Classical Numerical Analysis, Chapter 24

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Chapter 24, Part 2 of 2

Finite Difference Methods for Elliptic Problems

Elliptic Problems in One Dimension

General Elliptic Problems of Dirichlet Type

Let us consider here more general elliptic problems in one dimension and their finite difference approximation. We will focus on the Dirichlet problem

$$\begin{cases} Lu = f, & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $f \in C([0,1])$ and the operator L is a second-order elliptic operator to be specified below. We will be interested in constructing finite difference operators L_h that are consistent and have stencil $\{-1, 0, 1\}$, so that the finite difference problem will read: Find $w \in \mathcal{V}_0(\bar{\Omega}_h)$ such that

$$L_h w = f_h, \quad \text{in } \Omega_h, \qquad L_h w_i = -A_i w_{i-1} + C_i w_i - B_i w_{i+1}; \tag{2}$$

here, $f_h \in \mathcal{V}(\Omega_h)$ is, as usual, defined as $f_h(ih) = f(ih)$.

Divergence Form Operators

Here, we consider a differential operator in *divergence form*, i.e.,

$$Lu(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) + c(x)u(x), \tag{3}$$

where we assume that the coefficients satisfy $a \in C^1([0,1])$, $0 \le c \in C([0,1])$; in addition, there are constants $\lambda, \Lambda \in \mathbb{R}$ such that

$$0 < \lambda \le a(x) \le \Lambda, \quad \forall\, x \in [0,1].$$

In this setting, the operator is elliptic in the sense of what is defined in Chapter 23. Existence, uniqueness, and stability of the solution to (1) is discussed therein.

Finite Difference Method

We now wish to construct the difference method, i.e., find the coefficients A_i, B_i, C_i. We do so arguing from consistency considerations. Namely, we consider, for $v \in C^4([0,1])$ such that $v(0) = v(1) = 0$, the consistency error

$$\mathcal{E}_h[v] = L_h \pi_h v - \pi_h(Lv),$$

where π_h is the sampling operator, and require that it satisfies $\|\mathcal{E}_h[v]\|_{L_h^\infty} \leq Ch^2$.

We begin by introducing the following change of notation. Setting $\alpha_i = h^2 A_i$, $\beta_i = h^2 B_i$, and $\gamma_i = h^2 C_i$, we get

$$L_h v_i = -\frac{1}{h}\left[\beta_i \frac{v_{i+1} - v_i}{h} - \alpha_i \frac{v_i - v_{i-1}}{h}\right] + \kappa_i v_i = -\frac{1}{h}\left[\beta_i \delta_h v_i - \alpha_i \bar{\delta}_h v_i\right] + \kappa_i v_i,$$

where $\kappa_i = h^{-2}(\gamma_i - \beta_i - \alpha_i)$. Notice that, at least symbolically, the finite difference operator begins to resemble the divergence form operator L.

Consistency

Now, to achieve consistency, we must have

$$\mathcal{E}_h[v]_i = -\frac{1}{h}\left[\beta_i \delta_h v(x_i) - \alpha_i \bar{\delta}_h v(x_i)\right] + \kappa_i v(x_i) + (a(x_i) v'(x_i))' - c(x_i) v(x_i) = \mathcal{O}(h^2).$$

From Taylor expansions, we know that

$$\delta_h v(x_i) = v'(x_i) + \frac{1}{2} v''(x_i) h + \frac{1}{6} v'''(x_i) h^2 + \mathcal{O}(h^3),$$
$$\bar{\delta}_h v(x_i) = v'(x_i) - \frac{1}{2} v''(x_i) h + \frac{1}{6} v'''(x_i) h^2 + \mathcal{O}(h^3),$$

so that, substituting in $\mathcal{E}_h[v]_i$, we get

$$\mathcal{E}_h[v]_i = \left(a'(x_i) - \frac{\beta_i - \alpha_i}{h} \right) v'(x_i) + \left(a(x_i) - \frac{\alpha_i + \beta_i}{2} \right) v''(x_i)$$
$$- \frac{\beta_i - \alpha_i}{6} h v'''(x_i) + (\kappa_i - c(x_i)) v(x_i) + \mathcal{O}(h^2).$$

Consistency and Divergence Form

Thus, we require

$$\frac{\beta_i - \alpha_i}{h} = a'(x_i) + \mathcal{O}(h^2), \quad \frac{\alpha_i + \beta_i}{2} = a(x_i) + \mathcal{O}(h^2), \quad \kappa_i = c(x_i) + \mathcal{O}(h^2). \tag{4}$$

