Lecture : 10

The Four Fundamental Subspaces :-

① Correct error in Lect. 9 !!

② four fundamental subspaces (for matrix A). $\boxed{A = m \times n}$

㊉ 4 subspaces :

ⓐ Column space $\boxed{C(A)}$ → in $\mathbb{R}^m$

ⓑ Null space $\boxed{N(A)}$ → in $\mathbb{R}^n$ ✓

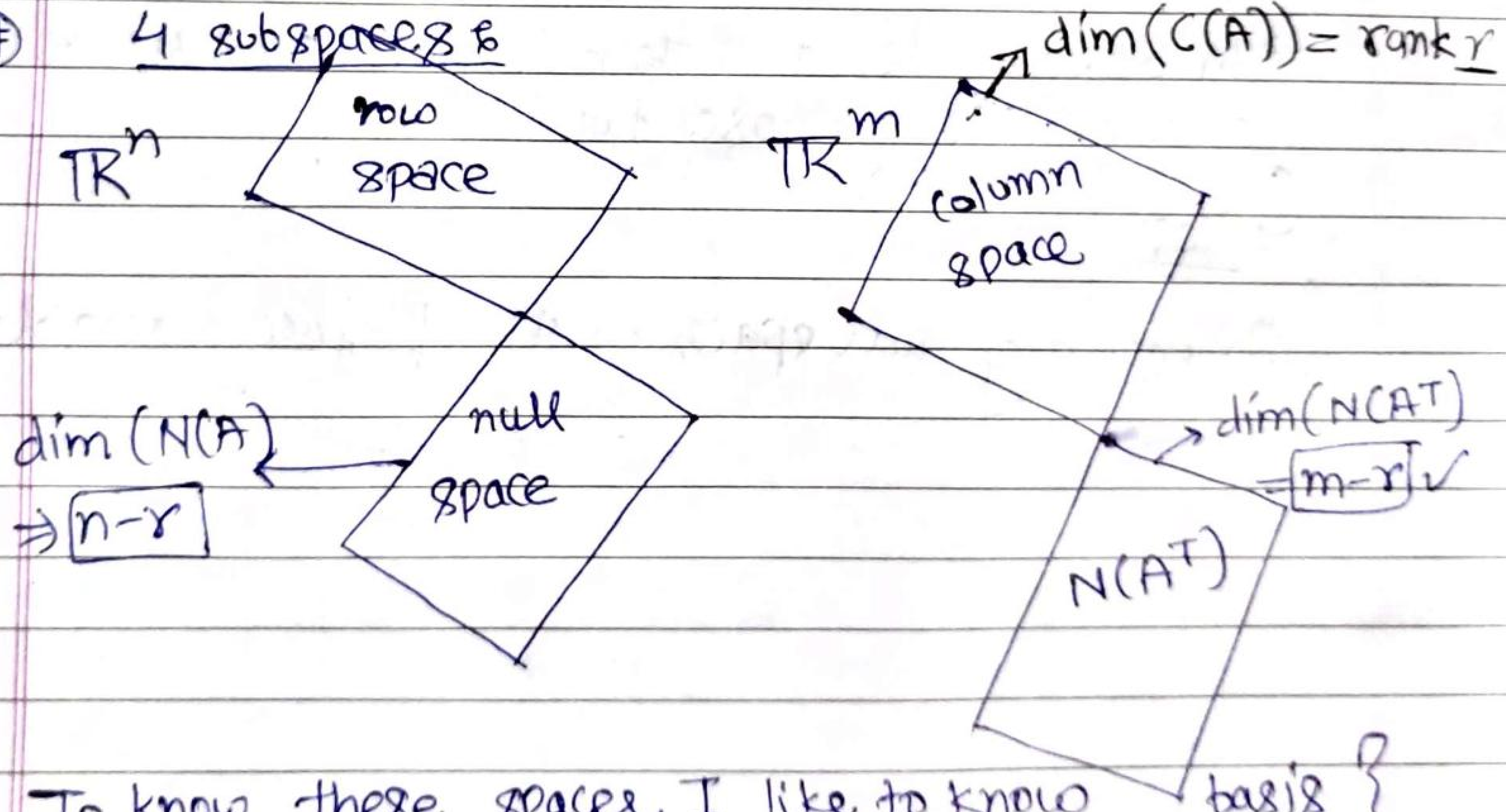
ⓒ Row space = all combinations of rows.

