CHAPTERS 10 & 11

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LINEAR ALGEBRA

- Underlying all of applied math is the subject of Linear (or Vector) Algebra
  - It is just the extension of **algebra** to other sorts of objects
  - You begin by defining what $+$, $-$, and $\times$ or $\cdot$ (multiplication) mean
  - The first two tend to be easy, the last one is where you get into trouble!!
    - There are several multiplications: scalar multiplication, matrix multiplication, cross-product, and the dot product
  - The dot product is just one of these multiplications that is defined for vectors
- Vectors (the basic objects) are often just ordered lists of numbers
  - **But not really!**
  - The algebra operations are relatively simple:

LINEAR ALGEBRA

Vector addition (and subtraction) are easy (and obvious) with "scalar multiplication" being a useful if not immediately obvious idea:

addition

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and

scalar multiplication

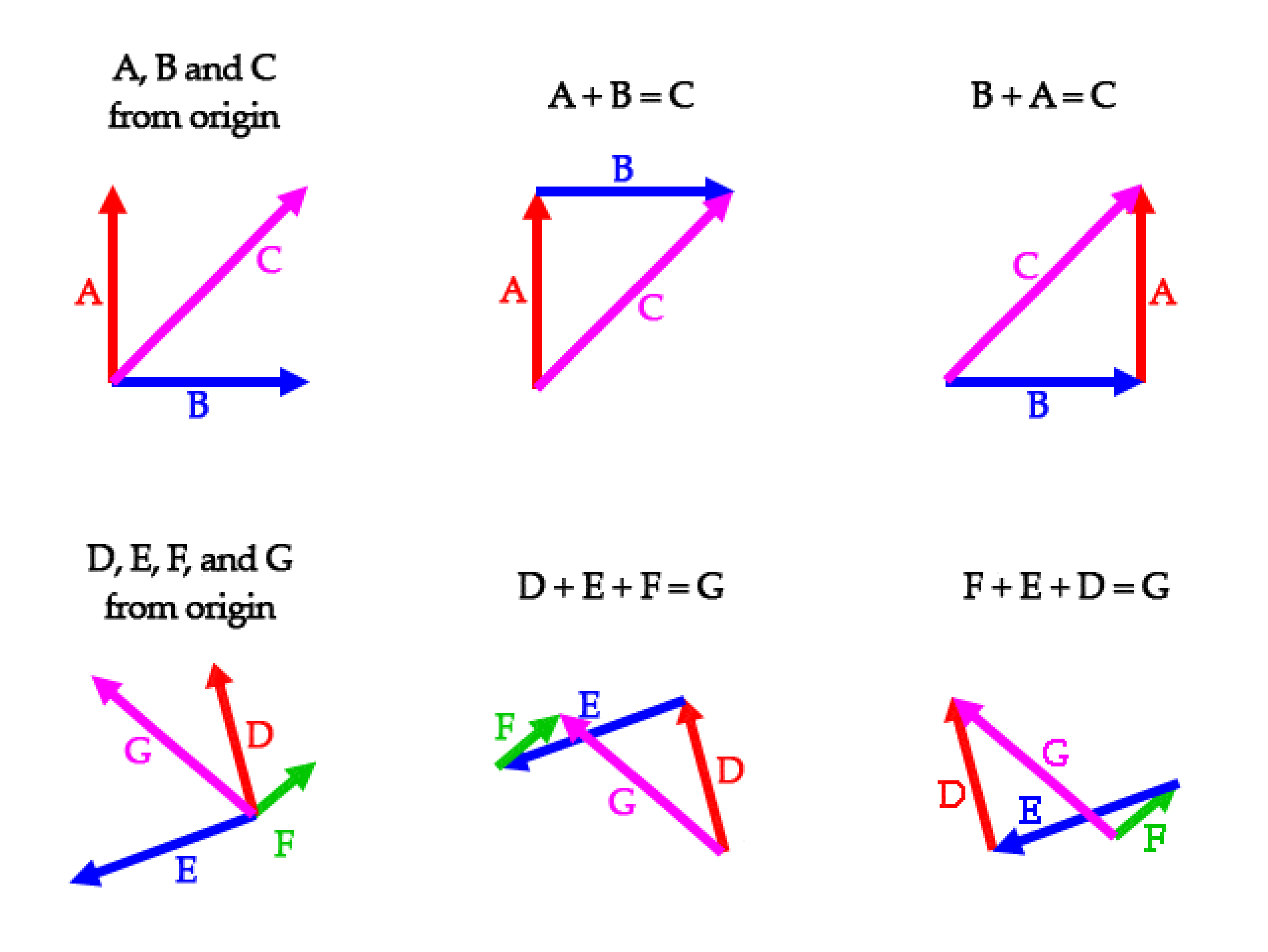
$$a \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a \cdot x_1 \\ a \cdot x_2 \\ \vdots \\ a \cdot x_n \end{bmatrix}$$

