Contents

List of Abbreviations

EOS **E**quation **o**f **S**tate
GR **G**eneral **R**elativity
LE **L**ane-**E**mden
TOV **T**olman **O**ppenheimer **V**olkoff

List of Symbols

Just a test

Units

This thesis uses the natural number system if not otherwise explicitly mentioned. This means $\hbar = c = G = k_B = 1$.

Abstract

This is not a good example for an abstract.

1 Contact Formulation of Thermodynamics

1.1 Motivation

Concepts and basic definitions can be read in the book [**eschrigTopologyGeometryPhysics2011**] by Eschrig. The work in this chapter is mainly motivated by [**mrugalaGeometricalFormulationEqui** **mrugalaContactStructureThermodynamic1991**, **weinholdMetricGeometryEquilibrium1975**
Throughout this section we will if not otherwise mentioned assume smoothness of manifolds and mappings. Also the Einstein summing convention will be used.
The theory of thermodynamics relates extensive quantities (eg. volume V, particle number N or entropy S) to a potential function Φ that resembles a form of energy of the thermodynamic system. In thermal equilibrium, the first law of thermodynamics states that

$$dU = \delta Q + \delta W \tag{1.1.1}$$

where U refers to the internal energy, Q to the heat of the system and W the work done. In physicists notation the δ-symbol is used to indicate that the 1-form considered here is not a exact differential. On the other hand, dU is postulated to be a exact differential of the internal energy U. In a thermodynamic system with variables (S, V, U, T, p) the first law can be expressed as

$$dU = TdS - pdV. \tag{1.1.2}$$

By writing down this equation, we see that $\partial U/\partial S = T$ and $\partial U/\partial V = -p$. This relates intensive parameters like p, T to extensive parameters V, S via derivatives. Considering the canonical ensemble, a potential function is given by the free energy

$$F = U - TS \tag{1.1.3}$$

from which all intensive parameters can be obtained by differentiating, for example

$$\frac{\partial F}{\partial V} = -p. \tag{1.1.4}$$

If we want to consider the intensive parameters as independent variables, we run into problems, since if U depends on X_n extensive and P_n intensive parameters, we have

$$dU = \frac{\partial U}{\partial X^i} dX^i + \frac{\partial U}{\partial P^i} dP^i = P_i dX^i + \frac{\partial U}{\partial P^i} dP^i. \tag{1.1.5}$$

But since the first law requires $dU = P_i dX^i \Rightarrow U = P_i X^i$, we obtain

$$0 = X_i dP^i \Rightarrow X_i = 0. \tag{1.1.6}$$

This poses the question on which space and in what manner these concepts can be defined rigidly. Clearly, since we are dealing with total derivatives, the aim is to consider a manifold on which the variables live. When considering variables $(U, X^1, \ldots, X^n, P^1, \ldots, P^n)$, one promptly notices that the system is overdefined since U is directly dependent on the other variables. However this only holds in thermal equilibrium and thus thermal equilibrium is a submanifold of the collection of these variables. Mathematical rigor of these statements is the aim of the next section.

1.2 Formal Definitions

Definition 1.1 - Distribution, Contact Element
A smooth distribution Δ of dimension m on a manifold M^n assigns at each point $p \in M$ a m-dimensional ($m \leq n$) subspace $\Delta_p \subseteq T_pM$ in such a way that for a neighborhood U of p there exist smooth vector fields $X_1, \ldots, X_m$ such that for every point $y \in U$ the subspace $\Delta_y = \mathrm{span}(X_1(y), \ldots, X_m(y))$. The set of smooth vector fields is called a local base of Δ. A tangent vector X is said to belong to Δ if $X(p) \in \Delta_p$ for all $p \in M$.
A smooth distribution of dimension $n-1$ on a n-dimensional Manifold is called a contact element.

Definition 1.2 - Involutive Distribution, Integral Manifold
A distribution is called involutive if for two vector fields X, Y belonging to Δ, the Lie bracket $[X, Y]$ also belongs to Δ. A connected submanifold N with its natural embedding $\iota : N \hookrightarrow M$ is called integral manifold of a distribution Δ on N if $\iota^p_*(T_pM) = \Delta_{\iota(p)}$. That means that at any point $p \in N$, the tangent space T_pN which is a subspace of T_pM is given by $\Delta_{\iota(p)} = \Delta_p$ (since $\iota(p) = p$ for all $p \in N$).

Corollary 1.3
Every contact element can be given locally by the kernel of a 1-form ω. As a direct consequence every distribution of dimension $m < n$ can be given locally by the intersection of the kernels of m linear independent 1-forms. On the other hand a (global) 1-form defines a distribution at any point $p \in M$.

Example 1.4
In the space of thermodynamic variables M^{2n+1} that is locally given by the coordinates $(\Phi, X^1, \ldots, X^n, P^1, \ldots, P^n)$ the 1-form

$$\theta = d\Phi - P_i dX^i \tag{1.2.1}$$

defines a contact element via $\theta = 0$. The contact element is the subspace of T_pM on which

$$d\Phi = P_i dX^i. \tag{1.2.2}$$

We see that the 1-form chosen satisfies the important relation $\theta \wedge (d\theta)^n \neq 0$ this will be used to define what a contact form is on a manifold.

The next statement will be the Frobenius theorem. It will be the justification why a geometric theory of thermodynamics must be formulated with contact manifolds.

Theorem 1.5 - Frobenius Theorem
Let Δ be a m-dimensional distribution on a manifold M^n with $1 \leq m \leq n$. There is a uniquely defined maximal connected integral manifold (N_p, ι_p) through every point $p \in M$ if and only if Δ is involutive.

Theorem 1.6 - "Dual" Frobenius Theorem
Given a m-dimensional distribution Δ in a sufficiently small neighborhood U of p there is an $(n-m)$-dimensional annihilator subspace of T^*_pM given by the dual of the orthogonal complement of Δ_p. This space $D^\perp_p$ is spanned by $n-m$ linearly independent differential 1-forms $\omega_{m+1}, \ldots, \omega_n$ which can be completed by addition of forms to form a complete basis of T_pM for points $p \in U$. The set $D^\perp_p$ can be characterised by the equation

$$\omega^i_j(p) dx^j = 0 \tag{1.2.3}$$

for all $i > m$ and $p \in U$. The statement of the theorem then says that this system of equations describes a submanifold $\iota : N \hookrightarrow M$ if and only if for $i > m$

$$d\omega^i \wedge \omega^{m+1} \wedge \cdots \wedge \omega^n = 0. \tag{1.2.4}$$

In this case the system is said to be completely integrable. A proof of both forms of this theorem can be found in [**eschrigTopologyGeometryPhysics2011**].

Example 1.7 - Adiabatic Submanifolds
The heat exchange 1-form of a thermodynamical system M^{2n+1} with local coordinates $(\Phi, S, X^2, \ldots, X^n, T, P^2, \ldots, P^n)$ for $n-1$ other (in principle here not relevant) extensive and intensive quantities X^i, P^i is given by

$$\delta Q = TdS. \tag{1.2.5}$$

Adiabaticity of a thermodynamic system means no exchange of heat $\delta Q = 0$. This characterises a distribution of dimension $2n$ and simultaneously states that the differential forms defining $D_p^{\perp}$ are given by this equation and in this case is solely $TdS = 0$. To obtain a basis of T_pM we take the canonical differential 1-forms dX^i, dT, dP^i. Then, it is clear that

$$d\delta Q \wedge dX^2 \wedge \cdots \wedge dX^n \wedge dT \wedge dP^2 \wedge \cdots \wedge dP^n = 0 \tag{1.2.6}$$

since $d\delta Q = dT \wedge dS$ and dT appears again in the $\wedge$-product, the whole term vanishes. Thus the requirements for the Frobenius theorem are satisfied and there exists a $2n$-dimensional submanifold $N \subset M$ on which adiabatic processes take place.

Another important theorem is the Darboux Theorem [**darbouxProblemePfaffPar1882**] which gives us a standard coordinate representation for any contact structure later introduced.

Theorem 1.8 - Darboux's Theorem
To a 1-form ω on a n-dimensional manifold M^n that satisfies

$$\omega \wedge (d\omega)^p = \omega \wedge d\omega \wedge \cdots \wedge d\omega = 0 \tag{1.2.7}$$

there exist local coordinates $(x^1, \ldots, x^{n-p}, y^1, \ldots, y^p)$ such that ω can be written as

$$\omega = x^i dy^i = x^1 dy^1 + \cdots + x^p dy^p. \tag{1.2.8}$$

A 1-form that fulfills

$$\omega \wedge (d\omega)^p \neq 0 \tag{1.2.9}$$

can be equipped with local coordinates $(x^0, \ldots, x^{n-p-1}, y^1, \ldots, y^p)$ such that

$$\omega = dx^0 + x^i dy^i = dx^0 + x^1 dy^1 + \cdots + x^p dy^p. \tag{1.2.10}$$

Lemma 1.9
A 1-form on a $2n+1$-dimensional manifold M that satisfies

$$\theta \wedge (d\theta)^n \neq 0 \tag{1.2.11}$$

does not define a submanifold via $\theta = 0$.

