Test 2 Josh Boehm

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Joshua Boehm December 1, 2022 Math 3423 Homework 4

1 Libraries/Imports

[ ]:
```
import numpy as np
import scipy as sp
from numpy.linalg import solve
from numpy.linalg import qr
from numpy.linalg import svd
from scipy.linalg import diagsvd
from scipy.linalg import pinv
from scipy.optimize import minimize
from statistics import mean
import matplotlib.pyplot as plt
```

2 Question 1

Given the data points

$$(2,1),(5,2),(7,3),(8,3)$$

derive the equation of the least-squares line

$$y = mx + b$$

that best fits the given data points. Solve the system using the QR factorization, the SVD, and the Normal Equations.

[ ]:
```
A = np.matrix([[2,1],[5,1],[7,1],[8,1]])
b = np.matrix([1,2,3,3]).T
```

2.1 QR Factorization

The matrix created from the data points:

$$\begin{bmatrix} 2,1 \\ 5,1 \\ 7,1 \\ 8,1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

To solve using QR factorization, we first factor A as QR. Next, we pre-multiply with Q^T, giving $Rx = Q^T b$ ($Q^T Q$ is the identity matrix) Now, we can call `solve()` with R and $Q^T b$

[ ]:
```
Q, R = qr(A)

x = solve(R, Q.T @ b)
print(x)
```

```
[[0.35714286]
 [0.28571429]]
```

2.1.1 Solution

$$x = \begin{bmatrix} 0.35714286 \\ 0.28571429 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} \\ \frac{2}{7} \end{bmatrix}$$

2.2 SVD Factorization

To solve using SVD factorization, we first factor A as SVD.
Now, we create the pseudo-inverse by taking the transpose of the the parts in reverse order, except with Σ, we take the take the reciprocals before transposing.
Last, we can call pre-multiply b with $A^\dagger$, as $A^\dagger A$ is the identity matrix

$$Ax = bU\Sigma V^T x = bA^\dagger = V\Sigma^\dagger U^T A^T A = IIx = A^\dagger bx = A^\dagger b \text{or} V\Sigma^\dagger U^T b$$

[ ]:
```
U, sigma, VT = svd(A)
Sigma = diagsvd(sigma, len(A), len(A.T))
PseudoA = (VT).T @ pinv(Sigma) @ U.T

x = PseudoA @  b
print(x)
```

```
[[0.35714286]
 [0.28571429]]
```

2.2.1 Solution

$$x = \begin{bmatrix} 0.35714286 \\ 0.28571429 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} \\ \frac{2}{7} \end{bmatrix}$$

2.3 Normal Equations

The normal equation is that which minimizes the sum of the square differences between the left and right sides:

$$A^T A x = A^T b$$

In this case, I would just use `solve(A.T @ A, A.T @ b)` and see. You could still utilize the previous factorizations, if desired.

2.3.1 Solution

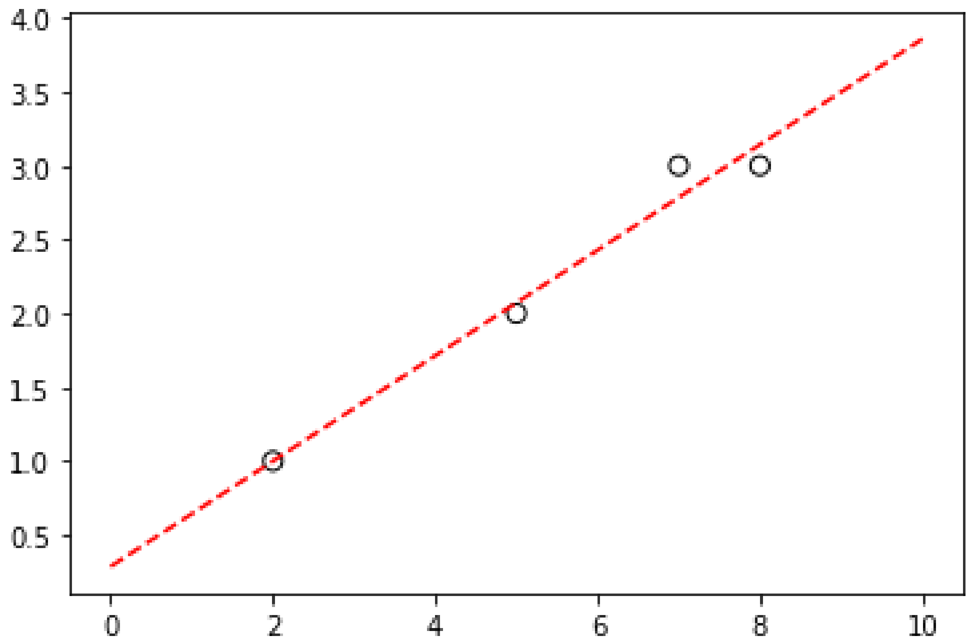
[ ]:
```
solvex = solve(A.T @ A, A.T @ b)
solveqr = solve(R.T @ Q.T @ Q @ R, R.T @ Q.T @ b)
solvesvd = solve(VT.T @ Sigma.T @ U.T @ U @ Sigma @ VT, VT.T @ Sigma.T @ U.T @␣
 ↪b)

print(f"{solvex}\n\n{solveqr}\n\n{solvesvd}")
```

```
[[0.35714286]
 [0.28571429]]

[[0.35714286]
 [0.28571429]]

[[0.35714286]
 [0.28571429]]
```

