GROUP EQUIVARIANT CNNs AND LOW COHERENCE MLPs

Advanced Deep Learning

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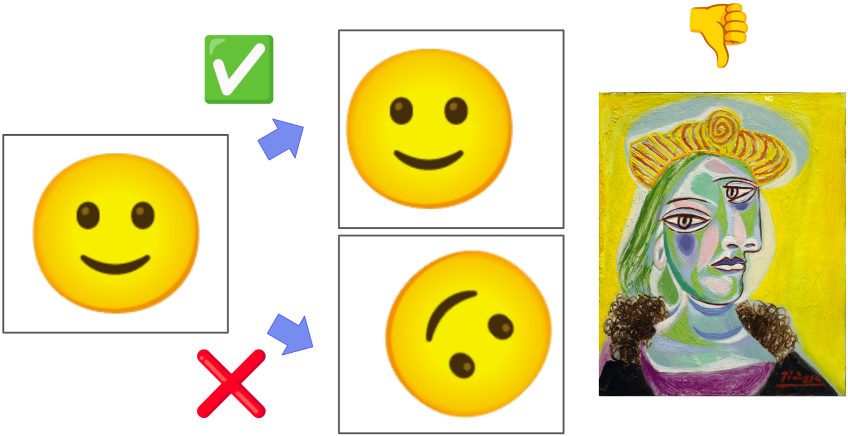
Introduction

Convolutional Neural Networks are equivariant to translations but not to other transformations.



Outline

Part I: Group Equivariant Convolutional Neural Networks

Convolutional Neural Networks (CNNs)

Notation

Here we model images and signals in general as functions $f : \mathbb{R}^d \to \mathbb{R}^K$, with K the number of channels of an image or general signal. For simplicity we take $K = 1$, unless specified. We will focus on the space of square integrable functions, $L^2(\mathbb{R}^d)$.

Convolution Operation

$$(k * f)(x) = \int_{\mathbb{R}^d} \mathrm{d}x' \; k(x - x')f(x') \tag{1}$$

Cross-Correlation Operation

$$(k \star f)(x) = \int_{\mathbb{R}^d} \mathrm{d}x' \; k(x' - x)f(x') = (\mathcal{K}f)(x) \tag{2}$$

where

- $k \in L^1(\mathbb{R}^d)$ is a kernel function and,
- $\mathcal{K} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a linear and bounded operator, which will be useful later.

Note: Here we refer to the convolution operation as the cross-correlation operation, which is more common in the literature. Furthermore, these two operations are intertwined in the forward and backward passes of a CNN.

Convolution Operation as Template Matching

The convolution (correlation) operator can be seen as defined by the left representation of the translation group $\mathscr{L}_x \in \mathrm{Hom}(\mathbb{R}^d, \mathcal{U}(L^2(\mathbb{R}^d)))$. Furthermore, the convolution operation can be interpreted as a form of template matching, where we slide a kernel k over the input function f to produce a new function.

$$\begin{aligned}(k * f)(x) &= \int_{\mathbb{R}^d} \mathrm{d}x' \; k(x' - x) f(x') \\ &= \int_{\mathbb{R}^d} \mathrm{d}x' \; k(x^{-1}x') f(x') \\ &= \langle \mathscr{L}_x \, k | f \rangle_{L^2(\mathbb{R}^d)} \\ &= (\mathcal{K} f)(x)\end{aligned} \tag{3}$$

where we leverage the fact that $\mathbb{R}^d$ is a G-space of the $(\mathbb{R}^d, +)$ group, and that k and f are functions defined on $\mathbb{R}^d$ and thus it makes sense that the left regular representation $\mathscr{L}_x$ acts on k.

Note: Here we use $*$ because the (cross-)correlation operation is nothing more that the convolution operation with a reflected kernel, i.e., $k(x) = k(-x) = k^*(x)$. This operation is called involution.

$$k * f = k^* \star f$$

Equivalence of the convolution operation

The convolution operation is an equivariant map for the translation group

$$
\begin{aligned}
(k *_{\mathbb{R}^d} \mathscr{L}_t f)(x) &= \int_{\mathbb{R}^d} \mathrm{d}x' \; k(x^{-1}x')\mathscr{L}_t f(x') \\
&= \langle \mathscr{L}_x \, k | \mathscr{L}_t f \rangle_{L^2(\mathbb{R}^d)} \\
&= \langle \mathscr{L}_{t^{-1}} \mathscr{L}_x \, k | f \rangle_{L^2(\mathbb{R}^d)} = \langle \mathscr{L}_{t^{-1}\,x} \, k | f \rangle_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} \mathrm{d}x' \; k(x^{-1}t \, x') f(x') \\
&= (k *_{\mathbb{R}^d} \, f)(t^{-1}x) \\
&= [\mathscr{L}_t (k *_{\mathbb{R}^d} \, f)](x)
\end{aligned}
$$

Remark: $\mathrm{d}x' = \mathrm{d}(tx')$ because $\mathrm{d}x'$ is the left Haar measure, which is an invariant measure on the group.

Note: The convolution operation is not an equivariant map for the group of rotations because a representation $\mathscr{L}_t \in \mathrm{Hom}(SO(d), \mathcal{U}(L^2(SO(d)))$ whereas the kernel function k and the input function f are defined on $\mathbb{R}^d$. Thus, the convolution operation does not preserve the equivariance property for rotations.

Group Equivariant Neural Networks

Group Equivariant Convolutional Neural Networks (G-CNNs)

$$
\begin{aligned}
(k *_G f)(g) &= \int_G \mathrm{d}g' \; k(g^{-1}g')f(g') \\
&= \langle \mathscr{L}_g \, k | f \rangle_{L^2(G)} = (\mathcal{K}f)(g)
\end{aligned} \tag{4}
$$

with $\mathcal{K} = L^2(G) \to L^2(G)$ the group correlation operator.

