1 Preliminary results from Calculus and Analysis

2 Ordinary differential equations (ODEs)

2.1 Classification

2.2 Existence and uniqueness of solutions

2.3 Solution methods

2.3.1 Linear ODEs

Inhomogeneous

We will consider the linear case first with non-constant forcing.

Theorem 2.1

Let f, g be two continuous functions, then the ODE

$$y'(x) + f(x)y(x) = g(x)\,,$$

has a general solution

$$y(x) = \frac{C + \int e^{\int f(x)dx} g(x)dx}{e^{\int f(x)dx}}\,.$$

Proof. Suppose there exist $h(x)$ (often called the **integrating factor**) s.t. $h'(x) = h(x)f(x)$. We can multiply both sides of the ODE by such function to get

$$g(x)h(x) = y'(x)h(x) + \underbrace{h(x)f(x)}_{h'(x)}y(x) = \underbrace{y'(x)h(x) + h'(x)y(x)}_{\text{product rule}} = (y(x)h(x))'\,.$$

Integrating both sides gives

$$\int (y(x)h(x))'dx = y(x)h(x) + C_1 = \int g(x)h(x)dx \quad \Rightarrow \quad y(x) = \frac{\int g(x)h(x)dx - C_1}{h(x)}\,.$$

Now in order to get an explicit form for $h(x)$ we use the initial requirement for its definition to get

$$h'(x) = h(x)f(x) \quad \Rightarrow \quad f(x) = \frac{h'(x)}{h(x)} = (\log(h(x)))'\,.$$

Integrating both sides yields

$$\int f(x)dx = \int (\log(h(x)))'dx = \log(h(x)) + C_2 \quad \Rightarrow \quad h(x) = \underbrace{e}_{C_3}{}^{-C_2} + \int f(x)dx = C_3\, e^{\int f(x)dx}\,.$$

Plugging this expression into the above form for the solution concludes the proof

$$y(x) = \frac{\int C_3\, e^{\int f(x)dx} g(x)dx - C_1}{C_3\, e^{\int f(x)g(x)dx}} = \frac{C + \int\, e^{\int f(x)dx} g(x)dx}{e^{\int f(x)g(x)dx}}\,, \quad C := -\frac{C_1}{C_3}\,.$$

$\square$

Highlight

The solution strategy for linear, first-order ODEs goes as follows:

1. put the ODE in standard form (i.e. compute f and g);
2. compute the integrating factor h;
3. multiply both sides of the ODE by h and simplify the product rule on the LHS;
4. integrate both sides and use the FTC to isolate the solution y in terms of h and g;
5. determine the integration constant C using the BCs.

Example 2.1

Let us consider the following Cauchy IVP

$$\begin{cases} xy' + 3y = 4x^2 - 3x\,, & x \in (0,1] \\ y(1) = 0\,, \end{cases}$$

we now follow the steps of the solution strategy introduced above to find a general integral (or family of solutions) of the ODE and a particular solution for the IVP:

1. we put the ODE in standard form by diving both sides by x

$$y' + \underbrace{\frac{3}{x}}_{f(x)} y = \underbrace{4x - 3}_{g(x)} ;$$

2. we compute the integrating factor

$$h(x) = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3 ;$$

3. we multiply both sides of the ODE in step 1. by the integrating factor and simplify the product rule

$$x^3 y' + 3x^2 y = (x^3 y)' = 4x^4 - 3x^3 ;$$

4. we integrate both sides of the ODE above

$$\int (x^3 y)' dx = x^3 y + C_1 = \int 4x^4 dx - \int 3x^3 dx = \frac{4}{5}x^5 + C_2 - \frac{3}{4}x^4 + C_3 ,$$

   and isolate y on the LHS

$$y = \frac{4}{5}x^2 - \frac{3}{4}x + \frac{C}{x^3} , \quad C := C_2 + C_3 - C_1 ,$$

   which is now our family of solutions of the ODE parametrised by C;

5. we impose the BC $y(1) = 0$ to determine the value of C and extract the unique solution that satisfy the IVP

$$0 = \frac{4}{5} - \frac{3}{4} + C \;\Rightarrow\; C = \frac{3}{4} - \frac{4}{5} = -\frac{1}{20} ,$$