Proof. We choose local coordinates as given by the Darboux theorem above. Since $\theta = 0$ characterises a contact element, we only have one single defining equation and thus equation 1.2.4 can in our setting be expressed as

$$\theta \wedge (dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n) = 0 \tag{1.2.12}$$

for a manifold to exist. Here we completed θ to the set $\{\theta, dx^i, dy^i\}$ which is linearly independent. But since $d\theta = dx_i \wedge dy^i$ and $(dx^i \wedge dy^i)$ and $(dx^j \wedge dy^j)$ commute, we also have

$$\theta \wedge (d\theta)^n = n\theta \wedge (dx^1 \wedge \cdots \wedge dy^n) \neq 0. \tag{1.2.13}$$

From this it can be already read of that equation 1.2.4 can not be fulfilled. □

Remark
As we have already forewarned, it is not possible to find a suitable submanifold of M^{2n+1} such that the thermodynamic system in equilibrium can be treated globally. However as we have already seen this is still the case for other processes of such thermodynamic systems. In particular considering processes that involve only 1 coordinate (eg. isobaric $dp = 0$). In this case a distribution is always involutive since $d(dp) = 0$ and thus it is easily integrable.

Definition 1.10 - Contact Form and Manifold
A smooth 1-form θ that satisfies

$$\theta \wedge (d\theta)^n \neq 0 \tag{1.2.14}$$

(meaning the nth $\wedge$-product of $d\theta$) is called a contact form. A $2n+1$-dimensional manifold with such a form (M, θ) is called a contact manifold. A contact form defines at any point $p \in M$ a contact element via $\theta = 0$. Such a contact element is also called contact structure. Note that a contact form is only defined up to a multiplicative constant $\lambda \neq 0$.

Definition 1.11 - Equilibrium (Legendre) Submanifold
A Legendre submanifold of a $2n+1$-dimensional contact manifold (M, θ) is a n-dimensional submanifold $\iota : N \hookrightarrow M$ such that

$$\iota^*(\theta) = 0. \tag{1.2.15}$$

Since this corresponds to a thermodynamical system in equilibrium, we often refer to this as beeing a equilibrium (sub)manifold.

Corollary 1.12
Since locally, a contact form is always given by standard coordinates, a Legendre submanifold is specified by equations

$$x^0 = \Phi(y^1, \ldots, y^n) \qquad x^i = \frac{\partial \Phi}{\partial y^i}. \tag{1.2.16}$$

For a thermodynamical system with extensive variables X^i and intensive variables P^i and potential Φ this means

$$\Phi = \Phi(X^1, \ldots, X^n) \qquad P^i = \frac{\partial \Phi}{\partial X^i} \tag{1.2.17}$$

where we abused notation in the first equation to simultaneously label the function Φ and coordinate Φ with the same symbol. On this Legendre submanifold the usual thermodynamic equations for deriving expressions for intesive variables with respect to the potential hold.

Definition 1.13 - Contact transformations
Given two contact manifolds (M, θ) and (N, ω), a diffeomorphism $\phi : N \to M$ is called a contact diffeomorphism if it preserves the contact structure, meaning

$$\phi^*(\omega) = f\theta \tag{1.2.18}$$

where f is a nonvanishing function on M. If $f = 1$, we call ϕ a strict contact transformation.

Corollary 1.14
Since pullback and exterior derivative commute, we have that given a contact form θ, a contact transformation implies

$$\begin{aligned}
\phi^*(\omega \wedge (d\omega)^n) &= \phi^*(\omega) \wedge \phi^*((d\omega)^n) \\
&= \phi^*(\omega) \wedge (\phi^*(d\omega))^n \\
&= \phi^*(\omega) \wedge (d(\phi^*\omega))^n \\
&= f\theta \wedge (d(f\theta)^n) \\
&= f\theta \wedge (df \wedge \theta + f d\theta)^n
\end{aligned}$$

Now whenever the product of n forms contains the 1-form $theta$, the total product vanishes since $\theta \wedge \theta = 0$. Thus the only term that survives is

$$\phi^*(\omega \wedge (d\omega)^n) = f\theta \wedge (f d\theta)^n = f^{n+1}\theta \wedge (d\theta)^n \tag{1.2.19}$$

and for $f \neq 0$ and by injectivity of ϕ^* (since ϕ was a diffeomorphism), we immediately see that if ω (or resp. θ) is a contact form, so is the other.

The next section aims to rigorously define what the phase space of thermodynamic coordinates is and gives the definitions from this section a physical interpretation.

1.3 Thermodynamic Phase Space and Postulates

Postulate - Thermodynamic Phase Space
To every thermodynamic system corresponds a thermodynamic phase space (TPS) which is a $2n+1$-dimensional contact manifold (M, θ) with intensive and extensive variables as coordinates. A equilibrium state of such a thermodynamic system is represented by a Legendre submanifold of the contact manifold (M, θ).

Example 1.15 - Canonical Ensemble
In the canonical ensemble, the Free Energy is the potential. We choose coordinates $(\mathcal{F}, -S, V, N, T, -p, \mu)$ and the contact form

$$\theta = dF + SdT + pdV - \mu dN. \tag{1.3.1}$$

The heat exchange 1-form δQ is then given by $\delta Q = SdT$. For applications in Physics, the free energy in thermal equilibrium often times is given by a partition function $\mathcal{Z}$ via the relation

$$\mathcal{F}(T, V, N) = -k_B T \log(\mathcal{Z}(T, V, N)). \tag{1.3.2}$$

This means on the Legendre submanifold where $\theta = 0$, we have

$$S = -\frac{\partial \mathcal{F}}{\partial T} = k_B \left(1 + T\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial T}\right) \tag{1.3.3}$$

$$p = -\frac{\partial \mathcal{F}}{\partial V} = k_B T\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial V} \tag{1.3.4}$$

$$\mu = \frac{\partial \mathcal{F}}{\partial N} = k_B T\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial N}. \tag{1.3.5}$$

The partition function $\mathcal{Z}$ can be calculated explicitly from the microscopic behaviour of the system given by the Hamilton Mechanics of the particles/fluids involved. Such a direct calculation is done in the next chapter. Often times the particle number N is assumed to be fixed and thus not be taken as a coordinate but simply as a whole number $N \in \mathbb{N}$.

Example 1.16 - Microcanonical Ensemble
In the microcanonical ensemble, the potential is given by the entropy S of the system. Thus we can choose coordinates $(S, E, V, N, 1/T, p/T, -\mu/T)$ and obtain the contact form

$$\theta = dS - \frac{1}{T}dE - \frac{p}{T}dV + \frac{\mu}{T}dN. \tag{1.3.6}$$

The transformation $\kappa : (x_0, x_1, x_2, x_3, p_1, p_2, p_3) \mapsto (x_1, x_0, x_2, x_3, 1/p_1, -p_2/p_1, -p_3/p_1)$ has the differential

$$D\kappa = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{p_1^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p_2}{p_1^2} & -\frac{1}{p_1} & 0 \\ 0 & 0 & 0 & 0 & \frac{p_3}{p_1^2} & 0 & -\frac{1}{p_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -T^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & pT & -T & 0 \\ 0 & 0 & 0 & 0 & \mu T & 0 & -T \end{bmatrix}. \tag{1.3.7}$$

Clearly, the mapping is a diffeomorphism and since $\kappa^*(\theta) = dE - TdS + pdV - \mu dN$, the mapping is a contact transformation and on the Legendre submanifold, we have $dE = TdS - pdV + \mu dN$.

2 The Internal Energy calculated explicitly

The canonical ensemble was introduced in its mathematical form in the previous section. On the equilibrium submanifold of the TPS B, we can write the free energy as a function of the partition function $\mathcal{Z}$. The internal energy is then obtained by a change of coordinates or simply the equation $\mathcal{F} = \mathcal{U} + TS$. Microscopically the partition function is given by the behaviour of the N particles determined by the Hamiltonian H. Concepts and definitions of Hamilton Mechanics can be read in [**eschrigTopologyGeometryPhysics2011**, **grosseAllgemeineRelativitaetstheorie2019**, **choquet-bruhatGeneralRelativityEinstein2009**, **choquet-bruhatIntroductionGeneralRelativity2015**, **choquet-bruhatAnalysisManifoldsPhys** **spivakPhysicsMathematiciansMechanics2010**]. In general we assume $H : TM^N \to \mathbb{R}$ to be a positive smooth function on the manifold of space-time M. This explicitly takes form in the well known equations

$$
\begin{aligned}
\mathcal{Z}(T,V,N) &= \int\limits_{TM^N} \exp\left(-\frac{H(x_1,\dots,x_N,p_1,\dots,p_N)}{k_BT}\right)\frac{dx_1dp_1\dots dx_Ndp_N}{N!h^{3N}} && (2.0.1)\\
\mathcal{F}(T,V,N) &= -k_BT\log\left(\mathcal{Z}(T,V,N)\right) && (2.0.2)\\
\mathcal{U}(T,V,N) &= \mathcal{F} + T\mathcal{S} && (2.0.3)\\
&= \mathcal{F} - T\frac{\partial\mathcal{F}}{\partial T} && (2.0.4)\\
&= k_BT^2\frac{1}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial T}. && (2.0.5)
\end{aligned}
$$

where $x_i \in M$ and $p_i \in T_{x_i}M$. The last equation shows that to fully calculate U it is necessary to obtain a readable output from $\mathcal{Z}$.

