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Computads for weak ω-categories as an inductive type

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Abstract

We give a new description of computads for weak globular ω-categories by giving an explicit inductive definition of the free words. This yields a new understanding of computads, and allows a new definition of ω-category that avoids the technology of globular operads. Our framework permits direct proofs of important results via structural induction, and we use this to give new proofs that every ω-category is equivalent to a free one, and that the category of computads with variable-to-variable maps is a presheaf topos, giving a direct description of the index category. We prove that our resulting definition of ω-category agrees with that of Batanin and Leinster and that the induced notion of cofibrant replacement for ω-categories coincides with that of Garner.

1 Introduction

Motivation. Given any type of algebraic structure in mathematics, a basic question is to understand the data from which one can generate it freely. Just as we generate a free monoid from a set, or a free category from a directed graph, we may generate a free n-category from an *n-computad*, in a way which is described by an adjunction between appropriate categories:

$$\mathsf{Set} \rightleftarrows \mathsf{Mon} \qquad \mathsf{DGraph} \rightleftarrows \mathsf{Cat} \qquad n\mathsf{Comp} \underset{R_n}{\overset{F_n}{\rightleftarrows}} \omega\mathsf{Cat}$$

Today these computads play a substantial role in higher category theory and its applications, including in rewriting [25], word problems and homology [11, 21, 23], homotopy theory [18] and topological quantum field theory [5].

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Initiated by Street [28] with the study of 2-computads, the theory of n-computads is now well-developed, with key contributions by Batanin [6], Garner [18] and others. Modern treatments typically proceed as follows [18, Section 5]. First, a category $\omega\mathsf{Cat}$ of ω-categories and strict ω-functors is defined as the category of algebras for a finitary monad on globular sets. Then, a theory of n-computads is developed inductively: a 0-computad is declared to be a set, and an $(n+1)$-computad declared as a triple (C, V, ϕ), where C is an underlying n-computad, V is a set of generating variables, and ϕ is a function assigning to each $v \in V$ its source and target in $F_n(C)$, the free ω-category generated by C. The final step is to define F_{n+1}, which is accomplished by a colimit construction in $\omega\mathsf{Cat}$. This is guaranteed to be successful thanks to general theorems about cocompleteness properties of categories of algebras of finitary monads [1, 2]. By induction, the construction is therefore complete.

This powerful method quickly establishes that computads are well-defined mathematical objects. However, the resulting definition is not fully explicit: while general theory tell us the necessary colimits must exist, their construction involves an infinite sequence of pushouts [1, Section I.5], which exhibits the set of free words in each dimension as an intricate quotient of a larger space. This technology exhibits the final space via a universal property, but does not give a straightforward description of the resulting quotient space.

In our work we show that this quotient space does in fact admit a direct description in each dimension, as a family of inductive sets. This yields an elementary and fully explicit definition of computads for weak globular ω-categories.

Since our definition of computad is concrete and elementary, it allows us to demonstrate known results about computads in new and simple ways. We show that our definition reproduces the universal cofibrant replacement of Garner, giving an explicit description of this construction for the case of ω-categories, and showing in particular that every ω-category is equivalent to one which is free on a computad. Furthermore, we show that the category of computads with structure-preserving maps is a presheaf topos, and give what we believe is the first explicit description of the index category. We also show that the definition of ω-category arising from our work precisely agrees with the notion of weak ω-category described by Batanin and Leinster.

Inductive structure. Our definition of n-computads is *inductive* in each dimension n, in the following sense. As the definition proceeds, all the necessary sets are described via *constructors*, which allow us to uniquely exhibit any particular element, without requiring that we pass to a quotient. Here we make use of *structural induction*, a powerful technique which enjoys wide use in theoretical computer science, and which generalizes the ordinary mathematical notion of induction over the natural numbers, allowing in cases such as ours more efficient definitions and proofs. We give an introduction to this concept in Appendix A.

Since the standard approach to computads requires constructing quotient spaces, it follows that our definition is different to the standard one obtained via the theory of locally presentable categories, although the definitions are certainly isomorphic. To illustrate this point, consider the situation with free monoids. For a set X, we can describe the free monoid on X as having an underlying set given by binary trees with leaves in $X \coprod \{\star\}$, modulo a quotient given by the the associativity and unit laws. This is of course well-defined, but not purely inductive, due to the requirement to take a quotient. Nonetheless, the resulting quotient *does* admit a direct inductive description, up to isomorphism—the set of words on X—and this inductive structure is convenient for many purposes.

The same question can be asked for many sorts of algebraic structure: does the underlying set of a free instance admit an inductive description? For 2-categories this is much harder to achieve, perhaps impossible, due to the complex nature of the quotient induced by the interchange law. For n-categories for $n > 2$ it is harder still, whether strict or weak, due to the complex set of equational constraints that must be admitted, and there seems no good reason to expect a positive answer.

Our demonstration of this freeness property for ω-categories may therefore be surprising. Furthermore, since every ω-category is equivalent to a free one (a phenomenon we discuss below), something even stronger is true: the entire theory of ω-categories and functors can be handled in terms of completely free algebraic structures.

That such a presentation of the theory of computads might exist was suggested by the recent development of a purely syntactic, type-theoretic presentation of the theory of ω-categories [15]. This theory was, in turn, itself motivated by deep connections between homotopy theory, higher category theory and dependent type theory [30]. We feel the present work is a nice example of how ideas from type theory can lead to insight even in classical mathematics.

Computad-first. Our approach yields a category of computads Comp as the primary object of study, which we construct directly, without passing via a preexisting definition of ω-category. We then exhibit an adjunction between Comp and the category Glob of globular sets, yielding a monad on Glob, and we define $\omega\mathsf{Cat}$ as the category of algebras for this monad. This gives a new definition of $\omega\mathsf{Cat}$ that avoids the standard technology of globular operads and contractions.

This is in contrast to other approaches to the theory of computads as discussed above, which begin with a definition of $\omega\mathsf{Cat}$, and use it as a building block in the definition of computad. Our work therefore presents a shift in perspective, which generates many of the benefits of our approach. Indeed, we propose that computads, as syntactic objects, are fundamentally simpler structures than ω-categories, and that our approach brings out this simplicity.

Computadic replacement. In a locally finitely presentable category, given a set of morphisms I, usually referred to as *generating cofibrations*, we can apply Quillen's small object argument to obtain a weak factorisation system $(\mathcal{L}, \mathcal{R})$, where $\mathcal{R} := I^{\pitchfork}$ and $\mathcal{L} := {}^{\pitchfork}\mathcal{R}$.[1] We say an object X is *cofibrant* when the initial morphism $0 \to X$ is in $\mathcal{L}$, and furthermore that X is a *cofibrant replacement* for Y when there exists an $\mathcal{R}$-morphism $X \to Y$. It was shown by Garner [18, Section 2] that one can then obtain a *universal cofibrant replacement comonad* (Q, μ, η). This monad generates the cofibrant replacement structure, in the following sense: for any object Y, the object QY is cofibrant, and the counit $\eta_Y : QY \to Y$ is the necessary $\mathcal{R}$-map exhibiting QY as a cofibrant replacement for Y. He shows that this comonad is universally determined by a certain lifting property.

For the globular ω-categories that we study here, we take our generating cofibrations I to be the inclusions $\iota_n : \mathbb{S}^{n-1} \to \mathbb{D}^n$ of spheres as the boundaries of disks. Garner's lifting condition then takes the following form:

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "A"] \arrow[d, "\iota_n"'] & QX \arrow[d, "r_X"] \\
\mathbb{D}^n \arrow[r, "x"'] \arrow[ur, dotted, "\phi"] & X
\end{tikzcd}$$

We give a direct proof of this lifting property by structural induction on our definition of computad. Our proof is elementary, and avoids any use of high-powered categorical machinery. It follows that our notion of computad generates the universal cofibrant replacement of Garner.

To justify our choice of generating cofibrations, let us recall that a basic recursive definition of categorical equivalence may begin as as follows [4, Section 3]: a 0-functor is a 0-equivalence when it is a bijection of sets, and an $(n+1)$-functor is an $(n+1)$-equivalence when it is essentially surjective on objects, and an n-equivalence on hom-n-categories. This definition is not complete, since the "essentially surjective" property requires an understanding of which 1-morphisms are "invertible". However, *identity* 1-morphisms should certainly count as invertible, and so this leads us to the strictly stronger notion of *local equivalence*: a 0-functor is a local 0-equivalence when it is a bijection, and an $(n+1)$-functor is a local $(n+1)$-equivalence when it is surjective on objects, and a local n-equivalence on hom-n-categories. A striking property of this definition is that we check surjectivity at every level, but injectivity is only relevant at level n. For the limiting case of $n = \omega$, a local ω-equivalence then simply involves verifying a surjection at every level, and so we may also call it a *local ω-surjection.* It is now a simple matter to check that the maps which have the right lifting property with respect to the inclusions $\iota_n : \mathbb{S}^{n-1} \to \mathbb{D}^n$ are exactly the local ω-surjections. Since we prove the lifting property, it follows that every ω-category X is indeed equivalent to QX, the free ω-category on the underlying ω-computad of X.

Presheaf structure. The study of presheaf structure on computads has a rich history. A 1-computad is a directed graph, and so the category of 1-computads

[1] For a set of morphisms S, we write $S^{\pitchfork}$ and ${}^{\pitchfork}S$ for the sets of morphisms with the right and left lifting property with respect to S, respectively.

is certainly a presheaf category. The category of 2-computads is also a presheaf category, as shown by Carboni and Johnstone [12], and given a modern presentation by Leinster [22, Section 7.6]; this holds in both the strict and weak forms. However, strict 3-computads do not form a presheaf category, as first shown by Makkai and Zawadowski [24], and further elucidated by Cheng [13] and Leinster [22, Section 7.6]. On the other hand, weak n-computads do form a presheaf category, as shown by Batanin [8], in the general setting of algebras for globular operads satisfying a normalization condition.

We use our definition to give an explicit proof of the presheaf property for the category of n-computads, including a direct construction of the index category. We believe this is the first time the index category has been explicitly presented. Roughly speaking, a representable has a unique top-level "dominating variable with generic type". Each cell in the terminal computad induces a canonical diagram of representables in lower levels. The dominating variables appearing in this diagram induce a canonical generic type in its colimit.

In a different direction, Henry [20] identifies subcategories of strict n-computads that are presheaf categories. He then gives for such subcategories a bijection between variables of the terminal computad and *plexes*, the objects of the index category. These plexes are strict computads satisfying a number of properties [20, Proposition 2.2.3], and are analogous to the representable computads that we study.

Comparison with other definitions. The work of Grothendieck, Maltsiniotis, Batanin and Leinster has led to a notion of weak ω-category, as an algebra for the initial contractible globular operad. By work of Berger [9], this operad can be seen as a *globular theory*; that is a category whose objects are Batanin trees (globular pasting diagrams) where morphisms admit a certain factorisation. We give an explicit description of these factorisations for morphisms out of a computad generated by a Batanin tree. In this way we obtain a homogeneous globular theory. Furthermore, we show that this globular theory satisfies the universal property of the globular theory corresponding to the initial contractible globular operad. It follows that the weak ω-categories corresponding to our computads are exactly Batanin-Leinster weak ω-categories.

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2 Batanin trees

Our definition of ω-categorical computad will be based on the existence of a distinguished collection of such objects parameterized by rooted planar trees. We will adopt the name *Batanin trees* here to emphasize their interpretation as parameterizing globular pasting diagrams. As we will be working extensively with inductively generated sets in what follows, we give an inductive characterization of trees as well as their *positions*. For those readers who may find it useful, we present a gentle introduction to inductive sets in Appendix A.

Definition 2.1. The set of Batanin trees is generated inductively by

- For every list $[B_1, \ldots, B_n]$ of Batanin trees, a tree $\mathsf{br}\,[B_1, \ldots, B_n]$.

Equivalently, the set of Batanin trees is an initial algebra for the endofunctor taking a set to the set of lists with elements from it. It be computed as the increasing union of the sequence of sets defined by

$$\begin{aligned} \mathsf{Bat}_0 &= \{\mathsf{br}\,[\,]\} \\ \mathsf{Bat}_{k+1} &= \{\mathsf{br}\,[B_1, \ldots, B_n] \,:\, n \in \mathbb{N},\, B_i \in \mathsf{Bat}_k\} \end{aligned}$$

We define the *dimension* of a Batanin tree B to be the least $k \in \mathbb{N}$ such that $B \in \mathsf{Bat}_k$. It is inductively given by

$$\begin{aligned} \mathsf{dim}\,(\mathsf{br}\,[\,]) &= 0 \\ \mathsf{dim}\,(\mathsf{br}\,[B, B_1, \ldots, B_n]) &= \mathsf{max}\,(\mathsf{dim}\ B + 1, \mathsf{dim}\,(\mathsf{br}\,[B_1, \ldots, B_n])) \end{aligned}$$

Each Batanin tree can be assigned a *globular set* of positions, corresponding to the cells in the pasting diagram it represents. Recall that a globular set X consists of a set X_n of n-*cell*s for every natural number n, together with source and target maps

$$\mathsf{src}, \mathsf{tgt} : X_n \to X_{n-1}$$

satisfying the *globularity conditions*

$$\mathsf{src} \circ \mathsf{src} = \mathsf{src} \circ \mathsf{tgt}, \qquad\qquad \mathsf{tgt} \circ \mathsf{src} = \mathsf{tgt} \circ \mathsf{tgt}\,.$$

They form a category Glob where morphisms $f : X \to Y$ are sequences of functions $f_n : X_n \to Y_n$ preserving the source and target maps. Equivalently, Glob is the category of set-valued presheaves on the category $\mathbb{G}$ with objects natural numbers and morphisms generated by the source and target inclusions

$$s, t : [n] \to [n+1]$$

under the duals of the globularity conditions.

Definition 2.2. For a Batanin tree B, we define the set $\mathsf{Pos}_n(B)$ of *positions of B of dimension n* inductively as follows:

- For every Batanin tree B, a position $\mathsf{ob\text{-}cns}(B) \in \mathsf{Pos}_0(B)$.

- For every $n \in \mathbb{N}$ and every triple (B, L, p) where
  - B is a Batanin tree,
  - $L = [B_1, \ldots, B_n]$ is a list of Batanin trees,
  - $p \in \mathsf{Pos}_n(\mathsf{br}\ L)$ is an n-position of $\mathsf{br}\ L$,

  a position $\mathsf{inr}\,(B, L, p) \in \mathsf{Pos}_n(\mathsf{br}\,[B, B_1, \ldots, B_n])$.
- For every $n \in \mathbb{N}$ and every triple (B, L, q) where
  - B is a Batanin tree,
  - $L = [B_1, \ldots, B_n]$ is a list of Batanin trees,
  - $q \in \mathsf{Pos}_n(B)$ is an n-position of B,

  a position $\mathsf{inl}\,(B, L, q) \in \mathsf{Pos}_{n+1}(\mathsf{br}\,[B, B_1, \ldots, B_n])$.

Every positive dimensional position may be assigned a source and a target, representing the globular structure of the pasting diagram represented by a Batanin tree. For a given Batanin tree B, the source and target maps

$$\begin{aligned}\mathsf{src\text{-}pos} &: \mathsf{Pos}_{n+1}(B) \to \mathsf{Pos}_n(B)\\ \mathsf{tgt\text{-}pos} &: \mathsf{Pos}_{n+1}(B) \to \mathsf{Pos}_n(B)\end{aligned}$$

are defined inductively by the following equations:

$$\begin{aligned}
\mathsf{src\text{-}pos}\,(\mathsf{inr}\,(B, L, q)) &= \mathsf{inr}\,(\mathsf{src\text{-}pos}\ q)\\
\mathsf{src\text{-}pos}\,(\mathsf{inl}\,(B, L, p)) &= \begin{cases}\mathsf{ob\text{-}cns} & \text{if } n = 0,\\ \mathsf{inl}\,(\mathsf{src\text{-}pos}\ p) & \text{if } n \geq 1,\end{cases}\\
\mathsf{tgt\text{-}pos}\,(\mathsf{inr}\,(B, L, q)) &= \mathsf{inr}\,(\mathsf{tgt\text{-}pos}\ q)\\
\mathsf{tgt\text{-}pos}\,(\mathsf{inl}\,(B, L, p)) &= \begin{cases}\mathsf{inr}\,(\mathsf{ob\text{-}cns}) & \text{if } n = 0\\ \mathsf{inl}\,(\mathsf{tgt\text{-}pos}\, p) & \text{if } n \geq 1\end{cases}
\end{aligned}$$

It is not difficult to show by induction the globularity conditions hold, so the positions of B form a globular set $\mathsf{Pos}\,(B)$.

Example 2.3. The *globes* are the Batanin trees defined inductively by

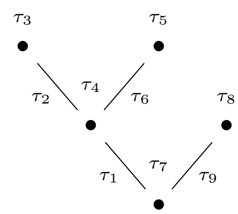
$$D_0 = \mathsf{br}\,[\,], \qquad\qquad D_{n+1} = \mathsf{br}\,[D_n].$$

It is easy to see by induction that the globular set of positions of D_n is isomorphic to the representable globular set $\mathbb{G}(-, [n])$.

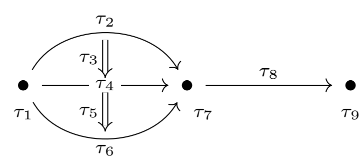
Example 2.4. Batanin trees of dimension at most one are sequences of the form $B_n = \mathsf{br}\,[D_0, \overset{n}{\ldots}, D_0]$ for some natural number $n \in \mathbb{N}$. The globular set of positions of B_n is a sequence of n composable arrows.

$$\bullet \xrightarrow{\mathsf{inl(ob\text{-}cns)}} \bullet \longrightarrow \cdots \longrightarrow \bullet \xrightarrow{\mathsf{inr}^{n-1}\,\mathsf{inl(ob\text{-}cns)}} \bullet$$

Example 2.5. Consider the tree $B = \mathsf{br}\,[\mathsf{br}\,[D_0, D_0],\, D_0]$. It has 9 positions, labelled in the diagram below. Position τ_1 corresponds to $\mathsf{ob\text{-}cns}$, while positions $\tau_2, \ldots, \tau_6$ correspond to the positions of $\mathsf{br}\,[D_0, D_0]$. Finally, $\tau_7, \ldots, \tau_9$ correspond to the positions of $D_1 = \mathsf{br}\,[D_0]$.



The source and target of each position are given by the positions adjacent to the edge below it, so that in particular the globular set $\mathsf{Pos}\,(B)$ corresponds to the following pasting diagram.



Remark 2.6. Batanin trees are in bijection with the zigzag sequences described by Weber [31, Section 4], hence also to the trees defined by Batanin [7]. The sequence corresponding to $\mathsf{br}\,[B_1, \ldots, B_n]$ is obtained from the sequences corresponding to the trees B_i by raising their entries by 1 and concatenating them with a 0 in between. One can easily show by induction that the globular set of positions of a tree B realizes the globular sum of the corresponding sequence.

The *boundary* $\partial_k B$ of a Batanin tree B at some $k \in \mathbb{N}$ is the tree obtained by removing all nodes of height more than k. The positions of the boundary may be included in those of B in two ways, identifying them with the positions in the source or target of the pasting diagram $\mathsf{Pos}\,(B)$. We now proceed to define the boundaries and the *source* and *target inclusions* more formally.

Definition 2.7. The *k-boundary* of a Batanin tree B and $k \in \mathbb{N}$, we define the *k-boundary of* B to be the Batanin tree $\partial_k B$ defined inductively by

$$\begin{aligned} \partial_0\, \mathsf{br}\,[B_1, \ldots, B_n] &= \mathsf{br}\,[\,] \\ \partial_{k+1}\, \mathsf{br}\,[B_1, \ldots, B_n] &= \mathsf{br}\,[\partial_k B_1, \ldots, \partial_k B_n] \end{aligned}$$

We define the source and target inclusions

$$s_k^B, t_k^B : \mathsf{Pos}\,(\partial_k B) \to \mathsf{Pos}\,(B)$$

recursively by

$$s_k^{\mathsf{br}\,[\,]} = t_k^{\mathsf{br}\,[\,]} = \mathsf{id},$$

and for every tree B and list of trees $L = [B_1, \dots, B_n]$,

$$
\begin{aligned}
s_0^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{ob\text{-}cns}) &= \mathsf{ob\text{-}cns}, & t_0^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{ob\text{-}cns}) &= \mathsf{inr}\,(t_0^{\mathsf{br}\ L}(\mathsf{ob\text{-}cns})),\\
s_{k+1}^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{ob\text{-}cns}) &= \mathsf{ob\text{-}cns}, & t_{k+1}^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{ob\text{-}cns}) &= \mathsf{ob\text{-}cns},\\
s_{k+1}^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{inl}\ p) &= \mathsf{inl}\,(s_k^B(p)), & t_{k+1}^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{inl}\ p) &= \mathsf{inl}\,(t_k^B(p)),\\
s_{k+1}^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{inr}\ p) &= \mathsf{inr}\,(s_{k+1}^{\mathsf{br}\ L}(p)), & t_{k+1}^{\mathsf{br}\,[B,B_1,\dots,B_n]}(\mathsf{inr}\ p) &= \mathsf{inr}\,(t_{k+1}^{\mathsf{br}\ L}(p)).
\end{aligned}
$$

It is easy to show by induction that s_k^B and t_k^B are morphisms of globular sets and that they coincide with the cosource and cotarget inclusions described by Weber. In particular, the following properties hold, which may also be verified directly by induction.

Proposition 2.8. *For every Batanin tree B and $k, l \in \mathbb{N}$,*

1. $\mathsf{dim}\,(\partial_k B) = \mathsf{min}\,(k, \mathsf{dim}\ B)$
2. *If $\mathsf{dim}\ B \leq k$, then $\partial_k B = B$ and $s_k^B = t_k^B = \mathsf{id}$.*
3. *If $l < k$, then $\partial_l \partial_k B = \partial_l B$ and*
$$
\begin{aligned}
s_k^B \circ s_l^{\partial_k B} &= t_k^B \circ s_l^{\partial_k B} = s_l^B\\
s_k^B \circ t_l^{\partial_k B} &= t_k^B \circ t_l^{\partial_k B} = t_l^B
\end{aligned}
$$
4. *The morphisms s_k^B and t_k^B are equal and bijective on positions of dimension less than k. Moreover, they are injective on positions of dimension k.*

The *strict ω-category monad* $(\mathsf{fc}^s, \eta^s, \mu^s)$ on the category of globular sets is defined in terms of Batanin trees. The endofunctor fc^s is the familially represented endofunctor sending a globular set X and $n \in \mathbb{N}$ to

$$\mathsf{fc}_n^s\, X = \coprod_{\mathsf{dim}\ B \leq n} \mathsf{Glob}(\mathsf{Pos}\,(B), X).$$

The source and target maps

$$\mathsf{src}, \mathsf{tgt} : \mathsf{fc}_{n+1}^s\, X \to \mathsf{fc}_{n+1}^s\, X$$

are given by precomposition with the source and target inclusions s_n^B and t_n^B respectively. The unit $\eta^s : \mathsf{id} \Rightarrow \mathsf{fc}^s$ sends an n-cell to the morphism out of $\mathsf{Pos}\,(D_n) \cong \mathbb{G}(-, [n])$ that it corresponds under the Yoneda embedding.

Before finishing this section, we introduce two families of positions of a Batanin tree that generate the image of the source and target inclusions respectively. Those positions will help us choose the types in our computads for which we postulate a canonical inhabitant exists.

Definition 2.9. Let B be a Batanin tree and $p \in \mathsf{Pos}_n(B)$ a position of dimension n. We say that p is a *source boundary position* if there does not exist any $q \in \mathsf{Pos}_{n+1}(B)$ such that $\mathsf{tgt\text{-}pos}\ q = p$. Conversely, we say that p is a *target boundary position* of B if there does not exist any $q \in \mathsf{Pos}_{n+1}(B)$ such that $\mathsf{src\text{-}pos}\ q = p$. We write $\partial_n^s(B)$ and $\partial_n^t(B)$ for the set of n-dimensional source and target positions respectively.

