Weak normalization of the simply typed λ-calculus

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Definition 1 (Simply typed λ calculus, à la Church)**.** The set of types is given by:

$$A ::= \alpha \mid A \to A$$

Typing rules are given by:

$$\frac{}{\vdash x^A : A}\,\texttt{ax} \qquad \frac{\vdash t : B}{\vdash \lambda x^A.\, t : A \to B}\,{\to}\texttt{I} \qquad \frac{\vdash t : A \to B \qquad \vdash s : A}{\vdash t\, s : B}\,{\to}\texttt{E}$$

Note that typable terms have unique type. Sometimes we write t^A to emphasize that t is a term of type A.

Definition 2 (Operations with multisets)**.** The letters $M, N, \ldots$ stand for multisets of non-negative integers. We write $M \uplus N$ for the *additive* union of multisets, *e.g.* $\{1,2,2\} \uplus \{2,3,3\} = \{1,2,2,2,3,3\}$. Given a multiset of non-negative integers M and a non-negative integer n, we write $M < n$ if $m < n$ for every $m \in M$. The binary relation $\succ^1$ between multisets of non-negative integers is defined as follows:

$$M \uplus \{n\} \succ^1 N \uplus \{m_1, \ldots, m_k\} \qquad \text{holds for every } n, k, m_1, \ldots, m_k \text{ such that } n > m_1, \ldots, m_k$$

The *multiset ordering* $M \succ N$ is defined as the transitive closure of $\succ^1$.

Theorem 3 (Dershowitz–Manna)**.** *The multiset ordering is well-founded.*

Definition 4. The *degree* $\delta(A)$ of a type A is its height, seen as a tree, that is:

$$\begin{array}{rcl} \delta(\alpha) & \stackrel{\text{def}}{=} & 0 \\ \delta(A \to B) & \stackrel{\text{def}}{=} & 1 + \max\{\delta(A), \delta(B)\} \end{array}$$

The *measure* $\#(t)$ of a term is the multiset of degrees of its redexes, that is:

$$\begin{array}{rcl} \#(x^A) & \stackrel{\text{def}}{=} & \varnothing \\ \#(\lambda x^A.\, t) & \stackrel{\text{def}}{=} & \#(t) \\ \#(t\, s) & \stackrel{\text{def}}{=} & \#(t) \uplus \#(s) \uplus \begin{cases} \{\delta(A \to B)\} & \text{if } t \text{ is of the form } t = \lambda x^A.\, t^B \\ \varnothing & \text{otherwise} \end{cases} \end{array}$$

Lemma 5. *Let* $\vdash t : A$ *and* $\vdash s : B$. *Suppose that* $n \geq 0$ *is such that* $\#(t) < n$ *and* $\#(s) < n$. *Then* $\#(t\{x^B := s\}) < \max\{n, 1 + \delta(B)\}$.

Proof. By induction on t:

1. Variable, $t = x^B$, with $A = B$: Then $\#(t\{x^B := s\}) = \#(s) < n \leq \max\{n, 1 + \delta(B)\}$.
2. Variable, $t = y^A \neq x$: Then $\#(t\{x^B := s\}) = \#(y) = \varnothing < n \leq \max\{n, 1 + \delta(B)\}$.
3. Abstraction, $t = \lambda y^C.\, u^D$, with $A = (C \to D)$: Note that $\#(u) = \#(\lambda y^C.\, u) < n$ by hypothesis, so $\#(t\{x^B := s\}) = \#(\lambda y^C.\, u\{x^B := s\}) = \#(u\{x^B := s\}) < \max\{n, 1 + \delta(B)\}$ by IH.
4. Application, $t = t_1^{C \to A}\, t_2^C$: Note that $\#(t_1) \subseteq \#(t) < n$ and similarly $\#(t_2) \subseteq \#(t) < n$ by hypothesis, so $\#(t_1) < n$ and $\#(t_2) < n$. By IH:

   $$\#(t_1\{x^B := s\}) < \max\{n, 1 + \delta(B)\} \qquad \text{and} \qquad \#(t_2\{x^B := s\}) < \max\{n, 1 + \delta(B)\}$$

   We consider two subcases, depending on whether $t_1\{x^B := s\}$ is an abstraction or not:

4.1 If $t_1\{x^B := s\}$ is not an abstraction, it is immediate to conclude, given that:

$$\#(t\{x^B := s\}) = \#(t_1\{x^B := s\}) \uplus \#(t_2\{x^B := s\}) < \max\{n, 1 + \delta(B)\}$$

4.2 If $t_1\{x^B := s\}$ is an abstraction, *i.e.* of the form $t_1\{x^B := s\} = \lambda y^C. p^A$: then there are two possibilities, either t_1 is an abstraction, or $t_1 = x$ and s is an abstraction:

4.2.1 If $t_1 = \lambda y^C. r^A$, then $\{\delta(C \to A)\} \subseteq \#(t_1\, t_2) = \#(t) < n$ by hypothesis. Hence:

$$\#(t\{x^B := s\}) = \#(t_1\{x^B := s\}) \uplus \#(t_2\{x^B := s\}) \uplus \{\delta(C \to A)\} < \max\{n, 1 + \delta(B)\}$$

4.2.2 If $t_1 = x$ and $s = \lambda y^C. p^A$, where $B = (C \to A)$, then $\delta(C \to A) = \delta(B) < 1 + \delta(B)$. Hence:

$$\#(t\{x^B := s\}) = \#(t_1\{x^B := s\}) \uplus \#(t_2\{x^B := s\}) \uplus \{\delta(C \to A)\} < \max\{n, 1 + \delta(B)\}$$

□

Lemma 6. *Let t be a typable term, and let $t \to t'$ be a β-step that results from contracting the rightmost redex of maximum degree in t. Then $\#(t) > \#(s)$.*

Proof. By induction on t.