Lifting-Correlation (The link)

$$
\begin{aligned}
(k *_G f)(g) &= \int_{\mathbb{R}^d} \mathrm{d}x' \; k(g^{-1}x')f(x') \\
&= \int_{\mathbb{R}^d} \mathrm{d}x' \; k(h^{-1}x^{-1}x')f(x') \\
&= \int_{\mathbb{R}^d} \mathrm{d}x' \; \mathscr{L}_h \, k(x^{-1}x')f(x') \\
&= \langle \mathscr{L}_x \mathscr{L}_h \, k | f \rangle_{L^2(\mathbb{R}^d)} = (\mathcal{K}f)(g)
\end{aligned} \tag{5}
$$

with $g = (x, h)$ and $\mathcal{K} = L^2(\mathbb{R}^d) \to L^2(G)$ the lifting correlation operator.

Summary of Equivariant Layers

Theorem

A linear layer between feature maps is equivariant if and only if it is a group convolutions.
Reference: [1]

Summary

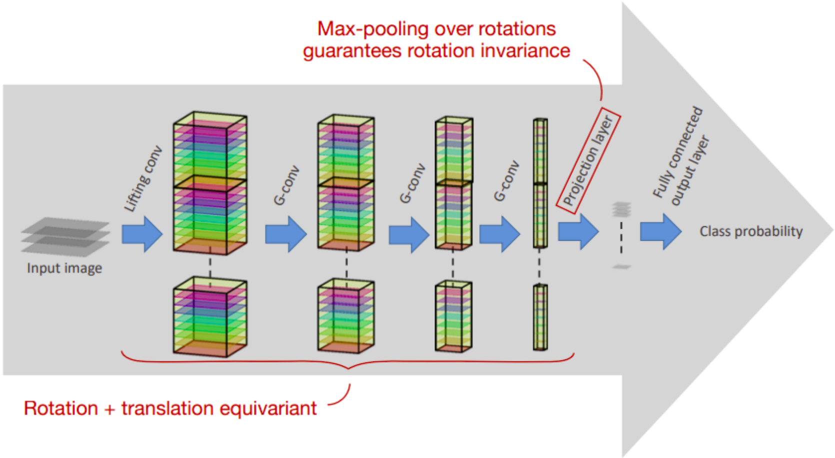
In a nutshell, we can differentiate the types of equivariant layers by:

- convolution operator $\mathcal{K} : L^2(X) \to L^2(Y)$
- the linear part H of the group G, which adds restrictions to the kernel function k

Limitations of G-CNNs

- The group G must be finite and the convolutions need to be discrete.

Workflow for working with GCNNs



Reference: [1]

Practical Convolution Filter Implementation

Planar Convolution

For the translation group in $G = (\mathbb{Z}^2, +)$, we have:

- input feature maps shape at layer l: $K^\ell \times H \times W$
- output feature maps shape at layer $l+1$: $K^{\ell+1} \times H' \times W'$
- layer filter weights shape at layer l : $K^{\ell+1} \times K^\ell \times \tilde{H} \times \tilde{W}$

Group Convolution

For an affine group $G = \mathbb{R}^d \rtimes H$, we have:

- input feature maps shape at layer l: $K^\ell \times S^\ell \times H \times W$
- output feature maps shape at layer $l+1$: $K^{\ell+1} \times S^{\ell+1} \times H' \times W'$
- layer filter weights shape at layer l : $K^{\ell+1} \times S^\ell \times K^\ell \times S^\ell \times \tilde{H} \times \tilde{W}$

where:

- K^ℓ is the number of feature maps at layer ℓ,
- $H \times W$ is the spatial dimensions of the input feature maps,
- S^ℓ is dimension of the group linear part H of the feature maps at layer ℓ,
- $S^\ell \times H \times W$ **is the group space dimensions of the feature maps at layer** ℓ**. Group convolutions just add the dimension of the group linear part** H **to the feature maps.**

Implementation Results

Architectures

- MLP: 1 hidden layer, 128 hidden units, dropout 0.2, batch norm, ReLU activation
- CNN: 4 hidden layers, 32 filters, 5x5 kernel, 1x1 stride, 0x0 padding
- GCNN: 4 hidden layers, 16 filters, 5x5 kernel, 1x1 stride, 0x0 padding, C_4 group

Model	**MNIST**	**Fashion-MNIST**	**CIFAR-10**
MLP	0.34, 0.98	0.19, 0.89	0.24, 0.52
CNN	0.46, **0.99**	0.22, **0.91**	0.29, **0.68**
GCNN	**0.92**, 0.98	**0.46**, 0.86	**0.35**, 0.49

Table: Test Accuracy (T.A.) Results for Different Models and Datasets. Entries: (T.A. with data augmentation, T.A. without data augmentation)

Note: The diehidral group D_4 gave issues in the `InterpolativeGroupKernel` implementation for the `GroupConvolution` layer. D_4 is a 2-dimensional group, C_4 is a 1-dimensional group.

Reference: [2]

Part II: Low Coherence MLPs

Frames and Coherence

Definition: Frame

A **frame** is a set of vectors $\{x_i\}_{i=1}^n$ in a Hilbert space H such that there exist constants $A, B > 0$ (called frame bounds) satisfying:

$$A\|x\|^2 \leq \sum_{i=1}^{N} |\langle x_i, x\rangle|^2 \leq B\|x\|^2 \quad \forall\, x \in H$$

A frame generalizes the notion of a basis but allows redundancy.
Let $H = \mathbb{R}^m$ and $X \in \mathbb{R}^{m\times n}$ be the matrix whose i-th column is x_i, i.e., $X = [x_1\ x_2\ \ldots\ x_n]$. Then the frame condition can be written as:

$$A\|x\|^2 \leq x^T X X^T x \leq B\|x\|^2.$$

Special Frame Types

- **Unit-norm frame**: Each frame vector has unit norm: $\|x_i\| = 1$
- **Equiangular frame**: All pairwise inner products are equal in absolute value: $|\langle x_i, x_j\rangle| = c$ for $i \neq j$
- **Tight frame**: $A = B$ and $XX^T = \frac{n}{m} I_m$

References: [3], [4]

Coherence and Equivariance

Definition: Frame Coherence

The **coherence** of a frame is a measure of how closely packed the frame vectors are. For a matrix X with frame vectors as columns, it is defined as:

$$\mu(X) = \max_{i \neq j} |\langle x_i, x_j \rangle|$$

Lower coherence means better **spread** of vectors and **less redundancy**.

