Endogenous Production Networks under Uncertainty*

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Abstract

Supply chain disturbances can lead to important increases in production costs. When looking for new suppliers, firms therefore worry about the risks associated with potential trading partners. We investigate the impact of beliefs and uncertainty on the structure of the production network and, through that channel, on the level and volatility of macroeconomic aggregates. To do so, we construct a model of endogenous network formation in which uncertainty about future outcomes affects the structure of the production network. In the model, when uncertainty rises producers prefer to purchase from more stable suppliers, even though they might sell at higher prices. The resulting reorganization of the network leads to less macroeconomic volatility, but at the cost of a decline in aggregate output. We calibrate the model on U.S. data and find that the mechanism can account for a sizable decline in expected GDP during periods of high uncertainty like the Great Recession.

JEL Classifications: E32, C67, D57, D80, D85

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1 Introduction

Firms rely on complex supply chains to provide intermediate inputs needed for production. These chains can be disrupted by natural disasters, trade barriers, changes in regulations, congestion in transportation links, etc. Such shocks to individual firms or sectors can propagate to the rest of the economy through input-output linkages, resulting in aggregate fluctuations. However, individual firms may also take steps that mitigate such propagation by reducing their reliance on risky suppliers. In this paper, we study how this kind of mitigating behavior affects an economy's production network and, through that channel, macroeconomic aggregates.

Business managers are well aware of supply chain risks as one of the key challenges that they face in operating their business, and firms devote substantial resources in mitigating these risks. In a survey by Wagner and Bode (2008), business executives in Germany report that supply chains issues were responsible for significant disruptions to production. Similarly, the Zurich Insurance Group (2015) conducted a global survey of executives in small and medium enterprises and found that, of all the respondent, 39% report that losing their main supplier would adversely affect their operation, and 14% report that they would need to significantly downsize their business, require emergency support or shut down. In addition, there is a large literature in operations research that documents the important impact of supply chain risk on firms' operations (see Ho et al. (2015) for a review).

The COVID-19 pandemic provides a good example of how uncertainty can affect supply relationships. In the aftermath of the pandemic, many firms realized that their supply chains were exposed to substantially more risk than they thought. In a recent survey of business executives, seventy percent agreed that the pandemic has pushed companies to favor higher supply chain resiliency instead of purchasing from the lowest-cost supplier. Many also reported that they plan to diversify their supply chains across suppliers and geographies to mitigate risk.[1]

To investigate whether the concerns expressed by firm managers in these surveys translate into actions, we combine data about firm-to-firm input-output relationships in the United States with measures of stock price volatility, which serve as a proxy for uncertainty. We then regress a dummy variable that equals one in the last year of a relationship on the change in the supplier's stock price volatility. The results are presented in column (1) of Table 1. In column (2), we follow Alfaro, Bloom, and Lin (2019) and address potential endogeneity concerns by instrumenting with industry-level exposure to ten aggregate sources of uncertainty shocks. Finally, in column (3), we use volatility implied by option prices as measure of uncertainty shocks. In all cases, we find a positive and statistically significant relationship between supplier volatility and the destruction of the supply relationship, which is consistent with buyers moving away from riskier suppliers. The effect is also economically large with a doubling in volatility associated with a 13 percentage point

[1] Survey by Foley & Lardner LLP, available online at https://www.foley.com/-/media/files/insights/publications/2020/09/foley-2020-supply-chain-survey-report-1.pdf.

	Dummy for last year of supply relationship		
	(1): OLS	(2): IV	(3): IV
ΔVolatility$_{t-1}$ of supplier	0.023**	0.113***	0.149**
	(0.011)	(0.032)	(0.067)
1st moment 10IV$_{t-1}$ of supplier	No	Yes	Yes
Type of volatility	Realized	Realized	Implied
Fixed effects	Yes	Yes	Yes
Observations	28,687	28,687	21,124
F-statistic	—	67.0	30.6

Notes: Table presents OLS and 2SLS annual regression results of firm-level voltality. The dependent variable is a dummy variable that equals one in the last year of a supply relatonship and zero otherwise. Supplier ΔVolatility$_{t-1}$ is the 1-year lagged change in supplier-level volatility. Realized volatility is the 12-month standard deviation of daily stock returns from CRSP. Implied volatility is the 12-month average of daily (365-day horizon) implied volatility of at-the-money-forward call options from OptionMetrics. As in Alfaro et al. (2019), "we address endogeneity concerns on firm-level volatility by instrumenting with industry-level (3SIC) non-directional exposure to 10 aggregate sources of uncertainty shocks. These include the lagged exposure to annual changes in expected volatility of energy, currencies, and 10-year treasuries (as proxied by at-the-money forward-looking implied volatilities of oil, 7 widely traded currencies, and TYVIX) and economic policy uncertainty from Baker et al. (2016). [...] To tease out the impact of 2nd moment uncertainty shocks from 1st moment aggregate shocks we also include as controls the lagged directional industry 3SIC exposure to changes in the price of each of the 10 aggregate instruments (i.e., 1st moment return shocks). These are labeled 1st moment 10IV$_{t-1}$." See Alfaro et al. (2019) for more details about the data and the construction of the instruments. All specifications include year×customer×supplier industry (3SIC) fixed effects. Standard errors (in parentheses) are two-way clustered at the customer and the supplier levels. F-statistics are Kleibergen-Paap. *,**,*** indicate significance at the 10%, 5%, and 1% levels, respectively.

Table 1: Link destruction and supplier volatility

increase in the likelihood that a relationship is destroyed according to the IV estimates.[2]

Motivated by this evidence, we construct a macroeconomic model of endogenous network formation to investigate how uncertainty affects firms' sourcing decisions and how, in turn, these decisions affect the macroeconomy. In the model, each firm produces a differentiated good that can be consumed by a representative household or be used as an intermediate input by another producer. Firms can produce their good in different ways, which we refer to as production techniques. A technique specifies which intermediate inputs to use and in what proportions. Techniques also differ in terms of productivity. The set of available techniques implies that firms can marginally adjust the importance of a supplier or drop that supplier altogether. As a result, adjustments to the production network happen at both the intensive and extensive margins.

Importantly, in our setting beliefs about firm-level productivities can influence the choice of technique and, thus, the structure of the production network. For instance, while a firm would generally prefer to purchase from a more productive firm, it might decide not to do so if this firm is also more risky. Such a firm would sell at a lower price on average, but it might suffer from a large negative productivity shock, in which case its price would rise substantially. Potential customers take this possibility into account and balance concerns about average productivity and stability when choosing a production technique.

As an example, consider a car manufacturer that must decide on what materials to use as inputs. If carbon fiber prices are expected to increase or to be more volatile, it may instead use

[2]The specifications in Table 1 are chosen following Alfaro et al. (2019). See note below the table and Appendix A.1 for more details about the data and this exercise.

steel for some components. If the change is large enough, it may switch away from using carbon fiber altogether, in which case the link between the car manufacturer and carbon fiber suppliers would disappear.

We prove that there always exists an efficient equilibrium in this environment. The implied equilibrium production network can thus be understood as resulting from a social planner maximizing the utility of the representative household, and firms acting as if they shared the representative household's risk aversion. The efficient equilibrium production network thus optimally balances a higher level of expected GDP against a lower variance. The relative importance of these two objectives is determined by the household's risk aversion. At the efficient equilibrium, the importance of a sector (as measured by its sales share or Domar weight) increases (i) if the expected value of its productivity increases, or (ii) if the variance of its productivity decreases.

One contribution of this paper is to highlight a novel mechanism through which uncertainty can lower expected aggregate output. In the model, in the presence of uncertainty, firms seek stability and, as a result, move away from the most productive (in expectation) suppliers in favor of producers that are less susceptible to risk. This flight to safety implies that less productive producers gain in importance, and aggregate productivity and GDP falls as a result. On the flip side, this reshuffling of the supply chains leads to a more resilient network and makes the aggregate economy more stable.

Our model also has some uncommon predictions about the impact of shocks on aggregate quantities. While an increase in expected productivity or a decline in volatility always have positive effect on welfare, their impact on GDP can be counterintuitive. For instance, an *increase* in expected productivity can lead to a *decline* in expected GDP, so that Hulten's (1978) theorem does not not hold in expectations, even as a first-order approximation. To understand why, consider a firm with low (on average) but stable productivity. Its high expected cost makes it unattractive as a supplier. But if its expected productivity increases, its risk-reward profile changes and other producers might begin to purchase from it. Doing so, they might move away from more productive—but also riskier—producers and, as a result, expected GDP might fall. We show that a similar mechanism is also at work for the variance of shocks, such that an increase in the volatility of a firm's productivity can lead to a decline in the variance of aggregate output.

To evaluate the quantitative importance of allowing firms to adjust their production techniques in response to changes in their beliefs, we calibrate the model using sectoral data for the United States. The model can match the broad features of the input-output structure of the US economy. We also show that the calibrated economy is able to replicate key data features that speak to the importance of beliefs for the structure of the production network; namely, the correlation between Domar weights and the mean and variance of sectoral productivity shocks.

We then evaluate the importance of the changing structure of the production network for macroeconomic aggregates. For this exercise, we first compare our baseline calibration with an alternative economy in which the production network is kept fixed, so that firms cannot move away from suppliers that become unproductive or volatile. We find that aggregate output is about 2.5% lower in this case, so that the endogenous response of the network to productivity shocks has an important impact on welfare. This finding also suggests that policies that impede the natural reorganization of the network (for instance, trade barriers) might have a sizable adverse effect.

To isolate the impact of uncertainty alone, we compare our calibrated model to an alternative economy in which firms no longer worry about risk when making sourcing decisions. While this economy is similar to the calibrated one during normal times, significant discrepancies appear during high-volatility periods, such as the Great Recession. During that episode, we find that firms respond to uncertainty by moving to safer but less productive suppliers. Taken together, these decisions leads to a 1% reduction in the volatility of GDP. The added stability comes however at the cost of a 0.3% decline in expected GDP. Interestingly, this increase in resilience pays off in our estimation as realized GDP in the economy in which firms disregard risk drops by an additional 1% compared to our baseline model.

Our work is related to a large literature that investigates the impact of uncertainty on macroeconomic aggregates (Bloom, 2009, 2014; Bloom et al., 2018). We contribute to that literature in two ways. First, we provide reduced-form evidence for the impact of uncertainty on the structure of the production network in the United States. Second, we propose a novel mechanism through which uncertainty can lower expected GDP. That mechanism operates through a flight to safety process in which firms facing higher uncertainty switch to safer but less productive suppliers.[3]

There is a large and growing literature that studies how shocks propagate through production networks, in the spirit of early contributions by Long and Plosser (1983), Dupor (1999) and Horvath (2000). For example, Acemoglu et al. (2012) derive conditions on input-output networks that allow firm or industry specific shocks to result in aggregate fluctuations even when the number of sectors is large.[4] Acemoglu et al. (2017) and Baqaee and Farhi (2019a) describe conditions under which production networks can generate fat-tailed aggregate output. Foerster et al. (2011) and Atalay (2017) study the empirical contributions of sectoral shocks for aggregate fluctuations. The mechanisms studied in these papers are also present in our model. Carvalho and Gabaix (2013) argue that the reduction in aggregate volatility during the so-called Great Moderation (and its potential recent undoing) can be explained by changes in the input-output network. In Carvalho and Gabaix's model, the production structure is taken as exogenous, and the volatility of sector-specific shocks is held fixed. In our model, the input-output network endogenously responds to changes in sector level volatility in a manner that, ceteris paribus, reduces aggregate volatility.

[3] Fernández-Villaverde et al. (2011) investigate the real impact of interest rate volatility for emerging economies. Jurado et al. (2015) provide econometric estimates of time-varying macroeconomic uncertainty. Baker et al. (2016) measure economic policy uncertainty based on newspaper coverage. Nieuwerburgh and Veldkamp (2006) and Fajgelbaum et al. (2017) develop models in which uncertainty can have long-lasting impacts on economic aggregates.

[4] Production networks are also one mechanism through which granular fluctuations can emerge (Gabaix, 2011).

In most of that literature, the propagation of shocks follows Hulten's (1978) theorem, so that sales shares are a sufficient statistics to predict the impact of microeconomic shocks on macroeconomic aggregates. In contrast, in our model firms can adjust production techniques ex ante and Hulten's theorem is not a useful guide to how this affects expected GDP, even as a first-order approximation. In fact, an increase in firm-level productivity can even have a negative impact on expected GDP.[5]

Our paper is not the first to study the endogenous formation of production networks. Oberfield (2018) considers an economy in which each firm must select one input and studies the emergence of star suppliers. Acemoglu and Azar (2020) build a model of endogenous network formation in which firms have multiple inputs and investigate its implications for growth. These papers focus on the extensive margin of the network (whether a link exists or not). Taschereau-Dumouchel (2020) and Acemoglu and Tahbaz-Salehi (2020) study economies in which the firms' decisions to operate or not shape the production network. Lim (2018) constructs a model to evaluate the importance of endogenous changes in the network for business cycles fluctuations but does not study how the network adjusts to changes in uncertainty. Dhyne et al. (2021) build a model of endogenous network formation and international trade. Boehm and Oberfield (2020) estimate a network formation model using Indian micro data to study misallocation in the inputs market. In our model, both the intensive and extensive margins are active. To the best of our knowledge, it is also the first model in which uncertainty directly affects the structure of the production network.

Several papers in the network literature endow firms with CES production functions, so that the input-output matrix varies with factor prices. Our model generates endogenous changes in the production network through a different mechanism, which is closer to Oberfield (2018) and Acemoglu and Azar (2020). In some circumstances links between sectors may be created or destroyed in our model, which cannot occur in the standard CES production network model. In addition, standard CES production network models do not allow for uncertainty and beliefs to play a role in shaping the production network, and introducing such mechanisms while keeping the model tractable is not straightforward.

The remainder of the paper is organized as follows. The next section introduces our model of network formation under uncertainty. In Section 3, we first characterize the equilibrium when the network is kept fixed. We then consider the full equilibrium with a flexible network in Section 4 and prove that an efficient equilibrium always exists.In Section 5, we describe the mechanisms at work in the environment and explain how shocks propagate through the network. In Section 6, we calibrate the model to U.S. data and quantitatively evaluate the importance of uncertainty on the macroeconomy through its impact on the production network. The last section concludes. All proofs are in the appendix.

[5] Baqaee and Farhi (2019a) investigate departures from Hulten's theorem. Recent work that has also studied production networks under frictions include Bigio and La'O (2020), Baqaee (2018) and Baqaee and Farhi (2019b).

2 A model of endogenous network formation under uncertainty

We study the formation of production networks under uncertainty in a multi-sector economy. Each sector is populated by a continuum of firms producing a differentiated good that can be used either as an intermediate input or for final consumption. To produce, each firm must choose a production technique, which specifies a set of inputs to use, the factor shares associated with these inputs, and an expected level of productivity. Firms are subject to sector-specific productivity shocks, and since firms choose production techniques before these shocks are realized, the distribution of shocks affects the input-output structure of the economy.

2.1 Firms and production functions

There are n industries, indexed by $i \in \mathcal{N} = \{1, \ldots, n\}$, each producing a differentiated good. In each industry, there is a large number of identical firms that behave competitively so that equilibrium profits are always zero. When this creates no confusion, we work with a representative firm for each industry, and use industry i, product i and firm i interchangeably. Unless specified otherwise, all vectors are column vectors.

Each firm in industry i has access to a set of production techniques $\mathcal{A}_i$. A technique $\alpha_i \in \mathcal{A}_i$ specifies the set of inputs that are used in production, the proportions in which these inputs are combined, and a productivity shifter $A_i(\alpha_i)$. We model these techniques as Cobb-Douglas technologies that can vary in terms of factor shares and total factor productivity. It is therefore convenient to identify a technique $\alpha_i \in \mathcal{A}_i$ with the intermediate input shares associated with that technique, $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, and to write the corresponding production function as

$$F(\alpha_i, L_i, X_i) = e^{\varepsilon_i} \zeta(\alpha_i) A_i(\alpha_i) L_i^{1-\sum_{j=1}^n \alpha_{ij}} \prod_{j=1}^n X_{ij}^{\alpha_{ij}}, \tag{1}$$

where L_i is labor and $X_i = (X_{i1}, \ldots, X_{in})$ is a vector of intermediate inputs. The term ε_i is the stochastic component of a firm's total factor productivity. Finally, $\zeta(\alpha_i)$ is a normalization to simplify future expressions.[6]

Since a technique α_i corresponds to a vector of factor shares, we define the set of feasible production techniques $\mathcal{A}_i$ for industry i as

$$\mathcal{A}_i = \left\{ \alpha \in [0,1]^n : \sum_{j=1}^n \alpha_j \leq \overline{\alpha}_i \right\}, \tag{2}$$

[6]This term is useful to simplify the expression of the unit cost of production, given by (9) below. It could be included in $A_i(\alpha_i)$ without any impact on the model. Its value is given by $\zeta(\alpha_i) = \left[\left(1 - \sum_{j=1}^n \alpha_{ij}\right)^{1-\sum_{j=1}^n \alpha_{ij}} \prod_{j=1}^n \alpha_{ij}^{\alpha_{ij}}\right]^{-1}$.

where the constant $0 \leq \overline{\alpha}_i < 1$ provides a lower bound on the share of labor in the production of good i.[7] We denote by $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ the Cartesian product of the sets $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$, such that an element $\alpha \in \mathcal{A}$ corresponds to a set of input shares for each firm. As such, it fully characterizes the production network and firms, through their choice of techniques, can influence the structure of this network. Importantly, the set $\mathcal{A}$ allows firms to adjust the importance of a supplier at the margin or to not use a particular input at all by setting the corresponding share to zero. The model is therefore able to capture network adjustments along both the intensive and extensive margins.[8]

The choice of technique α_i also influences the total factor productivity of firm i through the term $A_i(\alpha_i)$ in (1). We impose the following structure on $A_i(\alpha_i)$.

Assumption 1. *$A_i(\alpha_i)$ is smooth and strictly log-concave.*

This assumption is both technical and substantial in nature. The strict log-concavity ensures that there exists a unique technique that solves the optimization problem of the firm. It also implies that, for each industry i, there is a set of *ideal* input shares α_{ij}° that maximize A_i and that represent the most efficient way to combine intermediate inputs to produce good i. These ideal shares are given by nature and might differ across industries. When deciding on its optimal production technique the firm will take these ideal shares into account, but it will also evaluate how expensive and uncertain each input is.

Example. One example of a function $A_i(\alpha_i)$ that satisfies Assumption 1 and that we will use in the quantitative part of the paper is the quadratic form

$$\log A_i(\alpha_i) = -\sum_{j=1}^{n} \kappa_{ij} \left(\alpha_{ij} - \alpha_{ij}^{\circ}\right)^2 - \kappa_{i0} \left(\sum_{j=1}^{n} \alpha_{ij} - \sum_{j=1}^{n} \alpha_{ij}^{\circ}\right)^2, \tag{3}$$

where $\alpha_i^{\circ} = (\alpha_{i1}^{\circ}, \ldots, \alpha_{in}^{\circ})$ represents the ideal TFP-maximizing input shares. The parameter κ_{ij} determines the cost, in terms of productivity, of moving the j^{th} input share α_{ij} away from its ideal share α_{ij}°. The last term captures a penalty from deviating from an ideal labor share.

Beliefs about the random productivity shocks, the term ε_i in 1, play a crucial role in the model. We collect the productivities of all industries in the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, which we assume to be normally distributed $\varepsilon \sim \mathcal{N}(\mu, \Sigma)$. The vector μ captures the overall optimism of agents about sectoral productivities. Similarly, the matrix Σ captures how uncertain agents are about ε, as well

[7] We impose $\overline{\alpha}_i < 1$ to rule out pathological cases in which the economy could produce an infinite quantity of goods without using any labor.

[8] This is in stark contrast with standard network models with CES production functions. In those models, the observed share of an input can fluctuate but it can never reach zero. As a result, these models cannot generate the destruction or creation of links in the production network. The presence of both margins also highlights a distinguishing feature of our setup compared to Oberfield (2018) and Acemoglu and Azar (2020). In both of these setups, only the extensive margin is active.

as the perceived correlations between its elements. The vector ε is the only source of uncertainty in this economy.

In equilibrium, ε will have a direct impact on prices, and the beliefs (μ, Σ) will affect expectations about the price system. For instance, a firm with a high μ_i will have a low unit cost and sell at a low price, in expectation. Similarly, a high Σ_{ii} firm is subject to large productivity shocks which translate into a volatile price. Since production technique must be chosen before ε is realized, the beliefs (μ, Σ) affect the sourcing decisions of the firms. For instance, if carbon fiber prices are expected to increase or to be more volatile, a car manufacturer may switch to using steel instead for a few components. If the change is large enough, the manufacturer may switch away from using carbon fiber altogether, in which case the link with carbon fiber suppliers would disappear from the production network.

