

# Cascades and Fluctuations in an Economy with an Endogenous Production Network

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## Introduction

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- Production in modern economies involves a complex network of producers supplying and demanding goods from each other
- The shape of this network
  - ▶ is an important determinant of how micro shocks aggregate into macro fluctuations
  - ▶ is also constantly changing in response to micro shocks
    - For instance, after a severe shock a producer might shut down which might lead its neighbors to shut down as well, etc...
    - Cascade of shutdowns that spreads through the network

This paper proposes a

Theory of network formation and aggregate fluctuations

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## Overview of the Model

### Simple framework

- Set of firms that use inputs from connected suppliers
- Fixed cost to operate
  - ▶ Firms operate or not depending on economic conditions
  - Links between firms are active or not
  - Endogenously shape the network

Modeling choice motivated by the data

- U.S.:  $\approx 40\%$  of link destructions occur with exit of supplier or customer
- Theory also applies to link formation by thinking of links as special firms

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## Overview of the Results

**Key economic force:** Complementarities in operation decisions of nearby firms

Efficient organization of production

- Create tightly connected clusters centered around productive firms
- Small changes can trigger large reorganization of the network

Cascades of firm shutdowns

- Well-connected firms are hard to topple but create big cascades
- Elasticities of substitution matter for size and propagation of cascades

Aggregate fluctuations

- Recessions feature fewer well-connected firms and less clustering
- Allowing the network to adjust yields substantially smaller fluctuations

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## Along the way...

### Difficulty in solving the planner's problem

- The Karush-Kuhn-Tucker conditions do not apply
  1. Discrete choice about network formation
    - Constraint set is not convex
  2. Complementarities in decisions of nearby firms
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- Endogenous network formation
  - ▶ Atalay et al (2011), Oberfield (2018), Carvalho and Voigtländer (2014), Acemoglu and Azar (2018), Lim (2018)
- Network and fluctuations
  - ▶ Long and Plosser (1983), Horvath (1998), Dupor (1999), Acemoglu et al (2012), Baqaee (2018), Acemoglu et al (2016), Baqaee and Farhi (2018)
- Non-convex adjustments in networks
  - ▶ Bak, Chen, Woodford and Scheinkman (1993), Elliott, Golub and Jackson (2014)
- Measuring the propagation of shocks through networks
  - ▶ Barrot and Sauvagnat (2016), Carvalho et al (2017)
- Macro fluctuations from micro shocks
  - ▶ Jovanovic (1987), Gabaix (2011)

## I. Model

- There are  $n$  units of production (firm) indexed by  $j \in \mathcal{N} = \{1, \dots, n\}$ 
  - ▶ Each unit produces a differentiated good
  - ▶ Differentiated goods can be used to
    - produce a final good
    - produce other differentiated goods
- Representative household
  - ▶ Consumes the final good
  - ▶ Supplies  $L$  units of labor inelastically

$$Y \equiv \left( \sum_{j \in \mathcal{N}} \beta_j^{\frac{1}{\sigma}} c_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

## Model

- Firm  $j$  produces good  $j$

$$y_j = \frac{A}{\alpha_j^{\alpha_j} (1 - \alpha_j)^{1-\alpha_j}} z_j \theta_j \left( \sum_{i \in \mathcal{N}} \Omega_{ij}^{\frac{1}{\varepsilon_j}} x_{ij}^{\frac{\varepsilon_j - 1}{\varepsilon_j}} \right)^{\frac{\varepsilon_j}{\varepsilon_j - 1} \alpha_j} l_j^{1-\alpha_j}$$

- Firm  $j$  can only use good  $i$  as input if there is a *connection* from firm  $i$  to  $j$ 
  - ▶  $\Omega_{ij} > 0$  if connection and  $\Omega_{ij} = 0$  otherwise
  - ▶ A connection can be *active* or *inactive*
  - ▶ Matrix  $\Omega$  is *exogenous*
- A firm can only produce if it pays a fixed cost  $f_j$  in units of labor
  - ▶  $\theta_j = 1$  if  $j$  is operating and  $\theta_j = 0$  otherwise
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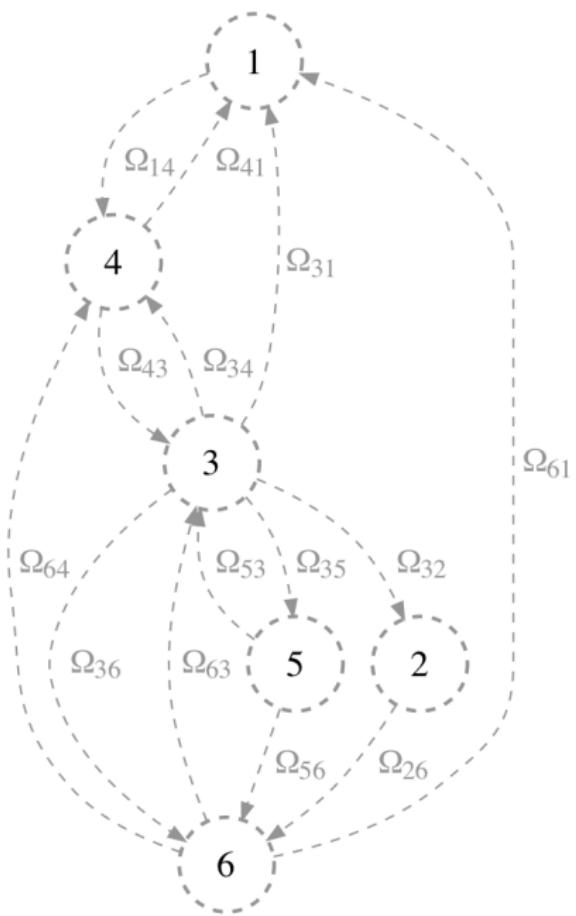
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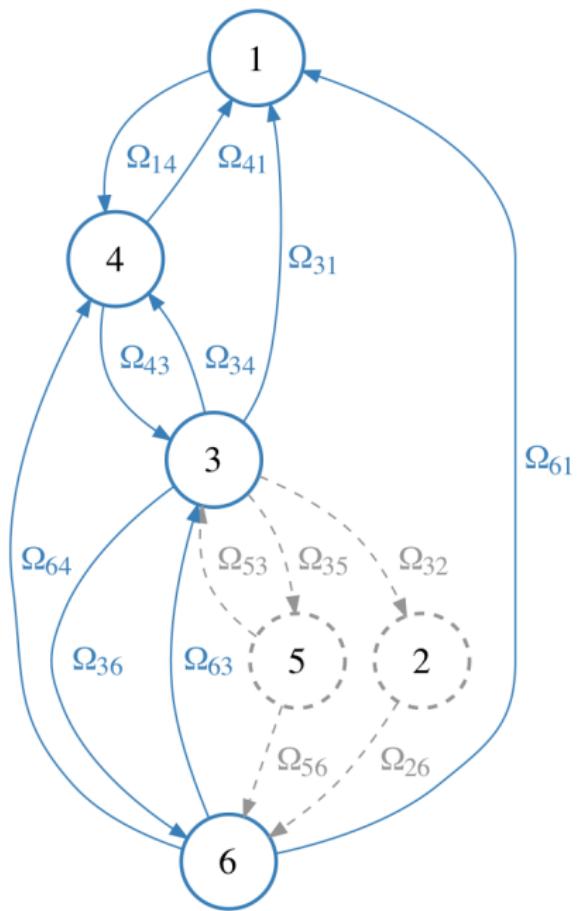
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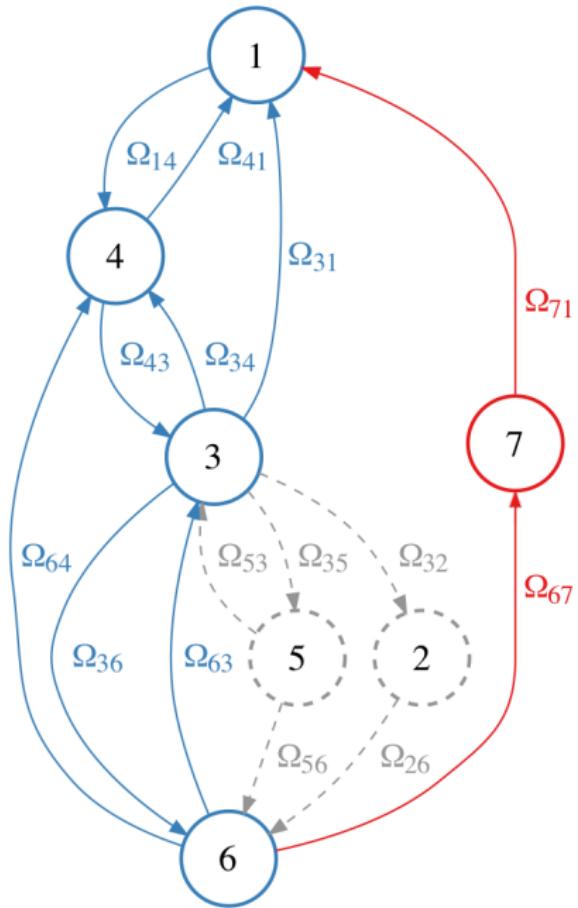
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Focus on the problem of a social planner, but...

