Classical Numerical Analysis, Chapter 01

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Chapter 01
Linear Operators and Matrices

Linear Systems

Why is linear algebra so important to numerical analysis? That is a fair question. The answer is that many algorithms in numerical analysis — for a broad range of problem types, interpolation, approximation of functions, approximating solutions to differential or integral equations — require, at some stage in the algorithm, the investigation of a system of linear equations:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = f_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = f_2, \\ \qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = f_m. \end{cases}$$

Many algorithms will require the solution of such systems. Others, by contrast, may need some or all of the eigenvalues or singular values of the associated coefficient matrix for the system.

Linear Operators and Matrices

Vector Spaces of Linear Operators

Definition (linear operator)

Let $\mathbb{V}$ and $\mathbb{W}$ be complex vector spaces. The mapping $A : \mathbb{V} \to \mathbb{W}$ is called a **linear operator** iff

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay, \quad \forall\, \alpha, \beta \in \mathbb{C}, \ \forall\, x, y \in \mathbb{V}.$$

The set of all linear operators from $\mathbb{V}$ to $\mathbb{W}$ is denoted by $\mathfrak{L}(\mathbb{V}, \mathbb{W})$ For simplicity, we denote by $\mathfrak{L}(\mathbb{V})$ the set of linear operators from $\mathbb{V}$ to itself. Suppose that $A, B \in \mathfrak{L}(\mathbb{V}, \mathbb{W})$ and $\alpha, \beta \in \mathbb{C}$ are arbitrary. We define, in a natural way, the object $\alpha A + \beta B$ via

$$(\alpha A + \beta B)x = \alpha Ax + \beta Bx, \quad \forall\, x \in \mathbb{V}.$$

Proposition (properties of $\mathfrak{L}(\mathbb{V}, \mathbb{W})$)

Let $\mathbb{V}$ and $\mathbb{W}$ be complex vector spaces. The set $\mathfrak{L}(\mathbb{V}, \mathbb{W})$ is a vector space using the natural definitions of addition and scalar multiplication given in the last definition. If $\dim(\mathbb{V}) = m$ and $\dim(\mathbb{W}) = n$, then $\dim(\mathfrak{L}(\mathbb{V}, \mathbb{W})) = mn$.

Matrices

Definition ($m \times n$ matrices)

Let $\mathbb{K}$ be a field. We define for any $m, n \in \mathbb{N}$,

$$\mathbb{K}^{m\times n} = \{\mathsf{A} = [a_{i,j}] \mid a_{i,j} \in \mathbb{K},\ i = 1, \ldots, m,\ j = 1, \ldots, n\}.$$

The object A *is called a* **matrix** *and the elements $a_{i,j} \in \mathbb{K}$ are called its* **components** *or* **entries**. *We call $\mathbb{C}^{m\times n}$ the set of* **complex** $m \times n$ **matrices**, *and $\mathbb{R}^{m\times n}$ the set of* **real** $m \times n$ **matrices**.

To extract the entry in the i–th row and j–th column of the $m \times n$ matrix $\mathsf{A} \in \mathbb{K}^{m\times n}$, we use the notation

$$[\mathsf{A}]_{i,j} = a_{i,j} \in \mathbb{K}.$$

We say that there are m rows and n columns in an $m \times n$ matrix A.

Vector Space of Matrices

We naturally define $m \times n$ matrix addition and scalar multiplication component–wise via

$$[\mathsf{A} + \mathsf{B}]_{i,j} = a_{i,j} + b_{i,j}, \quad [\alpha \mathsf{A}]_{i,j} = \alpha a_{i,j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n,$$

where $\mathsf{A}, \mathsf{B} \in \mathbb{K}^{m \times n}$ are arbitrary $m \times n$ matrices and $\alpha \in \mathbb{K}$ is an arbitrary scalar.

Proposition ($\mathbb{K}^{m \times n}$ is a vector space)

With addition and scalar multiplication defined as above, $\mathbb{K}^{m \times n}$ is a vector space over $\mathbb{K}$ and $\dim(\mathbb{K}^{m \times n}) = m \cdot n$.

Matrix-Matrix and Matrix-Vector Multiplication

Definition (matrix product)

Let $\mathsf{A} = [a_{i,k}] \in \mathbb{K}^{m\times p}$ and $\mathsf{B} = [b_{k,j}] \in \mathbb{K}^{p\times n}$. The **matrix product** $\mathsf{C} = \mathsf{AB}$ is a matrix in $\mathbb{K}^{m\times n}$ whose entries are computed according to the formula

$$[\mathsf{C}]_{i,j} = c_{i,j} = \sum_{k=1}^{p} a_{i,k} b_{k,j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

Definition (matrix–vector product)

Suppose that $\mathbf{x} = [x_s] \in \mathbb{K}^n$ and $\mathsf{A} = [a_{k,s}] \in \mathbb{K}^{m\times n}$ then the **matrix–vector product** $\mathbf{y} = \mathsf{A}\mathbf{x}$ is a vector in $\mathbb{K}^m$ whose components are computed via the formula:

$$[\mathbf{y}]_k = y_k = \sum_{s=1}^{n} a_{k,s} x_s, \quad k = 1, \ldots, m.$$

Matrices as Linear Mappings

Suppose that $\mathsf{A} \in \mathbb{C}^{m\times n}$. Then the (canonical) mapping

$$\mathsf{A} : \mathbb{C}^n \to \mathbb{C}^m$$

defined by

$$\mathbf{y} = \mathsf{A}\mathbf{x} \in \mathbb{C}^m, \quad \text{where} \quad \mathbf{x} \in \mathbb{C}^n$$

is linear. We can identify $\mathfrak{L}(\mathbb{C}^n, \mathbb{C}^m)$ with the space $\mathbb{C}^{m\times n}$ of matrices having m rows and n columns of complex entries. This says that all linear mappings from $\mathbb{C}^n$ to $\mathbb{C}^m$ are, essentially, matrices. This result can be generalized to identify $\mathfrak{L}(\mathbb{K}^n, \mathbb{K}^m)$ with $\mathbb{K}^{m\times n}$ for a generic field $\mathbb{K}$.