Vectors!

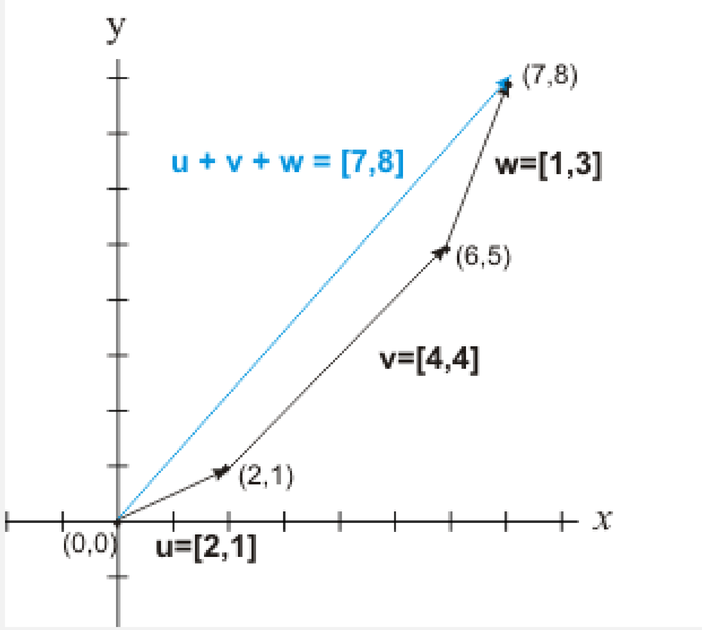
Let $a = -1$ to get subtraction!

$$\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}2\\4\\6\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}+2\begin{bmatrix}2\\4\\6\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\8\\12\end{bmatrix}=\begin{bmatrix}5\\10\\15\end{bmatrix}$$



A, B and C
from origin
A
B
C
A + B = C
A
B
C
B + A = C
A
B
C
D, E, F, and G
from origin
D
E
F
G
D + E + F = G
D
E
F
G
F + E + D = G
D
E
F
G



y
x
(0,0)
(2,1)
(6,5)
(7,8)
u=[2,1]
v=[4,4]
w=[1,3]
u + v + w = [7,8]

DOT PRODUCT

- The dot product is **one specific way** of multiplying two vectors that has proven useful in mathematics and science
- There are others that are also used!

- The dot product is defined between two vectors a and b (or just as often A and B). There are lots of different notations!

DOT PRODUCT FORMULA

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \bullet \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x.b_x + a_y.b_y + a_z.b_z$$

$$\begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_x B_x + A_y B_y + A_z B_z = \vec{A} \cdot \vec{B}$$

Book's Version (Yuck!):

$$dotproduct_{ab} = a \cdot b = \sum_{i=1}^{n} a_i b_i$$

DOT PRODUCT FORMULA

Book's Version (Yuck!):

$$dotproduct_{ab} = a \cdot b = \sum_{i=1}^{n} a_i b_i$$

a and b are the vectors, a_i and b_i represent the i-th components

HOW TO COMPUTE THE DOT PRODUCT

- Given two vectors (lists of numbers):

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

HOW TO COMPUTE IT IN MATLAB

- Given two vectors (lists of numbers):

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

- In MATLAB:

```
>> x = [1 2 3]; y = [4 5 6];
>> dot(x,y)
ans =
    32
>> sum(x.*y)
ans =
    32
```

The “dot” before * means do something “elementwise” not “linear algebrawise” ☺

```
>>
>> x = [1 2 3]; y = [4 5 6];
>> x*y
Error using  *
Inner matrix dimensions must agree.
>> x*y'
ans =
    32
>>
fx >> 
```

If you know a little bit about linear algebra, it seems like the dot product should just be multiplication.