It is easy to give an inductive characterization of the source and target boundary positions, similar to the one for all positions. The unique position of $\mathsf{br}\,[\,]$ is both source and target boundary, while for every Batanin tree B and list $L = [B_1, \ldots, B_n]$ of trees

- $\mathsf{ob\text{-}cns}(B, L)$ is source boundary.
- if $p \in \mathsf{Pos}_n(B)$ is source boundary or target boundary, so is $\mathsf{inl}\,(B, L, p)$.
- if $q \in \mathsf{Pos}_n(B)$ is target boundary, so is $\mathsf{inr}\,(B, L, q)$.
- if $q \in \mathsf{Pos}_n(B)$ is source boundary and $L \neq [\,]$, so is $\mathsf{inr}\,(B, L, q)$.

The converses of the last three clauses also hold, so this characterizes all source and target boundary positions of $\mathsf{br}\,[B, B_1, \ldots, B_n]$. In particular, we see that every Batanin tree has a unique source and target boundary position of dimension 0.

Proposition 2.10. *Let B a Batanin tree and $k \in \mathbb{N}$. The image of s_k^B is the globular subset of $\mathsf{Pos}\,(B)$ generated by the source boundary positions of dimension at most k. The image of t_k^B is the globular subset of $\mathsf{Pos}\,(B)$ generated by the target boundary positions of dimension at most k.*

Proof. Let S_{kl}^B and T_{kl}^B the sets of positions of dimension l in the globular subset of $\mathsf{Pos}\,(B)$ generated by the source and target boundary positions of dimension at most k respectively. Since $\mathbb{G}$ is a direct category, both sets are empty when $l > k$ and they satisfy for $l \leq k$ that

$$S_{kl}^B = (\partial_l^s B) \cup \left(\bigcup\nolimits_{p \in S_{k,l+1}^B} \{\mathsf{src\text{-}pos}\ p, \mathsf{tgt\text{-}pos}\ p\} \right),$$

$$T_{kl}^B = (\partial_l^s B) \cup \left(\bigcup\nolimits_{p \in S_{k,l+1}^B} \{\mathsf{src\text{-}pos}\ p, \mathsf{tgt\text{-}pos}\ p\} \right).$$

Since s_k^B and t_k^B are bijective on positions of dimension less than k, we need to show that

- a position of dimension k is in S_{kk}^B exactly when it is source boundary,
- a position of dimension k is in T_{kk}^B exactly when it is target boundary,
- $S_{kl}^B = T_{kl}^B = \mathsf{Pos}_l(B)$ for all $l < k$.

All of those statements follow trivially by the definitions when $B = D_0$ or $k = 0$, so let B_0 a Batanin tree and $L = [B_1, \ldots, B_n]$ a list of Batanin trees and suppose that the statements hold for B_0 and $\mathsf{br}\ L$, and that they hold for $B = \mathsf{br}\,[B_0, B_1, \ldots, B_n]$ when $k = 0$.

For any $k \in \mathbb{N}$, by definition of s^B_{k+1} on positions of dimension $k+1$, $S^B_{k+1,k+1}$ consists of positions of the form $\mathsf{inl}\,(B_0, L, s^{B_0}_k\, p)$ or $\mathsf{inr}\,(B_0, L, s^{\mathsf{br}\ L}_{k+1}\, q)$. By the inductive hypothesis and the inductive characterization above, those are precisely the source boundary positions of B of dimension $k+1$. The same argument shows that the second statement is also true for B.

We will prove the last statement by induction on $k - l$. Let $k \in \mathbb{N}$. We will first show that $S^B_{k+1,k}$ contains all positions of B of dimension k. It clearly contains all source boundary positions, so let $p \in \mathsf{Pos}_k(B)$ and suppose that it is not source boundary. Then one of the following holds.

- Suppose $k = 0$ and $p = \mathsf{inl}\,(B_0, L, p_0)$ for $p_0 \in \mathsf{Pos}_{k-1}(B_0)$ not source boundary. By the inductive hypothesis, $p_0 \in S^{B_0}_{k,k-1}$, so it is the source or target of some position $p_1 \in S^{B_0}_{kk} = \partial^s_k B_0$. Then p is the source or target of $\mathsf{inl}\ p_1 \in \partial^s_{k+1} B$, so it is in $S^B_{k+1,k}$.
- Suppose that $p = \mathsf{inr}\,(B_0, L, q)$ for some $q \in \mathsf{Pos}_k(\mathsf{br}\ L)$ that is not source boundary. Using the inductive hypothesis for $\mathsf{br}\ L$, we can conclude again as above that $p \in S^B_{k+1,k}$.
- Suppose finally that $k = 0$, $L = [\,]$ and $p = \mathsf{inr}\,(B_0, L, \mathsf{ob\text{-}nil})$. Then p is the target of $\mathsf{inl}\,(B_0, L, \mathsf{ob\text{-}nil})$, which is source boundary, so $p \in S^B_{k+1,k}$

We can similarly show that $T^B_{k+1,k}$ contains all positions of B of dimension k. It contains target boundary positions, so let $p \in \mathsf{Pos}_k(B)$ not target boundary. Splitting cases as above and using the same argument, it remains to show $p = \mathsf{ob\text{-}nil} \in T^B_{1,0}$. For that, let $p_0 \in \mathsf{Pos}_0(B_0)$ the unique target boundary 0-position of B_0. Then $\mathsf{inl}\ p_0$ is target boundary and has source p, so $p \in T^B_{1,0}$.

Finally, we are left to show for $l < k$ that $S^B_{kl} = \mathsf{Pos}_l(B) = T^B_{kl}$, assuming that this is true for $l+1$. Since every position of dimension l is either source boundary or the target of some position in $S^B_{k,l+1}$, the first equality holds. The second follows similarly. $\square$

Remark 2.11. By abuse of notation, we will often identity a tree with its globular set of positions, denoting the globular set $\mathsf{Pos}(B)$ simply by B.

3 Computads and their cells

In this section we will give our main definition of the category Comp of computads. We also give some simple examples showing how to build computads, and cells over computads. Throughout we make heavy use of the concepts of inductive set and recursively-defined function; for those readers who might appreciate an introduction to these important concepts, we supply one in Appendix A.

3.1 Main definition

Our definition begins by supplying some notation: we set $\mathsf{Comp}_{-1} = \star$, the terminal category, and define $\mathsf{Type}_{-1} : \star \to \mathsf{Set}$ as the functor picking out a singleton set. We then define by induction on $n \in \mathbb{N}$ the following:

- A category Comp_n of n-computads and n-homomorphisms.
- A forgetful functor $u_n : \mathsf{Comp}_n \to \mathsf{Comp}_{n-1}$, giving for every n-computad its underlying $(n-1)$-computad.
- A functor $\mathsf{Cell}_n : \mathsf{Comp}_n \to \mathsf{Set}$ giving for every n-computad its set of cells, and for every n-homomorphism its associated cell function.
- A functor $\mathsf{Type}_n : \mathsf{Comp}_n \to \mathsf{Set}$ giving for every n-computad its set of n-types, and for every n-homomorphism its associated type function. We will see that an n-type consists of a parallel pair of n-cells.
- A natural transformation $\mathsf{ty}_n : \mathsf{Cell}_n \to \mathsf{Type}_{n-1} \circ u_n$, giving for every cell of an n-computad its type in the underlying $(n-1)$-computad. Essentially, for each computad C, the function $\mathsf{ty}_{n,C}$ computes the boundary of each n-cell in C.
- For every globular set X, a computad $\mathsf{Free}_n\, X$ and a function $\mathsf{pos\text{-}type}_{n,X}$ giving an n-type of $\mathsf{Free}_n\, X$ from a pair $(a, b) \in X_n \times X_n$ with common source and target.
- For every n-computad C and n-cell $c \in \mathsf{Cell}_n(C)$, a set $\mathsf{fv}\,(c)$ of *free variables* of c.
- For every Batanin tree B, a distinguished subset of $\mathsf{Type}_n\, \mathsf{Free}_n(B)$ called the *full types*,
- For every n-computad C and n-cell $c \in \mathsf{Cell}_n(C)$, a finite number $\mathsf{cell\text{-}depth}\ c$, the *recursive depth* of c.
- for every n-homomorphism $f : C \to D$, an ordinal $\mathsf{mor\text{-}depth}\ c \leq \omega$, the *recursive depth* of f.

Base case. Our definition starts by giving the base case for the induction. This corresponds to the dimension $n = 0$. Here we proceed as follows:

- $\mathsf{Comp}_0 = \mathsf{Set}$
- $u_0 : \mathsf{Set} \to \star$ is the unique such functor
- $\mathsf{Cell}_0 : \mathsf{Set} \to \mathsf{Set}$ is the identity functor
- $\mathsf{Type}_0 : \mathsf{Set} \to \mathsf{Set}$ acts as $\mathsf{Type}_0(C) = C \times C$, the Cartesian product
- $\mathsf{ty}_0 : \mathsf{id}_{\mathsf{Set}} \to \mathsf{Type}_{-1} \circ u_0$ is the unique such natural transformation

- For a globular set X, we let $\mathsf{Free}_0\, X = X_0$ and we let $\mathsf{pos\text{-}type}_{0,X}$ be the identity function.
- For any 0-computad C and $c \in \mathsf{Cell}_0(C) = C$, we set the free variables of c to be $\mathsf{fv}(c) = \{c\}$,
- For a Batanin tree B, we declare a type $(a, b) \in \mathsf{Pos}_0(B) \times \mathsf{Pos}_0(B)$ to be full when $a \in \partial_0^s(B)$ and $b \in \partial_0^t(B)$, so every tree has exactly one full 0-type.
- Finally, we let the recursive depth of every 0-cell and every function be 0.

For the inductive step, we now fix $n > 0$ and suppose that we have defined all of the required data for every $0 \leq k < n$. The definition then proceeds as follows.

Objects. The objects of the category Comp_n are triples $C = (C_{n-1}, V_n^C, \phi_n^C)$, called *$n$-computads*, where

- $C_{n-1} \in \mathsf{Comp}_{n-1}$
- V_n^C is a set of *variables*
- $\phi_n^C : V_n^C \to \mathsf{Type}_{n-1}(C_{n-1})$ is a function assigning to each variable a *type* of C_{n-1}

The forgetful functor $u_n : \mathsf{Comp}_n \to \mathsf{Comp}_{n-1}$ acts on n-computads by projecting out the first component.

Free computads. For a globular set X, we define the n-computad $\mathsf{Free}_n\, X$ associated to X by

$$\mathsf{Free}_n\, X = (\mathsf{Free}_{n-1}\, X, X_n, \phi_n^{\mathsf{Free}\, X})$$

where

$$\phi_n^{\mathsf{Free}\, X}(p) = \mathsf{pos\text{-}type}_{n-1,X}\,(\mathsf{src}\ p, \mathsf{tgt}\ p).$$

For $n \geq 2$, the globularity conditions ensure that the positions $\mathsf{src}\ p$ and $\mathsf{tgt}\ p$ have common source and target, so that $\mathsf{pos\text{-}type}_{n-1,X}\,(\mathsf{src}\ p, \mathsf{tgt}\ p)$ is defined.

For every n-computad C, we now define mutually inductively the set $\mathsf{Cell}_n(C)$ of cells of C, the type of those cells, and the set of n-homomorphisms from an n-computad D to C.

Morphisms. An n-homomorphism $\sigma : (C_{n-1}, V_n^C, \phi_n^C) \to (D_{n-1}, V_n^D, \phi_n^D)$ is a pair (σ_{n-1}, σ_V) where

- $\sigma_{n-1} \in \mathsf{Comp}_{n-1}(C_{n-1}, D_{n-1})$
- $\sigma_V : V_n^C \to \mathsf{Cell}_n(D_{n-1}, V_n^D, \phi_n^D)$

such that for all $v \in V_C$ we have

$$\mathsf{ty}_{n,D}(\sigma_V(v)) = \mathsf{Type}_{n-1}(\sigma_{n-1})(\phi_n^C(v)).$$

The forgetful functor u_n is the evident one, sending σ to σ_{n-1}.

Cells and their type. For an n-computad $C = (C_{n-1}, V_n^C, \phi_n^C)$, the set $\mathsf{Cell}_n(C)$ is inductively generated as follows:

- For every element $v \in V_n^C$ an element $\mathsf{var}\, v \in \mathsf{Cell}_n(C)$.
- For every triple (B, A, τ) where
  - B is a Batanin tree with $\mathsf{dim}(B) \leq n$,
  - A is a full type in $\mathsf{Type}_{n-1}(\mathsf{Free}_{n-1}\, B)$,
  - $\tau = (\tau_{n-1}, \tau_V)$ is an n-homomorphism $\mathsf{Free}_n\, B \to C$

  an element $\mathsf{coh}\,(B, A, \tau) \in \mathsf{Cell}_n(C)$.

For every $c \in \mathsf{Cell}_n(C)$, the type $\mathsf{ty}_{n,C}(c)$ is defined recursively as follows:

- If $c = \mathsf{var}\, v$ for some $v \in V_n^C$, then
$$\mathsf{ty}_{n,C}\,(\mathsf{var}\, v) = \phi_n^C(v).$$
- If $c = \mathsf{coh}\,(B, A, \tau)$ is a coherence cell, then
$$\mathsf{ty}_{n,C}\,(\mathsf{coh}\,(B, A, \tau)) = \mathsf{Type}_{n-1}(\tau_{n-1})(A).$$

Recursive depth. Let $C = (C_{n-1}, V_n^C, \phi_n^C)$ an n-computad. The recursive depth of an n-cell $c \in \mathsf{Cell}_n(C)$ and that of an n-homomorphism with target C are defined as follows:

- If $c = \mathsf{var}\ v$ for some $v \in V_n^C$, then
$$\mathsf{cell\text{-}depth}\,(\mathsf{var}\ v) = 0.$$
- If $c = \mathsf{coh}\,(B, A, \tau)$ is a coherence cell, then
$$\mathsf{cell\text{-}depth}\,(\mathsf{coh}\,(B, A, \tau)) = \mathsf{mor\text{-}depth}\,(\tau) + 1.$$
- If $\sigma = (\sigma_{n-1}, \sigma_V) : (D_{n-1}, V_n^D, \phi_n^D) \to (C_{n-1}, V_n^C, \phi_n^C)$ is an n-homomorphism, then
$$\mathsf{mor\text{-}depth}\,(\sigma) = \mathsf{sup}\{\mathsf{cell\text{-}depth}\,(\sigma_V(v)) : v \in V_n^C\}$$

A priori, the depth of an n-cell and that of an n-homomorphism can be arbitrary ordinals. A simple induction shows though that the depth of an n-cell c is always finite. This is clearly the case if $c = \mathsf{var}\ v$ for some $v \in V_C$. On the other hand, if $c = \mathsf{coh}(B, A, \tau)$ is a coherence cell, and we assume that $\mathsf{cell\text{-}depth}\,(\tau_V(p))$ is finite for every position $p \in \mathsf{Pos}_n(B)$, then $\mathsf{mor\text{-}depth}(\tau)$ is finite. Indeed, in this case the set of variables of the domain is simply $\mathsf{Pos}_n(B)$ and hence $\mathsf{mor\text{-}depth}(\tau)$ is the supremum of a finite set of finite numbers. Consequently $\mathsf{cell\text{-}depth}\,(c)$ is also finite. As a consequence, the depth of an n-homomorphism must be at most ω, the least infinite ordinal. This observation allows us to view our construction as defining, by induction on $0 \leq d \leq \omega$:

- the n-cells $\mathsf{Cell}_n^{(d)}(C)$ of depth at most d
- the restriction of the typing function ϕ_n^C to $\mathsf{Cell}_n^{(d)}(C)$
- the set of morphisms $\mathsf{Comp}_n^{(d)}(D, C)$ of depth at most d

We define $\mathsf{Cell}_n(C)$ and $\mathsf{Comp}_n(D, C)$ to be $\mathsf{Cell}_n^{(\omega)}(C)$ and $\mathsf{Comp}_n^{(\omega)}(D, C)$ respectively. Since we always have that $\mathsf{cell\text{-}depth}\, c < \omega$ and $\mathsf{mor\text{-}depth}\, \tau \leq \omega$, this definition satisfies the claimed inductive property.

Composition and functoriality of cells. Let $C = (C_{n-1}, V_n^C, \phi_n^C)$ and $D = (D_{n-1}, V_n^D, \phi_n^D)$ be n-computads and let $\sigma = (\sigma_{n-1}, \sigma_V) : C \to D$ an n-homomorphism. We define the function $\mathsf{Cell}_n(\sigma)$ and post-composition by σ recursively as follows:

- If $c = \mathsf{var}\ v$ for some $v \in V_n^C$, we set
$$\mathsf{Cell}_n(\sigma)(\mathsf{var}\ v) = \sigma_V(v)$$
- If $c = \mathsf{coh}\,(B, A, \tau)$ is a coherence cell of C, we set
$$\mathsf{Cell}_n(\sigma)(\mathsf{coh}\,(B, A, \tau)) = \mathsf{coh}\,(B, A, \sigma \circ \tau) \tag{$*$}$$
- Given an n-homomorphism $\tau = (\tau_{n-1}, \tau_V) : (E_{n-1}, V_n^E, \phi_n^E) \to C$, we define the composition of σ and τ by
$$\sigma \circ \tau = (\sigma_{n-1} \circ \tau_{n-1}, v \mapsto \mathsf{Cell}_n(\sigma)(\tau_V(v))) \tag{**}$$

It is interesting to note that composition of homomorphisms $(**)$ is defined in terms of the action of homomorphisms on cells $(*)$, and vice versa. There is therefore a possible circularity here, in which case our recursion would fail to be well-founded. However, this issue does not in fact arise. To see why, observe that the definition of $\mathsf{Cell}_n(\sigma)(c)$ only requires the composition $\sigma \circ \tau$ to be defined when $\mathsf{mor\text{-}depth}\,(\tau) < \mathsf{cell\text{-}depth}\,(c)$. On the other hand, the definition of $\sigma \circ \tau$ requires $\mathsf{Cell}_n(\sigma)(c)$ to be defined when $\mathsf{cell\text{-}depth}(c) \leq \mathsf{mor\text{-}depth}\,(\tau)$ and that for such c,
$$\mathsf{ty}_{n,D}(\mathsf{Cell}_n(\sigma)(c)) = \mathsf{Type}_{n-1}(\sigma_0)(\mathsf{ty}_{n,C}(c)).$$

The fact that one of the inequalities above is strict shows that this mutually recursive definition terminates. Moreover, the last equality shows that ty_n is natural.

We may show that this definition makes Comp_n into a category and Cell_n into a functor by similar inductive arguments. For instance, given $\sigma : C \to D$ and $\tau : D \to E$ a pair of composable n-homomorphisms, we can easily show by induction that for every $c \in \mathsf{Cell}_n(C)$,

$$\mathsf{Cell}_n(\tau \circ \sigma)(c) = \mathsf{Cell}_n(\tau)(\mathsf{Cell}_n(\sigma)(c))$$

and that for every n-homomorphism ρ with target C,

$$(\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho).$$

With a similar argument, we can also show that for every n-computad C, the n-homomorphism given by

$$\mathsf{id}_C = (\mathsf{id}_{C_{n-1}}, v \mapsto \mathsf{var}\ v)$$

is the identity of C and that it acts trivially on $\mathsf{Cell}_n(C)$.

Types. For an n-computad $C = (C_{n-1}, V_n^C, \phi_n^C)$, the set $\mathsf{Type}_n(C)$ is defined to be the set of triples (A, a, b) where

- $A \in \mathsf{Type}_{n-1}(C_{n-1})$ is an $(n-1)$-type of C_{n-1}
- $a, b \in \mathsf{Cell}_n(C)$ are n-cells which satisfy

$$\mathsf{ty}_{n,C}(a) = \mathsf{ty}_{n,C}(b) = A$$

An n-homomorphism $\sigma = (\sigma_{n-1}, \sigma_V) : C \to D$ acts on n-types by

$$\mathsf{Type}_n(\sigma)(A, a, b) = (\mathsf{Type}_{n-1}(\sigma_{n-1})(A), \mathsf{Cell}_n(\sigma)(a), \mathsf{Cell}_n(\sigma)(b))$$

and this assignment is clearly functorial. We define also for $i = 1, 2$ a natural transformation $\mathsf{pr}_i : \mathsf{Type}_n \Rightarrow \mathsf{Cell}_n$ by the equations

$$\mathsf{pr}_{1,C}(A, a, b) = a \qquad\qquad \mathsf{pr}_{2,C}(A, a, b) = b$$

Free Computads (cont.) For a globular set X, the n-type associated to a pair $(p, q) \in X_n \times X_n$ with common source and target is given by

$$\mathsf{pos\text{-}type}_{n,X}\,(p,\ q) = (\mathsf{pos\text{-}type}_{n-1,X}\,(\mathsf{src\text{-}pos}\ p, \mathsf{tgt\text{-}pos}\ p), \mathsf{var}\ p, \mathsf{var}\ q).$$

Free variables. We can determine the free variables of a cell with a function

$$\mathsf{fv} : \mathsf{Cell}_n(C_{n-1}, V_n^C, \phi_n^C) \to \mathcal{P}(V_n^C)$$

defined by induction on the structure of $c \in \mathsf{Cell}_n(C_{n-1}, V_n^C, \phi_n^C)$, as follows.

- If $c = \mathsf{var}\ v$ for some variable $v \in V_n^C$, we set

$$\mathsf{fv}\,(\mathsf{var}\ v) = \{v\}$$

- If $c = \mathsf{coh}\,(B, A, \tau)$ is a coherence cell, we set

$$\mathsf{fv}\,(\mathsf{coh}\,(B, A, \tau)) = \bigcup_{p \in \mathsf{Pos}_n(B)} \mathsf{fv}\,(\tau_V(p))$$

Fullness. For a Batanin tree B, we can ask if a given type of $\mathsf{Free}_n\, B$ is full. We say that a type $(A, a, b) \in \mathsf{Type}_n(\mathsf{Free}_n\, B)$ is *full* when:

- $\mathsf{fv}(a) = \partial_n^s(B)$
- $\mathsf{fv}(b) = \partial_n^t(B)$
- $A \in \mathsf{Type}_{n-1}(\mathsf{Free}_{n-1}\, B)$ is full

Proposition 2.10 implies that for a full type (A, a, b), the globular set generated by the free variables of a, and its iterated sources and targets, is generated by the image of the n-boundary of B under the source inclusion s_n^B. Dually, the one generated by the free variables of b and its iterated sources and targets is the image of the n-boundary of B under the target inclusion t_n^B. The converse will be shown in Proposition 7.32. Thus, intuitively, fullness is the condition that a and b "cover" the entire n-dimensional source and target, respectively, of B. Note that this notion of fullness is directed, in the sense that it is not symmetric with respect to a and b when $\mathsf{dim}\, B < n$. This is necessary for the directed behaviour of the weak ω-categories that we are modelling.

Remark 3.1. In general, induction-recursion can lead the size of defined sets to increase rapidly [14]. However, this is not the case for us. Suppose that $U \in V$ are Grothendieck universes. Suppose that Comp_0 is V-small and locally U-small, and that Cell_0 lands in U-small sets. Then it follows by induction that the same is true of Comp_n and Cell_n respectively, for all $n > 0$.

This concludes the mutually inductive definition. Finally, we can also define infinite-dimensional computads as follows.

Definition 3.2. The category Comp of computads (or ω-computads) and homomorphisms is the limit of the forgetful functors:

$$\cdots \xrightarrow{u_{n+1}} \mathsf{Comp}_n \xrightarrow{u_n} \cdots \xrightarrow{u_2} \mathsf{Comp}_1 \xrightarrow{u_1} \mathsf{Comp}_0$$

A computad $C = (C_n)$ therefore comprises a choice for each n of an n-computad C_n, with the property that $C_{n+1} = (C_n, V^C_{n+1}, \phi^C_{n+1})$ for some variable sets V^C_{n+1} and typing functions ϕ^C_{n+1}. A homomorphism $\sigma : C \to D$ comprises similarly a choice for each n of an n-homomorphism $\sigma_n : C_n \to D_n$ such that $\sigma_{n+1} = (\sigma_n, \sigma_{n+1,V})$ for some functions $\sigma_{n+1,V} : V^C_{n+1} \to \mathsf{Cell}_{n+1}(D_{n+1})$. By definition of category Comp as a limit, there exist forgetful functors

$$U_n : \mathsf{Comp} \to \mathsf{Comp}_n,$$

discarding the structure of a computad above dimension n.

3.2 Examples

Here we unpack our main definition, demonstrating its properties and behaviour in some simple cases. We will describe how to extend n-computads to computads, and give an explicit description of 1-computads and their cells. In particular, we will describe composition of 1-cells, as well as 2-cells witnessing associativity.

Skeleton functors. We will describe for every $n \in \mathbb{N}$ a way to produce a computad out of an n-computad. We define first the *skeleton functor*

$$\mathsf{sk}_n : \mathsf{Comp}_n \to \mathsf{Comp}_{n+1}$$

on an n-computad C and on an n-homomorphism $\sigma : C \to D$ respectively by

$$\mathsf{sk}_n(C) = (C, \emptyset, \{\}) \qquad\qquad \mathsf{sk}_n(\sigma) = (\sigma, \{\})s$$

where $\{\}$ denotes the empty function. It follows immediately from the definition that

$$u_{n+1} \circ \mathsf{sk}_n = \mathsf{id}_{\mathsf{Comp}_n}$$

and that for every pair (C, D) where C is an n-computad and D is an $(n+1)$-computad the functor u_{n+1} induces a natural bijection

$$u_{n+1} : \mathsf{Comp}_{n+1}(\mathsf{sk}_n\, C, D) \xrightarrow{\sim} \mathsf{Comp}_n(C, u_{n+1} D),$$

whose inverses sends an n-homomorphism σ to $(\sigma, \{\})$. This shows that sk_n is left adjoint to u_{n+1} with unit the identity natural transformation. In particular, sk_n is fully faithful and injective on objects with image the replete full subcategory of $(n+1)$-computads with no variables of dimension $(n+1)$.

Repeatedly applying the skeleton functors, we get a functor

$$\mathsf{Sk}_n : \mathsf{Comp}_n \to \mathsf{Comp}$$

determined by

$$U_m \circ \mathsf{Sk}_n = \begin{cases} u_{m+1} \cdots u_n, & \text{if } m \leq n \\ \mathsf{sk}_{m-1} \cdots \mathsf{sk}_n, & \text{if } m > n. \end{cases}$$

As in the finite case, it follows easily that Sk_n is left adjoint to U_n with unit the identity, and that it is fully faithful with image the replete full subcategory of computads with no variables of dimension more than n.

Disks and spheres. For every globular set X, the n-computads $\mathsf{Free}_n\, X$ for all $n \in \mathbb{N}$ assemble into a computad $\mathsf{Free}\, X$. In particular, the n-globe D_n of Example 2.3 gives rise to a computad

$$\mathbb{D}^n = \mathsf{Free}\, D_n$$

which we will also call the *n-globe*.

The *∞-sphere* $\mathbb{S}^\infty = (\mathbb{S}^\infty_n)$ is another computad free on a globular set that has two generators in every dimension. Letting $V_n^{\mathbb{S}^\infty} = \{e_n^-, e_n^+\}$ for the set of generators, the typing functions $\phi_{n+1}^{\mathbb{S}^\infty} : V_{n+1}^{\mathbb{S}^\infty} \to \mathsf{Type}_n(\mathbb{S}^\infty_n)$ are defined inductively by the following equations:

$$\begin{aligned} \phi_1^{\mathbb{S}^\infty}(e_1^\pm) &= (e_0^-, e_0^+), \\ \phi_{n+2}^{\mathbb{S}^\infty}(e_{n+2}^\pm) &= (\phi_{n+1}^{\mathbb{S}^\infty}(e_{n+1}^-), \mathsf{var}\ e_{n+2}^-, \mathsf{var}\ e_{n+2}^+). \end{aligned}$$

The *k-sphere* $\mathbb{S}^k$ for $k \in \mathbb{N} \cup \{-1\}$ is defined similarly with two generators in every dimension up to n.

Understanding 1-computads. A 1-computad $C = (V_0, V_1, \phi : V_1 \to V_0, \times V_0)$ can be seen as a directed graph. We will denote a cell f such that $\phi(f) = (x, y)$ by an arrow

$$x \xrightarrow{f} y.$$

Recall that trees of dimension at most one are of the form $B_n = [D_0, \ldots, D_0]$ for $n \in \mathbb{N}$. Morphisms $\star_v : \mathsf{Free}_1\, B_0 \to C$ are in bijection to vertices $v \in V_0$, while for $n > 0$, morphisms $\sigma_{f_1,\ldots,f_n} : \mathsf{Free}_1\, B_n \to C$ correspond to sequences of composable 1-cells

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n.$$

Recall also that every Batanin tree has unique full 0-type. Those observations imply that the set of 1-cells of C is inductively generated by the following rules:

- For each edge $e \in V_1$ with $\phi(e) = (x, y)$, there exists a cell $\mathsf{var}\, e$ of type $\phi(e)$. That is,
$$x \xrightarrow{\mathsf{var}\, e} y.$$
- For every vertex $v \in V_0$, there exists an *identity cell*
$$v \xrightarrow{\mathsf{id}_v} v.$$
- For composable 1-cells $f_1, \ldots, f_n$ as above, there exists a *composite cell*
$$x_0 \xrightarrow{\mathsf{comp}(f_1,\ldots,f_n)} x_n.$$

Thus, each 1-cell of type (x, y) in C corresponds to a parenthesised path from x to y built out cells of edges of e and identity cells. Hence, a morphism of 1-computads $\sigma : C \to D$ amounts to a function $\sigma_0 : V_0^C \to V_0^D$, together with a function assigning every edge $e : x \to y$ of C, a parenthesised path from $\sigma_0(x)$ to $\sigma_0(y)$.

Some coherence cells. Having defined composition of 1-cells, we proceed to define some coherence 2-cells. For that, consider the 2-computad $\mathsf{Free}_2\, B_3$, which we may picture as follows

$$d_0 \xrightarrow{f} d_1 \xrightarrow{g} d_2 \xrightarrow{h} d_3.$$

The following 1-type is full, since it involves all variables of B_3,

$$A = ((d_0, d_3), \mathsf{comp}(f, \mathsf{comp}(g, h)), \mathsf{comp}(\mathsf{comp}(f, g), h)),$$

so we may form the 2-cell $\alpha = \mathsf{coh}(B_3, A, \mathsf{id})$. Given any 2-computad $E = (C, V_2, \psi)$, a 2-homomorphism $\mathsf{Free}_2\, B_3 \to E$ is the same as a 1-homomorphism $\mathsf{Free}_1\, B_3 \to C$, since B_3 has no 2-dimensional variables. Hence, for every triple of composable 1-cells f_1, f_2, f_3 of C, we may form the following cell

$$\mathsf{Cell}_2(\sigma_{f_1,f_2,f_3})(\alpha) : \mathsf{comp}(f_1, \mathsf{comp}(f_2, f_3)) \Rightarrow \mathsf{comp}(\mathsf{comp}(f_1, f_2), f_3)$$

that plays the role of an associator.

Different choices of Batanin tree and full type induce different coherence 2-cells. For example, let A' be the full 1-type defined by

$$A' = ((d_0, d_3), \mathsf{comp}(f, g, \mathsf{id}_{d_2}, h), \mathsf{comp}(f, g, h)).$$

Then $\upsilon = \mathsf{coh}(B_3, A', \sigma_{f_1,f_2,f_3})$ can be seen as an unbiased sort of *unitor*.

Simplicial sets. The non-degenerate simplices of a 2-simplicial set X assemble into a 2-computad $C(X)$. The underlying 1-computad C_1X is defined via the typing function

$$\phi_1^{C(X)} : X_1^{nd} \to \mathsf{Type}_0(X_0) = X_0 \times X_0$$

sending a non-degenerate 1-simplex to its faces

$$\phi_1^{C(X)}(x) = (d^1x, d^0x).$$

Every 1-simplex of X gives rise to a cell of C_1X via the function

$$\psi^X : X_1 \to \mathsf{Cell}_1(C_1X)$$

which sends a non-degenerate simplex to the variable cell it generates, and which sends a degenerate simplex $x = s^0d^1x = s^0d^0x$ to the coherence cell id_{d^1x} defined above. The computad C_2X is then defined via the typing function

$$\phi_2 : X_2^{nd} \to \mathsf{Type}_1(C_1X)$$

sending a 2-simplex x to

$$\phi_2(x) = \big((d^1d^1x, d^0d^1x), \mathsf{comp}(\psi(d^2x), \psi(d^0x)), \psi(d^1x)\big).$$

The fact that $\phi_2(x)$ is well-defined follows easily from the simplicial identities.

$$\begin{tikzcd}
& \bullet \arrow[dr, "d^0 x"] \arrow[d, phantom, "\Downarrow x"] & \\
\bullet \arrow[ur, "d^2 x"] \arrow[rr, "d^1 x"'] & {} & \bullet
\end{tikzcd}$$

Further work is needed to extend this construction to n-simplicial sets for $n > 2$, since the composition operations in our computads are only weakly associative and unital.

4 Defining ω-categories

4.1 The free computad adjunction

As we saw in the previous section, every globular set X gives rise to a computad $\mathsf{Free}\, X$ whose variables are the cells of X and the typing functions are given by the source and target functions of X. This process can easily be made functorial. Recursively on $n \in \mathbb{N}$, we will extend the assignment Free_n of the previous section to a functor

$$\mathsf{Free}_n : \mathsf{Glob} \to \mathsf{Comp}_n$$

by setting for a morphism of globular sets $f = (f_n) : X \to Y$,

$$\begin{aligned}
\mathsf{Free}_0\, f &= f_0 \\
\mathsf{Free}_{n+1}\, f &= (\mathsf{Free}_n\, f, x \mapsto \mathsf{var}(f_n\, x))
\end{aligned}$$

We let the *free computad functor*

$$\mathsf{Free} : \mathsf{Glob} \to \mathsf{Comp}$$

be the functor induced by this sequence.

On the other hand, recall that for each $n > 0$, an element of $\mathsf{Type}_n(C)$ is a triple (A, a, b) where $A \in \mathsf{Type}_{n-1}(C_{n-1})$ is an $(n-1)$-type of C_{n-1} and $a, b \in \mathsf{Cell}_n(C)$ are n-cells which satisfy $\mathsf{ty}_{n,C}(a) = \mathsf{ty}_{n,C}(b) = A$. It follows immediately that for every computad C, the sets of cells $\mathsf{Cell}_n(U_n C)$ of C form a globular set $\mathsf{Cell}\,(C)$ with source and target maps given by the functions

$$\mathsf{pr}_1\, \mathsf{ty}_n : \mathsf{Cell}_n(U_n C) \to \mathsf{Cell}_{n-1}(U_{n-1} C)$$

and

$$\mathsf{pr}_2\, \mathsf{ty}_n : \mathsf{Cell}_n(U_n C) \to \mathsf{Cell}_{n-1}(U_{n-1} C)$$

respectively. This construction may be extended to a functor

$$\mathsf{Cell} : \mathsf{Comp} \to \mathsf{Glob}$$

using that $\mathsf{Cell}_n\, U_n$ is a functor for every $n \in \mathbb{N}$, and the source and target maps are natural. Similarly, we may define a functor of types

$$\mathsf{Type} : \mathsf{Comp} \to \mathsf{Glob}$$

where the source and target of an n-type are both given by $\mathsf{ty}_n\, \mathsf{pr}_1 = \mathsf{ty}_n\, \mathsf{pr}_2$.

Proposition 4.1. *The functor* Cell *is right adjoint to* Free.

Proof. We will prove the proposition by constructing the unit and counit of the adjunction

$$\eta : \mathsf{id}_{\mathsf{Glob}} \Rightarrow \mathsf{Cell} \circ \mathsf{Free},$$
$$\varepsilon : \mathsf{Free} \circ \mathsf{Cell} \Rightarrow \mathsf{id}_{\mathsf{Comp}}$$

The component of the unit η on a globular set X is the morphism

$$\eta_X : X \to \mathsf{Cell}\,(\mathsf{Free}\, X)$$

defined on $x \in X_n$ by

$$\eta_{X,n}(x) = \begin{cases} x, & \text{if } n = 0 \\ \mathsf{var}\,(x), & \text{if } n \geq 1. \end{cases}$$

The component of the counit ε on a computad $C = (C_n)$ is the homomorphism given by the sequence

$$\varepsilon_{C,n} : \mathsf{Free}_n(\mathsf{Cell}\,(C)) \to C_n$$

defined by induction on $n \in \mathbb{N}$ as follows: The 0-homomorphism $\varepsilon_{C,0}$ is the identity of the set C_0. The n-homomorphism $\varepsilon_{C,n}$ for $n \geq 1$ is the pair $(\varepsilon_{C,n-1}, \varepsilon_{C,n,V})$ where

$$\varepsilon_{C,n,V} : V_n^{\mathsf{Free}\,(\mathsf{Cell}\ C)} = \mathsf{Cell}_n(C) \to \mathsf{Cell}_n(C)$$

is the identity function. Naturality of both constructions and the zigzag identities can be easily verified.

□

Corollary 4.2. *The functors* $\mathsf{Cell}_n\, U_n$, *and* $\mathsf{Type}_n\, U_n : \mathsf{Comp} \to \mathsf{Set}$ *are represented by the* n-globe $\mathbb{D}^n$ *and the* n-sphere $\mathbb{S}^n$ *respectively. Under those representations, the source and target of an* $(n+1)$*-cell is given by composition with the source and target inclusions* $\mathsf{Free}(s_n^{D_{n+1}})$ *and* $\mathsf{Free}(t_n^{D_{n+1}})$*, respectively.*

Proof. The computad $\mathbb{D}^n$ is free on D_n, which is isomorphic to the representable globular set on $[n]$. Hence for every computad C, there exists a natural isomorphism

$$\mathsf{Comp}\,(\mathbb{D}^n, C) \cong \mathsf{Glob}\,(\mathbb{G}(-,[n]), \mathsf{Cell}\ C) \cong \mathsf{Cell}_n(C).$$

This isomorphism sends a morphism $\sigma : \mathbb{D}^n \to C$ to its image of the unique variable of $\mathbb{D}^n$ of dimension n.

Let now S_n the globular set that the n-sphere is free on and let $j_n^{\pm} : D_n \to S_n$ be the morphism of globular sets which sends the position of D_n of dimension n to $e_n^{\pm}$, and let $i_n : S_{n-1} \to D_n$ be the morphism of globular sets which sends e_k^{-}

and e_k^+ to the iterated source and target of dimension k of that position. Then, the following square is a pushout in the category of globular sets:

$$\begin{tikzcd}
S_{n-1} \arrow[r, "i_n"] \arrow[d, "i_n"'] & D_n \arrow[d, "j_n^+"] \\
D_n \arrow[r, "j_n^-"'] & S_n. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Moreover,

$$s_n^{D_n+1} = i_{n+1} j_n^- \qquad\qquad t_n^{D_n+1} = i_{n+1} j_n^+$$

We will construct for every $n \geq -1$ a representation of $\mathsf{Type}_n\, U_n$ by $\mathbb{S}^n$ such that the natural transformations $\mathsf{pr}_1, \mathsf{pr}_2$ and ty_n are induced by j_n^-, j_n^+ and i_n respectively.

The functor Free is cocontinuous, so $\mathbb{S}^{-1} = \mathsf{Free}(S_{-1})$ is the initial computad. Since $\mathsf{Type}_{-1}\, U_{-1}$ is constant on a singleton set, it is represented by the (-1)-sphere. Moreover, $\mathsf{ty}_0\, U_0$ is induced by $\mathsf{Free}(i_0)$ being the only homomorphism out of the initial computad. Suppose therefore for $n \in \mathbb{N}$ that a representation of $\mathsf{Type}_{n-1}\, U_{n-1}$ by $\mathbb{S}^{n-1}$ has been given such that $\mathsf{ty}_n\, U_n$ is induced by $\mathsf{Free}(i_n)$. Then there exists a commutative diagram of the following form, where both the interior and exterior squares are pullbacks:

$$\begin{tikzcd}
\mathsf{Type}_n(C) \arrow[rrr, "\mathsf{pr}_2"] \arrow[dddd, "\mathsf{pr}_1"'] \arrow[dr, dashed, "\sim"] & & & \mathsf{Cell}_n(C) \arrow[dl, "\sim"] \arrow[dddd, "\mathsf{ty}_n"] \\
& \mathsf{Comp}(\mathbb{S}^n, C) \arrow[r, "(j_n^+)^*"] \arrow[d, "(j_n^-)^*"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{Comp}(\mathbb{D}^n, C) \arrow[d, "(\iota_n)^*"] & \\
& \mathsf{Comp}(\mathbb{D}^n, C) \arrow[r, "(\iota_n)^*"'] & \mathsf{Comp}(\mathbb{S}^{n-1}, C) & \\
& & & \\
\mathsf{Cell}_n(C) \arrow[rrr, "\mathsf{ty}_n"'] \arrow[uur, "\sim"] & & & \mathsf{Type}_{n-1}(C) \arrow[uul, "\sim"']
\end{tikzcd}$$

The universal properties of those squares gives a representation of $\mathsf{Type}_n\, U_n$ by $\mathbb{S}^n$.

To show that $\mathsf{ty}_{n+1}\, U_{n+1}$ is induced by $\mathsf{Free}(i_{n+1})$ and complete the proof, we need to show that if $A_n \in \mathsf{Type}_n(\mathbb{S}^n)$ is the universal n-type, then the type of the top dimensional position of D_{n+1} is $\mathsf{Type}_n\, \mathsf{Free}\,(i_{n+1})(A_n)$. The explicit description of $j_n^{\pm}$ and commutativity of the diagram above shows that A_n comprises of the cells e_n^- and e_n^+. Those are sent to the source and target of the top dimensional position of D_{n+1} under $\mathsf{Cell}_n\, \mathsf{Free}_n(i_{n+1})$, so $\mathsf{ty}_{n+1}\, U_{n+1}$ is indeed induced by $\mathsf{Free}(i_{n+1})$. $\square$

4.2 The free ω-category monad

The free computad adjunction

$$\mathsf{Free} : \mathsf{Glob} \underset{\longrightarrow}{\overset{\longleftarrow}{\scriptstyle\top}} \mathsf{Comp} : \mathsf{Cell}$$

induces a monad $(\mathsf{fc}^w, \eta^w, \mu^w)$ on the category of globular sets, whose endofunctor sends a globular set X to the cells $\mathsf{Cell}\,(\mathsf{Free}\,X)$ of the free computad on it.

Definition 4.3. We will call fc^w-algebras *ω-categories* and their morphisms *homomorphisms*. We will denote their category by $\omega\mathsf{Cat}$.

We note that these morphisms correspond to *strict* functors. (For a way to obtain weak homomorphisms, see the work of Garner [18].) We will also write

$$F^w : \mathsf{Glob} \to \omega\mathsf{Cat}, \qquad\qquad U^w : \omega\mathsf{Cat} \to \mathsf{Glob}$$

for the *free ω-category* and *underlying globular set* functors and

$$K^w : \mathsf{Comp} \to \omega\mathsf{Cat}$$

for the comparison functor making the following triangle commute in both directions:

$$\begin{tikzcd} \mathsf{Comp} \arrow[rr, "K^w"] \arrow[dr, shift left, "\mathsf{Cell}"] & & \omega\mathsf{Cat} \arrow[dl, shift left, "U^w"] \\ & \mathsf{Glob} \arrow[ul, shift left, "\mathsf{Free}"] \arrow[ur, shift left, "F^w"] & \end{tikzcd}$$

We proceed to show that $\omega\mathsf{Cat}$ is a locally finitely presentable category, and that homomorphisms of computads and free ω-categories coincide.

Proposition 4.4. *The monad fc^w is finitary, i.e. it preserves directed colimits.*

Proof. We may define for every $n \in \mathbb{N}$, every B Batanin tree of dimension at most n and $0 \le d \le \omega$, a functor

$$C^B_{d,n} : \mathsf{Glob} \to \mathsf{Set}$$

sending a globular set X to the subset of homomorphisms $\sigma : \mathsf{Free}\,B \to \mathsf{Free}\,X$ satisfying that $U_n\sigma$ has depth at most d. It suffices to show that those functors are finitary, since $\mathsf{fc}^w_n \cong C^{D_n}_{n,\omega}$. We proceed by induction on the pair (n, d).

The functor $C^{D_0}_{0,d}$ is isomorphic for every d to the one sending a globular set X to the set X_0, hence it is cocontinuous. Let therefore $n > 0$. The same argument shows that $C^{D_n}_{n,0}$ is cocontinuous. The functor $C^{D_k}_{n,0}$ for $k < n$ is finitary being isomorphic to $C^{D_k}_{n-1,\omega}$. Moreover, for arbitrary tree B of dimension at most n, we can easily see that the isomorphism

$$\mathsf{Comp}\,(\mathsf{Free}\,B, \mathsf{Free}\,-) \cong \lim_{\substack{k \le n,\\ p \in \mathsf{Pos}_k\,(B)}} \mathsf{Comp}\,(\mathbb{D}^k, \mathsf{Free}\,-)$$

given by the density lemma for presheaves and cocontinuity of Free restricts to an isomorphism

$$C^B_{n,0} \cong \lim_{k,p} C^{D_k}_{n,0}.$$

Since connected limits and filtered colimits of sets commute, this implies that $C^B_{n,0}$ is also finitary. For positive $0 < d < \omega$, every n-cell of depth at most d is either a variable or a coherence where the substitution has depth at most $d-1$, so

$$C^{D_n}_{n,d} \cong C^{D_n}_{n,0} \coprod_{\substack{\dim B \leq n \\ A \in \mathsf{Full}_{n-1}(B)}} C^B_{n,d-1}$$

Commutativity of colimits with colimits implies that $C^{D_n}_{n,d}$ is also finitary. The same arguments as above show that $C^B_{n,d}$ is also finitary for arbitrary tree B of dimension at most n. Finally, $C^B_{n,\omega}$ is the colimit of $C^B_{n,d}$ for finite d, so it is finitary since colimits commute with colimits. ☐

Corollary 4.5. *The category $\omega\mathsf{Cat}$ is locally finitely presentable.*

Proof. It is the category of algebras of a finitary monad on a locally finitely presentable category [2, Remark 2.76]. ☐

Proposition 4.6. *The comparison functor $K^w : \mathsf{Comp} \to \omega\mathsf{Cat}$ is fully faithful.*

Proof. Let C and D be computads and $f : K^w C \to K^w D$ a homomorphism. We will show by induction on $n \leq \omega$ that there exists unique n-homomorphism $\sigma_n : C_n \to D_n$ satisfying for every finite $k \leq n$ and every k-cell $c \in \mathsf{Cell}_k(C_n)$ that

$$\mathsf{Cell}_k(\sigma_n)(c) = f(c).$$

Then σ_ω must be the unique homomorphism such that $K^w \sigma_\omega = f$.

For $n = 0$, the equation above is satisfied only by the function $\sigma_0 = f_0$. Assuming then that unique σ_n exists for some $n \in \omega$, define σ_{n+1} to consist of σ_n and the composite

$$V^C_{n+1} \xrightarrow{\mathsf{var}} \mathsf{Cell}_{n+1}(C) \xrightarrow{f_{n+1}} \mathsf{Cell}_{n+1}(D).$$

This is the only morphism satisfying the equation above for cells of dimension at most n and for $(n+1)$-dimensional variables. Given a coherence cell $c = \mathsf{coh}\,(B, A, \tau)$ of dimension $n+1$, let $\tau^\dagger : B \to \mathsf{Cell}\,(C)$ the morphism of globular sets given by $p \mapsto \tau_V(p)$. Then

$$\tau = \varepsilon_{n+1,C} \circ \mathsf{Free}_{n+1}(\tau^\dagger)$$

and we may assume by structural induction on cells that

$$\mathsf{Cell}(\mathsf{Sk}_{n+1}\,\sigma_{n+1}) \circ \tau^\dagger = f\tau^\dagger.$$

Since f is a homomorphism of ω-categories, we may also observe that

$$\begin{aligned} f(c) &= f \circ \mathsf{Cell}(\varepsilon_C)(\mathsf{coh}\,(B, A, \mathsf{Free}_{n+1}\,\tau^\dagger)) \\ &= \mathsf{Cell}(\varepsilon_D \circ \mathsf{Free}\ f)(\mathsf{coh}\,(B, A, \mathsf{Free}_{n+1}\,\tau^\dagger)) \\ &= \mathsf{coh}\,(B, A, \varepsilon_{n+1,D} \circ \mathsf{Free}_{n+1}(f\tau^\dagger)). \end{aligned}$$

Combining the last two equations and using naturality of ε_C, we see that

$$f(c) = \mathsf{Cell}_{n+1}(\sigma_{n+1})(c),$$

so σ_{n+1} satisfies the claimed equation. Finally, we let $\sigma_\omega = (\sigma_n) : C \to D$. □

Remark 4.7. By abuse of notation, we will treat K^w as a subcategory inclusion, identifying a computad with the ω-category it generates.

Composition operations in ω-categories. Let B be a Batanin tree. In a strict ω-category, B-shaped diagrams of cells can be composed (uniquely) into a single cell of dimension $\mathsf{dim}\ B$, or more generally into a cell of any dimension $n \geq \mathsf{dim}\ B$ by also using identity operations. For weak ω-categories, we will now also define for every tree B and $n \geq \mathsf{dim}\ B$ an n-cell comp_n^B of B with a similar interpretation. A more general discussion of operations in ω-categories in terms of globular operads and theories follows in Section 7.3.

Intuitively, the cell corresponding to the unbiased composition over B is the coherence cell filling the type determined by the unbiased composite of the source and target of this diagram. Formally, we define inductively for every $n \in \mathbb{N}$ and Batanin tree B a full n-type $A_{B,n}$, and when $n \geq \mathsf{dim}\ B$ an n-cell comp_n^B. In the base case, the only Batanin tree of dimension 0 is D_0 and we let

$$\mathsf{comp}_0^{D_0} = \mathsf{ob\text{-}nil}$$

its unique cell. We let then for every Batanin tree B,

$$A_{B,0} = (s_0^B(\mathsf{comp}_0^{\partial_0 B}), t_0^B(\mathsf{comp}_0^{\partial_0 B}))$$

be its unique full 0-type.

Let now $n \in \mathbb{N}$ and suppose that the full types $A_{B,n}$ have been defined for every Batanin tree. Then we may let

$$\mathsf{comp}_{n+1}^{D_{n+1}} = \mathsf{var}\,(\mathsf{inl}^{n+1}(\mathsf{ob\text{-}nil}))$$

the unique top dimensional position of D_{n+1} and for every other Batanin tree B of dimension at most $n+1$,

$$\mathsf{comp}_{n+1}^B = \mathsf{coh}\,(B, A_{B,n}, \mathsf{id}).$$

In both cases, we have constructed a cell of type $A_{B,n}$, since the source and target of the top dimensional position of D_{n+1} are its unique source and target

n-positions. Finally, we define for arbitrary Batanin tree, the type $A_{B,n+1}$ by

$$\begin{aligned}\mathsf{pr}_1\, A_{B,n+1} &= \mathsf{Cell}_{n+1}\, \mathsf{Free}_{n+1}(s^B_{n+1})(\mathsf{comp}^{\partial_{n+1} B}_{n+1}),\\ \mathsf{pr}_2\, A_{B,n+1} &= \mathsf{Cell}_{n+1}\, \mathsf{Free}_{n+1}(s^B_{n+1})(\mathsf{comp}^{\partial_{n+1} B}_{n+1}).\end{aligned}$$

This type is full by Proposition 2.10.

As a particular example, binary composites of $(n+k+1)$-cells along a common k-dimensional boundary correspond to the Batanin tree $B_{k,n}$ inductively defined by $B_{0,n} = \mathsf{br}\,[D_n, D_n]$ and $B_{k+1,n} = \mathsf{br}\,[B_{k,n}]$. Note that this notion of composition operation implicitly also contains identity operations. Namely, the source and target of the cell $\mathsf{comp}^{D_n}_{n+1}$ are both the top dimensional position of D_n, allowing the operation corresponding to this cell to be interpreted as an identity operation.

5 The variable-to-variable subcategory

An important class of morphisms between computads is given by those maps which "preserve generating data".

Definition 5.1. A morphism of n-computads $\sigma : C \to D$ is said to be *variable-to-variable* when it satisfies the following inductively defined property:

- When $n = 0$, every morphism is variable-to-variable.
- When $n > 0$ is finite, a morphism σ is variable-to-variable when σ_{n-1} is variable-to-variable and for each variable $v \in V_C$, we have that $\sigma_V(v) = \mathsf{var}(w)$ for some variable $w \in V_D$.
- When $n = \omega$, a morphism is variable-to-variable when each of its projections, σ_k, is variable-to-variable.

It follows that a variable-to-variable morphism, σ, is uniquely determined by an assignment

$$\sigma'_k : V^C_k \longrightarrow V^D_k$$

from variables of C to variables of D for every finite $k \leq n$. It is easily verified that the class of variable-to-variable morphisms contains identities and is closed under composition.

Definition 5.2. For each n, we define $\mathsf{Comp}^{\mathsf{var}}_n$ to be the subcategory of Comp_n whose morphisms are variable-to-variable. We write $\zeta_n : \mathsf{Comp}^{\mathsf{var}}_n \hookrightarrow \mathsf{Comp}_n$ for the inclusion.

Remark 5.3. Suppose that we have the following commutative triangle of morphisms of computads:

$$\begin{tikzcd} & D \arrow[dr, "\tau"] & \\ C \arrow[ur, "\sigma"] \arrow[rr, "\rho"'] & & E \end{tikzcd}$$

where ρ is a variable-to-variable n-homomorphism. If a variable of C were sent to a coherence cell by σ, the same would be true for ρ, so σ must also be a variable-to-variable morphism. This cancellation property implies in particular that all isomorphisms of n-computads must be variable-to-variable.

Remark 5.4. The forgetful functors u_n and U_n can be restricted to give functors

$$\begin{aligned} u_n^{\mathsf{var}} &: \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}_{n-1}^{\mathsf{var}} \\ U_n^{\mathsf{var}} &: \mathsf{Comp}^{\mathsf{var}} \to \mathsf{Comp}_n^{\mathsf{var}} \end{aligned}$$

between the subcategories of variable-to-variable homomorphisms, so that the functors U_n^{var} form a limit cone above the diagram

$$\cdots \xrightarrow{u_{n+1}^{\mathsf{var}}} \mathsf{Comp}_n^{\mathsf{var}} \xrightarrow{u_n^{\mathsf{var}}} \cdots \xrightarrow{u_2^{\mathsf{var}}} \mathsf{Comp}_1^{\mathsf{var}} \xrightarrow{u_1^{\mathsf{var}}} \mathsf{Comp}_0^{\mathsf{var}}.$$

Example 5.5. Whenever $\sigma : C \to D$ is a variable-to-variable n-homomorphism, both $\mathsf{sk}_n\,\sigma$ and $\mathsf{Sk}_n\,\sigma$ are variable-to-variable homomorphisms, so we may restrict the two functors to get functors

$$\begin{aligned} \mathsf{sk}_n^{\mathsf{var}} &: \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}_{n+1}^{\mathsf{var}} \\ \mathsf{Sk}_n^{\mathsf{var}} &: \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}^{\mathsf{var}} \end{aligned}$$

left adjoint to u_{n+1}^{var} and U_n^{var} respectively. The functors u_{n+1}^{var} and U_n^{var} further admit right adjoints

$$\begin{aligned} \mathsf{cosk}_n^{\mathsf{var}} &: \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}_{n+1}^{\mathsf{var}} \\ \mathsf{coSk}_n^{\mathsf{var}} &: \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}^{\mathsf{var}}. \end{aligned}$$

The former extends an n-computad C by its set of n-types and an n-homomorphism $\sigma : C \to D$ by its action on n-types

$$\mathsf{cosk}_n^{\mathsf{var}}(C) = (C, \mathsf{Type}_n(C), \mathsf{id}), \qquad \mathsf{cosk}_n^{\mathsf{var}}(\sigma) = (\sigma, \mathsf{var} \circ \mathsf{Type}_n(\sigma)).$$

The latter is defined similarly to Sk_n by

$$U_m^{\mathsf{var}} \circ \mathsf{coSk}_n^{\mathsf{var}} = \begin{cases} u_{m+1}^{\mathsf{var}} \cdots u_n^{\mathsf{var}}, & \text{if } m \leq n \\ \mathsf{cosk}_{m-1}^{\mathsf{var}} \cdots \mathsf{cosk}_n^{\mathsf{var}}, & \text{if } m > n \end{cases}$$

Example 5.6. The image of the free computad functor $\mathsf{Free} : \mathsf{Glob} \to \mathsf{Comp}$ is contained in $\mathsf{Comp}_n^{\mathsf{var}}$. Thus, factoring through ζ_n, we obtain a functor

$$\mathsf{Free}^{\mathsf{var}} : \mathsf{Glob} \to \mathsf{Comp}^{\mathsf{var}}.$$

It is easily seen that $\mathsf{Free}^{\mathsf{var}}$ is fully faithful. Let X and Y be two globular sets and $\sigma = (\sigma_n) : \mathsf{Free}\,X \to \mathsf{Free}\,Y$ a variable-to-variable homomorphism. Then there exists for every $n \in \mathbb{N}$ a unique function $\sigma'_{n,V} : X_n \to Y_n$ such that $\sigma_0 = \sigma'_{0,V}$ and for every $n \geq 1$ and $x \in X_n$,

$$\sigma_{n,V}(x) = \mathsf{var}(\sigma'_{n,V} x).$$

For every such $x \in X_n$, we easily compute that

$$\begin{aligned}
\mathsf{pr}_1(\mathsf{Type}_{n-1}(\sigma_{n-1})(\phi_n^{\mathsf{Free}\ X} x)) &= \mathsf{var}(\sigma'_{n-1,V}(\mathsf{src}\ x)) \\
\mathsf{pr}_1(\mathsf{ty}_n(\sigma_{n,V}(x))) &= \mathsf{var}\,(\mathsf{src}\,(\sigma'_{n,V} x)) \\
\mathsf{pr}_2(\mathsf{Type}_{n-1}(\sigma_{n-1})(\phi_n^{\mathsf{Free}\ X} x)) &= \mathsf{var}(\sigma'_{n-1,V}(\mathsf{tgt}\ x)) \\
\mathsf{pr}_2(\mathsf{ty}_n(\sigma_{n,V}(x))) &= \mathsf{var}\,(\mathsf{tgt}\,(\sigma'_{n,V} x)).
\end{aligned}$$

On the other hand, we also know that

$$\mathsf{ty}_n(\sigma_{n,V}(x)) = \mathsf{Type}_{n-1}(\sigma_{n-1})(\phi_n^{\mathsf{Free}\ X} x),$$

since σ_n is an n-homomorphism, so

$$\mathsf{src}\,(\sigma'_{n,V} x) = \sigma'_{n-1,V}(\mathsf{src}\ x) \qquad \mathsf{tgt}\,(\sigma'_{n,V} x) = \sigma'_{n-1,V}(\mathsf{tgt}\ x)$$

In other words, the functions $\sigma'_{n,V}$ constitute a morphism of $\sigma' : X \to Y$, which one may easily check is the unique morphism satisfying that $\sigma = \mathsf{Free}(\sigma')$.

Proposition 5.7. *For each n, the category $\mathsf{Comp}_n^{\mathsf{var}}$ is cocomplete. Furthermore, the inclusion $\zeta_n : \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}_n$ preserves all colimits.*

Proof. We prove the statement by induction on n. The statement holds trivially for $n = -1$. Suppose that $n \in \mathbb{N}$. Given a diagram $J : \mathcal{D} \to \mathsf{Comp}^n_{\mathsf{var}}$, we set:

$$(\mathsf{colim}\ J)_{n-1} = \mathsf{colim}_{A \in D}\ (JA)_{n-1}, \qquad V_n^{\mathsf{colim}\ J} = \mathsf{colim}_{A \in D}\, V_n^{JA}$$

and we define the typing function $\phi_n^{\mathsf{colim}\ J} : V_n^{\mathsf{colim}\ J} \to \mathsf{Type}_{n-1}(\mathsf{colim}\ J)$ to be the following canonical morphism:

$$V_n^{\mathsf{colim}\ J} = \mathsf{colim}\ V_n^J \longrightarrow \mathsf{colim}\ \mathsf{Type}_{n-1}(u_{n-1}J) \longrightarrow \mathsf{Type}_{n-1}(\mathsf{colim}\, J)_{n-1}$$

The universal property of this n-computad in both Comp_n and $\mathsf{Comp}_n^{\mathsf{var}}$ is now easily verified. When $n = \omega$, we define colimits pointwise. □

Lemma 5.8. *For each n, the category $\mathsf{Comp}_n^{\mathsf{var}}$ has a terminal object $\top_n$.*

Proof. We proceed by induction on n. The result is trivial when $n = -1$. Hence suppose that $n \in \mathbb{N}$ and that $\mathsf{Comp}_{n-1}^{\mathsf{var}}$ has a terminal object $\top_{n-1}$. Then since $\mathsf{cosk}_{n-1}^{\mathsf{var}} : \mathsf{Comp}_{n-1}^{\mathsf{var}} \to \mathsf{Comp}_n^{\mathsf{var}}$ is a right adjoint, it preserves limits and so we may define $\top_n = \mathsf{cosk}_{n-1}^{\mathsf{var}}(\top_{n-1})$. Explicitly $(\top_n)_{n-1} = \top_{n-1}$, and $\top_n$ has a unique n-cell for each $(n-1)$-type in $\top_{n-1}$. Finally, we define $\top_\omega$ to be the computad $\mathsf{coSk}_0^{\mathsf{var}}(\top_0) = (\top_0, \ldots, \top_n, \ldots)$. □

5.1 $\mathsf{Comp}^{\mathsf{var}}$ is a presheaf topos

It turns out that $\mathsf{Comp}^{\mathsf{var}}$ is extremely well-behaved; in fact we will prove that it is a presheaf category. This is consistent with work of Batanin [8], although our approach is significantly different, as discussed in the Introduction. This presheaf property is striking since it is well known that the category of strict computads and generating data preserving maps is not a presheaf topos [24]. Thus, this result can be seen as evidence that weak higher categories are, in a sense, better behaved than strict higher categories.

The following construction will play a pivotal role in this section. Here we make reference to the *category of elements* or *Grothendieck construction* of a presheaf $X : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$. It is the category $\mathsf{el}\, X$ with class of objects the disjoint union of the sets $X(c)$ for $c \in \mathcal{C}$, and morphisms $f : (c, x) \to (c', x')$ the morphisms $f : c \to c'$ such that $X(f)(x') = x$.

Theorem 5.9. *Let $\mathsf{Cell}^{\mathsf{var}}$ and $\mathsf{Type}^{\mathsf{var}}$ be the restriction of the cell and type functors to the variable-to-variable subcategory, as follows:*

$$\mathsf{Cell}^{\mathsf{var}} = \mathsf{Cell} \circ \zeta : \mathsf{Comp}^{\mathsf{var}} \to \mathsf{Glob}$$
$$\mathsf{Type}^{\mathsf{var}} = \mathsf{Type} \circ \zeta : \mathsf{Comp}^{\mathsf{var}}_n \to \mathsf{Glob}$$

Then these functors are familially representable; there exist globular sets $\mathfrak{C}$, and $\mathfrak{T}$, together with functors from their category of elements

$$J^{\mathfrak{C}} : \mathsf{el}\, \mathfrak{C} \to \mathsf{Comp}^{\mathsf{var}} \qquad\qquad J^{\mathfrak{T}} : \mathsf{el}\, \mathfrak{T} \to \mathsf{Comp}^{\mathsf{var}}$$

such that we have isomorphisms, natural in $C \in \mathsf{Comp}^{\mathsf{var}}$ and $n \in \mathbb{G}$,

$$\mathsf{Cell}^{\mathsf{var}}_n(C) \cong \coprod_{c_0 \in \mathfrak{C}_n} \mathsf{Comp}^{\mathsf{var}}(J^{\mathfrak{C}}(c_0), C)$$
$$\mathsf{Type}^{\mathsf{var}}_n(C) \cong \coprod_{A_0 \in \mathfrak{T}_n} \mathsf{Comp}^{\mathsf{var}}(J^{\mathfrak{T}}(A_0), C).$$

Proof. We will show more generally that the functor giving the k-cells of an n-computad and the functor giving the k-types for $k \le n$ are familially representable. Let $\mathfrak{C}$ and $\mathfrak{T}$ the globular sets of cells and types of the terminal computad $\top_\omega$ respectively, and define for $c_0 \in \mathfrak{C}_k$ and $A_0 \in \mathfrak{T}_k$ the set *of cells of shape* c_0 of an n-computad C and that *of types of shape* A_0 by

$$\mathsf{Cell}\,(C; c_0) = \{c \in \mathsf{Cell}_k(C) \;:\; \mathsf{Cell}(!)(c) = c_0.\}$$
$$\mathsf{Type}\,(C; A_0) = \{A \in \mathsf{Type}_k(C) \;:\; \mathsf{Type}(!)(A) = A_0.\}$$

where ! denotes the unique variable-to-variable homomorphism to $\top_n$. It is easy to see that variable-to-variable homomorphisms preserve the shape of cells and types, so both definitions give rise to functors $\mathsf{Comp}^{\mathsf{var}}_n \to \mathsf{Set}$, for which we have natural isomorphisms,

$$\mathsf{Cell}^{\mathsf{var}}_k(\mathsf{Sk}_n\, C) = \coprod_{c_0 \in \mathfrak{C}_k} \mathsf{Cell}\,(C; c_0),$$
$$\mathsf{Type}^{\mathsf{var}}_k(\mathsf{Sk}_n\, C) = \coprod_{A_0 \in \mathfrak{T}_k} \mathsf{Type}\,(C; A_0).$$

The natural transformations pr_i for $i = 1, 2$, and the natural transformations ty, src and tgt for $k > 0$ may be restricted to natural transformations that we denote by the same names

$$\begin{aligned} \mathsf{pr}_i &: \mathsf{Type}\,(-; A_0) \Rightarrow \mathsf{Cell}\,(-; \mathsf{pr}_i\, A_0) \\ \mathsf{ty} &: \mathsf{Cell}\,(-; c_0) \Rightarrow \mathsf{Type}\,(-; \mathsf{ty}_k\, c_0), \\ \mathsf{src} &: \mathsf{Cell}\,(-; c_0) \Rightarrow \mathsf{Cell}\,(-; \mathsf{src}\, c_0), \\ \mathsf{tgt} &: \mathsf{Cell}\,(-; c_0) \Rightarrow \mathsf{Cell}\,(-; \mathsf{tgt}\, c_0). \end{aligned}$$

To prove the theorem, it suffices to show that the functors of cells of shape c_0 and types of shape A_0 are representable. Hence, we will construct by induction on $n \le \omega$, for every finite $k \le n$

1. for all $c_0 \in \mathfrak{C}_k$, an n-computad $J^{\mathfrak{C}}_n(c_0)$ and a canonical cell $\mathsf{self}_n(c_0)$ of shape c_0 of it such that evaluation at the canonical cell induces an isomorphism
$$\mathsf{Comp}^{\mathsf{var}}_n(J^{\mathfrak{C}}_n(c_0), -) \Rightarrow \mathsf{Cell}\,(-; c_0)$$

2. for all $A_0 \in \mathfrak{T}_k$, an n-computad $J^{\mathfrak{T}}_n(A_0)$ and a canonical type $\mathsf{self}_n(A_0)$ of shape A_0 of it such that evaluation at the canonical type induces an isomorphism
$$\mathsf{Comp}^{\mathsf{var}}_n(J^{\mathfrak{T}}_n(A_0), -) \Rightarrow \mathsf{Type}(-; A_0)$$

3. a pair of variable-to-variable n-homomorphisms for all $A_0 \in \mathfrak{T}_k$,
$$J_{n,A_0,i} : J^{\mathfrak{C}}_n(\mathsf{pr}_i\, A_0) \to J^{\mathfrak{T}}_n(A_0)$$
such that
$$\mathsf{Cell}\,(J_{n,A,i})(\mathsf{self}_n\, \mathsf{pr}_i\, A_0) = \mathsf{pr}_i(\mathsf{self}_n\, A_0)$$

4. for $k > 0$ and all $c_0 \in \mathfrak{T}_k$ variable-to-variable n-homomorphisms
$$\begin{aligned} J_{n,\partial c_0} &: J^{\mathfrak{T}}_n(\mathsf{ty}_k\, c_0) \to J^{\mathfrak{C}}_n(c_0) \\ J_{n,c_0,\mathsf{src}} &: J^{\mathfrak{C}}_n(\mathsf{src}\, c_0) \to J^{\mathfrak{C}}_n(c_0) \\ J_{n,c_0,\mathsf{tgt}} &: J^{\mathfrak{C}}_n(\mathsf{tgt}\, c_0) \to J^{\mathfrak{C}}_n(c_0) \end{aligned}$$
such that
$$\begin{aligned} \mathsf{Type}\,(J_{n,\partial c_0})(\mathsf{self}_n(\mathsf{ty}_k\, c_0)) &= \mathsf{ty}_k(\mathsf{self}_n\, c_0) \\ \mathsf{Cell}\,(J_{n,c_0,\mathsf{src}})(\mathsf{self}_n(\mathsf{src}\, c_0)) &= \mathsf{src}(\mathsf{self}_n\, c_0) \\ \mathsf{Cell}\,(J_{n,c_0,\mathsf{tgt}})(\mathsf{self}_n(\mathsf{tgt}\, c_0)) &= \mathsf{tgt}(\mathsf{self}_n\, c_0) \end{aligned}$$

and we will show that

1. for $k > 0$ and $A_0 = (A_0', a, b)$ the following square commutes

$$\begin{tikzcd} J_n^{\mathfrak{T}}(A_0') \arrow[r, "J_{n,\partial a}"] \arrow[d, "J_{n,\partial b}"'] & J_n^{\mathfrak{C}}(a) \arrow[d, "J_{n,A_0,1}"] \\ J_n^{\mathfrak{C}}(b) \arrow[r, "J_{n,A_0,2}"'] & J_n^{\mathfrak{T}}(A_0) \end{tikzcd}$$

2. for $k > 0$ and $c_0 \in \mathfrak{C}_k$ the n-homomorphisms associated to the source and target natural transformations are determined by

$$J_{n,c_0,\mathsf{src}} = J_{n,\mathsf{ty}_k\, c_0,1} J_{n,\partial_{c_0}}$$
$$J_{n,c_0,\mathsf{tgt}} = J_{n,\mathsf{ty}_k\, c_0,2} J_{n,\partial_{c_0}}$$

   and hence they satisfy for $k > 2$ the globularity conditions

$$J_{n,c_0,\mathsf{src}} J_{n,\mathsf{src}\, c_0,\mathsf{src}} = J_{n,c_0,\mathsf{tgt}} J_{n,\mathsf{tgt}\, c_0,\mathsf{src}}$$
$$J_{n,c_0,\mathsf{src}} J_{n,\mathsf{src}\, c_0,\mathsf{tgt}} = J_{n,c_0,\mathsf{tgt}} J_{n,\mathsf{tgt}\, c_0,\mathsf{tgt}}.$$

3. for $k < n < \omega$ and every $c_0 \in \mathfrak{C}_k$ and $A_0 \in \mathfrak{T}_k$, the following compatibility conditions hold

$$J_n^{\mathfrak{C}}(c_0) = \mathsf{sk}_n\, J_{n-1}^{\mathfrak{C}}(c_0), \qquad \mathsf{self}_n(c_0) = \mathsf{self}_{n-1}(c_0)$$
$$J_n^{\mathfrak{T}}(A_0) = \mathsf{sk}_n\, J_{n-1}^{\mathfrak{T}}(A_0). \qquad \mathsf{self}_n(A_0) = \mathsf{self}_{n-1}(A_0),$$

   while for $n = \omega$,

$$J^{\mathfrak{C}}(c_0) = \mathsf{Sk}_k\, J_k^{\mathfrak{C}}(c_0), \qquad \mathsf{self}\,(c_0) = \mathsf{self}_k(c_0)$$
$$J^{\mathfrak{T}}(A_0) = \mathsf{Sk}_k\, J_k^{\mathfrak{T}}(A_0). \qquad \mathsf{self}\,(A_0) = \mathsf{self}_k(A_0).$$

It is worth noting that if representability of $\mathsf{Cell}\,(-; c_0)$ is established, as well as that of $\mathsf{Cell}\,(-; \mathsf{src}\, c_0)$, $\mathsf{Cell}\,(-; \mathsf{tgt}\, c_0)$ and $\mathsf{Type}(-; \mathsf{ty}_k\, c_0)$, then existence and uniqueness of $J_{n,\partial c_0}$, $J_{n,c_0,\mathsf{src}}$, $J_{n,c_0,\mathsf{tgt}}$ and $J_{n,\mathsf{ty}\, c_0,i}$ with the given properties is a consequence of Yoneda's Lemma. Their uniqueness implies automatically that the various equations between them hold, since corresponding equations hold for the natural transformations they define. In light of that, it suffices to define the representations of each functor using those homomorphisms wherever needed.

