

运行说明

代码思路

有限元素法

1.数学推导

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$\varphi(x, 0) = \varphi(x, 1) = 0, \varphi(0, y) = \varphi(1, y) = 1$$

对于泊松方程 $\nabla^2 \varphi = \frac{-\rho}{\epsilon}$ 和第一类边界条件 $\varphi|_L = \varphi_0$

泛函极值问题：

$$\delta I(\varphi) = 0$$

$$I(\phi) = \int_{D(L)} \left[\frac{\epsilon}{2} \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) - \rho \varphi \right] dx dy$$

$$\varphi|_L = \varphi_0$$

对于此题可取 $\rho = 0, \epsilon = 2$, 则

$$I(\phi) = \int_{D(L)} \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right] dx dy$$

$$\text{整个区域 } I(\phi) = \sum_e \int_e \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right] dx dy = \sum I_e(\varphi)$$

对于每一个元素定义顶点函数值矩阵和三角形的型函数, 可求得泛函矩阵形式

$$I_e(\varphi) = \frac{1}{2} (\Phi)_e^T (K)_e (\phi)_e$$

对元素泛函求和并变分可得待求线性方程组 $(K)(\Phi) = 0$

2.对元素的划分

按三角形划分。考察第一类边界条件, 编号时前 n_0 个为内部节点, $n_0 + 1$ 到 n 个位边界节点。

代入边界条件 $\varphi_{n_0+i} = \varphi_0$ 后, 方程改写为 $(k_{11})(\Phi_1) = -(K_{12})(\Phi_2)$ 。

Φ_2 即为边界点对应值, 只需利用 K 求解内点值 (超松弛迭代)

3.超松弛迭代求解线性方程组

原理同上一次作业。

实现细节

在主函数中设置边界, 分割宽度等参数, 计算边界点个数及坐标。定义函数, 传入坐标等值, 进行有限元素分割, 计算矩阵 K 和 Ph_2 。返回元素编号和矩阵。定义函数, 传入编号和矩阵, 使用超松弛迭代计算各元素对应值, 根据编号和 x, y 坐标对应关系计算最终结果, 返回该矩阵。定义函数, 传入边界坐标分割和最终结果, 进行作图。

优化松弛参数: 改变 ω 值进行循环, 不断缩小范围至迭代次数最小。

使用该范围内的 ω 计算结果并作图。

运行结果

取松弛参数[1.0,1.9], 间距0.05:

```
omega = 1.000000, Converges in 114 steps
omega = 1.050000, Converges in 104 steps
omega = 1.100000, Converges in 94 steps
omega = 1.150000, Converges in 86 steps
omega = 1.200000, Converges in 77 steps
omega = 1.250000, Converges in 70 steps
omega = 1.300000, Converges in 62 steps
omega = 1.350000, Converges in 55 steps
omega = 1.400000, Converges in 48 steps
omega = 1.450000, Converges in 42 steps
omega = 1.500000, Converges in 34 steps
omega = 1.550000, Converges in 28 steps
omega = 1.600000, Converges in 31 steps
omega = 1.650000, Converges in 40 steps
omega = 1.700000, Converges in 42 steps
omega = 1.750000, Converges in 52 steps
omega = 1.800000, Converges in 64 steps
omega = 1.850000, Converges in 87 steps
omega = 1.900000, Converges in 133 steps
```

取松弛参数[1.50, 1.60], 间距0.005:

```
omega = 1.500000, Converges in 34 steps
omega = 1.505000, Converges in 33 steps
omega = 1.510000, Converges in 32 steps
omega = 1.515000, Converges in 31 steps
omega = 1.520000, Converges in 30 steps
omega = 1.525000, Converges in 29 steps
omega = 1.530000, Converges in 28 steps
omega = 1.535000, Converges in 27 steps
omega = 1.540000, Converges in 27 steps
omega = 1.545000, Converges in 27 steps
omega = 1.550000, Converges in 28 steps
omega = 1.555000, Converges in 27 steps
omega = 1.560000, Converges in 28 steps
omega = 1.565000, Converges in 29 steps
omega = 1.570000, Converges in 29 steps
omega = 1.575000, Converges in 29 steps
omega = 1.580000, Converges in 29 steps
omega = 1.585000, Converges in 30 steps
omega = 1.590000, Converges in 30 steps
omega = 1.595000, Converges in 31 steps
omega = 1.600000, Converges in 31 steps
```

由此可知最小收敛步数27, 不妨取 $\omega = 1.54$

作图:

