Classical Numerical Analysis, Chapter 17

Abner J. Salgado and Steven M. Wise

asalgad1@utk.edu swise1@.utk.edu
University of Tennessee

Chapter 17

Initial Value Problems (IVPs) For Ordinary Differential Equations

Existence of Solutions

Classical Solutions

To start, let us define precisely what we mean by a solution to an IVP.

In what follows, we will assume that the following are fixed: d is a positive integer; $\Omega \subseteq \mathbb{R}^d$ is an open set; $\mathbf{u}_0$ is a point in Ω; $I \subseteq \mathbb{R}$ is a closed interval; t_0 is a point in I; $S = I \times \overline{\Omega}$; and $\mathbf{f}\colon S \to \mathbb{R}^d$ is a given function, which we call the *slope function*.

The IVP that we consider seeks a function $\mathbf{u}\colon I \to \Omega$ that, in some sense, satisfies the *initial condition* $\mathbf{u}(t_0) = \mathbf{u}_0$ and the equation

$$\mathbf{u}'(t) = \mathbf{f}(t, \mathbf{u}(t)). \tag{1}$$

Definition (classical solution)

The function $\mathbf{u} \in C^1(I; \Omega)$ *is called a* **classical solution** *on* I *to the IVP if and only if* (1) *holds point-wise for all* $t \in I$ *and* $\lim_{t \to t_0} \mathbf{u}(t) = \mathbf{u}_0$.

Mild Solutions

Definition (mild solution)

We say that $\mathbf{u} \in C(I;\Omega)$ *is a* **mild solution** *on* I *to the IVP* (1) *if and only if, for all* $t \in I$*, we have*

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s))\mathrm{d}s. \tag{2}$$

Theorem (equivalence)

Assume that $\mathbf{f} \in C\left(S;\mathbb{R}^d\right)$*. A function is a mild solution on* I *to problem* (1) *if and only if it is a classical solution to problem* (1)*.*

Proof.

An exercise. □

u-Lipschitz and **Globally u-Lipschitz** Functions

Definition (**u**-Lipschitz)

We say that the slope function $\mathbf{f}\colon S \to \mathbb{R}^d$ *is* **u-Lipschitz** *on* S *if and only if there is a constant* $L > 0$ *such that*

$$\left\|\mathbf{f}(t,\mathbf{v}_1) - \mathbf{f}(t,\mathbf{v}_2)\right\|_2 \le L \left\|\mathbf{v}_1 - \mathbf{v}_2\right\|_2 \tag{3}$$

for all $t \in I$ *and for all* $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$*. If* (3) *holds with* $\Omega = \mathbb{R}^d$*, we say that* $\mathbf{f}$ *is* **globally u-Lipschitz**.

A Local Existence Result

Theorem (Picard–Lindelöf Theorem)

Suppose that there exist constants $\beta, \delta_0 > 0$*, such that* $I_0 = [t_0 - \delta_0, t_0 + \delta_0] \cap I \neq \emptyset$*, and* $\overline{B}(\mathbf{u}_0, \beta) \subset \Omega$*. Define*

$$S_0 = I_0 \times \overline{B}(\mathbf{u}_0, \beta).$$

Assume that $\mathbf{f} \in C\left(S_0; \mathbb{R}^d\right)$*; there is a constant* $M > 0$ *such that, for all* $(t, \mathbf{v}) \in S_0$, $\|\mathbf{f}(t, \mathbf{v})\|_2 \leq M$*; and* $\mathbf{f}$ *is* $\mathbf{u}$*-Lipschitz on* S_0 *with constant* $L > 0$*. Let*

$$\delta_1 = \min\left\{\delta_0, \frac{1}{2L}, \frac{\beta}{M}\right\}, \quad I_1 = [t_0 - \delta_1, t_0 + \delta_1].$$

Then there is a unique mild solution on I_1 *to* (1)*. Moreover,* $\mathbf{u} \in C(I_1; B(\mathbf{u}_0, \beta))$*.*

Proof.

The proof uses the Banach Fixed-Point Theorem. See the book. □

Solutions that Blow Up

Example

Suppose that $u_0 > 0$. Observe that $u(t) = \left(u_0^{-1} - t\right)^{-1}$ is a classical solution on the interval $[0, u_0)$ to the IVP

$$u'(t) = u^2(t), \quad u(0) = u_0.$$

The autonomous slope function $f(t, u(t)) = u^2(t)$ is not globally u-Lipschitz on $S = [0, T] \times \mathbb{R}$, regardless of the size of $T > 0$. Clearly, a global solution, i.e., a (classical or mild) solution on $\mathbb{R}$, cannot be guaranteed. In any case, the Picard–Lindelöf Theorem is applicable, and a unique solution, locally defined around $t = 0$, can be guaranteed.