2.2 Household preferences

A representative household supplies one unit of labor inelastically and chooses consumption $C = (C_1, \ldots, C_n)$ to maximize

$$u\left(\left(\frac{C_1}{\beta_1}\right)^{\beta_1} \times \cdots \times \left(\frac{C_n}{\beta_n}\right)^{\beta_n}\right), \tag{4}$$

where $\beta_i > 0$ for all i and $\sum_{i=1}^n \beta_i = 1$.[9] The utility function u is CRRA with a coefficient of risk aversion $\rho \geq 1$.[10] The household makes consumption decisions after uncertainty is revealed and so in each state of the world it faces the budget constraint

$$\sum_{i=1}^{n} P_i C_i \leq 1, \tag{5}$$

where P_i is the price of good i and where we use the wage as numeraire so that $W = 1$.[11]

Firms are owned by the representative household and maximize expected profits discounted by the household's stochastic discount factor Λ. In Appendix C.1, we show that Λ is given by

$$\Lambda = u'(Y) \times 1/\overline{P}, \tag{6}$$

[9] The model can handle $\beta_i = 0$ for some goods at the cost of extra complications in the proofs.

[10] The CRRA assumption is necessary for the stochastic discount factor to be log-normally distributed and, therefore, for the model to remain tractable. The case $0 < \rho < 1$ is straightforward to characterize but is somewhat unnatural since the household seeks to increase consumption uncertainty in this case. To see this, consider that since $\log Y$, where $Y = \prod_{i=1}^n \left(\beta_i^{-1} C_i\right)^{\beta_i}$, is normally distributed, maximizing $\mathrm{E}\left[C^{1-\rho}\right]$ amounts to maximizing $\mathrm{E}[\log Y] - \frac{1}{2}(\rho - 1)\mathrm{V}[\log Y]$ such that $\rho \lesseqgtr 1$ indicate whether the household likes uncertainty or not. This is a consequence of the usual increase in the mean of $\log Y$ from an increase in the standard deviation of Y coming from the lognormality.

[11] There is a different real wage associated with each state of the world (or, equivalently, per realization of ε). However, since P and W are both conditional on the state of the world and only the ratio P/W matters for outcomes, hence setting $W = 1$ is simply a normalization.

where $Y = \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i}$ is aggregate consumption and $\overline{P} = \prod_{i=1}^{n} P_i^{\beta_i}$ is the price index. The stochastic discount factor thus captures how much an extra unit of the numeraire contributes to the utility of the household in different states of the world. In our setting, aggregate consumption equals aggregate (real) GDP, so we also refer to Y as GDP in what follows.

From the optimization problem of the household it is straightforward to show that

$$y = -\beta' p, \tag{7}$$

where $y = \log Y$, $p = (\log(P_1), \ldots, \log(P_n))$ and $\beta = (\beta_1, \ldots, \beta_n)$.[12] This equation highlights that GDP is the negative of the sum of prices weighted by the household's consumption shares β. Intuitively, when prices are low relative to wages, the household can purchase more goods and aggregate consumption increases. Equation 7 also shows that it is sufficient to derive the vector of prices to determine GDP.

2.3 Unit cost minimization

We solve the problem of the firms in two stages. In the first stage, firms decide on which production technique to use. Importantly, this choice is made before the random productivity vector ε is realized. In contrast, consumption, labor and intermediate inputs are chosen (and their respective markets clear) in the second stage, after the realization of ε. This timing captures that production techniques take time to adjust, as they might involve retooling a plant, teaching new processes to workers, negotiating contracts with new suppliers, etc. We begin by deriving the optimal input choice of a firm in the second stage, with a given production technique α_i. The resulting expressions are then used to solve the firm's first-stage problem of choosing α_i.

Under a given production technique α_i, the cost minimization problem of the firm is

$$\begin{aligned} K_i(\alpha_i, P) = \min_{L_i, X_i} & \left(L_i + \sum_{j=1}^{n} P_j X_{ij} \right) \\ \text{subject to } & F(\alpha_i, L_i, X_i) \geq 1, \end{aligned} \tag{8}$$

where $P = (P_1, \ldots, P_n)$ is the price vector, L_i is the labor input and $X_i = (X_{i1}, \ldots, X_{in})$ is the vector of intermediate inputs.

This problem implicitly defines the unit cost of production $K_i(\alpha_i, P)$, which plays an important role in our analysis. Since, for a given α_i, the firm operates a constant returns to scale technology, K_i does not depend on the scale of the firm and is only a function of the (relative) prices P. It is straightforward to show (and we do so in Appendix C.2) that with the production function (1) the

[12] See Appendix C.1 for a derivation of that equation.

unit cost function is

$$K_i(\alpha_i, P) = \frac{1}{e^{\varepsilon_i} A_i(\alpha_i)} \prod_{j=1}^{n} P_j^{\alpha_{ij}}. \tag{9}$$

Equation (9) is the standard unit cost for a Cobb-Douglas production function. It states that the cost of producing one unit of good i is equal to the geometric mean of the individual input prices (weighed by their respective shares) and adjusted for the firm's total factor productivity. As such, the unit cost $K_i(\alpha_i, P)$ rises when inputs become more expensive and declines when the firm becomes more productive.

In equilibrium, competitive pressure between firms in the same industry will push prices to be equal to unit cost so that

$$P_i = K_i(\alpha_i, P) \text{ for all } i \in \{1, \ldots, n\}. \tag{10}$$

For a given network $\alpha \in \mathcal{A}$, this equation, together with (9), will allow us to fully characterize the price system as a function of the random productivity shocks ε.

2.4 Techniques choice

We can now turn to the first stage of the firm's problem, which is to pick a technique $\alpha_i \in \mathcal{A}_i$ to maximize expected profits,

$$\alpha_i^* \in \arg\max_{\alpha_i \in \mathcal{A}_i} \mathrm{E}\left[\Lambda Q_i \left(P_i - K_i(\alpha_i, P)\right)\right], \tag{11}$$

where Q_i is the equilibrium demand for good i and where the firm uses the stochastic discount factor Λ of the household to weigh profits in different states of the world.[13] Firms take prices P, demand Q_i and the stochastic discount factor Λ as given and so the only term in (11) over which the firm has any control is the unit cost $K_i(\alpha_i, P)$. The technique choice problem can therefore be written as

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathrm{E}\left[\Lambda Q_i K_i(\alpha_i, P)\right].$$

The firm thus selects a technique $\alpha_i \in \mathcal{A}_i$ to minimize the expected discounted value of the total cost of goods sold $Q_i K_i(\alpha_i, P)$, while taking into consideration that final consumption goods are valued differently across different states of the world, as captured by Λ.

2.5 Equilibrium conditions

An equilibrium is defined by the following conditions holding simultaneously.

[13] In equation 11 the representative firm in industry i faces the whole demand Q_i for good i. Because of the constant returns assumption, we could equivalently write that there are m firms, each supplying Q_i/m units of the good.

Definition 1. An equilibrium is a choice of technique for every firm $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$ and a stochastic tuple $(P^*, C^*, L^*, X^*, Q^*, \Lambda^*)$ such that

1. (Optimal technique choice) For each $i \in \{1, \ldots, n\}$, factor demand L_i^* and X_i^* are a solution to (8), and the technology choice $\alpha_i^* \in \mathcal{A}_i$ solves (11) given prices P^*, demand Q_i^* and the stochastic discount factor Λ^* given by (6).

2. (Consumer maximization) The consumption vector C^* maximizes (4) subject to (5) given prices P^*.

3. (Unit cost pricing) For each $i \in \{1, \ldots, n\}$,

$$P_i^* = K_i\left(\alpha_i^*, P^*\right), \tag{12}$$

   where $K_i\left(\alpha_i^*, P^*\right)$ is given by (9).

4. (Market clearing) For each $i \in \{1, \ldots, n\}$,

$$\begin{aligned} Q_i^* &= C_i^* + \sum_{j=1}^{n} X_{ji}^*, \\ Q_i^* &= F_i\left(\alpha_i^*, L_i^*, X_i^*\right), \\ \sum_{i=1}^{n} L_i^* &= 1. \end{aligned} \tag{13}$$

Condition 2, 3 and 4 correspond to the standard competitive equilibrium conditions for an economy with a fixed production network. They imply that firms and the household optimize in a competitive environment and that all markets clear given equilibrium prices. Condition 1 emphasizes that the production techniques, and hence the production network represented by the matrix α^*, are equilibrium objects that are affected by the primitives of the economy.

3 Equilibrium prices and GDP in a fixed-network economy

Before analyzing how the equilibrium production network α^* responds to changes in the environment, it is useful to first establish how prices and GDP depend on productivity under a fixed production network. This will allow us to characterize the uncertainty that a firm is facing when choosing its production technique.

To this end, we establish a first result that links the vector of firm-level productivities with prices and GDP.

Lemma 1. *For a given production network α,*

$$p(\alpha) = -\mathcal{L}(\alpha)(\varepsilon + a(\alpha)), \tag{14}$$

and

$$y = \beta' \mathcal{L}(\alpha)(\varepsilon + a(\alpha)) \tag{15}$$

where $a(\alpha) = (\log A_i(\alpha_i), \ldots, \log A_n(\alpha_n))$ and $\mathcal{L}(\alpha) = (I - \alpha)^{-1}$ is the Leontief inverse.

Proof. All proofs are in Appendix D. □

Equation (14) is obtained by combining (9) and (10). Equation (15) then follows from (7). Lemma 1 describes how prices and GDP depend on the vector of firm-level productivities and the production network. We will describe both of these channels in turn.

First, consider the impact of the vector of productivities $\varepsilon + a(\alpha)$. Since all elements of β and $\mathcal{L}(\alpha)$ are non-negative, increases in firm-level productivities have a negative impact on prices and a positive impact on GDP.[14] Intuitively, as firms become more productive, their unit costs decline and competition forces them to sell at lower prices. From the perspective of GDP, higher productivity implies that the available labor can be transformed into more consumption goods.

Next, we turn to the impact of the production network. As the lemma makes clear, α matters for prices and GDP through two channels. First, it has a direct impact on the productivity shifters a simply because different techniques have different productivities. For instance, if a firm deviates from its ideal input shares, its TFP declines which pushes for higher prices and lower GDP. Second, α also affects prices and GDP through its impact on the Leontief inverse. The matrix $\mathcal{L}(\alpha) = (I - \alpha)^{-1} = I + \alpha + \alpha^2 + \ldots$ implies that the price of good i depends not only on the productivity of i itself, but also on the productivity of all of its suppliers, and on the productivity of all of *their* suppliers, and so on. These higher-order connections also matter for GDP and so the impact of a firm's productivity depends on its importance as a direct and indirect supplier.

To better characterize which producers are important suppliers, we can rewrite (15) as

$$y = \omega' (\varepsilon + a(\alpha)) \tag{16}$$

where $\omega = (\omega_1, \ldots, \omega_n)$ is the vector of Domar weights $\omega_i = \beta' \mathcal{L}(\alpha) 1_i > 0$, which are also equal to the sales share of firm i in nominal GDP, so that $\omega_i = \frac{P_i Q_i}{P' C}$.[15] The Domar weights thus determine the relative importance of sectoral productivity changes in an economy with a fixed production network. As they depend only on β and α, Domar weights are constant in a fixed-network economy

[14] The non-negativity of $\mathcal{L}$ comes from the restrictions imposed on the sets of techniques $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. These restrictions also imply that $I - \alpha^*$ is always strictly diagonally dominant and therefore invertible.

[15] See the proof of Corollary 1 for a proof of that result. We use 1_i to denote an $n \times 1$ vector full of zeros except for a 1 at the i^{th} position.

but will vary when firms are free to adjust their sourcing decisions in response to changes in beliefs. In particular, a change in the production network that would make a given sector a more important supplier would also increase the importance of that sector's productivity for aggregate GDP.

Finally, Lemma 1 also show that the price vector p and GDP y are linear functions of the productivity vector ε and, as a result inherit the normality ε. This result is essential for the tractability of the model and allows us to compute the first and second moments of GDP as

$$\mathrm{E}\left[y\left(\alpha\right)\right] = \beta'\mathcal{L}\left(\alpha\right)\left(\mu + a\left(\alpha\right)\right), \tag{17}$$

$$\mathrm{V}\left[y\left(\alpha\right)\right] = \beta'\mathcal{L}\left(\alpha\right)\Sigma\mathcal{L}\left(\alpha\right)'\beta. \tag{18}$$

It is clear from these equations that the production network α matters for the mean and the variance of GDP. In addition, one important implication of 17 is that the variance Σ of ε has no impact on expected GDP, except through its influence on the structure of the network. It follows that whenever we discuss the response of expected GDP to a change in uncertainty, the mechanism of action is through the endogenous reorganization of the network.

We conclude this section with a corollary that describes the impact of firm-level shocks on the mean and the variance of GDP. In what follows, we use partial derivatives to emphasize that the network α is kept fixed.

Corollary 1. *For a fixed network α:*

1. *The impact of a change in firm-level expected TFP μ_i on expected GDP $\mathrm{E}\left[y\right]$ is given by*

$$\frac{\partial\,\mathrm{E}\left[y\right]}{\partial\mu_i} = \omega_i.$$

2. *The impact of a change in firm-level volatility Σ_{ij} on the variance of GDP $\mathrm{V}\left[y\right]$ is given by*[16]

$$\frac{\partial\,\mathrm{V}\left[y\right]}{\partial\Sigma_{ij}} = \begin{cases}\omega_i^2 & i=j,\\ 2\omega_i\omega_j & i\neq j.\end{cases}$$

The first part of the lemma demonstrates that for a fixed production network, Hulten's (1978) celebrated theorem also holds in expectational terms. That is, the change in expected GDP following a change in the expected productivity of an industry i is equal to that industry's sales share ω_i. The second part of the lemma establishes a similar result for changes in volatility. In this case, we see that the impact on the variance of GDP of an increase in the uncertainty of the TFP of a sector is equal to the square of that sector's sales share. As a result of the quadratic nature of this relationship, sectors with large ω_i's are disproportionately important for the variance of GDP. The

[16]For $i \neq j$, the following derivative simultaneously changes Σ_{ij} and Σ_{ji} to preserve the symmetry of Σ.

corollary also describes how aggregate volatility responds to a change in the correlation between two sectors. In this case, the increase in $\mathrm{V}[y]$ is proportional to the product of the two industries' sales shares. Since Domar weights are always positive, an increase in correlation always leads to higher aggregate volatility. Intuitively, positively correlated shocks do not offset each other, and their aggregate impact is therefore larger.

Finally, Corollary 1 emphasizes that for a fixed network, knowing the sales shares of every industry is sufficient to compute the impact of changes to μ and Σ on GDP. In the next section, we show that this is no longer true when firms can adjust their input shares in response to changes in the distribution of sectoral productivity. In fact, when the network is free to adjust, an increase in μ can even lead to a decline in expected GDP.

4 Equilibrium production networks

In the full equilibrium the production network endogenously responds to changes in beliefs. We begin by characterizing how firms select a production technique in this environment. We then establish that an equilibrium exists under general conditions. We also show that there exists a Pareto efficient equilibrium and that its associated production network is characterized by a trade-off between the expected level and the volatility of GDP.

4.1 Technique choice

In the previous section, we described prices under a given equilibrium network α^*. Here, we use that information to characterize the problem of an individual firm i that must choose a technique $\alpha_i \in \mathcal{A}_i$. To solve the firms' technique choice problem, it is convenient to work with the log of the stochastic discount factor $\lambda(\alpha^*) = \log \Lambda(\alpha^*)$, the log of the unit cost $k_i(\alpha_i, \alpha^*) = \log K_i(\alpha_i, P^*(\alpha^*))$ and the log of aggregate demand $q_i(\alpha^*) = \log Q_i(\alpha^*)$. The following lemma shows that all these objects are normally distributed and describes how they influence the firm's problem.

Lemma 2. *$\lambda(\alpha^*)$, $k_i(\alpha_i, \alpha^*)$ and $q_i(\alpha^*)$ are normally distributed and the technique choice problem of the firm can be written as*

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathrm{E}\left[k_i(\alpha_i, \alpha^*)\right] + \frac{1}{2}\mathrm{V}\left[k_i(\alpha_i, \alpha^*)\right] + \mathrm{Cov}\left[k_i(\alpha_i, \alpha^*), \lambda(\alpha^*) + q_i(\alpha^*)\right]. \tag{19}$$

This equation is central to the mechanisms of the model because it captures how beliefs and uncertainty affect the production network. The first term implies that the firm prefers to adopt techniques that provide, in expectation, a lower unit cost of production. Taking the expected value of the log of (9), we can write this term as

$$\mathrm{E}\left[k_i\left(\alpha_i, \alpha^*\right)\right] = \text{const} - a_i\left(\alpha_i\right) + \sum_{j=1}^{n} \alpha_{ij}\, \mathrm{E}\left[p_j\right],$$

so that, unsurprisingly, the firm prefers techniques that have high productivity a_i and that relies on inputs that are expected to be cheap.

The second term in (19) shows that the firm also prefers production techniques that lower the variance of that unit cost. Once again it is enlightening to use (9) to write

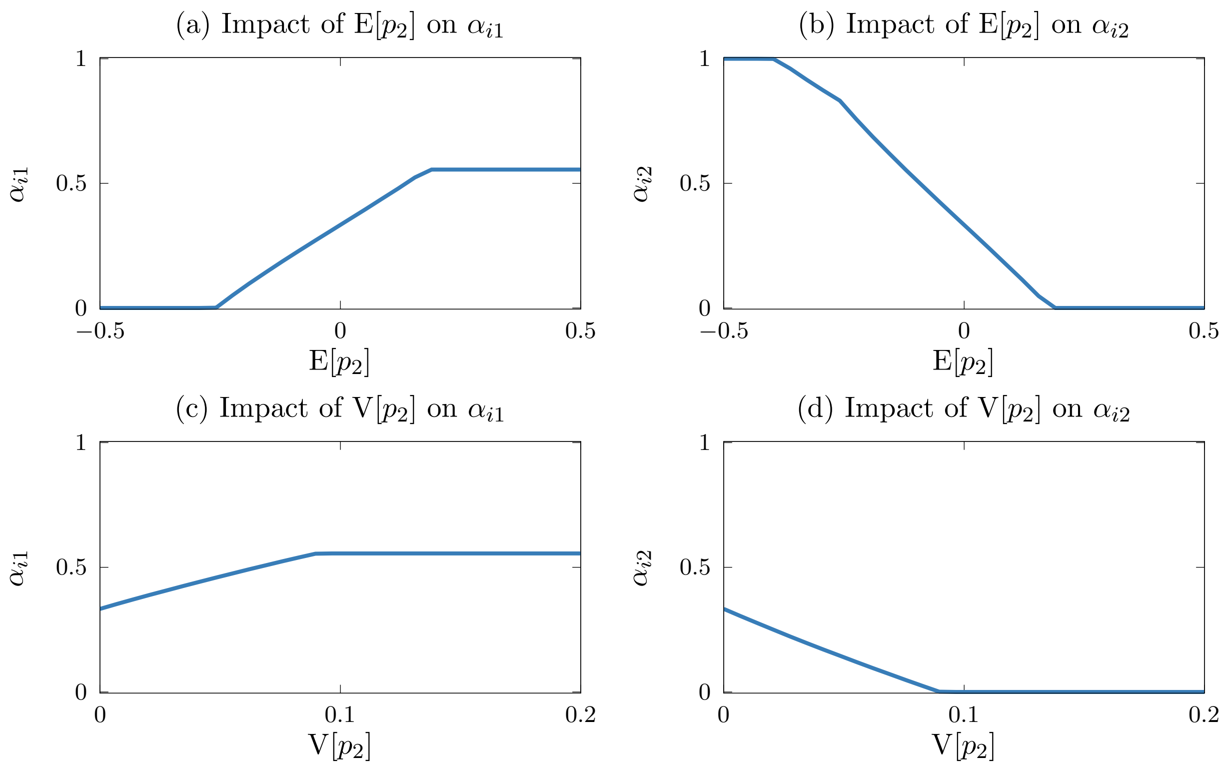
$$\mathrm{V}\left[k_i\left(\alpha_i, \alpha^*\right)\right] = \text{const} + \sum_{j=1}^{n} \alpha_{ij}^2\, \mathrm{V}\left[p_j\right] + \sum_{j \neq k} \alpha_{ij}\alpha_{ik}\, \mathrm{Cov}\left[p_j, p_k\right] + 2\, \mathrm{Cov}\left[-\varepsilon_i, \sum_{j=1}^{n} \alpha_{ij} p_j\right]. \tag{20}$$

This equation highlights three channels that affect the variance of the unit cost. First, the firm prefers to rely on inputs that are stable (first term in (20)). Second, the firm wants to avoid techniques that relies heavily on inputs with correlated prices (second term in (20)). These techniques might lead to large fluctuations in unit cost. Instead, the firm seeks to *diversify* its set of suppliers and adopt inputs whose variation in prices offset each other. Third, the firm prefer inputs whose prices are positively correlated with its own productivity shocks (third term in (20)). In this case, when the firm faces a low productivity the price of its inputs is low, which cushions the blow to its unit cost.

Finally, the third term in (19) captures the importance of *aggregate risk* for the firm's decision. It implies that the firm prefers suppliers whose products are cheap in states of the world in which the marginal utility of aggregate consumption is high or in which demand for the firm's goods is large. As a result, the coefficient of risk aversion ρ of the household indirectly determines how risk averse firms are.

We will explore in more details how beliefs affect the structure of the network in general equilibrium in the next section, but for now it is useful to highlight some of the key forces that affect the choice of technique of a firm by considering the following partial equilibrium example.

