Basic propositional logic

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In this notes we review the proofs of soundness and completeness for a Hilbert-style proof system for propositional logic. The proof of completeness relies on compactness.

1 Syntax

Definition 1 (Formulas). Let Var be a denumerable set of *propositional variables* $p_1, p_2, p_3, \ldots$. The set Form of formulas is given by the grammar:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A$$

We write $f\mathsf{v}(A)$ for the free variables of A. A context Γ is a set of formulas.

2 Semantics and compactness

Definition 2 (Valuations). A valuation is a function $V : \mathsf{Var} \to \mathbf{2}$. Valuations are extended to formulas $-^V : \mathsf{Var} \to \mathsf{Form}$ as follows:

$$
\begin{aligned}
p^V &\overset{\text{def}}{=} V(p) \\
\top^V &\overset{\text{def}}{=} 1 \\
\bot^V &\overset{\text{def}}{=} 0 \\
(\neg A)^V &\overset{\text{def}}{=} 1 - A^V \\
(A \wedge B)^V &\overset{\text{def}}{=} A^V \cdot B^V \\
(A \vee B)^V &\overset{\text{def}}{=} A^V \sqcup B^V \\
(A \to B)^V &\overset{\text{def}}{=} (1 - A^V) \sqcup B^V
\end{aligned}
$$

where $\cdot$ denotes the product and $\sqcup$ denotes the maximum.

Definition 3 (Logical consequence). We write "$\vDash$" for various notions of *logical consequence*:

- $V \vDash A$ holds if $A^V = 1$.

- $V \vDash \Gamma$ holds if $V \vDash A$ for every $A \in \Gamma$.

- $\vDash \Gamma$ holds if $V \vDash \Gamma$ for every valuation $V : \mathsf{Form} \to \mathbf{2}$.

- $\Gamma \vDash A$ holds if $V \vDash \Gamma$ implies $V \vDash A$ for every valuation $V : \mathsf{Form} \to \mathbf{2}$.

We say that Γ is *valid* or a *tautology* if $V \vDash \Gamma$ for every valuation $V : \mathsf{Form} \to \mathbf{2}$. We say that Γ is *satisfiable* if $V \vDash \Gamma$ for some valuation $V : \mathsf{Form} \to \mathbf{2}$, and *unsatisfiable* otherwise. Two formulas A, B are *logically equivalent* if for every valuation $V : \mathsf{Form} \to \mathbf{2}$ $V \vDash A$ holds if and only if $V \vDash B$ holds. Logical equivalence is defined similarly for contexts.

Remark 4. $\Gamma \vDash A$ holds if and only if $\Gamma, \neg A$ is unsatisfiable.

Theorem 5 (Compactness). *The following are equivalent:*

1. *Γ is satisfiable.*

2. *Γ' is satisfiable for every finite subset $\Gamma' \subseteq \Gamma$*

Proof. $(1 \implies 2)$ Immediate. $(2 \implies 1)$ For each $n \in \mathbb{N}$, let Γ_n be the set of formulas in Γ that use at most the first n propositional variables, *i.e.*

$$\Gamma_n \overset{\text{def}}{=} \{ A \in \Gamma \mid \mathsf{fv}(A) \subseteq \{p_1, \dots, p_n\} \}$$

The set Γ_n may be infinite, but up to logical equivalence it contains at most 2^{2^n} formulas. So for each n there is a finite set $\Gamma'_n \subseteq \Gamma_n$ logically equivalent to Γ_n. Since Γ'_n is satisfiable, there is a valuation V_n such that $V_n \vDash \Gamma_n$ for all $n \in \mathbb{N}$. Note moreover that if $i \geq n$ then $V_i \vDash \Gamma_n$.

Now for each $n \in \mathbb{N}_0$ we inductively define a set $I_n \subseteq \mathbb{N}$ in such a way that

- I_n is infinite;

- if $i, j \in I_n$ then V_i and V_j coincide on the first n propositional variables; and

- $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$.

To do so, set $I_0 \overset{\text{def}}{=} \mathbb{N}$ and

$$I_{n+1} \overset{\text{def}}{=} \begin{cases} E_n^1 & \text{if } E_n^1 \text{ is infinite} \\ E_n^0 & \text{otherwise} \end{cases}$$

where $E_n^b = \{ i \in I_n \mid i \geq n+1, \ V_i(p_{n+1}) = b \}$.