Column Vectors and Row Vectors

It will be helpful from this point on to always view $\mathbb{C}^k$ as a vector space of column k–vectors, that is $\mathbb{C}^{k\times 1}$. When we consider $\mathbf{x} \in \mathbb{C}^k$, we think

$$\mathbf{x} = \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}.$$

Upon introducing the *transpose* operation $\cdot^\intercal : \mathbb{C}^{k\times 1} \to \mathbb{C}^{1\times k}$ as mapping column k–vectors to row k–vectors, we will often express $\mathbf{x} \in \mathbb{C}^k$ inline as $\mathbf{x} = [x_1, \ldots, x_k]^\intercal$, i.e., as the transpose of a row vector.

Expressing Matrices as a Collection of Columns/Rows

Given a matrix $\mathsf{A} \in \mathbb{C}^{m\times n}$ we commonly wish to represent it in a column–wise format (as a collection of column vectors) via

$$\mathsf{A} = \begin{bmatrix} | & & | \\ \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ | & & | \end{bmatrix}, \quad \mathbf{c}_j \in \mathbb{C}^m,\ j = 1, \ldots, n,$$

or in a row–wise format (as a collection of row vectors) via

$$\mathsf{A} = \begin{bmatrix} - & \mathbf{r}_1^{\intercal} & - \\ & \vdots & \\ - & \mathbf{r}_m^{\intercal} & - \end{bmatrix}, \quad \mathbf{r}_i \in \mathbb{C}^n,\ i = 1, \ldots, m.$$

As a further short–hand, we will often write (inline) $\mathsf{A} = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$ and $\mathsf{A} = [\mathbf{r}_1, \ldots, \mathbf{r}_m]^{\intercal}$.

Matrix-Vector Multiplication as Weighted Column Sums

Suppose that $\mathsf{A} = [\mathbf{c}_1, \ldots, \mathbf{c}_n] \in \mathbb{C}^{m\times n}$ and $\mathbf{x} \in \mathbb{C}^n$. It is important to notice that if we view the matrix A in column–wise format, then the matrix–vector product $\mathbf{y} = \mathsf{A}\mathbf{x} \in \mathbb{C}^m$ is precisely

$$\mathbf{y} = \sum_{k=1}^{n} x_k \mathbf{c}_k .$$

In other words, the column vector $\mathbf{y}$ is a linear combination of the columns of A.

Range and Kernel

Definition (range and kernel)

Let $\mathsf{A} \in \mathbb{C}^{m\times n}$. The **image** (or **range**) of A is defined as

$$\operatorname{im}(\mathsf{A}) = \mathcal{R}(\mathsf{A}) = \{\mathbf{y} \in \mathbb{C}^m \mid \exists\, \mathbf{x} \in \mathbb{C}^n,\ \mathbf{y} = \mathsf{A}\mathbf{x}\} \subseteq \mathbb{C}^m.$$

The **kernel** (or **null space**) of A is

$$\ker(\mathsf{A}) = \mathcal{N}(\mathsf{A}) = \{\mathbf{x} \in \mathbb{C}^n \mid \mathsf{A}\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{C}^n.$$

Definition (row and column space)

Suppose that the matrix $A \in \mathbb{C}^{m\times n}$ is expressed column–wise as $A = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$ and row–wise as $A = [\mathbf{r}_1, \ldots, \mathbf{r}_m]^\intercal$. The **row space** of A is

$$\mathrm{row}(A) = \mathrm{span}(\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}) \leq \mathbb{C}^n,$$

and the **column space** of A is

$$\mathrm{col}(A) = \mathrm{span}(\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}) \leq \mathbb{C}^m.$$

The **row rank** of A is the dimension of $\mathrm{row}(A)$; similarly, the **column rank** is the dimension of $\mathrm{col}(A)$.

Theorem (row and column rank)

Suppose that $A \in \mathbb{C}^{m\times n}$. The row and column ranks of A are equal.

Definition (rank)

The **rank** of a matrix $A \in \mathbb{C}^{m\times n}$ is the dimension of its row/column space. We denote it by the symbol $\mathrm{rank}(A)$.

Theorem (range and column space)

Let $\mathsf{A} \in \mathbb{C}^{m\times n}$ *be represented column–wise as* $\mathsf{A} = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$. *Then*

$$\operatorname{im}(\mathsf{A}) = \operatorname{span}(\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}) = \operatorname{col}(\mathsf{A}).$$

In other words, the range of A *coincides with its column space.*

Proof.

($\subseteq$) Let $\mathbf{y} \in \operatorname{im}(\mathsf{A}) \subseteq \mathbb{C}^m$. Then, by definition there is an $\mathbf{x} \in \mathbb{C}^n$ for which $\mathbf{y} = \mathsf{A}\mathbf{x}$, or

$$\mathbf{y} = \sum_{k=1}^{n} x_k \mathbf{c}_k,$$

which implies that $\mathbf{y} \in \operatorname{col}(\mathsf{A})$.
($\supseteq$) On the other hand, if $\mathbf{y} \in \operatorname{col}(\mathsf{A})$, this implies that there are $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, n$, such that

$$\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{c}_i.$$

Define $\mathbf{x} = [\alpha_1, \ldots, \alpha_n]^{\intercal} \in \mathbb{C}^n$. The previous identity shows that $\mathbf{y} = \mathsf{A}\mathbf{x}$, so that $\mathbf{y} \in \operatorname{im}(\mathsf{A})$. $\square$

Rank and Nullity

Corollary (range and rank)

For any $\mathsf{A} \in \mathbb{C}^{m\times n}$,

$$\dim(\mathrm{im}(\mathsf{A})) = \mathrm{rank}(\mathsf{A}).$$

Definition (nullity)

Suppose $\mathsf{A} \in \mathbb{C}^{m\times n}$. *The* **nullity** *of* A *is the dimension of* $\ker(\mathsf{A})$*:*

$$\mathrm{nullity}(\mathsf{A}) = \dim(\ker(\mathsf{A})).$$

Properties of the Rank

Theorem (properties of the rank)