Base case. First suppose that $n = 0$. Then there is a unique 0-cell $\star \in \mathfrak{C}_0$, and we define

$$J_0^{\mathfrak{C}}(\star) = \top_0 = \{\star\} \in \mathsf{Comp}_0^{\mathsf{var}}$$

We set

$$\mathsf{self}_0 \star = \star.$$

There is a unique 0-type $(\star, \star)$ in $\mathfrak{T}_0$, and we define $J_0^{\mathfrak{T}}(\star, \star)$ to be the coproduct

$$J_0^{\mathfrak{C}}(\star) + J_0^{\mathfrak{C}}(\star) = \{\star_0, \star_1\},$$

and we set

$$\mathsf{self}_0\,(\star, \star) = (\star_0, \star_1).$$

We define also $J_{0,\star,i}$ to be the function picking $\star_i$. The functor of cells of shape $\star$ and that of types of shape $(\star_0, \star_1)$ are represented by those sets, since they are isomorphic to the identity and the power $(-)^2$ respectively.

Low dimensional cells and types. Let now $n > 0$ and suppose that the representations above have been defined for all $n' < n$. For $k < n$ and $c_0 \in \mathfrak{C}_k$, define the computad $J_n^{\mathfrak{C}}(c_0)$ and the cell $\mathsf{self}_n(c_0)$ so that the compatibility condition is satisfied. For finite n, this represents the functor $\mathsf{Cell}\,(-; c_0)$ on n-computads as well, since for every n-computad C the following square commutes

$$\begin{tikzcd}
\mathsf{Comp}_n^{\mathsf{var}}(J_n^{\mathfrak{C}}(c_0), C) \arrow[r, "\mathsf{self}_n(c_0)"] \arrow[d, "u_n^{\mathsf{var}}"', "\sim"] & \mathsf{Cell}\,(C; c_0) \arrow[d, equal] \\
\mathsf{Comp}_n^{\mathsf{var}}(J_{n-1}^{\mathfrak{C}}(c_0), u_n C) \arrow[r, "\sim", "\mathsf{self}_{n-1}(c_0)"'] & \mathsf{Cell}\,(u_n C; c_0)
\end{tikzcd}$$

For $n = \omega$, there is a similar commutative square involving U_k^{var}, proving that the defined computad represents $\mathsf{Cell}\,(-; c_0)$ again. For $A_0 \in \mathcal{T}_k$, the computad and the type given by the compatibility condition represent $\mathsf{Type}(-; A_0)$ for the same reason. It is easy to see that the canonical n-homomorphisms between the representing computads are also obtained by applying $\mathsf{sk}_n^{\mathsf{var}}$ or $\mathsf{Sk}_k^{\mathsf{var}}$ to the corresponding $(n-1)$-homomorphisms or k-homomorphisms respectively.

Top dimensional cells. Suppose further that $n < \omega$. The computad $J_n^{\mathfrak{C}}(c_0)$ for $c_0 \in \mathfrak{C}_n$ together with the canonical cell $\mathsf{self}_n(c_0)$ will be constructed by induction on c_0 and it will be shown that they represent $\mathsf{Cell}\,(-; c_0)$.

If $c_0 = \mathsf{var}\ A_0$ for some $A_0 \in V_n^{\top\omega} = \mathfrak{T}_{n-1}$, then let

$$J_n^{\mathfrak{C}}(c_0) = (J_{n-1}^{\mathfrak{T}}(A_0), \{\star\}, \{(\star \mapsto \mathsf{self}_n\, A_0)\})$$

and let

$$\mathsf{self}_n(c_0) = \mathsf{var}\ \star.$$

A variable-to-variable morphism $\sigma : J_n^{\mathfrak{C}}(c_0) \to C$ amounts to a type A of C of shape A_0, and a variable $v \in V_n^C$ of type A. A cell $c \in \mathsf{Cell}_n(C)$ is of shape c_0 exactly when it is of the form $\mathsf{var}\, v'$ for some variable such that

$$!_V(v') = \mathsf{var}\,(\mathsf{Type}(!)\,\mathsf{ty}_{n,C}(\mathsf{var}\ v')) = A_0,$$

so they are in bijection to pairs (v, A) as above. Therefore, $J_n^{\mathfrak{C}}(c_0)$ and $\mathsf{self}_n(c_0)$ represent the functor of cells of shape c_0.

Suppose now that $c_0 = \mathsf{coh}(B, A, \tau)$ for some Batanin tree B, full $(n-1)$-type A and morphism $\tau : \mathsf{Free}_n\, B \to \top_n$ and that for all $c_0' \in \mathfrak{C}_k$ of depth less than that of c_0, the representation of $\mathsf{Cell}(-; c_0')$ has been defined. Let $\mathsf{el}^*(\mathfrak{C})$ the full subcategory of the category of elements of $\mathfrak{C}$ consisting of cells of dimension at most $n-1$, and cells of dimension n and depth less than that of c_0. Then the representing objects assemble into a functor

$$J_n^{\mathfrak{C}} : \mathsf{el}^*(\mathfrak{C}) \to \mathsf{Comp}_n^{\mathsf{var}}$$

together with the n-homomorphisms $J_{n,c_0',\mathsf{src}}$ and $J_{n,c_0',\mathsf{tgt}}$.

The Batanin tree B must have dimension at most n, so transposing τ along $\mathsf{Sk}_{n+1} \vdash U_{n+1}$ and $\mathsf{Free} \vdash \mathsf{Cell}$, we get a unique morphism of globular sets

$$\tau^\dagger : B \to \mathfrak{C}$$

such that for all $k \leq n$ and k-position p of B,

$$\tau^\dagger(p) = \tau_{k,V}(p).$$

The image of the functor that $\tau^\dagger$ induces between the categories of elements is contained in $\mathsf{el}^*(\mathfrak{C})$, so we may define

$$J_n^{\mathfrak{C}}(c_0) = \mathsf{colim}(J_n^{\mathfrak{C}} \circ \mathsf{el}\ \tau^\dagger)$$

and define for every position p of B, n-homomorphisms

$$\psi_p : J_n^{\mathfrak{C}}(\tau^\dagger\, p) \to J_n^{\mathfrak{C}}(c_0)$$

forming a colimit cocone. The assignment

$$\sigma : B \to \mathsf{Cell}(\mathsf{Sk}_n\, J_n^{\mathfrak{C}}(c_0))$$

given by sending p to

$$\sigma(p) = \mathsf{Cell}(\mathsf{Sk}_n\, \psi_p)(\mathsf{self}_n\, \tau^\dagger p)$$

is easily checked to be a morphism of globular sets. Transposing σ as above, we get an n-homomorphism

$$\sigma^\dagger : \mathsf{Free}_n\, B \to J_n^{\mathfrak{C}}(c_0)$$

and we let

$$\mathsf{self}_n(c_0) = \mathsf{coh}(B, A, \sigma^\dagger).$$

It remains to show that this pair represents $\mathsf{Cell}(-; c_0)$. Variable-to-variable n-homomorphisms

$$\chi : J_{n+1}^{\mathfrak{C}}(c_0) \to C$$

are in bijection to families of variable-to-variable n-homomorphisms indexed by positions p of B

$$\chi_p : J_{n+1}(\tau^\dagger p) \to C$$

satisfying the boundary conditions

$$\begin{aligned}\chi_p \, J_{n,\tau^\dagger p,\mathsf{src}} &= \chi_{\mathsf{src}\,p}, \\ \chi_p \, J_{n,\tau^\dagger p,\mathsf{tgt}} &= \chi_{\mathsf{tgt}\,p}.\end{aligned}$$

Such families are in turn in bijection to families of cells $d_p \in \mathsf{Cell}\,(\mathsf{Sk}_n\, C)$ such that

$$\mathsf{Cell}(!)(d_p) = \tau^\dagger\, p$$

and they satisfy the boundary conditions

$$\begin{aligned}\mathsf{src}\ d_p &= d_{\mathsf{src}\,p}, \\ \mathsf{tgt}\ d_p &= d_{\mathsf{tgt}\,p}.\end{aligned}$$

Equivalently, they are in bijection with morphisms of globular sets

$$d : B \to \mathsf{Cell}(\mathsf{Sk}_n\, C)$$

such that

$$\mathsf{Cell}(!) \circ d = \tau^\dagger.$$

Transposing as above, we see that such morphisms correspond to n-homomorphisms

$$d^\dagger : \mathsf{Free}_n\, B \to C$$

such that

$$\tau = \,! \circ d^\dagger.$$

Finally, such morphisms correspond to cells of C of shape c_0, since any such cell must be of the form $\mathsf{coh}(B, A, d^\dagger)$ for such a morphism.

To establish that this bijection is the one given by evaluating at $\mathsf{self}_n(c_0)$, we need to show that for all pairs χ and $d^\dagger$ related by this bijection,

$$\mathsf{coh}(B, A, d^\dagger) = \mathsf{Cell}(\chi)(\mathsf{self}_n\, c_0) = \mathsf{coh}(B, A, \chi\sigma^\dagger),$$

that is $d^\dagger = \chi\sigma^\dagger$. Transposing this equality, it is equivalent to showing that

$$d = \mathsf{Cell}(\chi)\sigma : B \to \mathsf{Cell}(C).$$

For every position p,

$$\begin{aligned}d_p &= \mathsf{Cell}(\chi_p)(\mathsf{self}_n\, \tau^\dagger\, p) \\ &= \mathsf{Cell}(\chi)\, \mathsf{Cell}(\psi_p)(\mathsf{self}_n\, \tau^\dagger\, p) \\ &= \mathsf{Cell}(\chi)\sigma(p),\end{aligned}$$

so the final equality holds and hence the previous ones as well.

Top-dimensional types. Suppose that we have a type $A_0 = (A_0', a, b) \in \mathfrak{T}_n$. Then we define $J_n^{\mathfrak{T}}(A_0)$ and the variable-to-variable morphisms $J_{n,A_0,i}$ by the following pushout diagram

$$\begin{tikzcd}
J_n^{\mathfrak{T}}(A_0') \arrow[r, "J_{n,\partial a}"] \arrow[d, "J_{n,\partial b}"'] & J_n^{\mathfrak{C}}(a) \arrow[d, "J_{n,A_0,1}"] \\
J_n^{\mathfrak{C}}(b) \arrow[r, "J_{n,A_0,2}"'] & J_n^{\mathfrak{T}}(A_0)
\end{tikzcd}$$

By construction this square commutes as required. We define the type $\mathsf{self}_n(A_0)$ to be the one given by the parallel cells $\mathsf{Type}_n(\mathsf{Sk}_n\, J_{n,A_0,i})(\mathsf{self}_n(\mathsf{pr}_i\, A_0))$. For every n-computad C, the set of types of shape A_0 is the pullback of the sets of cells of shapes a and b respectively over the set of types of shape A_0'. From that, it follows easily that $J_n^{\mathfrak{T}}(A_0)$ with the type $\mathsf{self}_n(A_0)$ represent the functor of types of shape A_0. □

Corollary 5.10. *The functor* $\mathsf{Type}_n\, \zeta_n : \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Set}$ *preserves connected limits.*

Proof. This functor is familially representable by the previous proof, so this follows immediately, since coproducts in Set commute with connected limits. □

Corollary 5.11. *For each n, the category* $\mathsf{Comp}_n^{\mathsf{var}}$ *is complete. Furthermore, the inclusion* $\zeta_n : \mathsf{Comp}_n^{\mathsf{var}} \to \mathsf{Comp}_n$ *preserves all connected limits.*

Proof. Since $\mathsf{Comp}_n^{\mathsf{var}}$ has a terminal object, it suffices to show it has connected limits. Corollary 5.10 now allows us to proceed as we did for colimits. We prove the statement by induction on n. When $n = 0$, we have that $\mathsf{Comp}_n^{\mathsf{var}} = \mathsf{Comp}_n = \mathsf{Set}$ and so the statement holds trivially. Suppose that $0 < n < \omega$. Given a connected diagram $J : \mathcal{D} \to \mathsf{Comp}_n^{\mathsf{var}}$, we set:

$$(\mathsf{lim}\ J)_{n-1} = \lim_{A \in D} (JA)_{n-1}$$
$$V_n^{\mathsf{lim}\ J} = \lim_{A \in D} V_n^{JA}$$

and we define $\phi_n^{\mathsf{lim}\ J} : V_n^{\mathsf{lim}\, J} \to \mathsf{Type}_{n-1}(\mathsf{lim}\, J)$ to be the following canonical morphism:

$$V_n^{\mathsf{lim}\ J} = \mathsf{lim}\, V_n^J \longrightarrow \mathsf{lim}\, \mathsf{Type}_{n-1}\, J \cong \mathsf{Type}_{n-1}(\mathsf{lim}\, J)$$

The universal property of this computad in both Comp_n and $\mathsf{Comp}_n^{\mathsf{var}}$ is now easily verified. When $n = \omega$, we define connected limits pointwise. □

Corollary 5.12. *For all n, the inclusion ζ_n preserves and reflects monomorphisms and epimorphisms.*

Proof. Monomorphisms and epimorphisms can be characterised in terms of pullbacks and pushouts respectively. Since ζ preserves connected limits and colimits, it must preserve monomorphisms and epimorphisms. Moreover, it must reflect them, since it is faithful. □

Theorem 5.13. *The category $\mathsf{Comp}_n^{\mathsf{var}}$ is equivalent to the category of presheaves on $\mathcal{V}_n$, where $\mathcal{V}_n$ is its full subcategory on the n-computads of the form $J_n^{\mathcal{C}}(\mathsf{var}\ A_0)$ for some $-1 \leq k < n$ and $A_0 \in \mathfrak{T}_k$.*

Proof. In the proof of the previous theorem, we saw that for every such A_0 and every n-computad C,

$$\mathsf{Comp}_n^{\mathsf{var}}(J_n^{\mathcal{C}}(\mathsf{var}\ A_0), C) \cong \{v \in V_{k+1}^C \text{ whose type has shape } A_0\}.$$

From the construction of colimits of variable-to-variable n-homomorphisms, we can then conclude that the functor represented by $J_n^{\mathcal{C}}(\mathsf{var}\ A_0)$ is cocontinuous. Moreover, n-homomorphisms $\sigma : C \to D$ are isomorphisms exactly when they are variable-to-variable and the assignment on variables defining them is bijective. This is equivalent to being variable-to-variable and $\mathsf{Comp}_n^{\mathsf{var}}(J_n^{\mathcal{C}}(\mathsf{var}\ A_0), \sigma)$ being an isomorphism for every A_0. Those observations show that the theorem is a special case of the following proposition. $\square$

Proposition 5.14. *Let $\mathcal{D}$ a cocomplete category and $\mathcal{C}$ a small full subcategory of it such that the representable functors $\mathcal{D}(c, -)$ for $c \in \mathcal{C}$ are cocontinuous and jointly reflect isomorphisms. Then $\mathcal{D}$ is equivalent to the category of presheaves on $\mathcal{C}$.*

Proof. The inclusion of $\mathcal{C}$ into $\mathcal{D}$ defines a *nerve functor*

$$N : \mathcal{D} \to [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$$

given on $d \in \mathcal{D}$ and $c \in \mathcal{C}$ by

$$(Nd)(c) = \mathcal{D}(c, d).$$

By assumption, N reflects isomorphisms. Since $\mathcal{D}$ is cocomplete and $\mathcal{C}$ is small, the nerve functor admits a left adjoint

$$|-| : [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}] \to \mathcal{D}$$

given by the left Kan extension of the subcategory inclusion along the Yoneda embedding. More concretely, it is given on a presheaf $X : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ by the colimit formula

$$|X| = \operatorname*{\mathsf{colim}}_{(c,x) \in \mathsf{el}\, X} c.$$

The component of the unit η of the adjunction at a presheaf X is the morphism given at $c \in \mathcal{C}$ by the function

$$\eta_X : X(c) \to \mathcal{D}(c, |X|)$$

sending $x \in X(C)$ to the structure morphism of the colimit corresponding to the copy of c indexed by (c, x). This function may be factored as the composite of the isomorphisms

$$X(c) \xrightarrow{\sim} \operatorname*{\mathsf{colim}}_{(c',x) \in \mathsf{el}\, X} \mathcal{D}(c, c') \xrightarrow{\sim} \mathcal{D}(c, |X|)$$

given by the density theorem and cocontinuity of $\mathcal{D}(c, -)$, so η is invertible.

The counit ϵ of the adjunction satisfies the zigzag identity

$$(N\epsilon) \circ (\eta N) = \mathsf{id}_N,$$

so $N\epsilon$ is also invertible. As N reflects isomorphisms, ϵ is invertible, so $|-| \vdash N$ is an equivalence between $\mathcal{D}$ and the category of presheaves on $\mathcal{C}$. □

6 Computadic resolutions

Every set I of morphisms in a locally presentable category $\mathcal{C}$ generates an algebraic weak factorisation system $(\mathbb{L}, \mathbb{R})$ by Garner's algebraic small object argument [17]. Restricting the functorial factorisations to the slice $\emptyset/\mathcal{C} \cong \mathcal{C}$ below the initial object gives a comonad (Q, r, Δ), that Garner calls the *universal cofibrant replacement* of the weak factorisation system. This comonad may be identified by the following recognition principle.

Theorem 6.1. *[18, Proposition 2.6] For every object $X \in \mathcal{C}$, the morphism $r_X : QX \to X$ over X may be equipped with such a choice of liftings $\phi(j, h, k)$ for every lifting problem*

$$\begin{tikzcd} \mathsf{dom}(j) \arrow[r, "h"] \arrow[d, "j"'] & QX \arrow[d, "r_X"] \\ \mathsf{cod}(j) \arrow[r, "k"'] \arrow[ur, dashed, "\phi(j,h,k)"] & X \end{tikzcd}$$

where $j \in J$, such that (r_X, ϕ) is initial among objects of $\mathcal{C}/X$ equipped with a choice of liftings.

In particular, the set I of inclusions of spheres as boundaries of disks $\mathbb{S}^{n-1} \overset{\iota_n}{\rightarrowtail} \mathbb{D}^n$ generates a weak factorisation system on $\omega\mathsf{Cat}$. Our goal in this section is to explicitly describe the universal cofibrant replacement in this case using the language of computads.

We will first show that that for each computad C, the weak ω-category $K^w C$ is always cofibrant; that is the map $\emptyset \to K^w C$ is in the left class of the underlying weak factorisation system. In fact, we will prove that this map is *I-cellular*, i.e. in the closure of I under coproducts, pushouts and transfinite composition. We will then build a right adjoint $W : \omega\mathsf{Cat} \to \mathsf{Comp}^{\mathsf{var}}$ to the free ω-category functor $K^w\zeta : \mathsf{Comp}^{\mathsf{var}} \to \omega\mathsf{Cat}$ and show that the induced monad is the universal one, in analogy to [18, Proposition 5.7].

6.1 Immersions are cellular

We now define a simple class of morphisms of computads called *immersions*. We will show that immersions are I-cellular when viewed as morphisms of ω-categories, and conclude that free ω-categories on a computad are also I-cellular.

Definition 6.2. An *immersion* $\sigma : C \to D$ is a variable-to-variable homomorphism such that for all $n \in \mathbb{N}$ the assignment on variables $V_n^C \to V_n^D$ defining σ is injective.

Using representability of Cell and Corollary 5.12, we can easily see that a variable-to-variable homomorphism is an immersion if and only if it is monic in $\mathsf{Comp}^{\mathsf{var}}$ or equivalently in Comp.

We first show that certain immersions are pushouts of coproducts of maps in I. We will then show that every immersion is a transfinite composite of these particular immersions; consequently, every immersion is I-cellular. Restricting these results to morphisms out of the initial computad, we will conclude that free ω-categories on a computad are I-cellular and that they satisfy a freeness condition analogous to the one posed by Street [29, Section 4] for strict ω-categories.

Proposition 6.3. *Let $\sigma : C \to D$ be an immersion and suppose that there exists some $n \in \mathbb{N}$ such that, for all $k \neq n$, we have that $V_k^C = V_k^D$ and $\sigma_{k,V} = \mathsf{id}$. Let $V = V_n^D \setminus \sigma_{n,V}(V_n^C)$. Then, the following square is a pushout square:*

$$\begin{tikzcd}
\coprod_{e\in V} K^w\mathbb{S}^{n-1} \arrow[r, "\iota_n"] \arrow[d] & \coprod_{e\in V} K^w\mathbb{D}^n \arrow[d] \\
K^w C \arrow[r, "K^w\sigma"'] & K^w D
\end{tikzcd}$$

Here the vertical maps classify the variables in V and their types respectively.

Proof. The ω-categories $\mathbb{D}^n$ and $\mathbb{S}^n$ represent n-cells and pairs of parallel n-cells in respectively. Hence, the proposition amounts to showing that, for each ω-category X, each homomorphism $f : C \to X$, and each family of cells $\{x_e \in X_n\}_{e\in V}$ satisfying the boundary conditions,

$$\mathsf{src}\, x_e = f\,(\mathsf{src}\,\mathsf{var}\ e), \qquad \mathsf{tgt}\, x_e = f\,(\mathsf{tgt}\,\mathsf{var}\ e),$$

there exists a unique homomorphism $g : D \to X$ such that

$$f = g \circ \sigma, \qquad g(\mathsf{var}\, e) = x_e. \tag{$\star$}$$

Let $h : \mathsf{Cell}\, D \to U^w X$ the underlying morphism of globular sets for some such g. We claim that h satisfies the following conditions:

1. For each $k \in \mathbb{N}$, $v \in V_k^C$, we have that $h(\mathsf{var}\ v) = f(\mathsf{var}\ v)$.

2. For each $e \in V$, we have that $h(\mathsf{var}\ e) = x_e$.

3. Let $\alpha : \mathsf{fc}^w\, X \to X$ be the fc^w-algebra structure of X. Then, for each coherence cell, we have that

$$h(\mathsf{coh}\,(B, A, \tau)) = \alpha\,(\mathsf{coh}\,(B, A, \mathsf{Free}_{k+1}(h\tau^\dagger))).$$

   where $\tau^\dagger : B \to \mathsf{Cell}(D)$ is the morphism corresponding by adjointness to $\tau : \mathsf{Free}_{k+1}\, B \to D$.

