Cascades and Fluctuations in an Economy with an Endogenous Production Network

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Introduction

- Production in modern economies involves a complex network of producers supplying and demanding goods from each other
- The shape of this network is constantly changing in response to shocks
  - For instance, after a severe shock a producer might shut down ...
  - ... which might lead its neighbors to shut down as well
  - Cascade of shutdowns that spreads through the network
- These cascades
  - change the shape of the network
  - and influence the way micro shocks aggregate into macro fluctuation

This paper proposes a

Theory of network formation and aggregate fluctuations

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Overview of the Model

Key ingredients of the model

- Set of n firms that use inputs from connected firms
- Fixed cost to operate
  - Firms operate or not
  - Endogenously shape the network

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Key results from the model

- Efficient organization of production
  - Complementarities between operation decisions of nearby firms
  - Create tightly connected clusters centered around productive firms
  - Small changes can trigger large reorganization of the network
- Cascades of firm shutdowns
  - Well-connected firms are hard to topple but create big cascades
  - Elasticities of substitution matter for size and propagation of cascades
- Aggregate fluctuations
  - Recessions feature fewer well-connected firms and less clustering
  - Allowing the network to adjust yields substantially smaller fluctuations

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Key difficulty in solving the planner's problem

- The Karush-Kuhn-Tucker conditions do not apply
- Caused by two features of the environment
  - Binary choice (to operate or not)
  - Complementarities in decisions of nearby firms
- Standard approach
  - Brute force over all the 2^n potential networks
  - Impossible for more than a few firms
- Novel approach that relies on a *reshaping* of the problem
  - Theory: works if network is highly connected
  - Practice: works very well even for sparse networks
  - Methodological contribution that can be used elsewhere

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Literature Review

- Endogenous network formation
  - Atalay et al (2011), Oberfield (2013), Carvalho and Voigtländer (2014)
- Network of sectors and fluctuations
  - Horvath (1998), Dupor (1999), Acemoglu et al (2012), Baqaee (2016), Acemoglu et al (2016), Lim (2016)
- Non-convex adjustments in networks
  - Bak, Chen, Woodford and Scheinkman (1993), Elliott, Golub and Jackson (2014)

I. Model

Model

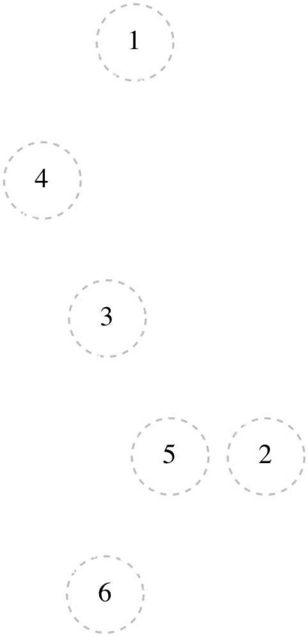
- There are n units of production (firm) indexed by $j \in \{1, \ldots, n\}$
  - Each unit produces a differentiated good
  - Differentiated goods can be used to
    - produce a final good
$$Y \equiv \left(\sum_{j=1}^{n} \left(y_j^0 \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$
    - produce other differentiated goods
- Representative household
  - Consumes the final good
  - Supplies L units of labor inelastically

Model

- Firm j produces good j

$$y_j = \frac{A}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z_j \left[\left(\sum_{i=1}^{n} x_{ij}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} \right]^\alpha l_j^{1-\alpha}$$

