Classical Numerical Analysis, Chapter 24

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu swise1@.utk.edu
The University of Tennessee

Chapter 24, Part 1 of 2

Finite Difference Methods for Elliptic Problems

Problem and Basic Assumptions

In this chapter, we explore finite difference methods (FDMs) for solving elliptic boundary value problems (BVPs), such as

$$-\Delta u = f \quad \text{in} \quad \Omega,$$

with

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

To present the essential ideas, without obscuring the discussion with technical details, we will assume that our grids are uniform; the domain is simple, say

$$\Omega = (0, 1)^d,$$

with $d \in \mathbb{N}$; and we will mostly discuss the one- $(d = 1)$ and two- $(d = 2)$ dimensional cases.

Finite Differences

As the name suggests, the main idea behind FDMs is to replace the derivatives appearing in a BVP by differences of values at nearby points, thus reducing the BVP to a system of algebraic equations (for the point values). This seems very natural. After all, if a function $v \colon \mathbb{R}^d \to \mathbb{R}$ is sufficiently smooth, for any $i = 1, \ldots, d$, we have

$$\frac{\partial v(\mathbf{x})}{\partial x_i} = \lim_{h \to 0} \frac{v(\mathbf{x} + h\mathbf{e}_i) - v(\mathbf{x})}{h},$$

where $\mathbf{e}_i$ is the ith canonical basis vector in $\mathbb{R}^d$, thus we can propose the following *finite difference approximations* of this partial derivative:

$$\frac{v(\mathbf{x} + h\mathbf{e}_i) - v(\mathbf{x})}{h}, \qquad \frac{v(\mathbf{x}) - v(\mathbf{x} - h\mathbf{e}_i)}{h}, \qquad h > 0.$$

Evidently, these are not the only two possibilities. For instance, the reader can easily verify that, for a sufficiently smooth function v, the quantities

$$\frac{v(\mathbf{x} + h\mathbf{e}_i) - v(\mathbf{x} - h\mathbf{e}_i)}{2h}, \qquad \frac{3v(\mathbf{x}) - 4v(\mathbf{x} - h\mathbf{e}_i) + v(\mathbf{x} - 2h\mathbf{e}_i)}{2h}$$

converge, as $h \downarrow 0$, to this partial derivative.

Consistency

Which finite difference approximation must be used then? The answer to this comes from trying to satisfy two, often contradictory, requirements, as follows.

Consistency: We wish the finite difference approximation to be as accurate as possible. The way this is usually verified is by assuming that the function v is sufficiently smooth. If this is the case, a Taylor expansion can show that, for instance,

$$\frac{v(\mathbf{x}+h\mathbf{e}_i)-v(\mathbf{x})}{h}=\frac{\partial v(\mathbf{x})}{\partial x_i}+\mathcal{O}(h),$$

$$\frac{v(\mathbf{x}+h\mathbf{e}_i)-v(\mathbf{x}-h\mathbf{e}_i)}{2h}=\frac{\partial v(\mathbf{x})}{\partial x_i}+\mathcal{O}(h^2),$$

as $h \downarrow 0$. For this reason, we say that the first finite difference approximation is *consistent to order one*, whereas the second one is *consistent to order two*. From this consideration alone, it seems that we need to find a finite difference approximation of the derivatives that is consistent to as high order as possible.

Stability

Stability: We are not trying to approximate the action of a derivative, but the solution to a BVP. For this reason, the solution of our FDM must be stable, not only with respect to perturbations of the data but, more importantly, in (discrete versions of) the same norms in which the solution to the continuous problem possesses its own stability properties. This will allow us to compare the continuous and discrete solutions (via consistency) and establish convergence.

Lax's Principle

It turns out that, at least for linear problems, the satisfaction of these two requirements is enough to obtain convergence. In the literature, this is commonly known as *Lax's Principle*:

$$\text{consistency} + \text{stability} \implies \text{convergence}.$$

Grid Functions and Finite Difference Operators

Let $N \in \mathbb{N}$, set $h = 1/(N+1)$, called the *grid size*, and define the *uniform grid*

$$\mathbb{Z}_h^d = \left\{ h\mathbf{z} \in \mathbb{R}^d \;\middle|\; \mathbf{z} \in \mathbb{Z}^d \right\}.$$

Definition (grid function)

A **grid domain** *in* $\mathbb{R}^d$ *is any nonempty set* $\mathcal{G}_h \subseteq \mathbb{Z}_h^d$*. The points* $\mathbf{x} \in \mathcal{G}_h$ *are the* **nodes** *or* **grid points***. A* **grid function** *on* $\mathcal{G}_h$ *is any function* $v\colon \mathcal{G}_h \to \mathbb{R}$*. For a grid function* v *and* $h\mathbf{i} \in \mathcal{G}_h$*, we set* $v_{\mathbf{i}} = v(h\mathbf{i})$*. The set of grid functions on* $\mathcal{G}_h$ *is denoted by*

$$\mathcal{V}(\mathcal{G}_h) = \{ v \mid v\colon \mathcal{G}_h \to \mathbb{R} \}.$$

This is a vector space. If $\#\mathcal{G}_h < \infty$*, this space is finite dimensional; otherwise, it is infinite dimensional. In a similar manner, if* $\mathbb{V}$ *is a, finite-dimensional, vector space, we set*

$$\mathcal{V}(\mathcal{G}_h; \mathbb{V}) = \{ v \mid v\colon \mathcal{G}_h \to \mathbb{V} \}.$$

Example

Let $d = 1$ and $\Omega = (0, 1)$ be the interval where we will usually consider our elliptic BVPs. For $N \in \mathbb{N}$ and $h = 1/(N + 1)$, we set

$$\bar{\Omega}_h = \bar{\Omega} \cap \mathbb{Z}_h = \{hi \mid i = 0, \dots, N + 1\} .$$

Notice that, in this case, $\mathcal{V}(\bar{\Omega}_h)$ is nothing but the collection of functions $v\colon \{0, \dots, N + 1\} \to \mathbb{R}$.

Notice also that $\partial\Omega = \{0, 1\}$ and, similarly, we can define the *discrete boundary* and *discrete interior* of $\bar{\Omega}_h$, respectively, as

$$\partial\Omega_h = \partial\Omega \cap \mathbb{Z}_h, \qquad \Omega_h = \bar{\Omega}_h \backslash \partial\Omega_h.$$

Example

The previous example can be easily generalized to any $d \in \mathbb{N}$. For instance, if $\Omega = (0,1)^2$, we can define

$$\bar{\Omega}_h = \bar{\Omega} \cap \mathbb{Z}_h^2 = \{(hi, hj) \mid i, j = 0, \dots, N+1\}.$$

The discrete interior is then

$$\Omega_h = \Omega \cap \mathbb{Z}_h^2 = \{(hi, hj) \mid i, j = 1, \dots, N\}.$$

The discrete boundary can also be defined accordingly. See the figure on the next slide. The space $\mathcal{V}(\bar{\Omega}_h)$ is defined accordingly.

