquad\qquad s_{n+1}t_n = t_{n+1}t_n.$$

This is clearly a direct category with dimension function the identity. Presheaves on it are called *globular sets* and they can be visualised, due to the globularity conditions, as collections of directed disks.

The arities for the signature Σ_{cat} are given by a family of globular sets indexed by rooted planar trees, which we call *Batanin trees* [7]. For alternative descriptions of this families, see [9, 14, 25]. This family consists of a globular set Bat with $s_n^* = t_n^* = \partial_n$. It consists moreover of a globular set $\mathrm{Pos}\,B$ for every Batanin tree B, and a pair of inclusions

$$s_n^B : \mathrm{Pos}(\partial_n B) \to \mathrm{Pos}\,B \qquad\qquad t_n^B : \mathrm{Pos}(\partial_n B) \to \mathrm{Pos}\,B$$

for every tree $B \in \mathrm{Bat}_{n+1}$ satisfying the globularity conditions.

This family of globular sets is well-studied, since it familially represents the free strict ω-category monad. While in a strict ω-category, diagrams of cells indexed by a Batanin tree admit a unique composite, in an arbitrary ω-category they only admit unique composite up to a higher coherence cell. This has be made precise in various equivalent ways using *contractible globular operads* [7, 25], *coherators* for ω-categories [27], and the type theory *CaTT* [15]. Equivalences between those approaches and the one below have already been established [4, 8, 14].

The signature $\Sigma_{\mathrm{cat}} = (\Sigma_{\mathrm{cat},n})_{n\in\mathbb{N}}$ is defined recursively. There are no function symbols of sort 0, since there should be no way to compose objects. The signature $\Sigma_{\mathrm{cat},n+1}$ has a family of function symbols $\mathrm{coh}_{B,A}$ where $B \in \mathrm{Bat}_{n+1}$ is a Batanin tree and $A = (a, b)$ is a pair of terms of $\mathrm{Pos}\,B$ of sort n such that

- the source and target of a and b coincide,
- $a = \mathrm{M}_{\Sigma_{\mathrm{cat},n}}(s_n^B)(a')$ for some term a' of $\mathrm{Pos}(\partial_n B)$ corresponding to an epimorphism,
- $b = \mathrm{M}_{\Sigma_{\mathrm{cat},n}}(t_n^B)(b')$ for some term b' of $\mathrm{Pos}(\partial_n B)$ corresponding to an epimorphism.

The arity of $\mathrm{coh}_{B,A}$ is given by $\mathrm{tr}_{n+1}\,\mathrm{Pos}\,B$ and its top dimensional boundary terms are given by

$$t_{\mathrm{coh}_{B,A},s_n} = a \qquad\qquad t_{\mathrm{coh}_{B,A},t_n} = b.$$

The rest of the boundary terms are determined uniquely by the cocycle conditions. The motivation for this choice of signature is that a', b' represents ways to compose the boundary of B, which should determine a canonical way to compose B. When $B \in \mathrm{Bat}_{n+1}$ satisfies that $s_n^B = t_n^B = \mathrm{id}$, that is it has dimension at most n, the a' and b' correspond to different ways to compose B and $\mathrm{coh}_{B,A}$ to a coherence cell between the composites.

Similarly, we can build a signature Σ_{grpd} for ω-groupoids by allowing as functional symbols $\mathrm{coh}_{B,A}$ all pairs of a Batanin tree $B \in \mathrm{Bat}_{n+1}$ and pairs of terms $A = (a, b)$ with common source and target. The motivation behind this definition being that the geometric realisation of the globular sets $\mathrm{Pos}\,B$ is contractible, so the free ω-groupoid on them should be trivially fibrant.

Multiple categories Strict multiple categories are an infinite dimensional generalisation of n-fold categories. They consist of a set of objects $X_\emptyset$, a set of

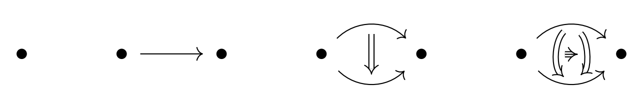

Figure 1: The first four representable globular sets.