### Proposition

*Every equilibrium is efficient.*

Key equilibrium concept is *stability* (Hatfield et al. 2013, Oberfield 2018).

- An allocation is *stable* if there exist no coalition of firms that wishes to deviate.

▶ Equilibrium Definition

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## Social Planner

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Problem  $\mathcal{P}_{SP}$  of a social planner

$$\max_{\substack{c, x, l \\ \theta \in \{0,1\}^n}} \left( \sum_{j \in \mathcal{N}} \beta_j^{\frac{1}{\sigma}} c_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

subject to

1. a resource constraint for each good  $j$

$$c_j + \sum_{k \in \mathcal{N}} x_{jk} \leq \frac{A}{\alpha_j^{\alpha_j} (1 - \alpha_j)^{1-\alpha_j}} z_j \theta_j \left( \sum_{i \in \mathcal{N}} \Omega_{ij}^{\frac{1}{\varepsilon_j}} x_{ij}^{\frac{\varepsilon_j-1}{\varepsilon_j}} \right)^{\alpha_j \frac{\varepsilon_j}{\varepsilon_j-1}} l_j^{1-\alpha_j}$$

2. a resource constraint for labor

$$\sum_{j \in \mathcal{N}} l_j + \sum_{j \in \mathcal{N}} f_j \theta_j \leq L$$

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**LM:**  $\lambda_j$

$$c_j + \sum_{k \in \mathcal{N}} x_{jk} \leq \frac{A}{\alpha_j^{\alpha_j} (1 - \alpha_j)^{1-\alpha_j}} z_j \theta_j \left( \sum_{i \in \mathcal{N}} \Omega_{ij}^{\frac{1}{\varepsilon_j}} x_{ij}^{\frac{\varepsilon_j-1}{\varepsilon_j}} \right)^{\alpha_j \frac{\varepsilon_j}{\varepsilon_j-1}} l_j^{1-\alpha_j}$$

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**LM:**  $w$

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## II. Social Planner with Exogenous $\theta$

## Social Planner with Exogenous $\theta$

---

Define  $q_j = w/\lambda_j$

- From the FOCs, output is  $(1 - \alpha_j) y_j = q_j l_j$
- $q_j$  is the *labor productivity* of firm  $j$

## Proposition

*In the efficient allocation,*

$$q_j = z_j \theta_j A \left( \sum_{i \in \mathcal{N}} \Omega_{ij} q_i^{\varepsilon_j - 1} \right)^{\frac{\alpha_j}{\varepsilon_j - 1}} \quad (1)$$

for all  $j \in \mathcal{N}$ . Furthermore, there is a unique vector  $q$  that satisfies (1) such that  $q_j > 0$  if firm  $j$  has access to a closed loop of active suppliers.

$$q_j = z_j \theta_j A \left( \sum_{i \in \mathcal{N}} \Omega_{ij} q_i^{\varepsilon_j - 1} \right)^{\frac{\alpha_j}{\varepsilon_j - 1}}$$

- Access to a larger set of inputs increases productivity  $q_j$
- Access to cheaper inputs ( $\text{lower } 1/q_i$ ) leads to a cheaper output
- Gains in productivity propagate downstream in the supply chain

Key Economic Force: Gains from input diversity

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Key Economic Force: **Gains from input diversity**

## Social Planner with Exogenous $\theta$

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Knowing  $q$ , we can solve for all other quantities easily.

### Lemma

Aggregate output is

$$Y = Q \left( L - \sum_{j \in \mathcal{N}} f_j \theta_j \right)$$

where  $Q \equiv \left( \sum_{j \in \mathcal{N}} \beta_j q_j^{\sigma-1} \right)^{\frac{1}{\sigma-1}}$  is aggregate labor productivity.

► Other quantities

### III. Social Planner with Endogenous $\theta$

## Social Planner with Endogenous $\theta$

---

Planner's problem is now

$$\max_{\theta \in \{0,1\}^n} Q \left( L - \sum_{j \in \mathcal{N}} f_j \theta_j \right)$$

with

$$q_j = z_j \theta_j A \left( \sum_{i \in \mathcal{N}} \Omega_{ij} q_i^{\varepsilon_j - 1} \right)^{\frac{\alpha_j}{\varepsilon_j - 1}}$$

Trade-off: making firm  $j$  produce ( $\theta_j = 1$ )

- increases labor productivity of the network ( $Q$ )
- reduces the amount of labor into production  $\left( L - \sum_{j \in \mathcal{N}} f_j \theta_j \right)$

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## Social Planner with Endogenous $\theta$

“Very hard problem” (MINLP — NP Hard)

1. The set  $\theta \in \{0, 1\}^n$  is not convex
2. Objective function is not concave

Naive approach: Exhaustive search

- For any vector  $\theta \in \{0, 1\}^n$  iterate on  $q$  and evaluate the objective function
- $2^n$  vectors  $\theta$  to try ( $\approx 10^6$  configurations for 20 firms)
- Guaranteed to find correct solution but infeasible for  $n$  large

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## Alternative approach

New solution approach: Find an alternative problem such that

- P1 The alternative problem is easy to solve
- P2 A solution to the alternative problem also solves  $\mathcal{P}_{SP}$

## Reshaping $\mathcal{P}_{SP}$

Consider the relaxed and reshaped problem  $\mathcal{P}_{RR}$

$$\max_{\theta \in \{0,1\}^n} Q \left( L - \sum_{j \in \mathcal{N}} f_j \theta_j \right)$$

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Parameters  $a_j > 0$  and  $b_{ij} \geq 0$  are *reshaping constants*

- Reshape the objective function away from optimum (i.e. when  $0 < \theta_j < 1$ )
  - ▶ For  $a_j$ : if  $\theta_j \in \{0, 1\}$  then  $\theta_j^{a_j} = \theta_j$
  - ▶ For  $b_{ij}$ :  $\{\theta_i = 0\} \Rightarrow \{q_i = 0\}$  and  $\{\theta_i = 1\} \Rightarrow \{\theta_i^{b_{ij}} q_i^{\varepsilon_j - 1} = q_i^{\varepsilon_j - 1}\}$

Reshaping constants:

$$\boxed{a_j = \frac{1}{\sigma - 1} \quad \text{and} \quad b_{ij} = 1 - \frac{\varepsilon_j - 1}{\sigma - 1}} \quad (*)$$

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with

$$q_j = z_j \theta_j^{a_j} A \left( \sum_{i \in \mathcal{N}} \Omega_{ij} \theta_i^{b_{ij}} q_i^{\varepsilon_j - 1} \right)^{\frac{\alpha_j}{\varepsilon_j - 1}}$$

Parameters  $a_j > 0$  and  $b_{ij} \geq 0$  are *reshaping constants*

- Reshape the objective function away from optimum (i.e. when  $0 < \theta_j < 1$ )
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$$a_j = \frac{1}{\sigma - 1} \quad \text{and} \quad b_{ij} = 1 - \frac{\varepsilon_j - 1}{\sigma - 1}$$

(\*)

## Reshaping $\mathcal{P}_{SP}$

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## Sufficiency of first-order conditions

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**P1** The alternative problem  $\mathcal{P}_{RR}$  is easy to solve

Proposition

*Let  $\varepsilon_j = \varepsilon$  and  $\alpha_j = \alpha$ . If  $\Omega_{jj} = c_i d_j$  for some vectors  $c$  and  $d$  then the KKT conditions are necessary and sufficient to characterize a solution to  $\mathcal{P}_{RR}$ .*

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Define  $\tilde{\Omega} = \omega(\mathbb{1} - I)$  where  $\mathbb{1}$  is the all-one matrix.