- Firm j can only use good i as input if there is a *connection* between firms i and j
  - $\Omega_{ij} = 1$ if connection and $\Omega_{ij} = 0$ otherwise
  - A connection can be *active* or *inactive*
  - Matrix Ω is *exogenous*
- A firm can only produce if it pays a fixed cost f in units of labor
  - $\theta_j = 1$ if j is operating and $\theta_j = 0$ otherwise
  - Vector θ is *endogenous*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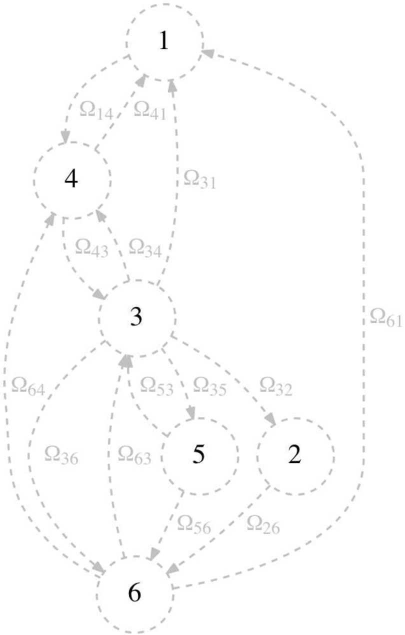
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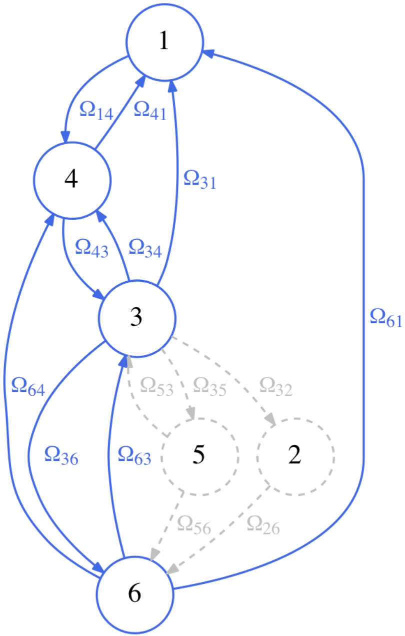
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Social Planner

Problem $\mathcal{P}_{SP}$ of a social planner

$$\max_{\substack{y^0,x,l\\ \theta\in\{0,1\}^n}} \left(\sum_{j=1}^{n}\left(y_j^0\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

subject to

1. a resource constraint for each good j

$$y_j^0+\sum_{k=1}^{n}x_{jk}\leq\frac{A}{\alpha^\alpha(1-\alpha)^{1-\alpha}}z_j\left(\sum_{i=1}^{n}x_{ij}^{\frac{\epsilon-1}{\epsilon}}\right)^{\alpha\frac{\epsilon}{\epsilon-1}}l_j^{1-\alpha}$$

2. a resource constraint on labor

$$\sum_{j=1}^{n}l_j+f\sum_{j=1}^{n}\theta_j\leq L$$

3. operation constraints: $\{\theta_j=0\}\Rightarrow\{l_j=0\}$
4. connection constraints: $\{\Omega_{ij}=0\}\Rightarrow\{x_{ij}=0\}$

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subject to

1. a resource constraint for each good j **(Lagrange multiplier: λ_j)**

$$y_j^0+\sum_{k=1}^{n}x_{jk}\leq\frac{A}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}}z_j\theta_j\left(\sum_{i=1}^{n}\Omega_{ij}x_{ij}^{\frac{\epsilon-1}{\epsilon}}\right)^{\alpha\frac{\epsilon}{\epsilon-1}}l_j^{1-\alpha}$$

2. a resource constraint on labor **(Lagrange multiplier: w)**

$$\sum_{j=1}^{n}l_j+f\sum_{j=1}^{n}\theta_j\leq L$$

3. ~~operation constraints: $\{\theta_j=0\}\Rightarrow\{l_j=0\}$~~
4. ~~connection constraints: $\{\Omega_{ij}=0\}\Rightarrow\{x_{ij}=0\}$~~

II. Social Planner with Exogenous θ

Social Planner with Exogenous θ

Define $q_j = w/\lambda_j$

- From the FOCs, output is $(1-\alpha)\, y_j = q_j l_j$
- q_j is the *labor productivity* of firm j

Proposition 1

In the efficient allocation,

$$q_j = z_j \theta_j A \left(\sum_{i=1}^{n} \Omega_{ij} q_i^{\epsilon - 1} \right)^{\frac{\alpha}{\epsilon - 1}} \tag{1}$$

Furthermore, there is a unique vector q that satisfies (1) such that $q_j > 0$ if $\theta_j = 1$.