It is.

Sort of…

You have to “transpose” the second vector, that is, stand it up:

$$\begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_x B_x + A_y B_y + A_z B_z = \vec{A} \cdot \vec{B}$$

General multiplication is much harder—the obvious choice of just multiplying corresponding elements in matrices does not have useful mathematical properties.

Instead:

$$\begin{array}{c} \\ \\ \text{row } i \hookrightarrow \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \ldots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \cdot \begin{array}{c} \begin{array}{cccccc} & & & \text{column } j & & \\ & & & \downarrow & & \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \ldots & b_{ij} & \ldots & b_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nj} & \ldots & b_{nn} \end{bmatrix} \end{array} =$$

$$= \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1j} & \ldots & c_{1n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \ldots & \boxed{\mathbf{c_{ij}}} & \ldots & c_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nj} & \ldots & c_{nn} \end{bmatrix} \begin{array}{c} \text{entry on row } i \\ \text{column } j \end{array}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \ldots + a_{in} \cdot b_{nj}$$

DOT PRODUCTS

- AKA:
  - Dot product
  - Scalar product
  - Inner product
- The dot product is a **single number**
  - It can be thought of as a **weighted sum** (one vector is the set of **numbers to be summed**, the other the **weights**)
  - It is the length of the **projection** of one vector onto the (*dimension of*) the other vector
  - It also relates to a **similarity measure**

DOT PRODUCTS

- AKA:
  - Dot product
  - Scalar product
  - Inner product
- Dot products also **define** *lengths* and *angles*:
  - $A \cdot B = \sum_{i=1}^{n} a_i b_i$ (Definition of the dot product)
  - $||A|| = \sqrt{A \cdot A}$ (Inner product definition of **length** of vector A)
  - $\cos(\theta) = \frac{A \cdot B}{||A||||B||}$ (Inner product definition of **angle $\boldsymbol{\theta}$** between A and B)

PROPERTIES OF DOT PRODUCTS

The dot product fulfills the following properties if **a**, **b**, and **c** are real vectors and *r* is a scalar.[1][2]

1. **Commutative:**

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a},$$

which follows from the definition (θ is the angle between **a** and **b**):

$$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta=\|\mathbf{b}\|\,\|\mathbf{a}\|\cos\theta=\mathbf{b}\cdot\mathbf{a}.$$

2. **Distributive over vector addition:**

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

3. **Bilinear**:

$$\mathbf{a}\cdot(r\mathbf{b}+\mathbf{c})=r(\mathbf{a}\cdot\mathbf{b})+(\mathbf{a}\cdot\mathbf{c}).$$

4. **Scalar multiplication:**

$$(c_1\mathbf{a})\cdot(c_2\mathbf{b})=c_1c_2(\mathbf{a}\cdot\mathbf{b}).$$

5. **Not associative** because the dot product between a scalar (**a** · **b**) and a vector (**c**) is not defined, which means that the expressions involved in the associative property, (**a** · **b**) · **c** or **a** · (**b** · **c**) are both ill-defined.[5] Note however that the previously mentioned scalar multiplication property is sometimes called the "associative law for scalar and dot product"[6] or one can say that "the dot product is associative with respect to scalar multiplication" because *c* (**a** · **b**) = (*c* **a**) · **b** = **a** · (*c* **b**).[7]

6. **Orthogonal:**

Two non-zero vectors **a** and **b** are *orthogonal* if and only if **a** · **b** = 0.

7. **No cancellation:**

Unlike multiplication of ordinary numbers, where if *ab* = *ac*, then *b* always equals *c* unless *a* is zero, the dot product does not obey the cancellation law:

If **a** · **b** = **a** · **c** and **a** ≠ **0**, then we can write: **a** · (**b** − **c**) = 0 by the distributive law; the result above says this just means that **a** is perpendicular to (**b** − **c**), which still allows (**b** − **c**) ≠ **0**, and therefore **b** ≠ **c**.

