MAT237 Notes

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'25 Fall - '26 Winter

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§1 Day 1: Administrative Stuff (Sept 2, 2025)

Everything is in the syllabus, but we did play with some blocks! There's many different ways to visualize the same thing. Went over classroom norms and whatnot, and then looked at syllabus, no math content today.

§2 Day 2: Speed and Velocity (Sept 4, 2025)

We want to differentiate and integrate functions $A \to B$, where $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$. Today we study the case where $m = 1$, so functions $A \subseteq \mathbb{R} \to \mathbb{R}^n$. Today, will mainly look at functions $\mathbb{R} \to \mathbb{R}^n$, as having a single parameter makes them much easier to work with.

Recall that distance is a scalar quantity, while velocity is a vector, meaning it has both magnitude and direction. Have $f : \mathbb{R} \to \mathbb{R}^n$ model some particle's position, and $||\cdot||$ be the euclidean norm. The average speed[1] over the time interval t_1 and t_2 ($t_1 < t_2$) is given by

$$\frac{||f(t_2) - f(t_1)||}{t_2 - t_1}$$

The instantaneous speed at time t is given by

$$\lim_{h \to 0} \left|\left| \frac{f(t+h) - f(t)}{h} \right|\right|$$

The average velocity between t_1 and t_2 is given by

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

and the instantaneous velocity at t by

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Theorem 2.1 (Absolute Homogeneity of Euclidean Norm)

The euclidean norm has the *absolute homogeneity* property.

For all $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$, a scalar λ

$$||\lambda v|| = |\lambda| \, ||v||$$

2.1 and various properties of norms were not mentioned in class.[2]

Proof.

$$\begin{aligned} ||\lambda v|| &= \left|\left| \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \right|\right| \\ &= \sqrt{\lambda^2 {v_1}^2 + \cdots + \lambda^2 {v_n}^2} \\ &= \sqrt{\lambda^2({v_1}^2 + \cdots + {v_n}^2)} \\ &= |\lambda| \sqrt{{v_1}^2 + \cdots + {v_n}^2)} \\ &= |\lambda| \, ||v|| \end{aligned}$$

□

[1] I don't believe we are concerned with the 'actual' speed over time in this course (as in the physics definition), since that would involve finding the arc length and even more trouble. Just assume it means 'magnitude of the displacement vector'.

[2] I included this for completeness because people didn't believe that $\frac{||\gamma(6+h) - \gamma(6)||}{|h|}$ and $\left|\left| \frac{\gamma(6+h) - \gamma(6)}{h} \right|\right|$ were the same quantity.

§3 Day 3: Graphs, Level sets, and Slices (Sept 8, 2025)

Office hours starting next week! Check Quercus for more details.

Last class, we looked at functions $\mathbb{R} \to \mathbb{R}^n$. Today we look at functions $\mathbb{R}^n \to \mathbb{R}$. We develop graphs, level sets, and slices because they offer new ways to analyze and study properties of such functions, that cannot be easily captured otherwise. For example, it is difficult to visualize a more than 3 dimensional vector.
Have $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^n$.

Definition 3.1 (Graph)**.** The graph of f is $\{(x, f(x)) : x \in A\}$

Note that when you plot a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, the graph would exist in $\mathbb{R}^{n+1}$.

Definition 3.2 (Level Set)**.** The level set of f at k is $\{x \in \mathbb{R}^n : f(x) = k\}$[3]

To produce the **slice** of a graph, we need to hold a coordinate x_i constant, which we set to a. We call the following a x_i-slice, where the c is at the i-th position in the tuple:

$$\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^n : (x_1, \cdots, a, \cdots, x_n) \in A, x_{n+1} = f(x_1, \cdots, a, \cdots, x_n)\}$$

where in the first tuple, x_i is omitted, with x_i is replaced by a in the second and third tuples. Note that the slice lives in $\mathbb{R}^n$, because we already have the information that $x_i = a$.

In this course, you always want to specify the domain and codomain of your function, to avoid confusion.