Note:

- (1) is not a contraction. Use Kennan (2001) instead.
- q can be solved by iterating on (1)

Social Planner with Exogenous θ

Knowing q we can solve for all other quantities

Lemma 1

Aggregate output is

$$Y = Q\left(L - f\sum_{j=1}^{n}\theta_j\right)$$

where $Q \equiv \left(\sum_{j=1}^{n} q_j^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$ *is aggregate labor productivity.*

Social Planner with Exogenous θ

Lemma 2

The optimal labor allocation satisfies

$$l = (1-\alpha)\underbrace{\left[I_n - \alpha\Gamma\right]^{-1}}_{(1)} \underbrace{\left(\frac{q}{Q}\right)}_{(2)}^{\circ(\sigma-1)} \left(L - f\sum_{j=1}^{n}\theta_j\right)$$

where I_n is the identity matrix and where Γ is an $n \times n$ matrix where $\Gamma_{jk} = \frac{\Omega_{jk} q_j^{\epsilon-1}}{\sum_{i=1}^{n} \Omega_{ik} q_i^{\epsilon-1}}$ captures the importance of j as a supplier to k.

Determinants of l_j

(1) Importance of j as a supplier

- Leontief inverse $\left([I_n - \alpha\Gamma]^{-1} = I_n + \alpha\Gamma + (\alpha\Gamma)^2 + \dots\right)$

(2) Relative efficiency

III. Social Planner with Endogenous θ

Social Planner with Endogenous θ

Planner's problem is now

$$\max_{\theta \in \{0,1\}^n} Q\left(L - f \sum_{j=1}^{n} \theta_j\right)$$

with

$$q_j = z_j \theta_j A \left(\sum_{i=1}^{n} \Omega_{ij} q_i^{\epsilon-1}\right)^{\frac{\alpha}{\epsilon-1}}$$

Trade-off: making firm j produce ($\theta_j = 1$)

- increases labor productivity of the network (Q)
- reduces the amount of labor into production $\left(L - f \sum_{j=1}^{n} \theta_j\right)$

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Social Planner with Endogenous θ

Very hard problem (MINLP — NP Hard)

- The set $\theta \in \{0,1\}^n$ is not convex
- Objective function is not concave

Naive approach

- For any vector $\theta \in \{0,1\}^n$ iterate on q and evaluate the objective function
- 2^n vectors θ to try ($\approx 10^6$ configurations for 20 firms)
- Impossible for n large

Social Planner with Endogenous θ

Consider the relaxed and reshaped problem $\mathcal{P}_{RR}$

$$\max_{\theta \in \{0,1\}^n} Q\left(L - f\sum_{j=1}^{n}\theta_j\right)$$

with

$$q_j = z_j \theta_j A \left(\sum_{i=1}^{n} \Omega_{ij} q_i^{\epsilon-1}\right)^{\frac{\alpha}{\epsilon-1}}$$

Parameters $a > 0$ and $b \geq 0$ are *reshaping constants*

- Reshape the objective function *away* from optimum (i.e. when $0 < \theta_j < 1$)
    - For a: if $\theta_j \in \{0,1\}$ then $\theta_j^a = \theta_j$
    - For b: $\{\theta_i = 0\} \Rightarrow \{q_i = 0\}$ and $\{\theta_i = 1\} \Rightarrow \left\{\theta_i^b q_i^{\epsilon-1} = q_i^{\epsilon-1}\right\}$
- Change marginal gain in productivity of letting a firm being "more active"

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Social Planner with Endogenous θ

Proposition 2

If θ^* solves $\mathcal{P}_{RR}$ and that $\theta_j^* \in \{0,1\}$ for all j, then θ^* also solves $\mathcal{P}_{SP}$.

Solution approach: Pick a and b such that $\mathcal{P}_{RR}$ has two key properties

P1 $\mathcal{P}_{RR}$ is easy to solve

- The Karush-Kuhn-Tucker conditions are necessary and sufficient

P2 The solution to $\mathcal{P}_{RR}$ also solves $\mathcal{P}_{SP}$

- Solution θ^* to $\mathcal{P}_{RR}$ is such that $\theta_j^* \in \{0,1\}$ for all j

Reshaping

Proposition 3

If $a \leq (\sigma - 1)^{-1}$, $b \leq 1 - a(\epsilon - 1)$ *and if the network of potential connections is complete* ($\Omega_{ij} = 1$ *for all* i, j)*, then the Karush-Kuhn-Tucker conditions are necessary and sufficient to characterize a solution to* $\mathcal{P}_{RR}$.