Let $\mathsf{A} \in \mathbb{C}^{m \times n}$*, then:*

1. $\operatorname{rank}(\mathsf{A}) \leq \min\{m, n\}$.
2. $\operatorname{rank}(\mathsf{A}) + \operatorname{nullity}(\mathsf{A}) = n$.
3. *For any* $\mathsf{B} \in \mathbb{C}^{n \times p}$ *we have* $\operatorname{rank}(\mathsf{AB}) \geq \operatorname{rank}(\mathsf{A}) + \operatorname{rank}(\mathsf{B}) - n$.
4. *For any* $\mathsf{C} \in \mathbb{C}^{m \times m}$ *with* $\operatorname{rank}(\mathsf{C}) = m$ *and any* $\mathsf{B} \in \mathbb{C}^{n \times n}$ *with* $\operatorname{rank}(\mathsf{B}) = n$ *it holds that*
$$\operatorname{rank}(\mathsf{CA}) = \operatorname{rank}(\mathsf{A}) = \operatorname{rank}(\mathsf{AB}).$$
5. $\operatorname{rank}(\mathsf{AB}) \leq \min\{\operatorname{rank}(\mathsf{A}), \operatorname{rank}(\mathsf{B})\}$.
6. $\operatorname{rank}(\mathsf{A} + \mathsf{B}) \leq \operatorname{rank}(\mathsf{A}) + \operatorname{rank}(\mathsf{B})$.

Definition (adjoint)

Suppose that $(\mathbb{V}, (\cdot, \cdot)_{\mathbb{V}})$ and $(\mathbb{W}, (\cdot, \cdot)_{\mathbb{W}})$ are inner product spaces over $\mathbb{C}$. Let $A \in \mathfrak{L}(\mathbb{V}, \mathbb{W})$. The **adjoint** of A is a linear operator $A^* \in \mathfrak{L}(\mathbb{W}, \mathbb{V})$ that satisfies

$$(Ax, y)_{\mathbb{W}} = (x, A^* y)_{\mathbb{V}}, \quad \forall x \in \mathbb{V}, \ y \in \mathbb{W}.$$

A linear operator $A \in \mathfrak{L}(\mathbb{V}) = \mathfrak{L}(\mathbb{V}, \mathbb{V})$ is called **self–adjoint** iff $A = A^*$.

Definition (matrix adjoint, conjugate transpose)

Let $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{m \times n}$. The **matrix adjoint** (or **conjugate transpose**) of A is the matrix $\mathsf{A}^{\mathsf{H}} \in \mathbb{C}^{n \times m}$ with entries

$$\left[\mathsf{A}^{\mathsf{H}}\right]_{i,j} = \bar{a}_{j,i}.$$

The **transpose** of A is the matrix $\mathsf{A}^{\intercal} \in \mathbb{C}^{n \times m}$ with entries

$$[\mathsf{A}^{\intercal}]_{i,j} = a_{j,i}.$$

A matrix $\mathsf{A} \in \mathbb{C}^{n \times n}$ is called **Hermitian** iff $\mathsf{A} = \mathsf{A}^{\mathsf{H}}$. A is called **skew-Hermitian** iff $\mathsf{A} = -\mathsf{A}^{\mathsf{H}}$. A matrix $\mathsf{A} \in \mathbb{R}^{n \times n}$ is called **symmetric** iff $\mathsf{A} = \mathsf{A}^{\intercal}$ and **skew–symmetric** if $\mathsf{A} = -\mathsf{A}^{\intercal}$.

Proposition (properties of matrix adjoints)

Let $\mathsf{A} \in \mathbb{C}^{m\times p}$ *and* $\mathsf{B} \in \mathbb{C}^{p\times n}$. *Then* $(\mathsf{AB})^{\mathsf{H}} = \mathsf{B}^{\mathsf{H}}\mathsf{A}^{\mathsf{H}}$, *and* $(\mathsf{A}^{\mathsf{H}})^{\mathsf{H}} = \mathsf{A}$.

Let $\mathbf{x} = [x_1, \ldots, x_n]^{\intercal} \in \mathbb{C}^n$. The *conjugate transpose* of $\mathbf{x}$ is defined as the row vector $\mathbf{x}^{\mathsf{H}} = [\bar{x}_1, \ldots, \bar{x}_n]$. This conforms to the definition above, provided we view any column n–vector as a matrix with n rows and one column. A direct computation shows that $(\mathbf{x}^{\mathsf{H}})^{\mathsf{H}} = \mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^n$. Moreover, upon identifying $\mathbb{C}^{1\times 1}$ with $\mathbb{C}$, if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$,

$$(\mathbf{x}, \mathbf{y})_{\ell^2(\mathbb{C}^m)} = (\mathbf{x}, \mathbf{y})_2 = \mathbf{y}^{\mathsf{H}}\mathbf{x} = \overline{\mathbf{x}^{\mathsf{H}}\mathbf{y}} = \overline{(\mathbf{y}, \mathbf{x})_2} = \overline{(\mathbf{y}, \mathbf{x})_{\ell^2(\mathbb{C}^m)}}.$$

Furthermore, if $\mathsf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$. Then it follows that

$$(\mathsf{A}\mathbf{x}, \mathbf{y})_{\ell^2(\mathbb{C}^m)} = \mathbf{y}^{\mathsf{H}}\mathsf{A}\mathbf{x} = \left(\mathsf{A}^{\mathsf{H}}\mathbf{y}\right)^{\mathsf{H}}\mathbf{x} = (\mathbf{x}, \mathsf{A}^{\mathsf{H}}\mathbf{y})_{\ell^2(\mathbb{C}^{\mathrm{n}})},$$

where $(\,\cdot\,,\,\cdot\,)_{\ell^2(\mathbb{C}^m)}$ is the Euclidean inner product on $\mathbb{C}^m$. For any $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^{\mathsf{H}} = \mathbf{x}^{\intercal}$, and for $\mathsf{A} \in \mathbb{R}^{m\times n}$ the conjugate transpose coincides with the transpose, $\mathsf{A}^{\intercal}$.

Theorem (properties of the conjugate transpose)

Let $\mathsf{A} \in \mathbb{C}^{m\times n}$. *Then*

1. $\operatorname{rank}(\mathsf{A}) = \operatorname{rank}(\mathsf{A}^{\mathsf{H}}) = \operatorname{rank}(\mathsf{A}^{\intercal})$.
2. $\ker(\mathsf{A}) = \operatorname{im}(\mathsf{A}^{\mathsf{H}})^{\perp}$.
3. $\operatorname{im}(\mathsf{A})^{\perp} = \ker(\mathsf{A}^{\mathsf{H}})$.