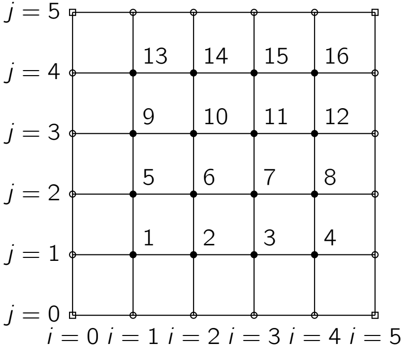

Figure: A uniform grid of size 5×5 and spacing $h = \frac{1}{5}$ covering $\Omega = (0,1)^2$. The discrete interior, Ω_h, is the collection of filled circles, and the discrete boundary, $\partial\Omega_h$, consists of the remaining ones. The space of grid functions, $\mathcal{V}(\bar{\Omega}_h)$, is the space of functions from all points in this grid into $\mathbb{R}$. The numbers next to the filled circles denote their lexicographical ordering, which is used to provide the isomorphism $\mathcal{V}(\bar{\Omega}_h) \longleftrightarrow \mathbb{R}^{16}$. The filled circles constitute the domain Ω_h^I for the two-dimensional discrete Laplacian, whereas the unfilled circles constitute Ω_h^B.

Definition (space $\mathcal{V}_0(\bar{\Omega}_h)$)

Let $d \in \mathbb{N}$, $\Omega = (0,1)^d$, $N \in \mathbb{N}$, $h = 1/(N+1)$, and $\bar{\Omega}_h = \bar{\Omega} \cap \mathbb{Z}_h^d$. To treat Dirichlet boundary conditions, we also introduce the space of grid functions that vanish on the boundary

$$\mathcal{V}_0(\bar{\Omega}_h) = \left\{ v \in \mathcal{V}(\bar{\Omega}_h) \;\middle|\; v_{\mathbf{i}} = 0, \ \forall \, h\mathbf{i} \in \partial\Omega_h \right\}.$$

Proposition (isomorphism)

Let $d, N \in \mathbb{N}$, $h = \frac{1}{N+1}$, $\Omega = (0,1)^d$, and $\bar{\Omega}_h = \bar{\Omega} \cap \mathbb{Z}_h^d$. There is a one-to-one correspondence between the space of grid functions $\mathcal{V}(\Omega_h)$ and $\mathbb{R}^{N^d}$. Likewise, there is a one-to-one correspondence between the space of grid functions $\mathcal{V}_0(\bar{\Omega}_h)$ and $\mathbb{R}^{N^d}$. Therefore, $\dim(\mathcal{V}(\Omega_h)) = \dim(\mathcal{V}_0(\bar{\Omega}_h)) = N^d$.

Proof.

Let $d = 1$. We make the canonical correspondence. If $v \in \mathcal{V}(\Omega_h)$, then define $\mathbf{v} \in \mathbb{R}^N$ component-wise via

$$[\mathbf{v}]_i = v_i, \quad ih \in \Omega_h.$$

Conversely, if $\mathbf{v} \in \mathbb{R}^N$ is given, define the grid function $v \in \mathcal{V}(\Omega_h)$ via

$$v_i = [\mathbf{v}]_i\,.$$

The grid functions with zero values on the boundary (those from $\mathcal{V}_0(\bar{\Omega}_h)$) are handled similarly.

Proof Cont.

To illustrate the general procedure, consider now $d = 2$. If $v \in \mathcal{V}(\Omega_h)$, then define $\mathbf{v} \in \mathbb{R}^{N^2}$ component-wise via

$$[\mathbf{v}]_{i+(j-1)N} = v_{i,j}, \quad (ih, jh) \in \Omega_h.$$

Conversely, if $\mathbf{v} \in \mathbb{R}^{N^2}$ is given, define the grid function $v \in \mathcal{V}(\Omega_h)$ component-wise via

$$v_{i,j} = [\mathbf{v}]_{i+(j-1)N}, \quad (ih, jh) \in \Omega_h.$$

The previous figure shows an illustration of this equivalence, which is commonly called *lexicographical ordering*. □

Remark (convention)

Because of this canonical (and trivial) correspondence, we need not be too careful about whether we are talking about a grid function or its vector representation. We will often write

$$\mathbf{v} \in \mathbb{R}^{N^d} \longleftrightarrow v \in \mathcal{V}(\Omega_h), \qquad \mathbf{v} \in \mathbb{R}^{N^d} \longleftrightarrow v \in \mathcal{V}_0(\bar{\Omega}_h)$$

to express the fact that our object may be viewed in either setting. We use the convention of denoting a grid function by a lowercase Greek or Roman character and its corresponding canonical vector representative by the boldface of the same Greek or Roman character.

Definition (shift operator)

Let $d \in \mathbb{N}$ *and* $\mathbf{e} \in \mathbb{Z}^d$. *The* **shift operator** *in the direction of* $\mathbf{e}$,

$$\mathcal{S}_{\mathrm{e}} \colon \mathcal{V}(\mathbb{Z}_h^d) \to \mathcal{V}(\mathbb{Z}_h^d),$$

is given, for $\mathbf{x} = h\mathbf{z} \in \mathbb{Z}_h^d$, *by*

$$(\mathcal{S}_{\mathrm{e}} v)(\mathbf{x}) = v(\mathbf{x} + h\mathbf{e}), \quad (\mathcal{S}_{\mathrm{e}} v)_{\mathrm{z}} = v_{\mathrm{z}+\mathrm{e}}.$$

Proposition (properties of shifts)

Let $d \in \mathbb{N}$. For any $\mathbf{e} \in \mathbb{Z}^d$, the shift operator satisfies $\mathcal{S}_{\mathrm{e}} \in \mathfrak{L}(\mathcal{V}(\mathbb{Z}_h^d))$. Moreover, this operator is invertible and its inverse is given by

$$\mathcal{S}_{\mathrm{e}}^{-1} = \mathcal{S}_{-\mathrm{e}}.$$

In particular, $\mathcal{S}_0$ is the identity operator.

Proof.

This is an exercise. $\square$

Definition (finite difference operator)

The mapping

$$\mathcal{F}_h \colon \mathcal{V}(\mathbb{Z}_h^d) \to \mathcal{V}(\mathbb{Z}_h^d)$$

is called a (linear) **finite difference (FD) operator** *if and only if it has the form*

$$(\mathcal{F}_h v)(\mathbf{x}) = \sum_{\mathrm{e} \in S} a_{\mathrm{e}}(\mathbf{x}, h)(\mathcal{S}_{\mathrm{e}} v)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}_h^d,$$

where the set $S \subset \mathbb{Z}^d$ is such that $\# S < \infty$, $\mathbf{0} \in S$, and it is called the **stencil** *of the operator $\mathcal{F}_h$. The* **stencil size** *is*

$$\max\left\{ \|\mathbf{j}\|_{\ell^\infty} \mid \mathbf{j} \in S \right\}.$$

Finally, for $\mathbf{e} \in S$, we have $a_{\mathrm{e}} \colon \mathbb{Z}_h^d \times \mathbb{R}_+ \to \mathbb{R}$.