### Proposition

Let  $\sigma = \varepsilon_j$  for all  $j$ . Suppose that the  $\{\beta_j\}$  are not too far from each other and that the fixed costs  $f_j > 0$  are not too big. If  $\Omega$  is close enough to  $\tilde{\Omega}$ , then the KKT conditions are necessary and sufficient to characterize a solution to  $\mathcal{P}_{RR}$ .

These propositions

- Only provides *sufficient* conditions
- Later: robustness of this approach

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## Equivalence between $\mathcal{P}_{RR}$ and $\mathcal{P}_{SP}$

**P2** A solution to the alternative problem also solves  $\mathcal{P}_{SP}$

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*If a solution  $\theta^*$  to  $\mathcal{P}_{RR}$  is such that  $\theta_j^* \in \{0, 1\}$  for all  $j$ , then  $\theta^*$  also solves  $\mathcal{P}_{SP}$ .*

We can check that this is verified, but...

Lemma

*The first-order conditions for the operating decision of firm  $j$  only depends on  $\theta_j$  through aggregates.*

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## Intuition

First-order condition on  $\theta_j$ :

$$\text{Marginal Benefit}(\theta_j, F_j(\theta)) - \text{Marginal Cost}(\theta_j, G_j(\theta)) = \bar{\mu}_j - \underline{\mu}_j$$

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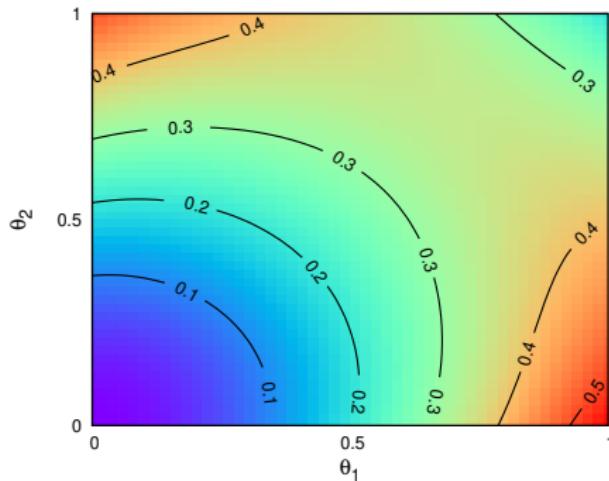
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## Example with two firms

Relaxed problem **without** reshaping

$$V(\theta) = Q(\theta) \left( L - \sum_{j \in \mathcal{N}} f_j \theta_j \right) \text{ with } q_j = z_j \theta_j A \left( \sum_{i \in \mathcal{N}} \Omega_{ij} q_i^{\varepsilon_j - 1} \right)^{\frac{\alpha_j}{\varepsilon_j - 1}}$$



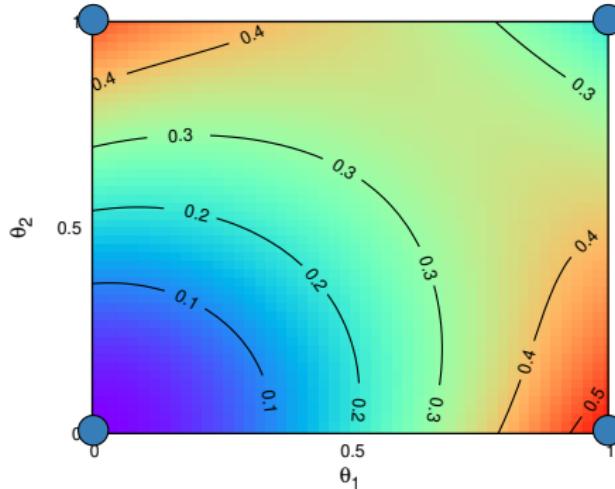
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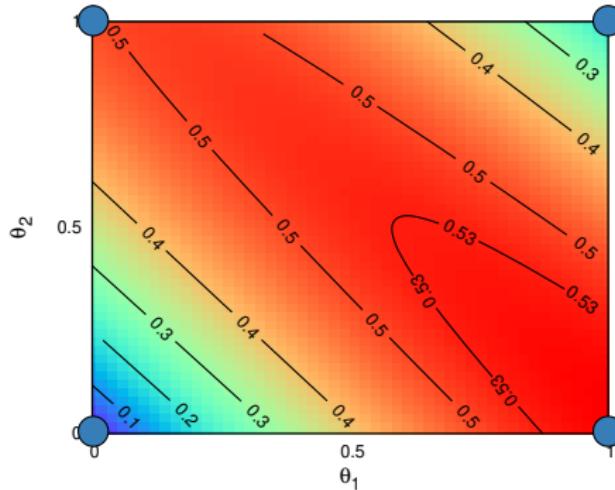
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**Problem:**  $V$  is now (quasi) concave

- ⇒ First-order conditions are necessary and sufficient
- ⇒ Numerical algorithm converges to global maximum

## Tests on Small Networks

For small networks we can solve  $\mathcal{P}_{SP}$  directly using exhaustive search

- Comparing solutions to  $\mathcal{P}_{RR}$  and  $\mathcal{P}_{SP}$ :

n	With reshaping		Without reshaping	
	Correct $\theta$	Error in C	Correct $\theta$	Error in C
8	99.9%	0.002%	89.0%	0.65%
10	99.9%	0.001%	87.0%	0.75%
12	99.9%	0.001%	86.8%	0.77%
14	99.9%	0.001%	86.1%	0.80%

► Notes   ► Break. by  $\Omega$    ► Homo. firms   ► Link by link   ► Large networks  
► Link by link large   ► Error FOCs

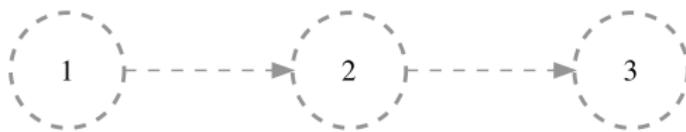
The errors come from

- firms that are particularly isolated
- two  $\theta$  configurations with almost same output

#### IV. Economic Forces at Work

### Proposition

*Operating a firm increases the incentives to operate its direct and indirect neighbors in  $\Omega$ .*