The first two conditions are easily verified. For 3, we observe that, since g is a homomorphism, we have that

$$\begin{aligned} h(\mathsf{coh}(B,A,\tau)) &= h(\mathsf{coh}(B,A,\varepsilon_{k+1,D}\circ \mathsf{Free}_{k+1}\,\tau^\dagger)) \\ &= (h\circ \mathsf{Cell}\,\varepsilon_D)(\mathsf{coh}(B,A,\mathsf{Free}_{k+1}\,\tau^\dagger)) \\ &= (\alpha\circ \mathsf{fc}^w\,h)(\mathsf{coh}(B,A,\varepsilon_{k+1,D}\circ\tau^\dagger)) \\ &= \alpha(\mathsf{coh}(B,A,\mathsf{Free}_{k+1}(h\circ\tau^\dagger)). \end{aligned}$$

These three conditions uniquely determine g. Consequently, it suffices to show that the map h, defined at each dimension by these conditions, is always a homomorphism of weak ω-categories. This, in turn, is equivalent to the following statements holding for each dimension k:

- If $k>0$, then for each k-cell c of D, the map h respects the source and target of c:

  $$\mathsf{src}\,h(c) = h(\mathsf{src}\,c), \qquad\qquad \mathsf{tgt}\,h(c) = h(\mathsf{tgt}\,c).$$

- For each k-cell c' of $\mathsf{Free}\,\mathsf{Cell}\,D$, we have that

  $$(h\circ \mathsf{Cell}(\varepsilon_D))(c') = (\alpha\circ \mathsf{fc}^w\,h)(c').$$

  Here ε_D is the counit of the adjunction $\mathsf{Cell}\dashv\mathsf{Free}$.

We proceed by induction on k. In dimension 0, this is straightforward. Hence, suppose the claim holds for some $k\in\mathbb{N}$. Suppose that c is a $(k+1)$-dimensional cell in D. If c is a variable, then 1 and 2 imply that h respects the source and target of h. Given a coherence $(k+1)$-cell $c=\mathsf{coh}\,(B,A,\tau)$, the inductive hypothesis implies that

$$\begin{aligned} \mathsf{src}\;h(c) &= \mathsf{src}\,\alpha(\mathsf{coh}\,(B,A,\mathsf{Free}_{k+1}(h\tau^\dagger))) \\ &= \alpha\,\mathsf{src}(\mathsf{coh}\,(B,A,\mathsf{Free}_{k+1}(h\tau^\dagger))) \\ &= (\alpha\circ\mathsf{fc}^w(h))(\mathsf{src}\;\mathsf{coh}\,(B,A,\mathsf{Free}_{k+1}\,\tau^\dagger)) \\ &= (h\circ\mathsf{Cell}(\varepsilon_D))(\mathsf{src}\;\mathsf{coh}\,(B,A,\mathsf{Free}_{k+1}\,\tau^\dagger)) \\ &= h(\mathsf{src}\;\mathsf{coh}\,(B,A,\varepsilon_{k+1,D}\,\mathsf{Free}_{k+1}\,\tau^\dagger)) \\ &= h(\mathsf{src}\;c). \end{aligned}$$

A similarly argument works for the target. Now suppose that c' is a $(k+1)$-dimensional cell of $\mathsf{Free}\,\mathsf{Cell}\,D$. First suppose that $c'=\mathsf{var}\,v$. Then the unit laws of fc^w and α imply that

$$(h\circ\mathsf{Cell}(\varepsilon_D))(\mathsf{var}\,v) = h(v) = \alpha(\mathsf{var}\,h(v)) = (\alpha\circ\mathsf{fc}^w\,h)(\mathsf{var}\,v).$$

Now suppose that $c'=\mathsf{coh}\,(B,A,\tau_0)$ is a coherence cell. Suppose inductively that h satisfies the inductive hypothesis on cells in the image of the transpose

$\tau_0^\dagger : B \to \mathsf{fc}^w\,\mathsf{Cell}\,D$. Then

$$\begin{aligned}
(h \circ \mathsf{Cell}(\varepsilon_D))(c') &= h(\mathsf{coh}(B, A, \epsilon_{k+1,D}\tau_0)) \\
&= \alpha(\mathsf{coh}(B, A, \mathsf{Free}_{k+1}(h(\varepsilon_{k+1,D} \circ \tau_0)^\dagger))) \\
&= \alpha \circ \mathsf{fc}^w(h \circ (\varepsilon_D \tau_0)^\dagger)(\mathsf{coh}\,(B, A, \mathsf{id})) \\
&= \alpha \circ \mathsf{fc}^w(h \circ \mathsf{Cell}(\varepsilon_D)\tau_0^\dagger)(\mathsf{coh}\,(B, A, \mathsf{id})) \\
&\overset{\text{ind}}{=} \alpha \circ \mathsf{fc}^w(\alpha \circ \mathsf{fc}^w\, h \circ \tau_0^\dagger)(\mathsf{coh}\,(B, A, \mathsf{id})) \\
&= (\alpha \circ \mathsf{fc}^w\, \alpha) \circ (\mathsf{fc}^w\, \mathsf{fc}^w\, h)(\mathsf{coh}\,(B, A, \mathsf{Free}_{k+1}\, \tau_0^\dagger)) \\
&\overset{\text{alg}}{=} (\alpha \circ \mathsf{Cell}(\varepsilon_{\mathsf{Free}\ X})) \circ (\mathsf{fc}^w\, \mathsf{fc}^w\, h)(\mathsf{coh}\,(B, A, \mathsf{Free}_{k+1}\, \tau_0^\dagger)) \\
&\overset{\text{nat}}{=} \alpha \circ \mathsf{Cell}(\mathsf{Free}\ h \circ \varepsilon_{\mathsf{fc}^w\ D})(\mathsf{coh}\,(B, A, \mathsf{Free}_{k+1}\, \tau_0^\dagger)) \\
&= (\alpha \circ \mathsf{fc}^w\, h)(c').
\end{aligned}$$

Thus, the map h satisfies the conditions required of a homomorphism of weak ω-categories. ☐

Theorem 6.4. *Every immersion is the transfinite composite of maps satisfying the hypotheses of the previous theorem. Hence, immersions are I-cellular.*

Proof. Let $\sigma : C \to D$ be an arbitrary morphism of computads. We will define, by induction, a tower of factorisations of σ,

$$\begin{tikzcd}
C \arrow[r, "\sigma^0"] \arrow[drrr, bend right, "\sigma"'] & P^0 \arrow[r, "\sigma^1"] \arrow[drr, "\rho^0"] & P^1 \arrow[r, "\sigma^2"] \arrow[dr, "\rho^1"] & \cdots \\
& & & D
\end{tikzcd}$$

where $U_n\rho^n = \mathsf{id}$. Intuitively, the computad P^n looks like D up to dimension n and like C above that dimension. The morphism ρ_n above dimension $n+1$ is given by σ. More precisely, the tower can be defined inductively starting from $P^{-1} = C$ and $\rho^{-1} = \sigma$. For the inductive step, define first

$$U_{n+1}P^{n+1} = D_{n+1}, \qquad U_{n+1}\rho^{n+1} = \mathsf{id}, \qquad U_{n+1}\sigma^{n+1} = U_{n+1}(\rho^n),$$

and then, for $k \geq n+1$, define

$$\begin{aligned}
U_{k+1}P^{n+1} &= (U_k P^{n+1}, V_{k+1}^{P^n}, \phi_{k+1}^{P^{n+1}}), \\
\phi_{k+1}^{P^{n+1}} &= \mathsf{Type}_k(U_k\sigma^{n+1})\phi_{k+1}^{P^n}, \\
U_{k+1}\sigma^{n+1} &= (U_k\sigma^{n+1}, v \mapsto \mathsf{var}\ v), \\
U_{k+1}\rho^{n+1} &= (U_k\rho^{n+1}, \rho_{k+1,V}^n).
\end{aligned}$$

It is easy to see inductively that the morphisms above are well-defined and make the diagram commute. Moreover, if σ is an immersion, then the σ^n satisfy the hypothesis of the previous proposition.

To finish the theorem, it remains to show that the ρ^n form a colimit cocone under the diagram of the σ^n in $\omega\mathsf{Cat}$. Since the cells of P^n of dimension at most n agree with those of D, it is easy to see that this becomes a colimit cocone after applying $\mathsf{Cell} = U^w K^w$. The forgetful functor U^w reflects filtered colimits, since the monad fc^w preserves them, so this is also a colimit of ω-categories. $\square$

Corollary 6.5. *Let C a computad. The free ω-category $K^w C$ is the colimit of its n-skeletons*

$$\emptyset = K^w \, \mathsf{Sk}_{-1} \, C_{-1} \rightarrowtail K^w \, \mathsf{Sk}_0 \, C_0 \rightarrowtail K^w \, \mathsf{Sk}_1 \, C_1 \rightarrowtail \cdots,$$

which for every $n \in \mathbb{N}$ fit in a pushout square

$$\begin{tikzcd} \coprod_{v \in V_n^C} K^w \mathbb{S}^{n-1} \arrow[r, tail, "\iota_n"] \arrow[d] & \coprod_{v \in V_n^C} K^w \mathbb{D}^n \arrow[d] \\ K^w \, \mathsf{Sk}_{n-1} \, C_{n-1} \arrow[r, tail] & K^w \, \mathsf{Sk}_n \, C_n \end{tikzcd}$$

In particular, $K^w C$ is I-cellular.

Proof. The tower of skeletons of C is exactly the tower described in the proof of Theorem 6.4 associated to the immersion $\emptyset \to C$. $\square$

6.2 The universal cofibrant replacement

The previous corollary describes how to define morphisms out of those ω-categories that are free on a computad. We will use this to define, for each $-1 \leq n \leq \omega$, a right adjoint $W_n : \omega\mathsf{Cat} \to \mathsf{Comp}_n^{\mathsf{var}}$ to the inclusion $K^w \, \mathsf{Sk}_n \, \zeta_n$ taking an n-computad to the free ω-category on it.

The functor W_{-1} is the unique functor to the terminal category, and it is right adjoint to $K^w \, \mathsf{Sk}_{-1} \, \zeta_{-1}$, since the latter sends the unique (-1)-computad to the initial ω-category. Suppose that, for some $n \in \mathbb{N}$, we have W_n, right adjoint to $K^w \, \mathsf{Sk}_n \, \zeta_n$. Let r_n be the counit of this adjunction. For an ω-category X, we define $W_{n+1} X = (W_n X, V_{n+1}^{WX}, \phi_{n+1}^{WX})$ using the following pullback square:

$$\begin{tikzcd} V_{n+1}^{WX} \arrow[rr, dashed, "k_{n+1,X}"] \arrow[dd, dashed, "\phi_{n+1}^{WX}"] & & X_{n+1} \arrow[d, "\wr"] \\ & & \omega\mathsf{Cat}(\mathbb{D}^{n+1}, X) \arrow[d, "(\iota_{n+1})^*"] \\ \mathsf{Type}_n(W_n X) \arrow[r, "\sim"] & \omega\mathsf{Cat}(\mathbb{S}^n, K^w \, \mathsf{Sk}_n \, \zeta_n W_n X) \arrow[r, "(r_n)_*"] & \omega\mathsf{Cat}(\mathbb{S}^n, X) \end{tikzcd}$$

The natural isomorphisms in this diagram are given by $\mathbb{D}^{n+1}$ being free on a representable, by $\mathbb{S}^n$ classifying types, and K^w being fully faithful respectively.

For a homomorphism $f : X \to Y$, we define $W_{n+1}f = (W_n f, (W_{n+1}f)_V)$ to be the variable-to-variable $(n+1)$-homomorphism given by

$$(W_{n+1}f)_V(A, x) = \mathsf{var}\,(\mathsf{Type}_n(W_n f)(A), f(x)).$$

In order to show that W_n is right adjoint to $K^w\,\mathsf{Sk}_{n+1}\,\zeta_{n+1}$, let C be an $(n{+}1)$-computad, and let X be an ω-category. A variable-to-variable $(n+1)$-homomorphism $\sigma : C \to W_{n+1}X$ consists of

- a variable-to-variable morphism $\sigma_n : C_n \to W_n X$,
- a function $\sigma_V^1 : V_{n+1}^C \to X_{n+1}$,
- a function $\sigma_V^2 : V_{n+1}^C \to \mathsf{Type}_n(W_n X)$,

such that

$$\begin{aligned} \mathsf{src}\ \sigma_V^1 &= r_{n,X}(\mathsf{pr}_1\,\sigma_V^2), \\ \mathsf{tgt}\ \sigma_V^1 &= r_{n,X}(\mathsf{pr}_2\,\sigma_V^2), \\ \sigma_V^2 &= \mathsf{Type}_n(\sigma_n)\phi_{n+1}^C. \end{aligned}$$

The first two equations are equivalent to $\sigma_V = (\sigma_V^1, \sigma_V^2)$ defining a function into V_{n+1}^{WX}, while the third is equivalent to (σ_n, σ_V) being an $(n{+}1)$-homomorphism. In the presence of this third equation, the function σ_V^2 is redundant and the other equations amount to

$$\begin{aligned} \mathsf{src}\ \sigma_V^1 &= r_n(K^w\,\mathsf{Sk}_n\,\zeta_n\sigma_n)(\mathsf{pr}_1\,\phi_n^C) = r_n(K^w\,\mathsf{Sk}_n\,\zeta_n\sigma_n)\ \mathsf{src} \\ \mathsf{tgt}\ \sigma_V^1 &= r_n(K^w\,\mathsf{Sk}_n\,\zeta_n\sigma_n)(\mathsf{pr}_2\,\phi_n^C) = r_n(K^w\,\mathsf{Sk}_n\,\zeta_n\sigma_n)\ \mathsf{tgt} \end{aligned}$$

By the inductive hypothesis, such data are in bijection to pairs of a homomorphism $\sigma_n^\dagger : \mathsf{Sk}_n\,C \to X$ and a function σ_V^1 as above, satisfying

$$\mathsf{src}\ \sigma_V^1 = \sigma_n^\dagger\ \mathsf{src}, \qquad\qquad \mathsf{tgt}\ \sigma_V^1 = \sigma_n^\dagger\ \mathsf{tgt},$$

which is the data of a homomorphism $\sigma_{n+1}^\dagger : \mathsf{Sk}_{n+1}\,C \to X$ by Corollary 6.5. It is easy to see that this bijection is natural in X and C. The component of the counit r_{n+1} at X is the homomorphism given by $r_{n,X}$ and the projection $k_{n+1,X}$.

Suppose finally that W_n has been defined for all $n < \omega$ and $u_{n+1}W_{n+1} = W_n$. For each ω-category X, we define the computad WX to consist of the n-computads $(W_n X)$, and we define f similarly on homomorphisms. A variable-to-variable morphism $\sigma : C \to WX$ consists of a variable-to-variable n-homomorphisms $\sigma_n : C_n \to W_n X$ for every $n \in N$, satisfying that $u_{n+1}\sigma_{n+1} = \sigma_n$. Using the explicit description of the counit above and the snake equations, it is easy to see that such families of n-homomorphisms are in bijection to morphisms $\sigma_n^\dagger : \mathsf{Sk}_n\,C \to X$ forming a cocone under the diagram in Corollary 6.5. Hence, they are in bijection to morphisms $\sigma^\dagger : C \to X$.

Definition 6.6. The *computadic resolution* (Q, r, Δ) is the comonad induced by the adjunction $K^w\zeta \dashv W$.

We will show that the pair (Q, r) satisfies the hypothesis of Theorem 6.1. Since initial objects are unique up to unique isomorphism, this implies that, for each ω-category X, the morphism $r_X : QX \to X$ is its universal cofibrant replacement. We need to construct, for each $n \in \mathbb{N}$, each type $A \in \mathsf{Type}_n(WX)$, and each variable $x \in X_n$, a lift of the following form:

$$\begin{tikzcd} K^w\mathbb{S}^{n-1} \arrow[r, "A"] \arrow[d, "K^w\iota_n"'] & QX \arrow[d, "r_X"] \\ K^w\mathbb{D}^n \arrow[r, "x"] \arrow[ur, dashed, "\phi(n,x,A)"] & X \end{tikzcd}$$

Commutativity of this square is equivalent to (A, x) being a variable of $W_{n+1}X$. Furthermore, the homomorphism r_X acts on variables like a projection; that is,

$$r_X(\mathsf{var}\,(A, x)) = x.$$

Hence, we may define the required lift by

$$\phi(n, A, x) = \mathsf{var}\,(A, x).$$

We claim that (r_X, ϕ) is initial among objects above X equipped with such a choice of lifts. Given $f : Y \to X$ with a choice of lifts ψ, we will define a sequence of homomorphisms $g_n : \mathsf{Sk}_n\, W_nX \to Y$ inductively. First, we define g_{-1} to be the unique homomorphism out of the initial ω-category. The homomorphism g_{n+1} is then given by g_n on cells of dimension at most n and sends the variable (x, A) to

$$g_{n+1}(\mathsf{var}\,(x, A)) = \psi(n + 1, g \circ A, x).$$

Finally, the homomorphism $g : QX \to Y$ is the one that agrees with g_n on cells of dimension at most n. The homomorphism g is the only one that can preserve the lifts strictly. We have that

$$g \circ \phi(n, A, x) = \psi(n, g \circ A, x).$$

Furthermore, any other homomorphism with this property would have to agree on all variables of QX with g. For the same reason, we have that $fg = r_X$. Hence, g is indeed the unique morphism above X preserving lifts. To summarise, Theorem 6.1 allows us to conclude the following proposition.

Proposition 6.7. *The pair (Q, r) underlies the universal cofibrant replacement comonad on $\omega\mathsf{Cat}$ that is generated by the set of inclusions of spheres into disks.*

7 Comparison to other notions of ω-categories

Our goal in this section is to show that our notion of ω-category coincides with that of Leinster [22]. We will first introduce a class of morphisms called *covers* and show that every morphism factors uniquely into a cover followed by an immersion. Using this observation, we will give an alternative characterisation of full types and show that morphisms of computads $\mathsf{Free}\,B \to \mathsf{Free}\,X$ also admit a *(generic, free) factorisation.* This implies that the monad fc^w has arities, in the sense of [10], in the full subcategory Θ_0 of Glob consisting of the globular sets of positions of Batanin trees. Furthermore, the globular theory Θ_w associated to this monad is homogeneous. We will finally show that under the equivalence of homogeneous theories over the theory Θ_s of strict ω-categories and Batanin's globular operads established in [9] and [10], the theory Θ_w corresponds to the initial contractible globular operad of Leinster [22]. This implies that fc^w is isomorphic to Leinster's monad, and hence that our notion of ω-category coincides with that of Leinster.

7.1 Free variables and covers

Since $\mathsf{Comp}^{\mathsf{var}}$ is a presheaf category, every variable-to-variable morphism can be factored into an epimorphism followed by a monomorphism. We will now extend this factorisation to an orthogonal factorisation system on the whole of Comp; every morphism of computads factors essentially uniquely into a *cover* followed by an *immersion.* Recall that immersion are the variable to variable monomorphisms. Covers will be defined below in terms of *free variables*, and we will see that, a posteriori, they coincide with epimorphisms of computads.

Definition 7.1. Suppose that $c \in \mathsf{Cell}_n(C)$ is a cell in a computad. For each $k \in \mathbb{N}$, we define the subset $\mathsf{fv}_k(c) \subseteq V_k^C$ of k-dimensional *free variables* in c inductively:

- If $k > n$, then
$$\mathsf{fv}_k(c) = \emptyset.$$
- If $n = 0$, and $k = 0$, then
$$\mathsf{fv}_0(c) = \{c\}.$$
- If $n > 0$, and $c = \mathsf{var}\,v$. Then, we define
$$\mathsf{fv}_n(c) = \{v\},$$
and, for each $k < n$, we define
$$\mathsf{fv}_k(c) = \mathsf{fv}_k(\mathsf{src}\ \mathsf{var}\ v) \cup \mathsf{fv}_k(\mathsf{tgt}\ \mathsf{var}\ v),$$
where the union is taken inside V_k^C.
- If $n > 0$, and $c = \mathsf{coh}(B, A, \tau)$ is a coherence cell, then we define for $k \leq n$,
$$\mathsf{fv}_k(c) = \bigcup_{\substack{m \in \mathbb{N},\\ p \in \mathsf{Pos}_m(B)}} \mathsf{fv}_k(\tau_{m,V}(p)).$$

The k-dimensional *free variables* of a morphism of computads $\sigma : D \to C$ is defined by

$$\mathsf{fv}_k(\sigma) = \bigcup_{\substack{n \in \mathbb{N}, \\ v \in V_n^D}} \mathsf{fv}_k(\sigma_{n,V}(v)).$$

Remark 7.2. This notion of k-dimensional free variables subsumes the one given in Section 3, in that, for each n-cell c, we have that $\mathsf{fv}_n(c) = \mathsf{fv}(c)$.

We can also define the k-dimensional variables of a cell c in an n-computad C by viewing c as a cell of $\mathsf{Sk}_n\, C$. Similarly, we can define the free variables of a morphism σ of n-computads to be the free variables of $\mathsf{Sk}_n\, \sigma$. The following lemmas, that can be easily shown by induction, demonstrate how free variables allows us construct lifts along immersions and epimorphisms of computads.

Lemma 7.3. *Let $c \in \mathsf{Cell}_n(C)$ a cell in a computad and $k \in \mathbb{N}$. Then*

$$\mathsf{fv}_k(\mathsf{src}\ c) \cup \mathsf{fv}_k(\mathsf{tgt}\ c) \subseteq \mathsf{fv}_k(c).$$

Moreover, for every morphism of computads $\sigma : C \to D$,

$$\mathsf{fv}_k(\mathsf{Cell}_n(\sigma)(c)) = \bigcup_{\substack{m \in \mathbb{N}, \\ v \in \mathsf{fv}_m(c)}} \mathsf{fv}_k(\sigma_{m,V}(c))$$

Lemma 7.4. *Let $\sigma : C \to D$ be a variable-to-variable morphism. The k-dimensional free variables of σ are those of the form $\sigma_{k,V}(v)$, for some variable $v \in V_k^C$.*

Lemma 7.5. *Let $\sigma : D' \to D$ be an n-immersion, and let $\tau : C \to D$ be an n-homomorphism, for some $n \leq \omega$. Then, there exists a unique $\tau' : C \to D'$ such that $\tau = \sigma\tau'$ if and only if $\mathsf{fv}_k(\tau) \subseteq \mathsf{fv}_k(\sigma)$ for all $k \in \mathbb{N}$.*

Lemma 7.6. *Two homomorphisms $\sigma, \sigma' : C \to D$ agree on a cell $c \in \mathsf{Cell}_n(C)$ if and only if they agree on its free variables.*

Remark 7.7. Lemma 7.4 implies in particular that the k-dimensional free variables of a variable-to-variable homomorphism σ only depend on its truncation $U_k\sigma$. This is not true in general. Consider for example the computad corresponding to the tree $B_2 = \mathsf{br}\,[D_0, D_0]$ of Example 2.4 consisting of two composable 1-cells $f : x \to y$ and $g : y \to z$. THe homomorphism $\sigma : \mathbb{D}^1 \to \mathsf{Free}\, B_2$ picking the composite $\mathsf{comp}(f, g)$ has y in its free variables. However, y is not in the free variables of σ_0.

Definition 7.8. We say that a morphism of computads $\sigma : D \to C$ is a *cover* when $\mathsf{fv}_k(\sigma) = V_k^C$ for all $k \in \mathbb{N}$. We say that a cell $c \in \mathsf{Cell}_n(C)$ *covers* C when the corresponding morphism $\mathbb{D}^n \to C$ is a cover.

Example 7.9. By Lemma 7.3, covers are closed under composition and every isomorphism is both a cover and an immersion.

Example 7.10. A coherence cell $\mathsf{coh}\,(B, A, \tau)$ is a cover if and only if τ is. In particular, cells of the form $\mathsf{coh}\,(B, A, \mathsf{id})$ are always covers.

Remark 7.11. A direct consequence of Lemma 7.6 is that covers are epimorphisms. Since $\mathsf{Comp}^{\mathsf{var}}$ is balanced, it follows that a morphism is both a cover and an immersion exactly when it is an isomorphism.

