§1 Optimization Workflow

Suppose you want to find maxima/minima of some $f : \mathbb{R}^n \to \mathbb{R}$, on the set $A \subseteq \mathbb{R}^n$. This is equivalent to finding maxima/minima of $f\big|_A : A \to \mathbb{R}$.

(i) Determine whether global extrema must exist.

Sufficient conditions for global extrema include:

- EVT Theorem 2.1: requires A to be compact, $f\big|_A$ to be continuous

(ii) Identify critical points on the interior.

Properties of critical points a on the interior:

- Local EVT Theorem 2.2: $\nabla f(a) = 0$ or $\nabla f(a)$ DNE

(iii) Check the boundary for extrema.

There are 2 main ways of checking the boundary:

Parameterization Find some $g : B \to A$, where $B \subseteq \mathbb{R}^m$ with $m < n$, such that $\operatorname{im}(g) = A$. This lets you solve the lower (m) dimensional optimization problem of $f \circ g$.

Lagrange Requires that f is C^1. Find some $g : \mathbb{R}^n \to \mathbb{R}$, such that $\partial A = g^{-1}(\{c\})$ for some $c \in \mathbb{R}$, with $\nabla g(p) \neq 0$ for all $p \in \partial A$.

Then for each local extrema a on S, there exists some $\lambda \in \mathbb{R}$ such that $\nabla f(a) = \lambda \nabla g(a)$.

(iv) Check all candidates

Interior and boundary points are only candidates. Global maximums are local maximums, and the same applies to the subsets case.

§2 Relevant Theorems

Theorem 2.1 (EVT)**.** If $A \subseteq \mathbb{R}^n$ is a non-empty compact set, and $f : A \to \mathbb{R}$ is continuous, then f attains maximum and minimum values at points of A.

Theorem 2.2 (Local EVT)**.** Let $A \subseteq \mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a real-valued function. If a is an interior point of A, and f has a local extremum at a, then $\nabla f(a) = 0$ or $\nabla f(a)$ DNE.

Theorem 2.3. Let $U \subseteq \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}$ be C^1. Suppose $S = f^{-1}(\{0\})$ is non-empty, and fix $p \in S$. If $\nabla f(p) \neq 0$, then S is a $(n-1)$ dimensional smooth manifold at p. A vector $v \in \mathbb{R}^n$ is a tangent vector of p if and only if $\nabla f(p) \cdot v = 0$.