8. **Product Rule:** If **a** and **b** are functions, then the derivative (denoted by a prime ′) of **a** · **b** is **a**′ · **b** + **a** · **b**′.

PROPERTIES OF DOT PRODUCTS (MATLAB)

The dot product fulfills the following properties if **a**, **b**, and **c** are real vectors and *r* is a scalar.[1][2]

1. **Commutative:**

   $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$

   which follows from the definition (θ is the angle between **a** and **b**):

   $$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta = \|\mathbf{b}\| \, \|\mathbf{a}\| \cos\theta = \mathbf{b} \cdot \mathbf{a}.$$

2. **Distributive over vector addition:**

   $$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

3. **Bilinear:**

   $$\mathbf{a} \cdot (r\mathbf{b} + \mathbf{c}) = r(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c}).$$

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   $$(c_1\mathbf{a}) \cdot (c_2\mathbf{b}) = c_1 c_2 (\mathbf{a} \cdot \mathbf{b}).$$

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   Two non-zero vectors **a** and **b** are *orthogonal* if and on…

7. **No cancellation:**

   Unlike multiplication of ordinary numbers, where if *ab* = … e dot product does not obey the cancellation law:
   If **a** · **b** = **a** · **c** and **a** ≠ **0**, then we can write: **a** · (**b** − **c**) = … this just means that **a** is perpendicular to (**b** − **c**), which still allows (**b** − **c**) ≠ **0**, and therefore **b** ≠ **c**.

8. **Product Rule:** If **a** and **b** are functions, then the derivative …

```
>> x = [1 2 3]; y = [4 5 6];
>> dot(x,y)
ans =
    32
>> dot(y,x)
ans =
    32
>>
```

```
>> x2 = 2*x
x2 =
     2     4     6
>> y3 = 3*y
y3 =
    12    15    18
>> dot(x2,y3)
ans =
   192
>> 2*3*dot(x,y)
ans =
   192
>>
```