A Global Existence Result

Theorem (global existence)

Assume that $S = [0, T] \times \mathbb{R}^d$ *and the slope function* $\mathbf{f} \in C\left(S; \mathbb{R}^d\right)$ *is globally* $\mathbf{u}$*-Lipschitz with constant* $L > 0$*. Then there is at least one mild solution on* $[0, T]$ *to* (1)*, which we denote by* $\mathbf{u} \in C\left([0, T]; \mathbb{R}^d\right)$*. Moreover, this solution satisfies the estimate*

$$\|\mathbf{u}(t) - \mathbf{u}_0\|_2 \leq \frac{M}{L}\left(e^{Lt} - 1\right), \quad \forall t \in [0, T],$$

where $M = \|\mathbf{f}(\cdot, \mathbf{u}_0)\|_{L^\infty(0,T)}$.

Proof.

The proof in the book. □

Uniqueness and Regularity of Solutions

Lemma (Grönwall-type inequalities)

Let $T > 0$, $K_1 \geq 0$, $K_2 \geq 0$, *and* $\Phi \in C^1([0, T])$. *If* $\Phi(0) = 0$, $\Phi(t) \geq 0$, *for all* $t \in [0, T]$, *and*

$$\Phi'(t) \leq K_1 \Phi(t) + K_2,$$

then

$$\Phi(t) \leq \frac{K_2}{K_1} \left[e^{K_1 t} - 1 \right]$$

and

$$\Phi'(t) \leq K_2 e^{K_1 t}.$$

Proof.

The solution to the IVP

$$\Phi'(t) - K_1 \Phi(t) = \alpha(t), \quad t \in [0, T], \quad \Phi(0) = 0$$

is

$$\Phi(t) = e^{K_1 t} \int_0^t \alpha(s) e^{-K_1 s} ds.$$

In the present case, $\alpha(t) \leq K_2$ for all $t \in [0, T]$.

Proof Cont.

Hence,

$$\begin{aligned}
\Phi(t) &\le e^{K_1 t} K_2 \int_0^t e^{-K_1 s} \mathrm{d}s \\
&= - \left. \frac{e^{K_1 t} K_2}{K_1} e^{-K_1 s} \right|_{s=0}^{s=t} \\
&= e^{K_1 t} \frac{K_2}{K_1} \left[1 - e^{-K_1 t} \right] \\
&= \frac{K_2}{K_1} \left[e^{K_1 t} - 1 \right].
\end{aligned}$$

Finally, using the last estimate,

$$\Phi'(t) = \alpha(t) + K_1 \Phi(t) \le K_2 + K_1 \frac{K_2}{K_1} \left[e^{K_1 t} - 1 \right] = K_2 e^{K_1 t},$$

and the proof is complete. □

Theorem (continuous dependence)

Let $\Omega_0 \subseteq \Omega$ (and one or both possibly equal to $\mathbb{R}^d$). Assume that $\mathbf{f} \in C\left(S; \mathbb{R}^d\right)$ is $\mathbf{u}$-Lipschitz on S with Lipschitz constant $L > 0$. Assume that, for each $\mathbf{q} \in \Omega_0$, there exists a classical solution, $\mathbf{u}(\,\cdot\,; \mathbf{q}) \in C^1([0, T]; \Omega)$, to the parameterized IVP

$$\mathbf{u}'(t; \mathbf{q}) = \mathbf{f}(t, \mathbf{u}(t; \mathbf{q})), \quad \mathbf{u}(0; \mathbf{q}) = \mathbf{q}. \tag{4}$$

Then, for all $\mathbf{q}_1, \mathbf{q}_2 \in \Omega_0$ and $t \in [0, T]$, we have

$$\left\| \mathbf{u}(t; \mathbf{q}_1) - \mathbf{u}(t; \mathbf{q}_2) \right\|_2 \leq \exp(Lt) \left\| \mathbf{q}_1 - \mathbf{q}_2 \right\|_2 . \tag{5}$$

Proof.

Owing to a previous Theorem, a classical solution is a mild solution. Thus, the corresponding parameterized mild solution satisfies, for all $t \in [0, T]$,

$$\mathbf{u}(t; \mathbf{q}) = \mathbf{q} + \int_0^t \mathbf{f}\left(s, \mathbf{u}(s; \mathbf{q})\right) \mathrm{d}s.$$

Proof Cont.