Example (Sourcing decisions in partial equilibrium)**.** Consider again the car manufacturer (firm i) that can use steel (good 1) and carbon fiber (good 2) as intermediate inputs in production. The firm must decide on the optimal shares α_{i1} and α_{i2} to pick. Assume that the input prices are $p = (p_1, p_2) \sim \mathcal{N}\left(\mathrm{E}\left[p\right], \mathrm{V}\left[p\right]\right)$ where the covariance matrix $\mathrm{V}\left[p\right]$ is diagonal. If we let the penalty function $A_i\left(\alpha_i\right)$ take the form (3), it is straightforward to solve the minimization problem (19), and we show in Figure 1 how the solution α_i^* is affected by changes in the mean and the variance of p_2. We see from panels (a) and (b) that, unsurprisingly, when good 2 is expected to be cheaper, firm i increases α_{i2} and lowers the share of good 1. A similar mechanism is at work when uncertainty about p_2 increases, as seen in Panels (c) and (d). When $\mathrm{V}\left[p_2\right]$ is large, the firm prefers to use a larger share of the relatively safer good 1. Notice that the share α_{i2} reaches zero when p_2 is expected to be sufficiently large or uncertain. In that case, firm i simply severs the link with the carbon fiber supplier and an input/output relationship disappears from the production network. In this example, both the intensive and extensive margins of network adjustment are thus active.



Notes: Parameters: $\rho = 5$, $\kappa = (1/3, 1/3, 1/3)$, $\alpha^\circ = (1/3, 1/3)$, $\mathrm{E}[p_1] = 0$ and $\mathrm{V}[p_1] = 0$. p_i is uncorrelated with (p_1, p_2).

Figure 1: Beliefs and input shares

4.2 Equilibrium existence and efficiency

In this section, we finally consider the full equilibrium with endogenous network. We first establish that there exists an equilibrium that satisfies the conditions in Definition 1 and that this equilibrium is efficient. Building on that insight, we further show that the production network in this equilibrium strikes an optimal balance between maximizing the mean level of aggregate output and minimizing its variance.

Existence of an equilibrium

Lemma 2 describes a self-map $\mathcal{K} : \mathcal{A} \to \mathcal{A}$ that can be used to define an equilibrium network α^*. At a fixed point of this mapping, we have that $\alpha_i^* = \mathcal{K}_i(\alpha^*)$ for all $i \in \mathcal{N}$, where $\mathcal{K}_i(\alpha^*)$ is the right-hand side of (19). Hence, such a fixed point describes an equilibrium network. Lemma 3 establishes that such a fixed point exists under general conditions.

Lemma 3. *There exists a production network α^* such that $\alpha^* = \mathcal{K}(\alpha^*)$.*

The proof first shows that $\mathcal{K}$ is a continuous mapping on the compact set $\mathcal{A}$. From Brouwer's fixed point theorem, we then know that there exists at least one element $\alpha^* \in \mathcal{A}$ such that $\alpha^* = \mathcal{K}(\alpha^*)$.[17]

With an equilibrium network α^* in hand, it is straightforward to compute prices from (15) and, from there, all other equilibrium quantities can be uniquely determined. The following proposition follows directly as a result.

Proposition 1. *An equilibrium exists.*

While Proposition 1 guarantees the existence of an equilibrium, it is silent about the number of such equilibria. However, the next subsection demonstrates that there always exists a Pareto efficient equilibrium, which provides a natural benchmark to study.

Pareto efficiency

There is a unique consumer in this economy, and hence finding the set of Pareto efficient allocations amounts to solving the problem of a social planner that maximizes the utility function (4) of the consumer subject to the resource constraints (13). We begin with a first result that describes the relationship between the Pareto efficient allocation and the set of equilibria. [18]

Proposition 2. *There exists an efficient equilibrium.*

This result is important for a few reasons. First, it provides a natural equilibrium selection device if multiple equilibria were to exist, and from here on, we will focus on the efficient equilibrium.[19] Second, it shows that the economy is undistorted by externalities or other market imperfections. The forces at work in the decentralized equilibrium are thus fundamental features of the environment that should not be distorted by policy makers. Third, Proposition 2 allows us to investigate the properties of the equilibrium by solving the problem of the social planner directly. Building on that insight, it is convenient to characterize the equilibrium network as the outcome of a welfare maximization problem, as the following lemma shows.

Corollary 2. *The equilibrium production network α^* solves*

$$\mathcal{W} \equiv \max_{\alpha \in \mathcal{A}} \mathrm{E}\left[y(\alpha)\right] - \frac{1}{2}(\rho - 1)\,\mathrm{V}\left[y(\alpha)\right], \tag{21}$$

[17]While the mapping $\mathcal{K}$ is in general not a contraction, iterating on that mapping turns out to be a convenient method for finding a fixed point. When this fails, an equilibrium can be found by solving the planner's problem, as we explain below. In the appendix, we group the proof of Lemma 3 with that of Proposition 1 below.

[18]We discuss conditions under which the solution to the planner's problem is generically unique at the end of Appendix (2). In particular, we establish a generic uniqueness result when $A_i(\alpha_i)$ takes the form (3), which we will adopt for our quantitative exercises.

[19]The Pareto efficient equilibrium is a natural benchmark to study, since any inefficient equilibrium would be the result of coordination failure among agents. While such coordination failures may exist in reality, they are not the focus of this paper.

where y is GDP as defined in (15).

Notice that (21) provides an expression for the welfare $\mathcal{W}$ of the representative household.

Corollary 2 follows directly from the fact that, by Proposition 2, the equilibrium network α^* must maximize the expected utility of the representative consumer. It is clear from the objective function (21) that the consumer prefers networks that strike a balance between maximizing expected GDP $\mathrm{E}[y(\alpha)]$ and minimizing aggregate uncertainty $\mathrm{V}[y(\alpha)]$, with the relative risk aversion ρ determining the importance of each term. Another consequence of Corollary 2 is that it casts a complicated network formation problem as a simple optimization problem. We will rely on this result in the next section to characterize how beliefs affect the structure of the production network.

The model studied above is relatively simple, but it is straightforward to extend it along several dimensions without losing tractability. For instance, we can generalize the set of techniques $\mathcal{A}_i$ to include lower and upper bounds on specific input shares. These bounds could be used to impose that certain sectors need a given input to produce or, inversely, can never use an input into production. It is also straightforward to extend the model to include multiple types of labor. In this case, we could separate firms between domestic and foreign ones, each using only one type of labor, and use the model to investigate the impact of beliefs and uncertainty on trade networks. In this case, trade costs could also be introduced by imposing that goods that are traded internationally transit through a fictitious "transportation" sector with a productivity less than one.[20]

5 Understanding the determinants of the production network

We now explore the economic forces at work in the model. In particular, we investigative how variations in the mean μ and the variance Σ of the productivity vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ affect the structure of the production network and how these changes, in turn, affect aggregate GDP and welfare.

5.1 Beliefs and Domar weights

In Section 3, we saw that Domar weights are key objects to understand how changes in μ and Σ affect GDP. In a fixed-network environment, these weights are fixed and do not respond themselves to changes in beliefs. In our model, the Domar weights are equilibrium objects and the next proposition shows how they respond to changes in beliefs about sectoral productivity.

Proposition 3. *The Domar weight ω_i of firm i is increasing in μ_i and decreasing in Σ_{ii}.*

[20]On the other hand, certain ingredients are essential to keep the model tractable. Here the key challenge comes from the fixed point between the choice of technique and the beliefs about equilibrium quantities. The Cobb-Douglas aggregators in (1) and (4), as well as the CRRA preferences and the normality of ε, are needed to keep the equilibrium beliefs normally distributed.

Proposition 3 shows that when the network is flexible, the Domar weights are increasing in the expected productivity of a firm and decreasing in its variance. This result can be understood intuitively both from an individual firm's perspective as well as from the perspective of the social planner. An individual firm relies more on suppliers whose prices are low and stable. As a result, these firms become more important suppliers and the Domar weights of the associated sectors increase. From the planner's perspective, recall from (16) that the Domar weight of a firm captures the contribution of its productivity to GDP. Since the planner wants to increase and stabilize GDP, it naturally increases the importance of more productive (larger μ_i) or less volatile (smaller Σ_{ii}) sectors in the production network. In Section 5.3 below we show that such an adjustment in the network is welfare-improving. But before doing so, we first discuss how changes in beliefs affect the precise structure of the equilibrium production network α.

5.2 Beliefs and the structure of the production network

Proposition 3 establishes that the Domar weights respond in an intuitive and unambiguous manner to changes in beliefs. The same is not true about individual elements of the matrix α that describes the complete structure of the production network. In fact, in some cases an increase in the expected productivity of a producer i can even lead some of its customer to lower their usage of input i. To shed light on the forces that affect the input shares, we first consider economies that satisfy a weak complementarity property, which we define below. Under this condition, we provide a sharp characterization of how the production network responds to changes in beliefs. We then discuss what happens when the structure of economy does not satisfy this property.

Response of the network when shares are complements

In this section, we consider economies in which the functions $(a_1, \dots, a_n)$ satisfy the following property.

Assumption 2 (Weak Complementarity)**.** *For all i, a_i satisfies $\frac{\partial^2 a_i(\alpha_i)}{\partial \alpha_{ij} \partial \alpha_{ik}} \geq 0$ for all $j \neq k$.*

Assumption 2 defines a weak complementarity property between the shares that a producer allocates to its suppliers. It states that as a firm increases the share of one input, the marginal benefit of increasing the share of the other inputs weakly increases as well. In the context of the function (3) described in our earlier example, weak complementarity is satisfied if $\kappa_{i0} \leq 0$.

The following lemma shows that the impact of μ and Σ on the equilibrium network is straightforward when Assumption 2 holds.

Lemma 4. *Let $\alpha^* \in int(\mathcal{A})$ be the equilibrium network and suppose that Assumption 2 holds. There exists a $\overline{\Sigma} > 0$ such that if $|\Sigma_{ij}| < \overline{\Sigma}$ for all i, j, there is a neighborhood around α^* in which*

(i) an increase in μ_j leads to an increase in the shares α^*_{kl} for all k, l;

(ii) an increase in Σ_{jj} leads to a decline in the shares α_{kl}^* for all k, l;

(iii) an increase in Σ_{ij} leads to a decline in the shares α_{kl}^* for all k, l.

Part (i) if this lemma shows that when μ_j increases there is a widespread increase in input shares throughout the economy. To understand this result, it is useful to decompose the impact of the change into three channels: 1) the direct impact, 2) the indirect impact, and 3) the complementarity effect. First, the increase in μ_j makes good j cheaper in expectation which pushes all of j's direct customers to increase their share of j in production. Second, all of j's customers now benefit from cheaper input prices, which makes their own goods cheaper through competition, and so other firms are also increasing their share of these goods into production (indirect effect). Finally, these increases in shares from the direct and indirect effects push firms to adopt techniques with higher input shares, because of the complementarities implied by Assumption 2. Taking these effects together, all shares α in the economy increase, and so the entire production sector moves away from labor.

Parts (ii) and (iii) of Lemma 4 provide similar results for increases in uncertainty and in correlations. As discussed in Section 4.1, firms prefer suppliers with stable and uncorrelated prices. As a result, the additional risk introduced by a higher Σ_{jj} pushes j's direct an indirect customers to reduce their exposure to j. An increase in correlation Σ_{ij} also pushes firms to avoid inputs i and j. The complementarity effect is also at work, and so firms overall move toward production techniques that are more labor intensive.[21]

Substitution between inputs

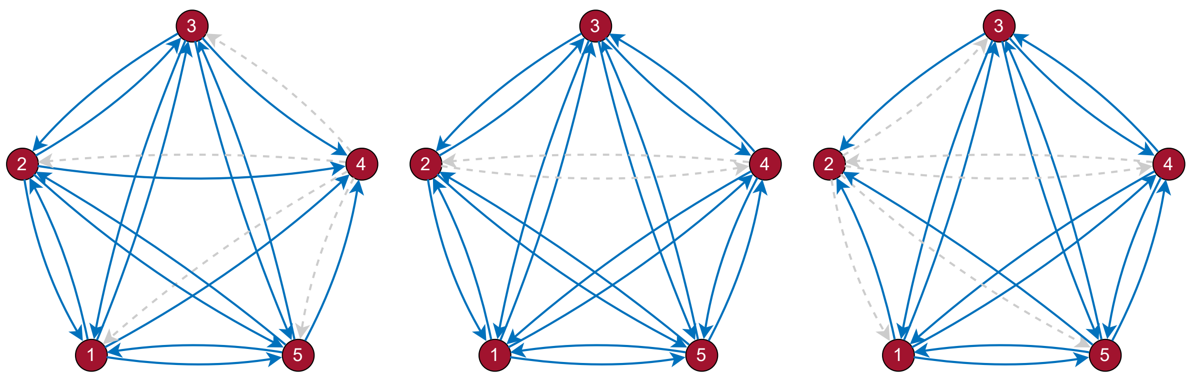
Lemma 4 makes sharp predictions about the structure of the network at the cost of some restrictions about the functions $(a_1, \ldots, a_n)$. These restrictions impose some form of complementarity between input shares, but the model can handle much richer substitution patterns. To give an example of what these patterns might involve, we can go back to our car manufacturer example. Suppose that the price of carbon fiber is expected to decrease (higher μ). The firm might respond by increasing the share of carbon fiber and decrease the share of steel it uses in production. At the same time, it might purchase additional equipment that is needed to handle carbon fiber. These changes highlight different substitution patterns (substitution between steel and carbon fiber, complementarity between steel and equipment) that can exist between inputs.

The theory, through the constraints embedded in the set $\mathcal{A}$ and the shape of the functions $(a_1, \ldots, a_n)$, is rich enough to accommodate some inputs that are complements while at the same

[21] The assumption $\alpha^* \in \operatorname{int}(\mathcal{A})$ in Lemma 4 is needed to avoid potential substitution patterns between firms. For instance, if $\sum_{k=1}^n \alpha_{ik} = \overline{\alpha}_i$ for a given firm i, an increase in μ_j might lead to a decline in some α_{ik}, $k \neq j$, to accommodate an increase in α_{ij}. The restriction on Σ is needed to prevent a strong uncertainty feedback. For example, if all firms increase their reliance on sector j (e.g., due to an increase in μ_j or a reduction in Σ_{jj}), the economy's exposure to j's risk may become so large that it will be optimal to reduce α_{kj} for some k instead. This does not happen when Σ is sufficiently small.

time others are substitutes. For instance, if the constraint $\sum_{k=1}^{n} \alpha_{ik} \leq \overline{\alpha}_i$ binds, firm i might need to lower the share of another input k to be able to increase α_{ij} after a decline in the expected price of j, in the shares of i and j would be substitutes. Similarly, the functional form of a can generate complementarities between inputs. As an example, consider the function $a(\alpha_i) = -(\alpha_{i1} - \alpha_{i1}^{\circ})^2 - (\alpha_{i1} - \alpha_{i2})^2$. In this case, any increase in α_{i1} will be accompanied by additional incentives to increase α_{i2}. The function a can also be specified to generate substitutabilities, for instance through the last term in (3) when $\kappa_{i0} > 0$.

Figure 2 provides an example of how substitution patterns might arise in equilibrium when we relax Assumption 2. Panel (a) shows the equilibrium network in an economy in which all firms are identical except that firm 4 is slightly less productive and, as a result, does not sell to other firms. Panel (b) shows the same economy except that ε_2 is now slightly more volatile. In response, other producers seek to diversify their set of suppliers and create new supply relationships with firm 4. In panel (c) ε_2 becomes much more volatile. As a result, all producers drop firm 2 as a supplier and reinforce their connection to firm 4. In this example, the substitution comes from the fact that firms do not want to deviate too much from an ideal labor share (last term in (3)).



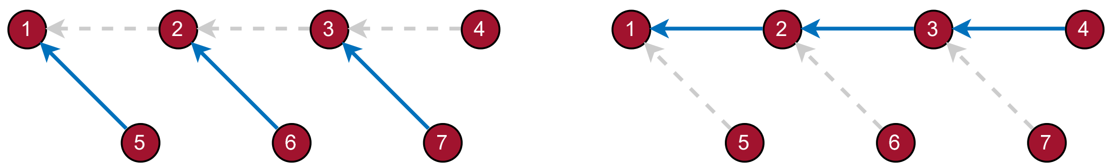
(a) Initial network (b) Small increase in Σ_{22} (c) Large increase in Σ_{22}

Figure 2: Uncertainty and the equilibrium network

Notes: Arrows represent the movement of goods: there is a blue solid arrow from j to i if $\alpha_{ij} > 0$. Dashed grey arrow indicate $\alpha_{ij} = 0$. a is as in (3) with the elements of κ equal to 1, except $\kappa_{ii} = \infty$ for all i. The elements of α° are 1/10 except $\alpha_{ii}^{\circ} = 0$ for all i. $\beta_i = 1/n$ for all i. $\mu = 0.1$ except for $\mu_4 = 0.0571$. $\Sigma = 0.3 \times I_{n \times n}$ in Panel (a). Panel (b): same as Panel (a) except $\Sigma_{22} = 0.35$. Panel (c): same as Panel (a) except $\Sigma_{22} = 1$. The risk aversion of the household is $\rho = 5$.

Cascading flight to safety

One consequence of the impact of uncertainty on the network when input shares are substitutes is that small changes in the volatility of a firm can push multiple producers to sequentially switch to safer suppliers. To give an example of that process, consider the simple economy depicted in Figure 3. Firms 4 to 7 can only use labor as an input, but firms 1 to 3 can each source inputs from two potential suppliers, indicated by the arrows. The model is parametrized such that shares of these suppliers are substitutes. When the productivity of firm 4 is uncertain (left figure), other producers avoid using it as a supplier. But as that uncertainty decreases, firm 3, seeking a stable supply of goods, switches to using good 4 as an input. As a result, firm 3's price becomes less volatile which pushes firm 2 to use good 3 in production. The same logic applies to firm 1, which also switches to the less volatile price provided by firm 2. As we can see, a change in the uncertainty associated with the productivity of a single firm can lead to a cascading movement to safety that reduces aggregate uncertainty.



(a) High uncertainty about ε_4 (b) Low uncertainty about ε_4

Figure 3: Cascading impact of Σ_{44}

Notes: Arrows represent the movement of goods: there is a blue solid arrow from j to i if $\alpha_{ij} > 0$. Dashed grey arrow indicate $\alpha_{ij} = 0$. a is as in (3) with the elements of κ equal to 0 if there is a potential link between two firms and infinity otherwise. Similarly, the elements of α° are one half when there is a potential link, 0 otherwise. $\mu = 0$ except for $\mu_4 = 0.1$. In the left figure, Σ is diagonal with each element equal to 0.1 except $\Sigma_{44} = 1$. In the right figure $\Sigma_{44} = 0$. The risk aversion of the household is $\rho = 2$. $\beta_i = 1/n$ for all i.

5.3 Implications for GDP and welfare

We now turn to the implications of the endogenous network mechanism for macroeconomic aggregates. We have already established in Corollary 1 how shocks to the mean μ and the variance Σ of productivity affect aggregate output when the production network is fixed. Here we generalize these results to our environment with an endogenous network, and further show that some shocks can have counterintuitive effects when the network itself responds to changes in the distribution of shocks.

Uncertainty lowers expected GDP

We begin with a general result that shows how GDP reacts to uncertainty.

Proposition 4. *Uncertainty lowers the expected value of GDP in equilibrium, such that* $\mathrm{E}[y]$ *is largest when* $\Sigma = 0$.

Proposition 4 follows directly from Corollary (2). When there is no uncertainty ($\Sigma = 0$), the variance $\mathrm{V}[y(\alpha)]$ of GDP is zero for all networks $\alpha \in \mathcal{A}$, so that the equilibrium network maximizes only the expected value of GDP. When, instead, the TFP vector is uncertain ($\Sigma \neq 0$), the equilibrium network also seeks to lower $\mathrm{V}[y(\alpha)]$, which necessarily leads to a lower expected GDP.

Proposition 4 highlights a novel mechanism through which uncertainty reduces expected GDP. To understand the intuition behind it, consider the perspective of the firms in equilibrium. When there is no uncertainty, firms do no worry about risk and simply buy inputs from the most productive suppliers. As a result, the aggregate economy is particularly productive, and GDP is large. When some suppliers become risky, firms worry that their inputs might become expensive and, to prevent large fluctuations in their own unit cost, start purchasing from more stable but less productive suppliers. As a result, the aggregate economy becomes less productive and expected GDP falls.

The endogenous response of the network is essential for the result of Proposition 4. Indeed, in our model uncertainty affects expected GDP only through the endogenous response of the firms' sourcing decisions. As a result, the mechanism through which uncertainty lowers expected GDP is only active when the production network is flexible. If instead the shares α were fixed, uncertainty would have no impact on $\mathrm{E}[y]$.

Welfare and the distribution of shocks

Proposition 4 establishes that any form of uncertainty has an adverse effect on expected GDP when the network adjusts in response to shocks. Here, we investigate how firm-level shocks affect GDP and welfare. As we will see, the endogenous response of the network matters here as well. Throughout this section, we again use partial differentiation to indicate that a derivative is taken keeping the network α fixed.