Welch Bound

For an m-dimensional unit-norm frame with n vectors, the **Welch bound** gives a theoretical lower bound on coherence:

$$\mu(X) \geq \sqrt{\frac{n-m}{m(n-1)}} = \mu_{\text{Welch}}$$

Equality is achieved iff the frame is **tight** and **equiangular**.

Connection to Equivariance and Stability

- **Low coherence frames** are more equivariant with respect to group actions, as the frame vectors are more uniformly distributed in the Hilbert space.

Low Coherence Optimization Problem

Upper Bound for Structured Frames

For unit-norm tight frames with κ distinct inner product values (each appearing equally often), the coherence μ satisfies:

$$\mu \leq \sqrt{\kappa}\mu_{\text{Welch}} = \mu_{\text{tight}}$$

Optimization Objective

We seek n unit-norm vectors in $\mathbb{R}^m$ (columns of matrix X) whose worst-case inner product is as small as possible:

$$L_{\text{coh}}(X) = \min_{X \in \mathbb{R}^{m \times n}} \max_{1 \leq i < j \leq n} \langle x_i, x_j \rangle \quad \text{subject to} \quad \|x_i\| = 1 \quad \forall i$$

Rationale

- Minimizing the maximum inner product in value (not magnitude) drives vectors to spread out maximally because lower inner products imply vectors are more widely separated.

Log-Sum-Exp Approximation

Smooth Surrogate Function

Since the maximum function is non-smooth (i.e., not differentiable) and we want to minimize it using gradient descent, we use the log-sum-exp surrogate:

$$L_{\text{coh}}(X) = \frac{1}{\lambda} \log \left(\sum_{i \neq j} \exp(\lambda \langle x_i, x_j \rangle) \right)$$

where $\lambda > 0$ is a parameter controlling approximation tightness.

Reference: [5]

Approximation Bounds

For $a_{ij} = \langle x_i, x_j \rangle$, the log-sum-exp satisfies:

$$\max_{i \neq j} a_{ij} \leq L_{\text{coh}}(X) \leq \max_{i \neq j} a_{ij} + \frac{\log(n)}{\lambda}$$

As $\lambda \to \infty$, $L_{\text{coh}}(X)$ converges to the true maximum.

Regularized Optimization

Regularized Loss Function

We combine the coherence loss with regularization terms to encourage both equiangularity and tightness:

$$L_{\text{total-coh}}(X) = L_{\text{coh}}(X) + \alpha \cdot L_{\text{equi}}(X) + \beta \cdot L_{\text{tight}}(X)$$

where:

- $L_{\text{equi}}(X) = \text{Var}\left(\{|\langle x_i, x_j \rangle| : i \neq j, i < j\}\right)$ encourages equiangularity
- $L_{\text{tight}}(X) = \|XX^T - \frac{n}{m} I_m\|_F^2$ encourages tightness
- $\alpha, \beta > 0$ are regularization hyperparameters

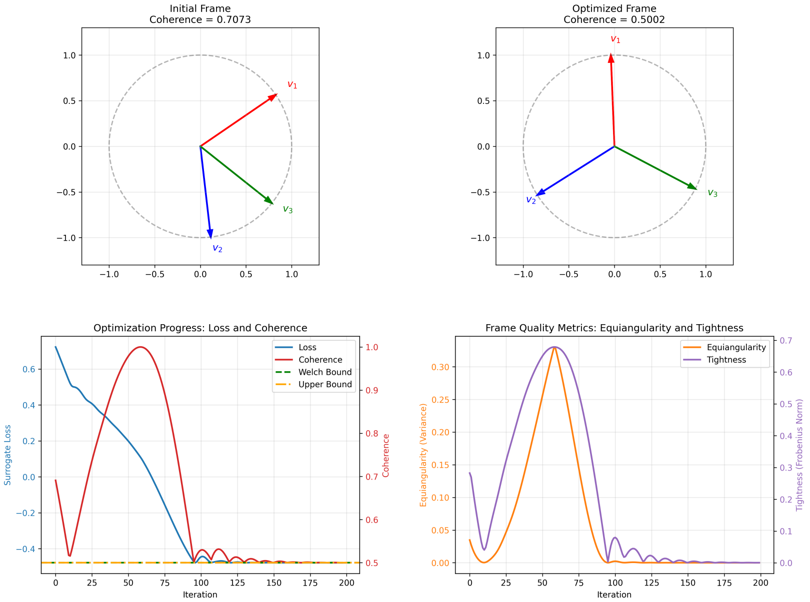
Gradient Descent Update

The optimization is performed via gradient descent:

$$x_i^{(k+1)} = x_i^{(k)} - \eta \frac{\partial L_{\text{total-coh}}(X^{(k)})}{\partial x_i}$$

where η is the learning rate and the unit-norm constraint is enforced after each step, i.e. $x_i^{(k+1)} = \frac{x_i^{(k)}}{\|x_i^{(k)}\|}$.

Optimization Example: Low Coherent Frames in 2D



Low Coherence MLPs Formulation

MLP Weight Matrices as Frames

For each linear layer in an MLP with weight matrix $W \in \mathbb{R}^{d_{\text{out}} \times d_{\text{in}}}$:

- When $d_{\text{in}} > d_{\text{out}}$: columns of W form frame vectors in $\mathbb{R}^{d_{\text{out}}}$
- Frame matrix: $X = W$ where $X \in \mathbb{R}^{m \times n}$ with $m = d_{\text{out}}$, $n = d_{\text{in}}$
- Goal: minimize coherence $\mu(W) = \max_{i \neq j} |\langle w_i, w_j \rangle|$ among weight columns

Combined Loss Function

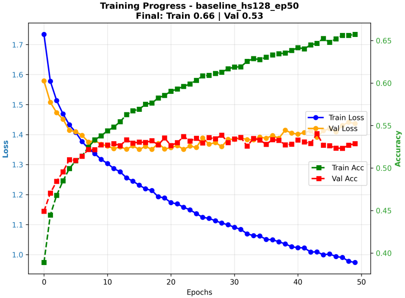
The total loss combines classification and coherence objectives:

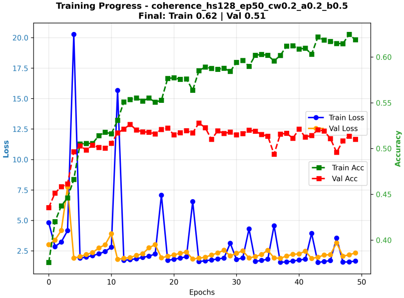
$$L_{\text{total}}(y, \hat{y}, W) = L_{\text{original}}(y, \hat{y}) + \gamma_{\text{coh}} \sum_{l} L_{\text{total-coh}}(W^{(l)}, \lambda, \alpha, \beta)$$

where:

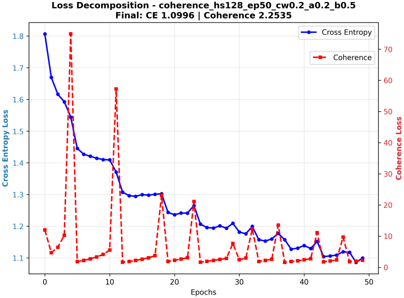
- L_{original} is the original loss function for the task at hand, e.g. cross-entropy
- $L_{\text{total-coh}}$ is the coherence loss function, see previous slides, and λ, α, β are shared hyperparameters for all layers
- $\gamma_{\text{coh}} > 0$ is the coherence regularization weight
- Sum is over all linear layers l with weight matrices $W^{(l)}$

Implementation Results: CIFAR10





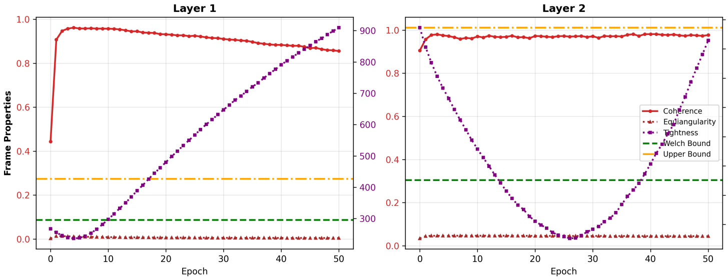
	Baseline	Coherence
Final Test Loss	1.44	2.31
Final Test Accuracy	0.52	**0.51**
Best Epoch	43	**24**
Best Val Accuracy	0.54	0.53



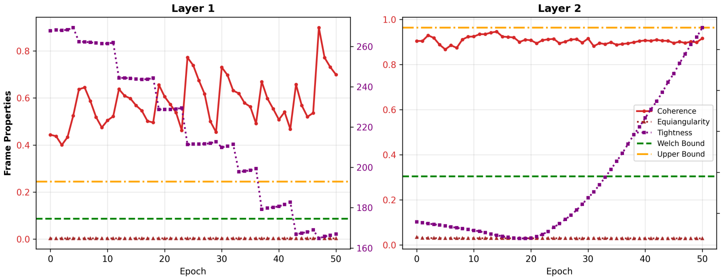
- A similar result is observed for MNIST, with better accuracy, but not for Fashion MNIST (see Annex).

Implementation Results: CIFAR10

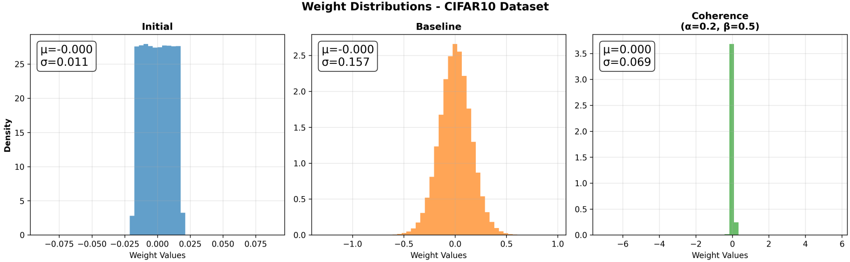
Per-Layer Frame Properties - baseline_hs128_ep50



Per-Layer Frame Properties - coherence_hs128_ep50_cw0.2_a0.2_b0.5

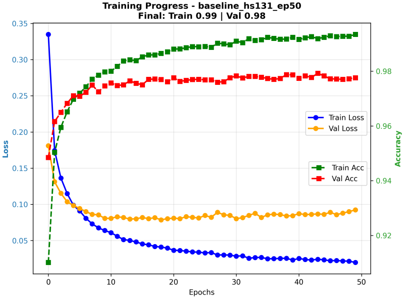


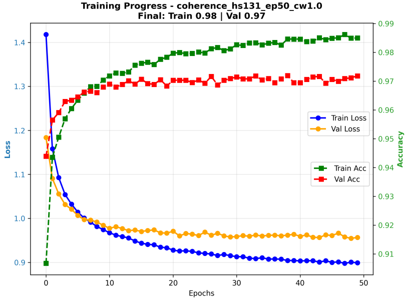
Implementation Results: CIFAR10



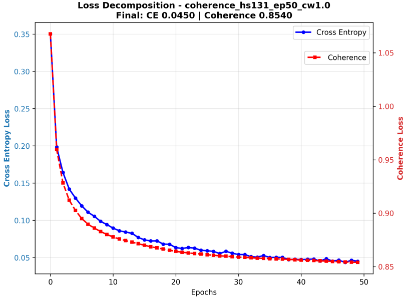
Remark: Regularization = Coherence + Equiangularity + Tightness.
Obs: The minimum coherence optimization slows down training.