Now we define a valuation $V : \mathsf{Var} \to \mathbf{2}$ as follows:

$$V(p_n) \overset{\text{def}}{=} V_i(p_n) \quad \text{for any } i \in I_n$$

noting that if $i \geq n$ then $V_i(p_n) = V(p_n)$. To conclude, we claim that $V \vDash \Gamma$. Indeed, let $A \in \Gamma$. Then $A \in \Gamma_n$ for some $n \in \mathbb{N}$. In particular, take $i \in I_n$. Then $V_i \vDash \Gamma_n$ because $i \geq n$, so also $V_i \vDash A$. Moreover $A^{V_i} = A^V$ because A uses only the first n variables, so $V \vDash A$ as required. $\qquad\square$

Corollary 6. *If $\Gamma \vDash A$ then there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vDash A$.*

Proof. By the previous remark, $\Gamma \vDash A$ holds if and only if the set $\Gamma, \neg A$ is unsatisfiable. So if $\Gamma \vDash A$ by Compactness there exists a finite unsatisfiable subset $\Gamma_0 \subseteq \Gamma, \neg A$. Without loss of generality we may assume that $\neg A \in \Gamma_0$ so $\Gamma_0 = (\Gamma', \neg A)$ is unsatisfiable. This in turn means that $\Gamma' \vDash A$. $\qquad\square$

3 Hilbert-style proof system

Definition 7 (Propositional proof system). The only deduction rule is *modus ponens*:

$$\frac{\vdash P : A \to B \qquad \vdash Q : A}{\vdash (P \cdot Q) : B}$$

Plus the following axioms, instantiated on arbitrary formulas A, B, …:

Implication
$$K \quad : \quad A \to (B \to A)$$
$$S \quad : \quad (A \to (B \to C)) \to (A \to B) \to (A \to C)$$

Truth and falsity
$$\text{trivial} \quad : \quad \top$$
$$\text{abort} \quad : \quad \bot \to A$$

Negation
$$\text{negi} \quad : \quad (A \to B) \to ((A \to \neg B) \to \neg A)$$
$$\text{nege} \quad : \quad A \to (\neg A \to B) \qquad \qquad \textbf{NB.} \text{ Redundant if dneg is allowed.}$$
$$\text{dneg} \quad : \quad \neg\neg A \to A \qquad \qquad \qquad \textbf{NB.} \text{ Classical.}$$

Conjunction
$$\text{pair} \quad : \quad A \to (B \to (A \wedge B))$$
$$\pi_1 \quad : \quad (A \wedge B) \to A$$
$$\pi_2 \quad : \quad (A \wedge B) \to B$$

Disjunction
$$\text{in}_1 \quad : \quad A \to (A \vee B)$$
$$\text{in}_2 \quad : \quad B \to (A \vee B)$$
$$\text{match} \quad : \quad (A \to C) \to ((B \to C) \to ((A \vee B) \to C))$$

We write $\Gamma \vdash A$ if A can be proved using *modus ponens*, the axioms, and hypotheses from Γ. We assume that $\to$ is right-associative and $\cdot$ is left-associative. Application $P \cdot Q$ is written $P\,Q$.

Lemma 8 (Identity). $\vdash A \to A$ *holds for any formula* A.

Proof. It suffices to take id $\overset{\text{def}}{=} S\,K\,K$:

$$\frac{\dfrac{\vdash S : (A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A \qquad \vdash K : A \to (A \to A) \to A}{\vdash SK : (A \to A \to A) \to A \to A} \qquad \dfrac{}{\vdash K : A \to A \to A}}{\vdash SKK : A \to A}$$

$\square$

Lemma 9 (Weakening)**.** *If $\Gamma \vdash A$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash A$.*

Proof. Straightforward by induction on the derivation of $\Gamma \vdash A$. □

Theorem 10 (Deduction theorem)**.** $\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \rightarrow B$.

