1 Discrete-time processes

1.1 Random walk

1.2 Markov chain

2 Stochastic calculus

2.1 Brownian motion

2.2 Ornstein-Uhlenbeck process (OUP)

Consider the OUP [1, Sub-section 4.4.4, p.106].

3 Fokker-Planck equation (FPE)

3.1 The adjoint equation

Consider the SDE [1, Sub-section 4.3.1, p.93]

$$dx = f(x,t)dt + \sqrt{g(x,t)}dW\,,$$

then the FPE [1, Sub-section 4.3.4, p.96] reads

$$\partial_t p(x,t|y,0) = -\partial_x\big(f(x,t)p(x,t|y,0)\big) + \frac{1}{2}\partial^2_{xx}\big(g(x,t)p(x,t|y,0)\big)\,.$$

The adjoint FPE [1, Section 3.6, p.55] reads

$$\partial_t p(x,0|y,t) = -f(y,t)\partial_y p(x,0|y,t) - \frac{1}{2}g(y,t)\partial^2_{yy}p(x,0|y,t)\,.$$

If both f and g are time-independent then the solution $x(t)$ is said to be *homogeneous* [1, Section 3.7, p.56] and as such its conditional transition probability is

$$p(x,t|y,0) = p(x,0|y,-t)\,,$$

$$\partial_t p(x,t|y,0) = \partial_t p(x,0|y,-t) = -\partial_{(-t)}p(x,0|y,-t)\,,$$

where the last equality is guaranteed by the chain-rule

$$g(x) = -x\,,\quad (f\circ g)(x) = g\big(f(x)\big) = -f(x)\,.$$

$$\frac{dg}{dx} = \frac{dg}{df}\frac{df}{dx} = -\frac{df}{dx}\,.$$

It follows immediately that the adjoint FPE can be rewritten as

$$\begin{aligned}
\partial_t p(x,0|y,t) &= -\partial_{(-t)}p(x,0|y,-t) = -\Big(-f(y)\partial_y p(x,0|y,-t) - \frac{1}{2}g(y)\partial^2_{yy}p(x,0|y,-t)\Big) = \\
&= f(y)\partial_y p(x,0|y,-t) + \frac{1}{2}g(y)\partial^2_{yy}p(x,0|y,-t) = \\
&= f(y)\partial_y p(x,t|y,0) + \frac{1}{2}g(y)\partial^2_{yy}p(x,t|y,0) = \\
&= -\partial_t p(x,t|y,0)\,.
\end{aligned}$$

3.2 Stationary solutions

Stationary solutions of the FPE are important when considering the asymptotic, steady-state distribution of a stochastic process.

Theorem 3.1

Let $f(x) = -V'(x)$ be a smooth vector field and $dx = -V'(x)\,dt + \sigma dW$ a non-linear SDE in Langevin form with additive diffusion, then the stationary solution of the associated FPE is $p_{\mathrm{eq}}(x) = N(x)\,e^{-\frac{V(x)}{D}}$ where $N(x) = N\Big(\int e^{\frac{V(x)}{D}}dx - C\Big)$, with $N>0$ being a normalisation constant, $C\in\mathbb{R}$ an integration constant and $D = \frac{\sigma^2}{2}$.

Proof. We first rewrite the FPE assuming stationarity of the pdf, i.e. $p(x,t) = p(x)$

$$\partial_t p(x) = 0 = \partial_x(V'(x)p(x)) + D\partial_{xx}p(x) = (V'(x)p(x))' + Dp''(x)\,,$$

where in the second step we used the notation of differentiation for ODEs. In the following we will drop the explicit dependence on x of the functions involved to simplify the notation. We now divide both sides by D and compactify the differentials as per the sum rule of differentiation

$$p'' + D^{-1}\,(V'p)' = (p' + D^{-1}\,V'p)' = 0\,,$$

We integrate both sides to get

$$p' + \frac{V'}{D}\,p = C_1\,.$$

The above is a linear, first-order ODE with constant forcing C_1: we can then compute the solution analytically [2, Theorem 2.1, p.1] using the integrating factor

$$h(x) = e^{\frac{1}{D}\int V'(x)dx} = e^{\frac{V(x)}{D}+\frac{c}{D}} = e^{\frac{V(x)}{D}}\underbrace{e^{\frac{c}{D}}}_{=:C_2} = C_2\,e^{\frac{V(x)}{D}}\,,$$

which, multiplied to both sides of the ODE leads to

$$C_2\,e^{\frac{V}{D}}p' + C_2\frac{V'}{D}e^{\frac{V}{D}}\,p = \cancel{C_2}\Big(e^{\frac{V}{D}}\,p\Big)' = C_1\cancel{C_2}\,e^{\frac{V}{D}}\,.$$

Integrating both sides again yields our (stationary) solution

$$\int\Big(e^{\frac{V}{D}}\,p\Big)'dx = \Big(e^{\frac{V}{D}}\,p\Big) + C_3 = C_1\int e^{\frac{V}{D}}\,dx\,,$$

$$p(x) = e^{-\frac{V(x)}{D}}\left(C_1\int e^{\frac{V(x)}{D}}dx - C_3\right) = e^{-\frac{V(x)}{D}}C_1\left(\int e^{\frac{V(x)}{D}}dx - \frac{C_3}{C_1}\right) = \underbrace{N\left(\int e^{\frac{V(x)}{D}}dx - C\right)}_{:=N(x)}e^{-\frac{V(x)}{D}}\,,$$

where $N := C_1$ and $C := \frac{C_3}{C_1}$. □

3.2.1 Stationary distribution of the OUP

We now use the result of Thorem 3.1 to derive the stationary distribution of the OUP.