Orthogonal Case

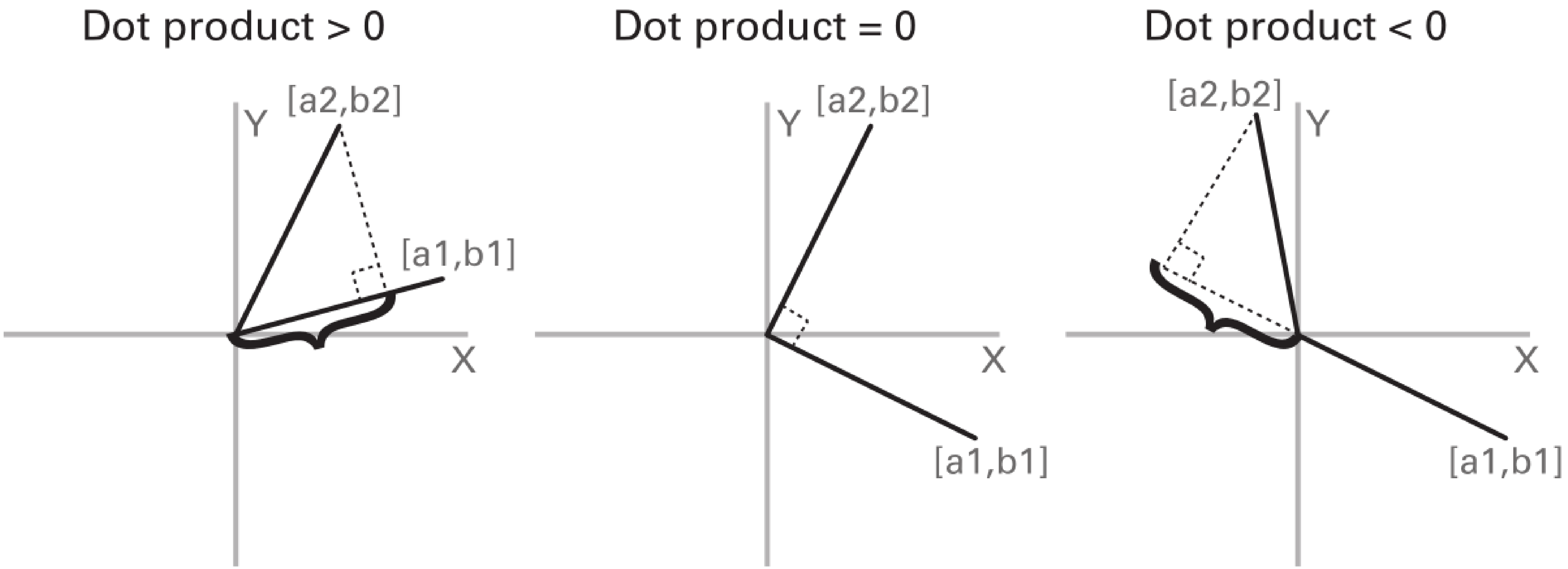


Figure 10.1

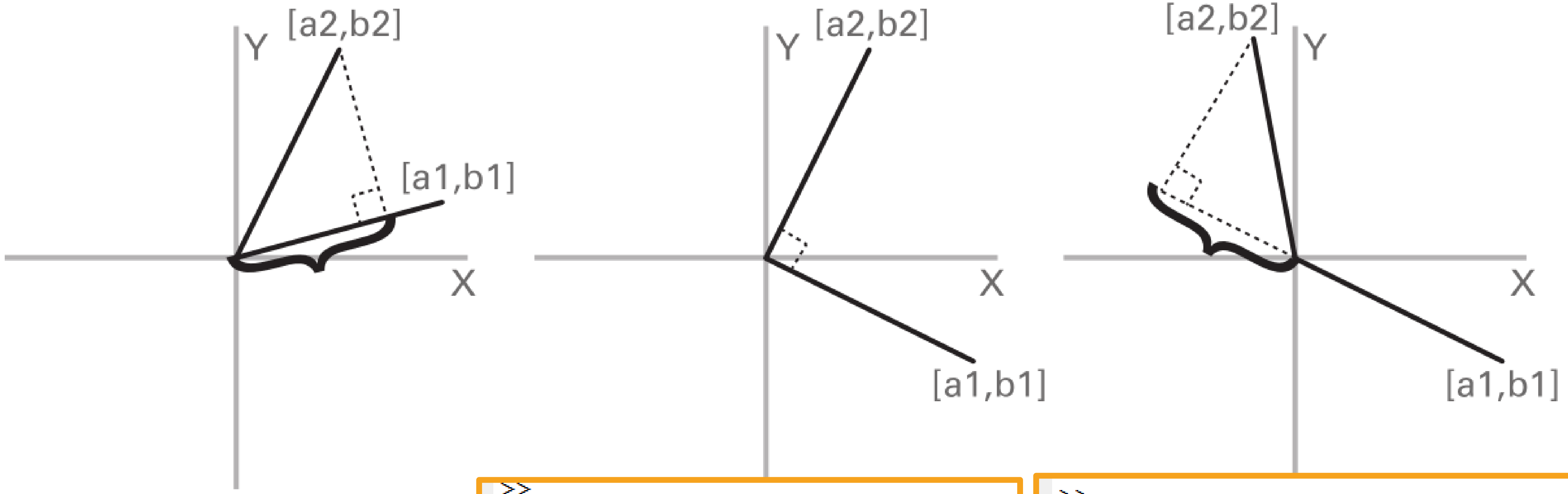
The dot product can be thought of as the length along one of the vectors of the **projection** of the other vector.