We begin by establishing a result that shows the impact of firm-level shocks on the expected welfare $\mathcal{W}$ of the representative consumer, as defined in (21).

Proposition 5. *When the network α is free to adjust to changes in μ and Σ, the following holds.*

1. *The impact of an increase in μ_i on expected welfare is given by*

$$\frac{d\mathcal{W}}{d\mu_i} = \frac{\partial \mathrm{E}[y]}{\partial \mu_i} = \omega_i. \tag{22}$$

2. *The impact of an increase in Σ_{ij} on expected welfare is given by*

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = \begin{cases} -\frac{1}{2}(\rho-1)\left(\frac{\partial \mathrm{E}[y]}{\partial \mu_i}\right)^2 = -\frac{1}{2}(\rho-1)\,\omega_i^2 & i=j, \\ -(\rho-1)\frac{\partial \mathrm{E}[y]}{\partial \mu_i}\frac{\partial \mathrm{E}[y]}{\partial \mu_j} = -(\rho-1)\,\omega_i\omega_j & i \neq j. \end{cases} \tag{23}$$

This proposition follows directly from applying the envelope theorem to (21). Its first part states that the impact of an increase in μ_i on welfare is equal to its marginal impact on expected GDP, *taking the network α as fixed.* By Corollary 1, this quantity is also equal to the Domar weight ω_i of firm i. Since Domar weights are positive, it follows that an increase in μ_i always has a positive impact on welfare. The second part of the proposition provides a similar result for an increase in

Σ_{ij}. In this case, the impact of the shock is proportional to the product of the Domar weights ω_i and ω_j. Again, (23) implies that an uncertainty shock must necessarily lower welfare when $\rho > 1$.

Amplification and dampening

One important consequence of the endogenous reorganization of the network is that shocks that are beneficial to welfare are amplified while shocks that are harmful are dampened. The following proposition establishes this result formally.

Proposition 6. *Let $\alpha^*(\mu, \Sigma)$ be the equilibrium production network under (μ, Σ) and let $\mathcal{W}(\alpha, \mu, \Sigma)$ be the welfare of the household under the network α. Then the change in welfare after a shock from (μ, Σ) to (μ', Σ') is larger under a flexible network than under a fixed network, in the sense that*

$$\mathcal{W}\left(\alpha^*\left(\mu', \Sigma'\right), \mu', \Sigma'\right) - \mathcal{W}\left(\alpha^*\left(\mu, \Sigma\right), \mu, \Sigma\right) \geq \mathcal{W}\left(\alpha^*\left(\mu, \Sigma\right), \mu', \Sigma'\right) - \mathcal{W}\left(\alpha^*\left(\mu, \Sigma\right), \mu, \Sigma\right). \quad (24)$$

This proposition shows that the impact of a shock on welfare is always better when firms can reorganize their supply chains. Intuitively, this extra margin of adjustment allows firms to produce more efficiently and to better mitigate risk, which translates into higher welfare for the household.

Shocks and GDP

Proposition 5 shows that changes in the mean and the variance of ε have intuitive impact on welfare. The same cannot be said about their impact on expected GDP when the production network is endogenous. To see this, it is helpful to decompose the impact of a shock in its direct and indirect impacts on expected GDP. For instance, for a shock μ_i we can write

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} = \underbrace{\frac{\partial\,\mathrm{E}[y]}{\partial \mu_i}}_{\substack{\text{direct impact with} \\ \text{fixed network}}} + \underbrace{\frac{\partial\,\mathrm{E}[y]}{\partial \alpha}\frac{d\alpha}{d\mu_i}}_{\text{network adjustment}}. \quad (25)$$

The first term on the right-hand side of (25) denotes the direct impact of the shock keeping the network α fixed. The second term captures the impact of the shock on the structure of the network α and the impact of that change in structure on expected GDP. When the network is fixed, this network adjustment term is zero and the full impact of the shock is simply equal to the direct impact. This is the situation that we explored in Corollary 1 which states that an increase in μ_i always has a positive impact on $\mathrm{E}[y]$. But under a flexible network the indirect effect can amplify, mitigate or even overwhelm the direct impact completely, in which case a positive shock to μ_i can lower expected GDP. When this happens, the Hulten-like theorem established in the last section (Corollary 1) ceases to work, even as a local approximation. A similar mechanism can also flip the impact of uncertainty shocks such that a positive shock to Σ_{ij} can lower the variance of aggregate

GDP.[22]

Example of counterintuitive response to shocks We now provide an example to show how the endogenous adjustment of the network can lead to counterintuitive responses to shocks. In the economy depicted in Figure 4, firms 4 and 5 use only labor to produce, while firms 1 to 3 can also use goods 4 and 5 as intermediate inputs. Firm 4 is more productive and volatile than firm 5 ($\mu_4 > \mu_5$ and $\Sigma_{44} > \Sigma_{55}$). Now consider the impact of a positive shock to μ_5. The solid blue lines in panels (a), (b) and (c) of Figure 5 illustrate the impact of that shock on $\mathrm{E}[y]$, $\mathrm{V}[y]$, and on welfare. Point O on the graphs represents the economy before the shock. As we can see, the initial increase in μ_5 has a negative impact on expected GDP. To understand why, notice that for a small increase in μ_5, firm 5 is still less productive (on average) than firm 4, but it now offers a better risk-reward trade-off given its lower variance. As a result, firms 1 to 3 increase their shares of good 5 and reduce their share of good 4. But since $\mu_4 > \mu_5$, this readjustment leads to a fall in expected GDP for a small increase in μ_5. At the same time, the variance of GDP also declines because firm 5 is less volatile than firm 4. The movements in $\mathrm{E}[y]$ and $\mathrm{V}[y]$ oppose each other in their impacts on welfare but the overall effect is positive, as predicted by Proposition 5, and as can be seen in panel (c). Of course, as μ_5 keeps increasing expected GDP eventually starts to increase.

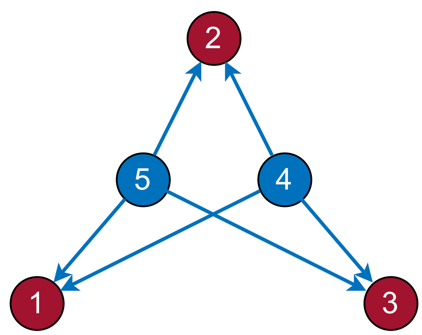


Figure 4: Network structure for the example

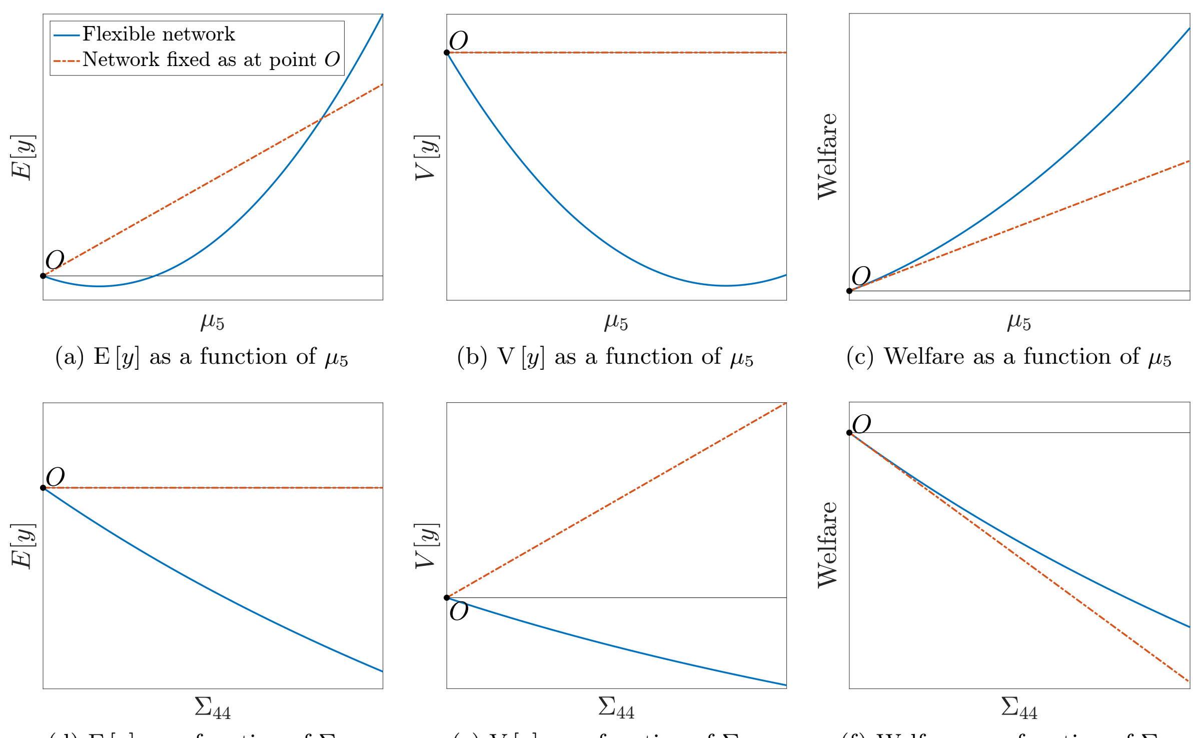
Notes: Arrows represent the movement of goods: there is a blue solid arrow from j to i if $\alpha_{ij} > 0$. a is as in (3) with the elements of κ equal to 0 if there is a potential link between two firms and very large otherwise. The elements of α° are one half when there is a potential link and zero otherwise. The labor share for firms 1, 2 and 3 is fixed at 0.5 and for firms 4 and 5 it is one (deviations are punished severely). The risk aversion of the household is $\rho = 2.5$. Household's utility weights are $\beta_1 = \beta_2 = \beta_3 = \frac{1}{3} - \epsilon$, $\beta_4 = \beta_5 = \frac{3}{2}\epsilon$, where ϵ is a very small positive number. $\mu = (0.1, 0.1, 0.1, 0.1, -0.04)$, Σ is diagonal, with $\mathrm{diag}(\Sigma) = (0.2, 0.2, 0.2, 0.3, 0.05)$.

To emphasize the importance of the flexible network for this mechanism, we also show the effect of the same increase in μ_5 when the network is kept fixed (dashed red lines in the same panels). Here, Corollary 1 holds so that the marginal impact of μ_5 on expected GDP is equal to its Domar weight and increasing μ_5 has a positive impact on $\mathrm{E}[y]$. At the same time, the variance of GDP is simply not affected by changes in μ. While an increase in μ_5 is welfare-improving in this case, the effect is less pronounced than in the flexible network economy. Indeed, in the latter case the

[22] Baqaee and Farhi (2019a) derive conditions under which Hulten's theorem does not hold globally in a standard efficient production network economy.

equilibrium network changes precisely to maximize the beneficial impact of the shock on welfare (Proposition (6)).

Finally, we can use the same economy to illustrate how a *positive* shock to an element of Σ can *lower* the variance of aggregate GDP, and simultaneously lower welfare. Start from the economy of Figure 4 (point O) and suppose that the volatility of firm 4 goes up. In response, firms 1 to 3 start to purchase from firm 5 more actively. Because firm 5 is less volatile (recall that $\Sigma_{55} < \Sigma_{44}$ initially), the variance of GDP declines (panel e). At the same time, expected GDP goes down because firm 5 is also less productive on average than firm 4 (panel d). The combined effect on welfare is negative, as predicted by Proposition 5 (panel f). In this case, the reorganization of the network mitigates the adverse effect of the increase in volatility on welfare. Instead, when the network is fixed an increase in Σ_{44} does not affect expected GDP but leads to a sharp increase in the variance of GDP. As a result, welfare drops more substantially than when the network flexible, as predicted by Proposition (6).



(a) $\mathrm{E}[y]$ as a function of μ_5 (b) $\mathrm{V}[y]$ as a function of μ_5 (c) Welfare as a function of μ_5

(d) $\mathrm{E}[y]$ as a function of Σ_{44} (e) $\mathrm{V}[y]$ as a function of Σ_{44} (f) Welfare as a function of Σ_{44}

Figure 5: The nonmonotone impact of firm-level shocks on GDP

The network structure and parameterization are detailed in Figure 4. In panels (a)-(c), μ_5 increases from -0.04 to 0.13. In panels (d)-(f), Σ_{44} increases from 0.3 to 0.4.

6 Endogenous network responses in a calibrated model

To better understand the quantitative importance of the mechanism, we now calibrate the model to the United States economy. We first describe our data sources and our calibration strategy. We then show that the calibrated model is able to replicate several important aspects of data. Finally, we use the calibrated model to evaluate the role of beliefs in shaping the production network and how the changing structure of the network influences welfare and the level and the volatility of aggregate output.

6.1 Data

We use several datasets to calibrate the model. First, we rely on sectoral input-output tables provided by the Bureau of Economic Analysis (BEA). Over time, the BEA has changed the number and the definition of the sectors included in the tables. We therefore rely on harmonized tables constructed by vom Lehn and Winberry (2021) that provide consistent annual data on intermediate input usage for $n = 37$ sectors over the period 1947—2018, as well as the final consumption associated with each sector. Table 2 provides the list of the sectors included in that dataset.

Table 2: The 37 sectors used in our analysis

Mining	Utilities
Construction	Wood products
Nonmetallic minerals	Primary metals
Fabricated metals	Machinery
Computer and electronic manufacturing	Electrical equipment manufacturing
Motor vehicles manufacturing	Other transportation equipment
Furniture and related manufacturing	Misc. manufacturing
Food and beverage manufacturing	Textile manufacturing
Apparel manufacturing	Paper manufacturing
Printing products manufacturing	Petroleum and coal manufacturing
Chemical manufacturing	Plastics manufacturing
Wholesale trade	Retail trade
Transportation and warehousing	Information
Finance and insurance	Real estate and rental services
Professional and technical services	Management of companies and enterprises
Administrative and waste management services	Educational services
Health care and social assistance	Arts and entertainment services
Accommodation	Food services
Other services	

Notes: Sectors are classified according to the NAICS-based BEA codes. See vom Lehn and Winberry (2021) for details of the data construction.

From these data, we can compute the input shares $\alpha_{ij,t}$ of each sector in each year t, which, as described below, we use as targets in our calibration. Overall, the average input share is 1.4% with an average cross-period standard deviation of 0.5%, indicating a substantial amount of variation over time.

To pin down the time series for the belief processes μ_t and Σ_t, which we assume to be time-varying, we use sectoral total factor productivity data, which also comes from vom Lehn and Winberry (2021). These sectoral TFP processes are computed as Solow residuals after removing the contributions of input factors from a sector's gross output.[23] Since we do not focus on long-run growth issues, we remove a common trend from these TFP processes.[24]

6.2 Calibration strategy

The three groups of parameters that we need to calibrate are 1) the households preferences, i.e. the consumption shares β and the risk-aversion ρ, 2) the parameters of the A_i functions, and 3) the parameters governing the process for the exogenous sectoral productivity shocks, i.e. μ and Σ. Some parameters have direct counterparts that can be computed directly from observable data. For the remaining parameters, we use a method of simulated moments and standard time-series methods to find values broadly consistent with how input shares and TFP evolved over our sample period. We describe how in detail below.

Preferences

In the household's utility function the different goods are combined through a Cobb-Douglas aggregator. This implies that an element i of the preference vector β corresponds to the share of good i in total consumption. We therefore use final consumption shares from the input-output tables to pin down β directly.

The CRRA parameter ρ plays an important role in our model as it determines to what extent firms are willing to trade off higher input prices for access to more stable suppliers. The literature uses a broad range of values for ρ and it is unclear a priori which one is best for our application. We therefore let the data pick ρ through the estimation procedure described below.

Total factor productivity

Sectoral TFP can be measured from the BEA's input-output tables and we use the same method as in vom Lehn and Winberry (2021) to do so. However, total TFP in our model is the product of the exogenous component e^{ε_t} and the endogenous component $A(\alpha_t)$. Here we describe how we take this into account when we parameterize the productivity processes.

[23] We make two departures from vom Lehn and Winberry (2021) in constructing TFP. First, to be consistent with our model, we let the input shares $\alpha_{ij,t}$ vary over time. Second, we do not smooth the resulting Solow residuals. We refer to vom Lehn and Winberry (2021) for the details of how the harmonized input-output table and TFP processes are computed.

[24] Specifically, we compute a quadratic trend for aggregate TFP and remove it from all sectoral TFP series. The results are similar if we use a linear trend instead but the fit to the data is a bit worse.

There are three objects related to TFP that we need to calibrate: the time series for the mean (μ_t) and the variance (Σ_t) of the process ε_t, and the parameters of the endogenous TFP function A_i. We assume that A_i takes the form (3), introduced earlier. We set the ideal share coefficients α_{ij}° to be equal to the time-average of the actual input shares in the data. We include the matrix κ, which describes how costly it is to deviate from the ideal shares, in the set of parameters that we estimate.

Here is a summary of our estimation procedure. For a given pair $\{\rho, \kappa\}$ we can use the structure of the model to back out the realized path followed by the stochastic process ε_t in the data. From this path, we then estimate the time series for μ_t and Σ_t. At this stage, the model is fully parametrized so that we can solve it and verify if the model outcomes are in line with their data counterparts. We then search for the parameters $\{\rho, \kappa\}$ that bring the model as close as possible to the data.

We now provide more details about each of the steps involve in that calibration process. First, note that from (1), the sectoral Solow residual of a given sector i, which is available in the data, corresponds to the term $e^{\varepsilon_{it}}\zeta(\alpha_{it}) A_i(\alpha_{it})$ in the model. It follows that for a given matrix κ, the function A_i is fully specified and we can use the observed input shares to extract the path followed by ε_{it}. We then use the realized vector ε_t to compute μ_t and Σ_t. To do so, we assume that ε_t follows a random walk with drift, such that

$$\varepsilon_t = \gamma + \varepsilon_{t-1} + u_t^{\varepsilon}, \tag{26}$$

where γ is an $n \times 1$ vector of deterministic sectoral drifts and $u_t^{\varepsilon} \sim \text{iid}\, \mathcal{N}(0, \Sigma_t)$ is a vector of shocks.[25] When making decisions in period t, we assume that firms know the past realizations of ε_t, so that the mean of their beliefs is given by $\mu_t = \gamma + \varepsilon_{t-1}$, and the variance of their beliefs Σ_t is given by the covariance matrix of u_t^{ε}. Given (26), we can estimate γ by computing the average of the innovations $\varepsilon_t - \varepsilon_{t-1}$ over time. Similarly, we compute Σ_t as the covariance of $\varepsilon_t - \varepsilon_{t-1}$. To capture the presence of uncertainty shocks, we assume that this covariance can change over time and we estimate it by using a rolling window that puts more weight on more recent observations.[26]

With these quantities in hand, we can compute for any $\{\rho, \kappa\}$ the model-implied input shares $\alpha_{ij,t}$ and the average standard deviation of GDP.[27] We then pick ρ and the matrix κ to minimize

[25] We adopt this random-walk specification since our sector-level productivity data shows clear signs of non-stationarity. Nelson and Plosser (1982) provide evidence suggesting that standard macroeconomic time series are not stationary even after removing a deterministic trend.

[26] To be precise, we compute the covariance between sectors i and j at time t as $\hat{\Sigma}_{ij,t} = \sum_{s=1}^{t-1} \lambda^{t-s-1} \Delta\varepsilon_{i,s} \Delta\varepsilon_{j,s}$ where $\Delta\varepsilon_{i,s} = \varepsilon_{i,s} - \varepsilon_{i,s-1}$ and where $0 < \lambda < 1$. The parameter λ captures how quickly volatility shocks vanish. We therefore pick its value by estimating a GARCH(1,1) on each sector's $\Delta\varepsilon_{i,s}$ and averaging the GARCH coefficients across sectors. The estimated value of λ is 0.41.

[27] We target an average annual standard deviation in annual gross domestic product of 4.75%, which corresponds to the data available from the Bureau of Economic Statistics since 1929. We use this longer time series because our goal is to capture the subjective riskiness of aggregate GDP. The rest of our data begins in 1948 and therefore excludes dramatic events such as the Second World War and the Great Depression. We adopt the view that agents believe that such "disaster" event can happen and we therefore include them, in this way, in our calibration. In that

the distance, in terms of these quantities, between the model and the data. Given that we have $n = 37$ sectors in our dataset, the matrix κ has $n \times (n+1) = 1406$ elements—an extremely large space to search in. We therefore impose some restrictions on the structure of κ to limit the number of free parameters. Namely, we assume that $\kappa = \kappa^i \kappa^j$ where κ^i is an $n \times 1$ column vector and κ^j is an $1 \times (n+1)$ row vector. The matrix κ can therefore be recovered from the more computationally reasonable $2n + 1$ parameters. Intuitively, an element k in vector κ^i is related to the cost for producer k of changing the share of any of its input. In contrast, an element l in κ^j is informative about the cost of changing the share of input l for all producers.

6.3 Overview of the calibrated economy

We now describe a few key features of the calibrated economy and evaluate its fit to the data.