$$x = \begin{bmatrix} 0.35714286 \\ 0.28571429 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} \\ \frac{2}{7} \end{bmatrix}$$

[ ]:
```
X1 = np.array(A[:,0])
Y1 = np.array(b)
x = np.linspace(0,10,50)
y = (5/14)*x+(2/7)
plt.scatter(X1,Y1, edgecolor='k',c='none',s=50)
plt.plot(x,y, "r--")
```

[ ]:
```
[<matplotlib.lines.Line2D at 0x17cce7430>]
```



3 Question 2

Let

$$f(x, y) = 3 + x + \frac{1}{2}y - xy + \frac{1}{2}x^2 + y^2$$

1. Find the Gradiant and the Hessian of this function.
2. Starting at $\begin{bmatrix} x^0 \\ y^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ apply ***one-step*** of Newtown's Method to find $\begin{bmatrix} x^1 \\ y^1 \end{bmatrix}$ and $f(x^1, y^1)$.
3. Use any two methods from `SCIPY.OPTIMIZE` to minimize this function.

3.0.1 Gradient

In order to derive the gradient, let's first define it:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \vdots \end{bmatrix}$$

First with regards to x:

$$\frac{\partial f}{\partial x}\left(3 + x + \frac{1}{2}y - xy + \frac{1}{2}x^2 + y^2\right) = x - y + 1$$

Next with regards to y:

$$\frac{\partial f}{\partial y}\left(3+x+\frac{1}{2}y-xy+\frac{1}{2}x^2+y^2\right)=-x+2y+\frac{1}{2}$$

Thus:

$$\nabla f=\begin{bmatrix} x-y+1 \\ -x+2y+\frac{1}{2} \end{bmatrix}$$

3.0.2 Hessian

Again, to derive the Hessian, it might help to define it.

$$\mathbf{H_f}=\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

So our second partials are as follows:

$$\frac{\partial^2 f}{\partial x^2}=1$$
$$\frac{\partial^2 f}{\partial y \partial x}=-1$$
$$\frac{\partial^2 f}{\partial y \partial x}=-1$$
$$\frac{\partial^2 f}{\partial y^2}=2$$

This makes the Hessian:

$$\mathbf{H_f}=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

3.0.3 Definitions

[ ]:
```
def q2(x):
   return 3 + x[0] + (1/2)*x[1] - x[0]*x[1] + (1/2)*x[0]**2 + x[1]**2

def q2_gradient(x):
   return np.array([x[0]-x[1]+1,
                    -x[0]+2*x[1]+(1/2)])

def q2_hessian(x):
   return np.array([[1,-1],
                    [-1,2]])
starting_points = [np.random.randint(-20,20, size = (1,2)) for i in range(3)]
section_break = "============================================================"
```

Apply one step of Newton's Method:

[ ]:

```

```

[ ]: array([0.79591837, 0.90816327])