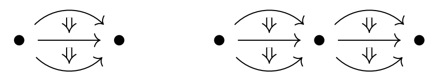

Figure 2: Batanin trees corresponding to vertical composition of 2-cells and the interchange axiom respectively.

arrows X_i for every direction $i \in \mathbb{N}$ and more generally a set of n-dimensional cubes X_I for every $I \subseteq \mathbb{N}$ of cardinality n, together with face maps and associative, unital composition operations

$$+_i : X_I \times_{X_{I \setminus \{i\}}} X_I \to X_I$$

for every $i \in I$ satisfying the usual interchange law. Weaker versions of them were recently introduced [17], where the composition $+_0$ is strictly associative and unital, while the rest are only associative and unital up to a higher cell. Here, we propose an alternative unbiased version of multiple category that is weak in all directions.

We start from the category $\mathbb{M}_+$ with objects finite subsets of the natural numbers and a morphism

$$\delta_J^\alpha : I \setminus J \to I$$

for $J \subseteq I$ and $\alpha : J \to \{0, 1\}$. Composition of morphisms is given by disjoint union

$$\delta_J^\alpha \delta_K^\beta = \delta_{J \cup K}^{(\alpha,\beta)}$$

where $(\alpha, \beta) : J \cup K \to \{0, 1\}$ is the map induced by the universal property of the disjoint union. Equivalently, morphisms in $\mathbb{M}_+$ are generated by the face maps

$$\delta_i^\alpha : I \setminus \{i\} \to I$$

for $i \in I$ and $\alpha \in \{0, 1\}$ under the commutativity condition

$$\delta_i^\alpha \delta_j^\beta = \delta_j^\beta \delta_i^\alpha.$$

Clearly, $\mathbb{M}_+$ is a direct category with dimension given by cardinality. We will call presheaves on it, *semi-multiple sets*.

Iterating the composition operations in a strict multiple category, one can compose grid-shaped arrays of cubes in a unique manner. Since grids are determined uniquely by their number of cubes in each direction, we let Grid_I be the set of sequences $G : I \to \mathbb{N}$ and we define for $J \subseteq I$,

$$d_J = (\delta_J^\alpha)^* : \mathrm{Grid}_I \to \mathrm{Grid}_{I \setminus J}$$

the function forgetting the values of a sequence at J. This assignment easily defined a semi-multiple set.

The semi-multiple set $\operatorname{Pos} G$ of positions of a grid $G : I \to \mathbb{N}$ consists of the cubes of that grid. Identifying each cube with its vertex closest to the origin,

$$
\begin{tikzcd}
\bullet \arrow[r, "{(0,1)}"] & \bullet \arrow[r, "{(1,1)}"] & \bullet \arrow[r, "{(2,1)}"] & \bullet \arrow[r, "{(3,1)}"] & \bullet \\
\bullet \arrow[r, "{(0,0)}"'] \arrow[u, "{(0,0)}"] \arrow[ur, Rightarrow, "{(0,0)}"] & \bullet \arrow[r, "{(1,0)}"'] \arrow[u, "{(1,0)}"] \arrow[ur, Rightarrow, "{(1,0)}"] & \bullet \arrow[r, "{(2,0)}"'] \arrow[u, "{(2,0)}"] \arrow[ur, Rightarrow, "{(2,0)}"] & \bullet \arrow[r, "{(3,0)}"'] \arrow[u, "{(3,0)}"] \arrow[ur, Rightarrow, "{(3,0)}"] & \bullet \arrow[u, "{(4,0)}"]
\end{tikzcd}
$$

Figure 3: The semi-multiple set of positions of the grid $G = (4, 1, 0, 0, \dots)$.

we obtain the following description. The set $\operatorname{Pos}_J G$ is empty unless $J \subseteq I$, in which case it consists of points $R \in \mathbb{N}^I$ such that $R(i) \le G(i)$ for all $i \in I$ and the inequality holds strictly when $i \in J$. The face $d_K^\alpha R \in \operatorname{Pos}_{J \setminus K} G$ is given by

$$
(d_K^\alpha R)(i) = \begin{cases} R(i) + \alpha(i), & \text{when } i \in K \\ R(i), & \text{when } i \in I \setminus K \end{cases}
$$