Estimated parameters

The estimated risk aversion is $\rho = 5.8$. This number might seem high given that most dynamic macroeconomic models with CRRA preferences use values for ρ that are close to one. But it is important to note that in these models ρ plays the dual role of pinning down the household's intertemporal elasticity of substitution (IES) in addition to its relative risk aversion.[28] Importantly, in our case the model is static and ρ's only role is to determine how risk averse the household is. It is therefore better to compare its value to previous work that estimate the household relative risk aversion as a separate object from the IES. Much of that work belongs to the asset pricing literature, and their estimates for ρ are quite a bit larger than unity. For instance, Bansal and Yaron (2004) find that a relative risk aversion of 10 is needed to explain several asset pricing puzzles. Using micro-level data, Vissing-Jørgensen and Attanasio (2003) find that values of ρ as low as 5-10 can explain the covariance of asset returns and consumption growth. Similar numbers are found in lab experiments (Barsky et al., 1997). We therefore view our estimate of ρ as reasonable in view of that literature.

Besides the risk-aversion parameter ρ, we also estimate the cost matrix κ. The overall mean of the elements of κ is 203 with a standard deviation of the individual point estimates is 48. In Appendix B.1, we provide more details about the κ's associated with each sector.

Estimated total factor productivity

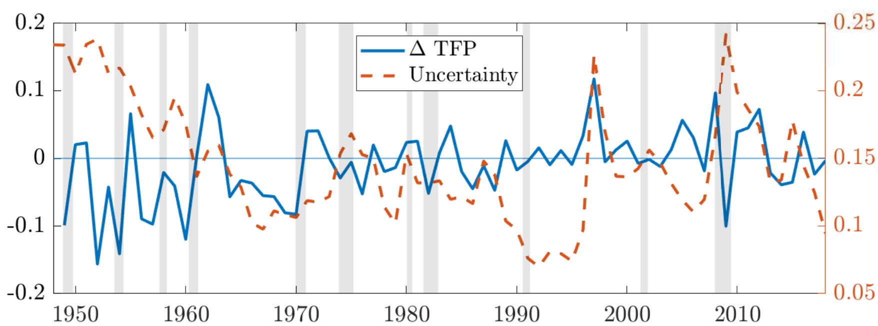
Having calibrated ρ and κ, we proceed to exploring what the model implies about total factor productivity. We find that the estimated drift vector γ features a lot of variation across sectors,

sense, our estimation procedure is in the spirit of the disaster risk literature as in Barro (2006) and Gourio (2012).

[28] Tallarini Jr (2000) shows that conventional business-cycle moments depend almost entirely on the IES and do not change much with the relative risk aversion.

indicating sizable dispersion in the trajectory of sectoral TFP. "Computer and electronic manufacturing" has the largest drift in our data (4.2% relative to trend) while "Food services" has the smallest one (−3.6%).

Figure 6 shows cross-sector averages of the estimated time series for μ_t and the diagonal of Σ_t. To provide a meaningful measure of how these shocks affect GDP, we use the sales shares ω_i as averaging weights. The red dashed line shows changes in average TFP over our sample. As expected, that measure tends to go below zero during NBER recessions and is positive during expansions. The solid blue line represents the average level of sector-level uncertainty. It is high in the post-war years at the beginning of our sample and stabilizes at a lower level after 1965. We also see large spikes in measured uncertainty around the late 1990s (dot-com bubble and the Asian financial crisis) and the Great Recession of 2007-2009.



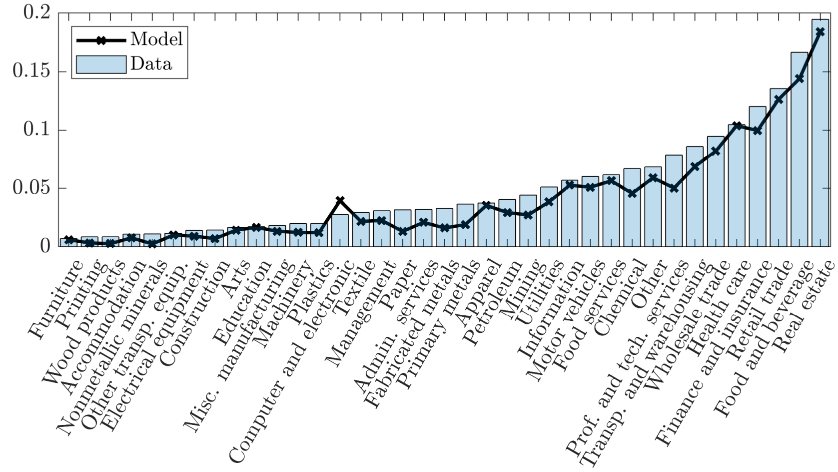
Notes: Red dashed line: sum of Domar-weighted changes in exogenous TFPs, $\sum_{j=1}^{n} \omega_{j,t-1} (\epsilon_{j,t} - \epsilon_{j,t-1})$. Blue solid line: sum of Domar-weighted diagonal elements of the estimated matrix $\sqrt{\Sigma}$, $\sum_{j=1}^{n} \omega_{j,t} \sqrt{\Sigma_{jj,t}}$. Shaded areas represent NBER recessions.

Figure 6: Domar-weighted uncertainty and TFP changes

Domar weights

We want our model to fit key features of the data that relates to the structure of the production network, how that network changes in response to shocks, and how these changes affect macroeconomic aggregates. As we have seen earlier, the Domar weights play a central role for these mechanisms and we now describe how the model fits these weights and their relationship with the beliefs (μ_t, Σ_t).

Figure 7 shows the average Domar weights of each sector in the data (blue bars) and in the model (black line). Reassuringly, we find that the calibrated model fits the data well along that dimension, so that the production network in the model and the data are similar along this dimension. The figure also shows that the sectors with the highest Domar weights are "Real estate", "Food and beverage", "Retail trade", "Finance and insurance" and "Health care". According to our theory, i.e. Proposition (5), changes in the expected level and uncertainty of productivity in those sectors will have most pronounced effects on welfare.

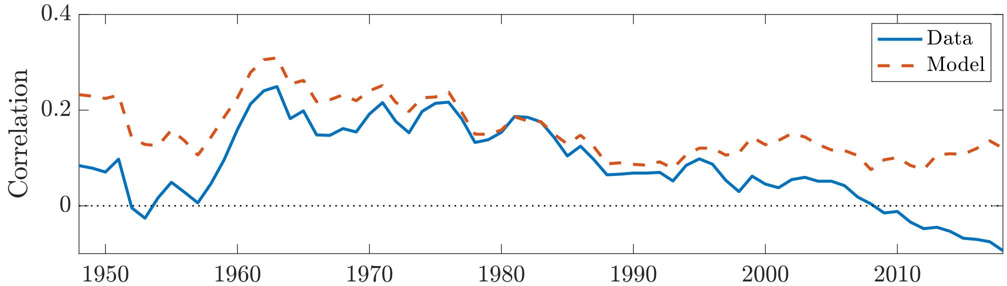


Notes: The Domar weights are computed for each sector in each year and then averaged over all time periods. 45-degree line in red.

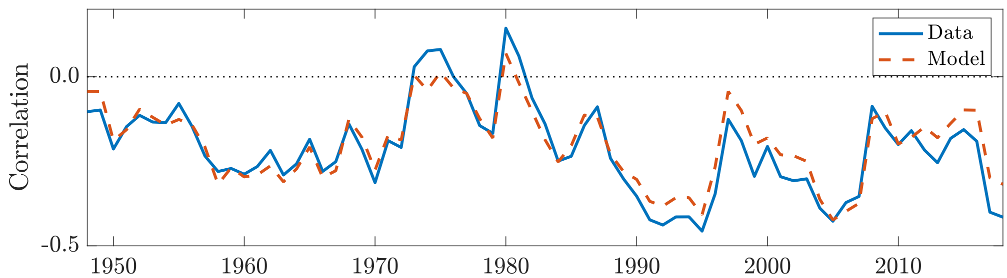
Figure 7: Sectoral Domar weights in the data and the model

The model also makes predictions about the relation between the Domar weights and changes in beliefs. In particular, after a decline in the expected productivity of a sector, or an increase in its variance, other firms reduce the importance of that sector as an input which leads to a decline in its Domar weight. Proposition 3 makes this point formally for a single shock to $\mu_{j,t}$ and $\Sigma_{jj,t}$. Of course, in the data multiple changes in μ_t and Σ_t occur at the same time so it is not possible to isolate the impact of a single shock on the Domar weights. Instead, we look at simple cross-sector correlations between the Domar weights $\omega_{j,t}$ and the first ($\mu_{j,t}$) and the second moments ($\Sigma_{jj,t}$) of sectoral TFPs, both in the data and in the model. These correlations provide a straightforward, albeit noisy, measure of the interrelations between ω, μ and Σ.

These correlations are plotted in Figure 8. In panel 8a, we see that the within-period correlation between ω_j and μ_j in the data (solid blue line) is mostly positive with an average of 0.09, suggesting that indeed firms tend to increase their input shares of sectors that are expected to be more productive. Reassuringly, the model (dashed red line) also features a positive correlation, but it is somewhat stronger than in the data (0.16 on average). The figure also shows in panel 8b the average correlation between ω_j and Σ_{jj} (solid blue line). This correlation is on average negative in the data (-0.21 on average), suggesting that one of the key forces explored in this paper—that firms tend to limit their usage of uncertain inputs—is indeed at work in reality. This correlation is also fairly volatile, with episodes in which uncertainty is strongly correlated with the Domar weights, while at other times the association is fairly weak. As we can see from the figure, the model (dashed red line) is able to replicate that correlation well, capturing both its overall negative level as well as its variation over time.



(a) Cross-sector correlation between a sector's Domar weight ω_j and the mean μ_j of its TFP



(b) Cross-sector correlation between a sector's Domar weight ω_j and the variance Σ_{jj} of its TFP

Figure 8: Correlation between Domar weights and TFP

6.4 Uncertainty, the structure of the network and GDP

In this section, we first evaluate the importance of the changing structure of the production network for GDP and overall welfare. To do so, we compare the equilibrium allocation with an alternative economy in which the production network is held fixed and so cannot respond to changes in μ_t and Σ_t. Specifically, we fix the input shares α to their averages in the calibrated model and recompute all other equilibrium quantities. We find that the economy with a fixed network is on average 2.55% less productive than the economy with a flexible network. The intuition for this large difference is straightforward. As some sectors of the economy become more productive, firms would like to take advantage of their cheaper inputs by relying on them more in production. In the flexible network economy this is allowed, and the aggregate economy becomes more productive as a result. In contrast, when the network is fixed, firms are stuck with less productive suppliers and the economy is overall less efficient. Perhaps surprisingly, the fixed network economy is also slightly more stable than its flexible counterpart with a standard deviation of GDP that is 0.07% smaller. In the flexible network economy, the planner tries to strike a balance between increasing expected value of GDP and lowering its variance. In the calibrated model, we see that the planner is willing to suffer a slight increase in variance for large gains in expected value. Overall, this comparison with a fixed network suggests that policy intervention that might impede or slow down the reorganization of supply chains might have a sizable impact on welfare.

These differences between the flexible and fixed network economies come from variations in both μ_t and Σ_t. Here, we provide an exercise to isolate the role of uncertainty alone in shaping the production network and affecting GDP. Namely, we consider a risk-neutral economy in which the household has a relative risk aversion of $\rho = 1$. In this case, firms do not respond to changes in Σ_t when making sourcing decisions. Since these decisions are the only ones taken under uncertainty, the only direct impact of this change in risk aversion is on the shape of the network. We then compare the allocation in this economy with the calibrated one. Table 3 reports long-run moments related to GDP and welfare, which is evaluated from the perspective of the risk-averse representative household of our benchmark model. We see that, in line with the theory, the baseline economy is slightly less productive and slightly less volatile than the alternative. When $\rho > 1$, it is worthwhile from the firms' perspective to use suppliers that are less productive but safer, which translates into the observed differences in $\mathrm{E}[y]$ and $\mathrm{V}[y]$. The differences are however fairly small. The reason for this is that there are long periods without much uncertainty in the economy, in which case firms simply select their inputs for the most productive suppliers without regards for any risk involved.[29]

	Baseline model compared to...	
	Fixed network	Risk neutral
Expected GDP $\mathrm{E}[y(\alpha)]$	+2.55%	−0.02%
Std. dev. of GDP $\sqrt{\mathrm{V}[y(\alpha)]}$	+0.07%	−0.08%
Welfare $\mathcal{W}$	+2.52%	+0.02%

Table 3: Uncertainty and GDP over the long run

Great Recession

The situation is however quite different when uncertainty spikes, for instance during the Great Recession. In this case firms move toward safer but less productive suppliers to avoid potentially disastrous increases in costs. Figure 9 shows how the baseline economy compares to the risk-neutral alternative (denoted with tildes in the figure) over the years 2006 to 2012.[30]

The top panel shows the difference in expected GDP over that period. We see that expected GDP in the baseline economy is about 0.25% lower in 2009 than in the risk-neutral economy. Uncertainty is large during the Great Recession (as can be seen from Figure 6). As a result, firms are worried about crippling increases in costs and move toward safer but more expensive suppliers. The result, in term of aggregate volatility, is visible in the second panel where we see that GDP is expected to be about 1% less volatile in 2009 in the baseline economy. While the amount of expected GDP sacrificed in the baseline economy is somewhat small, the reduction in variance that its leads to is important enough to generate an increase in welfare of about 0.3%, when compared

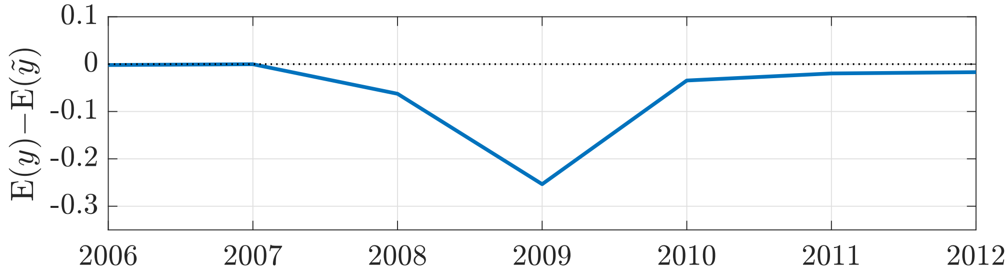
[29]As in Lucas (1987), the utility cost of business cycles fluctuations is, on average, small in our model and the planner does not want to sacrifice much in terms of the level of GDP for a reduction in volatility.

[30]In Appendix B.2, we also compare our baseline economy to the fixed-network alternative over the Great Recession.

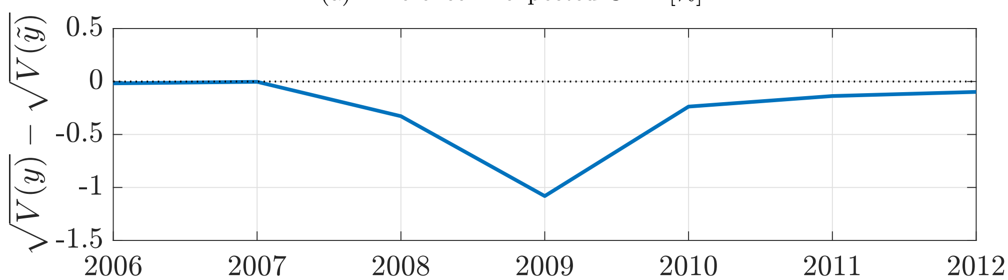
to the risk-neutral economy (third panel).[31] Interestingly, realized GDP, visible in the last panel, is quite a bit higher in the baseline economy. Essentially, firms were worried about bad draws from the TFP processes and opted for safer suppliers, and then their fears were realized. The year 2009 saw particularly bad TFP draws (as evident from Figure 6), and so the baseline economy fared about 1% better in terms of realized GDP.

Overall, our findings in this section are that allowing the production network to reorganize itself after shocks can lead to large gains in efficiency but that changes in uncertainty are, most of the time, not a major factor in shaping the production network. The situation is quite different however when uncertainty spikes. In this case, its impact on the structure of the network and, through that channel, on GDP and welfare can be sizable. Our results therefore highlight the importance for firms of reorganizing supply lines during turbulent periods. It also suggests that policies, such as trade barriers, that would slow down this reorganization might have significant side effects.

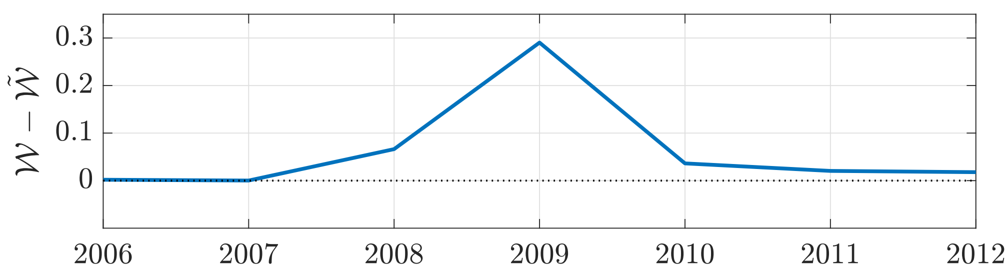
[31] We compute welfare in the risk-neutral economy from the perspective of the representative agent in the baseline economy.



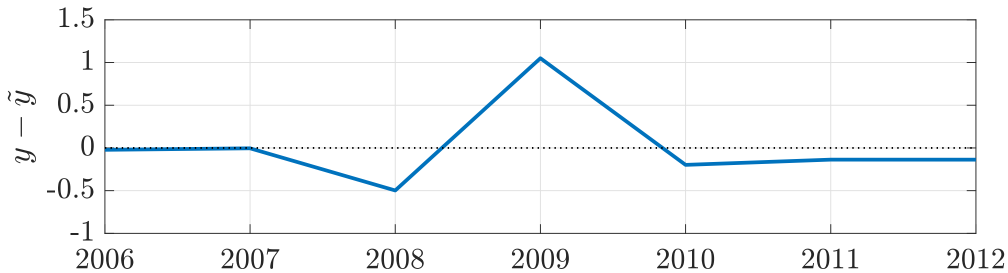
(a) Difference in expected GDP [%]



(b) Difference in expected standard deviation of GDP [%]



(c) Difference in expected welfare [%]



(d) Difference in realized GDP [%]

Notes: The differences between the series implied by the baseline model and the risk-neutral alternative ($\rho = 1$). Both economies are hit by the same shocks that are filtered out from the TFP data under our baseline model. All differences are expressed in percentage terms.

Figure 9: The role of uncertainty during the Great Recession

7 Conclusion

We construct a model in which agents' beliefs about fundamentals affect the structure of the production network and, through that channel, other macroeconomic aggregates such as output and welfare. We prove that there exists an equilibrium that is efficient and characterize how the equilibrium network changes with beliefs about the mean and the variance of productivity. We also describe how uncertainty, through its action on the network, lowers expected GDP. In our calibrated economy, the impact of uncertainty on the network can have a sizable effect on GDP and welfare during periods of high uncertainty like the Great Recession.

Several applications of the model might be worthwhile to investigate. First, the model could shed light on the impact of uncertainty on international trade networks. Recent events (introduction of trade barriers, lockdowns due to the COVID-19 pandemic, etc.) have highlighted the high uncertainty related to international supply chains and it would be interesting to use our model to quantify its effect. Another potentially fruitful application of the model would be to use firm-level data about the production network for quantitative exercises. Doubtlessly, uncertainty at that level is higher than at the sectoral level and so the mechanism could be more powerful. For such an exercise, it might also be more realistic to relax the assumption of perfect competition. Finally, introducing an extensive margin of production, so that firms can exit the economy after a large rise in uncertainty would be a natural extension that we leave for future research.

References

Acemoglu, D. and P. D. Azar (2020): "Endogenous production networks," *Econometrica*, 88, 33–82.

Acemoglu, D., V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi (2012): "The network origins of aggregate fluctuations," *Econometrica*, 80, 1977–2016.

Acemoglu, D., A. Ozdaglar, and A. Tahbaz-Salehi (2017): "Microeconomic Origins of Macroeconomic Tail Risks," *American Economic Review*, 107, 54–108.

Acemoglu, D. and A. Tahbaz-Salehi (2020): "Firms, Failures, and Fluctuations: The Macroeconomics of Supply Chain Disruptions," *Working Paper*.

Alfaro, I., N. Bloom, and X. Lin (2019): "The Finance Uncertainty Multiplier," .

Atalay, E. (2017): "How Important Are Sectoral Shocks?" *American Economic Journal: Macroeconomics*, 9, 254–280.

Baker, S. R., N. Bloom, and S. J. Davis (2016): "Measuring Economic Policy Uncertainty," *The Quarterly Journal of Economics*, 131, 1593–1636.

Bansal, R. and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *The Journal of Finance*, 59, 1481–1509.

Baqaee, D. R. (2018): "Cascading Failures in Production Networks," *Econometrica*, 86, 1819–1838.

Baqaee, D. R. and E. Farhi (2019a): "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hultens Theorem," *Econometrica*, 87, 1155–1203.

——— (2019b): "Productivity and Misallocation in General Equilibrium," *The Quarterly Journal of Economics*, 135, 105–163.

Barro, R. J. (2006): "Rare disasters and asset markets in the twentieth century," *The Quarterly Journal of Economics*, 121, 823–866.

Barsky, R. B., F. T. Juster, M. S. Kimball, and M. D. Shapiro (1997): "Preference parameters and behavioral heterogeneity: An experimental approach in the health and retirement study," *The Quarterly Journal of Economics*, 112, 537–579.

