Contents

List of Abbreviations

EoS   **E**quation **o**f **S**tate
GR   **G**eneral **R**elativity
LE   **L**ane-**E**mden
TOV   **T**olman **O**ppenheimer **V**olkoff

List of Symbols

$\mathcal{F}$ Free Energy
$\mathcal{U}$ Internal Energy
$\mathcal{Z}$ Partition Function
$\mathcal{S}$ Entropy
m Mass
p Pressure
ρ Density

Units

This thesis uses the geometrized unit system $G = c = 1$. If not otherwise mentioned any variable named G or c can be set to 1 and is only written for clarification.

Abstract

This is not a good example for an abstract.

1 Introduction

Test

2 Thermodynamical Calculation of an EoS

This chapter aims at developing an fully special relativistic equation of state of a noninteracting gas. We briefly summarise important concepts necessary for the derivation.

In the canonical ensemble, for which a introduction can be found in [1], the internal energy $\mathcal{U}$ is obtained by the relation $\mathcal{U} = \mathcal{F} - TS$ where all quantities present are either variables such as T or can be derived by means of the partition function $\mathcal{Z}$. Microscopically the partition function is given by the behaviour of the N particles determined by the Hamiltonian H. For concepts and definitions of Hamilton Mechanics the reader is referred to [2, 3, 4]. In general we assume $H : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R}$ to be a positive smooth function that assigns the positions and momenta of N particles an energy. This explicitly takes form in the well known equations

$$\mathcal{Z}(T, V, N) = \int_{\mathcal{V} \times \mathbb{R}^N} \exp\left(-\frac{H(x_1, \ldots, x_N, p_1, \ldots, p_N)}{k_B T}\right) \frac{dx_1 dp_1 \ldots dx_N dp_N}{N! h^{3N}} \tag{2.0.1}$$

$$\mathcal{F}(T, V, N) = -k_B T \log\left(\mathcal{Z}(T, V, N)\right) \tag{2.0.2}$$

$$\mathcal{U}(T, V, N) = \mathcal{F} + TS \tag{2.0.3}$$

$$= \mathcal{F} - T\frac{\partial \mathcal{F}}{\partial T} \tag{2.0.4}$$

$$= k_B T^2 \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial T}. \tag{2.0.5}$$

where $x_i \in \mathcal{V} \subset \mathbb{R}^N$ and $p_i \in \mathbb{R}^N$. The relation $V = \text{vol}(\mathcal{V})$ describes the total volume occupied by the medium while T and N are its Temperature and particle Number respectively. The last equation shows that to fully calculate U it is necessary to obtain a readable output from the partition function $\mathcal{Z}$.

2.1 The Ultra-Relativistic Case

The ultra-relativistic case is a textbook example and well known limit that we will use in order to verify our later calculated results. First we write down the ultra-relativistic Hamiltonian given by

$$H(x, p) = ||p||c. \tag{2.1.1}$$

The corresponding partition function for a N particle system then reads ($\beta = (k_B T)^{-1}$)

$$\mathcal{Z} = \frac{V^N}{N! h^{3N}} \left[\int_{\mathbb{R}^3} \exp\left(-\beta H(p)\right) d^3 p\right]^N \tag{2.1.2}$$

$$= \frac{V^N}{N! h^{3N}} \left[\int_0^\infty 4\pi p^2 \exp(-\beta p c) dp\right]^N \tag{2.1.3}$$

$$= \frac{V^N}{N! h^{3N}} \frac{(4\pi)^N (k_B T)^{3N}}{c^{3N}} \left[\int_0^\infty x^2 \exp(-x) dx\right]^N \tag{2.1.4}$$

$$\mathcal{Z} = \frac{1}{N!} \left(8\pi V \left(\frac{k_B T}{hc}\right)^3\right)^N \tag{2.1.5}$$

where from the first to second line we used usual spherical coordinates and afterwards used the integral transformation $x = \beta c p$. The remaining integral $\mathcal{U}$ can be solved to be exactly $= 2$ and the internal energy can now be written down and further approximated to subjective needs.

2.2 The Special-Relativistic Case

We will now turn our attention to the fully special-relativistic Hamiltonian given by

$$H = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}}. \tag{2.2.1}$$

In doing so, we keep in mind that the ultra relativistic limit can be obtained from this Hamiltonian by letting $m \to 0$. In this limiting case we should be able to recover the results from equation (2.1.5).

$$\mathcal{Z} = \frac{V^N}{N! h^{3N}} \left[\int_{\mathbb{R}^3} \exp\left(-\frac{mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}}}{k_B T} \right) d^3 p \right]^N \tag{2.2.2}$$

$$= \frac{V^N}{N! h^{3N}} \left[\int_0^\infty 4\pi p^2 \exp\left(-\beta mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} \right) dp \right]^N \tag{2.2.3}$$

$$= \frac{(4\pi V)^N}{N!} \left(\frac{mc}{h} \right)^{3N} \left[\int_0^\infty q^2 \exp\left(-\alpha \sqrt{1 + q^2} \right) dq \right]^N \tag{2.2.4}$$

$$= \frac{(4\pi V)^N}{N!} \left(\frac{mc}{h} \right)^{3N} \left(\int_0^\infty \sinh(x)^2 \cosh(x) \exp(-\alpha \cosh(x)) dx \right)^N \tag{2.2.5}$$

$$= \frac{(4\pi V)^N}{N!} \left(\frac{mc}{h} \right)^{3N} \left(\int_0^\infty \frac{\sinh(2x)}{2} \sinh(x) \exp(-\alpha \cosh(x)) dx \right)^N \tag{2.2.6}$$

$$= \frac{(4\pi V)^N}{N!} \left(\frac{mc}{h} \right)^{3N} \left(-\frac{\sinh(2x)}{2\alpha} \exp(-\alpha \cosh(x)) \Big|_0^\infty \right. \tag{2.2.7}$$

$$\left. + \frac{1}{\alpha} \int_0^\infty \cosh(2x) \exp(-\alpha \cosh(x)) dx \right)^N \tag{2.2.8}$$

$$= \frac{(4\pi V)^N}{N!} \left(\frac{mc}{h} \right)^{3N} \left(\frac{1}{\alpha} \int_0^\infty \cosh(2x) \exp(-\alpha \cosh(x)) dx \right)^N \tag{2.2.9}$$

$$= \frac{1}{N!} \left(8\pi V \left(\frac{k_B T}{hc} \right)^3 \frac{\alpha^2 K_2(\alpha)}{2} \right)^N \tag{2.2.10}$$

In the first step we used spherical coordinates followed by the substitution $qmc = p$ and $\alpha = \beta mc^2 = mc^2 / k_B T$. Afterwards we substituted $q = \sinh(x)$ and used the identity

$\cosh(x)\sinh(x) = \sinh(2x)/2$. Partial integration then leads to the last integral which can be identified [5] as the modified Bessel function of the 2nd kind $K_2(\alpha)$. The equation is then rewritten such that the ultra relativistic limit can be read off upon letting $\alpha \to 0$. We can now calculate the internal energy $\mathcal{U}$ from $\mathcal{Z}$ via equation (2.0.5)

$$\mathcal{U} = 3Nk_BT - Nk_BT\alpha \left(\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)} + 2 \right) \tag{2.2.11}$$

$$\mathcal{U} = 3Nk_BT - mc^2 \left(\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)} + 2 \right). \tag{2.2.12}$$

Again, it can be seen that the ultra relativistic limit can be obtained by letting $\alpha \to 0$ since the term written in brackets vanishes. It is immediate that the ultra relativistic equation of state is $\rho = 3p$ which is equivalent to a traceless energy momentum tensor.

From equation (2.2.10) and (2.0.2), we immediately derive the ideal gas equation via the definition of pressure in the canonical ensemble

$$p = \frac{\partial \mathcal{F}}{\partial V} = \frac{Nk_BT}{V}. \tag{2.2.13}$$

Note that in the case of the Hamiltonian H having a non-trivial dependence on the coordinates x_i, the partition function would differ by a factor involving V and T. This more general scenario does not yield the ideal gas law and would thus need to be treated separately.

2.3   The fully special relativistic equation of state

This section aims to develop an equation between the thermodynamic energy density $\rho = \mathcal{U}/V$ and the pressure p of the gas given by the ideal gas equation (2.2.13). We assume an additional constraint and thus further reduce the degrees of freedom of the thermodynamic system.