Proof. The ($\Leftarrow$) direction is immediate since by Weakening (Lem. 9) we have that $\Gamma, A \vdash A \rightarrow B$ so:

$$\frac{\Gamma, A \vdash A \rightarrow B \quad \Gamma, A \vdash A}{\Gamma, A \vdash B}$$

For the ($\Rightarrow$) direction we proceed by induction on the derivation of $\Gamma, A \vdash B$. There are three cases, either (1) we use an axiom or an assumption from Γ, (2) we use A, or (3) we use the *modus ponens* rule:

1. **Axiom or fixed assumption.** That is, the derivation of $\Gamma, A \vdash B$ is of the form:

$$\frac{}{\Gamma, A \vdash B}$$

   where B is one of the axioms (K, S, trivial, etc.) or $B \in \Gamma$. Then B can also be proved under Γ so we have:

$$\frac{\Gamma \vdash K : B \rightarrow A \rightarrow B \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B}$$

2. **Identity.** That is, the derivation of $\Gamma, A \vdash B$ is of the form:

$$\frac{}{\Gamma, A \vdash A}$$

   Then by the Identity lemma (Lem. 8) we have:

$$\frac{}{\Gamma \vdash A \rightarrow A}$$

3. **Modus ponens.** That is, the derivation of $\Gamma, A \vdash B$ is of the form:

$$\frac{\Gamma, A \vdash C \rightarrow B \quad \Gamma, A \vdash C}{\Gamma, A \vdash B}$$

   So by *i.h.* on each of the premises we have:

$$\frac{\dfrac{\Gamma \vdash S : (A \rightarrow (C \rightarrow B)) \rightarrow (A \rightarrow C) \rightarrow (A \rightarrow B) \quad \overset{i.h.}{\Gamma \vdash A \rightarrow (C \rightarrow B)}}{\Gamma \vdash (A \rightarrow C) \rightarrow (A \rightarrow B)} \quad \overset{i.h.}{\Gamma \vdash A \rightarrow C}}{\Gamma \vdash A \rightarrow B}$$

□

3.1 Basic facts

Following, we state and prove some basic principles.

1. **Contrapositive ($\Gamma, A \to B \vdash \neg B \to \neg A$).**
   By the deduction theorem, it suffices to show that $\Gamma, x : A \to B, y : \neg B \vdash \neg A$. Indeed:

$$\Gamma, x : A \to B, y : \neg B \vdash \mathsf{negi}\, x\, (\underbrace{K\, y}_{A \to \neg B}) : \neg A$$

2. **Explosion principle ($\Gamma, A, \neg A \vdash B$).** This is an immediate consequence of the deduction theorem and axiom nege. Moreover, if dneg is allowed, nege is redundant:

$$\Gamma, x : A, y : \neg A \vdash \mathsf{dneg}\, (\mathsf{negi}\, (\underbrace{\underbrace{K\, x}_{\neg B \to A}}\, \underbrace{(K\, y)}_{\neg B \to \neg A})) : B$$

$$\underbrace{\phantom{(\mathsf{negi}\, (K\, x)(K\, y))}}_{\neg\neg B}$$

3. **False antecedent ($\Gamma, \neg A \vdash A \to B$).** Immediate from the explosion principle and the deduction theorem.

4. **True consequent ($\Gamma, B \vdash A \to B$).** Immediate from the axiom $K : B \to A \to B$ and the deduction theorem.

5. **Disproving an implication ($\Gamma, A, \neg B \vdash \neg(A \to B)$).**
   By the true consequent property we have that $\Gamma, A, \neg B, A \to B \vdash B$ so by the deduction theorem $\Gamma, A, \neg B \vdash P : (A \to B) \to B$. Then:

$$\Gamma, x : A, y : \neg B \vdash \mathsf{negi}\, \underbrace{P}_{(A \to B) \to B}\, (\underbrace{K\, y}_{(A \to B) \to \neg B}) : \neg(A \to B)$$

6. **Conjunction introduction ($\Gamma, A, B \vdash A \wedge B$).** By the deduction theorem and the axiom pair.

7. **Negated conjunction introduction ($\Gamma, \neg A \vdash \neg(A \wedge B)$ and $\Gamma, \neg B \vdash \neg(A \wedge B)$).**

$$\Gamma, x : \neg A \vdash \mathsf{negi}\, \underbrace{\pi_1}_{(A \wedge B) \to A}\, (\underbrace{K\, x}_{(A \wedge B) \to \neg A}) : \neg(A \wedge B)$$

   Similarly:

$$\Gamma, y : \neg B \vdash \mathsf{negi}\, \underbrace{\pi_2}_{(A \wedge B) \to B}\, (\underbrace{K\, y}_{(A \wedge B) \to \neg B}) : \neg(A \wedge B)$$

8. **Disjunction introduction ($\Gamma, A \vdash A \vee B$ and $\Gamma, B \vdash A \vee B$).** By the deduction theorem and axioms $\mathsf{in}_1/\mathsf{in}_2$ respectively.