Proof.

We prove the second result and leave the first and last to exercises.
($\subseteq$) Let $\mathbf{x} \in \ker(\mathsf{A})$. By definition $\mathsf{A}\mathbf{x} = \mathbf{0} \in \mathbb{C}^m$. Let $\mathbf{z} \in \operatorname{im}(\mathsf{A}^{\mathsf{H}})$, i.e., $\exists \mathbf{y} \in \mathbb{C}^m$ for which $\mathbf{z} = \mathsf{A}^{\mathsf{H}}\mathbf{y}$. Now, compute

$$(\mathbf{z}, \mathbf{x})_2 = (\mathsf{A}^{\mathsf{H}}\mathbf{y}, \mathbf{x})_2 = (\mathbf{y}, \mathsf{A}\mathbf{x})_2 = 0,$$

which shows that $\mathbf{x} \in \operatorname{im}(\mathsf{A}^{\mathsf{H}})^{\perp}$.
($\supseteq$) Conversely, if $\mathbf{x} \in \operatorname{im}(\mathsf{A}^{\mathsf{H}})^{\perp}$ then $0 = (\mathbf{x}, \mathsf{A}^{\mathsf{H}}\mathbf{y})_2 = (\mathsf{A}\mathbf{x}, \mathbf{y})_2$ for every $\mathbf{y} \in \mathbb{C}^m$. Thus $\mathsf{A}\mathbf{x} = \mathbf{0}$. $\square$

Invertible Matrices

Definition (identity)

The matrix $\mathsf{I}_n \in \mathbb{C}^{n\times n}$, defined by

$$[\mathsf{I}_n]_{i,j} = \delta_{i,j},$$

is known as the **matrix identity** of order n.

Definition (inverse)

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$. If there is $\mathsf{B} \in \mathbb{C}^{n\times n}$ such that $\mathsf{AB} = \mathsf{BA} = \mathsf{I}_n$, then we say that A is **invertible** and call the matrix B an inverse of A.

We can prove that the inverse is unique. Consequently, we denote the inverse of A by A^{-1}.

Properties of Invertible Matrices

Theorem (properties of the inverse)

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$*. Then* A *is invertible iff* $\operatorname{rank}(\mathsf{A}) = n$*. Moreover, if* A *is invertible,*

1. A^{-1} *is invertible and* $(\mathsf{A}^{-1})^{-1} = \mathsf{A}$*.*
2. A^{H} *is invertible and* $(\mathsf{A}^{\mathsf{H}})^{-1} = (\mathsf{A}^{-1})^{\mathsf{H}}$*. In this case, we write*
$$\mathsf{A}^{-\mathsf{H}} = (\mathsf{A}^{\mathsf{H}})^{-1}.$$
3. $\mathsf{A}^{\intercal}$ *is invertible and* $(\mathsf{A}^{\intercal})^{-1} = (\mathsf{A}^{-1})^{\intercal}$*. In this case, we write*
$$\mathsf{A}^{-\intercal} = (\mathsf{A}^{\intercal})^{-1}.$$
4. *For all* $\alpha \in \mathbb{C}_\star = \mathbb{C} \setminus \{0\}$, $\alpha\mathsf{A}$ *is invertible and* $(\alpha\mathsf{A})^{-1} = \frac{1}{\alpha}\mathsf{A}^{-1}$*.*
5. *If* $\mathsf{B} \in \mathbb{C}^{n\times n}$ *is also invertible then the product* AB *is invertible and* $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}$*.*

Orthogonal and Unitary Matrices

Definition (unitary matrices)

Let $A \in \mathbb{R}^{m\times m}$. We say that A is **orthogonal** iff $A^{-1} = A^{\intercal}$. Similarly, for $A \in \mathbb{C}^{m\times m}$ we say that A is **unitary** iff $A^{H} = A^{-1}$.

Example

The matrix

$$Q = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

is an orthogonal matrix, that is, $Q^{\intercal}Q = I$. The set of columns and the set of rows of Q form a orthonormal bases for $\mathbb{R}^3$.

Matrix Norms

Matrix Norms

The space $\mathbb{C}^{m\times n}$ is a finite-dimensional vector space. Therefore, we can put a norm on this space, in much the same way we put a norm on $\mathbb{C}^n$ or $\mathbb{R}^n$.

Definition

A function $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ is called a **matrix norm** iff

1. (non-negativity): $\|A\| \geq 0$, for all $A \in \mathbb{C}^{m\times n}$;
2. (positive definiteness): $\|A\| = 0$ iff $A = O$;
3. (non-negative homogeneity): $\|\alpha A\| = |\alpha|\,\|A\|$, for all $\alpha \in \mathbb{C}$ and $A \in \mathbb{C}^{m\times n}$.
4. (triangle inequality): $\|A + B\| \leq \|A\| + \|B\|$, for all $A, B \in \mathbb{C}^{m\times n}$.

Example Matrix Norms

Definition (Frobenius norm)

Let $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{m \times n}$. The **Frobenius norm** is defined via

$$\|\mathsf{A}\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2.$$

Definition (max norm)

The **matrix max norm** is defined via

$$\|\mathsf{A}\|_{\max} = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} |a_{i,j}|,$$

for all $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{m \times n}$.

Induced Matrix Norms

Certain norms on $\mathfrak{L}(\mathbb{V}, \mathbb{W})$ are, in a sense, compatible with those of $\mathbb{V}$ and $\mathbb{W}$.

Definition (induced norm)

Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ and $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ be complex, finite–dimensional normed vector spaces. The **induced norm** on $\mathfrak{L}(\mathbb{V}, \mathbb{W})$ is

$$\|A\|_{\mathfrak{L}(\mathbb{V},\mathbb{W})} = \sup_{x \in \mathbb{V}_\star} \frac{\|Ax\|_{\mathbb{W}}}{\|x\|_{\mathbb{V}}}, \quad \forall\, A \in \mathfrak{L}(\mathbb{V}, \mathbb{W}),$$

where $\mathbb{V}_\star = \mathbb{V} \setminus \{0\}$. When $\mathbb{V} = \mathbb{W}$ it is understood that $\|\cdot\|_{\mathbb{V}} = \|\cdot\|_{\mathbb{W}}$, as well.