When assuming an adiabatic condition $\delta Q = 0$ and using the First Law of Thermodynamics [1] $dU = \delta Q + \delta W$, where $\delta W = -pdV$ and $dU = C_V dT$, we can relate pressure and temperature. This adiabatic condition is quite well satisfied since radiational losses compared to the total amount of mass can be neglected. Using equation 2.2.12 and 2.2.13, we obtain

$$-pdV = C_V dT \tag{2.3.1}$$

$$-\frac{Nk_BT}{V}dV = Nk_B \left[1 + \alpha^2 \left(\left(\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)} \right)^2 - \frac{\partial_\alpha^2 K_2(\alpha)}{K_2(\alpha)} \right) \right] dT \tag{2.3.2}$$

$$-\frac{dV}{V} = \left[1 - \alpha^2 \partial_\alpha \left(\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)} \right) \right] \frac{dT}{T} \tag{2.3.3}$$

$$= \left[1 - \alpha^2 \partial_\alpha^2 (\log K_2(\alpha)) \right] \frac{dT}{T} \tag{2.3.4}$$

This equation also shows explicitly the T dependence of the specific heat C_V for the non-ultra-relativistic case. Again, taking the ultra-relativistic-limit by taking $\alpha \to 0$, one can calculate that the right hand term in the first equation converges to $-2Nk_B$. This agrees with the expected specific heat for an ultra-relativistic gas $C_{V,\mathrm{ur}} = 3Nk_B$. With

the identity $d\alpha/\alpha = -dT/T$ (using $\alpha = mc^2/k_BT$), we can transform the equation and integrate it. After applying partial integration, the result is

$$\frac{dV}{V} = \left(1 - \alpha^2\partial_\alpha^2 \log K_2(\alpha)\right)\frac{d\alpha}{\alpha} \tag{2.3.5}$$

$$\log\left(\frac{V}{V_0}\right) = \log\left(\frac{\alpha}{\alpha_0}\right) - \int_{\alpha_0}^{\alpha} \alpha\frac{\partial^2}{\partial\alpha^2}\log(K_2(\alpha'))d\alpha' \tag{2.3.6}$$

$$= \log\left(\frac{\alpha}{\alpha_0}\right) + \log\left(\frac{K_2(\alpha)}{K_2(\alpha_0)}\right) - \left[\alpha\frac{\partial_\alpha K_2}{K_2}\right]_{\alpha_0}^{\alpha} \tag{2.3.7}$$

This equation enables us to write down a relation between volume and temperature (encapsulated in $\alpha = mc^2/k_BT$)

$$V(\alpha) = \frac{\alpha K_2(\alpha)}{C}\exp\left(\alpha\frac{K_3(\alpha) + K_1(\alpha)}{2K_2(\alpha)}\right) \tag{2.3.8}$$

where the constant C is defined by the equation beforehand and only depends on the integration boundaries α_0 and V_0. It is given by

$$C = \frac{\alpha_0 K_2(\alpha_0)}{V_0}\exp\left(\alpha_0\frac{K_1(\alpha_0) + K_3(\alpha_0)}{2K_2(\alpha_0)}\right). \tag{2.3.9}$$

Since the goal of this section is to obtain a readable output for an equation of state, it is necessary to construct a bijection relating p and T. This is made clear when writing down the energy density

$$\rho = \frac{\mathcal{U}}{V} = \frac{Nk_BT}{V} - \frac{Nk_BT}{V}\left(\alpha\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)}\right) \tag{2.3.10}$$

where $p = Nk_BT/V$ can be easily identified but the T dependence through α is not solved yet.

The pressure p can be rewritten to take the form

$$p = \frac{Nk_BT}{V} = CNmc^2\frac{1}{K_2(\alpha)\alpha^2}\exp\left(-\alpha\frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right). \tag{2.3.11}$$

At this point it is not reasonable to ask what happens in the ur-limit since C depends non-trivially on m and m is not fully substituted in α.

Interestingly, the pressure seems to be constant for very high temperatures. The limiting case is obtained when taking $T \to \infty$ (which corresponds to $\alpha \to 0$)

$$\lim_{\alpha\to0}\left[\frac{1}{K_2(\alpha)\alpha^2}\exp\left(-\alpha\frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right)\right] = \frac{1}{2e^2} \approx 0.006767. \tag{2.3.12}$$

The same argument then holds true for the density given by equation 2.3.10 and since

$$\lim_{\alpha\to0}\left[1 + \alpha\frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right] = 3 \tag{2.3.13}$$

we have

$$\lim_{\alpha\to0}\left(\frac{\rho(\alpha)}{Cmc^2}\right) = \frac{3}{2e^2} \approx 0.203003. \tag{2.3.14}$$

Theorem 2.1
The mapping $p : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \alpha \mapsto p(\alpha)$ written down in equation (2.3.11) is a bijection for any $N, m, c, C \neq 0$.

$$p = \frac{N k_B T}{V} = C N m c^2 \frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2 K_2(\alpha)}\right) \tag{2.3.15}$$

Proof. For this proof it suffices to show that the function $p(\alpha)$ has a strictly monotonous[1] behaviour. Without loss of generality, we assume $N, m, c, C > 0$. Now it is obvious that the first two terms $Nmc^2/CK_2(\alpha)$ and α^{-2} are strictly decreasing. This is easy to see when using [5]

$$\frac{\partial K_n}{\partial \alpha} = \frac{n}{\alpha} K_n - K_{n+1} = -\frac{K_{n-1} + K_{n+1}}{2}. \tag{2.3.16}$$

We then calculate the derivative of the third term and divide by the exponential (since it is positive)

$$\frac{1}{\exp(\alpha \partial_\alpha \log(K_2))} \frac{\partial}{\partial \alpha} \exp(\alpha \partial_\alpha \log(K_2)) \tag{2.3.17}$$

$$= \partial_\alpha \log(K_2) + \alpha \partial_\alpha^2 \log(K_2) \tag{2.3.18}$$

$$= \frac{\partial_\alpha K_2}{K_2} + \alpha \frac{\partial_\alpha^2 K_2}{K_2} + \alpha \left(\frac{\partial_\alpha K_2}{K_2}\right)^2 \tag{2.3.19}$$

$$= \frac{K_1 - \frac{2}{\alpha} K_2}{K_2} + \alpha \frac{\frac{1}{\alpha} K_1 - K_2 - \frac{2}{\alpha}\left(K_1 - \frac{2}{\alpha} K_2\right) + \frac{2}{\alpha^2} K_2}{K_2^2} + \alpha \frac{K_1^2 - \frac{1}{\alpha} K_1 K_2 - \frac{4}{\alpha^2} K_2^2}{K_2^2} \tag{2.3.20}$$

$$= -\alpha + \alpha \frac{K_1^2}{K_2^2} - 4 \frac{K_1}{K_2} \tag{2.3.21}$$

thus it is sufficient to show that

$$\alpha \frac{K_1^2}{K_2^2} < \alpha + 4\frac{K_1}{K_2}. \tag{2.3.22}$$

We quickly prove the more general result $K_\nu < K_{\nu+1}$. One possible definition[5] for the Bessel function K_ν is given by

$$K_\nu := \frac{\sqrt{\pi}}{(\nu - \frac{1}{2})!} \left(\frac{1}{2}z\right)^\nu \int\limits_1^\infty e^{-tz} \left(t^2 - 1\right)^{\nu - \frac{1}{2}} dt \tag{2.3.23}$$

We inspect the ratio

$$\frac{K_\nu}{K_{\nu+1}} = \frac{1}{\left(\nu + \frac{1}{2}\right)\left(\frac{1}{2}z\right)} \frac{\int e^{-tz} \left(t^2 - 1\right)^{\nu - 1/2} dt}{\int e^{-tz} \left(t^2 - 1\right)^{\nu + 1/2} dt} \tag{2.3.24}$$

and rewrite the demoninator with partial integration

$$\frac{1}{2}z \int\limits_1^\infty e^{-tz} \left(t^2 - 1\right)^{\nu + 1/2} dt = \left(\nu + \frac{1}{2}\right) \int\limits_1^\infty e^{-tz} t \left(t^2 - 1\right)^{\nu - 1/2} dt. \tag{2.3.25}$$

Now it is obvious that $K_{\nu+1} > K_\nu$. Thus in total, the function given by equation 2.3.26 can be inverted. $\qquad\square$

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[1]We omit this word in this proof for readability.

