MAT237 Notes

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Contents

§1 Day 1: Administrative Stuff (Sept 2, 2025)

Everything is in the syllabus, but we did play with some blocks! There's many different ways to visualize the same thing. Went over classroom norms and whatnot, and then looked at syllabus, no math content today.

§2 Day 2: Speed and Velocity (Sept 4, 2025)

We want to differentiate and integrate functions $A \to B$, where $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$. Today we study the case where $m = 1$, so functions $A \subseteq \mathbb{R} \to \mathbb{R}^n$. Today, will mainly look at functions $\mathbb{R} \to \mathbb{R}^n$, as having a single parameter makes them much easier to work with.

Recall that distance is a scalar quantity, while velocity is a vector, meaning it has both magnitude and direction. Have $f : \mathbb{R} \to \mathbb{R}^n$ model some particle's position, and $||\cdot||$ be the euclidean norm. The average speed[1] over the time interval t_1 and t_2 ($t_1 < t_2$) is given by

$$\frac{||f(t_2) - f(t_1)||}{t_2 - t_1}$$

The instantaneous speed at time t is given by

$$\lim_{h \to 0} \left|\left| \frac{f(t+h) - f(t)}{h} \right|\right|$$

The average velocity between t_1 and t_2 is given by

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

and the instantaneous velocity at t by

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Theorem 2.1 (Absolute Homogeneity of Euclidean Norm)

The euclidean norm has the *absolute homogeneity* property.

For all $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$, a scalar λ

$$||\lambda v|| = |\lambda| \, ||v||$$

2.1 and various properties of norms were not mentioned in class.[2]

Proof.

$$\begin{aligned} ||\lambda v|| &= \left|\left| \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \right|\right| \\ &= \sqrt{\lambda^2 {v_1}^2 + \cdots + \lambda^2 {v_n}^2} \\ &= \sqrt{\lambda^2 ({v_1}^2 + \cdots + {v_n}^2)} \\ &= |\lambda| \sqrt{{v_1}^2 + \cdots + {v_n}^2)} \\ &= |\lambda| \, ||v|| \end{aligned}$$

□

[1] I don't believe we are concerned with the 'actual' speed over time in this course (as in the physics definition), since that would involve finding the arc length and even more trouble. Just assume it means 'magnitude of the displacement vector'.

[2] I included this for completeness because people didn't believe that $\frac{||\gamma(6+h) - \gamma(6)||}{|h|}$ and $\left|\left| \frac{\gamma(6+h) - \gamma(6)}{h} \right|\right|$ were the same quantity.

§3 Day 3: Graphs, Level sets, and Slices (Sept 8, 2025)

Office hours starting next week! Check Quercus for more details.

Last class, we looked at functions $\mathbb{R} \to \mathbb{R}^n$. Today we look at functions $\mathbb{R}^n \to \mathbb{R}$. We develop graphs, level sets, and slices because they offer new ways to analyze and study properties of such functions, that cannot be easily captured otherwise. For example, it is difficult to visualize a more than 3 dimensional vector.
Have $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^n$.

Definition 3.1 (Graph)**.** The graph of f is $\{(x, f(x)) : x \in A\}$

Note that when you plot a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, the graph would exist in $\mathbb{R}^{n+1}$.

Definition 3.2 (Level Set)**.** The level set of f at k is $\{x \in \mathbb{R}^n : f(x) = k\}$[3]

To produce the **slice** of a graph, we need to hold a coordinate x_i constant, which we set to a. We call the following a x_i-slice, where the c is at the i-th position in the tuple:

$$\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^n : (x_1, \cdots, a, \cdots, x_n) \in A, x_{n+1} = f(x_1, \cdots, a, \cdots, x_n)\}$$

where in the first tuple, x_i is omitted, with x_i is replaced by a in the second and third tuples. Note that the slice lives in $\mathbb{R}^n$, because we already have the information that $x_i = a$.

In this course, you always want to specify the domain and codomain of your function, to avoid confusion.

Problem 3.3. Give a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level set at 0 is the set

$$\{(x, y) \in \mathbb{R}^2 : |x| = |y|\}$$

To solve these kinds of problems, you set $0 = |y| - |x|$ and you would get a candidate function $f(x, y) = |y| - |x|$. To show that two sets A and B are equal, you would typically prove that $A \subseteq B$, and $B \subseteq A$.

Alternatively you could define a function

$$g(x, y) = \begin{cases} 0 & \text{if } |x| = |y| \\ 1 & \text{otherwise} \end{cases}$$

which satisfies the requirements by construction.

[3]If $A \subseteq \mathbb{R}^2$, the level set is called a *contour*.

§4 Day 4: Vector Fields and Transformations (Sept 9, 2025)

So far, we have seen parametric curves ($\mathbb{R} \to \mathbb{R}^n$), real valued functions ($\mathbb{R}^n \to \mathbb{R}$). Today we look at functions $\mathbb{R}^n \to \mathbb{R}^n$, which are called vector fields or transformations[4].

Definition 4.1 (Vector Field)**.** A n dimensional vector field is a function $F : A \to B$ with $A, B \subseteq \mathbb{R}^n$.

Note that vector fields are capitalized as per convention.

(Not testable) Newton's law of gravity states that the force exerted by an object at the origin with mass m_1 on an object at (x, y, z) with mass m_2 is given by

$$F(x, y, z) = \frac{-Gm_1m_2}{||(x, y, z)||^2} \cdot \frac{(x, y, z)}{||(x, y, z)||}$$

The magnitude is controlled by the first part of the product (note that there are only scalars), and the direction is controlled by the unit vector (note that it is scaled down to have a magnitude of 1).

[4] very uncommon to call such functions transformations

§5 Day 5: Coordinate Transformations (Sept 11, 2025)

Definition 5.1 (Coordinate Transformation)**.** A **coordinate transformation** $f : A \to B$ is a continuous transformation that is usually bijective. A and the map f form a **coordinate system** for the codomain B.

We want to plot subsets of B and describe them using the coordinate system defined by f and A. We use (u, v) to describe the elements in A, and (x, y) for B, and would write

$$g(u, v) = (u^2 + v^2, v)$$

or simply

$$(x, y) = (u^2 + v^2, v)$$

Definition 5.2 (Polar Coordinate Transformation)**.** A map $T : \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T(r, \theta) = (r \cos\theta, r \sin\theta)$$

T describes a map from polar coordinates to cartesian coordinates. The radius can be negative. Two notable properties are $T(r, \theta) = T(-r, \theta + \pi)$ and $T(r, \theta) = T(r, \theta + 2\pi)$, following from trigonometry.

The set $\{(r, \theta) \in \mathbb{R}^2 : r = 2\}$ would describe the set $\{(2 \cos\theta, 2 \sin\theta) \in \mathbb{R}^2 : \theta \in \mathbb{R}\}$, which corresponds to a circle of radius 2 centered at the origin. Restricting both sets to $\theta \in \mathbb{R}^+$ or $\theta \in \mathbb{R}^-$ would still correspond to the same set, meaning that the polar coordinate transformation is not injective. Another way to show is to consider the case $r = 0$.

What follows was not covered in lecture:

Definition 5.3 (Cylindrical Coordinate Transformation)**.** A map $T : \mathbb{R}^3 \to \mathbb{R}^3$, with

$$T(r, \theta, z) = (r \cos\theta, r \sin\theta, z)$$

This is similar to the polar coordinate transformation except we add an additional z field which remains unchanged.

Definition 5.4 (Spherical Coordinate Transformation)**.** A map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$T(\rho, \theta, \phi) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$$

If this gets covered in class I'll make a writeup deriving the formula.