= all combs of the columns of A^T

= column space of $\boxed{C(A^T)}$ → $\mathbb{R}^{n}$

ⓓ Null space of A^T = $\boxed{N(A^T)}$ → $\mathbb{R}^m$ (left Nullspace of A)

㊉ 4 subspaces :



To know these spaces, I like to know basis ?

㊉ Basis } dimension ? ✓ $\boxed{\downarrow}$

		Dimension
C(A) ⇒ Basis ⇒ Pivot Cols;		rank (r)
row space C(A^T) ⇒		r $\boxed{C(A) = C(A^T)}$ w.r.t. ⓡ
Null / N(A) ⇒	Special Solutions	$\boxed{n-r}$ ↳ No of free variable

Note: Row space Dimension = Column Space Dimension. /

→ every special solution came from free variable.

Note: Matrix(A) → row operations → u [form] → rref(R)
↓
came from free variables → special solutions
↓
are in null space.
↓
Basis for Null space

Basis?

⇒ Got the basis for Column space = pivot cols.

⇒ row space → can be produced.
~~prod~~ ⇒ by transposing my matrix ? (?)
then doing → elimination → row reduction
→ checking out pivot columns

matrix $\boxed{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → elementry operation of row2 by (−1) & row1 − 2 row2

↓

$\rightarrow \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}}_{F} \\ \boxed{\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}} \end{bmatrix}$ → not contributing anything in rowspace.

what happend here: $\boxed{C(R) \neq C(A)}$ ✓ diff. colmns space.
Column space of R

→ Same Row Space

Basis for Row space in A!
↳ first 2 rows.

Basis for Row space (A or R) ⇒ first r rows of R //

Why? $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ & \boxed{0} & & \end{bmatrix}$ ⇒ why these are in row space?
because what all operation, we did on matrix A → ~~new~~ started with row operations & took combn of them. (rowx − rowy) → still in same row space. (rowx − c rowy) → same space, but basis are changing & ended by the best basis. /

Row Space is sitting in R → in cleanest form.

Left Nullspaces (4th space):- $N(A^T)$

$$\boxed{A^T y = 0}$$

then 'y' is in the null space of A transpose, of course.

⇒ $[\text{Matrix}]$ times $[\text{column}] = [0]$

⇒ Transpose of it:-

(reverse order)

$$y^T A^{TT} = 0^T$$

$$y^T A^{TT} = [\ 0\ ]$$

$$[\ y^T\ ]\begin{bmatrix} A \end{bmatrix} = [0]$$ ← that's why called left Nullspace

⇒ How to get basis for this?

ex: find a matrix $\boxed{E}$ which transformed $A \rightarrow R$.

Using Gauss Jordan

$$\underbrace{[A_{m\times n}\ I_{m\times m}]}_{} \xrightarrow{rref} [R_{m\times n}\ E_{m\times m}]$$

We multiplied something that took $A \rightarrow R$.

$I \rightarrow E_{m\times m}$

so:

$$E[A_{m\times n}\ I_{m\times n}] \rightarrow \underbrace{[R_{m\times n}\ E_{m\times m}]}_{}$$

$$\boxed{EA \rightarrow R}$$

Earlier matrix was square & we were finding its inverse

↓

Now its probably rectangular

↓

But still I can tack on I

Earlier. Chapter 2 for square matrix

↓

$\boxed{R \text{ was } I}$

then E was A^{-1}.

Here: A is rectangular hasn't got any inverse

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

#

A

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Apply all row operation of A to I./

$\longrightarrow \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \leftarrow \boxed{\text{I think its } E}$

if I do $\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow R$ ./

E A

↑ left null space ⇒ ~~rank~~ ⇒ $\boxed{m-r}$

dimension $(3-\underset{\uparrow}{2}) = 1$.,

↳ 1 dimensional ←

↓

Means One combination of those three rows that produces the zero row./

→ this is the vector; which produces → zero row,

$[-1 \;\; 0 \;\; 1] \Rightarrow -1R_1 + 0R_2 + 1R_3 = [0]$

Basis ;,

New vector space!