The proposition

- Provides conditions under which it is easy to solve $\mathcal{P}_{RR}$ (P1)
- Only provides *sufficient* conditions
  - The approach works for much more general Ω's

Reshaping

Solution θ^* to $\mathcal{P}_{RR}$ is such that $\theta_j^* \in \{0,1\}$ for all j (P2) if

- the network is large and sufficiently connected
- the reshaping parameters take the values

$$\boxed{a = \frac{1}{\sigma - 1} \quad \text{and} \quad b = 1 - \frac{\epsilon - 1}{\sigma - 1}} \tag{$\star$}$$

Key idea:

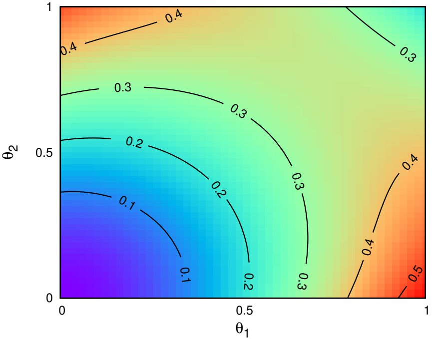
- Under $\star$ the marginal benefit of θ_j only depends on θ_j through aggregates
  - In a large connected network aggregates are essentially independent of θ_j
  - The marginal benefit of θ_j is either positive or negative, so $\theta_j \in \{0,1\}$

FOCs

Example with $n = 2$

Relaxed problem **without** reshaping

$$V(\theta) = Q(\theta)\left(L - f\sum_{j=1}^{n}\theta_j\right) \text{ with } q_j = z_j\theta_j A\left(\sum_{i=1}^{n}\Omega_{ij}q_i^{\epsilon-1}\right)^{\frac{\alpha}{\epsilon-1}}$$



Problem: V is not concave

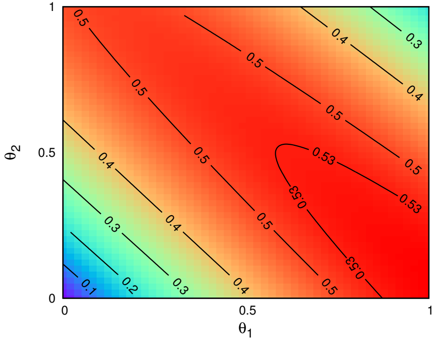
⇒ First-order conditions are not sufficient

⇒ Numerical algorithm can get stuck in local maxima

Example with $n = 2$

Relaxed problem **with reshaping constants** $\star$

$$V(\theta) = Q(\theta)\left(L - f\sum_{j=1}^{n}\theta_j\right) \text{ with } q_j = z_j\theta_j^{\frac{1}{\sigma-1}} A\left(\sum_{i=1}^{n}\Omega_{ij}\theta_i^{1-\frac{\epsilon-1}{\sigma-1}} q_i^{\epsilon-1}\right)^{\frac{\alpha}{\epsilon-1}}$$



~~Problem~~: V is now (quasi) concave

- $\Rightarrow$ First-order conditions are necessary and sufficient
- $\Rightarrow$ Numerical algorithm converges to global maximum

Testing the approach on small networks

For small networks we can solve $\mathcal{P}_{SP}$ directly by trying all possible vectors θ

- Comparing approaches for a million different economies:

	Number of firms n			
	8	10	12	14
A. With reshaping				
Firms with correct θ_j	99.9%	99.9%	99.9%	99.8%
Error in output Y	0.00039%	0.00081%	0.00174%	0.00171%
B. Without reshaping				
Firms with correct θ_j	84.3%	83.2%	82.3%	81.3%
Error in output Y	0.84%	0.89%	0.93%	0.98%

Notes: Parameters $f \in \{0.05/n, 0.1/n, 0.15/n\}$, $\sigma_z \in \{0.34, 0.39, 0.44\}$, $\alpha \in \{0.45, 0.5, 0.55\}$, $\sigma \in \{4, 6, 8\}$ and $\epsilon \in \{4, 6, 8\}$. For each combination of parameters 1000 different economies are created. For each economy, productivity is drawn from $\log(z_k) \sim$ iid $\mathcal{N}(0, \sigma_z)$ and Ω is drawn randomly such that each link Ω_{ij} exists with some probability such that a firm has on average five possible incoming connections. A network is kept in the sample only if the first-order conditions give a solution in which θ hits the bounds.