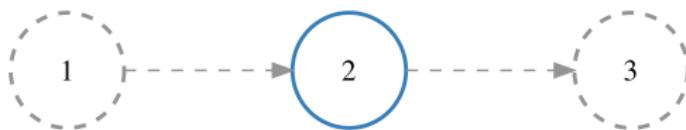
- Impact of operating 2 on the incentives to operate 1 and 3
  - ▶  $\theta_2 = 1 \rightarrow q_3$  is larger if 3 operates
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- Upstream and downstream complementarities in operating decisions
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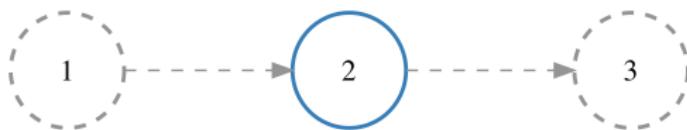
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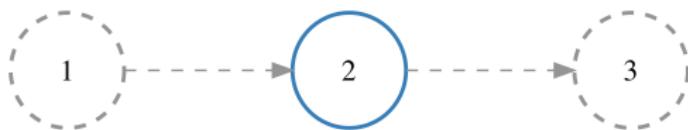
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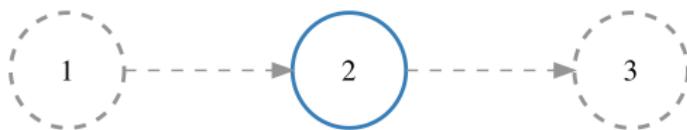
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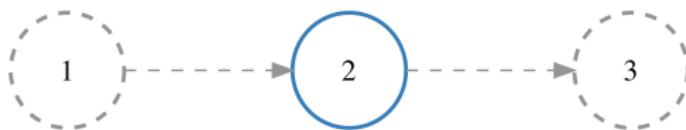
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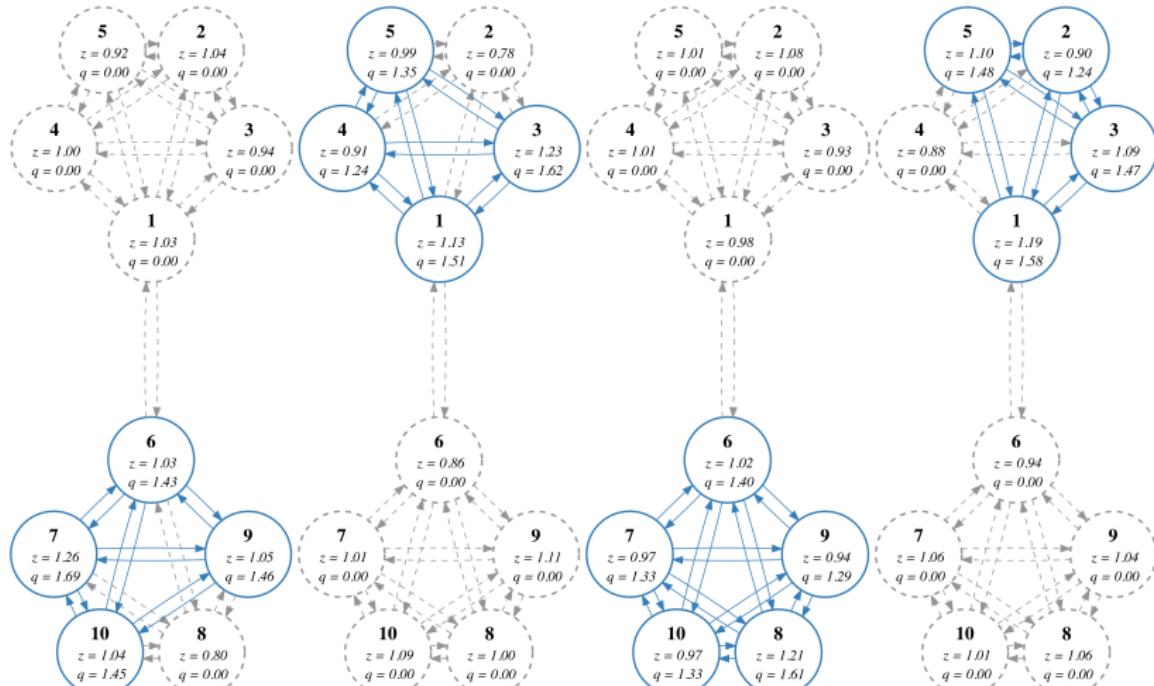


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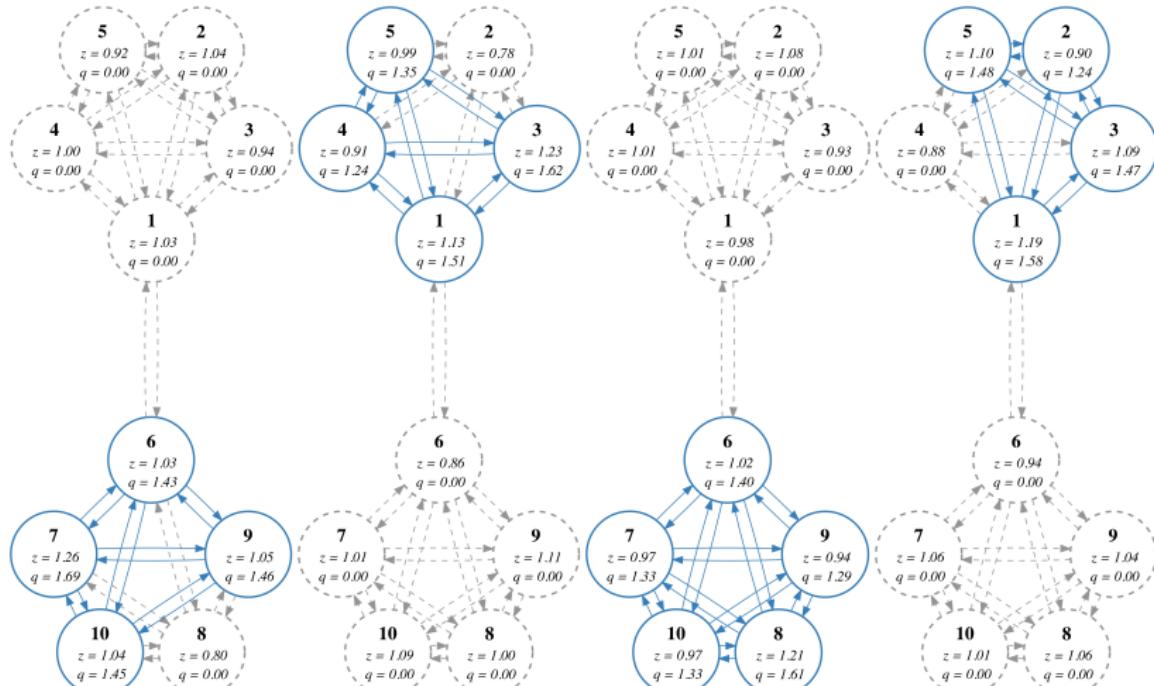
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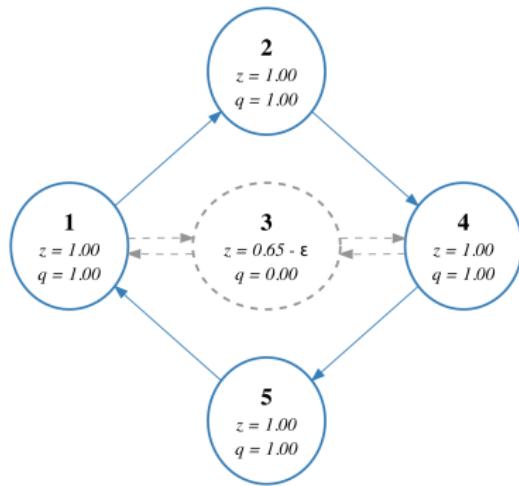
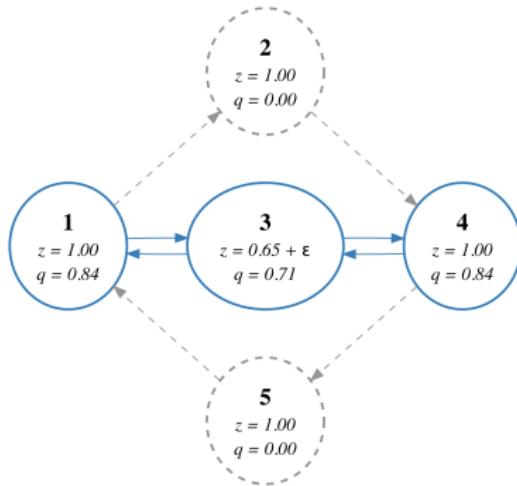
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## Large Impact of Small Shock

Non-convex nature of the economy:

- A small shock can lead to a large reorganization...