2.1 The Ultra-Relativistic Case

First we inspect the ultra-relativistic Hamiltonian given by

$$H(\vec{x},\vec{p}) = ||\vec{p}||c. \tag{2.1.1}$$

The corresponding partition function for a N particle system then reads ($\beta = (k_BT)^{-1}$)

$$
\begin{aligned}
\mathcal{Z} &= \frac{V^N}{N!h^{3N}}\left[\int\limits_{\mathbb{R}^3}\exp\left(-\beta H(\vec{p})\right)d^3p\right]^N && (2.1.2)\\
&= \frac{V^N}{N!h^{3N}}\left[\int\limits_0^\infty 4\pi p^2\exp(-\beta pc)dp\right]^N && (2.1.3)\\
&= \frac{V^N}{N!h^{3N}}\frac{(4\pi)^N(k_BT)^{3N}}{c^{3N}}\left[\int\limits_0^\infty x^2\exp(-x)dx\right]^N && (2.1.4)\\
\mathcal{Z} &= \frac{1}{N!}\left(8\pi V\left(\frac{k_BT}{hc}\right)^3\right)^N && (2.1.5)
\end{aligned}
$$

where from the first to second line we used usual spherical coordinates and afterwards used the integral transformation $x = \beta cp$. The remaining integral $\mathcal{U}$ can be solved to be exactly $= 2$ and the internal energy can now be written down and further approximated to subjective needs.

2.2 The Special-Relativistic Case

We will now turn our attention to the fully special-relativistic Hamiltonian given by

$$H = mc^2\sqrt{1+\frac{\vec{p}^2}{m^2c^2}}. \tag{2.2.1}$$

In doing so, we keep in mind that the ultra relativistic limit can be obtained from this Hamiltonian by letting $m \to 0$. In this limiting case we should be able to recover the results from equation (2.1.5).

$$\mathcal{Z} = \frac{V^N}{N!h^{3N}}\left[\int_{\mathbb{R}^3} \exp\left(-\frac{mc^2\sqrt{1+\frac{\vec{p}^2}{m^2c^2}}}{k_BT}\right) d^3p\right]^N \tag{2.2.2}$$

$$= \frac{V^N}{N!h^{3N}}\left[\int_0^\infty 4\pi p^2 \exp\left(-\beta mc^2\sqrt{1+\frac{p^2}{m^2c^2}}\right) dp\right]^N \tag{2.2.3}$$

$$= \frac{(4\pi V)^N}{N!}\left(\frac{mc}{h}\right)^{3N}\left[\int_0^\infty q^2 \exp\left(-\alpha\sqrt{1+q^2}\right) dq\right]^N \tag{2.2.4}$$

$$= \frac{(4\pi V)^N}{N!}\left(\frac{mc}{h}\right)^{3N}\left(\int_0^\infty \sinh(x)^2\cosh(x)\exp(-\alpha\cosh(x))dx\right)^N \tag{2.2.5}$$

$$= \frac{(4\pi V)^N}{N!}\left(\frac{mc}{h}\right)^{3N}\left(\int_0^\infty \frac{\sinh(2x)}{2}\sinh(x)\exp(-\alpha\cosh(x))dx\right)^N \tag{2.2.6}$$

$$= \frac{(4\pi V)^N}{N!}\left(\frac{mc}{h}\right)^{3N}\left(-\frac{\sinh(2x)}{2\alpha}\exp(-\alpha\cosh(x))\Big|_0^\infty\right. \tag{2.2.7}$$

$$\left.+\frac{1}{\alpha}\int_0^\infty \cosh(2x)\exp(-\alpha\cosh(x))dx\right)^N \tag{2.2.8}$$

$$= \frac{(4\pi V)^N}{N!}\left(\frac{mc}{h}\right)^{3N}\left(\frac{1}{\alpha}\int_0^\infty \cosh(2x)\exp(-\alpha\cosh(x))dx\right)^N \tag{2.2.9}$$

$$= \frac{1}{N!}\left(8\pi V\left(\frac{k_BT}{hc}\right)^3 2\alpha^2 K_2(\alpha)\right)^N \tag{2.2.10}$$

In the first step we used spherical coordinates followed by the substitution $qmc = p$ and $\alpha = \beta mc^2 = mc^2/k_BT$. Afterwards we substituted $q = \sinh(x)$ and used the identity

$\cosh(x)\sinh(x) = \sinh(2x)/2$. Partial integration then leads to the last integral which can be identified as the modified Bessel function of the 2nd kind $K_2(\alpha)$. The equation is then rewritten such that the ultra relativistic limit can be read off upon letting $\alpha \to 0$.
We can now calculate the internal energy $\mathcal{U}$ from $\mathcal{Z}$ via equation (2.0.5)

$$\mathcal{U} = 3Nk_BT - Nk_BT\left(\alpha\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)} + 2\right). \tag{2.2.11}$$

Again, it can be seen that the ultra relativistic limit can be obtained by letting $\alpha \to 0$ since the term written in brackets vanishes in this case.
From equation (2.2.10) and (2.0.2), we immediately derive the ideal gas equation via the definition of pressure in the canonical ensemble

$$p = \frac{\partial \mathcal{F}}{\partial V} = \frac{Nk_BT}{V}. \tag{2.2.12}$$

2.3 Deriving an equation of state

This section aims to develop an equation between the thermodynamic energy density $\rho = \mathcal{U}/V$ and the pressure p of the gas given by the ideal gas equation (2.2.12). This is done by assuming an additional constraint and thus further reducing the degrees of freedom of the thermodynamic system.
When assuming an adiabatic condition $\delta Q = 0$ and using the First Law of Thermodynamics $dU = \delta Q + \delta W$, where $\delta W = -pdV$ and $dU = C_V dT$, we can relate pressure and temperature. Using equation 2.2.11 and 2.2.12, we obtain

$$\begin{aligned}
-pdV &= C_V dT & (2.3.1)\\
-\frac{Nk_BT}{V}dV &= Nk_B\left[1+\alpha^2\left(\left(\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)}\right)^2 - \frac{\partial_\alpha^2 K_2(\alpha)}{K_2(\alpha)}\right)\right]dT & (2.3.2)\\
-\frac{dV}{V} &= \left[1-\alpha^2\partial_\alpha\left(\frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)}\right)\right]\frac{dT}{T} & (2.3.3)\\
&= \left[1-\alpha^2\partial_\alpha^2\left(\log K_2(\alpha)\right)\right]\frac{dT}{T} & (2.3.4)
\end{aligned}$$

This equation also shows explicitly the T dependence of the specific heat C_V for the non-ultra-relativistic case. Again, taking the ultra-relativistic-limit by taking $\alpha \to 0$, one can calculate that the right hand term in the first equation converges to $-2Nk_B$. This agrees with the expected specific heat for an ultra-relativistic gas $C_{V,\mathrm{ur}} = 3Nk_B$. With the identity $d\alpha/\alpha = -dT/T$ (using $\alpha = mc^2/k_BT$), we can transform the equation and integrate it. After applying partial integration, the result is

$$\begin{aligned}
\frac{dV}{V} &= \left(1-\alpha^2\partial_\alpha^2\log K_2(\alpha)\right)\frac{d\alpha}{\alpha} & (2.3.5)\\
\log\left(\frac{V}{V_0}\right) &= \log\left(\frac{\alpha}{\alpha_0}\right) - \int_{\alpha_0}^{\alpha}\alpha\frac{\partial^2}{\partial\alpha^2}\log(K_2(\alpha'))d\alpha' & (2.3.6)\\
&= \log\left(\frac{\alpha}{\alpha_0}\right) + \log\left(\frac{K_2(\alpha)}{K_2(\alpha_0)}\right) - \left[\alpha\frac{\partial_\alpha K_2}{K_2}\right]_{\alpha_0}^{\alpha} & (2.3.7)
\end{aligned}$$

This equation enables us to write down a relation between volume and temperature (encapsulated in $\alpha = mc^2/k_BT$)

$$V(\alpha) = \frac{\alpha K_2(\alpha)}{C} \exp\left(\alpha \frac{K_3(\alpha) + K_1(\alpha)}{2K_2(\alpha)}\right) \tag{2.3.8}$$

where the constant C is defined by the equation beforehand and only depends on the integration boundaries α_0 and V_0. It is given by

$$C = \frac{\alpha_0 K_2(\alpha_0)}{V_0} \exp\left(\alpha_0 \frac{K_1(\alpha_0) + K_3(\alpha_0)}{2K_2(\alpha_0)}\right). \tag{2.3.9}$$

Since the goal of this section is to obtain a readable output for an equation of state, it is necessary to construct a bijection relating p and T. This is made clear when writing down the energy density

$$\rho = \frac{\mathcal{U}}{V} = \frac{Nk_BT}{V} - \frac{Nk_BT}{V}\left(\alpha \frac{\partial_\alpha K_2(\alpha)}{K_2(\alpha)}\right) \tag{2.3.10}$$

where $p = Nk_BT/V$ can be easily identified but the T dependence through α is not solved yet.