The errors come from

- firms that are particularly isolated
- two θ configurations with almost same output

Testing the approach on large networks

For large networks we cannot solve $\mathcal{P}_{SP}$ directly by trying all possible vectors θ

- After all the 1-deviations θ are exhausted:

	With reshaping	Without reshaping
Firms with correct θ_j	99.8%	72.1%
Error in output Y	0.00028%	0.69647%

Notes: Simulations of 200 different networks Ω and productivity vectors z that satisfy the properties of the calibrated economy.

- Very few "obvious errors" in the allocation found by the approach

IV. Economic Forces at Work

Selection of firms

Two benefits from operating a firm

$$\max_{\theta \in \{0,1\}^n} Q\left(L - f\sum_{j=1}^{n}\theta_j\right) \text{ subject to } q_j = z_j\theta_j A\left(\sum_{i=1}^{n}\Omega_{ij}q_i^{\epsilon-1}\right)^{\frac{\alpha}{\epsilon-1}}$$

1. **Direct benefit:** more final goods are produced
2. **Indirect benefit:** improves production efficiency of customers

Translate into selection of firms based on

- TFP: firms with high z
- Number of connections
  - Firms with many *outgoing* connections increase q for their many customers
  - Firms with many *incoming* connections have a large q

But, number of connections is endogenous

Incentives to operate $\Leftrightarrow$ Production network

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Clustering

Operating a firm increases the incentives to operate its neighbors

$\Rightarrow$ Complementarity between operating decisions of nearby firms

$\Rightarrow$ Tendency to cluster economic activity around high-z firms

Tendency to cluster is more important if

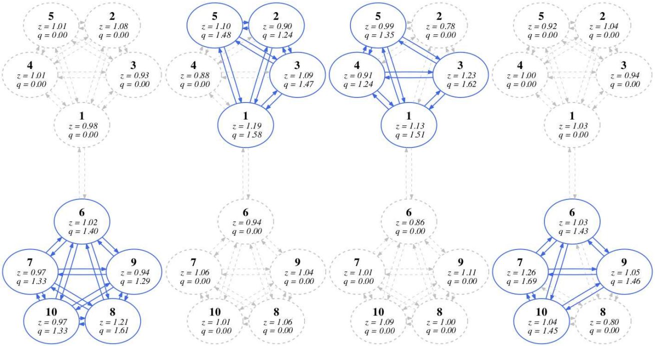
- Elasticity of substitution ϵ is small
- Share of intermediate goods α is large

Clustering

Operating a firm increases the incentives to operate its neighbors

⇒ Complementarity between operating decisions of nearby firms

⇒ Tendency to cluster economic activity around high-z firms



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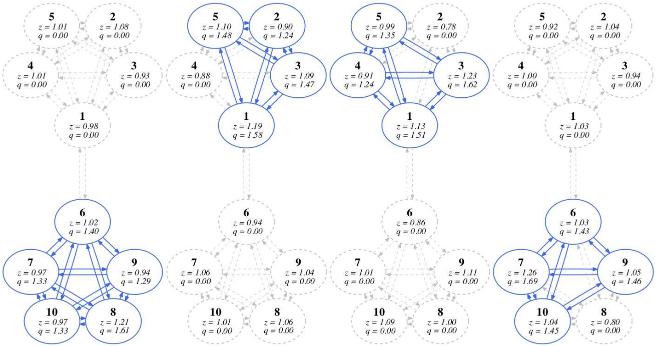

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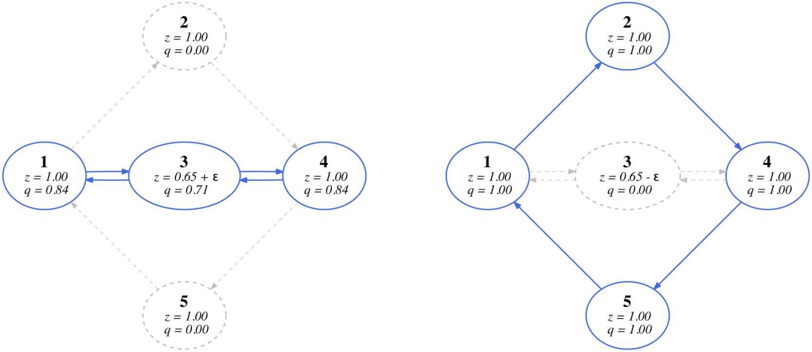
Non-convex nature of the economy:

- A small shock can lead to a large reorganization



V. Quantitative Exploration

Network data

- Two datasets that cover the U.S. economy
  - Cohen and Frazzini (2008) and Atalay et al (2011)
  - Both rely on Compustat data
    - Public firms must self-report customers that purchase more than 10% of sales
    - Use a fuzzy-text matching algorithm and manual matching to build network
  - Cover 1980 to 2004 and 1976 to 2009 respectively

Parameters

Parameters from the literature

- $\alpha = 0.5$ to fit the share of intermediate (Jorgenson et al 1987, Jones 2011)
- $\sigma = \epsilon = 6$ average of estimates (Broda et al 2006)
  - Robustness with smaller ϵ in the paper
- $\log(z_{it}) \sim \mathcal{N}\left(0, 0.39^2\right)$ from Bartelsman et al (2013)
- $f \times n = 5\%$ to fit employment in management occupations

Calibrate $n = 3000$ to match number of active firms in Atalay et al (2011)

Set of potential connections

Unobserved Ω:

- Power law in-degree distribution with tail coeff. 1 (Zipf's law) ...
  1. *Observed* in-degree distribution is power law
  2. Zipf's law arises naturally from network generating processes
  3. The calibrated observed in-degree distribution is closed to the data
- ... and add three potential connections to each firm
  1. More firms with few connections than power law (Bernard et al, 2015)
  2. Helps the algorithm to converge
  3. Minimal impact on highly-connected firms that drive fluctuations

▸ Data in-degree

Distribution of in-degree

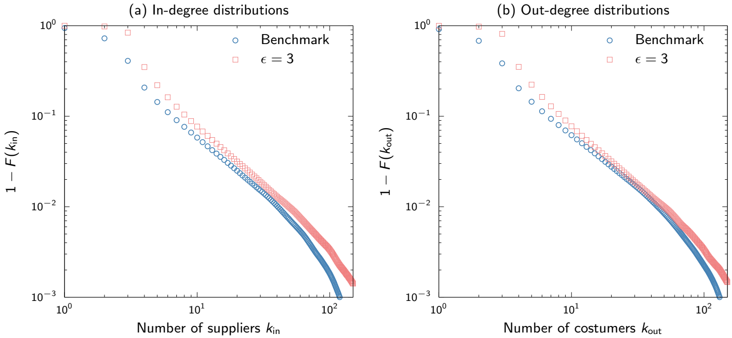


Figure: Distribution of the number of suppliers and the number of customers

In-degree power law shape parameter

- Calibration: 1.43
- Data: 1.37 (Cohen and Frazzini, 2008) and 1.3 (Atalay et al, 2011)

Data in-degree

Shape of the network

Compare **optimal network** to a completely **random network**

- Differences highlights how efficient allocation shapes the network

	Optimal network	Random network
A. Pareto shape parameters		
In-degree	1.43	1.48
Out-degree	1.37	1.48
B. Measures of proximity		
Clustering coefficient	0.027	0.018
Average distance between firms	2.26	2.64

Efficient allocation features

- Fatter tail of highly connected firms
- More clustering of firms

▸ Def. clust. coeff.

Firm-level outcomes

Regressing firm outcomes on in- and out-degree

Dependent variable	Employment l	Labor prod. q
In-degree	0.36	0.08
Out-degree	0.44	-0.05

Implications:

- More highly connected firms employ more workers (same as data)
- Firms with many suppliers have large q
- Firms with many customers operate *even with low* q

Firm-level distributions

In the efficient allocation:

- **Mitigation:** Low productivity firms do not operate
- **Magnification:** High productivity firms benefit from clustering

	Labor prod. q	Employment l
A. Optimal network economy		
Standard deviation	0.29	1.24
Skewness	0.39	0.85
Excess kurtosis	0.57	0.39
B. Random network economy		
Standard deviation	0.44	2.21
Skewness	-0.03	-0.05
Excess kurtosis	0.01	-0.06

Because of the optimal organization of the network

- Distributions are positively skewed ...
- ... and have fatter tails

Dist. of q

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Cascades of shutdowns

Because of the complementarities between firms

- Exit of a firm makes it more likely that its neighbors exit as well ...
- ... which incentivizes the second neighbors to exit as well ...
- ...