With this mapping $p : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \alpha \mapsto p(\alpha)$ and its inverse $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, p \mapsto \alpha(p)$, we can use 2.3.10 and finally write down the equation of state

$$\rho = \frac{\mathcal{U}}{V} = p \left(1 + \alpha(p) \frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))} \right). \tag{2.3.26}$$

Figure 1 is obtained by choosing numerical values and then interpolating and inverting equation 2.3.11. The constant factor

$$A := CNmc^2/p_0 \tag{2.3.27}$$

is substituted to obtain independence of $p_0{}^2$. Furthermore the graphs of the plotted EOS are normalised such that

$$\rho_i(p_0) = \rho_{0,i} \tag{2.3.28}$$

and can thus be compared with each other. Note that the ρ_0 of the plot is not a universal value but rather each function has been scaled individually and ρ_0 is a placeholder for the corresponding $\rho_{0,i}$. The polytropic EOS $\rho_{cla}(p) = Ap^{1/\gamma}$ when normalised as before the equation is independent of A and can be uniquely characterised by n. For further details, the interested reader is referred to [6].
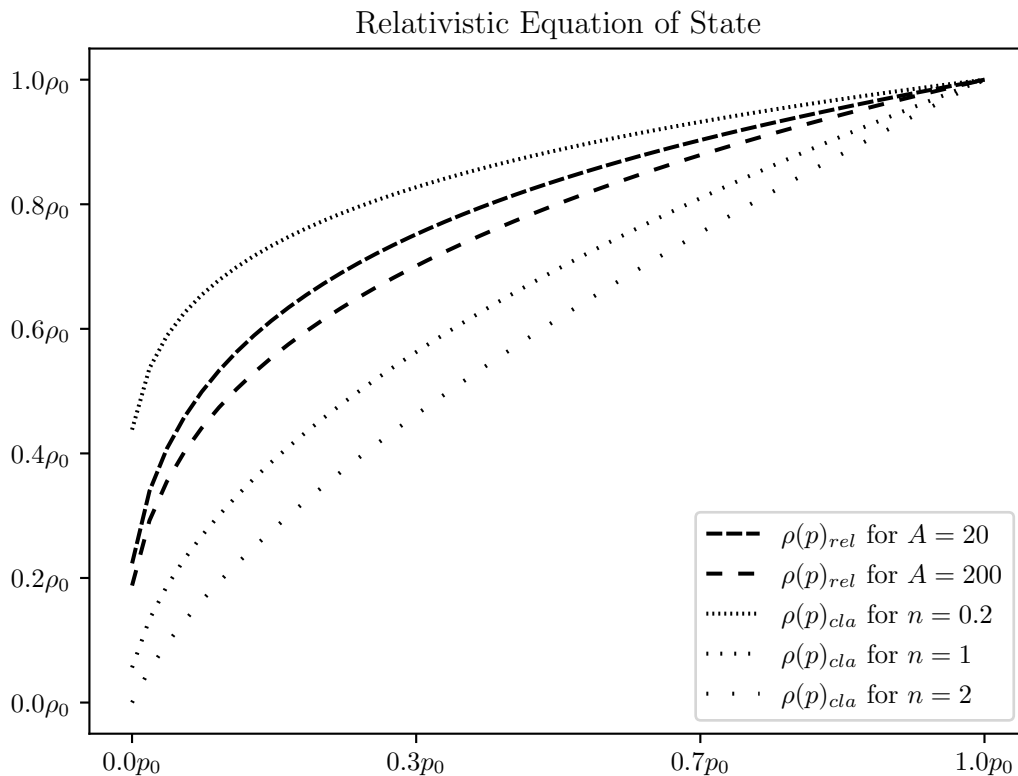

Figure 1: The plot shows the relativistic equation of state $\rho(p)$ normalised such that values can be compared with a polytropic EOS. Graphs for the relativistic version are independent of the exponent n which is a degree of freedom intrinsic to the polytropic equation of state. By normalisation, the graphs of the polytropic equation of state are independent of the factor A which is explained in equation 2.3.27.

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^{2}This will be the starting value for the pressure in the TOV equation. Since the pressure decreases from the inside of a star to the outside, this resembles our highest value of p.

3 Calculating the Mass of a Star with an EOS

3.1 Deriving the TOV-Equation

In this chapter, we consider a spherical-symmetric static Lorentz-Manifold (V, g) with charts such that the metric g can be written as

$$g = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{3.1.1}$$

The stress-energy tensor of an ideal fluid with density ρ and pressure p is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) \tag{3.1.2}$$

where u is the 4-velocity of the fluid. In the rest frame where $u^\mu = (-e^{-\nu/2}, 0, 0, 0)$, this equation simplifies to

$$T^\mu_\nu = \begin{bmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}. \tag{3.1.3}$$

The Christoffel symbols for this metric are

$$\Gamma^0_{\mu\nu} = \begin{bmatrix} 0 & \nu'/2 & & \\ \nu'/2 & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \tag{3.1.4}$$

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} \nu' e^{\nu-\lambda}/2 & & & \\ & \lambda'/2 & & \\ & & -re^{-\lambda} & \\ & & & -r\sin^2\theta e^{-\lambda} \end{bmatrix} \tag{3.1.5}$$

$$\Gamma^2_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & 0 & 1/r & \\ & 1/r & 0 & \\ & & & -\sin\theta\cos\theta \end{bmatrix} \tag{3.1.6}$$

$$\Gamma^3_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & 0 & 0 & 1/r \\ & 0 & 0 & \cos\theta/\sin\theta \\ & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix} \tag{3.1.7}$$

From these, the non-zero components of the Ricci-Tensor can be calculated

$$R_{11} = \frac{1}{4r} e^{-\lambda} \left[\left(2r\nu'' + r\nu'^2 \right) + (4 - r\lambda') \nu' \right] \tag{3.1.8}$$

$$R_{22} = -\frac{1}{4r} e^{-\lambda} \left[\left(2r\nu'' \right) + r\nu'^2 - r\lambda'\nu' - 4\lambda' \right] \tag{3.1.9}$$

$$R_{33} = -\frac{1}{2r^2} e^{-\lambda} \left(r\nu' - r\lambda' - 2e^{\lambda} + 2 \right) \tag{3.1.10}$$

$$R_{33} = R_{44} \tag{3.1.11}$$

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and with $R_{\mu\nu} - g_{\mu\nu}R/2 = G_{\mu\nu} = 8\pi T_{\mu\nu}$ ultimately yield the following field equations.

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \tag{3.1.12}$$

$$8\pi T_1^1 = 8\pi p = \nu'\frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \tag{3.1.13}$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{3.1.14}$$

Since $R_3^3 = R_2^2$, we omitted the last equation. From equation (3.1.12) we infer the relation.

$$e^{-\lambda} = 1 - \frac{2}{r}\int_0^r 4\pi\rho(r')r'^2 dr' = 1 - \frac{2m(r)}{r}. \tag{3.1.15}$$

The metric needs to be defined at every point in space and thus we can not have any additional integration constant in equation (3.1.15) since otherwise we would obtain a term a/r which is ill defined for $r \to 0$.

The property $m(r)$ can be recognized as the Newtonian Mass of the star (which is different to the proper mass). Since $e^{-\lambda} > 0$, we immediately see that $m(r) < r/2$.

In addition to the Field equations 3.1.12 to 3.1.14 the divergence of the Stress-Energy Tensor yields more information

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.1.16}$$

The following explicit calculation[3] shows how to obtain this additional restriction on the pressure and density.

$$\nabla_\mu T_\nu^\mu = \partial_\mu T_1^\mu + \Gamma_{\mu\sigma}^\mu T_\nu^\sigma - \Gamma_{\mu\nu}^\sigma T_\sigma^\mu \tag{3.1.17}$$

$$\nabla_\mu T_1^\mu = \frac{\partial p}{\partial r} + p\left(\Gamma_{01}^0 + \Gamma_{11}^1 + \Gamma_{21}^2 + \Gamma_{31}^3\right) - \Gamma_{\mu 1}^\sigma T_\sigma^\mu \tag{3.1.18}$$

$$= \frac{\partial p}{\partial r} + p\left(\frac{\nu' + \lambda'}{2} + \frac{2}{r}\right) + \rho\frac{\nu'}{2} - p\frac{\lambda'}{2} - p\frac{2}{r} \tag{3.1.19}$$

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2}\nu' \tag{3.1.20}$$

Together with equation 3.1.13 and the definition 3.1.15, we can write

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2}\left(\frac{8\pi p r + \frac{1 - e^{-\lambda}}{r}}{e^{-\lambda}}\right) \tag{3.1.21}$$

$$= -\frac{p + \rho}{2r}\left(\frac{8\pi p r + \frac{2m}{r^2}}{1 - \frac{2m}{r}}\right) \tag{3.1.22}$$

$$= -\frac{m\rho}{r^2}\left(1 + \frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m} + 1\right)\left(1 - \frac{2m}{r}\right)^{-1} \tag{3.1.23}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1 + \frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2} + 1\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \tag{3.1.24}$$

where in the last step the constants $c = G = 1$ were put back in.

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[3]Again assuming spherical symmetry.

