HW2

Exercise 0.1. Show that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if $ad(X)$ is a nilpotent endomorphism of $\mathfrak{g}$ for every $X \in \mathfrak{g}$.

Exercise 0.2. Show that a Lie algebra $\mathfrak{g}$ is semisimple if and only if every finite-dimensional representation is semisimple, i.e., every invariant subspace has a complement.

Exercise 0.3. A Lie algebra is called **reductive** if its radical is equal to its center. A Lie group is reductive if its Lie algebra is reductive. For example, $GL_n\mathbb{C}$ is reductive. Show that the following are true for a reductive Lie algebra $\mathfrak{g}$: (i) $\mathcal{D}\mathfrak{g}$ is semisimple; (ii) the adjoint representation of $\mathfrak{g}$ is semisimple; (iii) $\mathfrak{g}$ is a product of a semisimple and an abelian Lie algebra; (iv) $\mathfrak{g}$ has a finite-dimensional faithful semisimple representation. In fact, each of these conditions is equivalent to $\mathfrak{g}$ being reductive.

Exercise 0.4. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called **Cartan subalgebra** of $\mathfrak{g}$ if

1. $\mathfrak{h}$ is abelian
2. A generic element of $\mathfrak{h}$ is diagonalazable in the adjoint representation.
3. $\mathfrak{h}$ is a maximal subalgebra with above two property.

(i)Classify all Cartan subalgebras for the Lie algebras of dimension ≤ 3. Suppose G is simply connect connected group with the Lie algebra $\mathfrak{g}$. G acts on the set of Cartan subalgebras. (ii) In the same examples, classify all the orbits of this action.