All 3×3 matrices || → Matrices space = $\boxed{M}$

↳ Subspace: of $\boxed{M}$:,

All upper triangulars Matrices	symmetric matrices	diagonal matrices D

$\dim(D) = 3$

/

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

independent/
basis//

Lecture → 11

Matrix Spaces; Rank 1; Small World Graphs.

- Bases of new vectors spaces.
- Rank one matrices
- Small world graphs.

(dim M = 9)

Basis for M = all 3×3's matrices ⑨

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

→ ①

Subspaces like

Symmetric 3×3	upper triangular 3×3

dim? (S) = ⑥ dim? (U) = ⑥

M dim ? (whole space) ? = 9.

there's a natural basis for all three by three matrices.

Means ⇒ How many numbers does it take to specify that 3×3 matrix. ? = 9//

How many parameters do I choose in 3×3 matrices ?

- the diagonal (3) + (3) entries above the diagonal then I know what the three entries below./

Dimension of that space of all upper triangular ? (3×3)

⊙ Basis of ~~Upper~~ Ⓤ :- 6 of the -①

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

- This is accidental that big basis contains in it a basis for the subspace.

- But symmetric (S) → didn't have./

(#) $S \cap U$ = symm and uppertriangular.

if matrix is symmetric & also upper triangular ⇒ diagonal.

= Diagonal 3X3./ $(\dim(S \cap U) = \textcircled{3}$

→ taking vectors; that are in both.

(#) $S \cup U$:- symmetric or upper triangular.

↳ Not interested $\xrightarrow{bcoz}$ not in a subspace.

(#) To get the bigger spaces

(sum) S+U :- (combinations of things in S and things in U):
; I take anything in S + anything in U.

= : any ele of S + any element of U

=

If I take every symmetric matrix; take all symmetric matrices, and add them to all upper triangular matrices.
then I've got a whole lot of matrices and it is a subspace.

& what vector space would I then have?

= all matrices 3X3's

⑥ dim(S+U) = 9 = because we got all 3X3's //

(#) ⇒ $\boxed{\dim(S) + \dim(U) = \dim(S \cap U) + \dim(S+U)}$

(★) 6 + 6 = 3 + 9

⊕ Differential eq^n ($Ax=0$)

$$\frac{d^2y}{dx^2} + y = 0 \; ;$$

Sol^n to: this eq^n :- $y = \cos x , \; \sin x , \; \cancel{e^{ix}}$

Null space of diff. eq^n $\overset{i.e.}{\Rightarrow}$ solution space

Complete solutions

$$\boxed{y = c_1 \cos x + c_2 \sin x}$$ i.e. vector space.

↳ dim?
↳ basis

Basis : ? (means: all the guys in the space are combinations of these basis vectors)

$\Rightarrow \sin x , \; \cos x \leftarrow$ BASIS.

Dim : ? dim(sol^n space) = 2.

How many vector in the spaces? 2 → sin x, cos x
- don't look like vector
- look like function
- we can add, multiply then vector

⊕ Rank one Matrices

$$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$ ← multiple of row to have rank 1.

Basis of Row Space → 1st row.

Dim. Column Space → 1 = Rank = dim $C(A^T)$
(all column are multiple of one column).

$$= \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \; \underset{1\times 3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} \; \end{bmatrix}}$$

⊙ Every Rank 1 matrix,' has the form some column times some row.

$$\boxed{A = u\,v^T}$$

$\downarrow \ \downarrow$

23/5

⊙ rank one — building block..

M = all 5×17 matrices:-

subset of rank 4 matrices

⊙ if add 2 rank 4 matrices; result would be rank 4 ?

—Not usually;

sum will be rank could be five./

rank(A) + rank(B) = can't be more the rank(A) + rank(B).

= How big the rank be? = ⑤ ✓

- Subset of rank 1 matrices ?

