MAT244 Notes

Max Xu

January 19, 2026

Contents

§1 Day 1: Intro (Jan 05, 2026)

Our goal is to describe an unknown function using its derivatives, and hopefully obtain an explicit form.

Definition 1.1 (Proportional)**.** A is proportional to B if $A = cB$ where c is constant.

Problem 1.2

Model the following:

1. Researchers are studying a population of rabbits. Each year they measure the population, and they find that the population increases by 11% each year.
2. Based on field measurements, some researchers studying the population of rabbits in an area propose a model where at each moment, the population is increase at approximately 10.5% of its current size.
3. An object in free fall will accelerate towards the ground at $9.8m/s^2$ due to gravity, and will also be slowed down at a rate proportional to its velocity due to air resistance.

1. Let $P(t)$ denote the population at time t, with P_0 being the initial population. Some reasonable responses include:

   - $\frac{dP}{dt} = 0.11P$
   - $P(t) = P_0(1.11)^t$
   - $P(t) = 1.11P(t-1)$
   - $P(t) = 1.11P_0 t$
   - $P(t) = 0$

   Some only grow linearly, which may not model the population well. Some need to include the base case P_0. The question is intentionally vague, so the professor thought these were fine.

2. Responses could be

   - $\frac{dP}{dt} = \ln(1.105)P_0 1.105^t$
   - $\frac{dP}{dt} = 0.105P$

   The second is response is what we're looking for (it is autonomous).

3. Have $h(t)$ be the height (in meters) of the object at t seconds, where positive is 'up'. Slowing down is a colloquialism for some acceleration in another direction.

$$\begin{aligned} a(t) &= a_{\text{gravity}}(t) + a_{\text{airres}}(t) \\ &= -9.8 - c \cdot v(t) \\ h'' &= -9.8 - ch' \end{aligned}$$

Definition 1.3 (ODE)**.** An ordinary differential equation is an equation involving a single variable function and the independent variable (input) of that function, and also derivatives up to a finite order of that function.

Definition 1.4 (PDE)**.** Partial differential equations involve derivatives w.r.t. multiple variables.

An example is the wave equation, $\frac{\partial^2 u}{\partial t^2} = k\frac{\partial^2 u}{\partial x^2}$. They are not testable, and show up in MAT351/APM346.

In this course[1], a general differential equation of order n is written as

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})$$

A function $\phi(t)$ is a solution when $\phi(t) \in C^n$ (is at least n times continuously differentiable), is defined on an open interval, and satisfies the equation

$$\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t))$$

for all t in its domain. We usually have infinitely many solutions, but we can specify them using initial values.

General solutions contain one or more parameters.

Problem 1.5

Find the general solution to the DE $y' = -0.15y$. What if we add the constraint that $y(1) = -3$?

[1] so that solutions always exist

§2 Day 2: First Order Linear ODEs (Jan 07, 2026)

Sometimes, there is no better way to describe a function than as a solution of a DE. For example, e^{-x^2} has no elementary antiderivative.

Definition 2.1 (First Order Linear DE)**.** An equation of the form $y' + p(t)y = g(t)$ where p and g are some functions of t. It homogeneous if $g(t) = 0$ for all t.

When solving for A in $A'' = -9.8 - 0.17A'$, notice that A is not present. Make the substitution $V = A'$, giving $V' = -9.8 - 0.17V$, which is a first order linear DE.

Problem 2.2

Find the general solutions to each of $y' = ay$, and $y' = ay + b$.

Take $y = ce^{at}$. See that for arbitrary c, $\frac{d}{dt}ce^{at} = ace^{at} = ay$

Take $y = ce^{at} - \frac{b}{a}$.

Problem 2.3

Find the general solutions to each of $y' = g(t)$, and $y' = f(t)y$.

The general solution of $y = \int g(t)\,dt$ is acceptable.

The general solution is $c\,\mathrm{Exp}(\int f(t)\,dt)$

Theorem 2.4. The general solution of $y' = p(t)y + g(t)$ is

$$y = e^{P(t)}\left(c + \int e^{-P(t)}g(t)\,dt\right)$$

where $P(t) = \int_{t_0}^{t} p(s)\,ds$, and c is an arbitrary constant.

