Classical Numerical Analysis, Chapter 20

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Chapter 20, Part 2 of 2
Linear Multistep Methods

Zero Stability

Definition (zero stability)

Suppose that $\mathbf{f} \in \mathcal{F}^1(S)$ *and* $\mathbf{u} \in C^2([0, T]; \mathbb{R}^d)$ *is a classical solution to* (**??**). *Let, for* $i = 1, 2$, $\left\{\mathbf{w}_i^k\right\}_{k=0}^{K}$ *be approximations generated by the linear q-step method* (**??**) *with the starting values* $\left\{\mathbf{w}_i^k\right\}_{k=0}^{q-1}$, $i = 1, 2$, *respectively. The method is called* **zero stable** *if and only if there is a* $C > 0$ *independent of* $\tau > 0$ *and the starting values such that, for any* $k = q, \dots, K$,

$$\left\|\mathbf{w}_1^k - \mathbf{w}_2^k\right\|_2 \leq C \max_{m=0,\dots,q-1} \left\|\mathbf{w}_1^m - \mathbf{w}_2^m\right\|_2 .$$

Definition (root condition)

The linear q-step method (**??**) *satisfies the* **root condition** *if and only if:*

1. *All of the roots of the first characteristic polynomial* $\psi(z) = \sum_{j=0}^{q} a_j z^j$ *are inside the unit disk*

$$\{z \in \mathbb{C} \mid |z| \leq 1\} \subset \mathbb{C}.$$

2. *If* $\psi(\xi) = 0$ *and* $|\xi| = 1$, *then* ξ *is a simple root, i.e., its multiplicity is exactly one, i.e.,* $\psi'(\xi) \neq 0$.

Definition (homogeneous zero stability)

Suppose that $\mathbf{f} \equiv \mathbf{0}$ *and* $\mathbf{u}_0 = \mathbf{0}$, *so that the unique solution to* (**??**) *is* $\mathbf{u}(t) = \mathbf{0}$ *for all* $t \geq 0$. *Let* $\left\{\mathbf{w}^k\right\}_{k=0}^{K}$ *be the approximation generated by the linear q-step method* (**??**) *with the starting values* $\left\{\mathbf{w}^k\right\}_{k=0}^{q-1}$. *The method is called* **homogeneous zero stable** *if and only if there is a* $C > 0$ *independent of* $\tau > 0$ *and the starting values such that, for any* $k = q, \ldots, K$,

$$\left\|\mathbf{w}^k\right\|_2 \leq C \max_{m=0,\ldots,q-1} \left\|\mathbf{w}^m\right\|_2 .$$

Definition (stable solutions)

Suppose that $\{a_j\}_{j=0}^{q-1} \subset \mathbb{C}$ *are given. An equation of the form*

$$\zeta_{k+q} + \sum_{j=0}^{q-1} a_j \zeta_{k+j} = 0, \quad k = 0, 1, 2, \dots \tag{1}$$

is called a **homogeneous difference equation***. We say that solutions to* (1) *are* **stable** *if and only if, given any starting values* $\{\zeta_k\}_{k=0}^{q-1} \subset \mathbb{R}$*, the sequence* $\{\zeta_k\}_{k=0}^{\infty} \subset \mathbb{R}$ *is bounded by a constant* $C > 0$ *that only depends upon the starting values.*

Example

In this example, we exhibit a method that does not satisfy the root condition and is *not* homogeneously zero stable. Consider the method $q = 2$, $a_2 = 1$, $a_1 = -3$, $a_0 = 2$ and $b_2 = 0$, $b_1 = 0$, $b_0 = -1$. In other words,

$$\mathbf{w}^{k+2} - 3\mathbf{w}^{k+1} + 2\mathbf{w}^{k} = -\tau \mathbf{f}(t_k, \mathbf{w}^{k})$$

with the starting values $\mathbf{w}^0$, $\mathbf{w}^1$. The method is consistent. We find

$$C_0 = 0 = C_1, \quad C_2 = \frac{1}{2},$$

which implies that the method is consistent to order $p = 1$.

The first characteristic polynomial is

$$\psi(z) = z^2 - 3z + 2 = (z-1)(z-2).$$

Clearly, the method fails to satisfy the root condition.

Example (Cont.)

Since we are considering homogeneous zero stability, we take $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{u}_0 = \mathbf{0}$. The solution of the homogeneous linear constant coefficient difference equation,

$$\zeta_{k+2} - 3\zeta_{k+1} + 2\zeta_k = 0, \quad k = 0, 1, \ldots,$$

is precisely

$$\zeta_k = 2\zeta_0 - \zeta_1 + 2^k(\zeta_1 - \zeta_0).$$

This can be verified by a simple induction argument. For starting values, let us take $\zeta_0 = 0$, $\zeta_1 = \tau$. Then $\zeta_k = \tau(2^k - 1)$, $k = 0, 1, 2, \ldots$. Let us examine the approximation at time $T = 1$. In this case, $\tau = 1/K$ and we have, as $K \to \infty$,

$$w^K = \frac{2^K - 1}{K} \to \infty.$$

Thus, the method is not homogeneously zero stable.

Theorem (root condition and stability)

*Suppose that $q \in \mathbb{N}$. Consider a linear q-step method (**??**) with coefficients $a_j, b_j \in \mathbb{R}$, $j = 0, \ldots, q$, with $a_q = 1$ and $a_0 \neq 0$. The solutions to the corresponding homogeneous difference*

$$\zeta_{k+q} + \sum_{j=0}^{q-1} a_j \zeta_{k+j} = 0, \quad k = 0, 1, 2, \ldots$$

are bounded, i.e., stable, if and only if the root condition is satisfied.