There are several ways this can be achieved. For instance,

$$\beta_i = a(x_i + h/2), \qquad \alpha_i = a(x_i - h/2), \qquad \kappa_i = c(x_i), \tag{5}$$

$$\beta_i = \frac{a(x_{i+1}) + a(x_i)}{2}, \qquad \alpha_i = \frac{a(x_i) + a(x_{i-1})}{2}, \qquad \kappa_i = c(x_i) \tag{6}$$

are possible choices. Let us write the final operator with the first choice

$$L_h v_i = -\frac{1}{h}\left(a_{i+1/2}\delta_h v_i - a_{i-1/2}\bar{\delta}_h v_i\right) + c_i w_i = -\delta_h(\hat{a}_i \bar{\delta}_h v_i) + c_i w_i, \tag{7}$$

where $a_{i\pm 1/2} = a(x_i \pm h/2)$, $c_i = c(x_i)$, and $\hat{a}_i = a_{i-1/2}$. Notice the resemblance to the divergence form differential operator L.

Definition (H^1_h-seminorm)

*The H^1_h-**seminorm**, on $\mathcal{V}(\bar{\Omega}_h)$, is defined as*

$$\|v\|^2_{H^1_h} = h \sum_{i=1}^{N+1} |\bar{\delta}_h v_i|^2.$$

Notice that, indeed, this is not a norm, but only a seminorm. A grid function that takes constant, nonzero values satisfies $\|v\|_{H^1_h} = 0$. However, it turns out that on $\mathcal{V}_0(\bar{\Omega}_h)$ this is a norm.

Theorem (discrete Poincaré)

There is a constant, independent of $h > 0$, such that, for all $v \in \mathcal{V}_0(\bar{\Omega}_h)$, we have

$$\|v\|_{L^2_h} \leq C\|v\|_{H^1_h}.$$

Consequently, the quantity $\|\cdot\|_{H^1_h}$ is a norm on $\mathcal{V}_0(\bar{\Omega}_h)$.

Proof.

Homework exercise. $\square$

Theorem (stability)

There is a constant $C > 0$ that depends only on the coefficients a and c such that any solution to (2) with the operator defined as in (7) satisfies

$$\|w\|_{H_h^1} \leq C\|f_h\|_{L_h^2}.$$

As a consequence, the solution to this problem is unique and convergent with order $p = 2$ in the H_h^1-norm.

Proof.

Since the FDM is a square system of linear equations, the estimate implies uniqueness, and this in turn implies existence.

Let us now show the estimate. We can take the L_h^2-inner product of the method with w itself to obtain

$$-\left(\delta_h(\hat{a}\bar{\delta}_h w), w\right)_{L_h^2} + (cw, w)_{L_h^2} = (f_h, w)_{L_h^2} \leq \|f_h\|_{L_h^2}\|w\|_{L_h^2}.$$

Proof (Cont.)

Since, by assumption, $c \geq 0$, this inequality reduces to

$$-\left(\delta_h(\hat{a}\bar{\delta}_h w), w\right)_{L^2_h} \leq \|f_h\|_{L^2_h}\|w\|_{L^2_h}.$$

We now invoke the Abel transformation, a previous proposition in this chapter, to obtain, since $w \in \mathcal{V}_0(\bar{\Omega}_h)$,

$$-\left(\delta_h(\hat{a}\bar{\delta}_h w), w\right)_{L^2_h} = h\sum_{i=1}^{N} \hat{a}_i|\bar{\delta}_h w_i|^2 \geq \lambda\|w\|^2_{H^1_h},$$

where we used that $a_i = a(x_i - h/2) \geq \lambda$.

Finally, applying the discrete Poincaré inequality, and Young's inequality, we conclude that

$$\lambda\|w\|^2_{H^1_h} \leq C\|f_h\|_{L^2_h}\|w\|_{H^1_h} \leq \frac{C^2}{2\lambda}\|f_h\|^2_{L^2_h} + \frac{\lambda}{2}\|w\|^2_{H^1_h},$$

as we intended to show. $\square$

The Poisson Problem in Two Dimensions

The Poisson Problem in Two Dimensions

In this section, we introduce the two-dimensional Poisson problem on, for simplicity, a square domain $\Omega = (0,1)^2$. Recall that, for $v \in C^2(\Omega)$,

$$\Delta v(x_1, x_2) = \frac{\partial^2 v(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2 v(x_1, x_2)}{\partial x_2^2}.$$

Thus, we are trying to approximate the solution to

$$-\Delta u(x_1, x_2) = f(x_1, x_2), \quad (x_1, x_2) \in \Omega, \qquad u_{|\partial\Omega} = 0, \tag{8}$$

where $f \in C(\bar{\Omega})$ is given. The theory in Chapter 23 can be used to establish existence and uniqueness of a classical solution.