The pressure p can be rewritten to take the form

$$p = \frac{Nk_BT}{V} = CNmc^2 \frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right). \tag{2.3.11}$$

At this point it is not reasonable to ask what happens in the ur-limit since C depends non-trivially on m and m is not fully substituted in α.

Interestingly, the pressure seems to be constant for very high temperatures. The limiting case is obtained when taking $T \to \infty$ (which corresponds to $\alpha \to 0$)

$$\lim_{\alpha \to 0}\left[\frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right)\right] = \frac{1}{2e^2} \approx 0.006767. \tag{2.3.12}$$

The same argument then holds true for the density given by equation 2.3.10 and since

$$\lim_{\alpha \to 0}\left[1 + \alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right] = 3 \tag{2.3.13}$$

we have

$$\lim_{\alpha \to 0}\left(\frac{\rho(\alpha)}{Cmc^2}\right) = \frac{3}{2e^2} \approx 0.203003. \tag{2.3.14}$$

Theorem 2.1

The mapping $p : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \alpha \mapsto p(\alpha)$ written down in equation (2.3.11) is a bijection for any $N, m, c, C \neq 0$.

$$p = \frac{Nk_BT}{V} = CNmc^2 \frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right) \tag{2.3.15}$$

Proof. For this proof it suffices to show that the function $p(\alpha)$ has a strictly monotonous[1] behaviour. Without loss of generality, we assume $N, m, c, C > 0$. Now it is obvious that the first two terms $Nmc^2/CK_2(\alpha)$ and α^{-2} are strictly decreasing. This is easy to see when using [**ModifiedBesselFunction**]

$$\frac{\partial K_n}{\partial \alpha} = \frac{n}{\alpha} K_n - K_{n+1} = -\frac{K_{n-1} + K_{n+1}}{2}. \tag{2.3.16}$$

We then calculate the derivative of the third term and divide by the exponential (since it is positive)

$$\frac{1}{\exp(\alpha\partial_\alpha \log(K_2))} \frac{\partial}{\partial \alpha} \exp(\alpha \partial_\alpha \log(K_2)) \tag{2.3.17}$$

$$= \quad \partial_\alpha \log(K_2) + \alpha \partial_\alpha^2 \log(K_2) \tag{2.3.18}$$

$$= \quad \frac{\partial_\alpha K_2}{K_2} + \alpha \frac{\partial_\alpha^2 K_2}{K_2} + \alpha \left(\frac{\partial_\alpha K_2}{K_2} \right)^2 \tag{2.3.19}$$

$$= \quad \frac{K_1 - \frac{2}{\alpha} K_2}{K_2} + \alpha \frac{\frac{1}{\alpha} K_1 - K_2 - \frac{2}{\alpha} \left(K_1 - \frac{2}{\alpha} K_2 \right) + \frac{2}{\alpha^2} K_2}{K_2^2} + \alpha \frac{K_1^2 - \frac{1}{\alpha} K_1 K_2 - \frac{4}{\alpha^2} K_2^2}{K_2^2} \tag{2.3.20}$$

$$= -\alpha + \alpha \frac{K_1^2}{K_2^2} - 4 \frac{K_1}{K_2} \tag{2.3.21}$$

thus it is sufficient to show that

$$\alpha \frac{K_1^2}{K_2^2} < \alpha + 4 \frac{K_1}{K_2}. \tag{2.3.22}$$

We quickly prove the more general result $K_\nu < K_{\nu+1}$. One possible definition[**weissteinModifiedBesse** for the Bessel function K_ν is given by

$$K_\nu := \frac{\sqrt{\pi}}{(\nu - \frac{1}{2})!} \left(\frac{1}{2} z \right)^\nu \int_1^\infty e^{-tz} \left(t^2 - 1 \right)^{\nu - \frac{1}{2}} dt \tag{2.3.23}$$

We inspect the ratio

$$\frac{K_\nu}{K_{\nu+1}} = \frac{1}{\left(\nu + \frac{1}{2}\right) \left(\frac{1}{2} z\right)} \frac{\int e^{-tz} \left(t^2 - 1\right)^{\nu - 1/2} dt}{\int e^{-tz} \left(t^2 - 1\right)^{\nu + 1/2} dt} \tag{2.3.24}$$

and rewrite the demoninator with partial integration

$$\frac{1}{2} z \int_1^\infty e^{-tz} \left(t^2 - 1\right)^{\nu + 1/2} dt = \left(\nu + \frac{1}{2}\right) \int_1^\infty e^{-tz} t \left(t^2 - 1\right)^{\nu - 1/2} dt. \tag{2.3.25}$$

Now it is obvious that $K_{\nu+1} > K_\nu$. Thus in total, the function given by equation 2.3.26 can be inverted. □

[1] We omit this word in this proof for readability.

With this mapping $p : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \alpha \mapsto p(\alpha)$ and its inverse $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, p \mapsto \alpha(p)$, we can use 2.3.10 and finally write down the equation of state

$$\rho = \frac{\mathcal{U}}{V} = p\left(1 + \alpha(p)\frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))}\right). \tag{2.3.26}$$

Figure 1 is obtained by choosing numerical values and then interpolating and inverting equation 2.3.11. The constant factor

$$A := CNmc^2/p_0 \tag{2.3.27}$$

is substituted to obtain independence of ${p_0}^2$. Furthermore the graphs of the plotted EOS are normalised such that

$$\rho_i(p_0) = \rho_{0,i} \tag{2.3.28}$$

and can thus be compared with each other. Note that the ρ_0 of the plot is not a universal value but rather each function has been scaled individually and ρ_0 is a placeholder for the corresponding $\rho_{0,i}$. The polytropic EOS $\rho_{cla}(p) = Ap^{1/\gamma}$ when normalised as before the equation is independent of A and can be uniquely characterised by n. For further details, the interested reader is referred to [**pleyerGithubRepositoryJonas**].

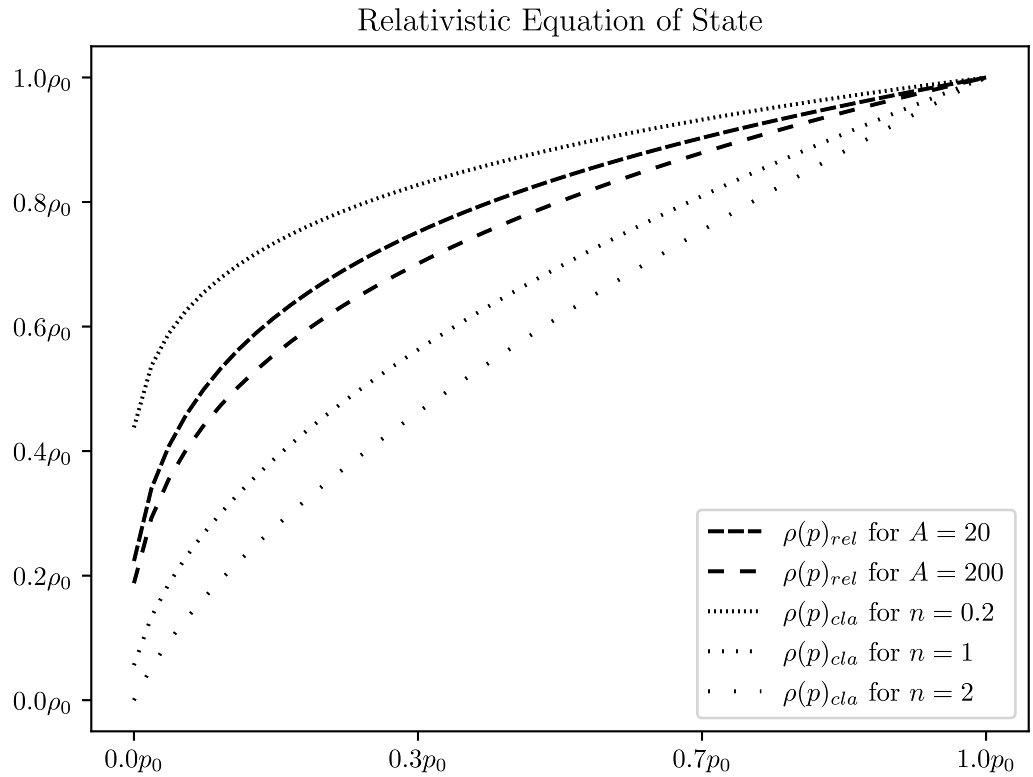

Figure 1: The plot shows the relativistic equation of state $\rho(p)$ normalised such that values can be compared with a polytropic EOS. Graphs for the relativistic version are independent of the exponent n which is a degree of freedom intrinsic to the polytropic equation of state. By normalisation, the graphs of the polytropic equation of state are independent of the factor A which is explained in equation 2.3.27.