3.2 Newtonian Limit

This section follows [7] and [8, pp. 89 sqq.]. Together with a polytropic equation of state $p = K\rho^{1+1/n}$ and the definition $\rho = \lambda\theta^n$, we expect to obtain the Newtonian behavior in the non-relativistic limit in the form of the Lane-Emden equation

$$\frac{K(n+1)\lambda^{1/n-1}}{4\pi}\Delta\theta + \theta^n = 0. \tag{3.2.1}$$

The usual non-relativistic limit is obtained from a Taylor expansion of equation 3.1.24 around $1/c^2$ in lowest order. The resulting equation then reads

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.2.2}$$

One could be tempted to make the assumption that $\frac{\partial p_{\text{TOV}}}{dr} \leq \frac{\partial p_{\text{LE}}}{dr}$. However this equation fails since the mass given in equation 3.1.24 is not the same as the one given in the LE equation. Using the previous relations for ρ and p and again setting $G = c = 1$, we can calculate

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}\left(K\rho^{1+1/n}\right) = K\lambda^{1+1/n}(n+1)\theta^n\frac{\partial\theta}{\partial r} = -\frac{m\lambda\theta^n}{r^2} \tag{3.2.3}$$

by using the definition of our polytropic equation of state and 3.2.2. Rearranging and taking the derivative of this equation and using $\partial m/\partial r = 4\pi\rho r^2$, we obtain

$$-\frac{\partial m}{\partial r} = K\lambda^{1/n}(n+1)\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -4\pi r^2\lambda\theta^n \tag{3.2.4}$$

$$\frac{K\lambda^{1/n-1}(n+1)}{4\pi}\Delta\theta + \theta^n = 0. \tag{3.2.5}$$

Upon redefining $\xi = r/\alpha$ where $4\pi\alpha^2 = (n+1)K\lambda^{1/n-1}$, one can obtain the mathematically cleaner looking equation

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) + \theta^n = 0 \tag{3.2.6}$$

Exact solutions are known for the cases $n = 0, 1, 5$. The derivation can be found in appendix A. Table 1 summarises them. Together with equation 3.2.6, it is immediate to see that for values $n \geq 5$, the equation does not yield solutions with a zero value.
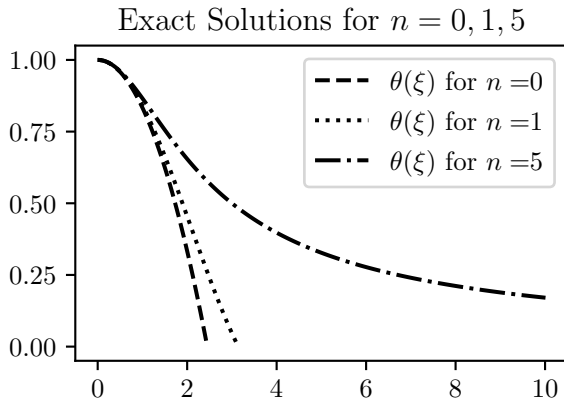

Figure 2: Graph of exact LE solutions.

n	Solution	ξ_0
$n = 0$	$1 - \dfrac{1}{6}\xi^2$	$\sqrt{6}$
$n = 1$	$\dfrac{\sin(\xi)}{\xi}$	π
$n = 5$	$\dfrac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}$	∞

Table 1: Lane Emden exact Solutions and their zero value.

3.3 Mass Bounds

We will first follow the approach given in [9]. The first assumptions will be that $d\rho/dr < 0$ and $\rho \geq 0$. Also we again consider a compact star, meaning $\rho(r) = 0$ for all $r > R$. We first state a useful Lemma for the proof of our next theorem.

Lemma 3.1
Given $\rho \in C^1(\mathbb{R}_{\geq 0})$ is monotonously increasing, the function $\rho_{av} = m/r^3$ has negative slope.

Proof. Taking the derivative of ρ_{av} and applying equation 3.1.15, we obtain

$$\frac{\partial}{\partial r}\left(\frac{m}{r^3}\right) = -3\frac{m}{r^4} + \frac{4\pi\rho}{r}. \tag{3.3.1}$$

By equation 3.1.24 we know that p has negative slope and since ρ is monotonously increasing, we have

$$m = \int\limits_0^r 4\pi\rho r^2 dr \geq \rho \int\limits_0^r 4\pi r^2 = \frac{4\pi r^3 \rho}{3} \tag{3.3.2}$$

which shows that $\frac{4\pi\rho r^3}{3} \leq m$ and completes the proof. $\qquad\square$

Theorem 3.2 - Mass Bound
The Mass of a spherically symmetric star is bound from above by

$$M < \frac{4}{9}R. \tag{3.3.3}$$

Proof. In our attempt to obtain an upper limit for the mass of a spherically symmetric star, we start by taking the difference of equation (3.1.13) and (3.1.14), we obtain

$$0 = \nu'\frac{e^{-\lambda}}{r} - \frac{1-e^{-\lambda}}{r^2} - \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{3.3.4}$$

$$= -\frac{2m(r)}{r^3} + \frac{\lambda' e^{-\lambda}}{2r} - \frac{e^{-\lambda}}{2}\left[\nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r} - \frac{\lambda'\nu'}{2}\right] \tag{3.3.5}$$

$$= r\frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) - \frac{e^{-\lambda}}{2}\left[\nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r} - \frac{\lambda'\nu'}{2}\right] \tag{3.3.6}$$

$$0 = \frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) - \frac{e^{-\lambda}}{2}\left[\frac{\nu''}{r} + \frac{\nu'^2}{2r} - \frac{\nu'}{r^2} - \frac{\lambda'\nu'}{2r}\right] \tag{3.3.7}$$

$$= \frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) - \frac{1}{2}e^{-\frac{\lambda+\nu}{2}}\frac{\partial}{\partial r}\left[\frac{1}{r}\nu' e^{\frac{\nu-\lambda}{2}}\right]. \tag{3.3.8}$$

Since $\partial_r\rho \leq 0$, also the average density $m(r)/r^3$ decreases with r. Thus we obtain

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\nu' \exp\left(\frac{\nu-\lambda}{2}\right)\right] \leq 0. \tag{3.3.9}$$

We integrate this equation from R to $r < R$

$$\frac{\nu'}{r}\exp\left(\frac{\nu-\lambda}{2}\right) \geq \frac{2\nu'(R)}{R}e^{-\frac{1}{2}\lambda(R)}\frac{\partial}{\partial r}e^{\frac{\nu}{2}}\bigg|_R \tag{3.3.10}$$

and use the Schwarzschild solution at $r = R$ for e^λ and e^ν. This is justified since we assumed $\rho(r) = 0$ for $r > R$ and thus we need to recover the vacuum solution for a spherically symmetric object which is given by the Schwarzschild solution. By continuity of the metric on every point of space, we can match

$$e^{-\lambda(r)}\big|_R = \left[1 - \frac{2M}{r}\right]_R = e^{\nu(r)}\big|_R \tag{3.3.11}$$

and with the explicit solution for $e^{-\lambda}$, we obtain

$$\frac{2m(r)}{r}\bigg|_R = \frac{2M}{R}. \tag{3.3.12}$$

When plugging this into equation (3.3.11), the result is

$$\frac{\nu'}{2r} \exp\left(\frac{\nu - \lambda}{2}\right) \geq \frac{(1 - 2M/R)^{1/2}}{R} \frac{\partial}{\partial r}\left(1 - \frac{2M}{r}\right)^{1/2}\bigg|_{r=R} = \frac{M}{R^3}. \tag{3.3.13}$$

Now we multiply by $r \exp(\lambda/2)$ and use the explicit solution for e^λ

$$\frac{\partial}{\partial r}\left(e^{\frac{\nu}{2}}\right) \geq \frac{M}{R^3} r e^{\frac{\lambda}{2}} = \frac{M}{R^3}\left(r - 2m(r)\right) \tag{3.3.14}$$

and integrate again this time from 0 to R

$$e^{\nu(0)/2} \leq \left(1 - \frac{2M}{R}\right)^{1/2} - \frac{M}{R^3} \int_0^R \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r\,dr. \tag{3.3.15}$$

As we have already noted, the average density $m(r)/r^3$ decreases, meaning explicitly $m(r)/r^3 \geq M/R^3$ and thus the integral with the previous equation can be written as

$$e^{\nu(0)/2} \leq \left(1 - \frac{2M}{R}\right)^{1/2} + \frac{1}{2}\left[1 - \frac{2Mr^2}{R^3}\right]^{1/2}\bigg|_{r=0}^{r=R} = \frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}. \tag{3.3.16}$$

The simple fact that $e^{\nu(0)/2} > 0$ then implies

$$\left(1 - \frac{2M}{R}\right)^{1/2} > \frac{1}{3} \tag{3.3.17}$$

which is equivalent to

$$M < \frac{4R}{9}. \tag{3.3.18}$$

$\square$

This shows that the mass of star has an upper limit under the assumptions $\rho \geq 0$, $\partial_r \rho \leq 0$ and $\rho(R) = 0$ for some $R \geq 0$.