This is the object that we will try to approximate via finite differences.

Definition (finite difference approximation)

Let $d = 2$, $f \in C(\bar{\Omega})$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution to the two-dimensional Poisson problem (8). Let $N \in \mathbb{N}$ and $h = \frac{1}{N+1}$. We call $w \in \mathcal{V}_0(\bar{\Omega}_h)$ a **finite difference approximation** *to u if and only if*

$$-\Delta_h w_{i,j} = f_{i,j}, \quad (ih, jh) \in \Omega_h, \tag{9}$$

where $f_{i,j} = f(ih, jh)$ and Δ_h denotes the two-dimensional discrete Laplace operator

$$\Delta_h w_{i,j} = \Delta_h^x w_{i,j} + \Delta_h^y w_{i,j}.$$

Definition (discrete L_h^p-norms)

Let $d = 2$ and $p \in [1, \infty)$. The L_h^p**-norm on** $\mathcal{V}_0(\bar{\Omega}_h)$ **or** $\mathcal{V}(\Omega_h)$ *is*

$$\|v\|_{L_h^p} = \left(h^2 \sum_{i,j=1}^{N} |v_{i,j}|^p \right)^{1/p} .$$

For $p = 2$, this norm comes from the L_h^2**-inner product**

$$(v, \phi)_{L_h^2} = h^2 \sum_{i,j=1}^{N} v_{i,j}\phi_{i,j}.$$

The L_h^∞**-norm** *on these spaces is*

$$\|v\|_{L_h^\infty} = \max_{i,j=1,\ldots,N} |v_{i,j}| .$$

Proposition (consistency)

The two-dimensional discrete Laplace operator is consistent, in $C_b(\mathbb{R}^2)$, to order exactly two with the Laplacian.

Proof.

An exercise. $\square$

Theorem (stiffness matrix)

Let $N \in \mathbb{N}$. Define $\mathsf{A}_N \in \mathbb{R}^{N\times N}$ via

$$\mathsf{A}_N = \begin{bmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & 4 & \ddots & & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & & -1 & 4 & -1 \\ 0 & \dots & 0 & -1 & 4 \end{bmatrix}.$$

Let $\mathsf{O}_N, \mathsf{I}_N \in \mathbb{R}^{N\times N}$ denote the zero and identity matrices, respectively. Define the matrix $\mathsf{A} \in \mathbb{R}^{N^2\times N^2}$ via

$$\mathsf{A} = \begin{bmatrix} \mathsf{A}_N & -\mathsf{I}_N & \mathsf{O}_N & \dots & \mathsf{O}_N \\ -\mathsf{I}_N & \mathsf{A}_N & \ddots & & \vdots \\ \mathsf{O}_N & \ddots & \ddots & -\mathsf{I}_N & \mathsf{O}_N \\ \vdots & & -\mathsf{I}_N & \mathsf{A}_N & -\mathsf{I}_N \\ \mathsf{O}_N & \dots & \mathsf{O}_N & -\mathsf{I}_N & \mathsf{A}_N \end{bmatrix}. \tag{10}$$

Theorem (Cont.)

The grid function $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^{N^2}$ *is a solution to the finite difference problem* (9) *if and only if it is a solution to the problem*

$$\mathsf{A}\mathbf{w} = h^2\mathbf{f}, \tag{11}$$

with $f \in \mathcal{V}(\Omega_h) \longleftrightarrow \mathbf{f} \in \mathbb{R}^{N^2}$. *By linearity, the error* $e \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{e} \in \mathbb{R}^{N^2}$ *and the consistency error* $\mathcal{E}_h[u] \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \boldsymbol{\mathcal{E}}_h[u] \in \mathbb{R}^{N^2}$ *are related by*

$$\mathsf{A}\mathbf{e} = h^2\boldsymbol{\mathcal{E}}_h[u]. \tag{12}$$

Proof.