Bigio, S. and J. La'O (2020): "Distortions in Production Networks," *The Quarterly Journal of Economics*, 135, 2187–2253.

Bloom, N. (2009): "The Impact of Uncertainty Shocks," *Econometrica*, 77, 623–685.

——— (2014): "Fluctuations in uncertainty," *Journal of Economic Perspectives*, 28, 153–76.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S. J. Terry (2018): "Really Uncertain Business Cycles," *Econometrica*, 86, 1031–1065.

Boehm, J. and E. Oberfield (2020): "Misallocation in the Market for Inputs: Enforcement and the Organization of Production," *The Quarterly Journal of Economics*, 135, 2007–2058.

Carvalho, V. and X. Gabaix (2013): "The Great Diversification and its Undoing," *American Economic Review*, 103, 1697–1727.

Cox, G. (2020): "Almost sure uniqueness of a global minimum without convexity," *The Annals of Statistics*, 48, 584–606.

Dhyne, E., A. K. Kikkawa, M. Mogstad, and F. Tintelnot (2021): "Trade and domestic production networks," *The Review of Economic Studies*, 88, 643–668.

Dupor, B. (1999): "Aggregation and irrelevance in multi-sector models," *Journal of Monetary Economics*, 43, 391–409.

Fajgelbaum, P. D., E. Schaal, and M. Taschereau-Dumouchel (2017): "Uncertainty Traps," *The Quarterly Journal of Economics*, 132, 1641–1692.

Fernández-Villaverde, J., P. Guerrón-Quintana, J. F. Rubio-Ramírez, and M. Uribe (2011): "Risk Matters: The Real Effects of Volatility Shocks," *American Economic Review*, 101, 2530–2561.

Foerster, A. T., P.-D. G. Sarte, and M. W. Watson (2011): "Sectoral versus Aggregate Shocks: A Structural Factor Analysis of Industrial Production," *Journal of Political Economy*, 119, 1–38.

Gabaix, X. (2011): "The Granular Origins of Aggregate Fluctuations," *Econometrica*, 79, 733–772.

Gourio, F. (2012): "Disaster risk and business cycles," *American Economic Review*, 102, 2734–66.

Ho, W., T. Zheng, H. Yildiz, and S. Talluri (2015): "Supply chain risk management: a literature review," *International Journal of Production Research*, 53, 5031–5069.

Horvath, M. (2000): "Sectoral shocks and aggregate fluctuations," *Journal of Monetary Economics*, 45, 69–106.

Hulten, C. R. (1978): "Growth Accounting with Intermediate Inputs," *The Review of Economic Studies*, 45, 511–518.

Jurado, K., S. C. Ludvigson, and S. Ng (2015): "Measuring Uncertainty," *American Economic Review*, 105, 1177–1216.

Lim, K. (2018): "Endogenous Production Networks and the Business Cycle," working paper.

Long, J. B. and C. I. Plosser (1983): "Real Business Cycles," *Journal of Political Economy*, 91, 39–69.

——— (1987): *Models of business cycles*, vol. 26, Basil Blackwell Oxford.

Nelson, C. R. and C. R. Plosser (1982): "Trends and random walks in macroeconmic time series," *Journal of Monetary Economics*, 10, 139–162.

Nieuwerburgh, S. V. and L. Veldkamp (2006): "Learning asymmetries in real business cycles," *Journal of Monetary Economics*, 53, 753–772.

Oberfield, E. (2018): "A Theory of Input-Output Architecture," *Econometrica*, 86, 559–589.

Tallarini Jr, T. D. (2000): "Risk-sensitive real business cycles," *Journal of Monetary Economics*, 45, 507–532.

Taschereau-Dumouchel, M. (2020): "Cascades and Fluctuations in an Economy with an Endogenous Production Network," *Working Paper*.

Vissing-Jørgensen, A. and O. P. Attanasio (2003): "Stock-Market Participation, Intertemporal Substitution, and Risk-Aversion," *American Economic Review*, 93, 383–391.

vom Lehn, C. and T. Winberry (2021): "The Investment Network, Sectoral Comovement, and the Changing U.S. Business Cycle," *The Quarterly Journal of Economics*.

Wagner, S. M. and C. Bode (2008): "An Empirical Examination of Supply Chain Performance Along Several Dimensions of Risk," *Journal of Business Logistics*, 29, 307–325.

Zurich Insurance Group (2015): "Potential business impact of loss of the main supplier for small and medium enterprises (SMEs) in 2015," .

Appendices

A Data appendix

A.1 Motivational evidence presented in the introduction

In this section, we provide more details about the regressions presented in Table 1. The data about the production network comes from the Factset Revere database and covers the period from 2003 to 2016. We limit the sample to relationships that have lasted at least five years and with at least one partner in the United States. The IV estimates remain significant when relationships of other lengths are considered. The data about firm-level uncertainty measures comes from Alfaro et al. (2019) and was downloaded from Nicholas Bloom's website at `https://nbloom.people.stanford.edu`. We thank the authors for sharing their data. Alfaro et al. (2019) describes how the data is constructed in details, and we only include here a summary of how the instruments are computed. The instruments are created by first computing the industry-level sensitivity to each aggregate shock c, where c is either the price of oil, one of seven exchange rates, the yield on 10-year US Treasury Notes and the economic policy uncertainty index of Baker et al. (2016). As Alfaro et al. (2019) explain, "for firm i in industry j, sensitivity$_j^c = \beta_j^c$ is estimated as follows

$$r_{i,t}^{riskadj} = \alpha_j + \sum_c \beta_j^c \cdot r_t^c + \epsilon_{i,t},$$

where $riskadj$ is the daily risk-adjusted return of firm i, r_t^c is the change in the price of commodity c, and α_j is industry j's intercept. [...] We allow these industry-level sensitivities to be time-varying by estimating them using 10-year rolling windows of daily data." The instruments $z_{i,t-1}^c$ are then computed as follows:

$$z_{i,t-1}^c = \left|\beta_{j,t-1}^c\right| \cdot \Delta\sigma_{t-1}^c,$$

where $\Delta\sigma_{t-1}^c$ denotes the volatility of the aggregate variable c. As in Alfaro et al. (2019), we also include in the IV regressions the first moments associated with each aggregate series c ("1st moment 10IV$_{i,t-1}$" in Table 1) to isolate the impact of changes in their 2nd moment alone.

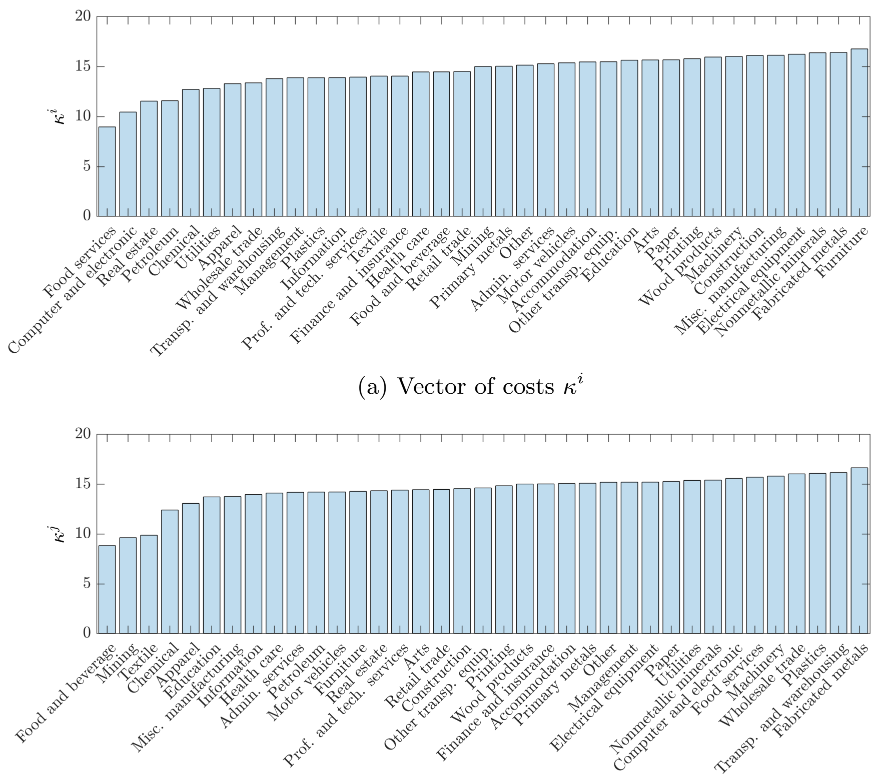
A.2 Data for the calibration

To calibrate the model we used data from vom Lehn and Winberry (2021). They have constructed harmonized labor, capital, investment, depreciation rate, intermediate inputs, consumption, and gross output tables for 37 sectors over a time period of 1947-2018. The TFP process is measured as the Solow residuals of real gross output net of labor, capital, and intermediate inputs. The TFP calculation in our paper differ from vom Lehn and Winberry (2021) in the following ways. We let the labor, capital, and intermediate inputs shares to vary over time. Also, there is no smoothing of the calculated Solow residuals.

B Additional results related to the calibrated economy

B.1 The matrix κ

The overall mean of the elements of the calibrated costs matrix κ is 203 with a standard deviation of 48. To better understand the structure of that matrix, Figure 10 shows for each sector the elements of the vectors κ^i and κ^j (recall that $\kappa = \kappa^i \kappa^j$). As we can see, the amount of variation across sectors is somewhat limited with the exception of a few sectors with particularly low κ^i's and κ^j's. The sectors with the smallest κ^i's are "Food services" and "Computer and electronics", indicating that it is particularly cheap for these sectors to deviate from their ideal input shares. Similarly, the sector with the smallest κ^j's are "Food and beverages", "Mining" and "Textile", so that all firms tend to find adjusting the shares of these sectors as an input to have a small impact on their productivity.



(a) Vector of costs κ^i

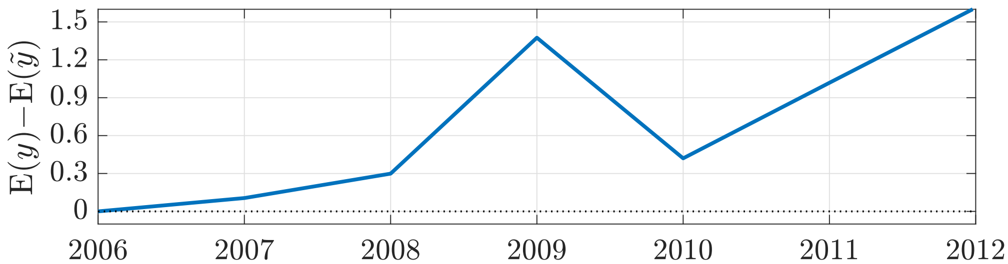
(b) Vector of costs κ^j

Figure 10: The calibrated costs of deviating from the ideal input shares

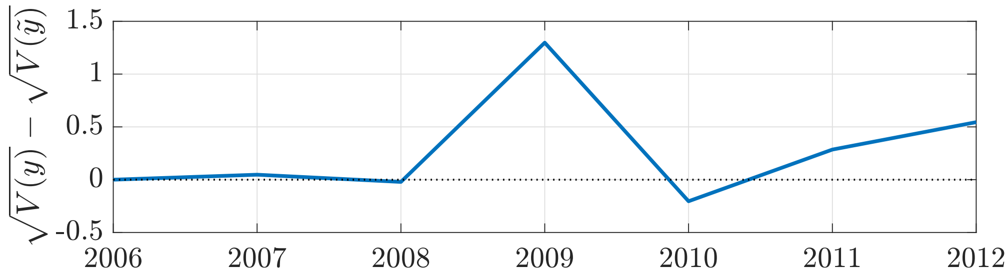
B.2 Great Recession: Flexible vs fixed network

In this section, we explore the role of network flexibility during the Great Recession—the period in which the economy was hit by large adverse shocks (Figure 6). Specifically, we fix the network α at its 2006 level and then hit the economy with the same shocks as in the baseline economy with endogenous network. Figure 11 shows how the baseline economy compares to the fixed-network alternative (denoted with tildes in the figure) over the years 2006 to 2012. We find that expected GDP (top panel) is higher under the flexible network. This is because firms are able to respond to changes in TFPs and move away from sectors that are expected to perform badly. When doing so, firms become exposed to more productive but also more volatile suppliers, which results in an increase in GDP volatility (second panel). However, the first effect dominates and welfare is quite substantially higher when the network is allowed to adjust (third panel). Interestingly, the differences in realized GDP (bottom panel) are quite small during the Great Recession years. As evident from the two top panels, firms optimally choose to be exposed to more productive but riskier suppliers. During the Great Recession, some of those risks were realized, pushing realized GDP down for the baseline case.

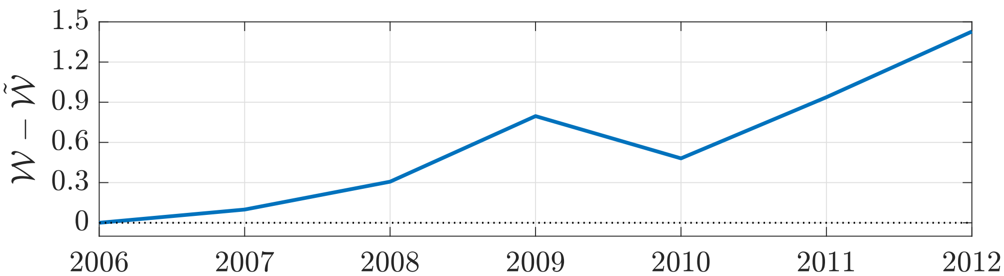
Finally, the differences between the baseline and the fixed-network models do not go to zero after the Great Recession. This is because in the latter scenario the network is fixed at its 2006 level, so that the differences are accumulated as sectoral TFPs keep evolving over time.



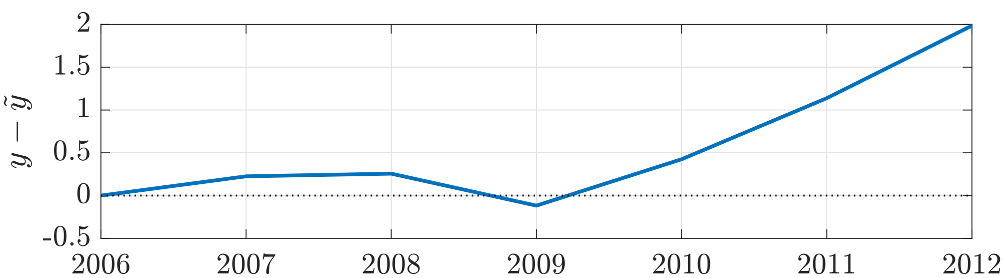
(a) Difference in expected GDP [%]



(b) Difference in expected standard deviation of GDP [%]



(c) Difference in expected welfare [%]



(d) Difference in realized GDP [%]

Notes: The differences between the series implied by the full model and the model in which the network is fixed at its 2006 level. All differences are expressed in percentage terms.

Figure 11: The role of network flexibility during the Great Recession

C Additional derivations

This appendix contains additional derivations that are used in the main text.

C.1 Derivation of the stochastic discount factor

The Lagrange multiplier on the budget constraint of the household captures the value of an extra unit of the numeraire and serves as stochastic discount factor for firms to compare profits across states of the world. The following lemma shows how to derive the expression in the main text.

Lemma 5. *The Lagrange multiplier on the budget constraint of the household* (5) *is*

$$\Lambda = \frac{u'(Y)}{\overline{P}},$$

where $Y = \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i}$ *and* $\overline{P} = \prod_{i=1}^{n} P_i^{\beta_i}$.

Proof. The household makes decisions after the realization of the state of the world ε. The state-specific maximization problem has a concave objective function and a convex constraint set so that first-order conditions are sufficient to characterize optimal decisions. The Lagrangian is

$$u\left(\left(\frac{C_1}{\beta_1}\right)^{\beta_1} \times \cdots \times \left(\frac{C_n}{\beta_n}\right)^{\beta_n}\right) - \Lambda\left(\sum_{i=1}^{n} P_i C_i - 1\right)$$

and the first-order condition with respect to C_i is

$$\beta_i u'(Y)\, Y = \Lambda P_i C_i. \tag{27}$$

Summing over i on both sides and using the binding budget constraint yields

$$u'(Y)\, Y = \Lambda, \tag{28}$$

which, together with (27), implies that

$$P_i C_i = \beta_i. \tag{29}$$

We can also plug back the first-order condition in $Y = \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i}$ to find

$$\begin{aligned} Y &= \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i} = \prod_{i=1}^{n} \left(\beta_i^{-1} \frac{\beta_i u'(Y)\, Y}{\Lambda P_i}\right)^{\beta_i} \\ \Lambda &= u'(Y) \prod_{i=1}^{n} P_i^{-\beta_i} \end{aligned} \tag{30}$$

which, combined with (28), yields :

$$Y = \prod_{i=1}^{n} P_i^{-\beta_i}. \tag{31}$$

This last equation implicitly defines a price index $\overline{P} = \prod_{i=1}^{n} P_i^{\beta_i}$ such that $\overline{P}Y = 1$. Combining that last equation with (28) yields the result. □

C.2 Derivation of the unit cost function

The cost minimization problem of the firm is

$$K_i(\alpha_i, P) = \min_{L_i, X_i} \left(L_i + \sum_{j=1}^{n} P_j X_{ij} \right)$$
$$\text{subject to } F(\alpha_i, L_i, X_i) \geq 1,$$

where F is given by (1). The first-order conditions are

$$L_i = \theta \left(1 - \sum_{j=1}^{n} \alpha_{ij} \right) F(\alpha_i, L_i, X_i),$$
$$P_j X_{ij} = \theta \alpha_{ij} F(\alpha_i, L_i, X_i),$$

where θ is the Lagrange multiplier. Plugging these expressions back into the objective function, we see that $K_i(\alpha_i, P) = \theta$ since $F(\alpha_i, L_i, X_i) = 1$ at the optimum. Now, plugging the first-order conditions in the production function we find

$$1 = e^{\varepsilon_i} A_i(\alpha_i)\, \theta \prod_{j=1}^{n} (P_j)^{-\alpha_{ij}},$$

which is the result.

C.3 Generic uniqueness of the efficient equilibrium

Consider the planner's objective function from (21): $\mathcal{W}(\alpha; z) = \beta' \mathcal{L}(\alpha)(\mu + a(\alpha, z)) + \frac{1}{2}(1-\rho)\beta' \mathcal{L}(\alpha)\Sigma\mathcal{L}(\alpha)'\beta$, where z is a vector of parameters, which includes μ, Σ, β, ρ and any additional parameters of the $a(\alpha, z)$ function. Define a space $\mathcal{Z}$ on the set of parameters z. We endow this space with an absolutely continuous probability measure $\mathbb{P}$. We will call the solution to that problem generically unique if the set $\mathcal{Z}^*$ for which $\mathcal{W}$ has multiple maximizers is almost surely empty, i.e. $\mathbb{P}(z \in \mathcal{Z}^*) = 0$.

Our proof strategy relies on Lemma 1 from Cox (2020).

Proposition 2. *Suppose that $A_i(\alpha_i)$ takes the form (3) and all elements of the κ matrix are* *positive.*[32] *Then the Pareto efficient equilibrium is generically unique.*

Proof. Lemma 1 of Cox (2020) requires that three properties be satisfied.

1. The set $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, where $\mathcal{A}_i$ is the set of feasible production technique for firm i given by (2), must be a disjoint union of finitely or countably many second-countable Hausdorff manifolds, possibly with boundary or corner. This assumption is satisfied in our case since $\mathcal{A}$ is a manifold in $\mathbb{R}^{n^2}$ of dimension $n^2 - n$.

2. We need $\mathcal{W}(\alpha, z)$ to be differentiable with respect to z and the derivative to be continuous with respect to α and z. This is satisfied in our case given the form (3).

3. It must be that for all $\alpha_1, \alpha_2 \in \mathcal{A}$ such that $\alpha_1 \neq \alpha_2$ we have $\frac{d\mathcal{W}(\alpha_1,z)}{dz} \neq \frac{d\mathcal{W}(\alpha_2,z)}{dz}$, where the derivative here indicates the gradient. We prove this by contraposition. For that purpose, take $\alpha_1, \alpha_2 \in \mathcal{A}$ such that $\frac{d\mathcal{W}(\alpha_1,z)}{dz} = \frac{d\mathcal{W}(\alpha_2,z)}{dz}$. We are going to show that it implies that $\alpha_1 = \alpha_2$. From Proposition 5, it must be that $\frac{d\mathcal{W}(\alpha_1,z)}{d\mu_i} = \omega_i(\alpha_1, z) = \omega_i(\alpha_2, z) = \frac{d\mathcal{W}(\alpha_2,z)}{d\mu_i}$. Since this is true for all i, it follows that the vector of Domar weights must be the same, that it $\omega(\alpha_1, z) = \omega(\alpha_2, z) > 0$. Next, differentiate $\mathcal{W}(\alpha, z)$ with respect to α_{il}° to write

$$\frac{d\mathcal{W}(\alpha, z)}{d\alpha_{il}^{\circ}} = 2\omega_i \left[\kappa_{il} \left(\alpha_{il} - \alpha_{il}^{\circ} \right) + \kappa_{i0} \left(\sum_{j=1}^{n} \alpha_{ij} - \sum_{j=1}^{n} \alpha_{ij}^{\circ} \right) \right].$$

   Suppose by contradiction that $\alpha_1 \neq \alpha_2$. Then there exists a pair i, l such that $(\alpha_{il})_1 \neq (\alpha_{il})_2$. Without loss of generality, suppose that $(\alpha_{il})_1 > (\alpha_{il})_2$. Then it must be that $\sum_{j=1}^{n} (\alpha_{ij})_1 < \sum_{j=1}^{n} (\alpha_{ij})_2$ for $\frac{d\mathcal{W}(\alpha_1,z)}{d\alpha_{il}^{\circ}} = \frac{d\mathcal{W}(\alpha_2,z)}{d\alpha_{il}^{\circ}}$ to hold. Therefore, there exists l' such that $(\alpha_{il'})_1 < (\alpha_{il'})_2$. But then it must be $\frac{d\mathcal{W}(\alpha_1,z)}{d\alpha_{il'}^{\circ}} < \frac{d\mathcal{W}(\alpha_2,z)}{d\alpha_{il'}^{\circ}}$. Therefore, we have a contradiction and $\alpha_1 = \alpha_2$.