   which gives us the curve

$$y(x) = \frac{4}{5}x^2 - \frac{3}{4}x - \frac{1}{20}x^{-3} .$$

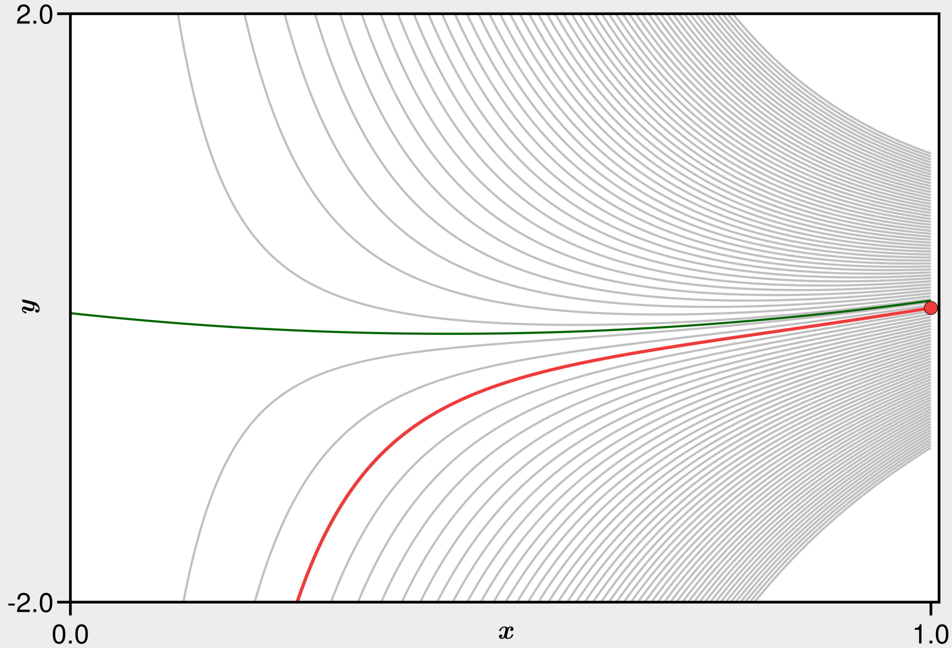

Figure 1: Family of solutions (gray lines) of the linear ODE in $(0,1]$ at different values of the integration constant $C \in [-1,1]$ and the solution of the IVP (red line) for the constant value $C = -\frac{1}{20}$ satisfying the BC $y(1) = 0$ (red dot). Note that the only bounded solution (green curve) is at $C = 0$.

Example 2.2

We repeat the same procedure to the find the solution of the following Cauchy IVP

$$\begin{cases} (x-2)y' + y = 3x^2 + 2x \,, \quad x \in [-1,1] \\ y(1) = -1 \; : \end{cases}$$

1. ;
2. ;
3. ;
4. $y = \frac{x^3+x^2+C}{x-2}$;
5. $C = -1$;

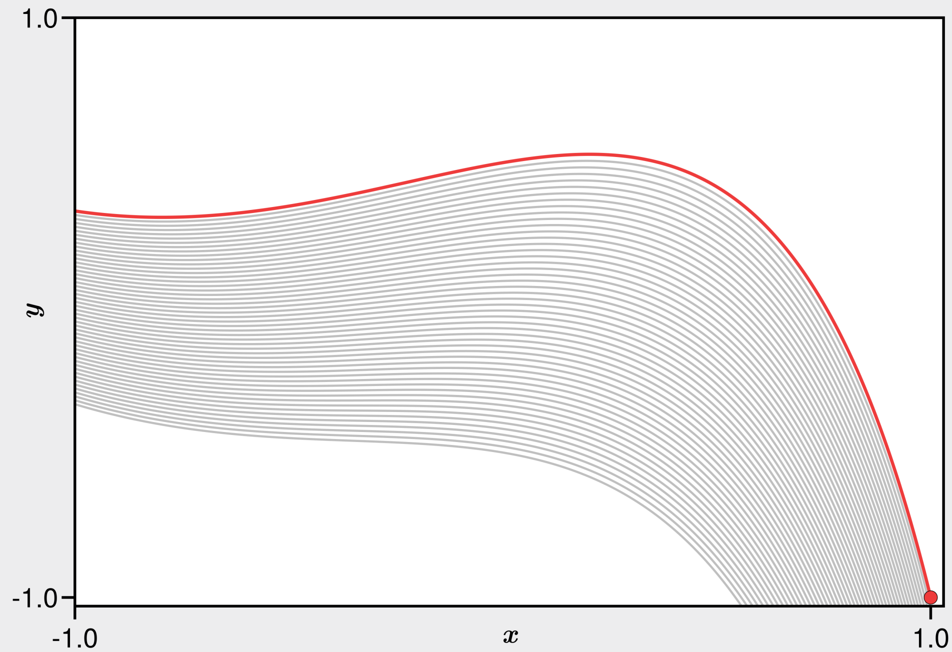

Figure 2: Family of solutions (gray lines) of the linear ODE in $[-1, 1]$ at different values of the integration constant $C \in [-1, 1]$ and the solution of the IVP (red line) for the constant value $C = -1$ satisfying the BC $y(1) = -1$ (red dot).