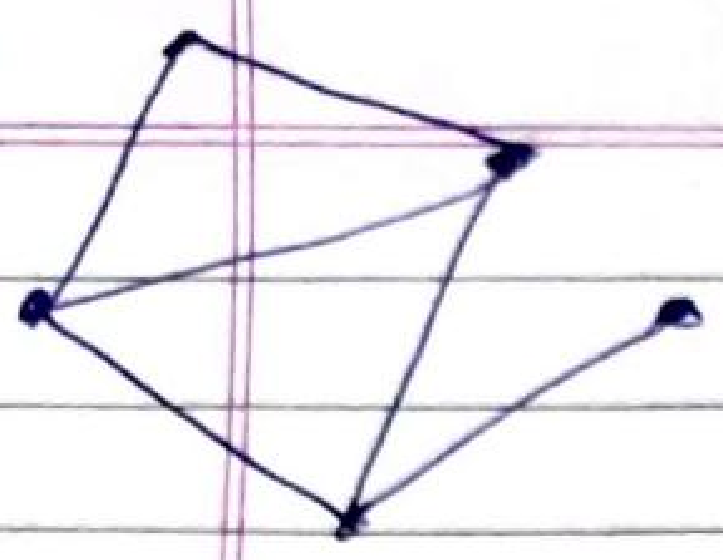
⇒ rank(1) + rank(1) = most likely rank(2)//

not a subspace./

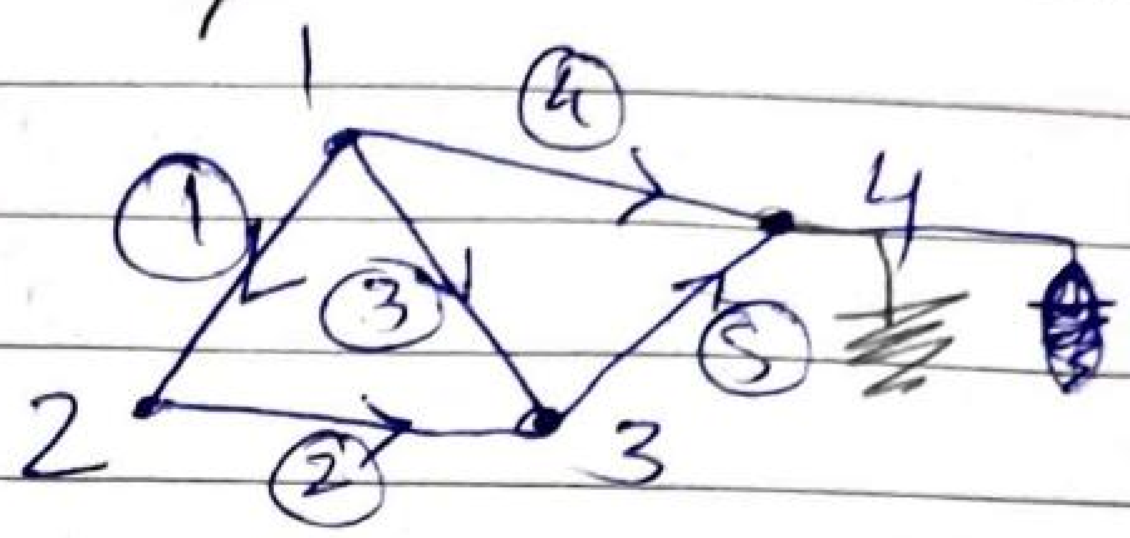
⊕ In R^4

Graphs {nodes, edges}

Lecture : 12



Graphs & N/w
Incidents Matrices
Kirchhoff's Laws.



n = 4 nodes.

- Row for every edges hence

m = 5 (matrix) (rows)

Incidence Matrix:-

- every row has only two non-zeros.
  ↳ very sparse matrix.
- The no. of non-zeros is exactly two times fives.

$$A = \begin{array}{cccc|cl} \text{node1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \text{edge} & \\ -1 & 1 & 0 & 0 & 1 & \\ 0 & -1 & 1 & 0 & 2 & \text{loop} \\ -1 & 0 & 1 & 0 & 3 & \\ -1 & 0 & 0 & 1 & 4 & \\ 0 & 0 & -1 & 1 & 5 & \end{array}$$

- Null space? → Are columns independent?

  if columns are independent, then what's in the null space?
  ↳ Only the zero vector.

  → tells us what combinations of colms. gets zero.

To find Nullspace: solve :- Ax = 0.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

↑ potential at nodes

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

↳ potential diff. across edges.