Definition (matrix p–norm)

Let $\mathsf{A} \in \mathbb{C}^{m\times n}$ *be given and* $p \in [1, \infty]$. *The induced* $\mathfrak{L}(\ell^p(\mathbb{C}^n), \ell^p(\mathbb{C}^m))$ *norm, called simply the* **induced matrix** p**–norm**, *is denoted* $\|\mathsf{A}\|_p$, *and is defined as*

$$\|\mathsf{A}\|_p = \sup_{\mathsf{x} \in \mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{x}\|_{\ell^p(\mathbb{C}^m)}}{\|\mathbf{x}\|_{\ell^p(\mathbb{C}^n)}}.$$

Matrix 1-Norm

Proposition (matrix 1–norm)

Let $\mathsf{A} = [a_{i,j}] = [\mathbf{c}_1, \ldots, \mathbf{c}_n] \in \mathbb{C}^{m\times n}$ *be arbitrary. The induced matrix 1–norm, which is, by definition,*

$$\|\mathsf{A}\|_1 = \sup_{\mathbf{x}\in\mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{x}\|_{\ell^1(\mathbb{C}^m)}}{\|\mathbf{x}\|_{\ell^1(\mathbb{C}^n)}},$$

may be calculated via the following formula:

$$\|\mathsf{A}\|_1 = \max_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{i,j}| \right).$$

Matrix 1-Norm, Cont.

Proof.

Given any $\mathbf{x} = [x_1, \ldots, x_n]^\intercal \in \mathbb{C}^n$,

$$\begin{aligned}
\|\mathsf{A}\mathbf{x}\|_{\ell^1(\mathbb{C}^m)} &= \left\| \sum_{j=1}^{n} x_j \mathbf{c}_j \right\|_{\ell^1(\mathbb{C}^m)} \\
&\leq \sum_{j=1}^{n} |x_j| \|\mathbf{c}_j\|_{\ell^1(\mathbb{C}^m)} \\
&\leq \max_{j=1}^{n} \|\mathbf{c}_j\|_{\ell_1(\mathbb{C}^m)} \sum_{j=1}^{n} |x_j| \\
&= \max_{j=1}^{n} \|\mathbf{c}_j\|_{\ell_1(\mathbb{C}^m)} \|\mathbf{x}\|_{\ell^1(\mathbb{C}^n)}.
\end{aligned}$$

This shows that

$$\|\mathsf{A}\|_1 \leq \max_{j=1}^{n} \|\mathbf{c}_j\|_{\ell_1(\mathbb{C}^m)} = \max_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{i,j}| \right).$$

Matrix 1-Norm, Cont.

Proof. cont.

$$\|\mathsf{A}\|_1 \leq \max_{j=1}^{n} \|\mathbf{c}_j\|_{\ell_1(\mathbb{C}^m)} = \max_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{i,j}| \right).$$

On the other hand, there must be an index j_0 where the maximum in the previous inequality is attained. Choose $\mathbf{x} = \mathbf{e}_{j_0}$, the j_0–th canonical basis vector, and notice then that

$$\|\mathsf{A}\mathbf{x}\|_{\ell^1(\mathbb{C}^m)} = \|\mathbf{c}_{j_0}\|_{\ell^1(\mathbb{C}^m)}.$$

It is not difficult to see that the supremum in the definition of induced norm is attained at this vector. This implies that the norm is the maximum absolute column sum, i.e.,

$$\|\mathsf{A}\|_1 = \max_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{i,j}| \right).$$

□

Sub-Multiplicativity and Consistency

Definition (sub–multiplicativity)

Suppose that $\|\cdot\| : \mathbb{C}^{n\times n} \to \mathbb{R}$ *is a matrix norm, i.e., a norm on the vector space* $\mathfrak{L}(\mathbb{C}^n)$. *We say that the norm is* **sub–multiplicative** *iff*

$$\|AB\| \leq \|A\| \, \|B\| \, , \quad \forall \, A, B \in \mathbb{C}^{n\times n}.$$

Definition (consistency)

Suppose $\|\cdot\|_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{R}$ *and* $\|\cdot\|_{\mathbb{C}^m} : \mathbb{C}^m \to \mathbb{R}$ *are norms, and* $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ *is a matrix norm. We say that* $\|\cdot\|$ *is* **consistent** *with respect to the norms* $\|\cdot\|_{\mathbb{C}^n}$ *and* $\|\cdot\|_{\mathbb{C}^m}$ *iff*

$$\|A\mathbf{x}\|_{\mathbb{C}^m} \leq \|A\| \, \|\mathbf{x}\|_{\mathbb{C}^n} \, ,$$

for all $A \in \mathbb{C}^{m\times n}$ *and* $\mathbf{x} \in \mathbb{C}^n$.

Properties of Induced Norms

Proposition (property of induced norms)

Suppose $\|\cdot\|_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{R}$ is a norm on $\mathbb{C}^n$, and $\|\cdot\| : \mathbb{C}^{n\times n} \to \mathbb{R}$ is the induced matrix norm

$$\|\mathsf{A}\| = \sup_{\mathbf{x}\in\mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}}, \quad \forall\, \mathsf{A} \in \mathbb{C}^{n\times n}.$$

Then $\|\cdot\|$ is a bona fide norm, that is, additionally, sub–multiplicative and consistent with respect to $\|\cdot\|_{\mathbb{C}^n}$.

Proof.

Let us prove the triangle inequality and sub-multiplicative properties. Let $\mathsf{A}, \mathsf{B} \in \mathbb{C}^{n\times n}$ be arbitrary. For all $\mathbf{x} \in \mathbb{C}^n_\star$, using the triangle inequality property of the base norm $\|\cdot\|_{\mathbb{C}^n}$, we have

$$\frac{\|(\mathsf{A}+\mathsf{B})\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} = \frac{\|\mathsf{A}\mathbf{x}+\mathsf{B}\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} \leq \frac{\|\mathsf{A}\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} + \frac{\|\mathsf{B}\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}}.$$

Proof, Cont.