[2] This will be the starting value for the pressure in the TOV equation. Since the pressure decreases from the inside of a star to the outside, this resembles our highest value of p.

3 Calculating the Mass of a Star with an EOS

3.1 Deriving the TOV-Equation

In this chapter, we consider a spherical-symmetric static Lorentz-Manifold (V, g) with charts such that the metric g can be written as

$$g = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.1.1}$$

The stress-energy tensor of an ideal fluid with density ρ and pressure p is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) \tag{3.1.2}$$

where u is the 4-velocity of the fluid. In the rest frame where $u^\mu = (-e^{-\nu/2}, 0, 0, 0)$, this equation simplifies to

$$T^\mu_\nu = \begin{bmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}. \tag{3.1.3}$$

The Christoffel symbols for this metric are

$$\Gamma^0_{\mu\nu} = \begin{bmatrix} 0 & \nu'/2 & & \\ \nu'/2 & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \tag{3.1.4}$$

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} \nu' e^{\nu-\lambda}/2 & & & \\ & \lambda'/2 & & \\ & & -re^{-\lambda} & \\ & & & -r\sin^2\theta e^{-\lambda} \end{bmatrix} \tag{3.1.5}$$

$$\Gamma^2_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & 0 & 1/r & \\ & 1/r & 0 & \\ & & & -\sin\theta\cos\theta \end{bmatrix} \tag{3.1.6}$$

$$\Gamma^3_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & 0 & 0 & 1/r \\ & 0 & 0 & \cos\theta/\sin\theta \\ & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix} \tag{3.1.7}$$

From these, the non-zero components of the Ricci-Tensor can be calculated

$$R_{11} = \frac{1}{4r} e^{-\lambda} \left[\left(2r\nu'' + r\nu'^2\right) + (4 - r\lambda')\,\nu' \right] \tag{3.1.8}$$

$$R_{22} = -\frac{1}{4r} e^{-\lambda} \left[(2r\nu'') + r\nu'^2 - r\lambda'\nu' - 4\lambda' \right] \tag{3.1.9}$$

$$R_{33} = -\frac{1}{2r^2} e^{-\lambda} \left(r\nu' - r\lambda' - 2e^{\lambda} + 2 \right) \tag{3.1.10}$$

$$R_{33} = R_{44} \tag{3.1.11}$$

and with $R_{\mu\nu} - g_{\mu\nu}R/2 = G_{\mu\nu} = 8\pi T_{\mu\nu}$ ultimately yield the following field equations.

$$-8\pi T^0_0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1-e^{-\lambda}}{r^2} \tag{3.1.12}$$

$$8\pi T^1_1 = 8\pi p = \nu'\frac{e^{-\lambda}}{r} - \frac{1-e^{-\lambda}}{r^2} \tag{3.1.13}$$

$$8\pi T^2_2 = 8\pi p = \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{3.1.14}$$

Since $R^3_3 = R^2_2$, we omitted the last equation. From equation (3.1.12) we infer the relation.

$$e^{-\lambda} = 1 - \frac{2}{r}\int_0^r 4\pi\rho(r')r'^2 dr' = 1 - \frac{2m(r)}{r}. \tag{3.1.15}$$

The metric needs to be defined at every point in space and thus we can not have any additional integration constant in equation (3.1.15) since otherwise we would obtain a term a/r which is ill defined for $r \to 0$.

The property $m(r)$ can be recognized as the Newtonian Mass of the star (which is different to the proper mass). Since $e^{-\lambda} > 0$, we immediately see that $m(r) < r/2$.

In addition to the Field equations 3.1.12 to 3.1.14 the divergence of the Stress-Energy Tensor yields more information

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.1.16}$$

The following explicit calculation[3] shows how to obtain this additional restriction on the pressure and density.

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_1 + \Gamma^\mu_{\mu\sigma} T^\sigma_\nu - \Gamma^\sigma_{\mu\nu} T^\mu_\sigma \tag{3.1.17}$$

$$\nabla_\mu T^\mu_1 = \frac{\partial p}{\partial r} + p\left(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31}\right) - \Gamma^\sigma_{\mu 1} T^\mu_\sigma \tag{3.1.18}$$

$$= \frac{\partial p}{\partial r} + p\left(\frac{\nu' + \lambda'}{2} + \frac{2}{r}\right) + \rho\frac{\nu'}{2} - p\frac{\lambda'}{2} - p\frac{2}{r} \tag{3.1.19}$$

$$\frac{\partial p}{\partial r} = -\frac{p+\rho}{2}\nu' \tag{3.1.20}$$

Together with equation 3.1.13 and the definition 3.1.15, we can write

$$\frac{\partial p}{\partial r} = -\frac{p+\rho}{2}\left(\frac{8\pi p r + \frac{1-e^{-\lambda}}{r}}{e^{-\lambda}}\right) \tag{3.1.21}$$

$$= -\frac{p+\rho}{2r}\left(\frac{8\pi p r + \frac{2m}{r^2}}{1 - \frac{2m}{r}}\right) \tag{3.1.22}$$

$$= -\frac{m\rho}{r^2}\left(1 + \frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m} + 1\right)\left(1 - \frac{2m}{r}\right)^{-1} \tag{3.1.23}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1 + \frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2} + 1\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \tag{3.1.24}$$

where in the last step the constants $c = G = 1$ were put back in.

[3] Again assuming spherical symmetry.

3.2 Newtonian Limit

This section follows [**weissteinLaneEmdenDifferentialEquation**] and [**chandrasekharChandrasel**
Together with a polytropic equation of state $p = K\rho^{1+1/n}$ and the definition $\rho = \lambda\theta^n$, we expect to obtain the Newtonian behavior in the non-relativistic limit in the form of the Lane-Emden equation

$$\frac{K(n+1)\lambda^{1/n-1}}{4\pi}\Delta\theta + \theta^n = 0. \tag{3.2.1}$$

The usual non-relativistic limit is obtained from a Taylor expansion of equation 3.1.24 around $1/c^2$ in lowest order. The resulting equation then reads

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.2.2}$$

Using the previous relations for ρ and p and again setting $G = c = 1$, we can calculate

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}\left(K\rho^{1+1/n}\right) = K\lambda^{1+1/n}(n+1)\theta^n\frac{\partial\theta}{\partial r} = -\frac{m\lambda\theta^n}{r^2} \tag{3.2.3}$$

by using the definition of our polytropic equation of state and 3.2.2. Rearranging and taking the derivative of this equation and using

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2, \tag{3.2.4}$$

we obtain

$$-\frac{\partial m}{\partial r} = K\lambda^{1/n}(n+1)\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -4\pi r^2\lambda\theta^n \tag{3.2.5}$$

$$\frac{K\lambda^{1/n-1}(n+1)}{4\pi}\Delta\theta + \theta^n = 0. \tag{3.2.6}$$

Upon redefining $\xi = r/\alpha$ where $4\pi\alpha^2 = (n+1)K\lambda^{1/n-1}$, one can obtain the mathematically cleaner looking equation

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) + \theta^n = 0 \tag{3.2.7}$$

Exact solutions are known for the cases $n = 0, 1, 5$. The derivation can be found in appendix A. Table 1 summarizes them. Together with equation 3.2.7, it is immediate to see that for values $n \geq 5$, the equation does not yield solutions with a zero value.

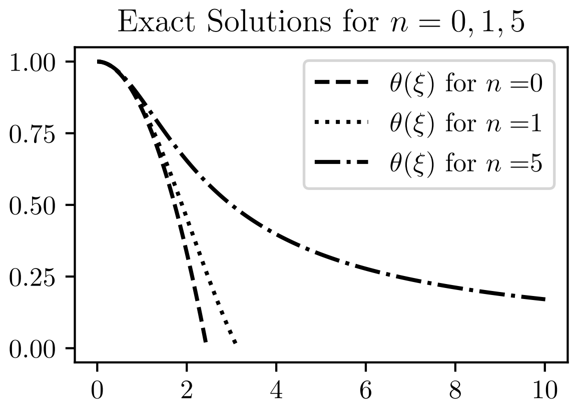

Figure 2: Graph of exact LE solutions.

n	Solution	ξ_0
$n = 0$	$1 - \frac{1}{6}\xi^2$	$\sqrt{6}$
$n = 1$	$\frac{\sin(\xi)}{\xi}$	π
$n = 5$	$\frac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}$	∞

Table 1: Lane Emden exact Solutions and their zero value.

3.3 Upper Limits

We will first follow mainly the approach given in [**waldGeneralRelativity1984**]. The first assumptions will be that $d\rho/dr < 0$ and $\rho \geq 0$. Also we again consider a compact star, meaning $\rho(r) = 0$ for all $r > R$. While we talk about the derivative of ρ it may not be differentiable at $r = R$. However differentiability of the metric demands at least continuity at every point. We first state a useful Lemma for the proof of our next theorem.