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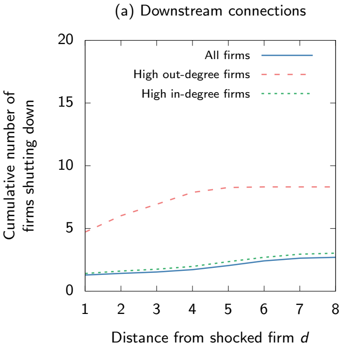

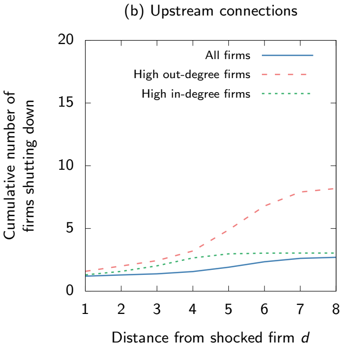

▸ Robustness

Resilience of firms

Magnitude of shock necessary to make a firm exit varies

	Probability of firm shut down after 1 std shock
All firms	92%
High out-degree firms	20%
High in-degree firms	56%

Implications:

- Highly-connected firms are hard to topple ...
- ... but upon shutting down they create large cascades

▸ Robustness

Note:

- Cascades are the manifestation of the efficient adjustment of the network in response to shocks
- Preventing them (bailout) can lead to substantially larger drops in output

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Aggregate fluctuations

The shape of the network changes with the business cycle

	Correlation with output		
	Model	Data	
		CF (2008)	AHRS (2011)
A. Power law shape parameters			
In-degree	-0.10	-0.10	-0.21
Out-degree	-0.31	-0.24	-0.13
B. Clustering coefficient	0.47	0.70	0.15

Implications:

- Recessions are periods with fewer highly-connected firms ...
- ... and in which clustering activity around most productive firms is costly

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Aggregate fluctuations

Size of fluctuations

$$Y = Q\left(L - f\sum_j \theta_j\right)$$

Table: Standard deviation of aggregates

	Output Y	Labor Prod. Q	Prod. labor $L - f\sum_j \theta_j$
Optimal network	**0.039**	0.039	0.0014
Random network	**0.054**	0.054	0

Implications:

- Substantially smaller fluctuations in optimal network economy ...
- ... comes from the reorganization of network after shocks

Intuition

Aggregate fluctuations

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Intuition

Conclusion

- Theory of network formation and aggregate fluctuations
- Propose an approach to solve these hard problems easily
- The optimal allocation features
    - Clustering of activity
    - Cascades of shutdowns/restarts
- Optimal network substantially limit the size of fluctuations

Details of reshaping

Simpler to consider

$$\mathcal{P}'_{RD}: \quad \max_{\theta\in[0,1]^n, q} \left(\sum_{j=1}^n q_j^{\sigma-1}\right)^{\frac{1}{\sigma-1}} \left(L - f\sum_{j=1}^n \theta_j\right)$$

$$q_j \leq A z_j \theta_j^a A B_j^\alpha \qquad \text{(LM: } \beta_j\text{)}$$

where $B_j = \left(\sum_{i=1}^n \Omega_{ij}\theta_i^b q_i^{\epsilon-1}\right)^{\frac{1}{\epsilon-1}}$.
First order condition with respect to θ_k:

$$\frac{\partial q_k}{\partial \theta_k}\frac{\partial Q}{\partial q_k}\left(L - f\sum_{j=1}^n \theta_j\right) - fQ + \sum_{j=1}^n \beta_j \left(\frac{\partial q_k}{\partial \theta_k}\frac{\partial B_j}{\partial q_k} + \frac{\partial B_j}{\partial \theta_k}\right)\frac{\partial q_j}{\partial B_j} = \overline{\mu}_k - \underline{\mu}_k$$

The terms are

$$\frac{\partial q_k}{\partial \theta_k}\frac{\partial Q}{\partial q_k} = z_k a \theta_k^{a-1} A B_k^\alpha \times \left(z_k \theta_k^a A B_k^\alpha\right)^{\sigma-2} Q^{2-\sigma}$$

$$\frac{\partial q_k}{\partial \theta_k}\frac{\partial B_j}{\partial q_k} + \frac{\partial B_j}{\partial \theta_k} = B_j \theta_k^{b-1}\Omega_{kj}\left(\frac{z_k\theta_k^a A B_k^\alpha}{B_j}\right)^{\epsilon-1}\left(a + \frac{b}{\epsilon-1}\right)$$