Remark (stencil)

The previous definition is sufficiently general for our purposes. However, some sources allow the stencil to depend on $\mathbf{x}$, *i.e.,*

$$(\mathcal{F}_h v)(\mathbf{x}) = \sum_{\mathrm{e} \in S(\mathrm{x})} a_{\mathrm{e}}(\mathbf{x}, h)(\mathcal{S}_{\mathrm{e}} v)(\mathbf{x}),$$

where, for each $\mathbf{x} \in \mathbb{Z}_h^d$, *the set* $S(\mathbf{x})$ *is a stencil according to our definition. We will not consider such operators.*

Example

The *forward difference* operator is defined as

$$\delta_h v(x) = \frac{v(x+h) - v(x)}{h} = \frac{v_{i+1} - v_i}{h},$$

where we assumed that $x = ih$. Its stencil is $\{0, 1\}$.

Example

The *backward difference* operator is defined as

$$\bar{\delta}_h v(x) = \frac{v(x) - v(x-h)}{h} = \frac{v_i - v_{i-1}}{h},$$

where we assumed that $x = ih$. Its stencil is $\{-1, 0\}$.

Example

The *centered difference* operator is defined as

$$\mathring{\delta}_h v(x) = \frac{v(x+h) - v(x-h)}{2h} = \frac{v_{i+1} - v_{i-1}}{2h},$$

where we assumed that $x = ih$. Its stencil is $\{-1, 0, 1\}$.

Example

The one-dimensional *discrete Laplace* operator is defined as

$$\Delta_h v(x) = \bar{\delta}_h \delta_h v(x) = \frac{v(x+h) - 2v(x) + v(x-h)}{h^2} = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2},$$

where we assumed that $x = ih$. Its stencil is $\{-1, 0, 1\}$.

Example

The two-dimensional discrete Laplace operator is defined, for $(i,j) \in \mathbb{Z}^2$, as

$$\Delta_h v_{i,j} = \frac{v_{i-1,j} + v_{i+1,j} + v_{i,j-1} + v_{i,j+1} - 4v_{i,j}}{h^2}.$$

Its stencil is the *five points* $\{(0,0),(0,\pm 1),(\pm 1,0)\}$.

Example

The discrete mixed derivative operator is defined, for $(i,j) \in \mathbb{Z}^2$, as

$$\delta_h^{\diamondsuit} v_{i,j} = \frac{v_{i-1,j-1} + v_{i+1,j+1} - v_{i+1,j-1} + v_{i-1,j-1}}{4h^2}.$$

Its stencil is the *five points* $\{(0,0),(\pm 1,\pm 1)\}$.

Example

The two-dimensional skew Laplacian operator is defined, for $(i, j) \in \mathbb{Z}^2$, as

$$\Delta_h^{\square} v_{i,j} = \frac{v_{i+1,j+1} + v_{i+1,j-1} + v_{i-1,j-1} + v_{i-1,j+1} - 4v_{i,j}}{2h^2}.$$

Its stencil is the *five points* $\{(0,0), (\pm 1, \pm 1)\}$.

Proposition (properties of difference operators)

Let $d = 1$ and consider the difference operators of the previous examples. For any $v, v_1, v_2 \in \mathcal{V}(\mathbb{Z}_h)$, we have the following identities:

1. *Product rule I:*

$$\delta_h(v_1 v_2)(x) = \delta_h v_1(x) v_2(x) + v_1(x+h)\bar{\delta}_h v_2(x+h).$$

2. *Product rule II:*

$$\bar{\delta}_h(v_1 v_2)(x) = \bar{\delta}_h v_1(x) v_2(x) + v_1(x-h)\delta_h v_2(x-h).$$

3. *Abel transformation:*

$$h\sum_{k=0}^{N-1} \delta_h v_1(kh) v_2(kh) = v_1(Nh)v_2(Nh) - v_1(0)v_2(0) - h\sum_{k=1}^{N} v_1(kh)\bar{\delta}_h v_2(kh).$$

4. *Symmetry:*

$$\Delta_h v(x) = \delta_h \bar{\delta}_h v(x).$$

Definition (interior and boundary grid points)

Let $\mathcal{G}_h \subseteq \mathbb{Z}_h^d$ be a grid domain and $\mathcal{F}_h$ be a finite difference operator with stencil S. The set of points

$$\mathcal{G}_h^I = \{\mathbf{x} \in \mathcal{G}_h \mid \mathbf{x} + \mathbf{s} \in \mathcal{G}_h, \ \forall\, \mathbf{s} \in S\}$$

is called the set of **interior grid points** *with respect to the operator $\mathcal{F}_h$. Similarly,*

$$\mathcal{G}_h^B = \mathcal{G}_h \backslash \mathcal{G}_h^I$$

is the set of **boundary grid points** *with respect to the operator $\mathcal{F}_h$.*

Example

The stencil of the forward difference operator, $\delta_h v(x)$, is $\{0, 1\}$. Thus,

$$\bar{\Omega}_h^I = [0, 1) \cap \mathbb{Z}_h = \{ih \mid i = 0, \ldots, N\}.$$

Example

The stencil of the backward difference operator, $\bar{\delta}_h v(x)$, is $\{-1, 0\}$. Thus,

$$\bar{\Omega}_h^I = (0, 1] \cap \mathbb{Z}_h = \{ih \mid i = 1, \ldots, N+1\}.$$

Example

The stencil of the centered difference operator, $\mathring{\delta}_h v(x)$, is $\{-1, 0, 1\}$. Thus,

$$\bar{\Omega}_h^I = \Omega_h.$$

Example

The stencil of the one-dimensional discrete Laplace operator, $\Delta_h v(x)$ is $\{-1, 0, 1\}$. Thus,

$$\bar{\Omega}_h^I = \Omega_h.$$

Example

The domains Ω_h^I and Ω_h^B for the two-dimensional discrete Laplace operator, , are illustrated in the following figure.

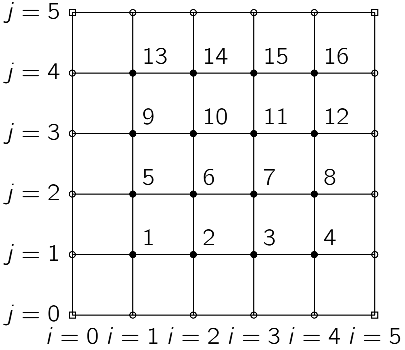

Figure: A uniform grid of size 5×5 and spacing $h = \frac{1}{5}$ covering $\Omega = (0,1)^2$. The discrete interior, Ω_h, is the collection of filled circles, and the discrete boundary, $\partial\Omega_h$, consists of the remaining ones. The space of grid functions, $\mathcal{V}(\bar{\Omega}_h)$, is the space of functions from all points in this grid into $\mathbb{R}$. The numbers next to the filled circles denote their lexicographical ordering, which is used to provide the isomorphism $\mathcal{V}(\bar{\Omega}_h) \longleftrightarrow \mathbb{R}^{16}$. The filled circles constitute the domain Ω_h^I for the two-dimensional discrete Laplacian, whereas the unfilled circles constitute Ω_h^B.