4   Numerical Solutions

4.1   Comparing TOV and LE results with a polytropic EOS

In this section the numerical solutions of the TOV equation

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{4.1.1}$$

$$\frac{\partial p}{\partial r} = -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1} \tag{4.1.2}$$

as derived previously in 3.1 will be discussed. To obtain numerical solvability a equation of state in the form $\rho(r,p)$ is supplied. In Figure 3 a plot of such a solution is presented. The density ρ is derived via the equation 4.1.1 and the integration is done with a 4th order Runge-Kutta Method. The integration is stopped once the pressure reaches values $p \leq 0$. The initial value of 4.1.1 is $\partial m/\partial r(r=0) = 0$. For 4.1.1, the initial value can be calculated when applying L'Hôpital's rule and combining them to obtain

$$\lim_{r\to 0}\frac{m}{r} = \lim_{r\to 0}\frac{\partial m}{\partial r} \quad = \lim_{r\to 0}\frac{4\pi\rho r^2}{1} = 0 \tag{4.1.3}$$

$$\lim_{r\to 0}\frac{m}{r^2} = \lim_{r\to 0}\frac{1}{2r}\frac{\partial m}{\partial r} \quad = \lim_{r\to 0}\frac{4\pi\rho r^2}{2r} = 0 \tag{4.1.4}$$

$$\lim_{r\to 0}\frac{m}{r^3} = \lim_{r\to 0}\frac{1}{3r^2}\frac{\partial m}{\partial r} \quad = \lim_{r\to 0}\frac{4\pi\rho r^2}{3r^2} = \frac{4\pi\rho_0}{3} \tag{4.1.5}$$

$$\lim_{r\to 0}\frac{\partial p}{\partial r} = 0 \tag{4.1.6}$$

Explicit code can be found in [6]. The Lane-Emden equation was obtained in section 3.2 as the non-relativistic limit of the TOV equation by neglecting terms of order $1/c^2$ and higher and setting $G = c = 1$. To obtain numerical results for the LE equation as given in equation 3.2.6, a substitution of the form $d\theta/d\xi = \chi$ was used.

$$\frac{d\theta}{d\xi} = \chi \qquad\qquad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0 \tag{4.1.7}$$

$$\frac{d\chi}{d\xi} = -\frac{2}{\xi}\chi - \theta^n \qquad \left.\frac{d\chi}{d\xi}\right|_{\xi=0} = -\theta^n \tag{4.1.8}$$

where the initial value for $d\chi/d\xi$ can be calculated with L'Hôpital's rule. With the conversion factor α derived in section 3.2, TOV and LE results can be compared. Figure 3 shows the solution of both equations for the parameters of table 2. Note that values in the last part are calculated instead of supplied to the solving routine. Additionally conversion equations to compare TOV and LE results are displayed.
Since the mass reads

$$m(r) = \int_0^r 4\pi r'^2\rho(r')dr', \tag{4.1.9}$$

we expect $\partial m/\partial r(R) = 0$ if $p(R) = 0$ when choosing a polytropic equation of state with $\gamma > 0$. The plot in figure 3 shows this expected behaviour for the Lane Emden equation at $r \approx 2.31$ and has the same behaviour for the TOV results at $r \approx 6.80$.[4]

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[4]For the purpose of nicely displaying the calculated result, the plot only shows result up to $r = 2.5$

TOV		LE	
EOS	$\rho = Ap^{1/\gamma}$	EOS	$p = K\rho^\gamma$
A	2		
$\gamma = 1 + \frac{1}{n}$	4/3	$n = 1/(\gamma-1)$	3
p_0	0.5	θ_0	1
m_0	0	$d\theta_0$	0
dr	0.01	$d\xi = dr/\alpha$	$0.01/0.3355 \approx 0.0298$
$\rho_0 = Ap_0^{1/\gamma}$	$2(2)^{\frac{4}{3}} \approx 1.1892$	$\lambda = \rho_0$	$2(2)^{\frac{4}{3}} \approx 1.1892$
		$K = A^{-1/\gamma}$	$2^{-3/4} \approx 0.5946$
		$\alpha^2 = ((n+1)K\lambda^{1/n-1})/(4\pi)$	≈ 0.1125

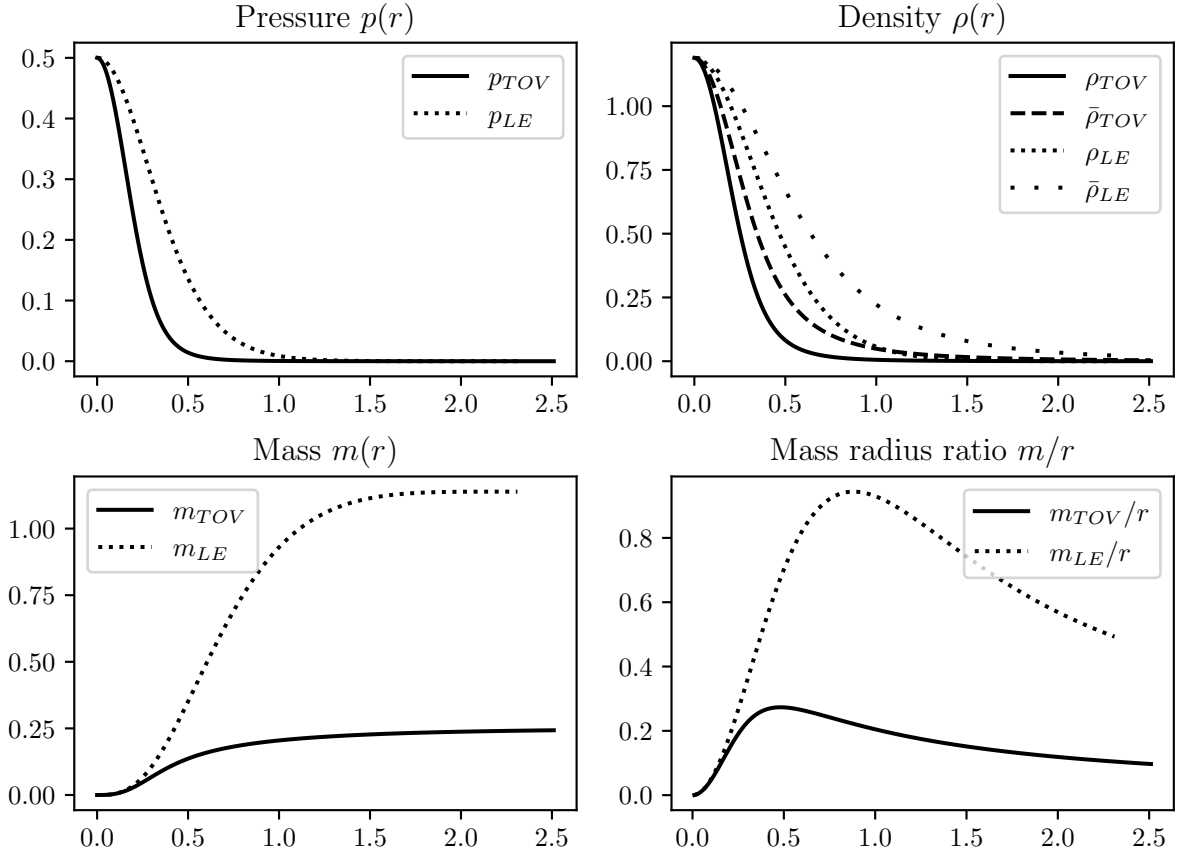Table 2: Parameters for numerical solving of the TOV and Lane-Emden equation.

Figure 3: Comparison of the fully relativistic TOV result and the classical Lane-Emden solution. The images show the plots for the parameters of table 2. First the pressure is presented. Afterwards the density calculated with the given EOS and the average density $\bar{\rho}_i = (4\pi/3)^{-1}m_i/r^3$ for the two solutions are beeing compared. In the second row, the mass and the ratio m_i/r can be seen.

4.2 Verifying the results

One can compare calculated LE results with already known exact solutions for certain exponents as given in table 1. All calculations are carried out with a chosen step size of $d\xi = 0.03$.