Problem 3.3. Give a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level set at 0 is the set

$$\{(x, y) \in \mathbb{R}^2 : |x| = |y|\}$$

To solve these kinds of problems, you set $0 = |y| - |x|$ and you would get a candidate function $f(x, y) = |y| - |x|$. To show that two sets A and B are equal, you would typically prove that $A \subseteq B$, and $B \subseteq A$.

Alternatively you could define a function

$$g(x, y) = \begin{cases} 0 & \text{if } |x| = |y| \\ 1 & \text{otherwise} \end{cases}$$

which satisfies the requirements by construction.

[3] If $A \subseteq \mathbb{R}^2$, the level set is called a *contour*.

§4 Day 4: Vector Fields and Transformations (Sept 9, 2025)

So far, we have seen parametric curves ($\mathbb{R} \to \mathbb{R}^n$), real valued functions ($\mathbb{R}^n \to \mathbb{R}$). Today we look at functions $\mathbb{R}^n \to \mathbb{R}^n$, which are called vector fields or transformations[4].

Definition 4.1 (Vector Field)**.** A n dimensional vector field is a function $F : A \to B$ with $A, B \subseteq \mathbb{R}^n$.

Note that vector fields are capitalized as per convention.

(Not testable) Newton's law of gravity states that the force exerted by an object at the origin with mass m_1 on an object at (x, y, z) with mass m_2 is given by

$$F(x,y,z) = \frac{-Gm_1m_2}{||(x,y,z)||^2} \cdot \frac{(x,y,z)}{||(x,y,z)||}$$

The magnitude is controlled by the first part of the product (note that there are only scalars), and the direction is controlled by the unit vector (note that it is scaled down to have a magnitude of 1).

[4]very uncommon to call such functions transformations

§5 Day 5: Coordinate Transformations (Sept 11, 2025)

Definition 5.1 (Coordinate Transformation)**.** A **coordinate transformation** $f : A \to B$ is a continuous transformation that is usually bijective. A and the map f form a **coordinate system** for the codomain B.

We want to plot subsets of B and describe them using the coordinate system defined by f and A. We use (u, v) to describe the elements in A, and (x, y) for B, and would write

$$g(u, v) = (u^2 + v^2, v)$$

or simply

$$(x, y) = (u^2 + v^2, v)$$

Definition 5.2 (Polar Coordinate Transformation)**.** A map $T : \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T(r, \theta) = (r \cos\theta, r \sin\theta)$$

T describes a map from polar coordinates to cartesian coordinates. The radius can be negative. Two notable properties are $T(r, \theta) = T(-r, \theta + \pi)$ and $T(r, \theta) = T(r, \theta + 2\pi)$, following from trigonometry.

The set $\{(r, \theta) \in \mathbb{R}^2 : r = 2\}$ would describe the set $\{(2\cos\theta, 2\sin\theta) \in \mathbb{R}^2 : \theta \in \mathbb{R}\}$, which corresponds to a circle of radius 2 centered at the origin. Restricting both sets to $\theta \in \mathbb{R}^+$ or $\theta \in \mathbb{R}^-$ would still correspond to the same set, meaning that the polar coordinate transformation is not injective. Another way to show is to consider the case $r = 0$.

What follows was not covered in lecture:

Definition 5.3 (Cylindrical Coordinate Transformation)**.** A map $T : \mathbb{R}^3 \to \mathbb{R}^3$, with

$$T(r, \theta, z) = (r \cos\theta, r \sin\theta, z)$$

This is similar to the polar coordinate transformation except we add an additional z field which remains unchanged.

Definition 5.4 (Spherical Coordinate Transformation)**.** A map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$T(\rho, \theta, \phi) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$$

If this gets covered in class I'll make a writeup deriving the formula.

§6 Day 6: Parametric, Implicit, and Explicit Form (Sep 15, 2025)

Have $n < m$ be positive integers.

Definition 6.1 (Parametric Form)**.** A set $S \subseteq \mathbb{R}^m$ can be written in parametric form (with n-variables) if there exists a set $A \subseteq \mathbb{R}^n$, and a continuous map $g : A \to \mathbb{R}^m$ such that

$$S = \{g(x) : x \in A\}$$

Definition 6.2 (Higher Dimensional Graphs)**.** Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function.

$$\operatorname{graph}(f) = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in A\}$$

In this course $\mathbb{R}^a \times \mathbb{R}^b = \mathbb{R}^{a+b}$, and we don't care whether the nesting of ordered pairs is done in the first or second entry.