Definition (approximation property)

Let $d \in \mathbb{N}$ and $\Omega = (0,1)^d$. For each $N \in \mathbb{N}$, define $h = 1/(N+1)$ and let $\bar{\Omega}_h = \bar{\Omega} \cap \mathbb{Z}_h^d$ be the corresponding grid domain. Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ be a normed space of functions on Ω, i.e.,

$$\mathbb{V} = \left\{ v \colon \bar{\Omega} \to \mathbb{R} \;\middle|\; \|v\|_{\mathbb{V}} < \infty \right\}.$$

Assume that, for each $N \in \mathbb{N}$, we endow the space $\mathcal{V}(\bar{\Omega}_h)$ with a norm $\|\cdot\|_h$. We say that the family $\{(\mathcal{V}(\bar{\Omega}_h), \|\cdot\|_h)\}_{h>0}$ possesses the **approximation property** *if there is a (linear) operator, called the* **grid projection operator**, *such that*

$$\pi_h \colon \mathbb{V} \to \mathcal{V}(\bar{\Omega}_h), \qquad \lim_{h \downarrow 0} \|\pi_h v\|_h = \|v\|_{\mathbb{V}}, \quad \forall\, v \in \mathbb{V}.$$

Example

Let $d = 1$, $\Omega = (0, 1)$, and $\mathcal{V}(\bar{\Omega}_h)$ be defined as before. Let us define, for $p \in [1, \infty)$, the norms

$$\|v\|_{L_h^p} = \left[h \sum_{i=1}^{N} |v_i|^p + \frac{h}{2} \left(|v_0|^p + |v_{N+1}|^p \right) \right]^{1/p}, \quad v \in \mathcal{V}(\bar{\Omega}_h).$$

These have the approximation property with respect to the norms of $L^p(0, 1)$, $p \in (1, \infty)$. Indeed, if $v \in L^p(0, 1)$, we define

$$(\pi_h v)_i = \begin{cases} \dfrac{1}{h} \displaystyle\int_{(i-1/2)h}^{(i+1/2)h} v(x) \mathrm{d}x, & i \in \{1, \ldots, N\}, \\ \dfrac{2}{h} \displaystyle\int_0^{h/2} v(x) \mathrm{d}x, & i = 0, \\ \dfrac{2}{h} \displaystyle\int_{1-h/2}^{1} v(x) \mathrm{d}x, & i = N+1, \end{cases}$$

which we call the *averaging operator*.

Example (Cont.)

Now

$$\|\pi_h v\|_{L_h^p}^p = h\sum_{i=1}^{N} |\pi_h v_i|^p + \frac{h}{2}\left(|\pi_h v_0|^p + |\pi_h v_{N+1}|^p\right),$$

and we consider, separately, the interior and boundary indices.
For the interior indices, using Young's inequality and setting $p' = p/(p-1)$, we have

$$\begin{aligned} h\sum_{i=1}^{N} |\pi_h v_i|^p &= h^{1-p}\sum_{i=1}^{N}\left|\int_{(i-1/2)h}^{(i+1/2)h} v(x)\mathrm{d}x\right|^p \\ &\leq h^{1-p-p/p'}\sum_{i=1}^{N}\int_{(i-1/2)h}^{(i+1/2)h} |v(x)|^p\,\mathrm{d}x \\ &= \sum_{i=1}^{N}\int_{(i-1/2)h}^{(i+1/2)h} |v(x)|^p\,\mathrm{d}x \\ &= \int_{h/2}^{1-h/2} |v(x)|^p\,\mathrm{d}x. \end{aligned}$$

Example (Cont.)

Similarly,

$$\frac{h}{2}|\pi_h v_0|^p \leq \int_0^{h/2} |v(x)|^p \, \mathrm{d}x, \qquad \frac{h}{2}|\pi_h v_{N+1}|^p \leq \int_{1-h/2}^{1} |v(x)|^p \, \mathrm{d}x,$$

so that, in conclusion, we have, for any $v \in L^p(0,1)$,

$$\|\pi_h v\|_{L^p_h} \leq \|v\|_{L^p(0,1)} \quad \Longrightarrow \quad \limsup_{h \downarrow 0} \|\pi_h v\|_{L^p_h} \leq \|v\|_{L^p(0,1)}.$$

The reverse inequality is the content of the so-called *Lebesgue Differentiation Theorem* and so we have the approximation property.

Example

The previous example, with few modifications, shows that $\mathcal{V}(\bar{\Omega}_h)$ with $\|\cdot\|_{L^1_h}$ has the approximation property with respect to $L^1(0,1)$. We leave the details to the reader as an exercise.

Example

The case $p = \infty$ requires a little modification. As usual, we define, for $v \in \mathcal{V}(\bar{\Omega}_h)$,

$$\|v\|_{L^\infty_h} = \max_{x \in \bar{\Omega}_h} |v(x)|$$

and

$$(\pi_h v)_i = v(ih), \quad i = 0, \ldots, N+1,$$

which we call the *sampling operator*. This has the approximation property on $C([0,1])$ with the norm $\|\cdot\|_{L^\infty(0,1)}$. We leave the details to the reader as an exercise.

Remark (scaling)

One may wonder what, for $p \in [1,\infty)$, is the purpose of the factor h in the definitions of the L^p_h-norms. After all, we could use, as norms of $\mathcal{V}(\bar{\Omega}_h)$, any of the norms $\|\cdot\|_{\ell^p(\mathbb{R}^N)}$, for $p \in [1,\infty]$. It is precisely to attain the approximation property. For instance, set $p = 1$ and consider the constant function $v \equiv \alpha > 0$ on $(0,1)$. Then the averaging operator yields $\pi_h v_i = \alpha$ for all i. Consequently,

$$\|\pi_h v\|_{L^1_h} = h \sum_{i=1}^{N} v_i + h\alpha = \frac{N}{N+1}\alpha + \frac{1}{N+1}\alpha = \|v\|_{L^1(0,1)}.$$

If we were to use the $\ell^1(\mathbb{R}^N)$-norm, we would not attain approximation

$$\|\boldsymbol{\pi}_h \mathbf{v}\|_1 = \sum_{i=0}^{N+1} v_i = (N+2)\alpha \to \infty, \qquad N \to \infty.$$

Definition (discrete inner products)

Define, on $\mathcal{V}(\bar{\Omega}_h)$, the bilinear forms

$$[v^{(1)}, v^{(2)}]_{L^2_h} = \frac{h}{2}\left(v_0^{(1)} v_0^{(2)} + v_{N+1}^{(1)} v_{N+1}^{(2)}\right) + h\sum_{i=1}^{N} v_i^{(1)} v_i^{(2)},$$

$$(v^{(1)}, v^{(2)})_{L^2_h} = h\sum_{i=1}^{N} v_i^{(1)} v_i^{(2)},$$

where $v^{(1)}, v^{(2)} \in \mathcal{V}(\bar{\Omega}_h)$.

Proposition (inner products)

The expression $[\cdot,\cdot]_{L_h^2}$*, introduced on the last slide, is an inner product on* $\mathcal{V}(\bar{\Omega}_h)$ *and it induces the* L_h^2*-norm. Similarly, the expression* $(\cdot,\cdot)_{L_h^2}$*, also introduced on the previous slide, is an inner product on* $\mathcal{V}_0(\bar{\Omega}_h)$ *and it induces the* L_h^2*-norm. In addition, we have*

$$(\delta_h v_1, v_2)_{L_h^2} = -\left(v_1, \bar{\delta}_h v_2\right)_{L_h^2}, \quad \forall\, v_1, v_2 \in \mathcal{V}_0(\bar{\Omega}_h); \tag{1}$$

as a consequence,

$$(-\Delta_h v_1, v_2)_{L_h^2} = \left(\bar{\delta}_h v_1, \bar{\delta}_h v_2\right)_{L_h^2} = (\delta_h v_1, \delta_h v_2)_{L_h^2}, \quad \forall\, v_1, v_2 \in \mathcal{V}_0(\bar{\Omega}_h).$$

Proof.