## V. Quantitative Exploration

## Network Data

---

Two datasets that cover the U.S. economy

- Compustat
  - ▶ Public firms must self-report important customers ( $>10\%$  of sales)
  - ▶ Cohen and Frazzini (2008) and Atalay et al (2011) use fuzzy-text matching algorithms to build the network
- Factset Revere
  - ▶ Includes public and private firms, and less important relationships
  - ▶ Analysts gather data from 10-K, 10-Q, annual reports, investor presentations, websites, press releases, etc

	Year	Firms/year	Links/year
Compustat			
Atalay et al (2001)	1976 - 2009	1,300	1,500
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## Parameters

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### Focus on the shape of the network and limit heterogeneity across firms

Parameters from the literature

- $\beta_j = 1$
- $\alpha_j = 0.5$  to fit share of intermediate (Jorgenson et al 1987, Jones 2011)
- $\sigma = \varepsilon_j = 5$  average of estimates (Broda et al 2006)
- Firm productivity follows AR1
  - ▶  $\log(z_{it}) \sim \text{iid } \mathcal{N}(0, 0.39^2)$  from Bartelsman et al (2013)
  - ▶  $\rho_z = 0.81$  from Foster et al (2008)
- $f_j \times n = 5\%$  to fit employment in management occupations
- Set  $n = 1000$  for high precision while limiting computations

Unobserved matrix  $\Omega$ :

- Picked to match the *observed* in-degree distribution
- Generate thousands of such  $\Omega$ 's and report averages
- All non-zero  $\Omega_{ij}$  are set to 1

▶  $\Omega$  ▶ Cal. econ.

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## Shape of the Network

What does an optimally designed network looks like?

- Compare **optimal networks** to completely **random networks**
- Differences highlights how efficient allocation shapes the network

	Model	
	Optimal network	Random network
Power law exponents		
In-degree distribution	1.07	1.22
Out-degree distribution	1.02	1.21
Global clustering coefficient	0.51	0.30

Notes: Clustering coeff. multiplied by the square roots of number of nodes for better comparison.

Efficient network features

- More highly connected firms
- More clustering of firms

► Firm level dist.   ► Def. clust. coeff.

## Cascades of Shutdowns

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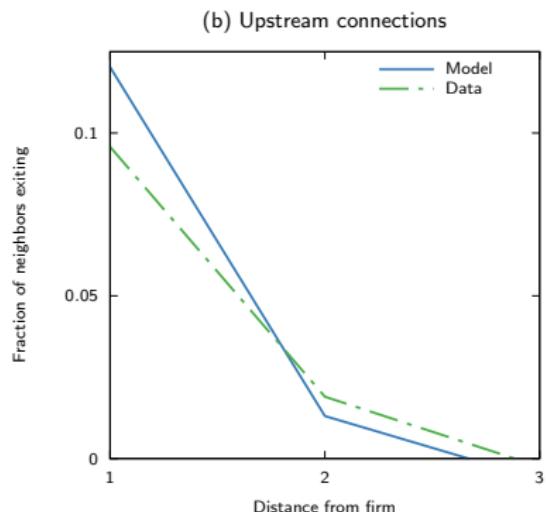
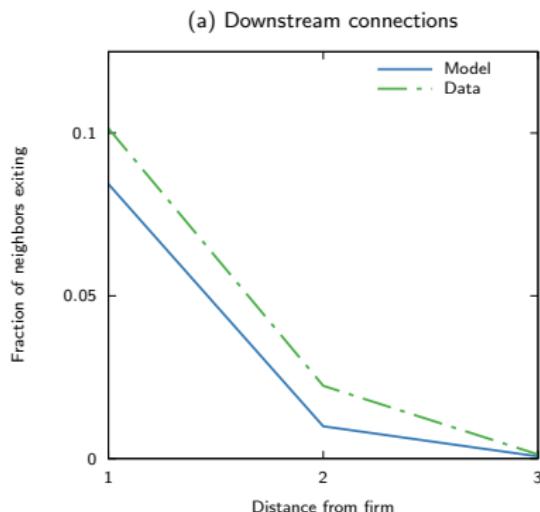
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### Size of cascades and probability of exit by degree of firm

	Size of cascades		Probability of exit	
	Model	Data	Model	Data
Average firm	0.4	0.9	16.3%	12.2%
High degree firm	5.9	7.9	2.3%	3.4%

*Notes:* Size of cascades refers to firm exits up to and including the third neighbors.  
High degree means above the 90th percentile.

- Highly-connected firms are hard to topple but upon shutting down they create large cascades

► Robustness

## Resilience of Firms

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## Aggregate Fluctuations

Static theory but  $z$  shocks move output and the shape of network together

Table: Correlations with aggregate output

	Model	Data		
		Factset	AHRS	CF
Power law exponents				
In-degree distribution	-0.57	-0.85	-0.35	-0.12
Out-degree distribution	-0.67	-0.94	-0.30	-0.11
Global clustering coefficient	0.46	0.68	0.17	0.20

- Recessions are periods with fewer highly-connected firms and in which clustering activity around most productive firms is costly

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Global clustering coefficient	0.46	0.68	0.17	0.20

- Recessions are periods with fewer highly-connected firms and in which clustering activity around most productive firms is costly

## Aggregate Fluctuations

Static theory but  $z$  shocks move output and the shape of network together

**Table:** Correlations with aggregate output

	Model	Data		
		Factset	AHRS	CF
Power law exponents				
In-degree distribution	-0.57	-0.85	-0.35	-0.12
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## Aggregate Fluctuations

Size of fluctuations

$$Y = Q \left( L - \sum_j f_j \theta_j \right)$$

Table: Standard deviations of aggregates

	Output $Y$	$\approx$	Labor Prod. $Q$	+	Prod. labor $L - \sum_j f_j \theta_j$
Optimal network	0.10		0.10		0.009
Fixed network	0.13		0.13		0

- Fluctuations are 30% smaller when network evolves endogenously
- The difference comes from changes in the shape of the network

» Intuition

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## Conclusion

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### Summary

- Theory of endogenous network formation and aggregate fluctuations
- The optimal network features complementarities between operating decisions of firms that lead to
  - ▶ clustering of activity
  - ▶ large impact of small changes
  - ▶ cascades of shutdowns/restarts
- Compared to U.S. data the model is able to replicate
  - ▶ intensity and occurrence of cascades of shutdowns
  - ▶ correlation between shape of network and business cycles
- The endogenous reorganization of the network limits the size of fluctuation
- Methodological contribution: approach to easily solve certain non-convex optimization problems

## Appendix

- Definitions

- ▶ A *contract* between  $i$  and  $j$  is a quantity shipped  $x_{ij}$  and a payment  $T_{ij}$ .
- ▶ An *arrangement* is a contract between all possible pairs of firms.
- ▶ A *coalition* is a set of firms  $J$ .
- ▶ A *deviation* for a coalition  $J$  consists of
  1. dropping any contracts with firms not in  $J$  and,
  2. altering any contract involving two firms in  $J$ .
- ▶ A *dominating deviation* is a deviation such that no firm is worse off and one firm is better off.
- ▶ An allocation is *feasible* if  $c_j + \sum_{k \in N} x_{jk} \leq y_j$  and  $\sum_j l_j + f_j \theta_j \leq L$ .

## Equilibrium

- Firm  $j$  maximize profits

$$\pi_j = p_j c_j - w l_j + \sum_{i \in \mathcal{N}} T_{ji} - \sum_{i \in \mathcal{N}} T_{ij} - w f_j \theta_j,$$

subject to  $c_j + \sum_{k \in \mathcal{N}} x_{jk} \leq y_j$  and  $c_j = \beta_j C (p_j/P)^{-\sigma}$ .

### Definition 1

A stable equilibrium is an arrangement  $\{x_{ij}, T_{ij}\}_{i,j \in \mathcal{N}}$ , firms' choices  $\{p_j, c_j, l_j, \theta_j\}_{j \in \mathcal{N}}$  and a wage  $w$  such that:

1. the household maximizes,
2. firms maximize,
3. markets clear,
4. there are no dominating deviations by any coalition, and
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### Definition 1

A stable equilibrium is an arrangement  $\{x_{ij}, T_{ij}\}_{i,j \in \mathcal{N}^2}$ , firms' choices  $\{p_j, c_j, l_j, \theta_j\}_{j \in \mathcal{N}}$  and a wage  $w$  such that:

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2. firms maximize,
3. markets clear,
4. there are no dominating deviations by any coalition, and
5. the equilibrium allocation is feasible.

