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Intro to Algorithms
Drill 1

Pick Your Constants: Big-O

1)

Definition: $T(n) = O\big(f(n)\big)\ iff\ T(n) \leq c\ \times\ f(n)\ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\ ,f(n) = n^2$

Prove: $T(n) = O\big(f(n)\big)$,

$or: 32n^2 + 17n + 1 \leq c \times n^2\ for\ n \geq n_0$

Constants: $n_0 = 1\ , c = 51$

Substitute: $32n^2 + 17n + 1 \leq 51n^2\ for\ n \geq 1$

Prove By Induction:

$Proof\ for\ \ n = 1:$

$(32 + 17 + 1 = 50) < 51$

$Proof\ for\ n + 1:$

$Assume: 32n^2 + 17n + 1 < 51n^2$

$Show\ that: 32(n + 1)^2 + 17(n + 1) + 1 < 51(n + 1)^2$

$After\ expansion\ we\ have:$

$32n^2 + 64n + 32 + 17n + 17 + 1 < 51n^2 + 102n + 51$

$After\ reordering\ we\ have:$

$(32n^2 + 17n + 1) + (64n + 33) < (51n^2) + (102n + 51)$

$Using\ assumption, and\ 64n + 33 < 102n + 51,$

$if\ \ a < c\ and\ b < d, then\ a + b < c + d,$

$therefore\ 32(n + 1)^2 + 17(n + 1) + 1 < 51(n + 1)^2$ ■

2)

--For n

Definition: $T(n) = O\big(f(n)\big)\ iff\ T(n) \leq c\ \times\ f(n)\ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\ , f(n) = n$

Prove: $T(n) \neq O\big(f(n)\big),$

$or, 32n^2 + 17n + 1 \leq c \times n\ for\ n \geq n_0\ \ cannot\ hold$

$for\ any\ c\ and\ n_0.$

$No\ matter\ what\ c\ is, for\ all\ n \geq c\ the\ inequality\ does\ not\ hold,$

$because\ 32n^2 \geq 32nc > nc\ for\ all\ c > 0.$ ■

--For nlogn

Definition: $T(n) = O\big(f(n)\big)\ iff\ T(n) \leq c\ \times\ f(n)\ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\ , f(n) = n\log(n)$

Prove: $T(n) \neq O\big(f(n)\big),$

$or, 32n^2 + 17n + 1 \leq c \times n\log(n)\ for\ n \geq n_0\ \ cannot\ hold$

$for\ any\ c\ and\ n_0.$

$No\ matter\ what\ c\ is, for\ n \geq 2^c\ the\ inequality\ does\ not$

$hold, because\ 32(2^c)^2 + 17(2^c) + 1 > c \times 2^c \log(2^c)$

$for\ all\ c \geq 0, as\ follows\ from\ simplifying:$

$32(2^c)^2 + 17(2^c) + 1 > c^2 \times 2^c$ → pull c out of log, log(2)=1

$32(2^c) + 17 + \frac{1}{2^c} > c^2$ → divide both sides by 2^c

$We\ know\ that\ this\ new\ inequality\ will\ hold\ if\ we\ prove$

$that\ 32(2^c) > c^2.\ Since, by\ the\ popular\ lemma,$

$2^c > c^2\ for\ c \geq 4, and\ for\ 0 \leq c < 4,\ \ 32(2^c) \geq 32 > 16 > c^2,$

$it\ follows\ that\ 32(2^c) > c^2 for\ all\ c \geq 0.$ ■

$\boldsymbol{Alternatively}, we\ can\ show\ it's\ not\ O(n\log(n))\ using\ limits,$

$by\ showing\ that\ no\ matter\ the\ c, when\ n\ gets\ large\ enough\ the$

$definition\ for\ being\ O(n\log(n))\ will\ fail:$

$32n^2 + 17n + 1 > n^2 > c \times n \log(n)$ → contradiction to T(n)=O(f(n))

$\frac{n}{\log(n)} > c$→ divide both sides by $n \log(n)$

$\lim_{n\to\infty} \frac{n}{\log(n)} = \lim_{n\to\infty} n \ln(b) = \infty > c$ → L'Hopital (b is the base of the log)

■

Pick Your Constants: Big-Omega

1)

--For n²

Definition: $T(n) = \Omega(f(n))\ iff\ T(n) \geq c \times f(n)\ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1, f(n) = n^2$

Prove: $T(n) = \Omega(f(n))$,

$or: 32n^2 + 17n + 1 \geq c \times n^2\ for\ n \geq n_0$

Constants: $n_0 = 1, c = 1$

Substitute: $32n^2 + 17n + 1 \geq n^2\ for\ n \geq 1$

$31n^2 + 17n + 1 > 0\ for\ n \geq 1$ subtract n^2 from both sides ■

--For n

Definition: $T(n) = \Omega(f(n))\ iff\ T(n) \geq c \times f(n)\ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1, f(n) = n$

Prove: $T(n) = \Omega(f(n))$,

$or: 32n^2 + 17n + 1 \geq c \times n \ \ for\ n \geq n_0$

Constants: $n_0 = 1\,, c = 1$

Substitute: $32n^2 + 17n + 1 \geq n\ for\ n \geq 1$

$32n^2 + 16n + 1 > 0\ for\ n \geq 1$ ■

2)