This is an exercise. □

Consistency and Stability of Finite Difference Methods

Generic Elliptic Equation

We assume that we have at hand the following problem. Let $\mathbb{V}$ be some normed space of functions defined on $\bar{\Omega} = [0,1]^d$, $d \in \mathbb{N}$, with norm $\|\cdot\|_{\mathbb{V}}$. We need to find $u \in \mathbb{V}$, such that

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ \ell u = g, & \text{in } \partial\Omega, \end{cases} \tag{2}$$

where L and ℓ are some differential operators. The operator L encodes the differential equation, whereas ℓ represents the boundary conditions. Here, the functions $f\colon \Omega \to \mathbb{R}$ and $g\colon \partial\Omega \to \mathbb{R}$ are assumed to be given and belong to some normed spaces $f \in \mathbb{F}$ and $g \in \mathbb{G}$, with norms $\|\cdot\|_{\mathbb{F}}$ and $\|\cdot\|_{\mathbb{G}}$, respectively.

Generic Finite Difference Approximation

Problem (2) is replaced by the finite difference problem: Find $w \in \mathcal{V}(\bar{\Omega}_h)$ such that

$$\begin{cases} L_h w = f_h, & \text{in } \Omega_h^I, \\ \ell_h w = g_h, & \text{in } \Omega_h^B. \end{cases} \tag{3}$$

Here, L_h and ℓ_h are finite difference operators; Ω_h^I and Ω_h^B are the interior and boundary grid points, respectively, with respect to the operator L_h; and $f_h \in \mathcal{V}(\Omega_h^I)$ and $g_h \in \mathcal{V}(\Omega_h^B)$ are assumed to be given.

Definition (stability)

Let $\{(\mathcal{V}(\bar{\Omega}_h), \|\cdot\|_h)\}_{h>0}$ *have the approximation property with respect to* $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$. *We say that the FDM* (3) *is* **stable** *if and only if there are constants* $h_0 > 0$ *and* $C > 0$ *such that, for all* $h \in (0, h_0]$, *problem* (3) *has a unique solution for any pair* $(f_h, g_h) \in \mathcal{V}(\Omega_h^I) \times \mathcal{V}(\Omega_h^B)$; *additionally, for* $h \in (0, h_0]$, *we have*

$$\|w\|_h \leq C \left(\|f_h\|_{\mathcal{V}(\Omega_h^I)} + \|g_h\|_{\mathcal{V}(\Omega_h^B)} \right),$$

where $\|\cdot\|_{\mathcal{V}(\Omega_h^I)}$ *and* $\|\cdot\|_{\mathcal{V}(\Omega_h^B)}$ *are norms that have the approximation property with respect to* $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ *and* $(\mathbb{G}, \|\cdot\|_{\mathbb{G}})$, *respectively.*

Definition (operator consistency)

Let $\{(\mathcal{V}(\mathbb{Z}_h^d), \|\cdot\|_h)\}_{h>0}$ have the approximation property with respect to $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$. Let $L\colon \mathbb{V} \to \mathbb{V}$ be a linear, not necessarily bounded, operator and $L_h\colon \mathcal{V}(\mathbb{Z}_h^d) \to \mathcal{V}(\mathbb{Z}_h^d)$ be a finite difference operator. We define the **consistency error** *at $v \in \mathbb{V}$ to be*

$$\mathcal{E}_h[L, v](\mathbf{x}) = \left(L_h \pi_h v - \pi_h(Lv)\right)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}_h^d.$$

We say that the finite difference operator L_h is **consistent to at least order** *$p \in \mathbb{N}$ with L if and only if there is $k \in \mathbb{N}$, usually $k > p$, such that there is a constant $h_1 > 0$ such that, whenever $h \in (0, h_1]$ and $v \in C^k(\mathbb{R}^d)$, there is $C > 0$, possibly depending on v, for which*

$$\|\mathcal{E}_h[L, v]\|_h \leq Ch^p.$$

Finally, we say that the operator is **consistent with exactly order** *p if there is at least one v for which the previous estimate cannot be improved upon, i.e., the value of p cannot be increased, by assuming that v is smoother.*

Definition (consistency)

Let $\{(\mathcal{V}(\bar{\Omega}_h), \|\cdot\|_h)\}_{h>0}$ have the approximation property with respect to $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$. We define the **consistency error** *of (3) to be the function $\mathcal{E}_h \in \mathcal{V}(\bar{\Omega}_h)$, defined as*

$$\mathcal{E}_h[u](\mathbf{x}) = \begin{cases} (L_h \pi_h u - f_h)(\mathbf{x}), & \mathbf{x} \in \Omega_h^I, \\ (\ell_h \pi_h u - g_h)(\mathbf{x}), & \mathbf{x} \in \Omega_h^B, \end{cases}$$

where $u \in \mathbb{V}$ solves (2). We say that the FDM (3) is **consistent to at least order** *$p \in \mathbb{N}$ with (2) if and only if there is $k \in \mathbb{N}$, usually $k > p$, such that if $u \in C^k(\bar{\Omega})$, then there are constants $h_1 > 0$ and $C > 0$ such that, whenever $h \in (0, h_1]$, we have*

$$\|\mathcal{E}_h[u]\|_{\mathcal{V}(\Omega_h^I)} + \|\mathcal{E}_h[u]\|_{\mathcal{V}(\Omega_h^B)} \leq Ch^p.$$

Finally, we say that the method is **consistent with exactly order** *p if the previous estimate cannot be improved upon, i.e., the value of p cannot be increased, by assuming that u is smoother.*

Remark (relation between notions)

Notice that we are calling consistency error two seemingly unrelated concepts. To show how they are related, let us assume, for the sake of illustration, that $f_h = \pi_h f$. *Then, for all points* $\mathbf{x} \in \Omega_h^I$, *we have*

$$
\begin{aligned}
(L_h \pi_h u - f_h)(\mathbf{x}) &= (L_h \pi_h u - \pi_h(Lu))(\mathbf{x}) + \pi_h(Lu)(\mathbf{x}) - f_h(\mathbf{x}) \\
&= \mathcal{E}_h[L, u](\mathbf{x}) + \pi_h(Lu - f)(\mathbf{x}) \\
&= \mathcal{E}_h[L, u](\mathbf{x}).
\end{aligned}
$$

In other words, when $f_h = \pi_h f$, *the consistency error of the method and the operator coincide (at interior points). In general, we see that the consistency error of a method has two components: the operator consistency error,* $\mathcal{E}_h[L, u](\mathbf{x})$, *and the grid projection error,* $\pi_h(Lu)(\mathbf{x}) - f_h(\mathbf{x})$. *A similar consideration can be made for the boundary points and the operator* ℓ.