Remark 7.12. Lemmas 7.3 and 7.5 imply that the free variables of a morphism $\sigma : C \to D$ form a computad fim_σ and that σ factors into a cover followed by an immersion via fim_σ. This factorisation is easily seen to be unique. Hence, covers and immersions form an orthogonal factorisation systems in Comp. One corollary of this result is that epimorphism are covers.

The following lemma and its converse, which is shown in Proposition 7.32, offers an equivalent characterisation of full types in terms of covers.

Lemma 7.13. *Let A a full n-type of a Batanin tree B. There exist unique cells a, b covering the boundary $\partial_n B$ such that*

$$\mathsf{pr}_1\, A = \mathsf{fc}^w(s_n^B)(a) \qquad\qquad \mathsf{pr}_2\, A = \mathsf{fc}^w(t_n^B)(b)$$

Proof. By the inductive definition of full types and Lemma 7.3, we can easily see that the k-dimensional free variables of $\mathsf{pr}_1\, A$ must contain all source boundary k-positions when $k \leq n$, and that equality holds when $k = n$. Lemma 7.3 and Proposition 2.10 imply then that the k-dimensional free variables of $\mathsf{pr}_1\, A$ and $\mathsf{Free}\,(s_n^B)$ coincide. The lifting lemma 7.5 implies existence and uniqueness of a, while 7.4 implies that a covers $\partial_n B$. The existence of b follows similarly. $\square$

7.2 Generic-free factorisations

Let Θ_0 the category with objects Batanin trees and morphisms of globular sets between their sets of positions

$$\Theta_0(B, B') = \mathsf{Glob}(B, B')$$

Our goal in this section is to show that the monad fc^w has arities in Θ_0 in the sense of [10]. To do that, we will construct a *generic*, variable-to-variable factorisation for every homomorphism of the form $\mathsf{Free}\, B \to \mathsf{Free}\, X$.

Definition 7.14. Let X, Y globular sets. We will say that a morphism of computads $\sigma : \mathsf{Free}\ X \to \mathsf{Free}\ Y$ is *generic* when for every solid commutative square of the form

$$\begin{tikzcd} \mathsf{Free}\ X \arrow[r, "\tau"] \arrow[d, "\sigma"'] & \mathsf{Free}\ Z \arrow[d, "g"] \\ \mathsf{Free}\ Y \arrow[r, "f"'] \arrow[ur, dashed, "h"] & \mathsf{Free}\ W \end{tikzcd}$$

with f, g variable-to-variable, there exists a unique h making the entire diagram commute.

Example 7.15. Every isomorphism is generic. If a variable-to-variable morphism $\sigma : \mathsf{Free}\ X \to \mathsf{Free}\ Y$ is generic then it is an isomorphism by the existence of a lift to the following square:

$$\begin{tikzcd} \mathsf{Free}\ X \arrow[r, equal] \arrow[d, "\sigma"'] & \mathsf{Free}\ X \arrow[d, "\sigma"] \\ \mathsf{Free}\ Y \arrow[r, equal] \arrow[ur, dashed] & \mathsf{Free}\ Y \end{tikzcd}$$

Remark 7.16. Generic homomorphisms correspond precisely to the fc^w-generic morphisms of Berger et al [10] under the adjunction $\mathsf{Free} \dashv \mathsf{Cell}$, since variable-to-variable homomorphisms correspond to free ones and Free is faithful.

Lemma 7.17. *Suppose that we have Batanin trees B_1, B_2, B_3 and a commutative square*

$$\begin{tikzcd} \mathsf{Free}\ B_1 \arrow[r, two heads, "\tau"] \arrow[d, "\sigma"'] & \mathsf{Free}\ B_3 \arrow[d, "g"] \\ \mathsf{Free}\ B_2 \arrow[r, "f"'] & \mathsf{Free}\ X \end{tikzcd}$$

where σ is generic, τ is a cover, and f and g are variable-to-variable. Then, $B_2 = B_3$, $f = g$ and $\sigma = \tau$.

Proof. By definition of σ being generic, there exists some $h : \mathsf{Free}\ B_2 \to \mathsf{Free}\ B_3$ such that $f = gh$ and $\tau = h\sigma$. By the first equation, h must be variable-to-variable, so $h = \mathsf{Free}\, h'$ for some morphisms of globular sets $h' : B \to B'$. By the second one, h must be a cover, so by Lemma 7.4, h' is epic. It follows from [9, Lemma 1.3] that h' must be an identity morphism. $\square$

Let Bat be the globular set whose n-cells are Batanin trees B of dimension at most n, and whose n-source and n-target are both given by the boundary operator ∂_n. The assignment $B \mapsto \mathsf{Pos}\, B$ can be extended to a functor $\mathsf{Pos} : \mathsf{el}(\mathsf{Bat}) \to \mathsf{Glob}$ out of the category of elements of Bat by sending the source and target morphisms $(n, \partial_n B) \to (n+k, B)$ to the morphisms s_n^B and t_n^B respectively. To define the (generic, free)-factorisation, we use the following proposition of Weber which says that the class of Batanin trees is closed under certain colimits.

Proposition 7.18. *[31, Proposition 4.7] Let $f : \mathsf{Pos}\,(B) \to \mathsf{Bat}$ a morphism where B is a Batanin tree. There exists (unique) Batanin tree B_f of dimension at most $\dim B$ such that*

$$\mathsf{colim}(\mathsf{Pos} \circ \mathsf{el}(f)) = \mathsf{Pos}\ B_f.$$

Moreover, this construction commutes with the source inclusions in that

$$\mathsf{colim}(\mathsf{Pos} \circ \mathsf{el}(f \circ s_n^B)) = \mathsf{Pos}\ \partial_n B_f$$

and the canonical morphism between the colimits is s_n^B, and similarly it commutes with target inclusions.

Proposition 7.19. *Suppose that B is a Batanin tree, X is globular set and $\sigma : \mathsf{Free}\ B \to \mathsf{Free}\ X$ a homomorphism. Then σ factors uniquely as*

$$\mathsf{Free}\ B \xrightarrow{\mathsf{gen}_\sigma} \mathsf{Free}\ B_\sigma \xrightarrow{\mathsf{fr}_\sigma} \mathsf{Free}\ X$$

for some Batanin tree B_σ of dimension at most $\mathsf{dim}\ B$, some generic gen_σ and some variable-to-variable morphism fr_σ.

Proof. Uniqueness of such factorisation is a direct consequence of the lifting property defining generics and [9, Lemma 1.3]. We will construct a generic, variable-to-variable factorisation for every homomorphism $\sigma : \mathsf{Free}\ ^B \to \mathsf{Free}\ X$ by induction on $\mathsf{dim}\, B$ and $\mathsf{mor\text{-}depth}\, \sigma_{\mathsf{dim}\ B}$. We will simultaneously show that the factorisation commutes with source and target inclusions; i.e. we have that $B_{\sigma \circ \mathsf{Free}\ s_k^B} = \partial_k B_\sigma$, and the following diagram commutes

$$\begin{tikzcd}
\mathsf{Free}\ \partial_k B \arrow[r, "\mathsf{gen}_{\sigma \circ \mathsf{Free}\ s_k^B}"] \arrow[d, "\mathsf{Free}\ s_k^B"'] & \mathsf{Free}\ \partial_k B_\sigma \arrow[dr, "\mathsf{fr}_{\sigma \circ \mathsf{Free}\ s_k^B}"] \arrow[d, dashed, "\mathsf{Free}\ s_k^{B_\sigma}"'] & \\
\mathsf{Free}\ B \arrow[r, "\mathsf{gen}_\sigma"] & \mathsf{Free}\ B_\sigma \arrow[r, "\mathsf{fr}_\sigma"'] & \mathsf{Free}\ B'
\end{tikzcd}$$

and a similar results holds for target inclusions.

Suppose first that $B = D_n$ is a globe and that $\mathsf{mor\text{-}depth}\, \sigma_n = 0$. In this case, σ must already be variable-to-variable, so we may set $\mathsf{gen}_\sigma = \mathsf{id}$ and $\mathsf{fr}_\sigma = \sigma$. Commutativity of this construction with source and target inclusions follows from $\sigma \circ \mathsf{Free}\ s_k^{D_n}$ and $\sigma \circ \mathsf{Free}\ t_k^{D_n}$ also being variable-to-variable morphisms out of a globe.

Suppose now that $B = D_n$ and $\mathsf{mor\text{-}depth}\, \sigma_n > 0$. Then σ corresponds to a coherence cell $\mathsf{coh}\,(B', A, U_n\tau)$ for unique morphism $\tau : \mathsf{Free}\ B' \to \mathsf{Free}\ X$. If we let $\tilde{\sigma} : \mathbb{D}^n \to \mathsf{Free}\ B'$ the cover corresponding to the cell $\mathsf{coh}(B', A, \mathsf{id})$, then $\sigma = \tau\tilde{\sigma}$, and we let

$$B_\sigma = B_\tau, \qquad \mathsf{gen}_\sigma = \mathsf{gen}_\tau\, \tilde{\sigma}, \qquad \mathsf{fr}_\sigma = \mathsf{fr}_\tau\,.$$

To show that gen_σ is generic, consider a commutative square where f, g are variable-to-variable.

$$\begin{tikzcd}
\mathbb{D}^n \arrow[r, "c"] \arrow[d, "\tilde{\sigma}"'] & \mathsf{Free}\, Y \arrow[dd, "f"] \\
\mathsf{Free}\ B' \arrow[ur, dashed, "\rho"] \arrow[d, "\mathsf{gen}_\tau"'] & \\
\mathsf{Free}\ B_\tau \arrow[r, "g"'] \arrow[uur, dashed, "h"] & \mathsf{Free}\, Z
\end{tikzcd}$$

Since f is variable-to-variable, the cell corresponding to c must be of the form $\mathsf{coh}(B, A, U_n\rho)$ for unique morphism ρ making the diagram above commute. The existence of unique h making the entire diagram commute follows then by gen_τ being generic.

When $k \geq n$, the source and target inclusions are identities, so this factorisation commutes trivially with them. let $k < n$ and consider the k-cell $a = \mathsf{src}^{(n-k-1)}(\mathsf{pr}_1\, A)$ of $\mathsf{Free}\, X$. This is the source of a full type and corresponds to the morphism $\tilde{\sigma} \circ \mathsf{Free}\ s_k^{D_n} : \mathbb{D}^k \to \mathsf{Free}\ B'$. Therefore, by Lemma 7.13, there exists unique cover $\tilde{a} : \mathbb{D}^k \to \mathsf{Free}\ \partial_k B'$ making the left square below commute.

$$
\begin{tikzcd}
\mathbb{D}^k \arrow[r, "\tilde{a}"] \arrow[d, "\mathsf{Free}\ s_k^{D_n}"'] & \mathsf{Free}\ \partial_k B' \arrow[r, "\mathsf{gen}_{\tau \circ \mathsf{Free}\ s_k^{B_0}}"] \arrow[d, "\mathsf{Free}\ s_k^{B_0}"'] & \mathsf{Free}\ \partial_k B_\tau \arrow[d, "\mathsf{Free}\ s_k^{B_\tau}"'] \arrow[dr, "\mathsf{fr}_{\tau \circ \mathsf{Free}\ s_k^{B_0}}"] & \\
\mathbb{D}^n \arrow[r, "\tilde{\sigma}"] & \mathsf{Free}\ B_0 \arrow[r, "\mathsf{gen}_\tau"] & \mathsf{Free}\ B_\tau \arrow[r, "\mathsf{fr}_\tau"] & \mathsf{Free}\, X
\end{tikzcd}
$$

The rest of the diagram commutes from the inductive hypothesis on τ. The top row is a cover, variable-to-variable factorisation of $\sigma \circ \mathsf{Free}\ s_k^{D_n}$. By Lemma 7.17, it must coincide with the generic, variable-to-variable factorisation, so constructed factorisation of σ commutes with source inclusions. A similar argument shows that it commutes with target inclusions.

Finally, let B be an arbitrary Batanin tree and suppose that for every position $(l, p) \in \mathsf{el}(\mathsf{Pos}\ B)$ we have constructed the generic, variable-to-variable factorisation of the morphism $\hat{p} : \mathbb{D}^l \to \mathsf{Free}\ X$ corresponding to the cell $\sigma_{l,V}(p)$.

$$
\mathsf{Free}\ D_l \xrightarrow{\mathsf{gen}_{\hat{p}}} \mathsf{Free}\ B_{\hat{p}} \xrightarrow{\mathsf{fr}_{\hat{p}} = \mathsf{Free}\ \mathsf{fr}'_{\hat{p}}} \mathsf{Free}\, X
$$

Since the generic, variable-to-variable factorisations commute with source and target inclusions, this factorisation is functorial in (l, p). Taking colimits, we obtain morphisms

$$
\begin{aligned}
\mathsf{colim}_{\mathsf{el}(\mathsf{Pos}\ B)}\, \mathsf{Free}\, \mathsf{Pos}\ D_k &\xrightarrow{\mathsf{gen}_\sigma} \mathsf{colim}_{\mathsf{el}(\mathsf{Pos}\ B)}\, \mathsf{Free}\, \mathsf{Pos}\ B_{\hat{p}} \\
&\xrightarrow{\mathsf{fr}_\sigma = \mathsf{Free}\ \mathsf{fr}'_\sigma} \mathsf{colim}_{\mathsf{el}(\mathsf{Pos}\ B)}\, \mathsf{Free}\, X.
\end{aligned}
$$

The first term is $\mathsf{Free}\ B$ by cocontinuity of Free and the density lemma. The third term is $\mathsf{Free}\, X$ since it is the colimit a constant diagram. The composite of the two morphisms is easily seen to be σ. Commutativity of generic, variable-to-variable factorisations with source and target inclusions implies that the middle term is a colimit of the form mentioned in Proposition 7.18, and so it is of the form $\mathsf{Free}\ B_\sigma$ for some Batanin tree B_σ. Moreover, this construction coincides with the one before in the case of globes, and commutes with the source and target inclusions by the last part of 7.18.

It remains to show that gen_σ is generic, so consider a commutative square as the one below where f, g are variable-to-variable

$$
\begin{tikzcd}
\mathbb{D}^l \arrow[r, dashed, "\mathsf{Free}\ i_p"] \arrow[d, "\mathsf{gen}_{\hat{p}}"'] & \mathsf{Free}\ B \arrow[r, "\tau"] \arrow[d, "\mathsf{gen}_\sigma"] & \mathsf{Free}\, Y \arrow[d, "g"] \\
\mathsf{Free}\ B_{\hat{\sigma}} \arrow[r, dashed, "\mathsf{Free}\ j_p"'] \arrow[urr, dashed, bend left=10, "h_p"] & \mathsf{Free}\ B_\sigma \arrow[r, "f"'] \arrow[ur, dashed, "h"] & \mathsf{Free}\ Z
\end{tikzcd}
$$

For every position (l, p), let i_p and j_p the morphisms of globular sets forming the colimit cocones over $\mathsf{Pos}\ B$ and $\mathsf{Pos}\ B_\sigma$ respectively. Since $\mathsf{gen}_{\hat{p}}$ is generic, there exists unique lift h_p to the diagram above. Uniqueness of those lifts shows that they are compatible with source and target inclusions, so they give a morphism h out of the colimit. Uniqueness of h follows from that of the lifts h_p. ☐

Corollary 7.20. *The monad* fc^w *has arities in* Θ_0.

Proof. By [10, Proposition 2.5], fc^w has arities in Θ_0 if and only if for every Batanin tree B, globular set X, and morphism $f : B \to \mathsf{fc}^w X$, a certain category of factorisations of f is connected. The (generic, variable-to-variable)-factorisation of the transpose $f^\dagger : \mathsf{Free}\ B \to \mathsf{Free}\ X$ corresponds to a factorisation of f into an fc^w-generic followed by a free morphism. This factorisation is by definition an initial object of the aforementioned factorisation category. Therefore fc^w has arities in Θ_0. ☐

Corollary 7.21. *Let B, B' be Batanin trees. A morphism* $\sigma : \mathsf{Free}\ B \to \mathsf{Free}\ B'$ *is a cover if and only if it is generic. In particular, generics between Batanin trees are closed under composition.*

Proof. Suppose that σ is a cover, and consider the following square:

$$\begin{tikzcd} \mathsf{Free}\ B \arrow[r, "\sigma"] \arrow[d, "\mathsf{gen}_\sigma"'] & \mathsf{Free}\ B' \arrow[d, "\mathrm{id}"] \\ \mathsf{Free}\ B_\sigma \arrow[r, "\mathsf{fr}_\sigma"'] & \mathsf{Free}\ B \end{tikzcd}$$

Lemma 7.17 implies that $\sigma = \mathsf{gen}_\sigma$ is generic. Conversely, suppose that σ is generic. By uniqueness of the generic-free factorisation, we have that $\sigma = \mathsf{gen}_\sigma$. However, it follows from the explicit description of gen_σ in the proof of Proposition 7.19 that gen_σ is a cover. ☐

7.3 Globular theories

Leinster's ω-categories [22] are algebras for the initial normalised *globular operad* with a contraction. By the equivalence of globular operads and *homogeneous globular theories* established in [3, 6.6.8], [9, Proposition 1.16] and [10, Theorem 3.13], we may equivalently speak of homogeneous globular theories with contractions. We equip the theory Θ_w, corresponding to fc^w, with a contraction. We will then show that Θ_w is the initial theory equipped with a contraction, and conclude that Leinster's operad is isomorphic to fc^w. We begin by recalling the relevant notions.

Definition 7.22. Let $F : \mathbb{G} \to \mathcal{C}$ a category under the category of globes. A *globular sum* in $\mathcal{C}$ is a colimit of a diagram of the form

$$\begin{tikzcd}[column sep=small] F([n_0]) & & \dots & & F([n_k]) \\ & F([m_1]) \arrow[ul, "F(t)"] \arrow[ur, "F(s)"'] & & F([m_k]) \arrow[ul, "F(t)"] \arrow[ur, "F(s)"'] & \end{tikzcd}$$

where n_i and m_j are elements of $\mathbb{G}$.

Example 7.23. Viewing Glob as a category under $\mathbb{G}$ using the Yoneda embedding, Remark 2.6 explains that globular sums coincide with Batanin trees.

Definition 7.24 ([9, Definition 1.5], [3, 2.2.6])**.** Let Θ_0 be the category with objects Batanin trees and arrows the morphisms between their globular sets of positions. A *globular theory* is a category Θ_A, together with a bijective on objects, faithful functor $j_A : \Theta_0 \hookrightarrow \Theta_A$ preserving globular sums. A *morphism of globular theories* $i : \Theta_A \to \Theta_B$ is a functor such that $j_B = j_A i$.

Example 7.25. Every faithful monad T on Glob induces a globular theory Θ_T, such that $\Theta_T(B, B') = T\text{-}\mathsf{Alg}(TB, TB')$. In particular, the free ω-category monad fc^w defines a globular theory $\Theta_w = \Theta_{\mathsf{fc}^w}$.

Definition 7.26 (See [9, Definition 1.15], [3, 2.7.1] and [10, Definition 3.9])**.** An *immersion* in a globular theory Θ_A is a morphism of Θ_0. We say that a morphism f of Θ_A is *homogeneous* if whenever $f = ig$ for some immersion i, we have that $i = \mathsf{id}$. We say that Θ_A is *homogeneous* if every morphism factors uniquely as $f = ic$ with i an immersion and c a cover.

In light of Proposition 4.6, Θ_w may be identified with the full subcategory of Comp whose objects are free on a Batanin tree. Since every morphism between the globular sets of positions of Batanin trees is monic (see [9, Lemma 1.3]), the two notions of immersions coincide. Furthermore, Proposition 7.19 and Corollary 7.21 imply that the two notions of covers also coincide. Hence, we have the following result:

Proposition 7.27. *The globular theory Θ_w is* homogeneous.

Suppose that Θ_s is the homogeneous globular theory whose arrows are morphisms between the *strict* ω-categories generated by Batanin trees. More concretely, for each Batanin tree B with $\mathsf{dim}\, B \leq n$, there is a unique cover $c_n^B : D_n \to B$. We follow [3, 2.7.6] and say that that a globular theory Θ_A is *homogeneous over* Θ when it comes equipped with a morphism $\Theta_A \to \Theta_s$ that preserves (and reflects) covers.

Remark 7.28. By [3, 6.6.8] the category of homogeneous globular theories over Θ_s is equivalent to category of *globular operads* in the sense of [6]. (See also [9, Proposition 1.16], [10, Theorem 3.13].) Given a homogeneous globular theory Θ_A over Θ_s, the corresponding globular operad O_A is defined so that, for each Batanin tree B and $n \in \mathbb{N}$, an operation of $O_{A,n}(B)$ corresponds exactly to a cover $\sigma : D_n \to B$ in Θ_A.

Proposition 7.29. *The globular theory Θ_w is homogeneous over Θ_s.*

Proof. We define an identity-on-objects functor $S : \Theta_w \to \Theta_s$. On immersions we set $S\,\mathsf{Free}\, i = i$. Suppose that $\sigma : \mathbb{D}^n \to \mathsf{Free}\; B$ is a cover in Θ_w. It follows from Proposition 7.19 that $\mathsf{dim}\, B \leq n$. Hence, we define $S\sigma = c_n^B$. Since every cover $B \to B'$ in Θ_w is a globular sum of covers of the form $\mathbb{D}^k \to B'$, and covers are closed under globular sums in both Θ_w and Θ_s, these data suffice to define S on all covers. Since every morphism in Θ_w factors uniquely as a cover followed by an immersion, this suffices to define S on all arrows in Θ_w. It is easily verified that this construction is functorial, and preserves and reflects covers. □

Definition 7.30. A homogeneous globular theory over Θ_s is *normalised* when there is a unique cover $D_0 \to D_0$, namely id_{D_0}. A *contraction* on a normalised homogeneous globular theory consists of, for each $n \in \mathbb{N}$, each Batanin tree B with $\dim B \leq n+1$, and each pair of covers $c, d : D_n \to \partial_n B$ that are parallel, i.e.

$$c \circ s_{n-1}^{D_n} = d \circ s_{n-1}^{D_n} \qquad \text{and} \qquad c \circ t_{n-1}^{D_n} = d \circ t_{n-1}^{D_n},$$

a choice of cover $l^{c,d} : D_{n+1} \to B$ such that

$$l^{c,d} \circ s_n^{D_{n+1}} = s_n^B \circ c \qquad \text{and} \qquad l^{c,d} \circ t_n^{D_{n+1}} = t_n^B \circ d.$$

Remark 7.31. Contractibility is usually viewed as a property of globular operads. Under the equivalence of Remark 7.28, the contractions described here correspond to the contractions of globular operads described by Leinster [22].

In order to equip Θ_w with a contraction and show that it is initial among globular theories with a contraction, we will need the following characterisation of full types.

Proposition 7.32. *Let A an n-type of a Batanin tree B. Then A is full if and only if there exist n-cells a, b that cover $\partial n B$ such that*

$$\mathsf{pr}_1\, A = \mathsf{fc}^w(s_n^B)(a), \qquad \mathsf{pr}_2\, A = \mathsf{fc}^w(s_n^B)(b).$$

Moreover, if $\dim B \leq n+1$, this is equivalent to the existence of an $(n+1)$-cell of type A that covers B.

Proof. Let $n \in \mathbb{N}$ and suppose that for all $m < n$ and all m-types of some Batanin tree, the proposition holds. Let B a Batanin tree and A a full n-type of it. The existence of covers a, b of $\partial_n B$ has then been shown in Lemma 7.13. When $\dim B \leq n+1$, a cover of B of type A is given by $\mathsf{coh}(B, A, \mathsf{id})$.