1. Variable, $t = x$: Vacuously true.
2. Abstraction, $t = \lambda x. s$: Immediate by IH.
3. Application, $t = t_1\, t_2$. We consider three subcases, depending on whether the step $t \to t'$ is at the root, internal to t_1, or internal to t_2:

3.1 Reduction at the root: then $t_1 = \lambda x^A. s^B$ and the step is of the form:

$$t = (\lambda x^A. s)\, t_2 \to s\{x^A := t_2\} = t'$$

Note that:

$$\#(t) = \#(s) \uplus \#(t_2) \uplus \{\delta(A \to B)\}$$

Note that the degree of the contracted redex is $\delta(A \to B)$ and, since it is the rightmost redex of maximum degree, $\#(s) < \delta(A \to B)$ and $\#(t_2) < \delta(A \to B)$. Hence by Lem. 5

$$\#(t') = \#(s\{x^A := t_2\}) < \max\{\delta(A \to B), 1 + \delta(A)\} = \delta(A \to B)$$

Therefore $\#(t) \succ \#(t')$.

3.2 Reduction internal to t_1: then the step is of the form:

$$t = t_1\, t_2 \to t_1'\, t_2 = t'$$

where $t_1 \to t_1'$ again results from contracting the rightmost redex of maximum degree in t_1. By IH we have that $\#(t_1) \succ \#(t_1')$. We consider two subcases, depending on whether t_1 is an abstraction or not:

3.2.1 If t_1 is an abstraction, *i.e.* $t_1 = \lambda x^A. p^B$. Then t_1' is also an abstraction and, by subject reduction, it must be of the form $t_1' = \lambda x^A. q^B$. Hence:

$$\#(t) = \#(t_1) \cup \#(t_2) \cup \{\delta(A \to B)\} \succ \#(t_1') \cup \#(t_2) \cup \{\delta(A \to B)\} = \#(t')$$

3.2.2 If t_1 is not an abstraction, we consider two further subcases, depending on whether t_1' is an abstraction or not:

3.2.2.1 If t_1' is an abstraction, *i.e.* $t_1' = \lambda x^A. q^B$, then note that, since t_1 is not an abstraction, it must be an application, and the step $t_1 \to t_1'$ must contract a redex at the root. This means that t_1 is of the form $t_1 = (\lambda y^C. p^{A \to B})\, r$, with $t_1' = p\{y^C := r\} = \lambda x^A. q^B$. Hence there are two possibilities, either p is an abstraction or $p = y^C$ and r is an abstraction.

- If p is an abstraction, then $p = \lambda z^A.\, q'^B$, so the step $t \to t'$ is of the form:

$$(\lambda y^C.\, \lambda z^A.\, q')\, r\, t_2 \to (\lambda z^A.\, q'\{y^C := r\})\, t_2$$

  Note that the degree of the contracted redex is $\delta(C \to (A \to B))$ and, since it is the rightmost redex of maximum degree, $\#(q') < \delta(C \to (A \to B))$ and $\#(r) < \delta(C \to (A \to B))$. Hence by Lem. 5

$$\#(q'\{y^C := r\}) < \max\{\delta(C \to (A \to B)), 1 + \delta(C)\} = \delta(C \to (A \to B))$$

  Therefore:

$$\begin{array}{rcl} \#(t) & = & \#(q') \uplus \#(r) \uplus \#(t_2) \uplus \{\delta(C \to (A \to B))\} \\ & \succ & \#(q'\{y^C := r\}) \uplus \#(t_2) \uplus \{\delta(A \to B)\} \\ & = & \#(t') \end{array}$$

- If $p = y^C$ and $r = \lambda x^A.\, q^B$ then $C = (A \to B)$ and the step $t \to t'$ is of the form:

$$t = (\lambda y^{A \to B}.\, y^{A \to B})\, (\lambda x^A.\, q^B)\, t_2 \to (\lambda x^A.\, q^B)\, t_2 = t'$$

  Then:

$$\#(t) = \#(q^B) \uplus \#(t_2) \uplus \{\delta((A \to B) \to A \to B)\} \succ \#(q^B) \uplus \#(t_2) \uplus \{\delta(A \to B)\} = \#(t')$$

3.2.2.2 If t'_1 is not an abstraction, then it is immediate to conclude, as:

$$\#(t) = \#(t_1) \cup \#(t_2) \succ \#(t'_1) \cup \#(t_2) = \#(t')$$

3.3 Reduction internal to t_2: then the step is of the form:

$$t = t_1\, t_2 \to t_1\, t'_2 = t'$$

where $t_2 \to t'_2$ again results from contracting the rightmost redex of maximum degree in t_2. By IH we have that $\#(t_2) \succ \#(t'_2)$. From this we conclude that $\#(t_1\, t_2) \succ \#(t_1\, t'_2)$, as required.

□

Theorem 7. *The simply typed λ-calculus is weakly normalizing.*

Proof. An easy corollary of Thm. 3 and Lem. 6. □