Hence,

$$\mathbf{u}(t;\mathbf{q}_1) - \mathbf{u}(t;\mathbf{q}_2) = \mathbf{q}_1 - \mathbf{q}_2 + \int_0^t [\mathbf{f}(s,\mathbf{u}(s;\mathbf{q}_1)) - \mathbf{f}(s;\mathbf{u}(s;\mathbf{q}_2))]\mathrm{d}s,$$

by the triangle inequality, and the fact that $\mathbf{f}$ is $\mathbf{u}$-Lipschitz,

$$\|\mathbf{u}(t;\mathbf{q}_1) - \mathbf{u}(t;\mathbf{q}_2)\|_2 \le \|\mathbf{q}_1 - \mathbf{q}_2\|_2 + L\int_0^t \|\mathbf{u}(s;\mathbf{q}_1) - \mathbf{u}(s;\mathbf{q}_2)\|_2 \,\mathrm{d}s. \tag{6}$$

Define

$$\Phi(t) = \int_0^t \|\mathbf{u}(s;\mathbf{q}_1) - \mathbf{u}(s;\mathbf{q}_2)\|_2 \,\mathrm{d}s;$$

in which case,

$$\Phi'(t) = \|\mathbf{u}(t;\mathbf{q}_1) - \mathbf{u}(t;\mathbf{q}_2)\|_2 .$$

Proof Cont.

By estimate (6), for all $t \in [0, T]$, we have that

$$\Phi'(t) - L\Phi(t) \leq \|\mathbf{q}_1 - \mathbf{q}_2\|_2 .$$

The final result now follows from the second Grönwall inequality:

$$\|\mathbf{u}(t;\mathbf{q}_1) - \mathbf{u}(t;\mathbf{q}_2)\|_2 = \Phi'(t) \leq \|\mathbf{q}_1 - \mathbf{q}_2\|_2 \, e^{Lt},$$

as we intended to show. □

Corollary (uniqueness)

Assume that $S = [0, T] \times \mathbb{R}^d$ and the slope function $\mathbf{f} \in C\left(S; \mathbb{R}^d\right)$ is globally $\mathbf{u}$-Lipschitz with constant $L > 0$. Then the solution $\mathbf{u} \in C^1([0, T]; \mathbb{R}^d)$ is unique.

Proof.

Use the previous results. The details are left for an exercise. □

Definition (classes of slope functions)

Assume that $S = [0, T] \times \overline{\Omega}$, with $\Omega \subseteq \mathbb{R}^d$ open. Let $\mathbf{f} \in C\left(S; \mathbb{R}^d\right)$ be a slope function. We say that $\mathbf{f} \in F^1(S)$ if and only if $\mathbf{f} \in C^1\left(S; \mathbb{R}^d\right)$, and there is a real number $A > 0$ such that, for any $i, j = 1, \ldots, d$, and all $(t, \mathbf{v}) \in S$,

$$\left|\partial_{u_j} f_i(t, \mathbf{v})\right| \leq A.$$

We also define, for $m \in \mathbb{N}$,

$$\mathcal{F}^m(S) = F^1(S) \cap C^m(S; \mathbb{R}^d).$$

Proposition (F^1 implies Lipschitz)

Assume that $S = [0, T] \times \overline{\Omega}$, with $\Omega \subseteq \mathbb{R}^d$ open and convex. Let $\mathbf{f}$ be a slope function in $F^1(S)$. Then, $\mathbf{f}$ is $\mathbf{u}$-Lipschitz on S. Moreover, if the hypotheses hold with $\Omega = \mathbb{R}^d$, then $\mathbf{f}$ is globally $\mathbf{u}$-Lipschitz.

Proof.

A homework exercise. □

Remark (simplification)

The assumption that $\mathbf{f} \in F^1(S)$, when $S = [0, T] \times \mathbb{R}^d$, is not often verified in practice. In fact, it often fails to be true. If $\mathbf{f} \in F^1([0, T] \times \mathbb{R}^d)$, then $\mathbf{f}$ would be globally $\mathbf{u}$-Lipschitz. In many important, real-world problems the slope function is only locally Lipschitz, at best. For example, consider the autonomous differential equation

$$u'(t) = -u^3 + u, \quad t \in [0, \infty)$$

with $u(0) = u_0 \in \mathbb{R}$. In this case, $f(t, u) = -u^3 + u$. Clearly, the first derivative of the slope function f with respect to u is unbounded; consequently, $f \notin F^1(S)$. Yet, this autonomous ODE has a bounded classical solution on $[0, \infty)$. In fact, one can show that $\lim_{t\to\infty} u(t) = 1$, if $u_0 > 0$, and $\lim_{t\to\infty} u(t) = -1$, if $u_0 < 0$.