We have shown that the three properties required by Lemma 1 of Cox (2020) are satisfied. It follows that $\mathbb{P}(z \in \mathcal{Z}^*) = 0$ and the planner's solution is generically unique. As a result, there is a generically unique efficient equilibrium. ∎

D Proofs

This section contains the proofs of the formal results from the main text.

[32]This is the functional form we use for the quantitative analyses. In our calibrated model, all elements of the κ matrix are positive.

D.1 Proofs of Section 2

Lemma 1. *For a given production network α,*

$$p(\alpha) = -\mathcal{L}(\alpha)(\varepsilon + a(\alpha)), \tag{14}$$

and

$$y = \beta' \mathcal{L}(\alpha)(\varepsilon + a(\alpha)), \tag{15}$$

where $a(\alpha) = (\log A_i(\alpha_i), \ldots, \log A_n(\alpha_n))$ and $\mathcal{L}(\alpha) = (I - \alpha)^{-1}$ is the Leontief inverse.

Proof. Combining the unit cost equation (9) with the equilibrium condition (12) and taking the log we find that, for all i,

$$p_i = -\varepsilon_i - a_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} p_j,$$

where $a_i(\alpha_i) = \log(A_i(\alpha_i))$. This is a system of linear equations whose solution is (14). The log price vector is also normally distributed since it is a linear transformation of normal random variable. Combining with (7) yields (15). □

Corollary 1. *For a fixed network α:*

1. *The impact of a change in firm-level expected TFP μ_i on expected GDP $\mathrm{E}[y]$ is given by*

$$\frac{\partial\, \mathrm{E}[y]}{\partial \mu_i} = \omega_i.$$

2. *The impact of a change in firm-level volatility Σ_{ij} on the variance of GDP $\mathrm{V}[y]$ is given by*[33]

$$\frac{\partial\, \mathrm{V}[y]}{\partial \Sigma_{ij}} = \begin{cases} \omega_i^2 & i = j, \\ 2\omega_i\omega_j & i \neq j. \end{cases}$$

Proof. (17) implies that $\frac{\partial\, \mathrm{E}[y(\alpha)]}{\partial \mu_i} = \beta' \mathcal{L}(\alpha) 1_i$. Since $P'C = WL = 1$ by the household's budget constraint, we need to show that $\beta' \mathcal{L}(\alpha) 1_i = P_i Q_i$ to complete the proof of the first result. From (29), we know that $P_i C_i = \beta_i$. Using Shepard's Lemma together with the marginal pricing equation (12), we can find the firm's factor demands equations

$$\begin{aligned} P_j X_{ij} &= \alpha_{ij} P_i Q_i \\ L_i &= \left(1 - \sum_{j=1}^{n} \alpha_{ij}\right) P_i Q_i. \end{aligned} \tag{32}$$

[33] For $i \neq j$, the following derivative simultaneously changes Σ_{ij} and Σ_{ji} to preserve the symmetry of Σ.

Using these results, we can write the market clearing condition (13) as

$$P_i Q_i = \beta_i + \sum_{j=1}^{n} \alpha_{ji} P_j Q_j.$$

Solving the linear system implies

$$\beta' \mathcal{L}(\alpha) 1_i = P_i Q_i, \tag{33}$$

which proves the first part of the proposition.

For the second part of the result, differentiating (18) with respect to Σ_{ij} and holding Σ symmetric yields

$$\frac{\partial \operatorname{V}[y(\alpha)]}{\partial \Sigma_{ij}} = \begin{cases} \beta' \mathcal{L}(\alpha) 1_i 1_i' \mathcal{L}(\alpha)' \beta & i = j, \\ \beta' \mathcal{L}(\alpha) \left[1_i 1_j' + 1_j 1_i'\right] \mathcal{L}(\alpha)' \beta & i \neq j \end{cases} = \begin{cases} \omega_i^2 & i = j, \\ 2\omega_i \omega_j & i \neq j, \end{cases}$$

which is the result. □

Lemma 2. $\lambda(\alpha^*)$, $k_i(\alpha_i, \alpha^*)$ *and* $q_i(\alpha^*)$ *are normally distributed and the technique choice problem of the firm can be written as*

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \operatorname{E}[k_i(\alpha_i, \alpha^*)] + \frac{1}{2} \operatorname{V}[k_i(\alpha_i, \alpha^*)] + \operatorname{Cov}[k_i(\alpha_i, \alpha^*), \lambda(\alpha^*) + q_i(\alpha^*)]. \tag{19}$$

Proof. We first consider the stochastic discount factor. (31) shows that aggregate consumption can be written as a function of prices. Combining that equation with (6) we can write $\lambda = \log(\Lambda)$ as

$$\lambda(\alpha^*) = -(1-\rho) \sum_{i=1}^{n} \beta_i p_i(\alpha^*) \tag{34}$$

Taking the log of (9) yields

$$k_i(\alpha_i, \alpha^*) = -(\varepsilon_i + a(\alpha_i)) + \sum_{j=1}^{n} \alpha_{ij} p_j(\alpha^*). \tag{35}$$

Both $\lambda(\alpha^*)$ and $k_i(\alpha_i, \alpha^*)$ are normally distributed since they are linear combinations of ε and the log price vector, which is normally distributed by Lemma 1.

Turning to the problem of the firm, we can write (11) as

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \operatorname{E}[\Lambda Q_i K_i(\alpha_i, P)], \tag{36}$$

or, taking the logs

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}\left[\exp\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right) + k_i\left(\alpha_i, \alpha^*\right)\right]\right],$$

where $q_i(\alpha^*) = \log Q_i(\alpha^*)$ and where we emphasize that λ and q_i depend only on the equilibrium technique choice α^*. From (33), q_i is normally distributed and so are all the terms in the exponential. We can therefore use the expression for the expected value of a lognormal distribution and write

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \exp\left\{\mathrm{E}\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right) + k_i\left(\alpha_i, \alpha^*\right)\right] + \frac{1}{2}\mathrm{V}\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right) + k_i\left(\alpha_i, \alpha^*\right)\right]\right\}.$$

Taking away the exponentiation, as it is a monotone transformation, and $\mathbb{E}\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right)\right]$ since it does not affect the minimization yields

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}\left[k_i\left(\alpha_i, \alpha^*\right)\right] + \frac{1}{2}\mathrm{V}\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right) + k_i\left(\alpha_i, \alpha^*\right)\right].$$

We can expend that expression as

$$\begin{aligned}\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}\left[k_i\left(\alpha_i, \alpha^*\right)\right] + \frac{1}{2}\mathrm{V}\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right)\right] + \frac{1}{2}\mathrm{V}\left[k_i\left(\alpha_i, \alpha^*\right)\right] \\ + \mathrm{Cov}\left(k_i\left(\alpha_i, \alpha^*\right), \lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right)\right)\end{aligned}$$

The term $\mathrm{V}\left[\lambda\left(\alpha^*\right) + q_i\left(\alpha^*\right)\right]$ can be dropped as it does not affect the optimization and we find (19).

For later derivations, it is also convenient to write (36) in terms of P_i as

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathrm{E}\left[\Lambda \frac{\beta' \mathcal{L}\left(\alpha^*\right) 1_i}{P_i} K_i\left(\alpha_i, P\right)\right]$$

where we have used (33). We can drop $\beta' \mathcal{L}(\alpha^*) 1_i \geq 0$ since it is deterministic and does not depend on α_i. Going through the same steps as above, the firm's problem becomes

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}\left[k_i\left(\alpha_i, \alpha^*\right)\right] + \frac{1}{2}\mathrm{V}\left[\lambda\left(\alpha^*\right) - p_i\left(\alpha^*\right) + k_i\left(\alpha_i, \alpha^*\right)\right]. \tag{37}$$

□

Proposition 1. *An equilibrium exists.*

Proof. We group here the proofs of Lemma 3 and Proposition 1. We proceed in three steps. First we show that there is a unique technique α_i that solves the problem of the firm, i.e. $\mathcal{K}_i$ is a function. Second, we show that that function is continuous. Finally, we use a fixed-point theorem to show the existence of an equilibrium.

Step 1. We show that the right-hand side of (37) is a strictly concave function. First, note that from (35) we can write

$$\mathrm{E}\left[k_i\left(\alpha_i,\alpha^*\right)\right]=\mathrm{E}\left[-\left(\varepsilon_i+a\left(\alpha_i\right)\right)+\sum_{j=1}^{n}\alpha_{ij}p_j\left(\alpha^*\right)\right]$$
$$=-a\left(\alpha_i\right)+\mathrm{E}\left[-\varepsilon_i-\alpha_i'\mathcal{L}\left(\alpha^*\right)\left(\varepsilon+a\left(\alpha^*\right)\right)\right]$$

which is strictly convex in α_i since $a\left(\alpha_i\right)=\log A_i\left(\alpha_i\right)$ is strictly concave by Assumption 1.

Similarly, combining (34) and (35) we can write

$$\frac{1}{2}\mathrm{V}\left[\lambda\left(\alpha^*\right)-p_i\left(\alpha^*\right)+k_i\left(\alpha_i,\alpha^*\right)\right]=\frac{1}{2}\mathrm{V}\left[-\left(\varepsilon_i+a\left(\alpha_i\right)\right)-p_i\left(\alpha^*\right)+\sum_{j=1}^{n}\left(\alpha_{ij}-\left(1-\rho\right)\beta_j\right)p_j\left(\alpha^*\right)\right].$$

We can remove the term $a\left(\alpha_i\right)$ from the variance as it is not stochastic. Combining with the equilibrium price equation (14), we get

$$\frac{1}{2}\mathrm{V}\left[\lambda\left(\alpha^*\right)-p_i\left(\alpha^*\right)+k_i\left(\alpha_i,\alpha^*\right)\right]=\frac{1}{2}\mathrm{V}\left[-\varepsilon_i+1_i'\mathcal{L}\left(\alpha^*\right)\left(\varepsilon+a\left(\alpha^*\right)\right)-\left(\alpha_i-\left(1-\rho\right)\beta\right)'\mathcal{L}\left(\alpha^*\right)\left(\varepsilon+a\left(\alpha^*\right)\right)\right]$$
$$=\frac{1}{2}\mathrm{V}\left[-\varepsilon_i-\left(\alpha_i-1_i-\left(1-\rho\right)\beta\right)'\mathcal{L}\left(\alpha^*\right)\left(\varepsilon+a\left(\alpha^*\right)\right)\right]$$

where 1_i is a column vector full of zeros, except for a 1 at location i. Once again we can drop the term in $a\left(\alpha^*\right)$ as it is non stochastic. Define the row vector B as

$$B\left(\alpha_i,\alpha^*\right)=-\left(\alpha_i-1_i-\left(1-\rho\right)\beta\right)'\mathcal{L}\left(\alpha^*\right)-1_i',$$

where $\beta=\left(\beta_1,\ldots,\beta_n\right)$ is a column vector. Then

$$\mathrm{V}\left[\lambda\left(\alpha^*\right)-p_i\left(\alpha^*\right)+k_i\left(\alpha_i,\alpha^*\right)\right]=B\left(\alpha_i,\alpha^*\right)\Sigma B\left(\alpha_i,\alpha^*\right)',$$

where Σ is the covariance matrix of ε. The right-hand side will have a term that is linear in α_i, and that therefore does not affect the concavity of the expression, and the quadratic term

$$\alpha_i'\mathcal{L}\left(\alpha^*\right)\Sigma\mathcal{L}\left(\alpha^*\right)'\alpha_i.$$

The matrix $\mathcal{L}\left(\alpha^*\right)\Sigma\mathcal{L}\left(\alpha^*\right)'$ is positive semi-definite since Σ, as it is covariance matrix, is positive semi-definite. To see this, note that for any column vector $x\in\mathbb{R}^n$ we have

$$x'\mathcal{L}\left(\alpha^*\right)\Sigma\mathcal{L}\left(\alpha^*\right)'x=y'\Sigma y\geq 0$$

where $y=\mathcal{L}\left(\alpha^*\right)'x$ and the last inequality follows from the fact that Σ is positive semi-definite. The expression $\mathrm{V}\left[\lambda\left(\alpha^*\right)-p_i\left(\alpha^*\right)+k_i\left(\alpha_i,\alpha^*\right)\right]$ is therefore convex in α_i. Since the sum of a strictly convex function and a convex function is strictly convex, the expression

$$\mathrm{E}\left[k_i\left(\alpha_i,\alpha^*\right)\right]+\frac{1}{2}\,\mathrm{V}\left[\lambda\left(\alpha^*\right)-p_i\left(\alpha^*\right)+k_i\left(\alpha_i,\alpha^*\right)\right]$$

is strictly convex in the vector α_i.

To complete this first step, note that the set of techniques $\mathcal{A}$ is convex. Since the problem of the firm involves the minimization of strictly convex function on a convex set it has a unique minimizer. The mapping $\mathcal{K}_i\left(\alpha^*\right)$ is therefore a function for every i and every $\alpha^*\in\mathcal{A}$.

Step 2. We now show that κ_i is continuous. To simplify the notation, define

$$g_i\left(\alpha,\alpha^*\right)=\mathbb{E}\left[k_i\left(\alpha,\alpha^*\right)\right]+\frac{1}{2}\mathbb{V}\left[\lambda\left(\alpha^*\right)-p_i\left(\alpha^*\right)+k_i\left(\alpha,\alpha^*\right)\right].$$

where we have temporarily removed the subscript i on the column vector α_i to avoid cluttering the notation.

We will first show that $g_i\left(\alpha,\alpha^*\right)$ is continuous. From (34) and (35), λ and k are continuous functions of α and linear functions of $p\left(\alpha^*\right)$. It therefore suffices to show that $p\left(\alpha^*\right)$ is continuous. From (14), we see that $p\left(\alpha^*\right)$ is continuous since $\mathcal{L}\left(\alpha^*\right)$, as a matrix inverse, in continuous and $a\left(\alpha^*\right)$ is continuous by Assumption 1. So $g_i\left(\alpha,\alpha^*\right)$ is continuous.

We now turn to the proof of the continuity of $\mathcal{K}_i$. We have already shown that g is strictly convex in α so there is a unique minimizer $\mathcal{K}_i\left(\alpha^*\right)=\arg\min_\alpha g_i\left(\alpha,\alpha^*\right)$. Take a sequence $\alpha^*_k\to\alpha^*_\star$ and let $\alpha_k=\mathcal{K}_i\left(\alpha^*_k\right)$ and $\alpha_\star=\mathcal{K}_i\left(\alpha^*_\star\right)$. Choose any subsequence $I\subset\mathbb{N}$, then α_k has an accumulation point α'_k since $\mathcal{A}$ is compact. Since $g\left(\alpha_k,\alpha^*_k\right)\le g\left(\alpha,\alpha^*_k\right)$ for all $\alpha\in\mathcal{A}$ and $k\in I$ we have, by continuity of g, that $g_i\left(\alpha'_k,\alpha^*\right)\le g_i\left(\alpha,\alpha^*\right)$ for all $\alpha\in\mathcal{A}$ and since the minimizer is unique it must be that $\alpha'_k=\alpha_\star$. As a result, $\alpha_k\to\alpha_\star$ and κ_i is continuous.

Step 3. We have shown that the mapping $\mathcal{K}_i\left(\alpha^*\right)$ is continuous for all $i=1,\ldots,n$. Define the mapping $\mathcal{K}\left(\alpha^*\right)=\left(\mathcal{K}_1\left(\alpha^*\right),\ldots,\mathcal{K}_n\left(\alpha^*\right)\right)$. Then $\mathcal{K}\left(\alpha^*\right)$ is a continuous mapping from $\mathcal{A}$ (a compact and convex set) to itself. Therefore, by Brouwer's fixed-point theorem $\mathcal{K}$ has a fixed point and an equilibrium exists. □

Proposition 2. *There exists a Pareto efficient equilibrium.*

Proof. Since we only have one agent in the economy, any Pareto efficient allocation must maximize the utility of the representative household. Under a given network and a given productivity shock ε the first welfare theorem applies and the equilibrium is efficient. The consumption of the planner is therefore given by (15). Taking a step back, the efficient production network must therefore solve

$$\begin{aligned}\max_{\alpha\in\mathcal{A}} \mathrm{E}\left[u\left(Y\right)\right] &= \max_{\alpha\in\mathcal{A}} \frac{1}{1-\rho}\,\mathrm{E}\left[\exp\left(\left(1-\rho\right)\log Y\right)\right] \\ &= \max_{\alpha\in\mathcal{A}} \frac{1}{1-\rho}\exp\left(\left(1-\rho\right)\mathrm{E}\left[\log Y\right]+\frac{1}{2}\left(1-\rho\right)^{2}\mathrm{V}\left[\log Y\right]\right) \\ &= \max_{\alpha\in\mathcal{A}} \mathrm{E}\left[\log Y\right]-\frac{1}{2}\left(\rho-1\right)\mathrm{V}\left[\log Y\right]\end{aligned} \tag{38}$$

where we have used the fact that $\log Y$ is normally distributed. The rest of the proof compares the first-order conditions of the planner and of the equilibrium.