We can make the right hand side larger by taking suprema: for all $\mathbf{x} \in \mathbb{C}^n_\star$,

$$\frac{\|(\mathsf{A}+\mathsf{B})\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} \le \sup_{\mathbf{w}\in\mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{w}\|_{\mathbb{C}^n}}{\|\mathbf{w}\|_{\mathbb{C}^n}} + \sup_{\mathbf{z}\in\mathbb{C}^n_\star} \frac{\|\mathsf{B}\mathbf{z}\|_{\mathbb{C}^n}}{\|\mathbf{z}\|_{\mathbb{C}^n}}.$$

Now, look at the right hand side. It is independent of $\mathbf{x}$, and gives an upper bound for objects on the left hand side. Thus,

$$\sup_{\mathbf{x}\in\mathbb{C}^n_\star}\frac{\|(\mathsf{A}+\mathsf{B})\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} \le \sup_{\mathbf{w}\in\mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{w}\|_{\mathbb{C}^n}}{\|\mathbf{w}\|_{\mathbb{C}^n}} + \sup_{\mathbf{z}\in\mathbb{C}^n_\star} \frac{\|\mathsf{B}\mathbf{z}\|_{\mathbb{C}^n}}{\|\mathbf{z}\|_{\mathbb{C}^n}}.$$

Likewise, using consistency, which must be shown first, for any $\mathbf{x} \in \mathbb{C}^n_\star$,

$$\frac{\|(\mathsf{AB})\,\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} = \frac{\|\mathsf{A}\,(\mathsf{B}\mathbf{x})\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} \le \frac{\|\mathsf{A}\|\,\|\mathsf{B}\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} \le \|\mathsf{A}\| \sup_{\mathbf{z}\in\mathbb{C}^n_\star} \frac{\|\mathsf{B}\mathbf{z}\|_{\mathbb{C}^n}}{\|\mathbf{z}\|_{\mathbb{C}^n}} = \|\mathsf{A}\|\,\|\mathsf{B}\|\,.$$

Since the far right hand side is independent of $\mathbf{x}$,

$$\|\mathsf{AB}\| = \sup_{\mathbf{z}\in\mathbb{C}^n_\star} \frac{\|(\mathsf{AB})\,\mathbf{x}\|_{\mathbb{C}^n}}{\|\mathbf{x}\|_{\mathbb{C}^n}} \le \|\mathsf{A}\|\,\|\mathsf{B}\|\,.$$

$\square$

The Vector and Matrix 2-Norms

Recall that, for $\mathbf{x} \in \mathbb{C}^n$,

$$\|\mathbf{x}\|_2 = \|\mathbf{x}\|_{\ell^2(\mathbb{C}^n)} = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\mathbf{x}^{\mathrm{H}}\mathbf{x}}.$$

Thus, for any $\mathsf{A} \in \mathbb{C}^{m\times n}$,

$$\|\mathsf{A}\|_2 = \sup_{\mathrm{x}\in\mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{x}\|_{\ell^2(\mathbb{C}^m)}}{\|\mathbf{x}\|_{\ell^2(\mathbb{C}^n)}} = \sup_{\mathrm{x}\in\mathbb{C}^n_\star} \frac{\sqrt{\mathbf{x}^{\mathrm{H}}\mathsf{A}^{\mathrm{H}}\mathsf{A}\mathbf{x}}}{\sqrt{\mathbf{x}^{\mathrm{H}}\mathbf{x}}}.$$

Suppose that $\mathsf{Q} \in \mathbb{C}^{n\times n}$ is unitary, that is, $\mathsf{Q}^{\mathrm{H}}\mathsf{Q} = \mathsf{I}$. Then

$$\|\mathsf{Q}\mathbf{x}\|_2 = \sqrt{\mathbf{x}^{\mathrm{H}}\mathsf{Q}^{\mathrm{H}}\mathsf{Q}\mathbf{x}} = \sqrt{\mathbf{x}^{\mathrm{H}}\mathbf{x}} = \|\mathbf{x}\|_2\,.$$

Unitary Invariance of the 2-Norm

Proposition (norm of a unitary matrix)

Let $\mathsf{A} \in \mathbb{C}^{m \times n}$ *be arbitrary and* $\mathsf{Q} \in \mathbb{C}^{m \times m}$ *be unitary. Then, we have*

$$\|\mathsf{QA}\|_2 = \|\mathsf{A}\|_2.$$

Proof.

Recall that, for any unitary matrix, Q, we have $\|\mathsf{Q}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$. The result follows from this fact. $\square$

Eigenvalues and Spectral Decomposition

Eigenvalues and Spectrum

Definition

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$. We say that $\lambda \in \mathbb{C}$ is an **eigenvalue** of A iff there exists a vector $\mathbf{x} \in \mathbb{C}^n_\star = \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that

$$\mathsf{A}\mathbf{x} = \lambda\mathbf{x}.$$

This vector is called an **eigenvector** of A associated to λ. The **spectrum** of A, denoted by $\sigma(\mathsf{A})$, is the collection of all eigenvalues of A. The pair $(\lambda, \mathbf{x})$ is called an **eigenpair of** A.

Properties of the Spectrum

Theorem

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$*. Then*

1. $\lambda \in \sigma(\mathsf{A})$ *iff* $\bar{\lambda} \in \sigma(\mathsf{A}^{\mathsf{H}})$.
2. A *is invertible iff* $0 \notin \sigma(\mathsf{A})$.
3. *The eigenvectors corresponding to distinct eigenvalues are linearly independent.*
4. $\lambda \in \sigma(\mathsf{A})$ *iff* $\chi_{\mathsf{A}}(\lambda) = 0$, *where* χ_{A} *is a polynomial of degree* n, *defined via*
$$\chi_{\mathsf{A}}(\lambda) = \det(\lambda \mathsf{I}_n - \mathsf{A}).$$
χ_{A} *is called the characteristic polynomial.*
5. *There are at most* n *distinct complex–valued eigenvalues of* A.

Algebraic Versus Geometric Multiplicity

The Fundamental Theorem of Algebra implies that the characteristic polynomial can be written as a product of factors, that is

$$\chi_{\mathsf{A}}(\lambda) = \prod_{i=1}^{L} (\lambda - \lambda_i)^{m_i}, \tag{1}$$

with $n = \sum_{i=1}^{L} m_i$.