Proposition (consistency)

Let $d = 1$. We have:

1. *The forward difference operator is consistent, on $C_b(\mathbb{R})$, to exactly order one with*
$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \quad x \in \mathbb{R}.$$
2. *The backward difference operator is consistent, on $C_b(\mathbb{R})$, to exactly order one with*
$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \quad x \in \mathbb{R}.$$
3. *The centered difference operator is consistent, on $C_b(\mathbb{R})$, to exactly order two with*
$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \quad x \in \mathbb{R}.$$

Proposition (consistency)

4. *The one-dimensional discrete Laplacian operator is consistent, on $C_b(\mathbb{R})$, to exactly order two with*

$$\Delta w(x) = \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \quad x \in \mathbb{R}.$$

As a consequence, if $P \in \mathbb{P}_1$,

$$\Delta_h P(x) = 0, \quad \forall\, x \in \Omega_h.$$

5. *The operator*

$$(\mathrm{BDF}_2\, w)_i = \frac{1}{2h}\left(3w_i - 4w_{i-1} + w_{i-2}\right)$$

is consistent, on $C_b(\mathbb{R})$, to exactly order two with

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \quad x \in \mathbb{R}.$$

Proof.

We will prove (4). Suppose that $w \in C^4([0,1];\mathbb{R})$ and $N \in \mathbb{N}$. Let $h = \frac{1}{N+1}$ and $p_i = i \cdot h$. As usual, we write $w_i = w(p_i)$. Then

$$\Delta_h w(p_i) = \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}.$$

By Taylor's Theorem,

$$w_{i\pm1} = w_i + w'(p_i)(\pm h) + \frac{1}{2}w''(p_i)h^2 + \frac{1}{6}w'''(p_i)(\pm h)^3 + \frac{1}{24}w^{(4)}(\eta_{i,\pm})h^4,$$

for some points $\eta_{i,\pm}$ in (p_{i-1}, p_{i+1}). Thus,

$$\begin{aligned}\Delta_h w(p_i) &= \frac{w''(p_i)h^2 + \frac{h^4}{24}\left(w^{(4)}(\eta_{i,+}) + w^{(4)}(\eta_{i,-})\right)}{h^2} \\ &= w''(p_i) + \frac{h^2}{24}\left(w^{(4)}(\eta_{i,+}) + w^{(4)}(\eta_{i,-}).\right)\end{aligned}$$

It follows that

$$\left|w''(p_i) - \Delta_h w(p_i)\right| \le Ch^2,$$

for some $C > 0$, independent of i and h, for all $1 \le i \le N$. $\square$

Definition (convergence)

Let $\{(\mathcal{V}(\bar{\Omega}_h), \|\cdot\|_h)\}_{h>0}$ *have the approximation property with respect to* $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$. *Let* $u \in \mathbb{V}$ *and* $w \in \mathcal{V}(\bar{\Omega}_h)$ *solve* (2) *and* (3), *respectively. The* **error** *of the method is the function* $e \in \mathcal{V}(\bar{\Omega}_h)$, *defined by*

$$e = \pi_h u - w.$$

We say that the FDM (3) *is* **convergent** *if and only if*

$$\|e\|_h \to 0, \qquad h \downarrow 0.$$

We say that the method is **convergent with rate** $p \in \mathbb{N}$ *if there is* $C > 0$ *such that, for sufficiently small* $h > 0$, *we have*

$$\|e\|_h \leq Ch^p.$$

Theorem (Lax)

Let problem (3) be stable and consistent. Then it is convergent. In addition, if (3) is consistent to order $p \in \mathbb{N}$, then the approximation is convergent with rate p.

Proof.

This is nothing but an exercise in notation. By definition, we have

$$\begin{cases} L_h \pi_h u(\mathbf{x}) = f_h(\mathbf{x}) + \mathcal{E}_h[u](\mathbf{x}), & \mathbf{x} \in \Omega_h^I, \\ \ell_h \pi_h u(\mathbf{x}) = g_h(\mathbf{x}) + \mathcal{E}_h[u](\mathbf{x}), & \mathbf{x} \in \Omega_h^B. \end{cases}$$

By linearity, we have

$$\begin{cases} L_h(\pi_h u - w)(\mathbf{x}) = \mathcal{E}_h[u](\mathbf{x}), & \mathbf{x} \in \Omega_h^I, \\ \ell_h(\pi_h u - w)(\mathbf{x}) = \mathcal{E}_h[u](\mathbf{x}), & \mathbf{x} \in \Omega_h^B. \end{cases}$$

Cont.

Stability then implies that

$$\|\pi_h u - w\|_h \leq C \left(\|\mathcal{E}_h[u]\|_{\mathcal{V}(\Omega_h^I)} + \|\mathcal{E}_h[u]\|_{\mathcal{V}(\Omega_h^B)} \right) .$$

Thus, convergence, even with the prescribed rate, follows from consistency. □

The Poisson Problem in One Dimension

The Poisson Problem in One Dimension

Let $\Omega = (0,1)$. We wish to approximate the solution to

$$-\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x), \ x \in \Omega, \qquad u(0) = u_L, \quad u(1) = u_R, \tag{4}$$

where $u_L, u_R \in \mathbb{R}$ and $f \in C(\bar{\Omega})$.

Definition (finite differences)

Suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a classical solution to (4). Let $N \in \mathbb{N}$. We call $\tilde{w} \in \mathcal{V}(\bar{\Omega}_h)$ a **finite difference approximation** *to (4) if and only if*

$$-\Delta_h \tilde{w}(x) = f_h(x), \quad x \in \Omega_h, \qquad \tilde{w}(0) = u_L, \ \tilde{w}(1) = u_R, \tag{5}$$

where $f_h \in \mathcal{V}(\Omega_h)$ is defined as $f_h = \pi_h f$ and π_h is the sampling operator.

Removing Inhomogeneous Boundary Conditions

One of our first concerns should be to determine whether our finite difference approximation is well defined. We can get rid of the Dirichlet boundary conditions via a change of variables. Indeed, let $P \in \mathcal{V}(\bar{\Omega}_h)$ be given by

$$P_i = u_L + (u_R - u_L)ih \qquad \Longrightarrow \qquad P_0 = u_L, \; P_{N+1} = u_R.$$

But, we have $\Delta_h P = 0$. Let us define $w = \tilde{w} - P$. Then $w \in \mathcal{V}_0(\bar{\Omega}_h)$ and it solves

$$-\Delta_h w(x) = f_h(x), \quad x \in \Omega_h, \qquad w(0) = 0, \; w(1) = 0, \tag{6}$$

if and only if $\tilde{w}$ solves (5).

An Equivalent Matrix Problem

Now, to find w, we use the standard identification to realize that problem (6) can be equivalently rewritten as

$$\mathsf{A}\mathbf{w} = h^2\mathbf{f},$$

where $\mathbf{w} \in \mathbb{R}^N \longleftrightarrow w \in \mathcal{V}_0(\bar{\Omega}_h)$, $\mathbf{f} \in \mathbb{R}^N \longleftrightarrow f_h \in \mathcal{V}(\Omega_h)$, and the matrix $\mathsf{A} \in \mathbb{R}^{N\times N}$ is the so-called *stiffness matrix*, which is defined as

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}. \tag{7}$$

Proposition (consistency)

Let $u \in C^4(\Omega)$ be a classical solution of (4) with $u_L = u_R = 0$. Let $w \in \mathcal{V}_0(\bar{\Omega}_h)$ be its finite difference approximation defined by (6). This method is consistent, in $C(\bar{\Omega})$, to exactly order $p = 2$.