Return

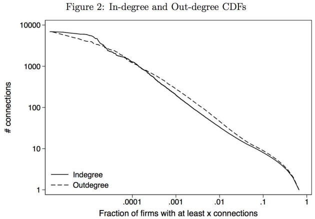


Figure: Distribution of in-degree and out-degree in Bernard et al (2015)

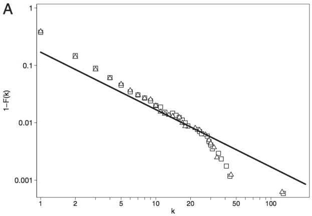


Figure: Distribution of in-degree in Atalay et al (2011)

Return Calib. Return Dist.

Clustering coefficient

- Triplet: three connected nodes (might be overlapping)
- Triangles: three fully connected nodes (3 triplets)

$$\text{Clustering coefficient} = \frac{3 \times \text{number of triangles}}{\text{number of triplets}}$$

Return

Firm-level distributions

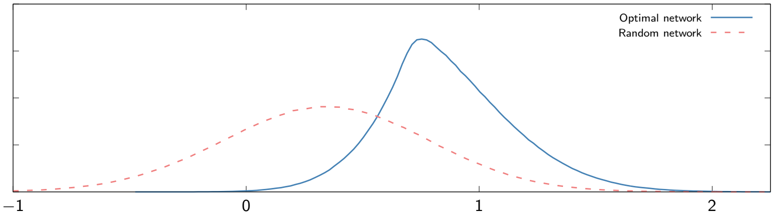
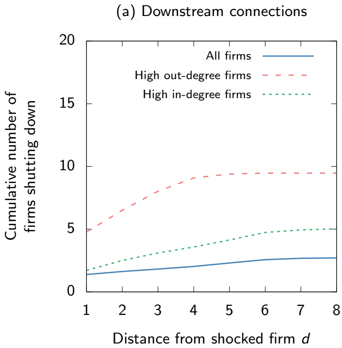


Figure: Distributions of $\log(q)$

return

Cascades of shutdowns



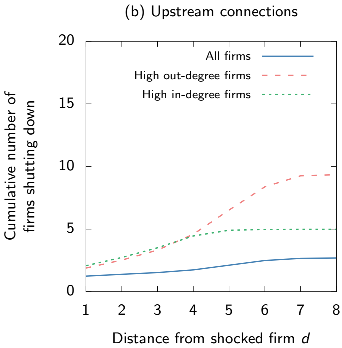


Figure: $\alpha = 0.75$

return

Cascades of shutdowns

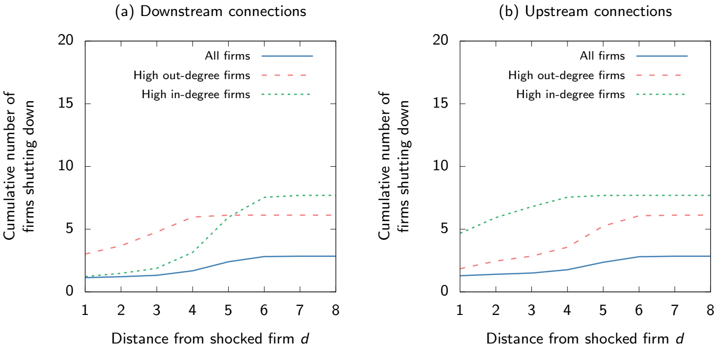


Figure: $\epsilon = 3$

return

Resilience

	Probability of firm shutdown		
	Benchmark	$\alpha = 0.75$	$\epsilon = 3$
All firms	92%	82%	32%
High out-degree firms	20%	8%	0%
High in-degree firms	56%	19%	15%

return

Aggregate fluctuations

Aggregate fluctuations are smaller in the optimal network economy

- The planner compares the 2^n potential networks
- Output for each network k is a random variable Y_k
- Maximization

$$Y = \max_k Y_k$$

- For n large, $\mathrm{Var}(Y)$ declines rapidly with n

Return