9. **Negated disjunction introduction ($\Gamma, \neg A, \neg B \vdash \neg(A \vee B)$).** Recall that $\vdash$ nege $: A \to \neg A \to B$, so by the deducion theorem, we may construct $\vdash$ nege$' : \neg A \to A \to B$ for arbitrary formulas A, B.

   Now let C be any provable formula, *e.g.* $C = \top$ or $C = (z \to z)$, and let $\vdash P : C$. Then:

$$\Gamma, x : \neg A, y : \neg B \vdash \text{negi} \, (\underbrace{K \, P}_{(A \vee B) \to C}) \, (\text{match} \, \underbrace{(\text{nege}' x)}_{A \to C} \, \underbrace{(\text{nege}' y)}_{B \to C})$$

$$\underbrace{\phantom{(\text{match} (\text{nege}' x) (\text{nege}' y))}}_{(A \vee B) \to \neg C}$$

10. **Double negation introduction ($\Gamma, A \vdash \neg\neg A$).**

$$\Gamma, x : A \vdash \text{negi} \, (\underbrace{K \, x}_{\neg A \to A}) \, \underbrace{\text{id}}_{\neg A \to \neg A} \, : \neg\neg A$$

11. **Proof by cases (if $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$ then $\Gamma \vdash B$).** By the deduction theorem and **Contrapositive** we have that $\Gamma \vdash P : \neg B \to \neg A$ and $\Gamma \vdash Q : \neg B \to \neg\neg A$.

$$\Gamma \vdash \text{dneg} \, (\underbrace{\text{negi} \, P \, Q}_{\neg\neg B}) : B$$

4   Soundness and completeness

Proposition 11 (Soundness). *If $\Gamma \vdash A$ then $\Gamma \vDash A$.*

Proof. By induction on the derivation of $\Gamma \vdash A$. There are three cases:

1. **Axiom.** It is routine to check that all the axioms are valid. For example, for the axiom K, let $V :$ Var $\to \mathbf{2}$ be any valuation. Then $(A \to (B \to A))^V = (1 - A^V) \sqcup (1 - B)^V \sqcup A^V = 1$.

2. **Assumption.** Let $V :$ Var $\to \mathbf{2}$ be any valuation such that $V \vDash \Gamma$. In particular $V \vDash A$ because $A \in \Gamma$ so we are done.

3. **Modus ponens.** The proof is of the form:

$$\frac{\Gamma \vdash B \to A \quad \Gamma \vdash B}{\Gamma \vdash A}$$

   By *i.h.* we have that $\Gamma \vDash B \to A$ and $\Gamma \vDash B$. Let $V :$ Var $\to \mathbf{2}$ be any valuation such that $V \vDash \Gamma$. Then $1 = (B \to A)^V = (1 - B^V) \sqcup A^V$ and $1 = B^V$, so we must have $A^V = 1$.

$$\square$$

Lemma 12. *Let A be a formula depending (at most) on the first n propositional variables $\{p_1, \ldots, p_n\}$. Let Γ be a context consisting of n formulas, the i-th of which is either p_i or $\neg p_i$. Then either $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.*

Proof. By induction on A.

1. **Variable, $A = p_i$.** Then either $\Gamma \vdash p_i$ or $\Gamma \vdash \neg p_i$.

2. **Truth.** Γ : trivial : $\top$.

3. **Falsity.** Γ : negi $\underbrace{\text{abort}}_{\bot \to \top}$ $\underbrace{\text{abort}}_{\bot \to \neg\top}$: $\neg\bot$.