$X = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$ suppose x is potential

↳ potential are constant

Basis for null space (1D) :- $\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$; Whole Nullspace = $c\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$

- if potentials are same, then there is no current flow. /
- if x_4 is grounded :-
- Any three potentials are independent, good variables. The fourth potential is not, we need to set, & typically we ground that node.

$\boxed{\text{Rank} = 3}$ ✓

$C(A)$. ; $\underline{N(A^T)} = ?$

free variables

$A^T = n \times m$; (4×5)

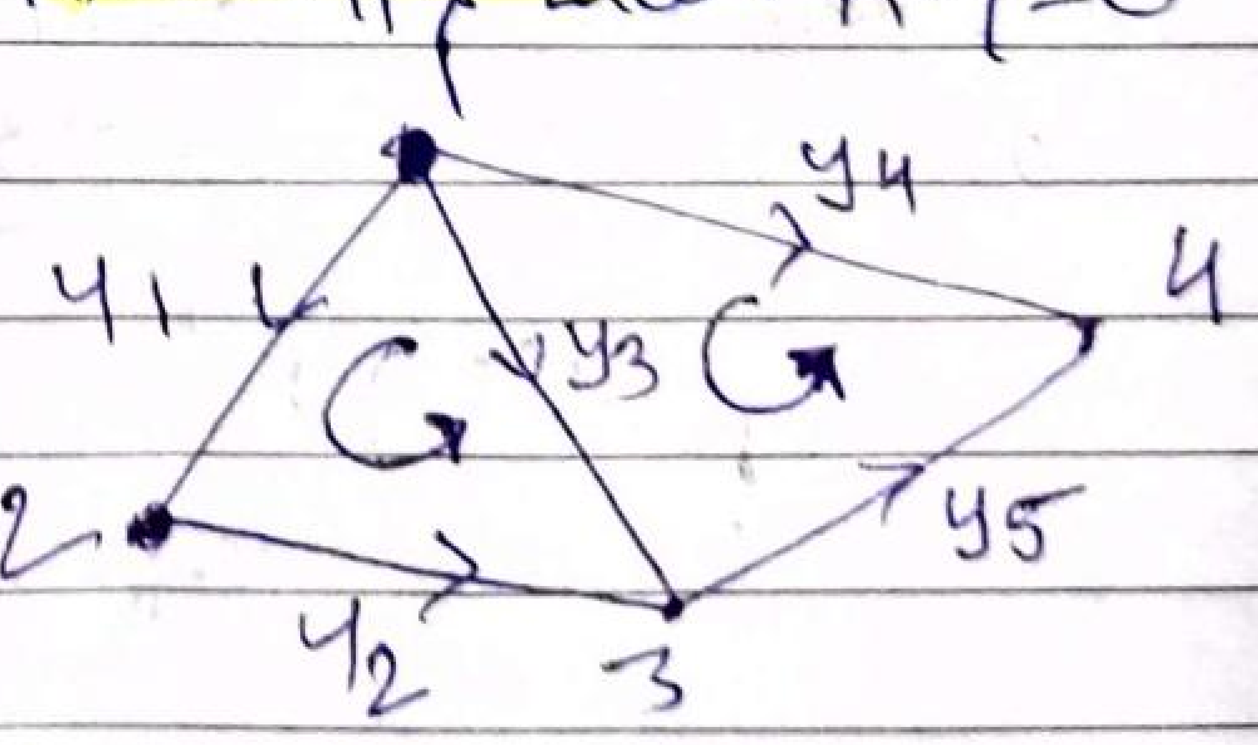
$A^T y = 0$ dimension : $\boxed{m-r}$ ✓

$$\begin{bmatrix}-1 & 0 & -1 & -1 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

↖ 5 current on edges & satisfy Kirchhoff's Law, $A^T y = 0$

$-y_1 - y_3 - y_4 = 0$ ← node 1
$y_1 - y_2 = 0$ ← node 2
$y_2 + y_3 - y_5 = 0$ ← node 3
$y_4 + y_5 = 0$ ← node 4

↑ charge doesn't accumulate at nodes;



$N(A^T)$:- How could current travel around this n/w without collecting any charge at the nodes?