Lemma 3.1
The function $\rho_{av}(r) = m(r)/r^3$ has negative slope for $\partial_r \rho(r) \leq 0$.

Proof. Taking the derivative of ρ_{av} we obtain

$$\frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) = -\frac{m(r)}{r^4} + \frac{4\pi\rho(r)}{r}. \tag{3.3.1}$$

For this equation to be ≤ 0, it is sufficient to have

$$\rho(r)r^3 \leq \frac{m(r)}{4\pi} = \int\limits_0^r \rho(r')r'^2 dr'. \tag{3.3.2}$$

Since for $r = 0$ both sides of the equations match, we can take another derivative and obtain

$$\rho(r)r^2 \geq \partial_r\rho(r)r^3 + \rho(r)r^2 \Leftrightarrow 0 \geq \partial_r\rho(r)r^2 \tag{3.3.3}$$

which is always true since ρ has negative slope by definition. □

Theorem 3.2 - Mass Bound
The Mass of a spherically symmetric star is bound from above by

$$M < \frac{4}{9}R. \tag{3.3.4}$$

Proof. In our attempt to obtain an upper limit for the mass of a spherically symmetric star, we start by taking the difference of equation (3.1.13) and (3.1.14), we obtain

$$0 = \nu' \frac{e^{-\lambda}}{r} - \frac{1-e^{-\lambda}}{r^2} - \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{3.3.5}$$

$$= -\frac{2m(r)}{r^3} + \frac{\lambda' e^{-\lambda}}{2r} - \frac{e^{-\lambda}}{2}\left[\nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r} - \frac{\lambda'\nu'}{2}\right] \tag{3.3.6}$$

$$= r\frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) - \frac{e^{-\lambda}}{2}\left[\nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r} - \frac{\lambda'\nu'}{2}\right] \tag{3.3.7}$$

$$0 = \frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) - \frac{e^{-\lambda}}{2}\left[\frac{\nu''}{r} + \frac{\nu'^2}{2r} - \frac{\nu'}{r^2} - \frac{\lambda'\nu'}{2r}\right] \tag{3.3.8}$$

$$= \frac{\partial}{\partial r}\left(\frac{m(r)}{r^3}\right) - \frac{1}{2}e^{-\frac{\lambda+\nu}{2}}\frac{\partial}{\partial r}\left[\frac{1}{r}\nu' e^{\frac{\nu-\lambda}{2}}\right]. \tag{3.3.9}$$

Since $\partial_r \rho \leq 0$, also the average density $m(r)/r^3$ decreases with r. Thus we obtain

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\nu' \exp\left(\frac{\nu-\lambda}{2}\right)\right] \leq 0. \tag{3.3.10}$$

We integrate this equation from R to $r < R$

$$\frac{\nu'}{r}\exp\left(\frac{\nu-\lambda}{2}\right) \geq \frac{2\nu'(R)}{R} e^{-\frac{1}{2}\lambda(R)} \frac{\partial}{\partial r} e^{\frac{\nu}{2}}\Big|_R \tag{3.3.11}$$

and use the Schwarzschild solution at $r = R$ for e^{λ} and e^{ν}. This is justified since we assumed $\rho(r) = 0$ for $r > R$ and thus we need to recover the vacuum solution for a spherically symmetric object which is given by the Schwarzschild solution. By continuity of the metric on every point of space, we can match

$$e^{-\lambda(r)}\big|_R = \left[1 - \frac{2M}{r}\right]_R = e^{\nu(r)}\big|_R \tag{3.3.12}$$

and with the explicit solution for $e^{-\lambda}$, we obtain

$$\frac{2m(r)}{r}\bigg|_R = \frac{2M}{R}. \tag{3.3.13}$$

When plugging this into equation (3.3.12), the result is

$$\frac{\nu'}{2r}\exp\left(\frac{\nu-\lambda}{2}\right) \geq \frac{(1-2M/R)^{1/2}}{R}\frac{\partial}{\partial r}\left(1 - \frac{2M}{r}\right)^{1/2}\Bigg|_{r=R} = \frac{M}{R^3}. \tag{3.3.14}$$

Now we multiply by $r\exp(\lambda/2)$ and use the explicit solution for e^{λ}

$$\frac{\partial}{\partial r}\left(e^{\frac{\nu}{2}}\right) \geq \frac{M}{R^3} r e^{\frac{\lambda}{2}} = \frac{M}{R^3}(r - 2m(r)) \tag{3.3.15}$$

and integrate again this time from 0 to R

$$e^{\nu(0)/2} \leq \left(1 - \frac{2M}{R}\right)^{1/2} - \frac{M}{R^3}\int_0^R \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r dr. \tag{3.3.16}$$

As we have already noted, the average density $m(r)/r^3$ decreases, meaning explicitly $m(r)/r^3 \geq M/R^3$ and thus the integral with the previous equation can be written as

$$e^{\nu(0)/2} \leq \left(1 - \frac{2M}{R}\right)^{1/2} + \frac{1}{2}\left[1 - \frac{2Mr^2}{R^3}\right]^{1/2}\Bigg|_{r=0}^{r=R} = \frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}. \tag{3.3.17}$$

The simple fact that $e^{\nu(0)/2} > 0$ then implies

$$\left(1 - \frac{2M}{R}\right)^{1/2} > \frac{1}{3} \tag{3.3.18}$$

which is equivalent to

$$M < \frac{4R}{9}. \tag{3.3.19}$$

□

This shows that the mass of star has an upper limit under the assumptions $\rho \geq 0$, $\partial_r \rho \leq 0$ and $\rho(R) = 0$ for some $R \geq 0$. The case in which $M = 4R/9$ would be achieved by using a constant density of

$$\rho = \frac{1}{4\pi} \frac{M}{R^3} \tag{3.3.20}$$

from which the mass $m(r)$ can be obtained

$$m(r) = \frac{r^3}{R^3} M. \tag{3.3.21}$$

However this case is actually forbidden as was just shown. Physically, the assumption that $\rho(R) = 0$ defines a radius for the stellar object since it limits the physical dimension. In general stellar objects need not to fulfill this condition. Physically it is however necessary to have $p(R) = 0$ while the density may have discontinuities at this point.

4 Numerical Solutions

4.1 Comparing TOV and LE results with a polytropic EOS

In this section the numerical solutions of the TOV equation

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{4.1.1}$$

$$\frac{\partial p}{\partial r} = -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1} \tag{4.1.2}$$

as derived previously in 3.1 will be discussed. To obtain numerical solvability a equation of state in the form $\rho(r, p)$ is supplied. In Figure 3 a plot of such a solution is presented. The density ρ is derived via the equation 4.1.1 and the integration is done with a 4th order Runge-Kutta Method. The integration is stopped once the pressure reaches values $p \leq 0$. The initial value of 4.1.1 is $\partial m/\partial r(r=0) = 0$. For 4.1.1, the initial value can be calculated when applying L'Hôpital's and combining them to obtain

$$\lim_{r\to 0}\frac{m}{r} = \lim_{r\to 0}\frac{\partial m}{\partial r} = \lim_{r\to 0}\frac{4\pi\rho r^2}{1} = 0 \tag{4.1.3}$$

$$\lim_{r\to 0}\frac{m}{r^2} = \lim_{r\to 0}\frac{1}{2r}\frac{\partial m}{\partial r} = \lim_{r\to 0}\frac{4\pi\rho r^2}{2r} = 0 \tag{4.1.4}$$

$$\lim_{r\to 0}\frac{m}{r^3} = \lim_{r\to 0}\frac{1}{3r^2}\frac{\partial m}{\partial r} = \lim_{r\to 0}\frac{4\pi\rho r^2}{3r^2} = \frac{4\pi\rho_0}{3} \tag{4.1.5}$$

$$\lim_{r\to 0}\frac{\partial p}{\partial r} = 0 \tag{4.1.6}$$

Explicit code can be found in [**pleyerGithubRepositoryJonas**]. The Lane-Emden equation was obtained in section 3.2 as the non-relativistic limit of the TOV equation by neglecting terms of order $1/c^2$ and higher and setting $G = c = 1$. Equation 3.2.7 was used to obtain numerical values for the Lane-Emden equation. Afterwards the factor α was reinserted with the corresponding numerical values given to the TOV equation to obtain comparable results.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{4.1.7}$$

one could be tempted to make the assumption that $\frac{\partial p_{\mathrm{TOV}}}{dr} \leq \frac{\partial p_{\mathrm{LE}}}{dr}$. However this equation fails since the mass given in equation 4.1.1 is not the same as the one given in the LE equation. Figure 3 shows the solution of both equations for the parameters of table 2. Note that values in the last part are calculated instead of supplied to the solving routine. Additionally conversion equations to compare TOV and LE results are displayed.
Since the mass reads

$$m(r) = \int_0^r 4\pi s^2 \rho(s) ds, \tag{4.1.8}$$

we expect $\partial m/\partial r(R) = 0$ if $p(R) = 0$ when choosing a polytropic equation of state with $\gamma > 0$. The plot in figure 3 shows this expected behaviour for the Lane Emden equation at $r \approx 2.31$ and has the same behaviour for the TOV results at $r \approx 6.80$.[4]