The use of the class $F^1([0, T] \times \mathbb{R}^d)$ is merely for convenience, as the assumption $\mathbf{f} \in F^1([0, T] \times \mathbb{R}^d)$ makes the analysis much simpler.

Regularity

Theorem (higher differentiability)

Assume that $S = [0,T] \times \overline{\Omega}$, with $\Omega \subseteq \mathbb{R}^d$ open and convex. Let $m \in \mathbb{N}$. Suppose that the slope function satisfies $\mathbf{f} \in \mathcal{F}^m(S)$. Assume that $\mathbf{u} \in C^1(I, \Omega)$ is a classical solution to (1). Then $\mathbf{u} \in C^{m+1}(I; \Omega)$.

Proof.

As $\mathbf{u}$ is a classical solution, for all $t \in I$, we have

$$\mathbf{u}'(t) = \mathbf{f}(t, \mathbf{u}(t)).$$

The right-hand side of this identity is differentiable on I; in particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{f}(t, \mathbf{u}(t))\right] = \partial_t \mathbf{f}(t, \mathbf{u}(t)) + D_{\mathrm{u}}\mathbf{f}(t, \mathbf{u}(t))\mathbf{f}(t, \mathbf{u}(t)),$$

where $D_{\mathrm{u}}\mathbf{f} = \left[\partial_{u_j} f_i\right]_{i,j=1}^{d}$ is the $d \times d$ matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{u}$. Consequently, $\mathbf{u}''(t)$ exists and is continuous on I. The higher order derivatives exist and are continuous on I, as may be seen via an induction argument. □

The Flow Map and the Alekseev–Gröbner Lemma

Definition (flow map)

Assume, for simplicity, that $S = [0, T] \times \mathbb{R}^d$. Suppose that, for some $m \in \mathbb{N}$, the slope function satisfies $\mathbf{f} \in \mathcal{F}^m(S)$. *The* **flow map of $\mathbf{z}'(t) = \mathbf{f}(t, \mathbf{z}(t))$**, *denoted*

$$\mathbf{U} \colon I \times \mathbb{R}^d \times I \to \mathbb{R}^d,$$

is defined by

$$\mathbf{U}(s, \mathbf{v}, t) = \mathbf{u}_{s,\mathrm{v}}(t),$$

where $\mathbf{u}_{s,\mathrm{v}} \in C^1\left(I; \mathbb{R}^d\right)$ is the unique solution to the ODE problem

$$\frac{\mathrm{d}\mathbf{u}_{s,\mathrm{v}}}{\mathrm{d}t}(t) = \mathbf{f}(t, \mathbf{u}_{s,\mathrm{v}}(t)), \quad \mathbf{u}_{s,\mathrm{v}}(s) = \mathbf{v}. \tag{7}$$

Proposition (properties of the flow map)

Assume, for simplicity, that $S = [0, T] \times \mathbb{R}^d$. Suppose that, for some $m \in \mathbb{N}$, the slope function satisfies $\mathbf{f} \in \mathcal{F}^m(S)$. Denote by $\mathbf{U}$ the flow map of $\mathbf{z}'(t) = \mathbf{f}(t, \mathbf{z}(t))$. Then

$$\mathbf{U}(s, \mathbf{v}, s) = \mathbf{v}, \quad \forall\, (s, \mathbf{v}) \in S.$$

For any $t_1, t_2 \in I$ and all $\mathbf{v} \in \mathbb{R}^d$, we have

$$\mathbf{U}(s, \mathbf{v}, t_2) = \mathbf{U}\left(t_1, \mathbf{U}(s, \mathbf{v}, t_1), t_2\right).$$

In addition, for any $s, t \in I$ and all $\mathbf{v} \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t}\mathbf{U}(s, \mathbf{v}, t) = \mathbf{f}\left(t, \mathbf{U}(s, \mathbf{v}, t)\right).$$

Proposition (properties of the flow map (Cont.))

