§1 Derivatives

1.1 Single-variable

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}^m$ be a function, with $a \in A^o$.

Definition 1.1 (Derivative).

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Physical $f'(a)$ is the (instantaneous) velocity

Geometric the 'tangent line' is given by

$$\{f(a) + hf'(a) : h \in \mathbb{R}\}$$

Analytic The linear approximation of f at a is the function $\ell : \mathbb{R} \to \mathbb{R}^m$ given by

$$\ell(x) = f(a) + f'(a)(x - a)$$

Algebraic Differentiability is the existence of a linear map $L : \mathbb{R} \to \mathbb{R}^m$ where

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0$$

1.2 Multi-variable

Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$, $a \in A^o$. Have $1 \leq j \leq n$, $\{e_1, \cdots, e_n\}$ be the standard basis for $\mathbb{R}^n$.

Definition 1.2 (Partial Derivative). The j-th partial derivative at a is given by

$$\partial_j f(a) := \lim_{h \to 0} \frac{f(a + he_j) - f(a)}{h}$$

Definition 1.3 (Directional Derivative). Fix some $v \in \mathbb{R}^n$. The directional derivative of f at a in the direction v is given by

$$D_v f(a) \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}$$

Definition 1.4 (Gradient). The gradient of f at a is written $\nabla f(a)$, when all such partial derivatives exist

$$\nabla f(a) = \begin{pmatrix} \partial_1 f(a) \\ \vdots \\ \partial_n f(a) \end{pmatrix}$$

Definition 1.5 (Differentiable). f is differentiable at a if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0$$

L is called the differential of f at a, written df_a.

Definition 1.6 (Jacobian). The Jacobian of f at a is the $m \times n$ matrix $Df(a)$ given by

$$Df(a) = [\partial_j f_i(a)]_{i,j} = \begin{pmatrix} \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(a) & \cdots & \partial_n f_m(a) \end{pmatrix}$$

Theorem 1.7

If f is differentiable at a
- For all $v \in \mathbb{R}^n$, $D_v f(a)$ exists and $df_a(v) = D_v f(a)$
- $df_a(v) = Df(a)v$

Definition 1.8 (Continuously Differentiable). f is continuously differentiable at a (or C^1 at a) if each $\partial_1 f, \cdots, \partial_n f$ are defined on some open set containing a, and are continuous at a.