Basis for $N(A^T)$; it's a 2 D space

My basis should have 2 vectors.

$$L_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

→ edge 1 → 1 unit of Amp travel to y 2

→ edge 2 ~~current~~

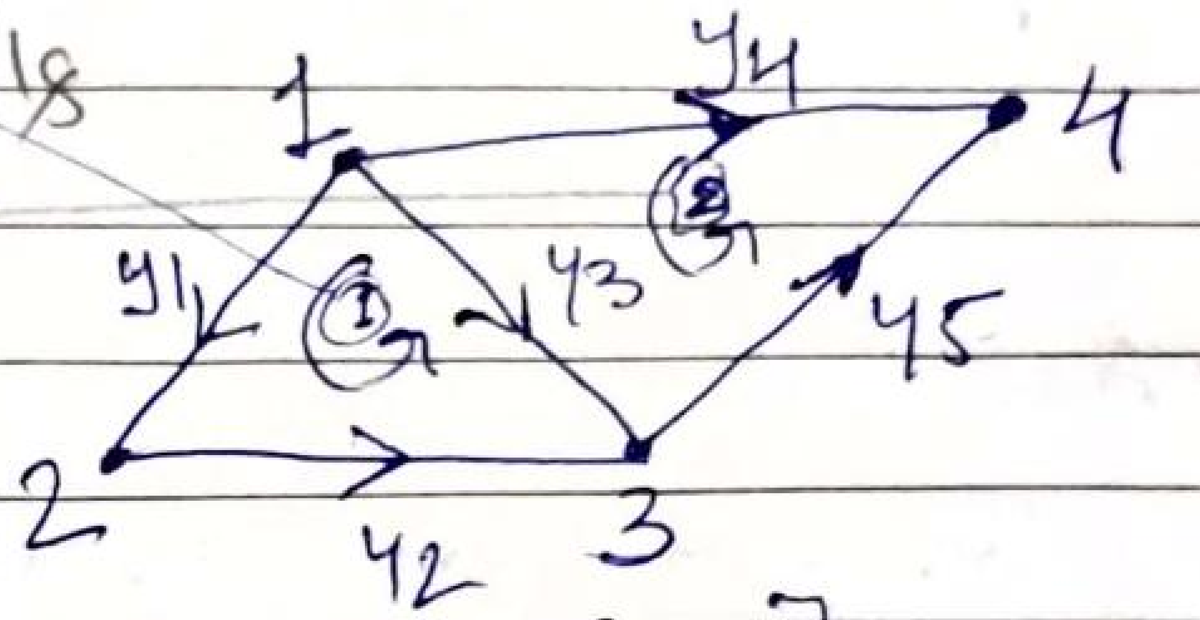
— edge 3

— edge 4

— ed 5

Any current around the loop satisfies current law

└ ~~stus~~ satisfies Kirchoff's current Law.



$$L_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

what about whole circuit ?

= is that in the null space of A^T ?

Sure ✓

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

✓ So why don't we now have a third vector in the basis ? because it's not independent, i.e. sum of $L_1 + L_2$

㊀ Row Space of (A)

Dimension = ?

— ⑤ row — ③ independent

$$\begin{array}{ccccc} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ \uparrow & \uparrow & & \uparrow & \\ 1 & 2 & & 4 & \end{array}$$

Pivot

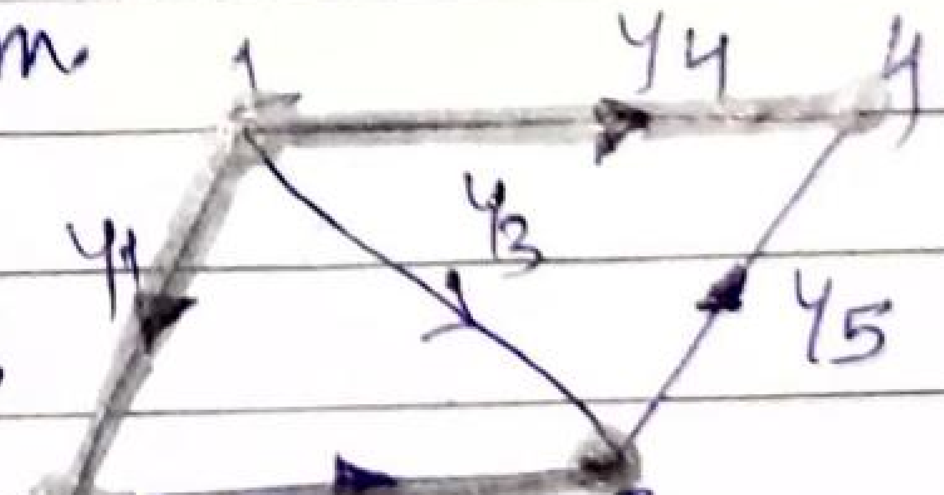
✲ The first three columns, are they the pivot columns of the matrix → NO