3.1 Use any two methods from SCIPY.OPTIMIZE to minimize this function.

3.1.1 Method 1: Nelder-Mead

[ ]:
```
for x in enumerate(starting_points):
    result = minimize(q2, x[1], method = 'Nelder-Mead', tol = 1.e-7)

    print(section_break)
    print('Test Run', x[0] + 1, ':')
    print(section_break)

    print('Starting Value Used: ', x[1])
    print("The Minimum Occurs at (x, y) = ", result.x)
    print("The Minimum Value = ", q2(result.x).round(3))

    print("Other Statistics:")
    print(result)
    print(section_break)
    print(section_break)
    print('\n')
```

```
==========================================================
Test Run 1 :
==========================================================
Starting Value Used:  [[-15   4]]
The Minimum Occurs at (x, y) =  [-2.5        -1.50000002]
The Minimum Value =  1.375
Other Statistics:
 final_simplex: (array([[-2.5       , -1.50000002],
       [-2.50000009, -1.50000007],
       [-2.50000003, -1.49999994]]), array([1.375, 1.375, 1.375]))
           fun: 1.3750000000000009
       message: 'Optimization terminated successfully.'
          nfev: 145
           nit: 75
        status: 0
       success: True
             x: array([-2.5       , -1.50000002])
==========================================================
==========================================================



==========================================================
Test Run 2 :
```

```
============================================================
Starting Value Used:  [[-2 12]]
The Minimum Occurs at (x, y) =  [-2.49999995 -1.49999996]
The Minimum Value =  1.375
Other Statistics:
 final_simplex: (array([[-2.49999995, -1.49999996],
       [-2.5       , -1.50000003],
       [-2.50000004, -1.50000002]]), array([1.375, 1.375, 1.375]))
           fun: 1.3750000000000009
       message: 'Optimization terminated successfully.'
          nfev: 121
           nit: 63
        status: 0
       success: True
             x: array([-2.49999995, -1.49999996])
============================================================
============================================================



============================================================
Test Run 3 :
============================================================
Starting Value Used:  [[  5 -15]]
The Minimum Occurs at (x, y) =  [-2.50000001 -1.49999999]
The Minimum Value =  1.375
Other Statistics:
 final_simplex: (array([[-2.50000001, -1.49999999],
       [-2.5000001 , -1.50000007],
       [-2.49999995, -1.50000005]]), array([1.375, 1.375, 1.375]))
           fun: 1.375
       message: 'Optimization terminated successfully.'
          nfev: 139
           nit: 71
        status: 0
       success: True
             x: array([-2.50000001, -1.49999999])
============================================================
============================================================
```

3.1.2 Method 2: Powell

[ ]:
```
for x in enumerate(starting_points):
    result = minimize(q2, x[1], method = 'Powell',tol = 1.e-7)

    print(section_break)
```

```python
    print('Test Run', x[0] + 1, ':')
    print(section_break)

    print('Starting Value Used: ', x[1])
    print("The Minimum Occurs at (x, y) = ", result.x)
    print("The Minimum Value = ", q2(result.x).round(3))

    print("Other Statistics:")
    print(result)

    print(section_break)
    print(section_break)
    print('\n')
```

```
==========================================================
Test Run 1 :
==========================================================
Starting Value Used:  [[-15   4]]
The Minimum Occurs at (x, y) =  [-2.5 -1.5]
The Minimum Value =  1.375
Other Statistics:
   direc: array([[ 0.    ,  1.    ],
       [-2.75 , -1.375]])
     fun: 1.3749999999999996
 message: 'Optimization terminated successfully.'
    nfev: 89
     nit: 3
  status: 0
 success: True
       x: array([-2.5, -1.5])
==========================================================
==========================================================



==========================================================
Test Run 2 :
==========================================================
Starting Value Used:  [[-2 12]]
The Minimum Occurs at (x, y) =  [-2.5        -1.49999998]
The Minimum Value =  1.375
Other Statistics:
   direc: array([[ 0.    ,  1.    ],
       [-6.75 , -3.375]])
     fun: 1.375
 message: 'Optimization terminated successfully.'
    nfev: 100
     nit: 3
```

```
  status: 0
 success: True
       x: array([-2.5       , -1.49999998])
==========================================================
==========================================================


==========================================================
Test Run 3 :
==========================================================
Starting Value Used:  [[  5 -15]]
The Minimum Occurs at (x, y) =  [-2.49999999 -1.50000001]
The Minimum Value =  1.375
Other Statistics:
   direc: array([[0.   , 1.   ],
       [6.75 , 3.375]])
     fun: 1.375
 message: 'Optimization terminated successfully.'
    nfev: 131
     nit: 3
  status: 0
 success: True
       x: array([-2.49999999, -1.50000001])
==========================================================
==========================================================
```

4 Question 3

Let

$$f(x, y) = 7 + 5x + 6y + 4xy + 3(x^2 + y^2)^2$$

1. Find the Gradiant and the Hessian of this function.
2. Use any two optimization methods from `SCIPY.OPTIMIZE` to minimize this function.