Implementation Results: MNIST

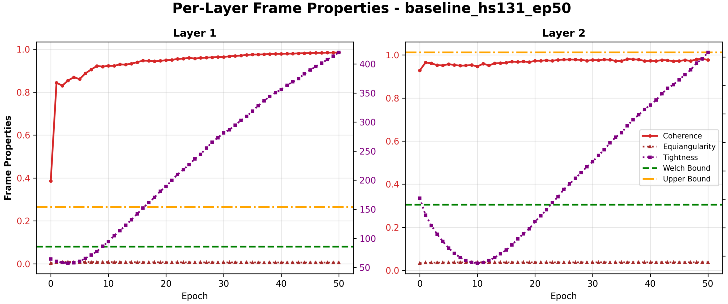


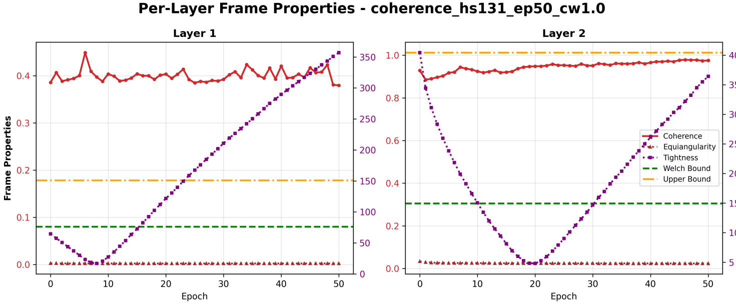


	Baseline	Coherence
Final Test Loss	0.08	0.94
Final Test Accuracy	0.98	**0.97**
Best Epoch	43	**38**
Best Val Accuracy	0.98	0.97

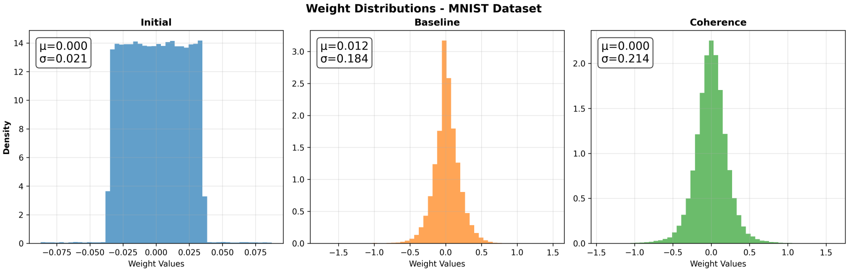


Implementation Results: MNIST





Implementation Results: MNIST



Remark: Regularization = Coherence.

Thank you for your

$$\textbf{softmax}\left(\frac{\boldsymbol{QK}^\top}{\sqrt{\boldsymbol{d_k}}}\odot \boldsymbol{M}\right)\boldsymbol{V}$$

References I

[1] E. Bekkers, "UvA - an introduction to group equivariant deep learning," (2025), [Online]. Available: https://uvagedl.github.io/.

[2] D. Knigge and P. Lippe, "Regular group convolutions," University of Amsterdam Deep Learning Tutorials. (2024), [Online]. Available: https://uvadlc-notebooks.readthedocs.io/en/latest/tutorial_notebooks/DL2/Geometric_deep_learning/tutorial1_regular_group_convolutions.html (visited on 01/27/2025).

[3] M. Thill and B. Hassibi, *Low-coherence frames from group fourier matrices*, Sep. 18, 2015. DOI: 10.48550/arXiv.1509.05739. arXiv: 1509.05739[cs]. [Online]. Available: http://arxiv.org/abs/1509.05739 (visited on 07/06/2025).

[4] M. Thill and B. Hassibi, "Group frames with few distinct inner products and low coherence," *IEEE Transactions on Signal Processing*, vol. 63, no. 19, pp. 5222–5237, Oct. 2015, ISSN: 1053-587X, 1941-0476. DOI: 10.1109/TSP.2015.2450195. arXiv: 1509.05087[cs]. [Online]. Available: http://arxiv.org/abs/1509.05087 (visited on 07/06/2025).

[5] Wikipedia contributors, "LogSumExp," Wikipedia, The Free Encyclopedia. (Jun. 23, 2024), [Online]. Available: https://en.wikipedia.org/wiki/LogSumExp (visited on 01/27/2025).

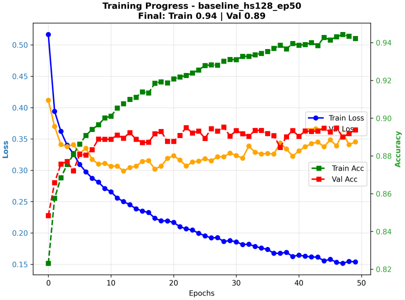
References II

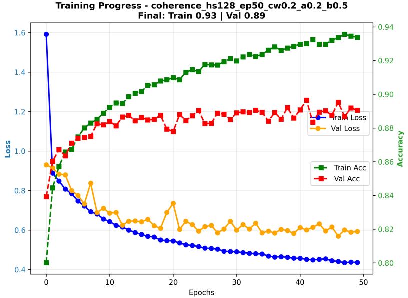
[6] L. Veefkind, "A probabilistic approach to learning the degree of equivariance in steerable CNNs,", 2024.

[7] E. J. Bekkers, "An introduction to equivariant convolutional neural networks for continuous groups,", 2021. [Online]. Available: https://uvagedl.github.io/GroupConvLectureNotes.pdf.

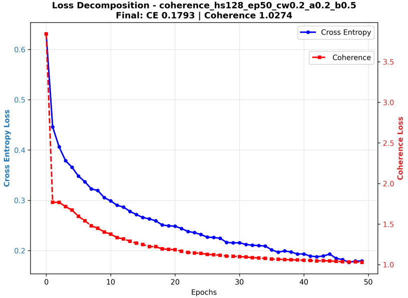
Annex

Implementation Results: Fashion MNIST

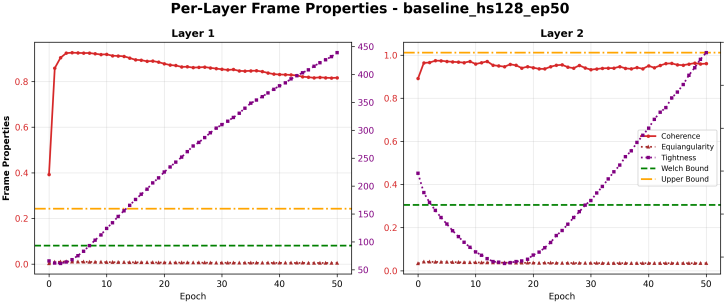


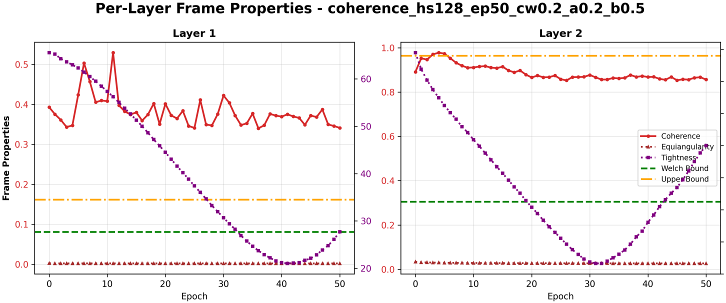


	Baseline	Coherence
Final Test Loss	0.37	0.61
Final Test Accuracy	0.89	**0.89**
Best Epoch	28	**41**
Best Val Accuracy	0.90	0.90