Suppose now A is a type for which such a, b exist. If $n = 0$, then A is full by definition of full 0-types, so let $n > 0$. The n-cell a covers $\partial_n B$, so by the inductive hypothesis, its type is full and there exist cells a', b' that cover $\partial_{n-1} B =$ such that

$$\mathsf{src}\, a = \mathsf{fc}^w(s_{n-1}^{\partial_n B})(a'), \qquad \mathsf{tgt}\, a = \mathsf{fc}^w(t_{n-1}^{\partial_n B})(b').$$

By the inductive hypothesis, we conclude that the type of $\mathsf{pr}_1\, A$ must be full, since

$$\mathsf{src}\,(\mathsf{pr}_1\, A) = \mathsf{fc}^w(s_n^B s_{n-1}^{\partial_n B})(a') = \mathsf{fc}^w(s_{n-1}^B)(a'),$$
$$\mathsf{tgt}\,(\mathsf{pr}_1\, A) = \mathsf{fc}^w(s_n^B t_{n-1}^{\partial_n B})(b') = \mathsf{fc}^w(t_{n-1}^B)(b').$$

The n-dimensional free variables of $\mathsf{pr}_1\, A$ and $\mathsf{pr}_2\, B$ must be those of $\mathsf{Free}\ s_n^B$ and $\mathsf{Free}\ t_n^B$ respectively, hence the source and target boundary n-positions. Therefore, the type A is full.

Let now again $n \in \mathbb{N}$ arbitrary, B a tree of dimension at most $n+1$ and $c \in \mathsf{Cell}_{n+1}(\mathsf{Free}\ B)$ a cover. It remains to show that there exist covers a, b of $\partial_n B$ such that

$$\mathsf{src}\ c = \mathsf{fc}^w(s_n^B)(a) \qquad\qquad \mathsf{tgt}\ c = \mathsf{fc}^w(t_n^B)(b).$$

It follows from Lemmas 7.5 and 7.4 that existence of those cells is equivalent to the k-dimensional free variables of $\mathsf{src}\ c$ and $\mathsf{tgt}\ c$ being precisely those of $\mathsf{Free}\ s_n^B$ and $\mathsf{Free}\ t_n^B$ respectively for all $k \in \mathbb{N}$.

We can easily see that it suffices for the k-dimensional free variables of $\mathsf{src}\ c$ to contain all source boundary k-positions for all $k \leq n$, and similarly for $\mathsf{tgt}\ c$. By Proposition 2.10 and Lemma 7.3, if this is the case then the k-dimensional free variables of $\mathsf{src}\ c$ contain those of $\mathsf{Free}\ s_n^B$ for all $k \in \mathbb{N}$. Since for $k < n$, the free variables of $\mathsf{Free}\ s_n^B$ contain all k-positions, it suffices to show that the n-dimensional free variables of $\mathsf{src}\ c$ do not contain any positions that are not source boundary. To see that consider the generic, variable-to-variable factorisation

$$\mathbb{D}^n \xrightarrow{\mathsf{gen}_a} \mathsf{Free}\ B_a \xrightarrow{\mathsf{fr}_a} \mathsf{Free}\ B$$

of the homomorphism corresponding to the cell a and recall that $\mathsf{dim}\, B_a \leq n$ and that fc_a must be an immersion. Since every position is parallel to a source boundary one, if the n-dimensional free variables of $\mathsf{src}\ c$ contained some non-source boundary n-position, they would contain two parallel n-positions. Then by the factorisation above, the Batanin tree B_a would have two non-parallel top dimensional positions, which we can see inductively is impossible.

We are left to show that the k-dimensional free variables of $\mathsf{src}\ c$ contain all source boundary k-positions for all $k \leq n$ when $c \in \mathsf{Cell}_{n+1}(\mathsf{Free}\ B)$ is a cover, and a similar statement for $\mathsf{tgt}\ c$. We will instead prove the following stronger claim: If $c \in \mathsf{Cell}_{n+1}(\mathsf{Free}\ B)$ is an arbitrary cell and $p \in \partial_k^s(B)$ for $k \leq n$ is in the free variables of c, then it is in the free variables of $\mathsf{src}\ c$. We will do so by induction on the cell c.

If $c = \mathsf{var}\ p'$ is a variable, then its source and target are parallel variable cells as well so for $k < n$,

$$\mathsf{fv}_k(c) = \mathsf{fv}_k(\mathsf{src}\ c) = \mathsf{fv}_k(\mathsf{tgt}\ c).$$

Moreover, $\mathsf{tgt}\ p'$ can not be source boundary, so the only n-position in the free variables of c that may be source boundary is $\mathsf{src}\ p' \in \mathsf{fv}_n(\mathsf{src}\ c)$. Therefore, the statement holds for variable cells.

Suppose now that $c = \mathsf{coh}(B', A', \tau)$ is a coherence cell and fix $k \leq n$. Then by definition of the free variables of c, we are left to show that for all $(l, q) \in \mathsf{el}(\mathsf{Pos}\ B')$,

$$\mathsf{fv}_k(\tau_{l,V}\, q) \cap \partial_k^s B \subseteq \mathsf{fv}_k(\mathsf{src}\ c).$$

As the type A' is full, the free variables of $\mathsf{pr}_1\, A'$ must be those of $s_n^{B'}$, so by Lemma 7.3,

$$\mathsf{fv}_k(\mathsf{src}\ c) = \bigcup_{\substack{l \in \mathbb{N} \\ q \in \mathsf{fv}_l(\mathsf{Free}\ s_n^{B'})}} \mathsf{fv}_k(\tau_{l,V}\, q)$$

Proposition 2.10 shows that $\mathsf{fv}_l(\mathsf{Free}\ s_n^{B'})$ contains all l-positions for $l < n$ and exactly the source boundary n-positions of B'. Hence, we are left to show the inclusion above when $l = n+1$, or when $l = n$ and q is not source boundary.

Let $q_0 \in \mathsf{Pos}_n(B')$ not source boundary. Then there exists some position $q \in \mathsf{Pos}_{n+1}(B')$ with target q_0. Let $q_1 \in \mathsf{Pos}_n(B')$ the source of q. By induction, we may assume that the claim holds for the cell $\tau_{V,l}(q)$, so

$$\partial_k^s B \cap \mathsf{fv}_k(\tau_{n,V}\, q_0) \subseteq \partial_k^s B \cap \mathsf{fv}_k(\tau_{n+1,V}\, q) \subseteq \cap\, \mathsf{fv}_k(\tau_{n,V}\, q_1).$$

If q_1 is source boundary, then we are done. Otherwise, we may repeat this process to get a position q_2 and so forth. This process is guaranteed to terminate in finitely many steps, since the positions of a Batanin tree are well-ordered and $p_1 < p_0$ [31, Section 4]. Finally, let $q' \in \mathsf{Pos}_{n+1}(B')$, then

$$\partial_k^s B \cap \mathsf{fv}_k(\tau_{V,n+1}\, q') \subseteq \partial_k^s B \cap \mathsf{fv}_k(\tau_{V,n}(\mathsf{src}\, q)) \subseteq \mathsf{fv}_k(\mathsf{src}\ c).$$

This shows that the claim also holds for coherence cells. □

Proposition 7.33. *The globular theory Θ^w can be equipped with a contraction.*

Proof. The globular theory Θ^w is clearly normalised. We now equip Θ^w with a contraction. Suppose that $n > 0$, and that B is a Batanin tree with $\mathsf{dim}\, B \le n$. Let $c, d : \mathbb{D}^{n-1} \to \mathsf{Free}\ \partial B$ parallel pair of covers. Then $\mathsf{Free}(s_n^B)(c)$ and $\mathsf{Free}(t_n^B)(d)$ constitute a full $(n-1)$-type A of B. We define $l^{c,d} : D_n \to B$ to be the morphism corresponding to the cell $\mathsf{coh}(B, A, \mathsf{id})$. This choice satisfies the required properties by construction. □

The following result now follows immediately from this definition together with the proof of Proposition 7.19.

Proposition 7.34. *Suppose that X is a globular set. Suppose that $n > 0$, and that $\sigma : \mathbb{D}^n \to \mathsf{Free}\, X$ is a morphism of computads. Then either σ is variable-to-variable, or σ can be uniquely written as a composite*

$$\mathbb{D}^n \xrightarrow{l^{c,d}} \mathsf{Free}\ B' \xrightarrow{\sigma'} \mathsf{Free}\, X$$

for some unique covers c, d, some unique Batanin tree B', and some unique morphism e' such that $\mathsf{mor\text{-}depth}\, \sigma' < \mathsf{mor\text{-}depth}\, \sigma$.

Theorem 7.35. *The globular theory Θ_w with the above choice of contraction is the initial normalised homogeneous globular theory over Θ_s with contraction.*

Proof. Suppose that $i : \Theta_0 \to \Theta$ is a normalised homogeneous globular theory over Θ_s with contraction. We now define a functor $F : \Theta^w \to \Theta$. Since Free is injective on objects, we may define $F\,\mathsf{Free}\,B = iB$ for each object B in Θ_w. Similarly, since Free is faithful, for each immersion $\sigma : B \to B'$, we may define $F\,\mathsf{Free}\,\sigma = i\sigma$. Since every morphism in Θ_w factors into a cover followed by an immersion, it remains to define F on covers.

Suppose that $\sigma : B \to B'$ is a cover. If $\mathsf{dim}\,B = 0$, then we must have that $\sigma = \mathsf{id}_{D^0}$, since Θ_w is normalised. In this case $F\sigma = \mathsf{id}_{FD^0}$. Hence, suppose that $\mathsf{dim}\,B = n > 0$. We define $F\sigma$ by induction on n, and we simultaneously show that for each morphism $\tau : B' \to B''$, we have that $F\tau \circ F\sigma = F(\tau \circ \sigma)$. Suppose that we have defined F on covers whose domain has dimension less than n. Then we define $F\sigma$ by induction on $\mathsf{mor\text{-}depth}\,\sigma$. We simultaneously show that the assignment F respects sources and targets:

$$F\sigma \circ Is^B_{n-1} = F(\sigma \circ \mathsf{Free}\,s^B_{n-1}).$$

and similarly for targets.

First suppose that $B = D^n$. When $\mathsf{mor\text{-}depth}\,\sigma = 0$, since B' is a globular set, the morphism σ is variable-to-variable, and so we have already defined $F\sigma$. Sources and targets are trivially preserved in this case. Hence, suppose that $\sigma = \sigma' \circ l^{c,d}$ for some unique morphism σ' with $\mathsf{mor\text{-}depth}\,\sigma' < \mathsf{mor\text{-}depth}\,\sigma$. Since σ is a cover and $l^{c,d}$ is a cover, we have that σ' is also a cover. Hence, we define

$$F\sigma = F\sigma' \circ l^{Fc,Fd}.$$

This is well defined by the uniqueness part of Proposition 7.34. Consider the following commutative diagram:

$$\begin{tikzcd}
ID^{n-1} \arrow[r, two heads, "Fc"] \arrow[d, tail, "Is^{D_n}_{n-1}"'] & I\partial B'' \arrow[d, tail, "Is^{B''}_{n-1}"'] \arrow[dr, tail, "F(\sigma' \circ \mathsf{Free}\,s^{\partial B''}_{n-1})"] & \\
ID^n \arrow[r, two heads, "l^{Fc,Fd}"'] & IB'' \arrow[r, tail, "F\sigma'"'] & IB'
\end{tikzcd}$$

The left hand square commutes by definition of $l^{Fc,Fd}$. The right hand triangle commutes by inductive hypothesis. The composite of the bottom row is $F\sigma$ by definition. Since, $n-1 < n$, the inductive hypothesis implies that the composite of the top row is $F(c \circ \sigma' \circ \mathsf{Free}\,s^{\partial B''}_{n-1})$. However,

$$F(c \circ \sigma' \circ \mathsf{Free}\,s^{\partial B''}_{n-1}) = F(c \circ \mathsf{Free}\,s^{B''}_{n-1} \circ \sigma') = F(Is^{D_n}_{n-1} \circ l^{c,d} \circ \sigma').$$

Thus, F respects sources. A similar argument shows that F respects targets.

Now suppose that B is an arbitrary Batanin tree with $\mathsf{dim}\,B = n$. Suppose that $\sigma : \mathsf{Free}\,B \to \mathsf{Free}\,B'$ is a cover. For each $(k, x) \in \mathsf{el}(B)$, we have

$$\mathsf{mor\text{-}depth}(\mathsf{gen}_{\sigma \circ \mathsf{Free}\,x}) = \mathsf{mor\text{-}depth}(\sigma \circ \mathsf{Free}\,x) \leq \mathsf{mor\text{-}depth}\,\sigma.$$

The equality follows from the fact that $\mathsf{fr}_{\mathsf{cov}_{\sigma \circ \mathsf{Free}\,x}}$ is variable-to-variable. The inequality follows from the definition of $\mathsf{mor\text{-}depth}$, and the Yoneda Lemma.

Hence, by the representable case, we can assume that we have defined

$$F(\sigma \circ \mathsf{Free}\, x) = F\, \mathsf{inc}_{\sigma \circ \mathsf{Free}\, x} \circ F\, \mathsf{cov}_{\sigma \circ \mathsf{Free}\, x}$$

However, $B = \mathsf{colim}_{(k,x) \in \mathsf{el}(B)} D^k$ is a canonical globular sum of representables. It follows that $iB = \mathsf{colim}_{(k,x) \in \mathsf{el}(B)} iD^k$. Hence, since F respects sources and targets, we may define $F\sigma : iB \to iB'$ to be the unique arrow in Θ such that, for each $(k,x) \in \mathsf{el}(B)$, we have that $F\sigma \circ ix = F(\sigma \circ \mathsf{Free}\, x)$.

Now suppose that $\tau : B' \to B''$. A straightforward induction on $\mathsf{mor\text{-}depth}\, \sigma$ now implies that $F\tau \circ F\sigma = F(\tau \circ \sigma)$. This completes the inductive definition of F.

In order to show that F is functorial, suppose that $f : A \to B$ and $g : B \to C$ are arrows in Θ_w. If both f and g are covers, or both f and g are immersions, or if f is a cover and g is an immersion, then functoriality follows from the definition of F. By homogeneity it now suffices to check the case where f is an immersion and g is a cover. Note that, by definition, we have that $F(g \circ f) = F\, \mathsf{inc}_{g \circ f} \circ F\, \mathsf{cov}_{g \circ f}$. Now suppose that $x : ID^n \to A$ is an n-cell of A. Let j and d be an immersion and a cover respectively such that $\mathsf{cov}_{g \circ f} \circ x = jd$. Consider the following commutative diagram:

$$\begin{tikzcd}
ID^n \arrow[r, two heads, "d"] \arrow[d, tail, "x"'] & \bullet \arrow[d, tail, "j"] \\
A \arrow[r, two heads, "\mathsf{cov}_{g \circ f}"] \arrow[d, tail, "f"'] & \bullet \arrow[d, tail, "\mathsf{inc}_{g \circ f}"] \\
B \arrow[r, two heads, "g"'] & C
\end{tikzcd}$$

Since g is a cover, we have that $Fg \circ F(f \circ x) = F(\mathsf{inc}_{g \circ f} \circ j) \circ Fd$. Hence, since F is functorial on immersions, we have that

$$Fg \circ Ff \circ Fx = F\, \mathsf{inc}_{g \circ f} \circ Fj \circ Fd.$$

On the other hand, since $\mathsf{cov}_{g \circ f}$ is a cover, we have that $F\, \mathsf{cov}_{g \circ f} \circ Fx = Fj \circ Fd$. Precomposing with $F\, \mathsf{inc}_{g \circ f}$, we obtain

$$F(g \circ f) \circ Fx = F\, \mathsf{inc}_{g \circ f} \circ F\, \mathsf{cov}_{g \circ f} \circ Fx = F\, \mathsf{inc}_{g \circ f} \circ Fj \circ Fd.$$

Hence, for all cells $x : ID^n \to A$, we have that

$$F(g \circ f) \circ Fx = Fg \circ Ff \circ Fx.$$

Since maps of the form $Fx : ID^n \to FA$ assemble into a colimit cone over A, it follows that $F(g \circ f) = Fg \circ Ff$.

By construction, the functor F is over Θ_s and preserves immersions, covers and globular sums of immersions. Conversely, any functor over Θ_s preserving these data must satisfy all the properties defining F. Thus, Θ^w is initial. □

In conclusion, we establish the desired comparison result.

Corollary 7.36. *The monad fc^w is the one induced by Leinster's operad. In particular, our notion of ω-category coincides with that of Leinster [22].*

Proof. Let T be the monad on globular sets induced by Leinster's operad. By Theorem 7.35, we have that $\Theta_{\mathsf{fc}^w} = \Theta_w \cong \Theta_T$ as globular theories. Therefore, by the equivalence described in [3, 6.6.8] between homogeneous globular theories and globular operads, viewed as cartesian monads over the strict ω-category monad fc^s, the monad fc^w must be isomorphic to T. Since Leinster's ω-categories are T-algebras, the two notions of ω-category coincide. $\square$

A Structural induction

The purpose of this appendix is to give an informal and gentle introduction to *structural induction* aimed at mathematicians. We hope that some readers may find this useful, since structural induction is not yet standard in the mathematical community, despite being a precise and powerful tool with wide use across theoretical computer science. Our introduction here will be necessarily brief, and there are many good resources for reading further about this concept, including textbooks by Pierce [27, Section 3.3], Winskel [32, Chapter 3] and Mitchell [26, Chapter 1] where a careful development can be followed. Today there is considerable activity in mathematics using *polynomial functors* to formalize notions of structural induction (see in particular the influential work of Gambino and Kock [16]) although we do not take that approach here.

Fundamental idea. Ordinary mathematical induction allows us to establish correctness of a *well-ordered* family of propositions. To be well-ordered, the family must in particular be *well-founded*, meaning there are no infinitely descending chains, and also totally ordered. Every proposition in such a family is either the base case or a successor, and if an element has a successor, that successor is unique.

Structural induction generalizes this idea. The well-founded requirement is retained, but the total order requirement is dispensed with, allowing the elements to form a partial order in general. This gives significant additional flexibility for defining and reasoning about mathematical structures, while retaining the power and rigour of inductive arguments. In practical terms, this means we still have base cases and successors, but these will no longer necessarily be unique. The successor structure in particular can now be far more complex, and is described in general by *constructors*.

Inductive sets. In this paper we frequently employ this idea to define an *inductive set*, which is simply a set presented by structural induction. This allows us to present notions of "formal syntax" in a precise way. We illustrate this idea by example. One of the simplest inductive sets is isomorphic to the ordinary natural numbers, and we can define it as follows:

$$\mathsf{zero} \in \mathbb{N}; \text{ and for all } x \in \mathbb{N},\ \mathsf{succ}\,(x) \in \mathbb{N}$$

Applying this definition, we can exhibit elements of $\mathbb{N}$ such as "zero", "$\mathsf{succ}\,(\mathsf{zero})$", "$\mathsf{succ}\,(\mathsf{succ}\,(\mathsf{zero}))$", and so on. These syntactic expressions do not merely *denote* the elements of $\mathbb{N}$; according to this definition, they *are* the elements of $\mathbb{N}$. That is, the elements we can exhibit in this way exhaust $\mathbb{N}$, and two such elements are equal only when their syntactic representations are identical.

Here the words "zero" and "succ" are formal tokens called *constructors*, which we can invoke to exhibit new elements of $\mathbb{N}$. The elements of $\mathbb{N}$ exhibited in this way are also sometimes called the *terms of type* $\mathbb{N}$, and the elements provided in brackets are called the *arguments*. For the purposes of this paper we assume only finite induction, so any element of $\mathbb{N}$ can be exhibited in finite time, and we also assume that any given inductive set will have only finitely many constructors.

As a mild extension of this idea, we can construct multiple sets by simultaneous structural induction. For example, we can construct the sets $\mathbb{O}$ and $\mathbb{E}$ of odd and even natural numbers respectively as follows:

$$\mathsf{zero} \in \mathbb{E};\text{ and for all } x \in \mathbb{E},\ \mathsf{f}(x) \in \mathbb{O};\text{ and for all } y \in \mathbb{O},\ \mathsf{g}(y) \in \mathbb{E}$$

So $\mathbb{E}$ contains elements such as "zero", "$\mathsf{g}(\mathsf{f}(\mathsf{zero}))$", "$\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{zero}))))$", and $\mathbb{O}$ contains elements such as "$\mathsf{f}(\mathsf{zero})$", "$\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{zero})))$". As before, the words "zero", "f" and "g" are constructors. We will make heavy use of this mutual induction technique later on when giving our definition of computad.

Defining functions by recursion. Given sets S, T where S is an inductive set, we can define a function $f : S \to T$ by *recursion* on the structure of S. (The manner of definition of T is irrelevant, but in our case it will again often be an inductive set.) We do this by *case analysis* on the different possible elements of S which we might encounter. Since S has only finitely many constructors, this gives a compact way of expressing our function.

Illustrating this concept once again by example, we consider a recursive definition of the function $\mathsf{add} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, building on the definition of $\mathbb{N}$ as an inductive set given above. Our definition of add is as follows:

$$\begin{aligned}\mathsf{add}(\mathsf{succ}(x), y) &:= \mathsf{succ}(\mathsf{add}(x, y))\\ \mathsf{add}(\mathsf{zero}, x) &:= x\end{aligned}$$

Note that any particular call $\mathsf{add}(p, q)$ for $p, q \in \mathbb{N}$ will match the left-hand side of precisely one of the statements above, since p must either be of the form $\mathsf{succ}(x)$ for some $x \in \mathbb{N}$, or equal to the base case constructor zero. When we apply the corresponding rule to obtain the resulting expression on the right-hand side, we say that we have *computed* the function by a single step.

For this definition of add we have chosen to recurse in the first argument; an alternative definition could instead recurse on the second argument.

Proof by structural induction. When defining a recursive function in this way, it is usually important to prove that it will actually terminate; i.e. that for any particular invocation of the function, as we continue to unfold the definition recursively, this process will conclude in finite time. There are usually many other properties that we would also like to establish: for example, that for all $x, y, z \in \mathbb{N}$ we have $\mathsf{add}(x, \mathsf{add}(y, z)) = \mathsf{add}(\mathsf{add}(x, y), z)$; that for all $x \in \mathbb{N}$ we have $\mathsf{add}(x, \mathsf{zero}) = \mathsf{add}(\mathsf{zero}, x) = x$; and so on.

When our recursive functions are defined on inductive sets, we can use *proof by structural induction* to establish these properties. This proceeds as follows. First, we establish the property for the base case. Then, we establish the property for each constructor, on the assumption that it holds for each argument. We conclude by structural induction that the property therefore holds for all terms. Here we see the power of the approach compared to ordinary mathematical induction: we can induct on the entire structure of the term, rather than being restricted to induct only over the natural numbers.

As an example, let us prove that the recursive function defined above does indeed terminate.

Theorem. *The function* add *terminates in finite time on all inputs.*

Proof. We prove the claim by structural induction on subterms. For the base case, we consider $\mathsf{add}(\mathsf{zero}, x)$. This is defined to be x, and so termination in a single step is clear. For the inductive case, we consider the term $t = \mathsf{add}(\mathsf{succ}(x), y)$. This computes in a single step to $\mathsf{succ}\,(\mathsf{add}(x, y))$, and so it suffices to show that $\mathsf{add}\,(x, y)$ computes in finite time. But since x and y are subterms of t, this follows from the inductive hypothesis. $\square$

Terminology. We note briefly that there is some inconsistency in the literature with the usage of the terms “induction” and “recursion”. What we have called a “proof by structural induction” some might call a “recursive proof”; and where above we “define functions by recursion”, others might have written “define functions by induction”. There are good reasons for this diverse terminology; we simply warn the reader that care must be taken when reading other works on this topic.

Induction-recursion. When building inductive sets in this paper, we use an additional technique called *induction-recursion*. The idea here is that at the same time as building the inductive set, we simultaneously define a recursive function on that set, and use it as a “side condition” for validating when certain constructors can be applied. Induction-recursion is in principle very powerful, being able to construct large classes (such as set-theoretic universes) in a way which cannot be done with mere inductive sets. However here we use only *small induction-recursion*, which has precisely the same power as ordinary induction [19]. Our use of it here allows for more streamlined definitions to be given.

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