Definition 6.3 (Explicit Form)**.** A set S can be written in explicit form in n variables if $S \subseteq \mathbb{R}^m$ is $\operatorname{graph}(f)$ for some continuous function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Definition 6.4 (Implicit Form)**.** A set S can be written in implicit form in n variables if there exists a constant $c \in \mathbb{R}^m$, and a continuous function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that $S = f^{-1}(\{c\})$.

Implicit form is a generalized version of a level set. The level set is defined for functions $g : \mathbb{R}^n \to \mathbb{R}$, notice that $g^{-1}(\{c\}) = \{x \in \mathbb{R}^n : g(x) = c\}$ is the same as the level set of g at c.

§7 Day 7: Interior, Boundary, and Closure (Sep 16, 2025)

Let $A \subseteq \mathbb{R}^n$.

Definition 7.1 (Interior Point). $p \in \mathbb{R}^n$ is an interior point f A if there exists an $\epsilon > 0$ such that $B_\epsilon(p) \subseteq A$.

Definition 7.2 (Interior). The interior denoted A^o is the set of all interior points of A.

Definition 7.3 (Boundary Point). $p \in \mathbb{R}^n$ is a boundary point if for every $\epsilon > 0$, $B_\epsilon(p) \cap A$ and $B_\epsilon \cap A^c$ are non-empty.

The boundary point can be isolated, there may exist ϵ-balls at p such that p is the only point in said ball.

Definition 7.4 (Topological Boundary). The topological boundary of A, written ∂A.

Definition 7.5 (Limit Point). $p \in \mathbb{R}^n$ is a limit point of A if for every $\epsilon > 0$, $B_\epsilon \setminus \{p\}$ contains points in A. The set of all limit points is written A^*.

A set B containing points in A is the same as saying that $A \cap B \neq \emptyset$. Also to trip you up, sometimes questions will need you negate the definition, so do that if it seems unintuitive to explain.

Definition 7.6 (Closure). The closure of A is written $\overline{A}$, defined as $\overline{A} = A^* \cup A$.

Theorem 7.7. Every interior point of A is a limit point of A. That is $A^o \subseteq A^*$.

Proof. Let p be an interior point of A. By definition, exist $\epsilon > 0$, such that $B_\epsilon(p) \subseteq A$. Let $\epsilon' > 0$ be arbitrary. If $\epsilon' > \epsilon$, note that $B_\epsilon(p) \subseteq B_{\epsilon'}(p)$e can take any point in the open ball $B_\epsilon(p) \setminus \{p\}$, which is a subset of A following from the definition. If $\epsilon' \leq \epsilon$, we can take any point in $B_{\epsilon'}(p) \setminus \{p\} \subseteq B_\epsilon(p) \setminus \{p\} \subseteq A$. □

Note that this proof relies on the fact for $\epsilon > 0$, there exist points in $B_\epsilon(p) \setminus \{p\}$ that are not p. This is not true from some topologies.

Theorem 7.8.

$$
\begin{aligned}
A^o \subseteq A &\subseteq \overline{A} \\
A^o \cap \partial A &= \emptyset \\
\overline{A} &= A^o \cup \partial A \\
\partial A &= \overline{A} \setminus A^o
\end{aligned}
$$

You should be able to verify this by wrangling definitions.

§8 Day 8: Sequences (Sep 18, 2025)

Definition 8.1 (Sequence)**.** A sequence in in $\mathbb{R}^n$ is a function with domain $\{k \in \mathbb{Z} : k > k_0\}$ for some fixed $k_0 \in \mathbb{Z}$ and codomain $\mathbb{R}^n$.

Definition 8.2 (Subsequence)**.** Let $x : \mathbb{N}^+ \to \mathbb{R}^n$ be a sequence and $m : \mathbb{N}^+ \to \mathbb{N}^+$ be a strictly increasing function. The sequences $\{x(m(k))\}_{k=1}^{\infty}$ is a subsequence of the sequences $\{x(k)\}_{k=1}^{\infty}$.