Orthogonal Case

Dot product > 0

Dot product = 0

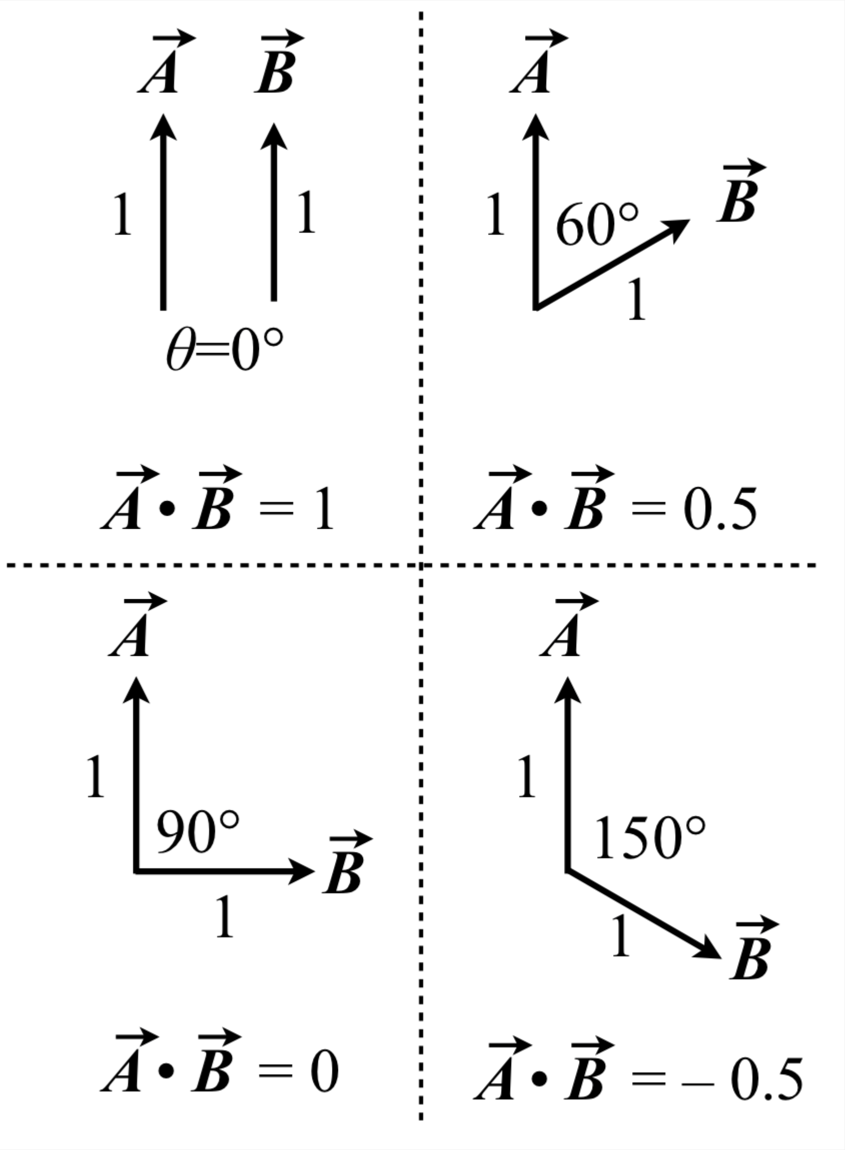
Dot product < 0




```
>>
>> a1b1 = [5 -2]; a2b2 = [2 5];
>> dot(a1b1, a2b2)
ans =
     0
>>
```

```
>>
>> a1b1 = [5 -2]; a2b2 = [-1 5];
>> dot(a1b1, a2b2)
ans =
   -15
>>
```

