Jason Caplan
Design and Analysis of Algorithms
Professor Leff
HW1

3 | Graph Properties

<u>Prove</u>: every graph with 2 or more vertices must include 2 vertices that have equal degrees.

<u>By Contradiction</u>:

- Assume the reverse, i.e. that there exists a graph with 2 or more vertices where all vertex degrees are distinct
- Notice that for a graph with n vertices, the minimum degree of a given vertex is 0 (not connected to any other vertex), and the maximum is n-1 (connected to every other vertex)
- We can then define the set of all the possible degree values as the integers between 0 and n-1, a total of n possible degree values
- However, the degree value 0 and the degree value n-1 are mutually exclusive:
    - If there is a vertex with degree 0 it is not connected to any other vertex, so no vertex can have degree n-1 (the maximum is n-2, since it cannot be connected to itself nor to the vertex with degree 0)
    - If there is a vertex with degree n-1, it is connected to every other vertex, so every other vertex has at least a degree of 1, i.e. there can be no vertex with degree 0
- Therefore, the possible degrees are not 0 ... $n-1$, but rather 0 ... $n-2$ OR 1 ... $n-1$, either way a total of n-1 possible degrees
- This leads to a contradiction: by the pigeonhole principle, we cannot assign n vertices distinct degree values from a set of n-1 possible degree values ∎

4 | Probability Properties

<u>Algorithm</u>: Perform pairs of coin flips. If you get (heads, tails), generate a 1. If you get (tails, heads), generate a 0. If you get doubles, i.e. (heads, heads) or (tails, tails), disregard this pair.

<u>Proof</u>:

- Assume the probability of getting heads or tails does not change within a given pair of flips
- Let N be the probability of getting heads on a single flip. It is some real number between 0 and 1 exclusive (we assume that both heads and tails have non-zero probability or occuring)
- It follows that the probability of getting tails is $(1 - N)$
- The probability of getting the exact sequence (heads, tails) is:

  $$N \times (1-N) = \boldsymbol{N - N^2}$$

- The probability of getting the exact sequence (tails, heads) is
  $(1-N) \times N = \boldsymbol{N - N^2}$
- Since the odds of getting (heads, tails) are equivalent to the odds of getting (tails, heads), there is an equal chance of generating a 0 and 1 respectively on a given pair of flips ∎