[4]For the purpose of nicely displaying the calculated result, the plot only shows result up to $r = 2.5$

TOV		**LE**	
EOS	$\rho = Ap^{1/\gamma}$	EOS	$p = K\rho^{\gamma}$
A	2		
$\gamma = 1 + \frac{1}{n}$	4/3	$n = 1/(\gamma - 1)$	3
p_0	0.5	θ_0	1
m_0	0	$d\theta_0$	0
dr	0.01	$d\xi = dr/\alpha$	$0.01/0.3355 \approx 0.0298$
$\rho_0 = Ap_0^{1/\gamma}$	$2(2)^{\frac{4}{3}} \approx 1.1892$	$\lambda = \rho_0$	$2(2)^{\frac{4}{3}} \approx 1.1892$
		$K = A^{-1/\gamma}$	$2^{-3/4} \approx 0.5946$
		$\alpha^2 = ((n+1)K\lambda^{1/n-1})/(4\pi)$	≈ 0.1125

Table 2: Parameters for numerical solving of the TOV and Lane-Emden equation.

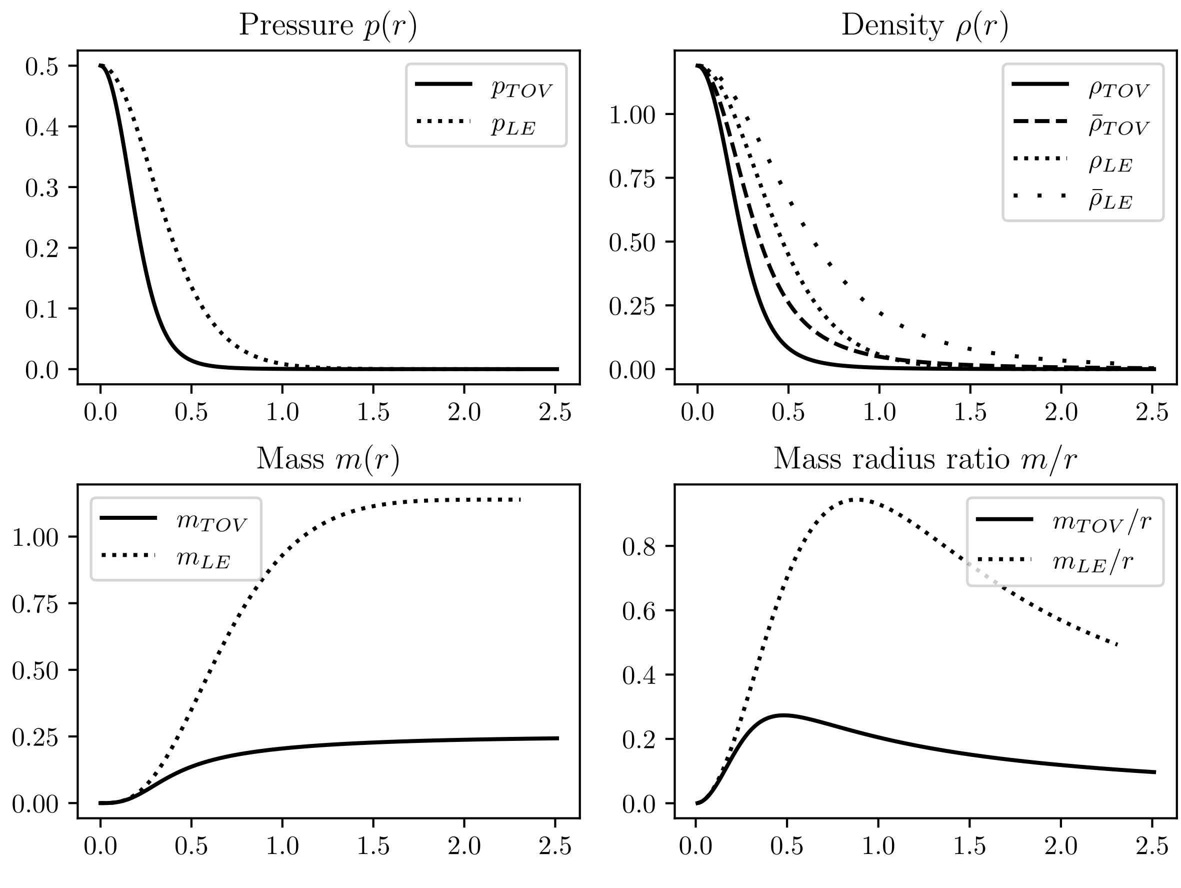

Figure 3: Comparison of the fully relativistic TOV result and the classical Lane-Emden solution. The images show the plots for the parameters of table 2. First the pressure is presented. Afterwards the density calculated with the given EOS and the average density $\bar{\rho}_i = (4\pi/3)^{-1} m_i/r^3$ for the two solutions are beeing compared. In the second row, the mass and the ratio m_i/r can be seen.

4.2 Verifying the results

Verification is done in different ways. First, one can compare calculated LE results with already known exact solutions for certain exponents as given in table 1. All calculations are carried out with a chosen stepsize of $d\xi = 0.03$.

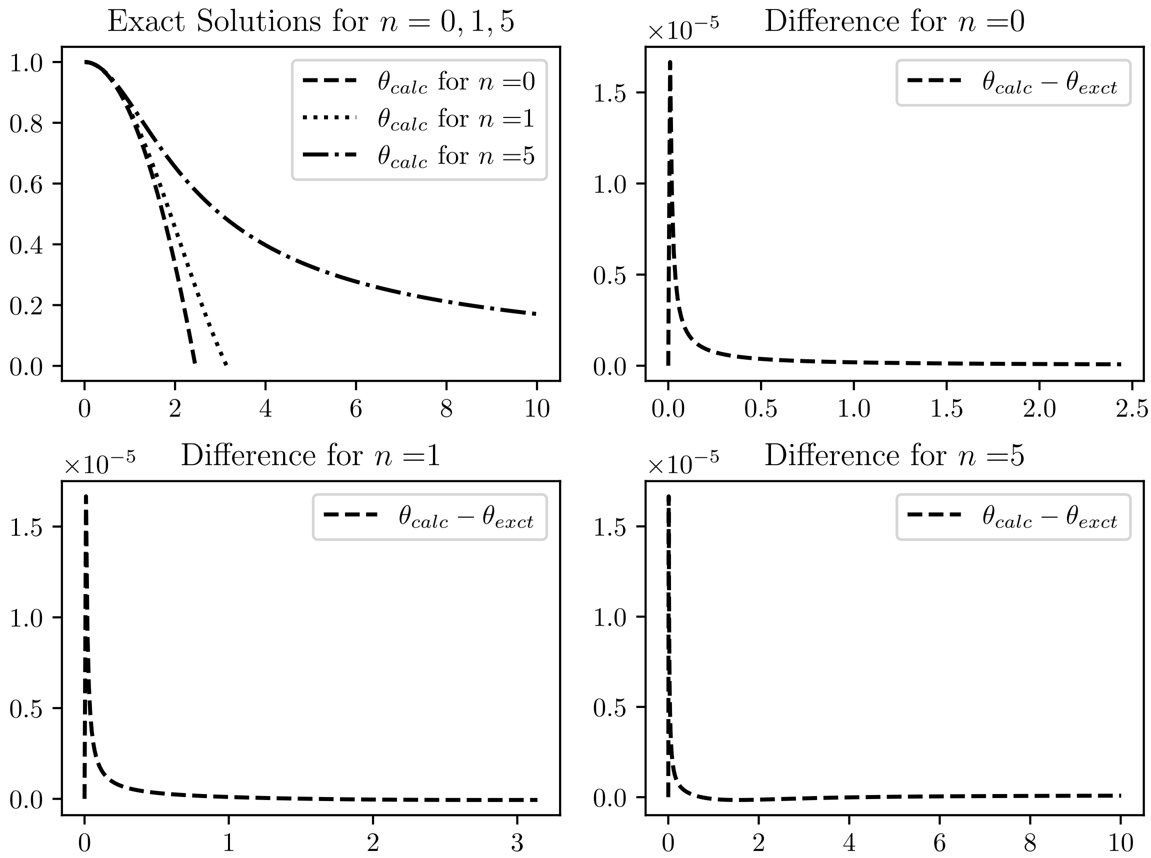

Figure 4: The plot shows the exact solutions for the LE equation as given by table 1. Afterwards the difference of the exact to the numerically calculated result is shown. The y-scale of each plot is multiplied by a factor of 10^{-5} which shows that the solutions agree up to very high precision.