Proof.

Clearly, $\mathcal{E}_h[u]_0 = \mathcal{E}_h[u]_{N+1} = 0$, so we only need to study the consistency of approximating the differential equation. By definition of the consistency error, we have, on Ω_h,

$$-\Delta_h \pi_h u = f_h + \mathcal{E}_h[u].$$

But we have seen that our finite difference approximation is of exactly order $p = 2$, that is,

$$\|\mathcal{E}_h[u]\|_{L_h^\infty} \leq C_2 h^2, \qquad \|\boldsymbol{\mathcal{E}}_h[u]\|_\infty \leq C_2 h^2.$$

□

Theorem (spectrum of A)

Let $N \in \mathbb{N}$. Suppose that $\mathsf{A} \in \mathbb{R}^{N \times N}$ is the stiffness matrix defined in (7). Consider the set of grid functions $\{\varphi_k\}_{k=1}^N \subset \mathcal{V}_0(\bar{\Omega}_h)$, defined by

$$[\boldsymbol{\varphi}_k]_i = \sin(k\pi ih), \quad i = 1, \ldots, N.$$

1. *The set $\{\boldsymbol{\varphi}_k\}_{k=1}^N \subset \mathbb{R}^N$ with $\boldsymbol{\varphi}_k \longleftrightarrow \varphi_k$ is an orthogonal set of eigenvectors of the stiffness matrix A.*
2. *The eigenvalue λ_k corresponding to the eigenvector $\boldsymbol{\varphi}_k$ is given by*

$$\lambda_k = 2\left(1 - \cos(k\pi h)\right) = 4\sin^2\left(\frac{k\pi h}{2}\right).$$

   Since $0 < \lambda_k < 4$, for all $k = 1, \ldots, N$, the stiffness matrix A is symmetric positive definite (SPD) and is, therefore, invertible.

3. *There is a constant $C_1 > 0$, independent of h, such that, if $0 < h < \frac{1}{2}$,*

$$\left\|\mathsf{A}^{-1}\right\|_2 = \frac{1}{4\sin^2\left(\frac{h\pi}{2}\right)} \leq C_1 h^{-2}.$$

Theorem

4. *The (spectral) condition number of* A *satisfies the estimate*

$$\kappa_2(\mathsf{A}) = \|\mathsf{A}\|_2 \left\|\mathsf{A}^{-1}\right\|_2 \leq 4C_1 h^{-2}.$$

Proof.

The first two statements were proved in Chapter 3. Let us then focus on the norm and condition number estimates. The largest eigenvalue of A is λ_N and the smallest is λ_1. Since the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, it follows that

$$\left\|\mathsf{A}^{-1}\right\|_2 = \frac{1}{\lambda_1} = \frac{1}{4\sin^2\left(\frac{\pi h}{2}\right)} = \frac{1}{2\left(1 - \cos(\pi h)\right)}.$$

Now, by Taylor's Theorem, for any $x \in (0, \pi/2)$, there is $\eta \in (0, x)$ such that

$$1 - \cos(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}\cos(\eta) \geq \frac{x^2}{2} - \frac{x^4}{24}.$$

Proof Cont.

Consequently, if $0 < \pi h < \pi/2$, or, equivalently, $0 < h < 1/2$,

$$2h^{-2}(1 - \cos(\pi h)) \geq \pi^2 - \frac{\pi^4 h^2}{12}.$$

Since

$$\frac{\sqrt{6}}{\pi} > \frac{1}{2} > h > 0,$$

it follows that

$$2h^{-2}(1 - \cos(\pi h)) \geq \pi^2 - \frac{\pi^4 h^2}{12} \geq \frac{\pi^2}{2}.$$

Equivalently,

$$\frac{2}{\pi^2 h^2} \geq \frac{1}{2(1 - \cos(\pi h))} = \frac{1}{\lambda_1}$$

and the norm estimate proof is completed upon taking $C_1 = \frac{2}{\pi^2}$. The condition number estimate follows immediately from

$$\kappa_2(\mathsf{A}) = \frac{\lambda_N}{\lambda_1} \leq 4 C_1 h^{-2}. \qquad \square$$

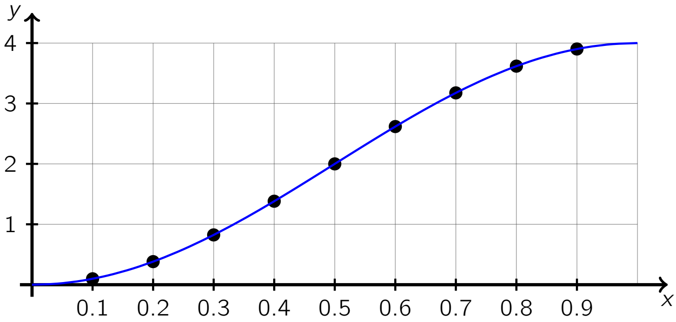

Figure: The eigenvalues of the stiffness matrix A are shown (filled circles) for $N = 9$. The horizontal axis is $x = kh$. The solid curve is the plot of $4\sin^2\left(\frac{\pi x}{2}\right)$. Observe that $0 < \lambda_k < 4$.

Corollary (stability)

For every $N \in \mathbb{N}$, the finite difference problem (6) *has a unique solution $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^N$, which, moreover, satisfies the estimate*

$$\|w\|_{L_h^2} \leq C_1 \|f_h\|_{L_h^2}.$$

Proof.

For every $N \in \mathbb{N}$, the matrix A is invertible; thus, there is a unique $\mathbf{w}$. Moreover, this satisfies

$$\|\mathbf{w}\|_2 \leq h^2 \left\|\mathsf{A}^{-1}\right\|_2 \|\mathbf{f}\|_2 \leq C_1 \|\mathbf{f}\|_2,$$

where we used the estimate on the norm of the inverse. Since $\|w\|_{L_h^2} = h^{1/2} \|\mathbf{w}\|_2$, the result follows. □

Remark (consistency)

In one dimension, $\{\tilde{\varphi}_k, \mu_k\}_{k\in\mathbb{N}}$ is the family of eigenpairs of the Laplacian:

$$-\tilde{\varphi}_k''(x) = \mu_k \varphi_k(x), \quad x \in (0,1), \qquad \tilde{\varphi}_k(0) = \tilde{\varphi}_k(1) = 0,$$

where

$$\tilde{\varphi}_k(x) = \sin(k\pi x), \quad \mu_k = k^2\pi^2.$$

The theorem shows that the matrix $h^{-2}\mathsf{A}$ has eigenvectors

$$\varphi_k = \pi_h \tilde{\varphi}_k,$$

where π_h is the sampling operator, and eigenvalues

$$\frac{1}{h^2}\lambda_k^2 = 4(N+1)^2 \sin^2\left(\frac{k\pi}{2(N+1)}\right)^2.$$

For $N \uparrow \infty$ $(h \downarrow 0)$, we have

$$\frac{1}{h^2}\lambda_k^2 = 4(N+1)^2\left(\frac{k^2\pi^2}{4(N+1)^2} + \mathcal{O}((N+1)^{-4})\right) = \mu_k + \mathcal{O}(h^2).$$

This shows that this method is spectrally consistent.