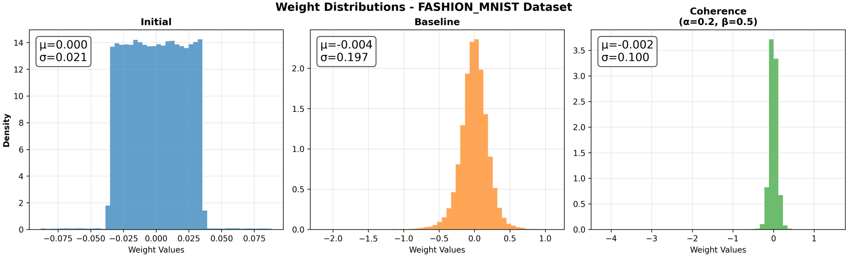


Implementation Results: Fashion MNIST





Implementation Results: Fashion MNIST



Remark: Regularization = Coherence + Equiangularity + Tightness.

Group Theory

Group

A group is a **set** G with a binary operation $\cdot : G \times G \to G$ that satisfies:

- **Closure**: $\forall\, g, h \in G,\ g \cdot h \in G$.
- **Associativity**: $\forall\, g, h, \xi \in G,\ (g \cdot h) \cdot \xi = g \cdot (h \cdot \xi)$.
- **Identity Element**: $\exists!\, e \in G$ st. $\forall\, g \in G,\ e \cdot g = g \cdot e = g$.
- **Inverse Element**: $\forall\, g \in G,\ \exists!\, g^{-1} \in G$ st. $g \cdot g^{-1} = g^{-1} \cdot g = e$.

(Abbuse of) Notation

$$G = (G, \cdot) \qquad , \qquad hg \equiv h \cdot g,\ \ \forall\, h, g \in G$$

Group Homomorphism

A map $\phi : G \to H$ between groups $G = (G, \cdot)$ and $H = (H, *)$ is a group **homomorphism** if:

$$\phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2) \quad \forall\, g_1, g_2 \in G$$

i.e., the group operation is preserved under the map ϕ. The “multiplication” of elements in G is mapped to the “composition” of elements in H.

Group Theory

Group Action and G-space

Given a group G, a (left) G-space X is a set equipped with a group action $G \times X \to X$, $(g, x) \mapsto g \cdot x$, i.e. a map satisfying the following axioms:

- Identity: $\forall x \in X$: $\quad e \cdot x = x$
- Compatibility: $\forall a, b \in G, \forall x \in X$: $\quad a \cdot (b \cdot x) = (a \cdot b) \cdot x$

If these axioms hold, G acts on the G-space X.

(Abbuse of) Notation

$$gx \equiv g \odot x, \;\; \forall\, g \in G, \; \forall\, x \in X$$

Orbit of a Group

Given a group G acting on a set X, the **orbit of an element** $x \in X$ under the group action is the set of all elements that can be reached from x by applying elements of G:

$$\mathcal{O}_x = \{g \cdot x : g \in G\} \subseteq X$$

An orbit is then the “path” traced by x under the action of G.
Reference: [6]

Group Theory

Direct Product Groups

Given two groups $(G, *)$ and $(H, +)$, the direct product group $(G \times H, \cdot)$ is defined as the Cartesian product $G \times H$ of the two sets, along with the group law:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 * g_2, h_1 + h_2).$$

$K \times H$ is often used as the shorthand notation for the direct product between H and K.

Semi-direct Product Groups

Considering two groups $(N, *)$ and $(H, +)$ and a group action $\phi : H \times N \to N$ of H on N, the semi-direct product group $N \rtimes_\phi H$ is defined as the Cartesian product $N \times H$ with the following binary operation:

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 * \phi(h_1, n_2), h_1 + h_2)$$

and the inverse element:

$$(n, h)^{-1} = (\phi(h^{-1}, n^{-1}), h^{-1})$$

Group Theory

Homogeneous Space

A homogeneous space is a G-space with a transitive action on G, meaning that for every pair of points $x, y \in X$, there exists an element $g \in G$ such that the action of g on x moves x to y. Formally, this is expressed as:

$$\forall x, y \in X, \exists g \in G \text{ s.t. } g \cdot x = y,$$

Remark: All points are connected by a group action, and the whole space is just a single orbit. In other words, the space looks the same from any point.

Stabilizer Group

Given a group G acting on a set X, the **stabilizer** of an element $x \in X$ under the group action is the set of all group elements that fix x:

$$\mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\}$$

Lemma: Let X be a homogeneous space of a group G. Then, X can be identified with G/H, where $H = \mathrm{Stab}_G(x_0)$ for any $x_0 \in X$.

Group Theory

Linear Group Representation

A Linear Group Representation ρ of a group G on a vector space V is a group homomorphism from G to the general linear group $GL(V)$:

$$\rho : G \to GL(V) \quad \text{s.t.} \quad \rho(g_1 \cdot_G g_2)v = \rho(g_1) \cdot_{GL(V)} \rho(g_2)v \quad \forall g_1, g_2 \in G, \forall v \in V$$

$\rho(g)$ is an invertible linear transformation that preserves the structure of G, V.