Figure 10.1



Vectors in the same direction: 1
Vectors perpendicular (orthogonal): 0

Acute Angles: Positive
Obtuse Angles: Negative

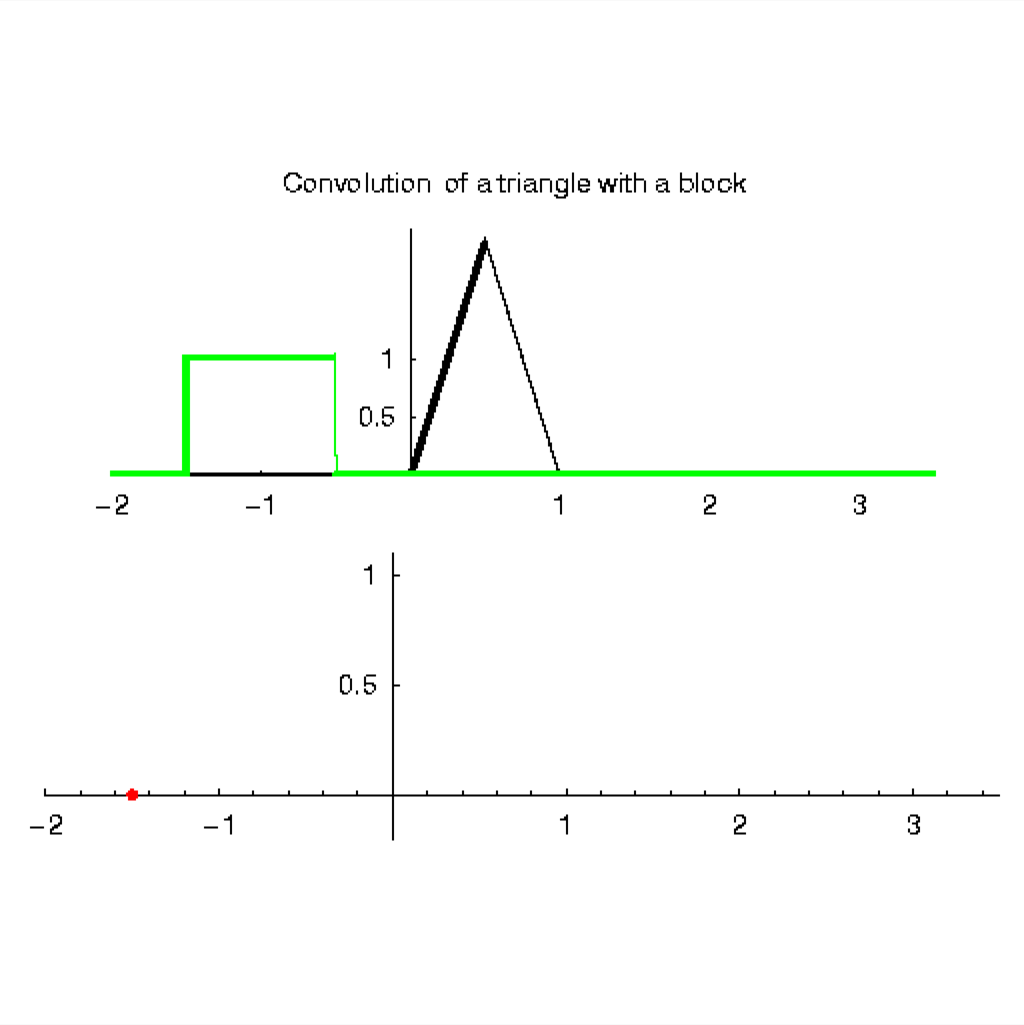
http://textbook.scotthill.us/01-Vectors/04-Dot.php See also the vector sections of: http://textbook.scotthill.us/ more generally.

WEIRD AT FIRST

- It takes some getting used to, but our vectors live in a **high dimensional space**:
  - A vector of length 2 is a point in **2-space** (the **Cartesian plane**)
  - A vector of length 3 is a point in **3-D space**
  - A vector representing a 2 second sample of EEG (for one electrode) with a sampling rate of 512 Hz is a vector of length 1024
    - It lives as a single point in a **1024-dimensional space**
    - Each number in the vector is one coordinate of the 1024-dimensional point

CONVOLUTION

- Has been called a “rolling together” of two vectors (or functions or signals)
- Think of it as smearing and shifting the signal



http://bmia.bmt.tue.nl/education/courses/fev/course/notebooks/triangleblockconvolution.gif

Formula for convolution for a point x:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\tau) \cdot g(x-\tau)\, d\tau$$