Moreover, for every grid $G \in \operatorname{Grid}_I$, subset $J \subseteq I$ and morphism $\delta_J^\alpha : I \setminus J \to I$, there exists an inclusion

$$
\begin{aligned}
\delta_J^{\alpha,G} &: \operatorname{Pos}(d_J G) \to \operatorname{Pos} G \\
(\delta_J^{\alpha,G} R)(i) &= \begin{cases} R(i), & \text{when } i \in I \setminus J \\ \alpha(i) \cdot G(i), & \text{when } i \in J \end{cases}
\end{aligned}
$$

Those data forms precisely a family of semi-multiple sets.

The signature $\Sigma_{\mathrm{mcat}} = (\Sigma_{\mathrm{mcat},n})_{n \in \mathbb{N}}$ is defined recursively, similar to Σ_{cat}. The signature $\Sigma_{\mathrm{mcat},0}$ contains no function symbols of sort $\emptyset$. The signature $\Sigma_{\mathrm{mcat},n}$ for $n > 0$ has one family of function symbols of arity I for every subset $I \subset \mathbb{N}$ of size n. The family consists of symbols $\operatorname{coh}_{G,A}$, where $G \in \operatorname{Grid}_I$ is a grid and $A = (t_i^\alpha)_{i \in I, \alpha \in \{0,1\}}$ is a collection of $\Sigma_{\mathrm{mcat},n}$-terms of $\operatorname{Pos} G$ of sort $I \setminus \{i\}$ such that

- $(\delta_j^\beta)^* t_i^\alpha = (\delta_i^\alpha)^* t_j^\beta$,
- $t_i^\alpha = \mathrm{M}_{\Sigma_{\mathrm{mcat},n}}(\delta_i^{\alpha,G})(t_i^{\alpha\prime})$ for some term $t_i^{\alpha\prime}$ corresponding to an epimorphism.

The arity of $\operatorname{coh}_{G,A}$ is given by $\operatorname{tr}_n \operatorname{Pos} G$ and its top dimensional boundary terms are given by

$$
t_{\operatorname{coh}_{G,A}, \delta_i^\alpha} = t_i^\alpha.
$$

The rest of its boundary terms are determined by the cocycle conditions.

A A factorisation system for computads

As shown in Theorem 10, the category of computads and variable-to-variable morphisms is a presheaf topos. As a consequence, every variable-to-variable morphism admits a factorisation into an epimorphism followed by a monomorphism. It turns out that this is true more generally in the category of computads and arbitrary morphisms: epimorphisms and variable-to-variable monomorphisms form an orthogonal factorisation system.

The inclusion $\zeta_\Sigma : \mathrm{Comp}_\Sigma^{\mathrm{var}} \hookrightarrow \mathrm{Comp}_\Sigma$ is faithful and preserves connected limits and colimits, so it preserves and reflects both epimorphisms and monomorphisms. Using that the nerve functor of Theorem 10 is an equivalence, we see that a variable-to-variable morphism $\sigma : C \to D$ is a monomorphism of computads exactly when the functions $\sigma_i : V_i^C \to V_i^D$ are injective for every sort i.

In order to characterise epimorphisms of computads, we introduce the *support* of a morphism of computads. Intuitively the support captures the generators of the target that are used in the definition of the morphism. We will see that a morphism is an epimorphism exactly when its support contains every generator of its target.