Matrix Representation

A matrix representation is a linear group representation, with a choice of basis, that acts on a finite dimensional vector space $\mathbb{R}^d$ such that all group elements are represented as concrete matrices $\mathbf{D}(g) \in GL(\mathbb{R}^d)$. **Used in practice.**

Left Regular Representation

For a group G and function $f : G \to Y$, with Y a field, the left-regular representation $\mathscr{L}_g$ **acts** on f by "shifting" its domain. For a group element $g \in G$, the action $\mathscr{L}_g$ on f results in a new function $\mathscr{L}_g f : G \to Y$, $\mathscr{L}_g f \in L^2(G)$, defined by:

$$(\mathscr{L}_g f)(h) = f(g^{-1}h) \quad \forall\, h, g \in G$$

Group Theory

Affine Groups

Affine groups $G = \mathbb{R}^d \rtimes H$ are a class of groups that are the semidirect product of a group $H \subseteq GL(\mathbb{R}^d)$ acting on $\mathbb{R}^d$, with $GL(\mathbb{R}^d)$ the group of general linear transformations acting on $\mathbb{R}^d$.

Example: Roto-translation Group

The **roto-translation group** $SE(2)$ is the Special Euclidean Group in 2D, consisting of all combinations of translations and rotations in the plane. It can be defined as:

$$SE(2) = \mathbb{R}^2 \rtimes SO(2)$$

Each element is represented as (x, θ) where $x \in \mathbb{R}^2$ is a translation vector and $\theta \in SO(2)$ is a rotation angle. The group operation is:

$$(x_1, \theta_1) \cdot (x_2, \theta_2) = (\mathbf{R}_{\theta_1} x_2 + x_1, \theta_1 + \theta_2)$$

where $\mathbf{R}_\theta$ is the 2D rotation matrix for angle θ.

Group Theory

Equivariance

Given a group G and two sets X and Y that are acted on by G, a map (e.g., a layer in a neural network) $f : X \to Y$ is equivariant iff

$$\forall x \in X, \forall g \in G : \quad f(g \cdot x) = g \cdot f(x)$$

- Equivariance guarantees that input transformations result in predictable output transformations.
- Input transformations shift information rather than losing it, so no information is lost.
- Enables weight sharing across group transformations - features detected at one location/pose are equally detected at others.

Invariance

Given a group G and two sets X and Y that are acted on by G, a map $f : X \to Y$ is invariant iff

$$\forall x \in X, \forall g \in G : \quad f(g \cdot x) = f(x)$$

- Output remains constant for all inputs related by group actions.

Equivariant Neural Networks

In the basic case, a neural network is a function that maps an input f to an output function g via a series of linear maps $\mathcal{N}_i$ followed by non-linear activation functions, which can be expressed as:

$$g = \mathcal{N}(f) = \mathcal{N}_L \circ \mathcal{N}_{L-1} \circ \ldots \circ \mathcal{N}_1(f) \tag{6}$$

Theorem: Dunford-Pettis: Linear bounded maps are integral transforms

Let $K : L^2(X) \to L^2(Y)$ be a linear and bounded operator that maps between spaces of feature maps $L^2(X)$ and $L^2(Y)$. Then there exists a kernel $k \in L^1(Y \times X)$ such that K is an integral transform via:

$$(Kf)(y) = \int_X k(y,x) f(x)\, d\mu_X(x)$$

with $f \in L^2(X)$, and $d\mu_X$ a "Radon measure" on X.

Vectors:

$$f^{l+1} = Wx \quad \leftrightarrow \quad f_j^{l+1} = \sum_{i=1}^{N_l} w_{ji} x_i^l \tag{7}$$

Signals:

$$f^{l+1} = Kf^l \quad \leftrightarrow \quad f^l(y) = \int_X k(y,x) f^l(x) d\mu_X(x) \tag{8}$$

Types of Equivariant Neural Networks

Theorem: Equivariant linear layers on homogeneous spaces

Let $\mathcal{K} : L^2(X) \to L^2(Y)$ be a linear and bounded operator. Let X, Y be homogeneous spaces on which a (Lie) group G acts transitively, and let $d\mu_X$ be a "Radon measure" on X. Suppose $\mathcal{K}$ is equivariant to the group G via $L_g^Y \circ \mathcal{K} = \mathcal{K} \circ L_g^X$, with L_g^X and L_g^Y are the left-regular representations of G on $L^2(X)$ and $L^2(Y)$, respectively. **Then:**

1. $\mathcal{K}$ is a group convolution given by:

$$(\mathcal{K}f)(y) = \int_X \mathrm{d}\mu_X(x) \frac{\mathrm{d}\mu_X(g_y^{-1}x)}{\mathrm{d}\mu_X(x)} k(g_y^{-1}x) f(x)$$

   for any $g_y \in G$ such that $y = g_y y_0$ for some fixed origin $y_0 \in Y$.

2. The kernel must satisfy the symmetry constraint:

$$k(x) = \frac{\mathrm{d}\mu_X(g_y^{-1}x)}{\mathrm{d}\mu_X(x)} k(h^{-1}x)$$

   for any $h \in H = \mathrm{Stab}_G(y_0)$, and any $x \in X$.

Reference: [7]

Types of Equivariant Neural Networks

Isotropic $\mathbb{R}^d$ Convolutions ($X = Y = \mathbb{R}^d$)

An isotropic $\mathbb{R}^d$ convolution layer maps between planar signals $L_2(\mathbb{R}^d)$ with $\mathcal{K}$ a planar correlation given by

$$(\mathcal{K}f)(y) = \int_{\mathbb{R}^d} \frac{1}{|\det h|} k(\mathbf{x} - \mathbf{y}) f(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{9}$$

and in which k satisfies

$$\forall h \in H : \quad k(\mathbf{x}) = \frac{1}{|\det h|} k(h^{-1}\mathbf{x})$$

Lifting Layer ($X = \mathbb{R}^d$, $Y = G$)

A lifting layer maps from planar signals $L_2(\mathbb{R}^d)$ to signals $L_2(G)$ on the group G, with $\mathcal{K}$ a lifting correlation given by

$$(\mathcal{K}f)(g) = \int_{\mathbb{R}^d} \frac{1}{|\det h|} k(g^{-1}\tilde{\mathbf{x}}) f(\tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} \tag{10}$$

Group Convolution Layer ($X = Y = G$)

A group convolution layer maps between G-feature maps in $L_2(G)$, with $\mathcal{K}$ a group correlation given by

$$(\mathcal{K}f)(g) = \int_G k(g^{-1}\tilde{g}) f(\tilde{g}) \mathrm{d}\tilde{g} \tag{11}$$

with $\mathrm{d}\tilde{g}$ the left "Haar" measure on G.