§2.1 Integrating Factors

The product rule says that $f(t) = \frac{d}{dt}(\mu(t)y(t)) = \mu(t)y'(t) + \mu'(t)y(t)$. Dividing both sides by $\mu(t)$, see that this is a first order linear DE,

$$y' + \frac{\mu'}{\mu}y = y' + p(t)y = g(t) = \frac{f}{\mu}$$

Solve for μ in $\frac{\mu'}{\mu} = p(t)$. Finally, get $y(t) = \frac{\int f(t)\,dt + c}{\mu(t)} = \frac{\int g(t)\mu(t)\,dt}{\mu(t)}$.

§3 Day 3: Separable ODEs (Jan 12, 2026)

Recall that y is a function of x.

Definition 3.1 (Separable)**.** A first order ODE is separable if there exist functions M, N such that $y' = \frac{M(x)}{N(y)}$.

See that $\int N(y)\,dy = \int M(x)\,dx$ implicitly gives a solution.

Problem 3.2

1. Find the general solution to $y' = \frac{-x}{y}$, and the solution when $y(3) = -4$.
2. Find the general solution to $y' = \frac{x^2}{y^2-1}$
3. Find the general solution to $y' = 1 + y^2$

(1) $N(y) = y$, and $M(x) = -x$.

$$\begin{aligned} \int y\,dy &= \int -x\,dx \\ \frac{y^2}{2} &= c - \frac{x^2}{2} \\ y &= \pm\sqrt{2c - x^2} \end{aligned}$$

For the IVP, we choose the negative branch. $-4 = -\sqrt{c - 3^2}$, $c = 25$. $y = -\sqrt{25 - x^2}$.

(2) $\frac{y^3}{3} - y = \frac{x^3}{3} + c$ implicitly gives a solution.[2]

(3) Check that $\tan(x + c)$ works.

Generalizing (1), we find that the solution to the initial value problem $y' = \frac{M(x)}{N(y)}$ with $y(x_0) = y_0$ is given by

$$\int_{y_0}^{y} N(s)\,ds = \int_{x_0}^{x} M(t)\,dt$$

We can quickly check that it satisfies both the initial condition and the differential equation.

[2]You can use the cubic formula to get an explicit form (not testable)

§4 Day 4: Euler's Method (Jan 14, 2026)

For some differentiable function f, $f(a+\Delta x) \approx f(a) + f'(a)\Delta x$, which is called the first order approximation.

Euler's method performs a sequence of linear approximations to estimate values $y(t)$ of an IVP. Given $y' = f(t, y)$, $y(a) = y_0$, we want to estimate $y(b)$.

Create an increasing sequence $a = t_0, t_1, \cdots, t_n = b$, and define

$$y_{i+1} = y_i + (t_{i+1} - t_i) f(t_i, y_i)$$

This is reminiscent of the slope field tracing activity. For a better approximation, choose t_i such that $t_i - t_{i-1}$ is small.

Usually, each step of the approximation tends to move further away from the original solution curve, hence the error tends to accumulate. Euler's method can get an arbitrarily good approximation of y on some finite interval $[a, b]$ as step size goes to 0, no guarantee exists for unbounded intervals.

Remark 4.1

Euler's method is mostly a pedagogical tool, there are better tools developed (Runge-Kutta) that converge faster.

§5 Day 5: Autonomous and Exact ODEs (Jan 19, 2026)

Definition 5.1 (Autonomous)**.** An ODE is autonomous if y' doesn't depend on t.

In other words, $y' = g(y)$. By setting $N(x) = 1$, $M(y) = \frac{1}{g(y)}$, we see that all autonomous ODEs are separable. Solving first order autonomous ODEs using the template for first order separable ODEs may miss the equilibrium (constant) solutions though.

In general, the equilibrium solutions of $y' = f(y)$ are $y = c$, where $f(c) = 0$.

Definition 5.2

- The **critical points** of y_0 are values where $f(y_0) = 0$.
- y_0 is asymptotically stable for nearby values y_1, if the solutions of $y' = f(y)$, $y(0) = y_1$, have.