Finally, $\mathbf{U}$ *is continuously differentiable with respect to its second variable, i.e., the initial condition. This derivative, denoted* $D_{\mathrm{v}}\mathbf{U}$*, is a* $d \times d$ *matrix at every point* $(s, \mathbf{v}, t) \in I \times \mathbb{R}^d \times I$*, and we have*

$$[D_{\mathrm{v}}\mathbf{U}(s, \mathbf{v}, t)]_{i,j} = \frac{\partial U_i}{\partial v_j}(s, \mathbf{v}, t).$$

Furthermore, $D_{\mathrm{v}}\mathbf{U}$ *is differentiable with respect to its third argument,* t*, and satisfies the differential equation*

$$\frac{\partial}{\partial t} D_{\mathrm{v}}\mathbf{U}(s, \mathbf{v}, t) = D_{\mathrm{u}}\mathbf{f}(t, \mathbf{U}(s, \mathbf{v}, t)) D_{\mathrm{v}}\mathbf{U}(s, \mathbf{v}, t)$$

subject to the initial data

$$D_{\mathrm{v}}\mathbf{U}(s, \mathbf{v}, s) = \mathrm{I}_d,$$

where $D_{\mathrm{u}}\mathbf{f}$ *is the derivative of* $\mathbf{f}$ *with respect to its second argument.*

Theorem (Alekseev–Gröbner Lemma)

Let $t_0 \in I$ and, for some $m \in \mathbb{N}$, $\mathbf{f}, \mathbf{g} \in \mathcal{F}^m(S)$. Denote by $\mathbf{U}$ the flow map of $\mathbf{z}'(t) = \mathbf{f}(t, \mathbf{u}(t))$. Suppose that $\mathbf{u}, \mathbf{v} \in C^1(I; \mathbb{R}^d)$ are the unique classical solutions on I of

$$\begin{aligned} \mathbf{u}'(t) &= \mathbf{f}(t, \mathbf{u}(t)), && t \in I, \quad \mathbf{u}(t_0) = \mathbf{u}_0, \\ \mathbf{v}'(t) &= \mathbf{f}(t, \mathbf{v}(t)) + \mathbf{g}(t, \mathbf{v}(t)), && t \in I, \quad \mathbf{v}(t_0) = \mathbf{u}_0. \end{aligned}$$

Then

$$\mathbf{v}(t) = \mathbf{u}(t) + \int_{t_0}^{t} D_{\mathrm{v}} \mathbf{U}(s, \mathbf{v}(s), t) \mathbf{g}(s, \mathbf{v}(s)) \mathrm{d}s. \tag{8}$$

Dissipative Equations

Definition (monotonicity)

Suppose that $(\cdot,\cdot)$ *is an inner product on* $\mathbb{C}^d$ *and* $\|\cdot\|$ *is the induced norm. Let* $\mathbf{f}\colon [0,T]\times\mathbb{C}^d \to \mathbb{C}^d$*. We say that* $\mathbf{f}$ *is* **monotone** *with respect to* $(\,\cdot\,,\,\cdot\,)$*, if and only if*

$$\Re\left[(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{f}(t,\mathbf{v}_1) - \mathbf{f}(t,\mathbf{v}_2))\right] \leq 0,$$

for all $t \in [0,T]$ *and every* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^d$*.*

Theorem (dissipativity)

Assume that $\mathbf{f} \colon [0,T] \times \mathbb{C}^d \to \mathbb{C}^d$ *is monotone with respect to* $(\cdot,\cdot)$. *Let* $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{C}^d$. *Assume that* $\mathbf{f}$ *is such that there are unique classical solutions on* $[0,T]$ *to the problems*

$$\mathbf{u}'(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{v}'(t) = \mathbf{f}(t,\mathbf{v}(t)), \quad \mathbf{v}(0) = \mathbf{v}_0.$$

In this setting, we have that, for all $t \in [0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{u}(t) - \mathbf{v}(t)\|^2 \leq 0.$$

Furthermore, for all $0 \leq t_1 \leq t_2 \leq T$,

$$\|\mathbf{u}(t_2) - \mathbf{v}(t_2)\| \leq \|\mathbf{u}(t_1) - \mathbf{v}(t_1)\| \leq \|\mathbf{u}_0 - \mathbf{v}_0\| .$$

Proof.

Set

$$E(t) = \frac{1}{2} \|\mathbf{u}(t) - \mathbf{v}(t)\|^2 .$$

Proof Cont.

Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \Re\left[\left(\mathbf{u}(t)-\mathbf{v}(t),\mathbf{u}'(t)-\mathbf{v}'(t)\right)\right] \\ &= \Re\left[\left(\mathbf{u}(t)-\mathbf{v}(t),\mathbf{f}(t,\mathbf{u}(t))-\mathbf{f}(t,\mathbf{v}(t))\right)\right] \\ &\leq 0.\end{aligned}$$

This proves that the function E is nonincreasing. Thus, the second inequality follows. □

Example

Let $A \in \mathbb{C}^{d\times d}$ be such that if $\lambda \in \sigma(A)$, then $\Re\lambda \leq 0$. The slope function

$$\mathbf{f}(t, \mathbf{v}) = A\mathbf{v}$$

is monotone.