Definition 23. Let C a Σ-computad and i a sort. The *support of sort i* of a term $t \in \mathrm{Term}_{\Sigma,j}(C)$ of some sort j of dimension $\beta \le \alpha$ is defined recursively by

$$\begin{aligned}
\mathrm{supp}_i(\mathrm{var}\, v) &= \{v \ : \ i = j\} \cup \bigcup_{\mathrm{id} \neq \delta : k \to j} \mathrm{supp}_i(\phi_\delta^C(v)) \\
\mathrm{supp}_i(f[\tau]) &= \bigcup_{\substack{k \in \mathcal{I}_\beta \\ b \in B_{f,k}}} \mathrm{supp}_i(\tau_k(b))
\end{aligned}$$

The *support of sort i* of a morphism $\sigma : D \to C$ is defined by

$$\mathrm{supp}_i(\sigma) = \bigcup_{\substack{j \in \mathcal{I}_\alpha \\ v \in V_j^D}} \mathrm{supp}_i(\sigma_j(v)).$$

Recursively, we can see that the support of sort i of a term of sort j is empty unless there exists a morphism $\delta : j \to i$, which in particular implies that $\dim j \le \dim i$. The following lemma shows that the support is closed under boundary maps, which implies in particular that the support of a term and the morphism classifying it coincide. Its proof is a simple mutual induction, left to the reader.

Lemma 24. *Let C a computad, i a sort, and $t \in \mathrm{Term}_{\Sigma,j}(C)$ a term of C of some sort j of dimension $\beta \le \alpha$. For every face map $\delta : k \to j$,*

$$\mathrm{supp}_i(\delta^* t) \subseteq \mathrm{supp}_i(t),$$

while for every morphism of Σ-computads $\sigma : C \to D$,

$$\mathrm{supp}_i(\mathrm{Term}_\Sigma(\sigma)(t)) = \bigcup_{\substack{k \in \mathcal{I}_\beta \\ v \in \mathrm{supp}_k(t)}} \mathrm{supp}_i(\sigma_k(v)).$$

Proposition 25. *Two morphisms of computads agree on a term if and only if they agree on every generator of its support.*

Proof. Let $\sigma, \sigma' : C \to D$ morphisms of computads. By induction on the dimension of the signature, it suffices to show the result for terms t of C of sort j of dimension α. Suppose first that $t = \operatorname{var} v$ is a generator. Since $v \in \operatorname{supp}_j(v)$, one direction obviously holds. For the converse, suppose that σ and σ' agree on t. Then they must also agree on $\phi^C_\delta(v)$ for every $\delta : k \to j$. By the inductive hypothesis, they agree on the support of all $\phi^C_\delta(v)$, hence on the support of t as well.

Suppose now that $t = f[\tau]$. Then σ and σ' agree on t if and only if $\sigma\tau = \sigma\tau'$. This amounts to them agreeing on $\tau_k(b)$ for every $k \in \mathcal{I}_{\dim j}$ and $b \in B_{f,k}$. By the inductive hypothesis, this is equivalent to agreeing on the support of each $\tau_k(b)$. Equivalently, they agree on the union of those supports, which is the support of t. ☐

Corollary 26. *If the support of a morphism contains every generator, then it is an epimorphism.*

The converse of this corollary also holds. We will deduce it by showing that every morphism of computads factors uniquely as one whose support contains all generators followed by one a variable-to-variable monomorphism. the following lemma concerning the support of a variable-to-variable morphism can easily be shown again by induction on depth.

Lemma 27. *The support of a variable-to-variable morphism $\sigma : C \to D$ consists precisely of the generators in the images of the functions σ_i.*

Lemma 28. *Let $\rho : D \to E$ a variable-to-variable monomorphism and let $\sigma : C \to D$ arbitrary morphism. Then $\sigma = \rho\sigma'$ for some $\sigma' : C \to D$ if and only if the support of σ is contained in that of ρ. Moreover, the factorisation is unique.*

Proof. Uniqueness of the factorisation follows by ρ being a monomorphism, while necessity of this condition for the existence of a factorisation follows by Lemma 24. By induction on the dimension of the signature, we may assume that this condition is sufficient for morphisms of Σ_β-computads for all $\beta < \alpha$.

We proceed by induction on the depth of σ. Suppose first that $C = \mathbb{D}^i_\Sigma$ and that σ classifies a generator $t = \operatorname{var} v$. By Lemma 27, there exists $v' \in V^D_i$, mapped via ρ to t. The morphism corresponding to $\operatorname{var} v'$ gives the claimed factorisation. If σ classifies a term $t = f[\tau]$, then

$$\operatorname{supp}_i(\sigma) = \operatorname{supp}_i(t) = \operatorname{supp}_i(\tau),$$

so by the inductive hypothesis, there exists a factorisation $\tau = \rho\tau'$. The morphism corresponding to $f[\tau']$ gives a factorisation of σ via ρ.