Definition 8.3 (Convergence)**.** We fix a $p \in \mathbb{R}^n$. A sequence $\{x(k)\}_k$ in $\mathbb{R}^n$ converges to p if for every $\epsilon \in \mathbb{R}^+$, there exists a $K \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, if $k \geq K$ then $||x(k) - p|| < \epsilon$.

Theorem 8.4. A sequence $\{x(k)\}$ converges to a point p if and only if for every $\epsilon > 0$, the set of indices $\{k \in \mathbb{N}^+ : x(k) \notin B_\epsilon(p)\}$ is finite.

Proof. ($\implies$) Suppose $\{x(k)\}$ converges to p. Then we take an arbitrary $\epsilon > 0$. There exists a $K \in \mathbb{N}^+$ such that for $k \in \mathbb{N}^+$, $k \geq K$, $x(k)$ must belong to $B_\epsilon(p)$. Then points $x(j)$ not belonging to $B_\epsilon(p)$ must be of the form $x(j)$, where $j \in \mathbb{N}^+, j < k$. Since $j \in \{1, 2, \cdots, k-1\}$, we have finitely many such points.

($\impliedby$) Suppose that for any $\epsilon > 0$ there exists only finitely many indices such that their corresponding point is not in $B_\epsilon(p)$. Let S denote the aforementioned set of indices. In the case where this is the empty set, we can take $K = 1$. If the set is non-empty, then take K as the largest such index plus one.

In both cases, all indices $k \geq K$ have $x(k) \in B_\epsilon(p)$. We can prove this by contradiction: Suppose there exists some $j \geq K$, $x(j) \notin B_\epsilon(p)$. If $S = \emptyset$, then this is immediately a contradiction. Otherwise, then the existence of j contradicts the maximality of K, since K is defined as the largest such index plus one.

As such a j does not exist, we have that for all $k \in \mathbb{N}^+$, if $k \geq K$ then $x(k) \in B_\epsilon(p)$. □

We cannot say that $\{x(k)\}$ converges to p is equivalent to for all $\epsilon > 0$, the set $\{x(k) : k \in \mathbb{N}^+, x(k) \notin B_\epsilon(p)\}$ is finite. Consider the sequence $k \in \mathbb{N}^+$, $x(k) = (-1)^k$. This sequence has only finitely many points in its image, being $\{1, -1\}$, but is clearly not convergent to any point $p \in \mathbb{R}$ (picking $\epsilon < \frac{1}{2}$ would work).

§9 Day 9: Open and Closed Sets (Sep 22, 2025)

You should know that there exist sequential formulations equivalent to the definition of closed and open sets.

We want sets that have nice properties: Let $A \subseteq \mathbb{R}^n$. We want a class of sets such that if a sequence in $\mathbb{R}^n$ converges to $a \in A$, then the *tail* of the sequence belongs in A.

Definition 9.1 (Open Set)**.** A is open if every point of A is an interior point of A.

This is equivalent to saying $A = A^o$, and $A \cap \partial A = \emptyset$.

Definition 9.2 (Cloesd Set)**.** A is closed if every limit point of A belongs to A.

That is, $A^* \subseteq A$.

Theorem 9.3

A set A is open if and only if its complement $A^c = \mathbb{R}^n \setminus A$ is closed.

§10 Day 10: Compactness (Sep 23, 2025)

Definition 10.1 (Compactness)**.** A set $A \subseteq \mathbb{R}^n$ is compact if every sequence of A has a subsequence which converges to a point lying inside A.

This definition is actually for sequentially compact sets. For this reason, our theorem relating compactness to being closed and bounded will actually be the Bolzano-Weierstrass theorem instead of the usual Heine-Borel.

Definition 10.2 (Boundedness)**.** $A \subseteq \mathbb{R}^n$ is bounded if there exists a $R > 0$, $A \subseteq \{x \in \mathbb{R}^n : ||x|| < R\}$.

Note that the complement of an unbounded set may not be bounded. Yet the complement of a bounded set is always unbounded.

Theorem 10.3 (Bolzano-Weierstrass)

A set being compact if and only if it is both closed and bounded.