4.0.1 Gradient

As before, the Gradient definition is:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \vdots \end{bmatrix}$$

First with regards to x:

$$\frac{\partial f}{\partial x}\left(7 + 5x + 6y + 4xy + 3(x^2 + y^2)^2\right) = 12x(x^2 + y^2) + 4y + 5$$

Next with regards to y:

$$\frac{\partial f}{\partial y}\left(7 + 5x + 6y + 4xy + 3(x^2 + y^2)^2\right) = 4x + 12y(x^2 + y^2) + 6$$

Thus:

$$\nabla f = \begin{bmatrix} 12x(x^2 + y^2) + 4y + 5 \\ 4x + 12y(x^2 + y^2) + 6 \end{bmatrix}$$

4.0.2 Hessian

Again, to derive the Hessian, it might help to define it.

$$\mathbf{H_f} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

So our second partials are as follows:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 36x^2 + 12y^2 \\ \frac{\partial^2 f}{\partial y \partial x} &= 24xy + 4 \\ \frac{\partial^2 f}{\partial y \partial x} &= 24xy + 4 \\ \frac{\partial^2 f}{\partial y^2} &= 12x^2 + 36y^2 \end{aligned}$$

This makes the Hessian:

$$\mathbf{H_f} = \begin{bmatrix} 36x^2 + 12y^2 & 24xy + 4 \\ 24xy + 4 & 12x^2 + 36y^2 \end{bmatrix}$$

4.0.3 Definitions

[ ]:

```
def q3(x):
    return 3 + x[0] + (1/2)*x[1] - x[0]*x[1] + (1/2)*x[0]**2 + x[1]**2

def q3_gradient(x):
    return np.array([12*x[0]*(x[0]**2 + x[1]**2) + 4*x[1] + 5,
                     4*x[0] +12*x[1]*(x[0]**2 + x[1]**2) + 6])

def q3_hessian(x):
    return np.array([[36*x[0]**2 + 12*x[1]**2,24*x[0]*x[1]+4],
                     [24*x[0]*x[1]+4,12*x[0]**2 + 36*x[1]**2 ]])

section_break = "==========================================================="
```

4.0.4 Method 1: Nelder-Mead

[ ]:
```
for x in enumerate(starting_points):
    result = minimize(q3, x[1], method = 'Nelder-Mead', tol = 1.e-7)

    print(section_break)
    print('Test Run', x[0] + 1, ':')
    print(section_break)

    print('Starting Value Used: ', x[1])
    print("The Minimum Occurs at (x, y) = ", result.x)
    print("The Minimum Value = ", q3(result.x).round(3))

    print("Other Statistics:")
    print(result)
    print(section_break)
    print(section_break)
    print('\n')
```

```
==========================================================
Test Run 1 :
==========================================================
Starting Value Used:  [[-15   4]]
The Minimum Occurs at (x, y) =  [-2.5        -1.50000002]
The Minimum Value =  1.375
Other Statistics:
 final_simplex: (array([[-2.5       , -1.50000002],
       [-2.50000009, -1.50000007],
       [-2.50000003, -1.49999994]]), array([1.375, 1.375, 1.375]))
           fun: 1.3750000000000009
       message: 'Optimization terminated successfully.'
          nfev: 145
           nit: 75
        status: 0
       success: True
             x: array([-2.5       , -1.50000002])
==========================================================
==========================================================



==========================================================
Test Run 2 :
==========================================================
Starting Value Used:  [[-2 12]]
The Minimum Occurs at (x, y) =  [-2.49999995 -1.49999996]
The Minimum Value =  1.375
Other Statistics:
 final_simplex: (array([[-2.49999995, -1.49999996],
```

```
       [-2.5       , -1.50000003],
       [-2.50000004, -1.50000002]]), array([1.375, 1.375, 1.375]))
           fun: 1.3750000000000009
       message: 'Optimization terminated successfully.'
          nfev: 121
           nit: 63
        status: 0
       success: True
             x: array([-2.49999995, -1.49999996])
==========================================================
==========================================================



==========================================================
Test Run 3 :
==========================================================
Starting Value Used:  [[  5 -15]]
The Minimum Occurs at (x, y) =  [-2.50000001 -1.49999999]
The Minimum Value =  1.375
Other Statistics:
 final_simplex: (array([[-2.50000001, -1.49999999],
       [-2.5000001 , -1.50000007],
       [-2.49999995, -1.50000005]]), array([1.375, 1.375, 1.375]))
           fun: 1.375
       message: 'Optimization terminated successfully.'
          nfev: 139
           nit: 71
        status: 0
       success: True
             x: array([-2.50000001, -1.49999999])
==========================================================
==========================================================
```