Let now C and σ be arbitrary. By the inductive hypothesis, there exist unique $\sigma'_\beta : C_\beta \to D_\beta$ such that $\sigma_\beta = \rho_\beta\sigma'_\beta$ for all $\beta < \alpha$ and unique terms

$\sigma'_i(v) \in \mathrm{Term}_\Sigma(D)$ for every generator $v \in V_i^C$ of dimension α such that

$$\mathrm{Term}_\Sigma(\rho)(\sigma'_i(v)) = \sigma_i(v).$$

By uniqueness, this data assembles into a morphism $\sigma' : C \to D$ such that $\sigma = \rho\sigma'$. $\square$

Proposition 29. *Every morphism can be factored as a morphism whose support contains all generators, followed by a variable-to-variable monomorphism.*

Proof. Let $\sigma : C \to D$ arbitrary morphism. We will construct recursively a variable-to-variable monomorphism $\iota_\sigma : \operatorname{supp}\sigma \to D$ where the generators of $\operatorname{supp}\sigma$ are given by $V_i^{\operatorname{supp}\sigma} = \operatorname{supp}_i \sigma$, and σ_i is the obvious subset inclusion. The gluing maps of $\operatorname{supp}\sigma$ are defined recursively: given a face map $\delta : j \to i$, and some $v \in \operatorname{supp}_i \sigma$, by Lemma 28, there exists unique term $\phi_\delta^{\operatorname{supp}\sigma}(v)$ of $\operatorname{supp}\sigma$ satisfying that

$$\mathrm{Term}_{\Sigma_\beta}(\iota_{\sigma,\beta})\phi_\delta^{\operatorname{supp}\sigma}(v) = \phi_\delta^C(v),$$

where $\beta = \dim j$. Uniqueness implies the cocycle condition $\delta^*\phi_{\delta'}^{\operatorname{supp}\sigma} = \phi_{\delta'\delta}^{\operatorname{supp}\sigma}$, so this data defines a computad $\operatorname{supp}\sigma$ and a variable-to-variable monomorphism out of it. Moreover, by construction,

$$\operatorname{supp}_i(\iota_\sigma) = \operatorname{supp}_i(\sigma),$$

so by the same lemma, there exists unique

$$\pi_\sigma : C \to \operatorname{supp}\sigma$$

such that $\sigma = \pi_\sigma\iota_\sigma$. A simple recursive argument shows that the support of π_σ contains all generators of $\operatorname{supp}\sigma$. $\square$

Corollary 30. *The support of epimorphisms contains all generators.*

Proof. If σ is an epimorphism, then the variable-to-variable monomorphism ι_σ is also epic. Since $\mathrm{Comp}_\Sigma^{\mathrm{var}}$ is a topos, ι_σ must be invertible, hence bijective on generators. Therefore, its support contains all generators. Such morphisms are closed under composition by Lemma 24, so the support of $\sigma = \pi_\sigma\iota_\sigma$ must contain all generators as well. $\square$

Corollary 31. *Epimorphisms and variable-to-variable monomorphisms form an orthogonal factorisation system.*

Proof. Both classes contain isomorphisms and they are closed under composition. Moreover, every morphism factors as an epimorphism followed by a variable-to-variable monomorphism, so it remains to show tha the factorisation is unique up to unique isomorphism [2, Proposition 14.7]. Uniqueness follows from the left class being epimorphism, so it remains to show existence. For that, let $\sigma : C \to D$ an epimorphism, $\rho : D \to E$ a variable-to-variable monomorphism and consider the commutative square

$$\begin{tikzcd} C \arrow[r, "\pi_{\rho\sigma}"] \arrow[d, "\sigma"'] & \operatorname{supp}\sigma \arrow[d, "\iota_\sigma"] \\ D \arrow[r, "\rho"] \arrow[ur, dashed, "\chi"] & E \end{tikzcd}$$