## Other quantities

- Labor allocation

$$l = \left[ (I_n - \Gamma) \operatorname{diag} \left( \frac{1}{1-\alpha} \right) \right]^{-1} \left( \beta \circ \left( \frac{q}{Q} \right)^{\circ(\sigma-1)} \frac{Y}{Q} \right)$$

- Output

$$(1 - \alpha_j) y_j = q_j l_j$$

- Consumption

$$c_j = \beta_j \left( \frac{q_j}{w} \right)^\sigma Y$$

- Intermediate goods flows

$$x_{ij} \lambda_i^{\varepsilon_j} = \lambda_j^{\varepsilon_j} \alpha_j \left( A z_j \theta_j \left( \frac{\lambda_j}{w} \right)^{1-\alpha_j} \right)^{\frac{\varepsilon_j-1}{\alpha_j}} \delta_{ij} \Omega_{ij}^{\varepsilon_j} y_j.$$

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## Tests Details

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### Aggregates parameters

- $\sigma \in \{4, 6, 8\}$
- $\log(z_k) \sim \text{iid } \mathcal{N}(0, 0.25^2)$
- $\Omega$  randomly drawn such that firms have on average 3,4,5,6,7 or 8 *potential* incoming connections
  - ▶ The corresponding average number of *active* incoming connections is 2.1, 3.0, 3.8, 4.5, 5.3, and 5.8, respectively.
  - ▶ For each non-zero:  $\Omega_{ij} \sim \text{iid } U([0, 1])$

### Individual parameters

- $f_j \sim \text{iid } U([0, 0.2/n])$
- $\alpha_j \sim \text{iid } U([0.25, 0.75])$
- $\varepsilon_j \sim \text{iid } U([4, \sigma])$
- $\beta_j \sim \text{iid } U([0, 1])$

For each possible combination of aggregate parameters, 200 networks  $\Omega$  and productivity vectors  $z$  are drawn. An economy is kept in the sample only if the first-order conditions yield a solution for which  $\theta$  hits the bounds  $\{0, 1\}$ . More than 90% of the economies are kept in the sample.

## Breakdown by $\Omega$

---

$n$	All matrices $\Omega$		More connected $\Omega$		Less connected $\Omega$	
	Correct $\theta$	Error in $C$	Correct $\theta$	Error in $C$	Correct $\theta$	Error in $C$
8	99.9%	0.002%	100%	0%	99.7%	0.005%
10	99.9%	0.001%	100%	0%	99.8%	0.002%
12	99.8%	0.002%	$\approx$ 100%	$\approx$ 0%	99.7%	0.004%
14	99.8%	0.002%	$\approx$ 100%	$\approx$ 0%	99.6%	0.005%

- Less connected  $\Omega$ : firms have 2, 3 or 4 potential incoming connections
- More connected  $\Omega$ : firms have 6, 7 or 8 potential incoming connections

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## Homogeneous Firms

	Number of firms $n$			
	8	10	12	14
<b>A. With reshaping</b>				
Firms with correct $\theta$	99.9%	99.8%	99.8%	99.8%
Error in output $Y$	0.001%	0.002%	0.002%	0.002%
<b>B. Without reshaping</b>				
Firms with correct $\theta$	87.2%	85.8%	84.7%	83.8%
Error in output $Y$	0.71%	0.79%	0.85%	0.89%

Notes: Random networks with parameters  $f \in \{0.05/n, 0.1/n, 0.15/n\}$ ,  $\sigma_z = 0.25$ ,  $\alpha \in \{0.45, 0.5, 0.55\}$ ,  $\sigma \in \{4, 6, 8\}$ ,  $\varepsilon \in \{4, 6, 8\}$  and networks  $\Omega$  randomly drawn such that firms have on average 2, 4, 5, 6, 7 to 8 potential incoming connections. Each non-zero  $\Omega_{ij}$  is set to 1. For each combination of the parameters, 200 different economies are created. For each economy, productivity is drawn from  $\log(z_k) \sim \text{iid } \mathcal{N}(0, \sigma_z^2)$ . An economy is kept in the sample only if the first-order conditions yield a solution for which  $\theta$  hits the bounds. More than 90% of the economies are kept in the sample.

## Link by link

- Real firms:  $f_j = 0$ ,  $\alpha_j = 0.5$ ,  $\sigma = \varepsilon_j = 6$  and  $\sigma_z = 0.25$
- Link firms:  $\beta_j = 0$ , only one input and one output,  $f_j \sim \text{iid } U([0, 0.1/n])$ ,  $\alpha_j \sim \text{iid } U([0.5, 1])$ ,  $\sigma_z = 0.25$
- $\Omega$ : between any two real firm, there is a link firm with probability  $p \in \{0.7, 0.8, 0.9\}$

m	With reshaping		Without reshaping	
	Correct $\theta$	Error in C	Correct $\theta$	Error in C
3	99.9%	0.001%	94.1%	0.17%
4	99.7%	0.003%	91.3%	0.25%
5	99.7%	0.006%	89.2%	0.31%

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## Large Networks

For large networks we cannot solve  $\mathcal{P}_{SP}$  directly by trying all possible vectors  $\theta$

- After all the welfare-improving 1-deviations  $\theta$  are exhausted:

n	With reshaping		Without reshaping	
	Correct $\theta$	Error in C	Correct $\theta$	Error in C
1000	≈ 100%	≈ 0%	69.3%	0.696%

Notes: 200 different  $\Omega$  and  $z$  that satisfy the properties of the calibrated economy.

- No guarantee that the solution has been found but very few “obvious errors”

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## Link by link

- Same parameters as before
- After all the welfare-improving 1-deviation in  $\theta$  are exhausted:

m	With reshaping		Without reshaping	
	Correct $\theta$	Error in $C$	Correct $\theta$	Error in $C$
10	99.7%	0.005%	83.8%	0.46%
25	99.9%	0.001%	80.5%	0.55%
40	99.9%	0.001%	79.5%	0.57%

- $\theta_j$  converges on  $\{0, 1\}$  for all  $j$  in about 60-85% of the tests
  - ▶ Even without convergence small error in output and few errors in  $\theta$

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## Solution away from corners

- Sometimes the first-order conditions do not converge on a corner.
- Without excluding these simulations:

n	Reshaping?	Error in $C$		
		All matrices $\Omega$	More connected $\Omega$	Less connected $\Omega$
8	Yes	0.027%	0.010%	0.045%
	No	0.642%	0.591%	0.692%
10	Yes	0.036%	0.010%	0.063%
	No	0.740%	0.688%	0.792%
12	Yes	0.030%	0.010%	0.049%
	No	0.768%	0.701%	0.828%
14	Yes	0.041%	0.012%	0.069%
	No	0.806%	0.773%	0.839%

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## Clustering coefficient

---

- $\Omega$  is drawn randomly so that joint distribution of in-degree and out-degree is a bivariate power law of the first kind

$$f(x_{in}, x_{out}) = \xi(\xi - 1)(x_{in} + x_{out} - 1)^{-(\xi+1)}$$

where  $\xi$  is calibrated to 1.85. The marginals for  $x_{in}$  and  $x_{out}$  follow power law with exponent  $\xi$ .

- Correlation between observed in-degree and out-degree
  - ▶ Model: 0.67
  - ▶ Data: 0.43

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## Calibrated Network

	Model	Data		
		Factset	AHRS	CF
<b>Power law exponents</b>				
In-degree distribution	1.07	0.95	1.13	1.32
Out-degree distribution	1.02	0.81	2.24	2.22
Global clustering coefficient	0.51	0.64	0.013	0.014

*Notes:* Global clustering coefficients are multiplied by the square roots of the number of nodes for better comparison.