4.0.5 Method 2: Powell

[ ]:
```
for x in enumerate(starting_points):
    result = minimize(q3, x[1], method = 'Powell',tol = 1.e-7)

    print(section_break)
    print('Test Run', x[0] + 1, ':')
    print(section_break)

    print('Starting Value Used: ', x[1])
    print("The Minimum Occurs at (x, y) = ", result.x)
    print("The Minimum Value = ", q3(result.x).round(3))
```

```python
    print("Other Statistics:")
    print(result)

    print(section_break)
    print(section_break)
    print('\n')
```

```
==========================================================
Test Run 1 :
==========================================================
Starting Value Used:  [[-15   4]]
The Minimum Occurs at (x, y) =  [-2.5 -1.5]
The Minimum Value =  1.375
Other Statistics:
   direc: array([[ 0.   ,  1.   ],
       [-2.75 , -1.375]])
     fun: 1.3749999999999996
 message: 'Optimization terminated successfully.'
    nfev: 89
     nit: 3
  status: 0
 success: True
       x: array([-2.5, -1.5])
==========================================================
==========================================================



==========================================================
Test Run 2 :
==========================================================
Starting Value Used:  [[-2 12]]
The Minimum Occurs at (x, y) =  [-2.5        -1.49999998]
The Minimum Value =  1.375
Other Statistics:
   direc: array([[ 0.   ,  1.   ],
       [-6.75 , -3.375]])
     fun: 1.375
 message: 'Optimization terminated successfully.'
    nfev: 100
     nit: 3
  status: 0
 success: True
       x: array([-2.5       , -1.49999998])
==========================================================
==========================================================
```

```
===========================================================
Test Run 3 :
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Starting Value Used:  [[  5 -15]]
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       [6.75 , 3.375]])
     fun: 1.375
 message: 'Optimization terminated successfully.'
    nfev: 131
     nit: 3
  status: 0
 success: True
       x: array([-2.49999999, -1.50000001])
===========================================================
===========================================================
```
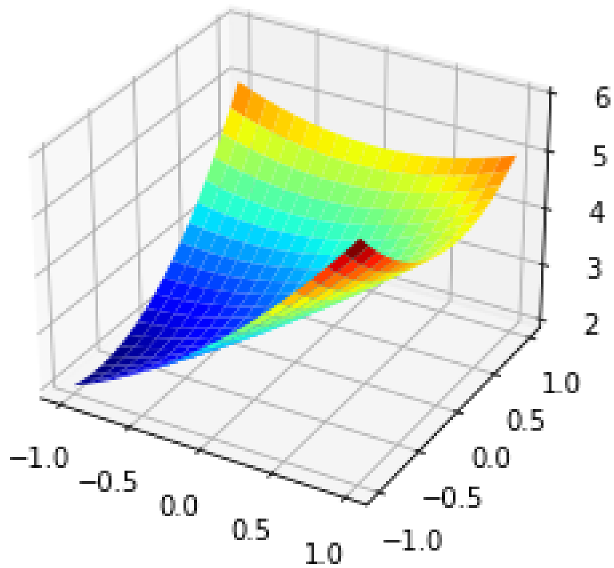

[ ]:

```

```

[ ]:

```python
from numpy import linspace, meshgrid
import matplotlib.pyplot as plt
from matplotlib import cm

fig = plt.figure()
ax = fig.gca(projection='3d')

x = linspace( -1, 1, 20)
y = linspace( -1, 1, 20)

x, y = meshgrid(x,y)
z= 3 + x + (1/2)*y - x*y + (1/2)*x**2 + y**2

surf = ax.plot_surface(x, y, z,
       rstride=1, cstride=1, cmap=cm.jet, linewidth=0)
plt.show()
```

```
/var/folders/90/b16ybj8s2gv__lhxf5ttmvj40000gn/T/ipykernel_34318/1248200888.py:6
: MatplotlibDeprecationWarning: Calling gca() with keyword arguments was
deprecated in Matplotlib 3.4. Starting two minor releases later, gca() will take
no keyword arguments. The gca() function should only be used to get the current
axes, or if no axes exist, create new axes with default keyword arguments. To
create a new axes with non-default arguments, use plt.axes() or plt.subplot().
  ax = fig.gca(projection='3d')
```


6
5
4
3
2
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0