Since σ is epic, the supports of ρ, $\sigma\rho$ and $\iota_{\sigma\rho}$ coincide. Two applications of Lemma 28 give a diagonal lift χ as in the diagram and a lift χ^{-1} in the opposite direction. By definition,

$$\iota_{\rho\sigma}\chi\chi^{-1} = \iota_{\rho\sigma} \qquad\qquad \rho\chi^{-1}\chi = \rho.$$

Since ρ and $\iota_{\rho\sigma}$ are monic, the morphism χ is an isomorphism with inverse χ^{-1}. $\square$

Corollary 32. *The category of computads is Cauchy-complete.*

Proof. Let $\sigma : C \to C$ an idempotent morphism of computads. By Corollary 31, there exists a factorisation $\sigma = \iota\pi$ with π epic and ι monic. Since $\sigma\sigma = \sigma$, it follows that $\pi\iota = \mathrm{id}$. Therefore, every idempotent morphism of computads splits. $\square$

B Some properties of the term monad

We conclude with some technical properties of the term monad, namely that it is cartesian and accessible. From the latter, we will deduce that the category of Σ-algebras is locally presentable, hence complete, cocomplete, and every set of morphisms cofibrantly generates a weak factorisation system. As usually, Σ will denote an $\mathcal{I}$-sorted signature of some dimension α.

Proposition 33. *The term monad is cartesian.*

Proof. Let $\mathrm{Cptd}^{\mathrm{var}}_\Sigma : [\mathcal{I}^{\mathrm{op}}_\alpha, \mathrm{Set}] \to \mathrm{Comp}^{\mathrm{var}}_\Sigma$ the restriction of the inclusion of presheaves into computads to the subcategory of variable-to-variable morphisms. The composite $V^\bullet_i \circ \mathrm{Cptd}^{\mathrm{var}}_\Sigma$ is cocontinuous for every sort i, since it sends a presheaf X to the set X_i. By the decomposition of $V^\bullet_i$ in Section 3, the functor $\mathcal{T}_p \circ \mathrm{Cptd}^{\mathrm{var}}_\Sigma$ preserve connected limits for every plex p. By Theorem 10, $\mathrm{Cptd}^{\mathrm{var}}_\Sigma$ must preserve them as well. Corollary 8 finally implies that the composite $\mathrm{M}_\Sigma = \mathrm{Term}^{\mathrm{var}}_\Sigma \mathrm{Cptd}^{\mathrm{var}}_\Sigma$ must preserve them as well.

It remains to show that η_Σ and μ_Σ are cartesian natural transformations. For the former, let $\sigma : X \to Y$ a morphism of presheaves and consider the naturality square

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\eta_{\Sigma,X}"] & Y \arrow[d, "\eta_{\Sigma,Y}"] \\ \mathrm{M}_\Sigma(X) \arrow[r, "\mathrm{M}_\Sigma(\sigma)"] & \mathrm{M}_\Sigma(Y) \end{tikzcd}$$

Pullbacks in categories of presheaves are computed object-wise, so we need to show for every sort i that pairs $t \in \mathrm{M}_{\Sigma,i}(X)$ and $y \in Y_i$ that satisfy the compatibility condition

$$\mathrm{M}_\Sigma(\sigma)(t) = \eta_{\Sigma,Y}(y) = \operatorname{var} y$$

can be lifted uniquely to an element of X_i. From the compatibility condition, we deduce that t must be a generator, so $t = \eta_{\Sigma,X}(x)$ for unique $x \in X_i$. Substituting t into the compatibility condition, we get then that $y = \sigma(x)$, so the square is a pullback.