An exercise in manipulation. □

Theorem (spectrum of A)

Let $N \in \mathbb{N}$*. Suppose that* $\mathsf{A} \in \mathbb{R}^{N^2 \times N^2}$ *is the stiffness matrix defined in* (10)*. Consider the set of vectors*

$$S = \left\{ \boldsymbol{\varphi}_{k+(n-1)N} \;\middle|\; (kh, nh) \in \Omega_h \right\},$$

where the components of $\boldsymbol{\varphi}_{k+(n-1)N}$*, for* $(ih, jh) \in \Omega_h$*, are*

$$\left[\boldsymbol{\varphi}_{k+(n-1)N}\right]_{i+(j-1)N} = \varphi_{k+(n-1)N,i+(j-1)N} = \sin(k\pi ih)\sin(n\pi jh).$$

Then we have:

1. S *is an orthogonal set of eigenvectors of* A*.*
2. *The eigenvalue* $\lambda_{k+(n-1)N}$ *corresponding to the eigenvector* $\boldsymbol{\varphi}_{k+(n-1)N}$ *is given by*

$$\begin{aligned}\lambda_{k+(n-1)N} &= 2\left(2 - \cos(k\pi h) - \cos(n\pi h)\right)\\ &= 4\sin^2\left(\frac{k\pi h}{2}\right) + 4\sin^2\left(\frac{n\pi h}{2}\right).\end{aligned}$$

   Therefore, $0 < \lambda_{k+(n-1)N} < 8$*, for all* $(kh, nh) \in \Omega_h$*; consequently,* A *is an SPD matrix.*

Theorem (Cont.)

3. *There is a constant* $C_1 > 0$*, independent of* h*, such that, if* $0 < h < \frac{1}{2}$*,*

$$\left\|\mathsf{A}^{-1}\right\|_2 = \frac{1}{8\sin^2\left(\frac{h\pi}{2}\right)} \le C_1 h^{-2}.$$

4. *The spectral condition number of* A *satisfies the estimate*

$$\kappa_2(\mathsf{A}) = \|\mathsf{A}\|_2 \left\|\mathsf{A}^{-1}\right\|_2 \le 8C_1 h^{-2}.$$

Proof.

Homework exercise. □

Corollary (well-posedness)

For every $N \in \mathbb{N}$, there is a unique solution $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^{N^2}$ to the finite difference problem (9).

Theorem (convergence)

Let $u \in C^4(\overline{\Omega})$ be a classical solution to the two-dimensional Poisson problem (8). *Let $N \in \mathbb{N}$. Suppose that $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^{N^2}$ is a solution to the finite difference problem* (9). *Let $e \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{e} \in \mathbb{R}^{N^2}$ be its error. Then there is a constant $C_2 > 0$, independent of h, such that*

$$\|\mathcal{E}_h[u]\|_{L_h^\infty} \leq C_2 h^2.$$

Furthermore, if $0 < h < \frac{1}{2}$,

$$\|e\|_{L_h^2} \leq C_1 C_2 h^2,$$

where $C_1 > 0$ is the constant from the previous theorem.

Proof.

The consistency estimate follows from a previous proposition. To obtain convergence, we recall that the consistency error $\mathcal{E}_h[u]$ and error e are related by

$$A\mathbf{e} = h^2 \boldsymbol{\mathcal{E}}_h[u].$$

Proof (Cont.)

Therefore, $\mathbf{e} = h^2 \mathsf{A}^{-1} \boldsymbol{\mathcal{E}}_h[u]$ and

$$\|\mathbf{e}\|_2 \leq h^2 \left\|\mathsf{A}^{-1}\right\|_2 \|\boldsymbol{\mathcal{E}}_h[u]\|_2 \leq C_1 \|\boldsymbol{\mathcal{E}}_h[u]\|_2 \leq C_1 h^{-1} \|\boldsymbol{\mathcal{E}}_h[u]\|_\infty \leq C_1 C_2 h.$$

Using the fact that $\|e\|_{L_h^2} = h \|\mathbf{e}\|_2$, the result follows. $\square$

Remark (suboptimality)

Once again, we get a suboptimal error estimate in the L_h^∞-norm using our L_h^2 estimate:

$$\|e\|_{L_h^\infty} \leq \frac{1}{h} \|e\|_{L_h^2} \leq C_1 C_2 h.$$

We sharpen this estimate in the next section.

Theorem (Discrete Maximum Principle)

Let $d = 2$. Suppose that $v \in \mathcal{V}(\bar{\Omega}_h)$ is such that

$$-\Delta_h v(\mathbf{x}) \leq 0, \quad \forall\, \mathbf{x} \in \Omega_h.$$

Then

$$\max_{\mathbf{x}\in\bar{\Omega}_h} v(\mathbf{x}) \leq \max_{\mathbf{x}\in\partial\Omega_h} v(\mathbf{x}).$$

In other words, the maximum must occur on the boundary.

Proof.