Definition: $T(n) = \Omega(f(n))\ iff\ T(n) \geq c\ \times\ f(n)\ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\,, f(n) = n^3$

Prove: $T(n) \neq \Omega(f(n))$,

$or: 32n^2 + 17n + 1 \geq c \times n^3\ for\ n \geq n_0\ \ cannot\ hold\ for\ any$

$for\ any\ c\ and\ n_0$

$First\ we\ simplify:$

$32 + \frac{17}{n} + \frac{1}{n^2} \geq c \times n$ → divide both sides by n²

$Using\ limits, we\ will\ show\ that\ as\ n\ approaches\ \infty, the\ left$

$will\ be\ smaller\ than\ the\ right\ side, proving\ T(n) \neq \Omega(f(n))$:

$\lim_{n\to\infty} 32 + \frac{17}{n} + \frac{1}{n^2} = 32\ \ <\ \ \lim_{n\to\infty} c \times n\ = \infty\ (for\ c > 0)$ ■

Pick Your Constants: Big-Theta

1)

Definition: $T(n) = \theta(f(n))\ iff\ c_1 \times f(n) \leq T(n) \leq c_2\ \times\ f(n)\ \ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\,, f(n) = n^2$

Prove: $T(n) = \theta\big(f(n)\big)$,

$or\text{:}\ c_1 n^2 \leq 32n^2 + 17n + 1 \leq c_2 n^2\ \ for\ n \geq n_0$

Constants: $c_1 = 1, c_2 = 51, n_0 = 1$

Substitute: $n^2 \leq 32n^2 + 17n + 1 \leq 51n^2\ for\ n \geq 1$

Split it up:

$32n^2 + 17n + 1 \leq 51n^2\ \ for\ n \geq 1$

Using induction:

$Proof\ for\ \ n = 1$:

$(32 + 17 + 1 = 50) < 51$

$Proof\ for\ n + 1$:

$Assume: 32n^2 + 17n + 1 < 51n^2$

$Show\ that: 32(n + 1)^2 + 17(n + 1) + 1 < 51(n + 1)^2$

$After\ expansion\ we\ have$:

$32n^2 + 64n + 32 + 17n + 17 + 1 < 51n^2 + 102n + 51$

$After\ reordering\ we\ have$:

$(32n^2 + 17n + 1) + (64n + 33) < (51n^2) + (102n + 51)$

$Using\ assumption, and\ 64n + 33 < 102n + 51,$

$if\ a < c\ and\ b < d, then\ a + b < c + d,$

$therefore\ 32(n + 1)^2 + 17(n + 1) + 1 < 51(n + 1)^2$

$32n^2 + 17n + 1 \geq n^2\ for\ n \geq 1$

$31n^2 + 17n + 1 > 0\ for\ n \geq 1$ subtract n^2 from both sides ■

2)

--For n

Definition: $T(n) = \theta\big(f(n)\big)\ iff\ c_1 \times f(n) \leq T(n) \leq c_2\ \times\ f(n)\ \ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\ , f(n) = n$

Prove: $T(n) \neq \theta(f(n))$,

$or\colon\ c_1 n \leq 32n^2 + 17n + 1 \leq c_2 n\ \ for\ n \geq n_0\ cannot\ hold$

$for\ any\ c\ and\ n_0$

Simplify: $just\ need\ to\ show\ that\ 32n^2 + 17n + 1 \leq c_2 n\ for\ n \geq n_0$

$cannot\ hold\ for\ any\ c_2\ and\ n_0$

$No\ matter\ what\ c_2\ is, for\ n \geq c\ the\ inequality\ does\ not\ hold,$

$because\ 32n^2 + 17n + 1 \geq 32nc > nc\ for\ all\ c > 0.$ ■

--For n³

Definition: $T(n) = \theta(f(n))\ iff\ c_1 \times f(n) \leq T(n) \leq c_2\ \times\ f(n)\ \ for\ n \geq n_0$

Givens: $T(n) = 32n^2 + 17n + 1\ , f(n) = n^3$

Prove: $T(n) \neq\ \theta(f(n))$,

$or\colon\ c_1 n^3 \leq 32n^2 + 17n + 1 \leq c_2 n^3\ \ for\ n \geq n_0\ cannot\ hold$

$for\ any\ c\ and\ n_0$

Simplify: $just\ need\ to\ show\ that\ c_1 n^3 \leq 32n^2 + 17n + 1\ for\ n \geq n_0$

$cannot\ hold\ for\ any\ c_1\ and\ n_0$

$First\ we\ simplify$:

$32 + \frac{17}{n} + \frac{1}{n^2} \geq c_1 n$ → divide both sides by n^2

$Using\ limits, we\ will\ show\ that\ as\ n\ approaches\ \infty, the\ left$

$will\ be\ smaller\ than\ the\ right\ side, proving\ T(n) \neq \Omega(f(n))$:

$\lim_{n\to\infty} 32 + \frac{17}{n} + \frac{1}{n^2} = 32\ \ <\ \ \lim_{n\to\infty} c_1 n\ = \infty\ (for\ c_1 > 0)$ ■