The graphs overall show good numerical agreement. Initial values at $r = 0$ are identical since they were set to be. The spike occurring afterwards can be explained by equation 4.1.8. The initial value of all derivatives is identically 0 which means no change of value for θ or χ for the initial step of the numerical integration. On the other hand the exact results will see a change which explains the large discrepancy in the beginning. It is clear that the relative difference on the right hand side will spike again once smaller values for θ are reached if the absolute difference stays constant. This behaviour is best seen in the solution for $n = 5$ where a large range of $0 \leq r \leq 100$ for solving was chosen. The offset between the numerically calculated values and the exact result stays constant while the value of θ decreases inverse proportionally.
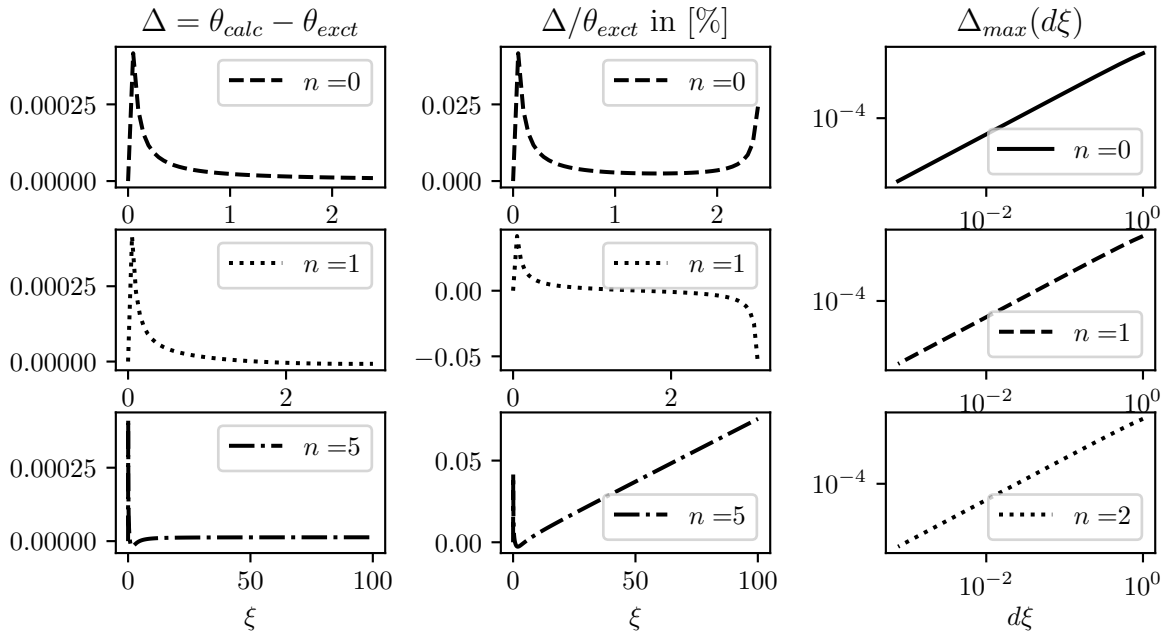

Figure 4: Validation of numerical LE results - The first two plots show absolute and relative difference of known exact and numerically calculated results while the third plot shows the behaviour of the maximum value of Δ as the stepsize decreases.

As can be seen in the last column of plots, the condition for convergence in this case is well fulfilled. The double logarithmic plot displays nicely that by lowering the integration stepsize, the maximum difference in this inverval decreases.

4.3 Relativistic EOS

In the previous discussion, we relied on the EOS given by 3. This is a versatile assumption, but one could ask, what would happen to a star in which the particles have no interaction but are near relativistic speed. The resulting EOS was calculated in the beginning 2.3.26

although not written down explicitly. Since explicit inversion of the given function is hard, we rely on numerical methods for calculation.

5 More Exact Results

5.1 LE equation

One can derive another exact solution for the LE equation at the exponent $n = 2$ when considering a series expansion of θ around the point $\xi = 0$ with initial conditions

$$\theta = \sum_{m=0}^{\infty} a_m \xi^m \qquad a_0 = \theta|_{\xi=0} = \theta_0 \qquad a_1 = \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0 \qquad (5.1.1)$$

Since the series is absolut convergent, we can plug it into the LE equation 3.2.6 and use the Cauchy Product formula.

$$\sum_{m=2}^{\infty} m(m-1)a_m \xi^{m-2} + \sum_{m=1}^{\infty} (2m)a_m \xi^{m-2} + \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_{m-k}a_k \xi^m = 0 \qquad (5.1.2)$$

Theorem 5.1
The odd coefficients a_{2m+1} of this series expansion vanish.

Proof. We rewrite the summations of equation 5.1.2 to start at the same index $m = 0$ and combine them

$$\sum_{m=0}^{\infty} \left((m+2)(m+1)a_{m+2}\xi^m + (2m+2)a_{m+1}\xi^{m-1} + \sum_{k=0}^{m} a_{m-k}a_k \xi^m \right) = 0 \qquad (5.1.3)$$

With equation 5.1.1, we can start the summation in the middle one index higher and separate the term ξ^m. This equation has to be true inside the radius of convergence of the series and thus needs to vanish for ambiguous ξ.

$$(m+2)(m+1)a_{m+2} + 2(m+2)a_{m+2} + \sum_{k=0}^{m} a_{m-k}a_k = 0 \qquad (5.1.4)$$

and upon further manipulation results in the recursive description for the coefficients of the series

$$a_{m+2} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^{m} a_{m-k}a_k. \qquad (5.1.5)$$

We show the statement by induction. For a_1 it is already true. Let the statement be true for all odd values $2k + 1 \leq 2m + 1$. Writing down a_{2m+3} gives us

$$a_{2m+3} = -\frac{1}{(2m+3)(2m+4)} \left(a_0 a_{2m+1} + a_1 a_{2m} + \cdots + a_{2m}a_1 + a_{2m+1}a_0 \right). \qquad (5.1.6)$$

It is clear that in the summation odd and even coefficients get paired and will thus vanish completely. $\qquad \square$

This proof shows that we can restrict ourselves to the subsequence $b_m = a_{2m}$ and the subseries with initial values given by

$$\theta = \sum_{m=0}^{\infty} b_m \xi^{2m} \qquad b_{m+1} = -\frac{1}{(2m+2)(2m+3)} \sum_{k=0}^{m} b_{m-k}b_k \qquad b_0 = \theta_0 \qquad (5.1.7)$$

Theorem 5.2

The series $\theta = \sum\limits_{m=0}^{\infty} b_m \xi^{2m}$ converges for $\xi \leq 1$.

Proof. We start by showing that $|b_{m+1}| \leq 1/(4m+6)$. Using the triangle inequality on 5.1.7, we obtain

$$|b_{m+1}| \leq \frac{1}{(2m+2)(2m+3)} \sum_{k=0}^{m} |b_{m-k} b_k|. \tag{5.1.8}$$

For $m = 0$, we have $b_1 = -1/6$ and thus $|b_1| \leq 1/3$ and so the statement is true for $m = 0$. If the statement holds for all $m < m+1$, we have

$$|b_{m+1}| \leq \frac{m}{(2m+2)(2m+3)} = \frac{1}{2}\frac{m}{(m+1)}\frac{1}{(2m+3)} \leq \frac{1}{2}\frac{1}{(2m+3)} \tag{5.1.9}$$

which proves the first statement. To see that b_m is a alternating sequence, we again inspect 5.1.7. The first element of the series is $b_0 = 1$ and the second is $b_1 = -1/6$ so the initial statement is again correct. Let b_m be alternating for all $m < 2m+1$. Then equation 5.1.7 for odd values reads

$$b_{2m+1} = -\frac{1}{(4m+4)(4m+5)} (b_{2m}b_0 + b_{2m-1}b_1 + \cdots + b_0 b_{2m}) \tag{5.1.10}$$

This shows that if all odd values b_{2k+1} are negative and all even ones are positive, then b_{2m+1} is negative too. The same holds true when the index $2m+2$ is even. By the Leibniz criterion the series converges if $\xi \leq 1$. $\qquad\square$
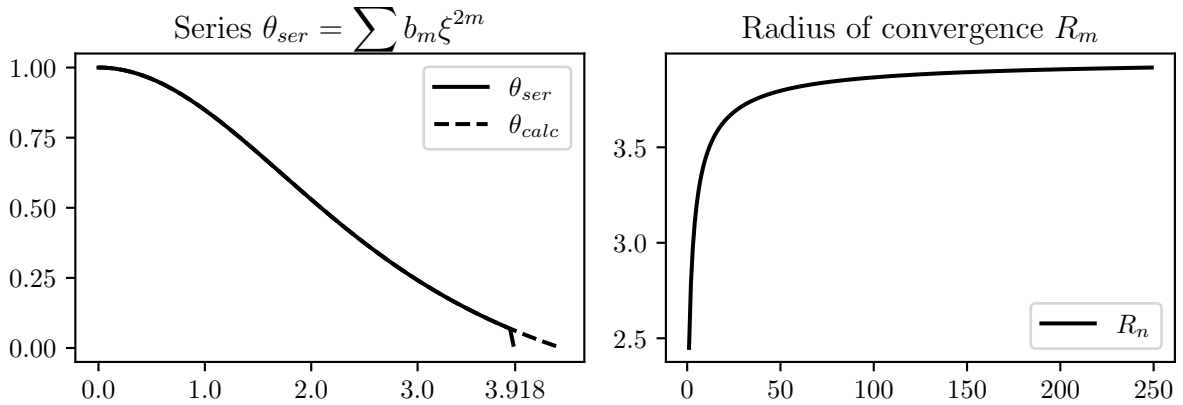

Figure 5: LE Solution for $n = 2$ - We see the power the series calculated with the coefficients explained above (the far right tick is the last calculated value for R_m where plotting is stopped) and the sequence $R_m = (|a_m|)^{-1/m}$ which approaches the radius of convergence.