Definition (algebraic multiplicity)

Let $\mathsf{A} \in \mathbb{C}^{n \times n}$ be given. The number m_i in (1) is called the **algebraic multiplicity** of the eigenvalue λ_i.

Definition (geometric multiplicity)

Let $\mathsf{A} \in \mathbb{C}^{n \times n}$ and $\lambda \in \sigma(\mathsf{A})$. Define the eigenspace

$$E(\lambda, \mathsf{A}) = \{\mathbf{x} \in \mathbb{C}^n \mid \mathsf{A}\mathbf{x} = \lambda \mathbf{x}\}.$$

This is a vector subspace of $\mathbb{C}^n$; its dimension, $\mathbf{dim}(E(\lambda, \mathsf{A}))$, is called the **geometric multiplicity** of λ.

Algebraic Versus Geometric Multiplicity

Theorem

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ *and* $\lambda \in \sigma(\mathsf{A})$. *The geometric multiplicity of* λ *is not larger than the algebraic multiplicity of* λ.

Example

Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

The algebraic multiplicity of $\lambda = 2$ is 3. But, the geometric multiplicity of $\lambda = 2$ is 1. In particular, there is only one linearly independent eigenvector in the subspace $E(2, \mathsf{A})$.

Diagonalizable Matrices

Definition (triangular matrices)

The square matrix $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{n \times n}$ is called **upper triangular** iff $a_{i,j} = 0$, for all $i > j$. A is called **lower triangular** iff $a_{i,j} = 0$, for all $i < j$. A matrix is called **triangular** iff it is either upper or lower triangular. A is called **diagonal** iff $a_{i,j} = 0$ for all $i \neq j$. A matrix $\mathsf{A} = [a_{i,j}] \in \mathbb{C}^{n \times n}$ is called **unit lower triangular** (**unit upper triangular**) iff it is lower (upper) triangular and $a_{i,i} = 1$, $i = 1, \ldots n$.

Definition (similarity)

Let $\mathsf{A}, \mathsf{B} \in \mathbb{C}^{n \times n}$. We say that A and B are **similar**, denoted by $\mathsf{A} \asymp \mathsf{B}$, iff there is an invertible matrix S, such that

$$\mathsf{A} = \mathsf{S}^{-1} \mathsf{B} \mathsf{S}.$$

We say that the matrix A is **diagonalizable** if it is similar to a diagonal matrix. A is **triangularizable** if it is similar to a triangular matrix.

Spectrum of Similar Matrices

Proposition

Let $\mathsf{A}, \mathsf{B} \in \mathbb{C}^{n\times n}$ *be such that* $\mathsf{A} \asymp \mathsf{B}$. *Then* $\chi_{\mathsf{A}} = \chi_{\mathsf{B}}$ *and, consequently,* $\sigma(\mathsf{A}) = \sigma(\mathsf{B})$. *Furthermore,* $\det(\mathsf{A}) = \det(\mathsf{B})$ *and* $\operatorname{tr}(\mathsf{A}) = \operatorname{tr}(\mathsf{B})$.

Proof.

Suppose $\mathsf{A} = \mathsf{S}^{-1}\mathsf{B}\mathsf{S}$. Then,

$$\begin{aligned}
\chi_{\mathsf{A}}(\lambda) &= \det(\lambda \mathsf{I}_n - \mathsf{A}) \\
&= \det(\mathsf{S}^{-1}\mathsf{S}) \det(\lambda \mathsf{I}_n - \mathsf{A}) \\
&= \det(\mathsf{S}^{-1}) \det(\lambda \mathsf{I}_n - \mathsf{A}) \det(\mathsf{S}) \\
&= \det\left(\mathsf{S}^{-1}(\lambda \mathsf{I}_n - \mathsf{A})\mathsf{S}\right) \\
&= \det(\lambda \mathsf{I}_n - \mathsf{B}) \\
&= \chi_{\mathsf{B}}(\lambda).
\end{aligned}$$

□

Diagonalizability Criterion

Definition (defective matrix)

A matrix $\mathsf{A} \in \mathbb{C}^{n\times n}$ is called **defective** iff there is an eigenvalue λ_k with geometric multiplicity strictly smaller than the algebraic multiplicity. Otherwise, the matrix is called **non–defective**.

Theorem

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ *be non–defective, then it is diagonalizable.*

Proof.

Let $\sigma(\mathsf{A}) = \{\lambda_k\}_{k=1}^{L}$, where $\lambda_k \neq \lambda_j$, $k \neq j$. For each k,

$$E(\lambda_k, \mathsf{A}) = \mathsf{span}(\{\mathbf{x}_1^{(k)}, \dots, \mathbf{x}_{m_k}^{(k)}\}) = \mathsf{span}(S_k),$$

where the set $S_k = \{\mathbf{x}_1^{(k)}, \dots, \mathbf{x}_{m_k}^{(k)}\}$ is linearly independent. Then $S = \cup_{k=1}^{L} S_k$ is a basis of $\mathbb{C}^n$. Indeed, a previous Theorem, the set S is linearly independent. Moreover, $\#(S) = \sum_{k=1}^{L} m_k = n$, since the matrix A is non–defective.

Diagonalizability Criterion, Cont.

Proof, Cont.

Now, set $\mathsf{D} = \mathbf{diag}\left(\lambda_1, \ldots, \lambda_1, \ldots, \lambda_L, \ldots, \lambda_L\right)$ and

$$\mathsf{X} = \begin{bmatrix} | & & | & & | & & | \\ \mathbf{x}_1^{(1)} & \cdots & \mathbf{x}_{m_1}^{(1)} & \cdots & \mathbf{x}_1^{(L)} & \cdots & \mathbf{x}_{m_L}^{(L)} \\ | & & | & & | & & | \end{bmatrix},$$

where in D each eigenvalue λ_k appears exactly m_k times. Notice now that, since all the columns of X are linearly independent, we have $\mathbf{rank}(\mathsf{X}) = n$ and this implies that X is invertible. Since, for all $j = 1, \ldots, m_k$, we have $\mathsf{A}\mathbf{x}_j^{(k)} = \lambda_k \mathbf{x}_j^{(k)}$, we see that

$$\mathsf{AX} = \mathsf{A}\left[\mathbf{x}_1, \ldots, \mathbf{x}_n\right] = \left[\mathsf{A}\mathbf{x}_1, \ldots, \mathsf{A}\mathbf{x}_n\right] \quad \text{and} \quad \mathsf{XD} = \left[\lambda_1\mathbf{x}_1, \ldots, \lambda_n\mathbf{x}_n\right].$$