First-order conditions of the planner. Using (17) and (18), we can write (38) as

$$\max_{\alpha\in\mathcal{A}} \beta'\mathcal{L}\left(\alpha\right)\left(\mu+a\left(\alpha\right)\right)+\frac{1}{2}\left(1-\rho\right)\beta'\mathcal{L}\left(\alpha\right)\Sigma\mathcal{L}\left(\alpha\right)'\beta.$$

The first-order conditions are

$$\begin{aligned}0 = {} & \beta'\left(\frac{\partial}{\partial\alpha_{ij}}\mathcal{L}\left(\alpha\right)\right)\left(\mu+a\left(\alpha\right)+\frac{1}{2}\left(1-\rho\right)\Sigma\mathcal{L}\left(\alpha\right)'\beta\right) \\ & +\beta'\mathcal{L}\left(\alpha\right)\left(\frac{\partial}{\partial\alpha_{ij}}a\left(\alpha\right)+\frac{1}{2}\left(1-\rho\right)\Sigma\left(\frac{\partial}{\partial\alpha_{ij}}\mathcal{L}\left(\alpha\right)\right)'\beta\right)+\underline{\mu}_{ij}-\gamma_{i}\end{aligned}$$

where $\underline{\mu}_{ij}$ are the Lagrange multipliers on the constraints on $\alpha_{ij}\geq 0$ and γ_i is the Lagrange multiplier on the constraint $\sum_j \alpha_{ij}\leq\overline{\alpha}_i$.[34] Now,

$$\frac{\partial}{\partial\alpha_{ij}}\mathcal{L}\left(\alpha\right)=\frac{\partial}{\partial\alpha_{ij}}\left(I-\alpha\right)^{-1}=-\left(I-\alpha\right)^{-1}\left[\frac{\partial}{\partial\alpha_{ij}}\left(I-\alpha\right)\right]\left(I-\alpha\right)^{-1} \tag{39}$$

$$=\left(I-\alpha\right)^{-1}\left[O_{ij}\right]\left(I-\alpha\right)^{-1}=\mathcal{L}\left(\alpha\right)O_{ij}\mathcal{L}\left(\alpha\right) \tag{40}$$

where $O_{ij}=1_i 1_j'$ is a matrix full of zero except for a one at element (i,j). Plugging back in and grouping terms yields

$$\begin{aligned}0 = {} & \beta'\mathcal{L}\left(\alpha\right)1_i 1_j'\mathcal{L}\left(\alpha\right)\left[\mu+a\left(\alpha\right)\right]+\beta'\mathcal{L}\left(\alpha\right)1_i\frac{\partial}{\partial\alpha_{ij}}a_i\left(\alpha\right) \\ & +\left(1-\rho\right)\beta'\mathcal{L}\left(\alpha\right)1_i 1_j'\mathcal{L}\left(\alpha\right)\Sigma\mathcal{L}\left(\alpha\right)'\beta+\underline{\mu}_{ij}-\gamma_i\end{aligned}$$

Since $\beta'\mathcal{L}\left(\alpha\right)1_i$ is a strictly positive scalar we can divide the whole equation by it to find

[34] Note that $\sum_j \alpha_{ij}\leq\overline{\alpha}_i<1$ implies that $\alpha_{ij}<1$, so we do not need to explicitly consider a constraint $\alpha_{ij}\leq 1$.

$$0 = 1_j' \mathcal{L}(\alpha)[\mu + a(\alpha)] + \frac{\partial}{\partial \alpha_{ij}} a_i(\alpha) + (1-\rho) 1_j' \mathcal{L}(\alpha) \Sigma \mathcal{L}(\alpha)' \beta \tag{41}$$

$$+ \left(\beta' \mathcal{L}(\alpha) 1_i\right)^{-1} \left(\underline{\mu}_{ij} - \gamma_i\right) \tag{42}$$

First-order conditions in the equilibrium. We can repeat similar steps for the equilibrium. Combining (37) with (14), (34) and (35), we find that firm i's problem can be written as

$$\begin{aligned}\alpha_i^* = \arg\min_{\alpha_i \in \mathcal{A}_i} & -a(\alpha_i) - \alpha_i' \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) \\ & + \frac{1}{2}\left((\alpha_i - 1_i - (1-\rho)\beta)' \mathcal{L}(\alpha^*) + 1_i'\right) \Sigma \left((\alpha_i - 1_i - (1-\rho)\beta)' \mathcal{L}(\alpha^*) + 1_i'\right)'.\end{aligned}$$

Differentiating with respect to α_{ij} we can write the first-order conditions as

$$\begin{aligned}0 = & -\frac{\partial a(\alpha_i)}{\partial \alpha_{ij}} - 1_j' \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) + \frac{1}{2}\left(1_j' \mathcal{L}(\alpha^*)\right) \Sigma \left((\alpha_i - 1_i - (1-\rho)\beta)' \mathcal{L}(\alpha^*) + 1_i'\right)' \\ & + \frac{1}{2}\left((\alpha_i - 1_i - (1-\rho)\beta)' \mathcal{L}(\alpha^*) + 1_i'\right) \Sigma \mathcal{L}(\alpha^*)' 1_j + \underline{\mu}_{ij}^e - \gamma_i^e\end{aligned}$$

or

$$\begin{aligned}0 = & -\frac{\partial a(\alpha_i)}{\partial \alpha_{ij}} - 1_j' \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) \\ & + \left((\alpha_i - 1_i - (1-\rho)\beta)' \mathcal{L}(\alpha^*) + 1_i'\right) \Sigma \mathcal{L}(\alpha^*)' 1_j + \underline{\mu}_{ij}^e - \gamma_i^e,\end{aligned}$$

where the Lagrange multipliers have a superscript e to indicate the equilibrium. In equilibrium $\alpha = \alpha^*$ and so

$$-\frac{\partial a(\alpha_i^*)}{\partial \alpha_{ij}} - 1_j' \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) + \left((\alpha_i^* - 1_i - (1-\rho)\beta)' \mathcal{L}(\alpha^*) + 1_i'\right) \Sigma \mathcal{L}(\alpha^*)' 1_j + \underline{\mu}_{ij}^e - \gamma_i^e = 0.$$

Finally, we can show that $(1_i - \alpha_i^*)' \mathcal{L}(\alpha^*) - 1_i' = 0$ by right-multiplying both sides by $(\mathcal{L}(\alpha^*))^{-1}$. As a result, the first-order conditions become

$$-\frac{\partial a(\alpha_i^*)}{\partial \alpha_{ij}} - 1_j' \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) - (1-\rho)\beta' \mathcal{L}(\alpha^*) \Sigma \mathcal{L}(\alpha^*)' 1_j + \underline{\mu}_{ij}^e - \gamma_i^e = 0.$$

Notice that these are the same first-order conditions (up to a normalization of the Lagrange multipliers) as the planner's (equation 41). The complementary slackness conditions are also the same in both problems. As a result, any equilibrium allocation also satisfied the planner's first-order conditions and vice versa. □

Corollary 2. *The equilibrium production network α^* solves*

$$\max_{\alpha\in\mathcal{A}} \mathrm{E}\left[y\left(\alpha\right)\right] - \frac{1}{2}\left(\rho-1\right)\mathrm{V}\left[y\left(\alpha\right)\right], \tag{21}$$

where GDP y is given by (15).

Proof. This is an intermediate result that was proven at (38) in the proof of Proposition 2. □

D.2 Proofs of Section 5

Proposition 3. *The Domar weight ω_i of firm i is increasing in μ_i and decreasing in Σ_{ii}.*

Proof. Fix the initial mean and variance-covariance matrix at μ^0 and Σ^0, and denote the optimal network by $\alpha^*\left(\mu^0,\Sigma^0\right)$. Now, consider an increase in μ_i from μ_i^0 to μ_i^1 (holding other elements of μ and Σ fixed). The welfare changes from $\mathcal{W}\left(\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right);\mu_i^0,\mu_{-i}^0,\Sigma^0\right)$ to $\mathcal{W}\left(\alpha^*\left(\mu_i^1,\mu_{-i}^0,\Sigma^0\right);\mu_i^1,\mu_{-i}^0,\Sigma^0\right)$, which, by Proposition 5, can be written as

$$\mathcal{W}\left(\alpha^*\left(\mu_i^1,\mu_{-i}^0,\Sigma^0\right);\mu_i^1,\mu_{-i}^0,\Sigma^0\right) = \mathcal{W}\left(\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right);\mu_i^0,\mu_{-i}^0,\Sigma^0\right) + \int_{\mu_i^0}^{\mu_i^1}\omega_i\left(\mu_i,\mu_{-i}^0,\Sigma^0\right)d\mu_i.$$

Now suppose instead that the network is fixed at its original value $\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right)$. From Equations (17) and (18), a change in μ_i affects welfare only through its impact on expected GDP. By Lemma 1, the change in welfare can be written as

$$\mathcal{W}\left(\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right);\mu_i^1,\mu_{-i}^0,\Sigma^0\right) = \mathcal{W}\left(\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right);\mu_i^0,\mu_{-i}^0,\Sigma^0\right) + \omega_i\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right)\left(\mu_i^1-\mu_i^0\right).$$

Since the initial network $\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right)$ is attainable at $\left(\mu_i^1,\mu_{-i}^0,\Sigma^0\right)$, it must be that $\mathcal{W}\left(\alpha^*\left(\mu_i^1,\mu_{-i}^0,\Sigma^0\right);\mu_i^1,\mu_{-i}^0,\Sigma^0\right) \geq \mathcal{W}\left(\alpha^*\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right);\mu_i^1,\mu_{-i}^0,\Sigma^0\right)$. Because μ_i^1 can be picked arbitrary close to μ_i^0, it must therefore be that $\omega_i\left(\mu_i^1,\mu_{-i}^0,\Sigma^0\right) \geq \omega_i\left(\mu_i^0,\mu_{-i}^0,\Sigma^0\right)$, or $\frac{d\omega_i}{d\mu_i}\geq 0$.

For the second part of the proposition, recall that $\frac{d\mathcal{W}}{d\Sigma_{ii}} = \left(1-\rho\right)\omega_i^2$ by Proposition 5. Using analogous steps, we then can establish the second part of this proposition. □

Lemma 4. *Let $\alpha^*\in int(\mathcal{A})$ be the equilibrium network and suppose that Assumption 2 holds. There exists a $\overline{\Sigma}>0$ such that if $|\Sigma_{ij}|<\overline{\Sigma}$ for all i,j, there is a neighborhood around α^* in which*

(i) an increase in μ_j leads to an increase in the shares α_{kl}^* for all k,l;

(ii) an increase in Σ_{jj} leads to a decline in the shares α_{kl}^* for all k,l;

(iii) an increase in Σ_{ij} leads to a decline in the shares α_{kl}^* for all k,l.

Proof. **Point (i).** Away from the constraints, the first-order conditions of the planner are

$$F_{jk} := \frac{\partial a_j}{\partial \alpha_{jk}} + 1_k' \mathcal{L} (\mu + \alpha) + (1 - \rho) \beta' \mathcal{L} \Sigma \mathcal{L}' 1_k = 0.$$

To investigate how α changes with μ_i, we use the implicit function theorem. First, differentiate F_{jk} with respect to μ_i to find

$$\frac{\partial F_{jk}}{\partial \mu_i} = 1_k' \mathcal{L} 1_i = \mathcal{L}_{ki}.$$

Then

$$\frac{\partial F}{\partial \mu_i} = \begin{pmatrix} \left(\frac{\partial F_{1\cdot}}{\partial \mu_i}\right)' \\ \left(\frac{\partial F_{2\cdot}}{\partial \mu_i}\right)' \\ \cdots \\ \left(\frac{\partial F_{n\cdot}}{\partial \mu_i}\right)' \end{pmatrix} = 1_{n \times 1} \otimes (\mathcal{L} 1_i),$$

where $1_{n\times 1}$ is an $n \times 1$ column vector of ones, $\frac{\partial F}{\partial \mu_i}$ is an $n^2 \times 1$ column vector which consists of the n column vectors $\left(\frac{\partial F_{j\cdot}}{\partial \mu_i}\right)'$ with elements $\left(\frac{\partial F_{jk}}{\partial \mu_i}\right)_{k=1,\ldots,n}$.

Next, differentiate F_{jk} with respect to α_{lm} to get

$$\begin{aligned} \frac{\partial F_{jk}}{\partial \alpha_{lm}} =& \frac{\partial^2 a_j}{\partial \alpha_{jk} \partial \alpha_{lm}} + 1_k' \mathcal{L} 1_l \frac{\partial a_l}{\partial \alpha_{lm}} + 1_k' \mathcal{L} \left(1_l 1_m'\right) \mathcal{L} (\mu + a) \\ &+ (1-\rho) 1_k' \mathcal{L} \Sigma \left(\beta' \mathcal{L} 1_l 1_m' \mathcal{L}\right)' + (1-\rho) \beta' \mathcal{L} \Sigma \left(1_k' \mathcal{L} 1_l 1_m' \mathcal{L}\right)' \\ =& \frac{\partial^2 a_j}{\partial \alpha_{jk} \partial \alpha_{lm}} + (1-\rho) 1_k' \mathcal{L} \Sigma \left(\beta' \mathcal{L} 1_l 1_m' \mathcal{L}\right)' + \mathcal{L}_{kl} \left[\underbrace{\frac{\partial a_l}{\partial \alpha_{lm}} + 1_m' \mathcal{L} (\mu + a) + (1-\rho) \beta' \mathcal{L} \Sigma \mathcal{L}' 1_m}_{=F_{lm}=0} \right], \end{aligned}$$

where we use the first-order condition to set the last term to 0.

Now, denote by A the $n^2 \times n^2$ block-diagonal matrix with the n blocks $A_1, A_2, \ldots, A_n$ along the main diagonal such that $(A_j)_{kl} = \left(\frac{\partial a_j}{\partial \alpha_{jk} \partial \alpha_{jl}}\right)_{k,l=1,\ldots,n}$. Denote by D the $n \times n^2$ matrix $(1-\rho) \left[\left(\beta^T \mathcal{L}\right) \otimes (\mathcal{L} \Sigma \mathcal{L}')\right]$. Then denote by B the $n^2 \times n^2$ matrix that consists of n copies of D, i.e. $B = 1_{n\times 1} \otimes D$. Then, by the implicit function theorem, we have

$$\begin{pmatrix} \left(\frac{\partial \alpha_{1\cdot}}{\partial \mu_i}\right)' \\ \left(\frac{\partial \alpha_{2\cdot}}{\partial \mu_i}\right)' \\ \cdots \\ \left(\frac{\partial \alpha_{n\cdot}}{\partial \mu_i}\right)' \end{pmatrix} = -(A+B)^{-1} \frac{\partial F}{\partial \mu_i}. \tag{43}$$

We will now show that when $\Sigma = 0$ (and so $B = 0$), all the elements on the left-hand side of (43) are positive. Since the right-hand side of (43) is continuous in the elements of Σ, the left-hand side will remain positive for small Σ.

We first establish that the elements of $-A^{-1}$ are positive. Since a_i is strictly concave by Assumption 1, A_i is strictly negative definite for all i. As, in addition, Weak Complementarity (Assumption 2) holds, $-A_i$ is a (non-singular) M-matrix and so its inverse $-A_i^{-1}$ is nonnegative. The diagonal elements of $-A_i^{-1}$ are also strictly positive. To see this, note that since A_i is Hermitian, so is A_i^{-1}, and we know from the Rayleigh quotient that

$$\lambda_{\min}\left(A_i^{-1}\right) \leq \frac{x' A_i^{-1} x}{x' x} \leq \lambda_{\max}\left(A_i^{-1}\right),$$

where $\lambda_{\min}\left(A_i^{-1}\right)$ and $\lambda_{\max}\left(A_i^{-1}\right)$ are the smallest and largest eigenvalues of A_i^{-1}, respectively, and where x is any nonzero vector. By setting $x = 1_t$, the tth basis vector we get $\lambda_{\min}\left(A_i^{-1}\right) \leq \left(A_i^{-1}\right)_{tt} \leq \lambda_{\max}\left(A_i^{-1}\right)$. Since the eigenvalues of A_i are strictly negative by Assumption 1, we know that $\lambda_{\min}\left(A_i^{-1}\right) = 1/\lambda_{\max}\left(A_i\right)$ and $\lambda_{\max}\left(A_i^{-1}\right) = 1/\lambda_{\min}\left(A_i\right)$. We therefore have that $\left(\lambda_{\max}\left(A_i\right)\right)^{-1} \leq \left(A_i^{-1}\right)_{tt} \leq \left(\lambda_{\min}\left(A_i\right)\right)^{-1}$, and so all diagonal elements of A_i^{-1} are strictly negative and bounded away from zero by some number $0 > \overline{A} \geq \left[A_i^{-1}\right]_{tt}$, and so the diagonal elements of $-A_i^{-1}$ are positive.

Now, due to the block-diagonal structure of A, it is true that $-A^{-1}$ is a matrix with all positive diagonal elements and nonnegative off-diagonal elements. Notice also that all elements of $\frac{\partial F}{\partial \mu_i}$ are elements of the Leontief inverse matrix $\mathcal{L} = I + \alpha + \alpha^2 + \ldots$ and are positive since $\alpha_i \in \operatorname{int}\left(\mathcal{A}_i\right)$ for all i.

Now, in the case of no uncertainty, $\Sigma = 0$, $B = 0$ and the right-hand side of 43 must be strictly positive and so is the vector of $\frac{\partial \alpha_{kl}}{\partial \mu_i}$. In this case, both parts of the Lemma hold. If there is uncertainty ($\Sigma = 0$), the result still holds if all the elements of Σ are sufficiently close to zero. Indeed, $-\left(A + B\right)^{-1}$ is continuous in Σ and, thus there exists $\overline{\Sigma} > 0$ such that if $\left|\Sigma_{ij}\right| < \overline{\Sigma}$ for all $i, j \in \mathcal{N}^2$ then elements of $-\left(A + B\right)^{-1} \frac{\partial F}{\partial \mu_i}$ have the same signs as the corresponding elements of $-A^{-1} \frac{\partial F}{\partial \mu_i}$.[35]

Point (ii). The proof is analogous to that of point (i). We differentiate the first order conditions with respect to a diagonal element of Σ

$$\frac{\partial F}{\partial \Sigma_{ii}} = (1-\rho)\left[1_{n\times 1} \otimes \left(\left(\beta' \mathcal{L} 1_i\right)\left(\mathcal{L}\iota_i\right)\right)\right] = (1-\rho)\left(\beta' \mathcal{L} 1_i\right)\frac{\partial F}{\partial \mu_i}.$$

Since $\rho > 1$ and $\omega_i = \beta' \mathcal{L} 1_i > 0$ for all $i \in \mathcal{N}$, the result follows from the same steps as in point (i)

Point (iii). The proof is analogous to that of point (i). We differentiate the first order

[35]Note that $(A+B)^{-1}$ exists for small Σ because A the eigenvalues of A are strictly negative (and so $\det(A) \neq 0$ and A is invertible) and that the determinant of $A + B$ is a continuous function of Σ. Note also that as we move Σ away from 0 the optimal matrix α changes and so do A and $\mathcal{L}$. But these changes are continuous so the strict inequality $-\left(A+B\right)^{-1} \frac{\partial F}{\partial \mu_i} > 0$ is preserved for small enough Σ.

conditions with respect to an off-diagonal element of Σ. To preserve the symmetry of Σ, we simultaneously change Σ_{ij} and Σ_{ji} to find

$$\frac{\partial F}{\partial \Sigma_{ij}} = (1-\rho)\left[1_{n\times 1}\otimes\left(\left(\beta'\mathcal{L}1_i\right)\left(\mathcal{L}\iota_j\right)+\left(\beta'\mathcal{L}1_j\right)\left(\mathcal{L}\iota_i\right)\right)\right] = (1-\rho)\left[\left(\beta'\mathcal{L}1_i\right)\frac{\partial F}{\partial \mu_j}+\left(\beta'\mathcal{L}1_j\right)\frac{\partial F}{\partial \mu_i}\right].$$

Since $\rho > 1$ and $\omega_i = \beta'\mathcal{L}1_i > 0$ for all $i \in \mathcal{N}$, the result follows from the same steps as in point (i). □

Proposition 4. *Uncertainty lowers the expected value of GDP, such that* $\mathrm{E}[y]$ *is largest when* $\Sigma = 0$.

Proof. The proof follows from Lemma 2. Without uncertainty ($\Sigma = 0$), the term $\mathrm{V}[c(\alpha)]$ is 0 for all α, and so α is set to maximize $\mathrm{E}[c(\alpha)]$. When uncertainty is introduced, the objective function also depends on $\mathrm{V}[c(\alpha)]$ and so $\mathrm{E}[c]$ is no longer maximized. □

Proposition 5. *When the network* α *is free to adjust to changes in* μ *and* Σ*, the following holds.*

1. *The impact of an increase in* μ_i *on expected welfare is given by*

$$\frac{d\mathcal{W}}{d\mu_i} = \frac{\partial\,\mathrm{E}[y]}{\partial \mu_i} = \omega_i. \tag{22}$$

2. *The impact of an increase in* Σ_{ij} *on expected welfare is given by*

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = \begin{cases} -\frac{1}{2}(\rho-1)\left(\frac{\partial\,\mathrm{E}[y]}{\partial\mu_i}\right)^2 = -\frac{1}{2}(\rho-1)\,\omega_i^2 & i=j, \\ -(\rho-1)\frac{\partial\,\mathrm{E}[y]}{\partial\mu_i}\frac{\partial\,\mathrm{E}[y]}{\partial\mu_j} = -(\rho-1)\,\omega_i\omega_j & i\neq j. \end{cases} \tag{23}$$

Proof. Recall from Lemma 2 that the equilibrium α^* solves the welfare-maximization problem

$$\mathcal{W}(\mu,\Sigma) = \max_{\alpha\in\mathcal{A}}\left\{\mathrm{E}[y(\alpha)] - \frac{1}{2}(\rho-1)\,\mathrm{V}[y(\alpha)]\right\}.$$

Since that the objective function and the constraints are continuously differentiable functions of α, we can apply the envelope theorem, such that

$$\frac{d\mathcal{W}}{d\mu_i} = \frac{\partial\,\mathrm{E}[y]}{\partial\mu_i} = \beta'\mathcal{L}(\alpha)\,1_i = \omega_i,$$

and

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho-1)\frac{\partial\,\mathrm{V}[y(\alpha)]}{\partial\Sigma_{ij}} = (1-\rho)\,\beta'\mathcal{L}(\alpha)\left(1_i 1_j'\right)\mathcal{L}(\alpha)'\,\beta = (1-\rho)\,\omega_i\omega_j,$$

where we used the expressions for the expectation and the variance of output given by (17) and (18). ☐

Proposition ??. *Let $\alpha^*(\mu, \Sigma)$ be the equilibrium production network under μ and Σ and let $\mathcal{W}(\alpha, \mu, \Sigma)$ be the welfare of the household under the network α. Then, the change in welfare after a shock to (μ, Σ) is larger under the flexible network than under the fixed network, in the sense that*

$$\mathcal{W}\left(\alpha^*\left(\mu', \Sigma'\right), \mu', \Sigma'\right) - \mathcal{W}\left(\alpha^*(\mu, \Sigma), \mu, \Sigma\right) \geq \mathcal{W}\left(\alpha^*(\mu, \Sigma), \mu', \Sigma'\right) - \mathcal{W}\left(\alpha^*(\mu, \Sigma), \mu, \Sigma\right),$$

where (μ', Σ') denotes the after-shock values.

Proof. By definition, the change in welfare under the flexible network is

$$\mathcal{W}\left(\alpha^*\left(\mu', \Sigma'\right), \mu', \Sigma'\right) - \mathcal{W}\left(\alpha^*(\mu, \Sigma), \mu, \Sigma\right).$$

By Proposition 2, $\alpha^*(\mu', \Sigma')$ maximizes welfare under (μ, Σ) so that

$$\mathcal{W}\left(\alpha^*\left(\mu', \Sigma'\right), \mu', \Sigma'\right) \geq \mathcal{W}\left(\alpha^*(\mu, \Sigma), \mu', \Sigma'\right).$$

Combining the two expression gives the result. ☐