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## Shape of Network

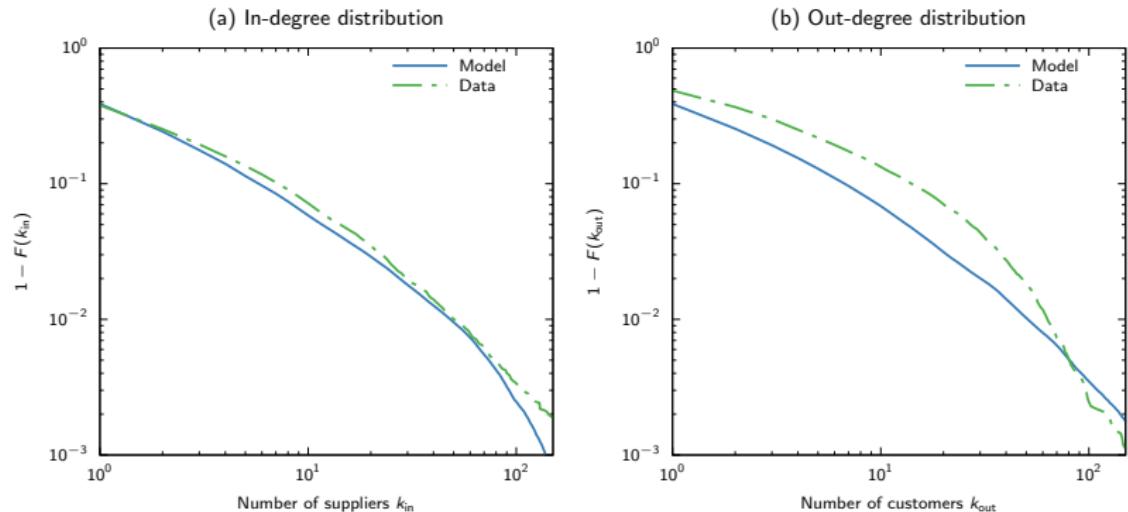


Figure: Model and Factset data for 2016

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## Clustering coefficient

- Triplet: three connected nodes (might be overlapping)
- Triangles: three fully connected nodes (3 triplets)

$$\text{Clustering coefficient} = \frac{3 \times \text{number of triangles}}{\text{number of triplets}}$$

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## Firm-level distributions

---

In the efficient allocation:

- **Selection:** Low productivity firms do not operate
- **Magnification:** High productivity firms benefit from clustering

	Optimal network economy		Random network economy	
	Productivity $q$	Empl. /	Productivity $q$	Empl. /
Mean	2.40	0.0018	1.75	0.0018
Std. dev.	0.30	1.26	0.46	1.94
Skewness	0.53	1.04	-0.05	0.05

Because of the optimal organization of the network

- Distributions are positively skewed

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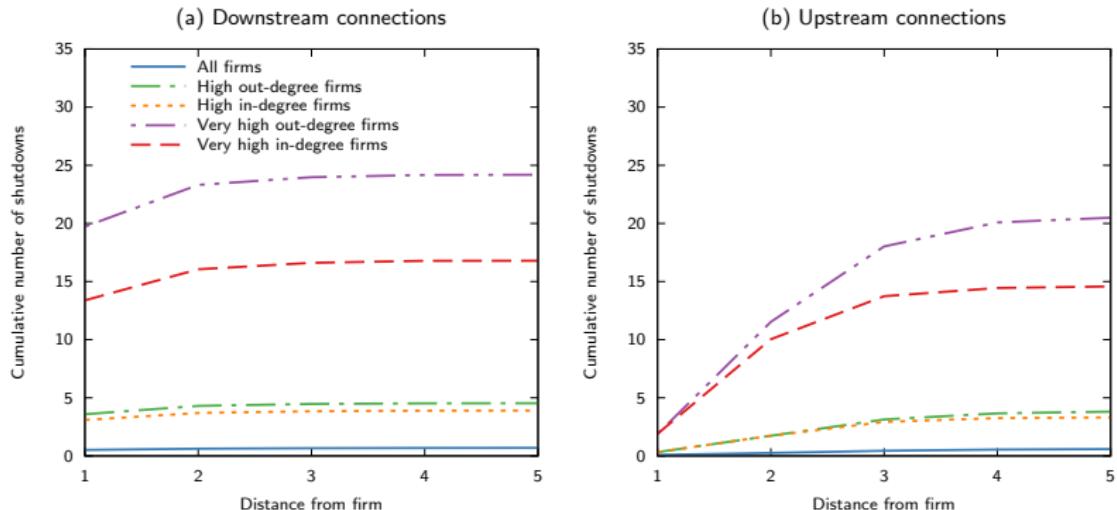
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## Cascades of shutdowns

Causal impact of a firm exits on its neighbors



- Cascades mostly propagate downstream
- Firms with higher degree create larger cascades

## Cascades of shutdowns

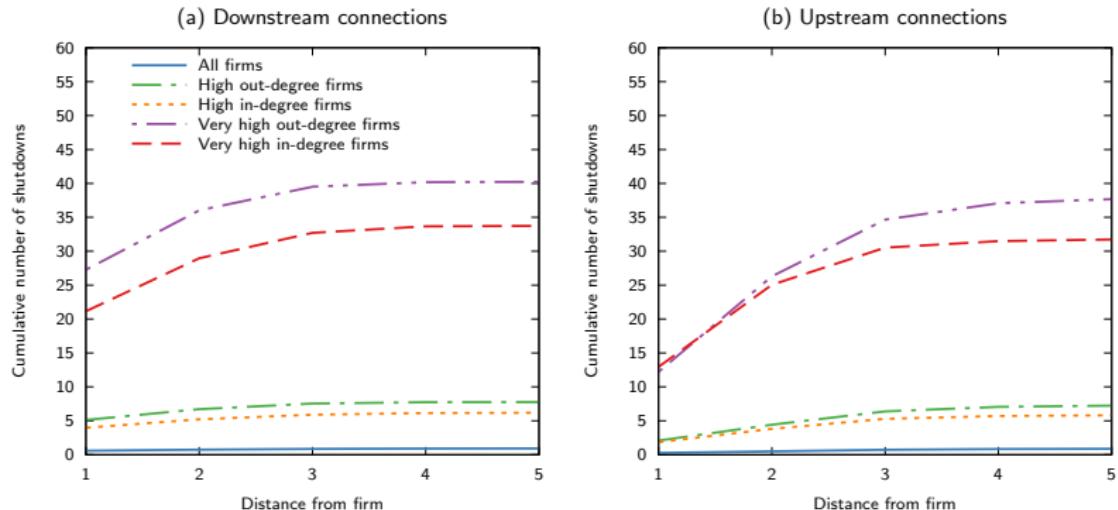


Figure:  $\epsilon = 3$

◀ return

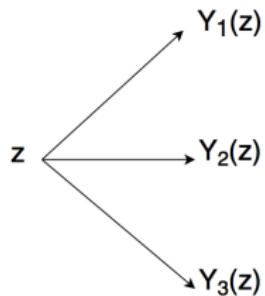
## Resilience Robustness

	Probability of exit			
	Benchmark	$\alpha = 0.75$	$\sigma = 7$	$\varepsilon = 4$
Average firm	16.3%	17.0%	25.2%	15.0%
High degree firm	2.3%	1.2%	2.8%	1.0%