This implies that $\mathsf{AX} = \mathsf{XD}$, or, since X is invertible, $\mathsf{A} = \mathsf{XDX}^{-1}$. In conclusion, A is diagonalizable. □

Schur Normal Form

Lemma

Let $\mathsf{A} \in \mathbb{C}^{n \times n}$*. There are, not necessarily unique, matrices* $\mathsf{U}, \mathsf{R} \in \mathbb{C}^{n \times n}$*, with* U *unitary and* R *upper triangular, such that*

$$\mathsf{A} = \mathsf{U}\mathsf{R}\mathsf{U}^{\mathsf{H}}.$$

Notice that, in the setting of the Lemma, we have that $\mathsf{A} \asymp \mathsf{R}$ and that, since R is upper triangular, its diagonal entries coincide with its spectrum.

Spectral Decomposition Theorem

Proposition

Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ *be self–adjoint (Hermitian), i.e.,* $\mathsf{A}^{\mathsf{H}} = \mathsf{A}$. *Then* $\sigma(\mathsf{A}) \subseteq \mathbb{R}$ *and there is a unitary* $\mathsf{U} \in \mathbb{C}^{n\times n}$ *such that*

$$\mathsf{A} = \mathsf{U}\mathsf{D}\mathsf{U}^{\mathsf{H}},$$

where the matrix $\mathsf{D} = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$. *Furthermore, there exists an orthonormal basis* $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *of eigenvectors of* A *for the space* $\mathbb{C}^n$ *and* $\mathsf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i$, $i = 1, \ldots, n$.

Proof.

Using the Schur Normal form, we are guaranteed that there is a unitary matrix $\mathsf{U} \in \mathbb{C}^{n\times n}$ and an upper triangular matrix $\mathsf{D} \in \mathbb{C}^{n\times n}$, such that

$$\mathsf{A} = \mathsf{U}\mathsf{D}\mathsf{U}^{\mathsf{H}}.$$

But, since A is self–adjoint,

$$\mathsf{A}^{\mathsf{H}} = \mathsf{U}\mathsf{D}^{\mathsf{H}}\mathsf{U}^{\mathsf{H}} = \mathsf{U}\mathsf{D}\mathsf{U}^{\mathsf{H}} = \mathsf{A}.$$

Spectral Decomposition Theorem, Cont.

Proof, Cont.

This implies that $\mathsf{D}^{\mathsf{H}} = \mathsf{D}$, that is, D is self–adjoint. Since D is triangular, it must be diagonal. Furthermore, the diagonal elements of D must be real. Otherwise, D could not be self–adjoint. Therefore, we have the desired factorization.

Now, the eigenvalues of a diagonal matrix, are precisely its diagonal entries. Since A is similar to the diagonal matrix D, the eigenvalues of A are precisely $\lambda_i = d_{i,i} \in \mathbb{R}$, $i = 1, \ldots, n$.

Finally, observe that the columns of U form an orthonormal basis for $\mathbb{C}^n$. Indeed, suppose that the k–th column of U is denoted $\mathbf{u}_k$. Then the $\mathsf{AU} = \mathsf{UD}$ iff

$$\mathsf{A}\mathbf{u}_k = d_{k,k}\mathbf{u}_k = \lambda_k \mathbf{u}_k .$$

Thus, the eigenvectors of A, namely, $\mathbf{u}_k$, $k = 1, \ldots, n$, form an orthonormal basis for $\mathbb{C}^n$: $(\mathbf{u}_k, \mathbf{u}_j)_2 = \mathbf{u}_j^{\mathsf{H}}\mathbf{u}_k = \delta_{k,j}$, $k, j = 1, \ldots, n$. □

Spectral Decomposition Theorem for Operators

Definition (eigenvalue)

Suppose that $\mathbb{V}$ is a complex vector space space and $A \in \mathfrak{L}(\mathbb{V})$. The scalar $\lambda \in \mathbb{C}$ satisfying

$$Aw = \lambda w, \quad \exists\, w \in \mathbb{V} \setminus \{0\},$$

is called an **eigenvalue** of A, and w is a corresponding **eigenvector**. The spectrum of A, $\sigma(A)$, is the set of all eigenvalues of A. The pair (λ, w) is called an **eigenpair of** A.

For self–adjoint operators we have the following general result.

Theorem (Spectral Decomposition Theorem)

Suppose that $(\mathbb{V}, (\,\cdot\,,\,\cdot\,))$ *is an* n*–dimensional complex inner product space and* $A \in \mathfrak{L}(\mathbb{V})$ *is self–adjoint. Then there are precisely* n *eigenvalues, counting multiplicities, and* $\sigma(A) \subseteq \mathbb{R}$*. Moreover, there is an orthonormal basis* $B = \{w_1, \ldots, w_n\}$ *of eigenvectors of* A *for the space* $\mathbb{V}$*:* $(w_i, w_j) = \delta_{i,j}$, $i, j = 1, \ldots, n$.

Definition (normal matrix)

The square matrix $\mathsf{A} \in \mathbb{C}^{n\times n}$ is called **normal** iff $\mathsf{A}^{\mathsf{H}}\mathsf{A} = \mathsf{A}\mathsf{A}^{\mathsf{H}}$.

Lemma (normal and triangular)

Suppose that $\mathsf{A} \in \mathbb{C}^{n\times n}$ *is normal and upper triangular. Then, it must be diagonal.*

Proof.

A homework problem. □

Theorem (diagonalization of normal matrices)

Suppose that $\mathsf{A} \in \mathbb{C}^{n\times n}$ *is normal. Then* A *is unitarily diagonalizable, i.e., there is a unitary matrix* $\mathsf{U} \in \mathbb{C}^{n\times n}$ *and a diagonal matrix* $\mathsf{D} \in \mathbb{C}^{n\times n}$ *such that*

$$\mathsf{A} = \mathsf{U}\mathsf{D}\mathsf{U}^{\mathsf{H}}.$$

Proof.

Use the Schur factorization and the last Lemma. □