Theorem 3.2

Let x_t be the solution of the OUP $dx = -\theta(x-\mu)\,dt + \sigma dW$ then $x_t \sim \mathcal{N}(\mu, \frac{\sigma^2}{2\theta})$ when $t \to +\infty$.

Proof. The proof can be derived by considering the solution of the OUP (see sub-section 2.2) conditioned on $x(0) = x_0$

$$x_t = \mu + (x_0 - \mu)\,e^{-\theta t} + \sigma\int_0^t e^{-\theta(t-s)}ds\,,$$

whose time-dependent distribution is

$$x_t \sim \mathcal{N}\Big(\mu + (x_0-\mu)\,e^{-\theta t}, \frac{\sigma^2}{2\theta}\Big(1 - e^{-2\theta t}\Big)\Big)\,,$$

and then taking the asymptotic limit $t \to +\infty$ to get

$$x_\infty \sim \mathcal{N}\Big(\mu, \frac{\sigma^2}{2\theta}\Big) .$$

Here instead we want to obtain the same result by solving the steady-state ODE and determining the constants involved in the $N(x)$ as per Theorem 3.1. For the OUP we write

$$f(x) = -\theta(x-\mu) = -V'(x) \;\Rightarrow\; V(x) = \theta\Big(\frac{x^2}{2} - \mu x\Big) ,$$

so that the stationary FPE reads

$$\Big(p' + \frac{2\theta(x-\mu)}{\sigma^2}\, p\Big)' = \Big(p' + \frac{x-\mu}{D}\, p\Big)' = 0 , \quad D := \frac{\sigma^2}{2\theta} .$$

In order for $p(x)$ to be a pdf a necessary condition is that it decays to 0 in the unbounded domain $(-\infty, +\infty)$, together with its first-order derivative. As such we specify the BCs

$$p(x), p'(x) \to 0 , \quad x \to \pm\infty ,$$

in order to formulate an IVP for our ODE and thus get a unique solution for $p(x)$. We therefore integrate the second-order ODE in $(-\infty, x]$ to get

$$\int_{-\infty}^{x} \Big(p'(y) + \frac{y-\mu}{D}\, p(y)\Big)' dy = \Big(p'(x) + \frac{x-\mu}{D}\, p(x)\Big) - \cancelto{0}{\lim_{x \to -\infty} \Big(p'(x) + \frac{x-\mu}{D}\, p(x)\Big)} = p'(x) + \frac{x-\mu}{D}\, p(x) = 0 ,$$

which is a (reduced) first-order, linear, homogeneous ODE. We rescale the independent variable

$$\tilde{x} := x - \mu ,$$

to ease the computations. The integrating factor is

$$h(\tilde{x}) = e^{\frac{1}{D}\int \tilde{x} dx} = C_1\, e^{\frac{\tilde{x}^2}{2D}} ,$$

which we multiply to both sides of our ODE to get

$$e^{\frac{\tilde{x}^2}{2D}} p'(\tilde{x}) + \frac{\tilde{x}}{D}\, e^{\frac{\tilde{x}^2}{2D}}\, p(\tilde{x}) = \Big(e^{\frac{\tilde{x}^2}{2D}}\, p(\tilde{x})\Big)' = 0 .$$

We integrate again to obtain

$$\int \Big(e^{\frac{\tilde{x}^2}{2D}}\, p(\tilde{x})\Big)' d\tilde{x} = e^{\frac{\tilde{x}^2}{2D}}\, p(\tilde{x}) + C_2 = 0 ,$$

which yields our unnormalised solution

$$p(\tilde{x}) = -C_2\, e^{-\frac{\tilde{x}^2}{2D}} = N\, e^{-\frac{\theta \tilde{x}^2}{\sigma^2}} , \quad N := -C_2 .$$

We then need to compute the normalisation constant $N > 0$ by imposing the identity between the integral of the pdf in $(-\infty, +\infty)$ and 1

$$1 = \int_{-\infty}^{+\infty} p(\tilde{x}) d\tilde{x} = N \int_{-\infty}^{+\infty} e^{-\frac{\theta(x-\mu)^2}{\sigma^2}} dx .$$

By recalling that

$$\int_{-\infty}^{+\infty} e^{-\alpha(x+\beta)^2} dx = \sqrt{\frac{\pi}{\alpha}}\,,$$

we get that the value of the normalisation constant is

$$N = \frac{1}{\int_{-\infty}^{+\infty} e^{-\frac{\theta}{\sigma^2}(x-\mu)^2} dx} = \sqrt{\frac{\theta}{\sigma^2 \pi}}\,,$$

and thus our stationary solution of the FPE for the OUP in fully explicit form

$$p(x) = \sqrt{\frac{\theta}{\sigma^2 \pi}}\, e^{-\frac{\theta}{\sigma^2}(x-\mu)^2}\,.$$

□

References

[1] Gardiner, C. *Handbook of stochastic methods for physics, chemistry and the natural sciences (3rd ed.)* Springer, 2003.

[2] Papapicco, D. *Notes on dynamical systems.* https://github.com/papadeiv/Pinakes/tree/master/notes/DynamicalSystems.