To verify the equality of both solutions, we calculate the LE result with the TOV solving algorithm by dropping terms from right to left in equation 4.1.2. These intermediate solutions have been numerically calculated and results can be seen in figure 5.

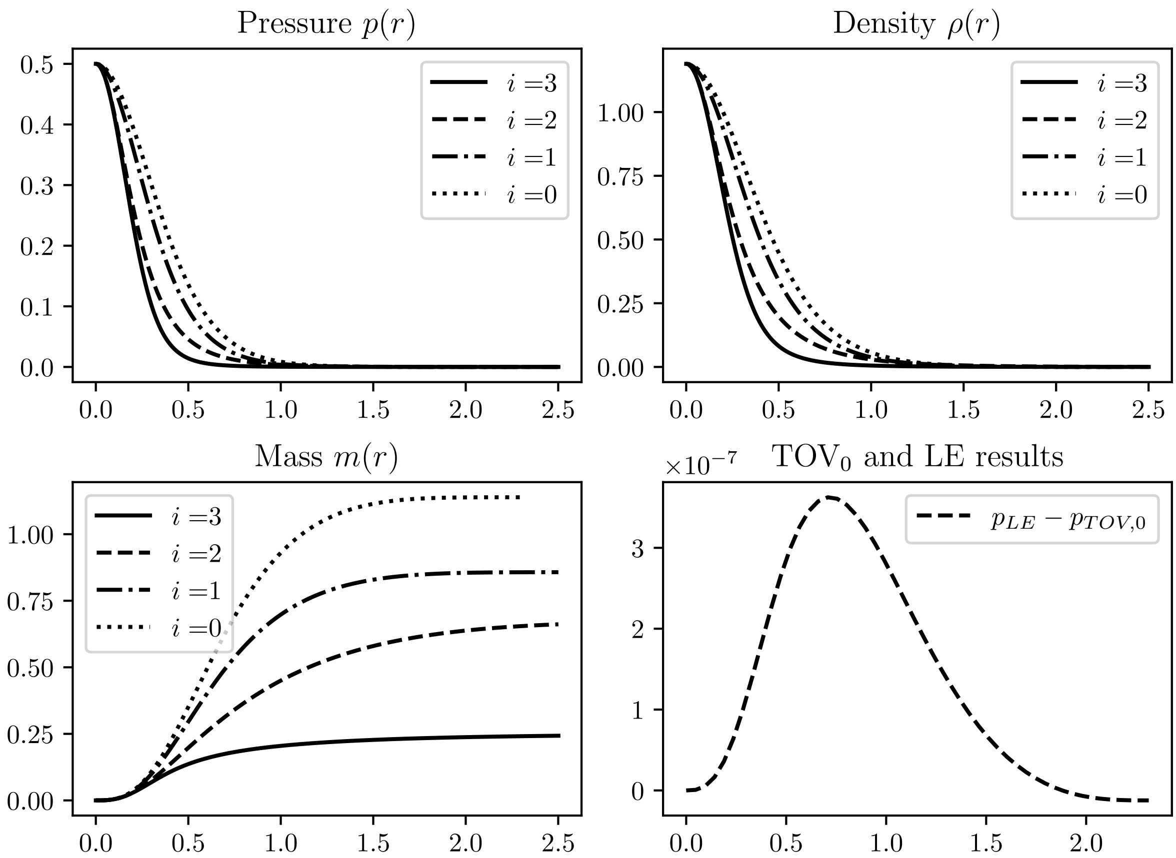

Figure 5: Comparison between the LE and TOV solutions while dropping terms from equation 4.1.2 from right to left. The last figure shows the difference between the TOV solution with 0 terms and the LE solution. The scale of the difference shows that the numerical differ only by values up to 3.5×10^{-7}. In order to achieve such a comparison, a polynomial fit of both pressure solutions had to be done. This should however not alter the result in any noticeable way. For further details see [**pleyerGithubRepositoryJonas**].

4.3 Relativistic EOS

In the previous discussion, we relied on the EOS given by 3. This is a versatile assumption, but one could ask, what would happen to a star in which the particles have no interaction but are near relativistic speed. The resulting EOS was calculated in the beginning 2.3.26 although not written down explicitly. Since explicit inversion of the given function is hard, we rely on numerical methods for calculation.

5 Ideas

Theorem 5.1 - Lane-Emden Finite Boundary
For every $0 \leq n \leq 5$, the Lane-Emden equation

$$\frac{d^2\theta}{d\xi^2} + \theta^n = 0 \tag{5.0.1}$$

with $\theta(\xi = 0) = 1$ has a zero value for a finite ξ_0. To prove this theorem, show the statements 5.2, 5.3 and 5.4.

Theorem 5.2 - Lane-Emden Finite Boundary 1
For every $n \geq 0$, the Lane-Emden equation 5.0.1 has a solution in a ϵ Ball around $r = 0$.

Theorem 5.3 - Lane-Emden Finite Boundary 2
For $n < 5$, the Lane-Emden equation 5.0.1 has no Solution defined on the total space $\mathbb{R}_{\geq 0}$.

Theorem 5.4 - Lane-Emden Finite Boundary 3
Let $n < 5$ and $\theta : [0, x) \to \mathbb{R}$ be a solution of 5.0.1. If $\theta(x) > 0$, then θ can be extended with Theorem 5.2 to $[0, x + \epsilon)$ where ϵ follows a growth condition $\epsilon(\xi) = ...$ such that if θ would have no zero point, there would be a solution on total $\mathbb{R}_{\geq 0}$ and thus a contradiction with 5.3.

Theorem 5.5 - Lane-Emden Finite Boundary 4
The Lane Emden equation has a zero value if the exponent function $n(\xi)$ fullfills the growth condition XYZ.

Theorem 5.6 - TOV Exact Solution
The TOV equation 4.1.1 with a polytropic EOS $\rho = Ap^{1/\gamma}$ has a well defined limiting case where $A \to 0$ with $m = 0$ and

$$p = \frac{1}{r^2 + 1} \bmod 4\pi. \tag{5.0.2}$$

Theorem 5.7 - TOV Finite Boundary
For each $A > 0$, the TOV equation 4.1.1 has a exponent $n > 0$ for which a solution p does not have any zero points.

Theorem 5.8 - TOV Finite Boundary 1
If p_1 and p_2 are two solutions of the TOV equation (for equal A) with p_i corresponding to n_i exponents, then if $n_1 < n_2$, the zero point (if it exists) of p_1 is smaller than that of p_2 (if it exists or is ∞).

Theorem 5.9 - TOV Finite Boundary 2
a

List of Figures

List of Tables

Appendix

A Exact solutions of the LE equation

This section partly relies on information in [**weissteinLaneEmdenDifferentialEquation**] and [**chandrasekharChandrasekharAnIntroductionStudy1958**]. The LE equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0 \tag{A.1}$$

which for $n=0$ transforms readily into

$$\int_0^{\xi}\frac{d}{d\xi}\left(\xi'^2\frac{d\theta}{d\xi}\right)d\xi'=-\int_0^{\xi}\xi'^2 d\xi'. \tag{A.2}$$

Both sides can be evaluated directly and then simplified further.

$$\xi^2\frac{d\theta}{d\xi}=-\frac{\xi^3}{3} \tag{A.3}$$

$$\frac{d\theta}{d\xi}=-\frac{\xi}{3} \tag{A.4}$$

$$\theta(\xi)=\theta(0)-\frac{\xi^2}{6} \tag{A.5}$$

With the initial condition $\theta(0)=1$, we obtain the desired result. For $n=1$, equation A.1 transforms into

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\xi^2\theta=0 \tag{A.6}$$

which is a spherical Bessel differential equation

$$\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)+\left[k^2r^2-m(m+1)\right]U=0 \tag{A.7}$$

when setting $m=0$ and $k=1$. The solution to this equation is [**weissteinSphericalBesselDifferential**]

$$U(r)=A\sqrt{\frac{\pi}{kr}}J_{m+1/2}(kr)+B\sqrt{\frac{\pi}{kr}}Y_{m+1/2}(kr) \tag{A.8}$$

$$=Aj_m(kr)+By_m(kr) \tag{A.9}$$

where J_n is the Bessel function of the first and Y_n of the second kind and j_n and y_n are the corresponding spherical Bessel functions. Thus in our case with $j_0(x)=\sin(x)/x$ and $Y_0(x)=-\cos(z)/z$, we obtain

$$\theta(\xi)=A\frac{\sin(\xi)}{\xi}-B\frac{\cos(k\xi)}{\xi}. \tag{A.10}$$

The need for a well defined limit at $\xi\to 0$ implies that $B=0$ and thus since $\sin(z)/z\to 1$ for $z\to 0$, we have $A=\theta(0)=1$ and

$$\theta(\xi)=\frac{\sin(\xi)}{\xi}. \tag{A.11}$$

For $n=5$

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Erklärung

Hiermit versichere ich, dass ich die eingereichte Masterarbeit selbständig verfasst habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Inhalte als solche kenntlich gemacht. Weiter versichere ich, dass die eingereichte Masterarbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens war oder ist.

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