These results can be used to calculate the power series of the LE equation numerically. Figure 5 shows those results. Good agreement is found between the two solutions as the difference $\Delta = \theta_{calc} - \theta_{ser}$ is not larger than 0.00015. The solutions differ for larger values around ≈ 3.918 The top right plot indicates that the radius of convergence might actually be larger than where solutions deviate. One important observation is that in this

procedure, the coefficients were calculated up to $b_{250} \approx -4.7723 \times 10^{-296}$ which is in close proximity to the floating point limit of python given by $\approx 2.2251 \times 10^{-308}$ and indicates that numerical errors may be the reason the series starts to diverge. Also in order to achieve sufficient numerical precision, we would need to calculate higher order terms of the series, which is supported by $(R_{250})^{500} \approx 3.2159 \times 10^{296}$ and shows that these parts of the series still yield large contributions to the value of θ at this point.

5.2 TOV equation

5.3 Existance of Solutions

6 Ideas

Theorem 6.1 - Lane-Emden Finite Boundary

For every $0 \leq n \leq 5$, the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{6.0.1}$$

with $\theta_0 = 1$ has a zero value for a finite ξ_0. To prove this theorem, show the statements 6.2, 6.3 and 6.4.

Theorem 6.2 - Lane-Emden Local Existence

For every $n \geq 0$, the Lane-Emden equation 6.0.1 has a solution in an ϵ Ball around $\xi = 0$.

Proof. This proof can be generalised to $m \geq 2$ dimensions. Following [10] we see that the LE differential equation can be written as

$$\frac{d}{d\xi}\left(\xi^{m-1}\frac{d\theta}{d\xi}\right) + \xi^{m-1}\theta^n = 0. \tag{6.0.2}$$

Integrating once and rearranging leads us to

$$\frac{d\theta}{d\xi} = \frac{d\theta}{d\xi}\bigg|_0 - \int_0^\xi \left(\frac{t}{\xi}\right)^{m-1}\theta^n dt. \tag{6.0.3}$$

By choosing the initial value of $d\theta/d\xi = 0$ we see that positive solutions to the LE equation need to have negative slope. This equation also shows that for $\theta_0 > 0$, the solution θ will be positive in a ϵ region around $\xi = 0$. We integrate both sides again and end up with

$$\theta = \theta_0 - \int_0^\xi \int_0^s \left(\frac{t}{s}\right)^{m-1}\theta^n dt ds. \tag{6.0.4}$$

We define C^* as the space of all real continuous functions $f \geq 0$ on $[0, \epsilon]$ that share the same initial value θ_0 and with negative slope. With the norm $||.||_\infty$ this space is complete. Inspecting the operator $T : C^* \to C^*$ defined by equation 6.0.4 we can apply the Banach fixed-point theorem.

$$||T(U) - T(V)||_\infty \leq ||U^n - V^n||_\infty \int_0^\xi \int_0^s \left(\frac{t}{s}\right)^{m-1} dt ds \tag{6.0.5}$$

$$= ||U^n - V^n||_\infty \frac{\xi^2}{2m} \tag{6.0.6}$$

Let $n \geq 1$. Since the function $x \mapsto x^n$ is differentiable for $n \geq 1$ it is also Lipschitz with constant L on the compact interval $I = [0, \theta_0]$. This means in particular that we have

$$||T(U) - T(V)||_\infty \leq ||U - V||_\infty \frac{L\xi^2}{2m} \tag{6.0.7}$$

Thus when choosing $\epsilon < \sqrt{2m/L}$ we have a contraction and the Banach fixed-point theorem then gives us a unique solution to equation 6.0.4.

Now let $0 \leq n < 1$. In this situation the preceding method does not work due to x^n not being differentiable at $x = 0$. For $\theta_0 \leq 1$, we know that $\theta^n \leq \theta_0^n$. We can now choose ϵ such that with

$$\int\limits_0^\xi \int\limits_0^s \left(\frac{t}{s}\right)^{m-1} \theta^n(t) dt ds \leq \frac{\xi^2}{2m} \theta_0^n \tag{6.0.8}$$

we have $\theta \geq \theta_0/2$ on $[0, \epsilon]$. If $\theta_0 > 1$, we have $\theta^n \leq \theta_0$ and obtain the same result. Then the same procedure as before can be carried through but with the interval $I = [\theta_0/2, \theta]$. $\quad\square$

Theorem 6.3 - Lane-Emden Finite Boundary 2
For $n < 5$, the Lane-Emden equation 6.0.1 has no Solution defined on the total space $\mathbb{R}_{\geq 0}$.

Proof. From [10] we know that solutions for equation 6.0.1 on the total space exist for $n \geq 5$.

$\quad\square$

Theorem 6.4 - Lane-Emden Finite Boundary 3
Let $n < 5$ and $\theta : [0, x) \to \mathbb{R}$ be a solution of 6.0.1. If $\theta(x) > 0$, then θ can be extended with Theorem 6.2 to $[0, x + \epsilon)$ where ϵ follows a growth condition $\epsilon(\xi) = \ldots$ such that if θ would have no zero point, there would be a solution on total $\mathbb{R}_{\geq 0}$ and thus a contradiction with 6.3.

Theorem 6.5 - Lane-Emden Finite Boundary 4
The Lane Emden equation has a zero value if the exponent function $n(\xi)$ fullfills the growth condition XYZ.

Theorem 6.6 - TOV Exact Solution
The TOV equation 4.1.1 with a polytropic EOS $\rho = Ap^{1/\gamma}$ has a well defined limiting case where $A \to 0$ with $m = 0$ and

$$p = \frac{p_0}{2\pi r p_0 + 1} \tag{6.0.9}$$

Proof. First we transform the TOV equation 3.1.24 using $p = y^a$ with $a > 0$ and $m = Arv$ and together with the polytropic EOS $\rho = Ap^{1/\gamma}$ obtain

$$\frac{\partial v}{\partial r} = 4\pi r y^{a/\gamma} - Av \tag{6.0.10}$$

$$ay^{a-1}\frac{\partial y}{\partial r} = -\frac{vA^2 y^{a/\gamma}}{r}\left(1 + \frac{y^a}{Ay^{a/\gamma}}\right)\left(\frac{4\pi r^2 y^a}{Av} + 1\right)\frac{1}{1 - 2vA} \tag{6.0.11}$$

Rearranging the second equation one obtains

$$\frac{\partial y}{\partial r} = -\frac{y^{a/\gamma - a + 1}}{ar}\left(A + y^{a - a/\gamma}\right)\left(4\pi r^2 y^a + Av\right)\frac{1}{1 - 2vA}. \tag{6.0.12}$$

Using $\gamma = 1 + 1/n$ with $n > 0$, we see that this equation is continuous in every variable $(r, y, v) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times [0, 1/2A)$. We restrict ourselves to a compact domain $r \in \overline{B_\delta(\tau)}$ and $v \in [0, 1/2A - \epsilon]$ where $0 < \epsilon < 1/2A$. We choose $0 < \delta$ and $0 \leq \tau$ in such a way that

$0 < \epsilon < 1/2A$ is satisfied for some ϵ which is possible since $m \to 0$ for $r \to 0$. To obtain Lipschitz continuity in (y, v), all of the following conditions need to be fulfilled.

$$\frac{a}{\gamma} - a + 1 \geq 1 \qquad a - \frac{a}{\gamma} \geq 1 \qquad a \geq 1 \tag{6.0.13}$$