The multiplication of the monad can be written as the following whiskered composite

$$\mu_\Sigma = \mathrm{Term}_\Sigma\, \varepsilon_\Sigma(\zeta_\Sigma\, \mathrm{Cptd}_\Sigma^{\mathrm{var}}).$$

The functor of Σ-terms is representable by $\mathbb{D}_\Sigma^i$, so it preserves pullbacks. Therefore, it suffices to show that $\varepsilon_\Sigma\zeta_\Sigma$ is cartesian. Let therefore $\rho : C \to D$ a variable-to-variable morphism of computads and suppose that a solid commutative diagram of the following form is given

$$\begin{tikzcd}
E \arrow[dr, dashed, "\tau"] \arrow[drr, bend left, "\tau^1"] \arrow[ddr, bend right, "\tau^2"'] & & \\
& \mathrm{Cptd}_\Sigma\, \mathrm{Term}_\Sigma\, C \arrow[r, "\mathrm{Cptd}_\Sigma\, \mathrm{Term}_\Sigma\, \rho"] \arrow[d, "\varepsilon_{\Sigma,C}"] & \mathrm{Cptd}_\Sigma\, \mathrm{Term}_\Sigma\, D \arrow[d, "\varepsilon_{\Sigma,D}"] \\
& C \arrow[r, "\rho"] & D
\end{tikzcd}$$

By induction on the dimension of the signature, we may assume that $\varepsilon_{\Sigma_\beta}\zeta_{\Sigma_\beta}$ is cartesian for all $\beta < \alpha$. We proceed by recursion on the depth of τ^1 to show that there exists unique τ making the entire diagram above commute.

Suppose first that $E = \mathbb{D}_\Sigma^i$ for some sort i of sort α, and suppose further that τ^1 classifies a generator $t_1 = \operatorname{var} t_1'$ where $t_1' \in \mathrm{Term}_{\Sigma,i}(D)$. Then τ^2 classifies some term $t_2 \in \mathrm{Term}_{\Sigma,i}(C)$ such that

$$\mathrm{Term}_\Sigma(\rho)(t_2) = t_1'.$$

The morphism τ must classify some generator for the lower triangle to commute, and that generator must by $\operatorname{var}(t_2)$ for the left triangle to commute. Conversely, the morphism classifying $\operatorname{var} t_2$ makes the diagram commute, so there exist unique τ making the diagram commute.

Suppose then that τ^1 classifies a composite term $t_1 = f[\hat{\tau}^1]$ and let t_2 the term classified by τ^2. Then

$$\mathrm{Term}_\Sigma(\rho)(t_2) = f[\varepsilon_{\Sigma,D}\hat{\tau}^1]$$

and ρ is variable-to-variable, so $t_2 = f[\hat{\tau}^2]$ for some $\hat{\tau}^2$ satisfying that

$$\rho\hat{\tau}^2 = \varepsilon_{\Sigma,D}\hat{\tau}^1.$$

By the recursive hypothesis, there exists unique $\hat{\tau} : B_f \to \mathrm{Cptd}_\Sigma \mathrm{Term}_\Sigma C$ such that

$$\hat{\tau}^1 = (\mathrm{Cptd}_\Sigma \mathrm{Term}_\Sigma \rho)\hat{\tau} \qquad\qquad \hat{\tau}^2 = \varepsilon_{\Sigma,C}\hat{\tau}.$$

The morphism τ corresponding to the term $f[\hat{\tau}]$ makes the diagram above commute, and it is easily seen to be unique.

Finally, let E be arbitrary. Then by the recursive hypothesis, for every $\beta < \alpha$, there exists unique $\tau_\beta : E_\beta \to \mathrm{tr}^\Sigma_\beta \mathrm{Cptd}_\Sigma \mathrm{Term}_\Sigma C$ making the obvious truncated versions of the diagram above and there exist for every $v \in V_i^E$ for i of dimension α, unique term $\tau_i(v)$ of $\mathrm{Cptd}_\Sigma \mathrm{Term}_\Sigma C$ such that

$$\mathrm{Term}_\Sigma(\varepsilon_{\Sigma,C})(\tau_i(v)) = \tau_i^2(v) \qquad (M_\Sigma \mathrm{Term}_\Sigma \rho)(\tau_i(v)) = \tau_i^1(v).$$

By uniqueness, we can easily deduce that those morphisms and terms constitute a morphism $\tau : C \to D$ making the diagram above commute. It is not hard to see that said τ is unique. $\square$