To obtain a contradiction, suppose that a strict maximum occurs in the interior. If this is true, there is some $(kh, \ell h) \in \Omega_h$ such that

$$v(kh, \ell h) = \max_{(ih,jh)\in\Omega_h} v(ih, jh) > \max_{(ih,jh)\in\partial\Omega_h} v(ih, jh).$$

For simplicity, let us suppose that $2 \leq k, \ell \leq N - 1$. Then

$$0 \geq -h^2 \Delta_h v_{k,\ell} = -v_{k-1,\ell} - v_{k+1,\ell} - v_{k,\ell-1} - v_{k,\ell+1} + 4v_{k,\ell} \geq 0.$$

Proof (Cont.)

This implies that $\Delta_h w_{k,\ell} = 0$ and

$$v_{k,\ell} = \frac{1}{4}(v_{k-1,\ell} + v_{k+1,\ell} + v_{k,\ell-1} + v_{k,\ell+1}).$$

The only way to satisfy the last equation and the fact that $v_{k,\ell} \geq v_{k\pm1,\ell}, v_{k,\ell\pm1}$ is to have $v_{k,\ell} = v_{k\pm1,\ell} = v_{k,\ell\pm1}$.

We can now repeat our argument at neighboring points, and we conclude that

$$v_{k,\ell} = v_{i,j}, \quad \forall\, (ih, jh) \in \tilde{\Omega}_h = \Omega_h \backslash \left\{(h,h),(h,Nh),(Nh,h),(Nh,Nh)\right\}.$$

Next to the left boundary, we have, assuming that $(h, jh) \in \tilde{\Omega}_h$,

$$0 \geq -h^2 \Delta_h v_{1,j} = -v_{0,j} - v_{2,j} - v_{1,j-1} - v_{1,j+1} + 4v_{1,j} > 0,$$

because $v_{1,j} > v_{0,j}$. This is a contradiction. The other possible cases are treated similarly. □

Theorem (stability)

Let $d = 2$. Given $f \in \mathcal{V}(\Omega_h)$, suppose that $w \in \mathcal{V}(\bar{\Omega}_h)$ satisfies

$$-\Delta_h w(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_h.$$

Then there is some constant $C > 0$, independent of h and w, such that

$$\|w\|_{L_h^\infty} \leq \max_{(ih,jh)\in\partial\Omega_h} |w_{i,j}| + C\|f\|_{L_h^\infty}.$$

Proof.

The strategy, as in previous cases, is to construct a comparison function. This time, the function $\Phi\colon [0,1]^2 \to \mathbb{R}$ is

$$\Phi(\mathbf{x}) = \left\| \mathbf{x} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\|_2^2 \geq 0.$$

Define the grid function $\Phi_{i,j} = \Phi(ih, jh)$. Then, for all $(ih, jh) \in \Omega_h$,

$$-\Delta\Phi(ih, jh) \equiv -4 = -\Delta_h \Phi_{i,j}.$$

Proof (Cont.)

Define the grid function

$$\Psi_{\pm} = \pm w + \frac{\|f\|_{L_h^\infty}}{4}\Phi.$$

Notice that, in Ω_h, we have

$$-\Delta_h \Psi_{\pm} = \pm f - \|f\|_{L_h^\infty} \leq 0.$$

By the Discrete Maximum Principle then, for all $(ih, jh) \in \Omega_h$,

$$\pm w_{i,j} \leq \Psi_{i,j} \leq \max_{\partial\Omega_h} \Psi \leq \max_{\partial\Omega_h} w + \frac{\|f\|_{L_h^\infty}}{8},$$

as we needed to show. □

Corollary (stability)

Let $d = 2$. Suppose that $v \in \mathcal{V}(\bar{\Omega}_h)$ satisfies

$$-\Delta_h v(\mathbf{x}) = 0, \quad \forall\, \mathbf{x} \in \Omega_h.$$

Then

$$\|v\|_{L^\infty_h} \leq \max_{\partial\Omega_h} |v|.$$

Corollary (convergence)

Suppose that $u \in C^4(\overline{\Omega})$ *is a classical solution to the two-dimensional Poisson problem* (8)*. Let* $N \in \mathbb{N}$ *and* $w \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{w} \in \mathbb{R}^{N^2}$ *be a solution to the finite difference problem* (9)*. Let* $e \in \mathcal{V}_0(\bar{\Omega}_h) \longleftrightarrow \mathbf{e} \in \mathbb{R}^{N^2}$ *be its error. Then there is a constant* $C_3 > 0$*, independent of* h*, such that*

$$\|e\|_{L_h^\infty} \leq C_2 C_3 h^2,$$

where $C_2 > 0$ *is the local truncation error constant from a previous theorem.*

Proof.

Repeat the proof of the $d = 1$ case. □