4. **Negation,** $A = \neg B$**.** By *i.h.* there are two cases:

   4.1 If $\Gamma \vdash B$, by **Double negation introduction** $\Gamma \vdash \neg\neg B = \neg A$.

   4.2 If $\Gamma \vdash P$: $\neg B = A$ it is immediate.

5. **Conjunction,** $A = B \wedge C$**.** By *i.h.* on B there are two cases:

   5.1 If $\Gamma \vdash B$, then by *i.h.* on C there are two subcases:

        5.1.1 If $\Gamma \vdash C$, then by axiom pair we have $\Gamma \vdash B \wedge C = A$

        5.1.2 If $\Gamma \vdash \neg C$, then by **Negated conjunction introduction** $\Gamma \vdash \neg(B \wedge C) = \neg A$.

   5.2 If $\Gamma \vdash \neg B$, then by **Negated conjunction introduction** $\Gamma \vdash \neg(B \wedge C) = \neg A$.

6. **Disjunction,** $A = B \vee C$**.** By *i.h.* on B there are two cases:

   6.1 If $\Gamma \vdash B$, then by axiom in_1 we have $\Gamma \vdash B \wedge C = A$.

   6.2 If $\Gamma \vdash \neg B$, then by *i.h.* on C there are two subcases:

        6.2.1 If $\Gamma \vdash C$, then by axiom in_2 we have $\Gamma \vdash B \vee C = A$

        6.2.2 If $\Gamma \vdash \neg C$, then by **Negated disjunction introduction** $\Gamma \vdash \neg(B \vee C) = \neg A$.

7. **Implication,** $A = B \to C$**.** By *i.h.* on B there are two cases:

   7.1 If $\Gamma \vdash B$, then by *i.h.* on C there are two cases:

        7.1.1 If $\Gamma \vdash C$, then by **True consequent** we have $\Gamma \vdash B \to C = A$.

        7.1.2 If $\Gamma \vdash \neg C$, then by **Disproving an implication** we have $\Gamma \vdash \neg(B \to C) = A$.

   7.2 If $\Gamma \vdash \neg B$, then by **False antecedent** we have $\Gamma \vdash B \to C = A$.

$\square$

The previous lemma can be generalized when the context consists of $k \leq n$ formulas.

Lemma 13. *Let A be a formula depending (at most) on the first n propositional variables $\{p_1, \dots, p_n\}$. Let Γ be a context consisting of $k \leq n$ formulas, the i-th of which is either p_i or $\neg p_i$. Moreover, suppose that $\vDash A$. Then either $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.*

Proof. By induction on k downwards from n to 0 (*i.e.* on $n - k$).

1. **Base case,** $k = n$**.** This is precisely the previous lemma (Lem. 12).

2. **Induction, "$k + 1 \implies k$".** Consider the context Γ extended with p_{k+1}. By *i.h.* there are two possibilities, $\Gamma, p_{k+1} \vdash A$ or $\Gamma, p_{k+1} \vdash \neg A$. The second case is not possible since, by Soundness (Prop. 11), we would have that $\Gamma, p_{k+1} \vDash \neg A$, *i.e.* there is a valuation V such that $V \vDash \Gamma, p_{k+1}$ and $A^V = 0$. But note that $\vDash A$ by hypothesis, so $A^V = 1$, a contradiction. Hence $\Gamma, p_{k+1} \vdash A$.

   Similarly, if we consider the context Γ extended with $\neg p_{k+1}$ we obtain by *i.h.* that $\Gamma, \neg p_{k+1} \vdash A$.

   Finally, note that $\Gamma, p_{k+1} \vdash A$ and $\Gamma, \neg p_{k+1} \vdash A$ together entail $\Gamma \vdash A$ using the principle of **Proof by cases**.

$\square$

Theorem 14 (Completeness)**.** *The following hold:*

1. **Completeness.** *If $\vDash A$ then $\vdash A$.*

2. **Implicational completeness.** *If $\Gamma \vDash A$ then $\Gamma \vdash A$.*

Proof. For item 1., by the previous lemma (Lem. 13) in the particular case in which $k = 0$, we have that either $\vdash A$ or $\vdash \neg A$. The second case is not possible since, by Soundness (Prop. 11), we would have that $\vDash \neg A$, *i.e.* that there is a valuation V such that $A^V = 0$. But note that $\vDash A$ by hypothesis, so $A^V = 1$, a contradiction. Hence $\vdash A$.

For item 2., suppose that $\Gamma \vDash A$. By the corollary of compactness (Coro. 6) there exists a finite subset $\{B_1, \ldots, B_n\} \subseteq \Gamma$ such that $B_1, \ldots, B_n \vDash A$. This in turn means that $\vDash B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A$. By item 1. of this theorem, we have $\vdash B_1 \rightarrow \ldots \rightarrow B_n \rightarrow A$. By the deduction theorem (Thm. 10) we have $\Gamma' \vdash A$. Finally, by weakening (Lem. 9), $\Gamma \vdash A$. $\square$