$1^{st} + 2^{nd}$ Column = 3^{rd} column

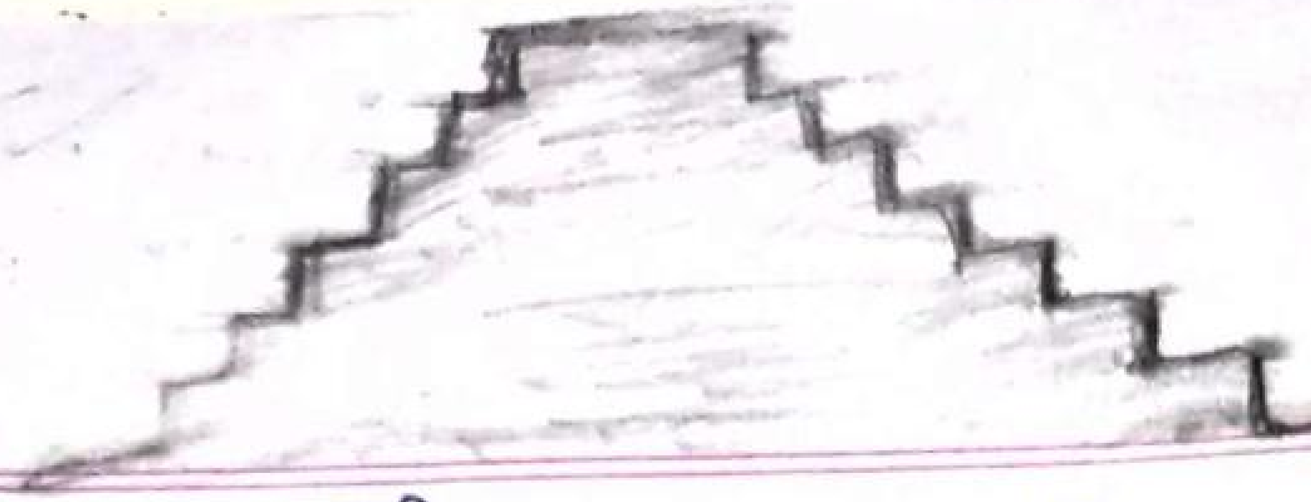
↳ they come from loop

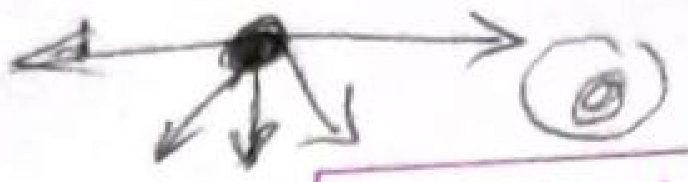
these edges ⇒ Independent Colm (Pivot)

has No loop



TREE (GRAPH WITH NO LOOPS)





dim $N(A^T) = m - r$

$$\boxed{\# \text{loops} = \# \text{edges} - (\# \text{nodes} - 1)}$$

(rank = n − 1)
nodes)

$$\boxed{\# \text{nodes} - \# \text{edges} + \# \text{loops} = 1}$$ → Euler's formula

0D   1D   2D

ex:

$5 - 7 + 3 = 1$

$\boxed{A^T y = 0}$ Kirchoff's

Steps: Matrix A

Potential $e = Ax$

potential diff $\xrightarrow[\text{OHM's LAW}]{y = Ce}$ current on edges $(y_1, y_2, y_3, y_4, y_8)$ $\xrightarrow{A^T}$ $A^T y = 0$

So $\boxed{A^T \boxed{C \boxed{Ax}}} = f$