MAT237 Notes

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Contents

§1 Day 1: Administrative Stuff (Sept 2, 2025)

Everything is in the syllabus, but we did play with some blocks! There's many different ways to visualize the same thing. Went over classroom norms and whatnot, and then looked at syllabus, no math content today.

§2 Day 2: Speed and Velocity (Sept 4, 2025)

We want to differentiate and integrate functions $A \to B$, where $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$. Today we study the case where $m = 1$, so functions $A \subseteq \mathbb{R} \to \mathbb{R}^n$. Today, will mainly look at functions $\mathbb{R} \to \mathbb{R}^n$, as having a single parameter makes them much easier to work with.

Recall that distance is a scalar quantity, while velocity is a vector, meaning it has both magnitude and direction. Have $f : \mathbb{R} \to \mathbb{R}^n$ model some particle's position, and $|| \cdot ||$ be the euclidean norm. The average speed[1] over the time interval t_1 and t_2 ($t_1 < t_2$) is given by

$$\frac{||f(t_2) - f(t_1)||}{t_2 - t_1}$$

The instantaneous speed at time t is given by

$$\lim_{h \to 0} \left|\left| \frac{f(t+h) - f(t)}{h} \right|\right|$$

The average velocity between t_1 and t_2 is given by

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

and the instantaneous velocity at t by

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Theorem 2.1 (Absolute Homogeneity of Euclidean Norm)

The euclidean norm has the *absolute homogeneity* property.

For all $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$, a scalar λ

$$||\lambda v|| = |\lambda| \, ||v||$$

2.1 and various properties of norms were not mentioned in class.[2]

Proof.

$$\begin{aligned} ||\lambda v|| &= \left|\left| \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \right|\right| \\ &= \sqrt{\lambda^2 {v_1}^2 + \cdots + \lambda^2 {v_n}^2} \\ &= \sqrt{\lambda^2({v_1}^2 + \cdots + {v_n}^2)} \\ &= |\lambda| \sqrt{{v_1}^2 + \cdots + {v_n}^2)} \\ &= |\lambda| \, ||v|| \end{aligned}$$

□

[1] TODO: I don't believe we are concerned with the 'actual' speed over time in this course, since that would involve finding the arc length and even more trouble. I'm unsure about this, for now I will assume it means 'magnitude of the displacement vector'.

[2] I included this for completeness because people didn't believe that $\frac{||\gamma(6+h)-\gamma(6)||}{|h|}$ and $\left|\left|\frac{\gamma(6+h)-\gamma(6)}{h}\right|\right|$ were the same quantity.

§3 Day 3: Graphs, Level sets, and Slices (Sept 8, 2025)

Office hours starting next week! Check Quercus for more details.

Last class, we looked at functions $\mathbb{R} \to \mathbb{R}^n$. Today we look at functions $\mathbb{R}^n \to \mathbb{R}$. We develop graphs, level sets, and slices because they offer new ways to analyze and study properties of such functions, that cannot be easily captured otherwise. For example, it is difficult to visualize a more than 3 dimensional vector.
Have $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^n$.

Definition 3.1 (Graph)**.** The graph of f is $\{(x, f(x)) : x \in A\}$

Note that when you plot a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, the graph would exist in $\mathbb{R}^{n+1}$.

Definition 3.2 (Level Set)**.** The level set of f at k is $\{x \in \mathbb{R}^n : f(x) = k\}$[3]

To produce the **slice** of a graph, we need to hold a coordinate x_i constant, which we set to a. We call the following a x_i-slice, where the c is at the i-th position in the tuple:

$$\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^n : (x_1, \cdots, a, \cdots, x_n) \in A, x_{n+1} = f(x_1, \cdots, a, \cdots, x_n)\}$$

where in the first tuple, x_i is omitted, with x_i is replaced by a in the second and third tuples. Note that the slice lives in $\mathbb{R}^n$, because we already have the information that $x_i = a$.

In this course, you always want to specify the domain and codomain of your function, to avoid confusion.

Problem 3.3. Give a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level set at 0 is the set

$$\{(x, y) \in \mathbb{R}^2 : |x| = |y|\}$$

To solve these kinds of problems, you set $0 = |y| - |x|$ and you would get a candidate function $f(x, y) = |y| - |x|$. To show that two sets A and B are equal, you would typically prove that $A \subseteq B$, and $B \subseteq A$.

Alternatively you could define a function

$$g(x, y) = \begin{cases} 0 & \text{if } |x| = |y| \\ 1 & \text{otherwise} \end{cases}$$

which satisfies the requirements by construction.

[3] If $A \subseteq \mathbb{R}^2$, the level set is called a *contour*.