◀ return

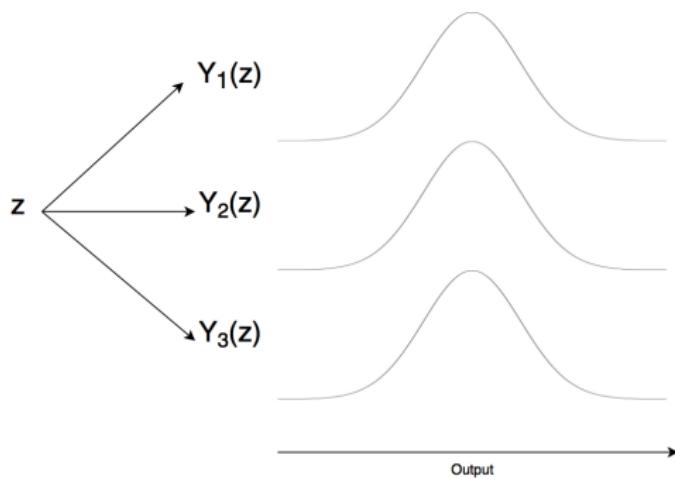
## Intuition

A given network  $\theta^k$  is a function that maps  $z \rightarrow Y_k(z)$



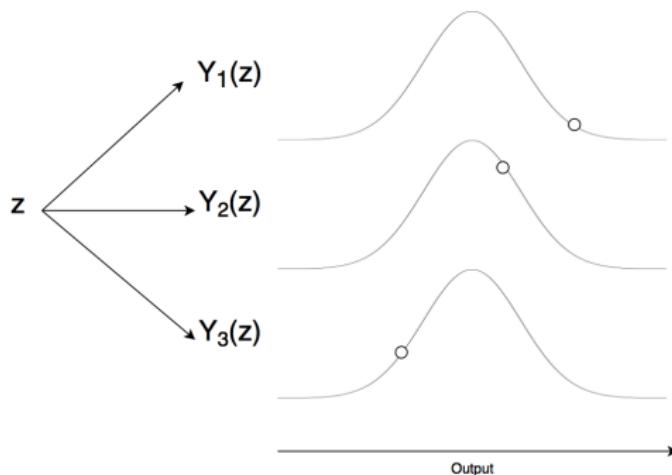
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From extreme value theory

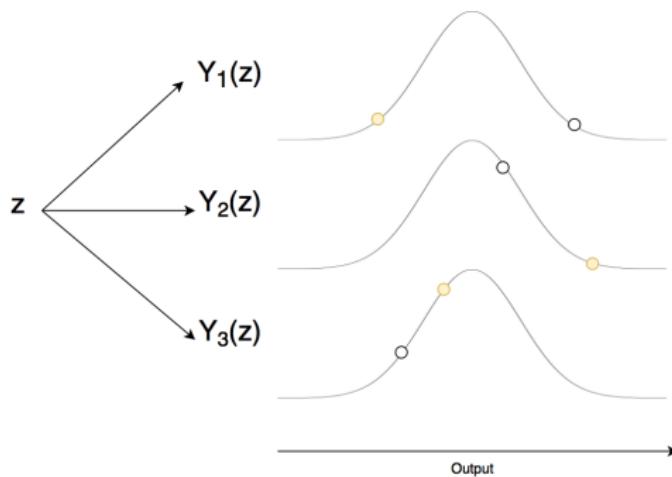
$$\text{Var}(Y) = \text{Var} \left( \max_{k \in \{1, \dots, 2^n\}} Y_k \right)$$

declines rapidly with  $n$

[◀ Return](#)

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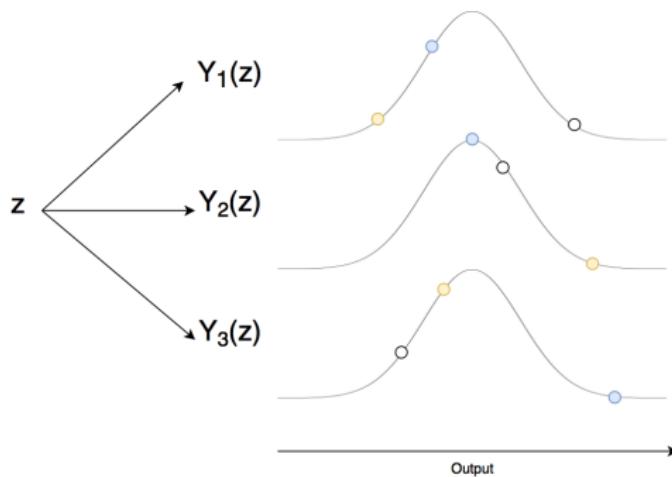
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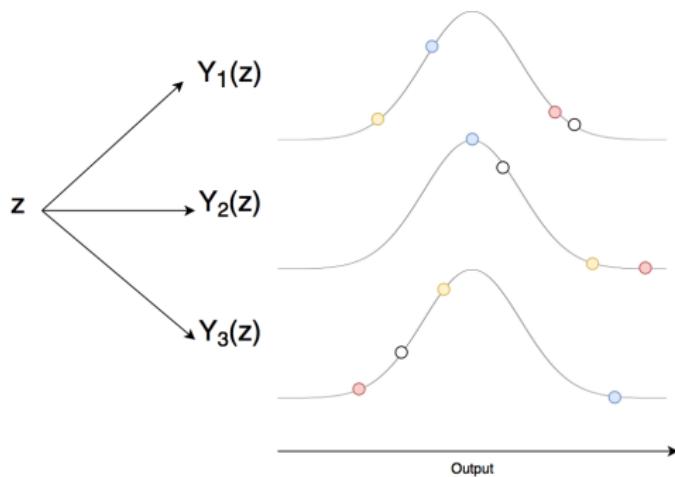
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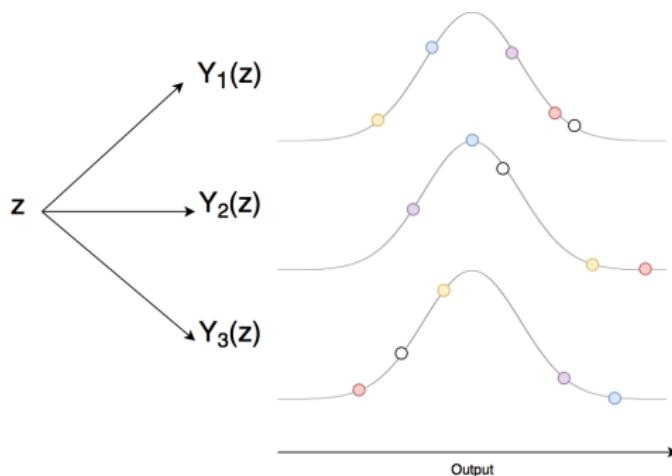
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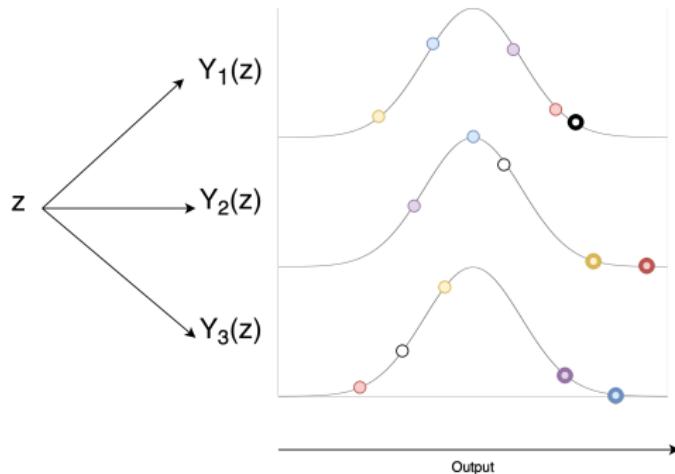
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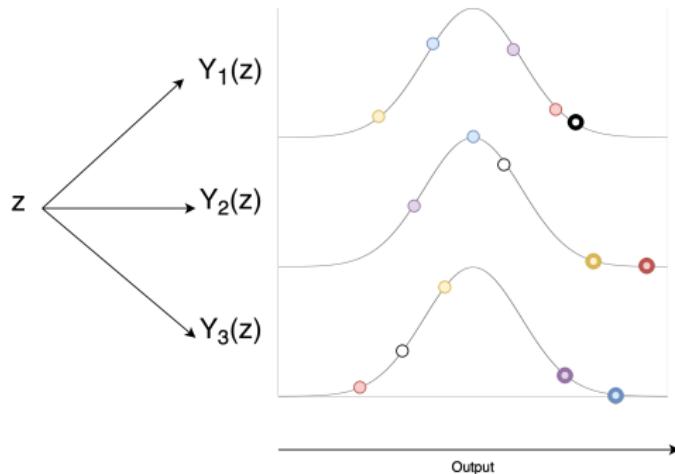
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