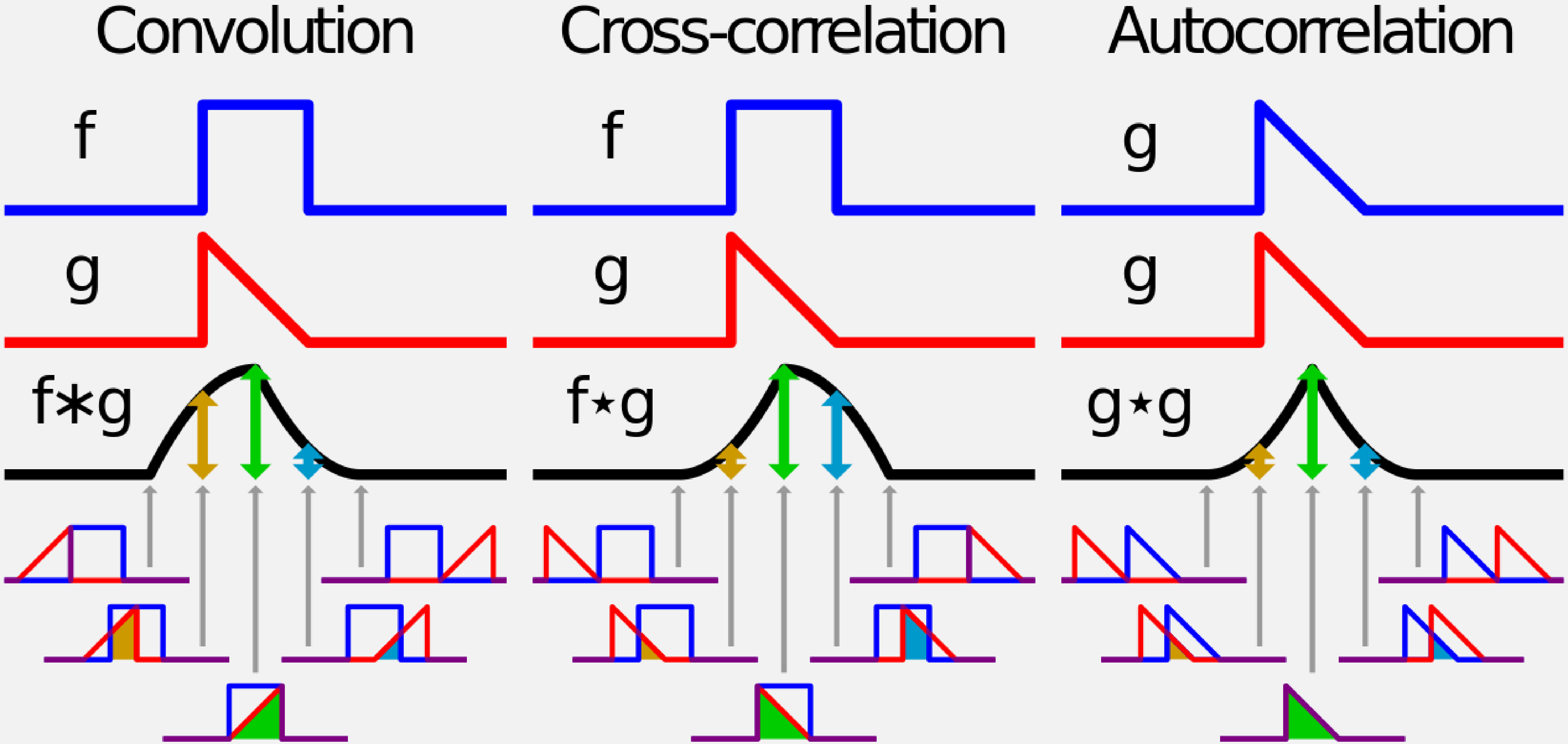
$$f[x] * g[x] = \sum_{k=-\infty}^{\infty} f[k] \cdot g[x-k]$$

Either of these formulae are applied to **each** value of x that you are interested in knowing about.

When things get small enough, we just stop (to deal with the $\pm\infty$.

Convolution is a **sliding**, **weighted-sum**, of function $f(\tau)$, with the weights specified by the weighting function $g(-\tau)$.

Sorry! The book calls my "x" here "k"!!


Convolution
Cross-correlation
Autocorrelation
f
g
f∗g
f
g
f⋆g
g
g
g⋆g

CONVOLUTION

- Has been called a “rolling together” of two vectors (or functions or signals)
- Think of it as smearing and shifting the signal
- What about the end points?
  - Zero padding
  - Trim excess points at the end (p. 115)

WHY CONVOLVE?

- Convolution is how you apply a filter to a data signal
- It can show the action of a physical system: the system's effect is called the "impulse response" and this is convolved with the input signal
  - In EEG, this is used in modeling the effect of the skull on the EEG, for instance
- We will use it to pick out frequency band specific activity from the EEG signal