Theorem (convergence)

Let $N \in \mathbb{N}$, $h = \frac{1}{N+1}$, $f \in C(\Omega)$, and $u \in C^4(\bar{\Omega})$ be a classical solution to the one-dimensional Poisson problem (4) with $u_L = u_R = 0$. Suppose that $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^N$ is the solution to the finite difference problem (6). Let $e \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{e} \in \mathbb{R}^N$ be its error. Then there is a constant $C_2 > 0$, independent of h, such that, if $0 < h < \frac{1}{2}$,

$$\|e\|_{L_h^2} \leq C_1 C_2 h^2,$$

where $C_1 > 0$ is the constant from before.

Proof.

We know that

$$\|\mathcal{E}_h[u]\|_{L_h^\infty} \leq C_2 h^2, \qquad \|\boldsymbol{\mathcal{E}}_h[u]\|_\infty \leq C_2 h^2.$$

We now proceed in the usual way. By definition of the consistency error, we have, on Ω_h,

$$-\Delta_h \pi_h u = f_h + \mathcal{E}_h[u].$$

Proof Cont.

Subtracting from the method (6), we obtain an equation that controls the error

$$-\Delta_h e = \mathcal{E}_h[u] \qquad \longleftrightarrow \qquad \mathsf{A}\mathbf{e} = h^2 \boldsymbol{\mathcal{E}}_h[u].$$

Thus, using the stability theorem,

$$\|e\|_{L^2_h} = h^{1/2}\|\mathbf{e}\|_2 \leq h^{5/2}\left\|\mathsf{A}^{-1}\right\|_2 \|\boldsymbol{\mathcal{E}}_h[u]\|_2 \leq C_1\sqrt{hN}\|\boldsymbol{\mathcal{E}}_h[u]\|_\infty \leq C_1 C_2 h^2,$$

where, for $v \in \mathcal{V}_0(\bar{\Omega}_h)$,

$$\|v\|_{L^2_h} = \sqrt{h\sum_{i=1}^{N}|v_i|^2}.$$

$\square$

Remark (suboptimal estimate)

We can immediately get an error estimate in the L_h^∞-norm using our L_h^2-estimate:

$$\|e\|_{L_h^\infty} \leq \frac{1}{\sqrt{h}} \|e\|_{L_h^2} \leq C_1 C_2 h^{\frac{3}{2}}.$$

As we will see, this estimate is suboptimal.

Theorem (Discrete Maximum Principle)

Suppose that $v \in \mathcal{V}(\bar{\Omega}_h)$ *satisfies*

$$-\Delta_h v(x) \le 0, \quad \forall\, x \in \Omega_h.$$

Then, for all $x \in \bar{\Omega}_h$,

$$v(x) \le \max\{v(0), v(1)\}.$$

In other words, the maximum must occur on the boundary.

Proof.

To obtain a contradiction, suppose that a strict maximum occurs in the interior. If this is true, there is some $x_k \in \Omega_h$ such that

$$v_k = \max_{x_j \in \Omega_h} v_j > \max\{v_0, v_{N+1}\}.$$

Let us suppose, for simplicity, that $2 \le k \le N-1$. Then

$$0 \ge -h^2 \Delta_h v_k = 2v_k - v_{k-1} - v_{k+1} \ge 0.$$

Proof Cont.

This implies that $-\Delta_h v_k = 0$ and, therefore,

$$v_k = \frac{1}{2}(v_{k-1} + v_{k+1}).$$

The only way to satisfy the last equation and the fact that $v_k \geq v_{k\pm1}$ is to have $v_k = v_{k\pm1}$.

We can now repeat our argument at all neighboring points; we conclude that

$$v_1 = \cdots = v_k = \cdots = v_N.$$

Next to the left boundary point, we have, for example,

$$0 \geq -h^2 \Delta_h v_1 = 2v_1 - v_0 - v_2 > 0,$$

since $v_1 > v_0$. This is a contradiction. $\square$

Theorem (stability)

Suppose that $w \in \mathcal{V}(\bar{\Omega}_h)$ *and* $f_h \in \mathcal{V}(\Omega_h)$ *and* $g_0, g_1 \in \mathbb{R}$ *are such that*

$$-\Delta_h w = f_h, \quad \Omega_h, \qquad w_0 = g_0, \quad w_{N+1} = g_1.$$

There is some constant $C > 0$, *independent of* h *and* w, *such that*

$$\|w\|_{L_h^\infty} \leq \max_{j \in \{0,1\}} |g_j| + C\|f_h\|_{L_h^\infty}.$$

Proof.

Define the comparison function $\Phi\colon [0,1] \to \mathbb{R}$ via

$$\Phi(x) = \left(x - \frac{1}{2}\right)^2 \geq 0.$$

Define the grid function $\phi = \pi_h \Phi$, where π_h is the sampling operator. Then, in Ω_h, we have

$$-\Phi'' \equiv -2 \equiv -\Delta_h \phi.$$

Proof Cont.

Now define the grid functions

$$\psi_\pm = \pm w + \frac{\|f_h\|_{L_h^\infty}}{2}\phi.$$

Observe that, in Ω_h, we have

$$-\Delta_h \psi_\pm = \pm f_h - \|f_h\|_{L_h^\infty} \le 0.$$

By the Discrete Maximum Principle, in Ω_h,

$$\pm w \le \psi_\pm \le \max\{\psi_\pm(0), \psi_\pm(1)\} \le \max_{j\in\{0,1\}} |g_j| + \frac{\|f_h\|_{L_h^\infty}}{8}.$$

Putting the two results together, we get

$$\|w\|_{L_h^\infty} \le \max_{j\in\{0,1\}} |g_j| + \frac{1}{8}\|f_h\|_{L_h^\infty}.$$

□

Corollary (convergence)

Let $f \in C(\Omega)$ and $u \in C^4(\bar{\Omega})$ be a classical solution to the one-dimensional Poisson problem (4) with $u_L = u_R = 0$. Let $N \in \mathbb{N}$. Suppose that $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^N$ is the solution to the finite difference problem (6). Then there is a constant $C_3 > 0$, independent of h, such that

$$\|e\|_{L_h^\infty} \leq C_2 C_3 h^2,$$

where $C_2 > 0$ is the consistency error constant.

Proof.

We found that the error $e \in \mathcal{V}_0(\bar{\Omega}_h)$ satisfies, in Ω_h,

$$-\Delta_h e = \mathcal{E}_h[u].$$

By the previous stability result, we have

$$\|e\|_{L_h^\infty} \leq \frac{1}{8}\|\mathcal{E}_h[u]\|_{L_h^\infty} \leq \frac{C_2}{8} h^2.$$

The result follows with $C_3 = \frac{1}{8}$. □