Proposition 34. *The monad* M_Σ *preserves* λ*-filtered colimits for some regular cardinal* λ*.*

Proof. Let λ a regular cardinal strictly greater than the cardinality of the disjoint union $\coprod_j B_{f,j}$ for every sort $i \in \mathcal{I}_\alpha$, and every function symbol $f \in \Sigma_{i,F}$. Such λ exists, since both the category of sorts and the collection of function symbols are small. To show that M_Σ preserves λ-filtered colimits, it suffices to show that $\mathrm{M}_{\Sigma,i}$ preserves them for every sort i. By induction on the dimension α of the signature, we may assume that $\mathrm{M}_{\Sigma_\beta}$ preserves λ-filtered colimits for all $\beta < \alpha$, so it remains to show that $\mathrm{M}_{\Sigma,i}$ preserves them for i of dimension α.

In order to do that, define for every ordinal $\gamma \le \lambda$,

$$\mathrm{M}^\gamma_{\Sigma,i} : [\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Set}$$

to be the functor sending a presheaf X to the set of terms of $\mathrm{Cptd}_\Sigma X$ of sort i and recursive depth at most γ. Define also for every function symbol $f \in \Sigma_i$, a functor

$$\mathrm{M}^\gamma_{\Sigma,i,f} : [\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Set}$$

sending a presheaf X to the set of morphisms $\mathrm{Cptd}_\Sigma(B_f) \to \mathrm{Cptd}_\Sigma(X)$ of depth at most β. The discussion on recursive depth shows that for $\beta = \lambda$, we recover the sets of all terms of sort i, and all morphisms respectively, so it suffices to show that those functors preserve λ-filtered colimits.

We proceed recursively on the ordinal γ. The functor $\mathrm{M}^0_{\Sigma,i}$ is cocontinuous, since it sends X to the set X_i. The functor $\mathrm{M}^{\gamma+1}_{\Sigma,i}$ for $\gamma < \lambda$ decomposes as

$$\mathrm{M}^{\gamma+1}_{\Sigma,i} = \mathrm{M}^0_{\Sigma,i} \amalg \coprod_{f \in \Sigma_i} \mathrm{M}^\gamma_{\Sigma,i,f}$$

while for $\gamma \le \lambda$ limit ordinal, we have that

$$\mathrm{M}^\gamma_{\Sigma,i} = \operatorname*{colim}_{\gamma' < \gamma} \mathrm{M}^{\gamma'}_{\Sigma,i}.$$

In both cases, we see that the functor preserves λ-filtered colimits by the inductive hypothesis and commutativity of colimits with colimits. Finally, given any function symbol $f \in \Sigma_i$ and $\gamma \leq \lambda$ arbitrary, consider the functor from the category of elements of the presheaf B_f sending $j \in \mathcal{I}_\alpha$ and $b \in B_{f,j}$ to the set $\mathrm{M}_{\Sigma,j}$ when $\dim j < \alpha$, and to $\mathrm{M}^\gamma_{\Sigma,j}$ otherwise. The domain of the functor is λ-small and its limit is $\mathrm{M}^\gamma_{\Sigma,i,f}$. By the inductive hypothesis and commutativity of λ-small limits with λ-filtered colimits, we see that $\mathrm{M}^\gamma_{\Sigma,i,f}$ preserves λ-filtered colimits. This concludes the induction. □

Corollary 35. *The category of algebras is locally presentable.*

Proof. Let λ a regular cardinal such that M_Σ preserves λ-filtered colimits. The category $[\mathcal{I}_\alpha^{\mathrm{op}}, \mathrm{Set}]$ is locally finitely presentable, being a presheaf category [3, Example 1.1.12], hence also λ-presentable [3, Remark 1.1.20]. The category of algebras is therefore the category of algebras of a λ-accessible monad on a locally λ-presentable category. Hence, it is also a locally λ-presentable category [3, Remark 2.2.78]. □

Acknowledgements

I would like to thank my supervisor Prof. Jamie Vicary for his support during this project. Furthermore, I would like to acknowledge funding from the Onassis foundation - Scholarship ID: F ZQ 039-1/2020-2021.

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