The second equation implies the first and the third. Thus we only need to choose $a \geq (1 - 1/\gamma)^{-1}$. With $\gamma = 1 + 1/n$ we can rewrite this equation to

$$a \geq n + 1 \tag{6.0.14}$$

which can be easily satisfied. This now shows with extension of the Picard-Lindelöf Theorem that there exists a unique solution for given initial values $\tau, y(\tau), v(\tau) \in B_\delta(\tau) \times \mathbb{R}_{\geq 0} \times [0, 1/2A - \epsilon)$ that especially continuously depends on A. Without loss of generality we can choose δ small enough such that solutions are positive in p. By transforming back equation 6.0.11 to

$$\frac{\partial p}{\partial r} = -\frac{1}{r^2} \left(A p^{1/\gamma} + p \right) \left(4\pi r^3 p + vA \right) (1 - 2vA)^{-1} \tag{6.0.15}$$

and letting $A \to 0$, we obtain

$$\frac{\partial p}{\partial r} = -4\pi r p^2 \tag{6.0.16}$$

which is then solved by

$$p = \frac{\tilde{p}}{2\pi \tilde{p}(r^2 - \tau^2) + 1} \tag{6.0.17}$$

where $\tilde{p}$ is the initial value at $r = \tau$. It is clear that this solution can be extended to $r \in [0, \infty)$ by choosing arbitrary small τ. In the limit, the result equals the hypothesis. $\square$

Theorem 6.7 - TOV Finite Boundary
For each $A > 0$, the TOV equation 4.1.1 has a exponent $n > 0$ for which a solution p does not have any zero points.

Theorem 6.8 - TOV Finite Boundary 1
If p_1 and p_2 are two solutions of the TOV equation (for equal A) with p_i corresponding to n_i exponents, then if $n_1 < n_2$, the zero point (if it exists) of p_1 is smaller than that of p_2 (if it exists or is ∞).

Theorem 6.9 - TOV Finite Boundary 2
a

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Appendix

A   Exact solutions of the LE equation

This section relies on information in [7] and [8]. The LE equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{A.1}$$

which for $n = 0$ transforms readily into

$$\int_0^\xi \frac{d}{d\xi}\left(\xi'^2\frac{d\theta}{d\xi}\right)d\xi' = -\int_0^\xi \xi'^2 d\xi'. \tag{A.2}$$

Both sides can be evaluated directly and then simplified further.

$$\xi^2\frac{d\theta}{d\xi} = -\frac{\xi^3}{3} \tag{A.3}$$

$$\theta(\xi) = \theta(0) - \frac{\xi^2}{6} \tag{A.4}$$

With the initial condition $\theta(0) = 1$, we obtain the desired result. For $n = 1$, equation A.1 transforms into

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \xi^2\theta = 0 \tag{A.5}$$

we start by substituting $U = \theta x$ and thus obtain (also multiplying by x^2)

$$0 = \frac{dU}{d\xi} + \xi\frac{d^2U}{d\xi^2} - \frac{dU}{d\xi} + \xi U \tag{A.6}$$

$$-U = \frac{d^2U}{d\xi^2} \tag{A.7}$$

and the last equation can be solved with a linear combination of $\cos(\xi)$ and $\sin(\xi)$. Transforming back to θ, we then have

$$\theta(\xi) = A\frac{\sin(\xi)}{\xi} - B\frac{\cos(k\xi)}{\xi}. \tag{A.8}$$

The need for a well defined limit at $\xi \to 0$ implies that $B = 0$ and thus since $\sin(z)/z \to 1$ for $z \to 0$, we have $A = \theta(0) = 1$ and

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}. \tag{A.9}$$

For $n = 5$, we start by making the two substitutions $x = 1/\xi$ and $\theta = ax^\omega$ as done in [8, pp. 94 sqq.] and then transforming the LE equation into

$$x^4\frac{d\theta}{d\xi} \qquad\quad + \theta^n \quad\ = 0 \tag{A.10}$$

$$a\omega(\omega - 1)x^{\omega+2} + a^n x^{n\omega} = 0. \tag{A.11}$$

From this we see that $\omega + 2 = n\omega$ and $a\omega(\omega - 1) = -a^n$ needs to be satisfied, since the equation needs to hold for all $x \in \mathbb{R}_{\geq 0}$. Rewriting these conditions, we obtain the singular solution for the LE equation

$$\theta(x) = \left(\frac{2(n-3)}{(n-1)^2}\right)^{1/(n-1)} x^{2/(n-1)} \tag{A.12}$$

since with $x = 1/\xi$, we have $\theta(\xi) \to \infty$ for $\xi \to 0$. Notice that the solution is only valid if $n \geq 3$. We can use this solution to the LE equation and perturb it to make the more general ansatz

$$\theta(x) = ax^\omega z(x). \tag{A.13}$$

If $n < 3$, the factor a has to be replaced with a more general one. The singular solution is obtained when taking $z = 1$. Using another transformation $1/x = \xi = \exp(-t)$, we obtain

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) \qquad\qquad + \theta^n \qquad\qquad = 0 \tag{A.14}$$

$$x^4 \frac{d^2\theta}{dx^2} \qquad\qquad + \theta^n \qquad\qquad = 0 \tag{A.15}$$

$$ax^{\omega+2}\left[x^2 \frac{d^2z}{dx^2} + 2\omega x \frac{dz}{dx} + \omega(\omega - 1)z\right] + a^n x^{n\omega} z^n \qquad = 0 \tag{A.16}$$

$$\frac{d^2z}{dt^2} + (2\omega - 1)\frac{dz}{dt} + \omega(\omega - 1)z + a^{n-1}z^n \qquad = 0 \tag{A.17}$$

$$\frac{d^2z}{dt^2} + \frac{5-n}{n-1}\frac{dz}{dt} + 2\frac{3-n}{(n-1)^2}z + 2\frac{(n-3)}{(n-1)^2}z^n = 0. \tag{A.18}$$

For $n = 5$, we obtain

$$\frac{d^2z}{dt^2} = \frac{1}{4}z(1 - z^4). \tag{A.19}$$

we multiply both sides with dz/dt and integrate

$$\frac{1}{2}\left(\frac{d^2z}{dt^2}\right) = \frac{1}{8}z^2 - \frac{1}{24}z^6 + D \tag{A.20}$$

where D is the integration constant. For $\xi \to 0$, we expect $\theta \to \theta_0$ and thus $z = \theta_0 e^{-\omega t}(1/a + \mathcal{O}(e^{-t}))$ as t approaches ∞. We immediately see that dz/dt exhibits a similar behaviour and thus the integration constant D has to vanish.

The right hand side of equation A.20 cannot get negative since otherwise z would take complex values which enables us to take the square root with a minus sign[5] and integrate again

$$\int \left(1 - \frac{1}{3}z^4\right)^{-1/2}\frac{dz}{z} = -\frac{1}{2}\int dt. \tag{A.21}$$

We change the integration by again substituting the variables $1/3z^4 = \sin^2(\alpha)$ and calculate $dz/z = 2\cos(\alpha)/\sin(\alpha)d\alpha$ as well as $-dt = d\alpha/\sin(\alpha)$. Now we rewrite the integral

$$\int \frac{1}{\sqrt{1 - \sin^2(\alpha)}}\frac{2\cos(\alpha)\,d\alpha}{\sin(\alpha)} = -\int dt \tag{A.22}$$

$$\int \frac{2d\alpha}{\sin(\alpha)} = -\int dt. \tag{A.23}$$

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[5]This only determines the direction in which t is defined, so it is arbitrary.

Evaluating those integrals leads us to

$$\log(\tan(\alpha/2)) + \log(1/C) = -t \tag{A.24}$$

where the integration constant has been chosen in advance to simplify the next expressions. From here, we further manipulate equation A.24 and combine it with our previous substitution $1/3z^4 = \sin^2(\alpha)$ to obtain

$$\frac{1}{3}z^4 = \sin^2(\alpha) = \frac{4\tan^2(\alpha/2)}{(1+\tan^2(\alpha/2))}^2 \tag{A.25}$$

and with the solution for out integral before and plugging in the substitution from the beginning $\xi = e^{-t}$, we have

$$z = \pm \left(\frac{12C^2\xi^2}{1 + C^2\xi^2} \right)^{1/4}. \tag{A.26}$$

For θ, we need to have $\theta \to \theta 0 = 1$ as $\xi \to 0$. This means that $C = 1$ and with $\theta = ax^\omega z$, we obtain

$$\theta = \frac{1}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}}. \tag{A.27}$$

We see that this equation has no zero value and tends to 0 as $\xi \to \infty$.

Erklärung

Hiermit versichere ich, dass ich die eingereichte Masterarbeit selbständig verfasst habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Inhalte als solche kenntlich gemacht. Weiter versichere ich, dass die eingereichte Masterarbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens war oder ist.

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