Example: Roto-translation lifting convolutional layer

Let $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$ be the roto-translation group in 2D. The lifting correlation of a convolutional layer represented by the linear operator $\mathcal{K}^l : \mathbb{R}^2 \to SE(2)$ on a function $f \in L^2(\mathbb{R}^2)$ is:

$$\begin{aligned} f^{l+1}(g) &= (\mathcal{K}^l f)(g) \\ &= (k *_{SE(2)} f)(g) \\ &= \left\langle \mathscr{L}_x \mathscr{L}_\theta k \middle| f^l \right\rangle_{L^2(\mathbb{R}^2)} \\ &= \left\langle \mathscr{L}_x k_\theta \middle| f^l \right\rangle_{L^2(\mathbb{R}^2)} \\ &= (k_\theta *_{\mathbb{R}^2} f)(x) \end{aligned}$$

where $f^{l+1} \in L^2(SE(2))$, $g = (x, \theta)$, $x \in \mathbb{R}^2$ and $\theta \in SO(2)$.

Note: the lifting convolution and G-convolutions can be obtained by first precomputing a filter bank of transformed kernels and then apply them to the input signal via the usual planar correlation (GPU optimized) on $\mathbb{R}^2$, **or $\mathbb{Z}^2$ to be precise in the discrete practical case.** Here, the transformed kernel is obtained via $k_\theta(x) = k(\mathbf{R}_\theta^{-1} x)$. Computationally, this says that to get the transformed kernel at the point x, we need to do a lookup in the original kernel k at the point $\mathbf{R}_\theta^{-1} x$, which is the unique point that gets mapped to x by θ.

Frame Theory

Matrix Definition of a Frame

The frame condition can be written as:

$$A\|x\|^2 \leq x^T X X^T x = \|X^T x\|^2 \leq B\|x\|^2.$$

This follows because:

$$\sum_{i=1}^{n} |\langle x, f_i \rangle|^2 = \sum_{i=1}^{n} (x^T f_i)^2 = x^T \left(\sum_{i=1}^{n} f_i f_i^T \right) x = x^T X X^T x = \|X^T x\|^2.$$

Frame Operator

Given a frame $\{f_i\}_{i=1}^n$, the **frame operator** $S : H \to H$ is defined by

$$Sx = \sum_{i=1}^{n} \langle x, f_i \rangle f_i.$$

Using Dirac notation, we can write:

$$S = \sum_{i=1}^{n} |f_i\rangle \langle f_i| \implies S |x\rangle = \sum_{i=1}^{n} \langle x|f_i\rangle |f_i\rangle$$

Frame Theory

Outer Product Decomposition of the Frame Operator

The frame operator can be written as

$$S = XX^T = \sum_{i=1}^{n} f_i f_i^T$$

Proof: Recall that the standard basis vector $e_i \in \mathbb{R}^n$ satisfies $(e_i)_j = \delta_{ij}$ such that $X = \sum_{i=1}^n f_i e_i^T$, because $f_i e_i^T$ produces a matrix with f_i in the i-th column and zeros elsewhere.
Now compute the frame operator:

$$\begin{aligned} XX^T &= \left(\sum_{i=1}^{n} f_i e_i^T\right)\left(\sum_{j=1}^{n} e_j f_j^T\right)^T \\ &= \left(\sum_{i=1}^{n} f_i e_i^T\right)\left(\sum_{j=1}^{n} e_j f_j^T\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} f_i (e_i^T e_j) f_j^T = \sum_{i=1}^{n} f_i f_i^T \quad \square \end{aligned}$$

Frame Theory

Tight Frame

A frame $\{f_i\}_{i=1}^n$ is called a **tight frame** if the frame bounds satisfy $A = B$. In this case, the frame inequality becomes:

$$\sum_{i=1}^{n} |\langle x, f_i \rangle|^2 = A\|x\|^2 \quad \forall x \in \mathcal{H}$$

Key Properties of Tight Frames

- **Optimal reconstruction**: Equal frame bounds provide stability
- **Energy preservation**: Perfect energy balance across all directions
- **Simplified analysis**: Frame operator becomes a scalar multiple of identity
- **Connection to equiangular frames**: Combined with equiangularity, achieves Welch bound

Frame Theory

Frame Operator of Tight Frames

Let $\{f_i\}_{i=1}^n$ be a tight frame in $\mathbb{R}^m$, and let $X \in \mathbb{R}^{m \times n}$ be the matrix of frame vectors. Then:

$$XX^T = AI_m$$

Proof

Let $S = XX^T$ be the frame operator. Since the frame is tight with bound A:

$$\langle Sx, x \rangle = \sum_{i=1}^{n} |\langle x, f_i \rangle|^2 = A\|x\|^2 = \langle Ax, x \rangle$$

Thus, for all $x \in \mathbb{R}^m$: $\langle (S - AI)x, x \rangle = 0$.
This implies $S = AI_m$ by the polarization identity and positive-definiteness of S. $\square$

Frame Theory

Tight Frame Bound for Unit Vectors

If the frame vectors are unit norm and the frame is tight, then:

$$A = \frac{n}{m}$$

Proof

Taking the trace of both sides of $XX^T = AI_m$:

$$\mathrm{Tr}(XX^T) = \sum_{i=1}^{n} \mathrm{Tr}(f_i f_i^T) = \sum_{i=1}^{n} \|f_i\|^2 = n$$
$$\mathrm{Tr}(AI_m) = A \cdot m$$

Equating: $Am = n \Rightarrow A = \frac{n}{m}$. $\square$

Geometric Interpretation

For unit-norm tight frames, each coordinate direction receives exactly $\frac{n}{m}$ units of total energy from all frame vectors, providing perfect energy balance.