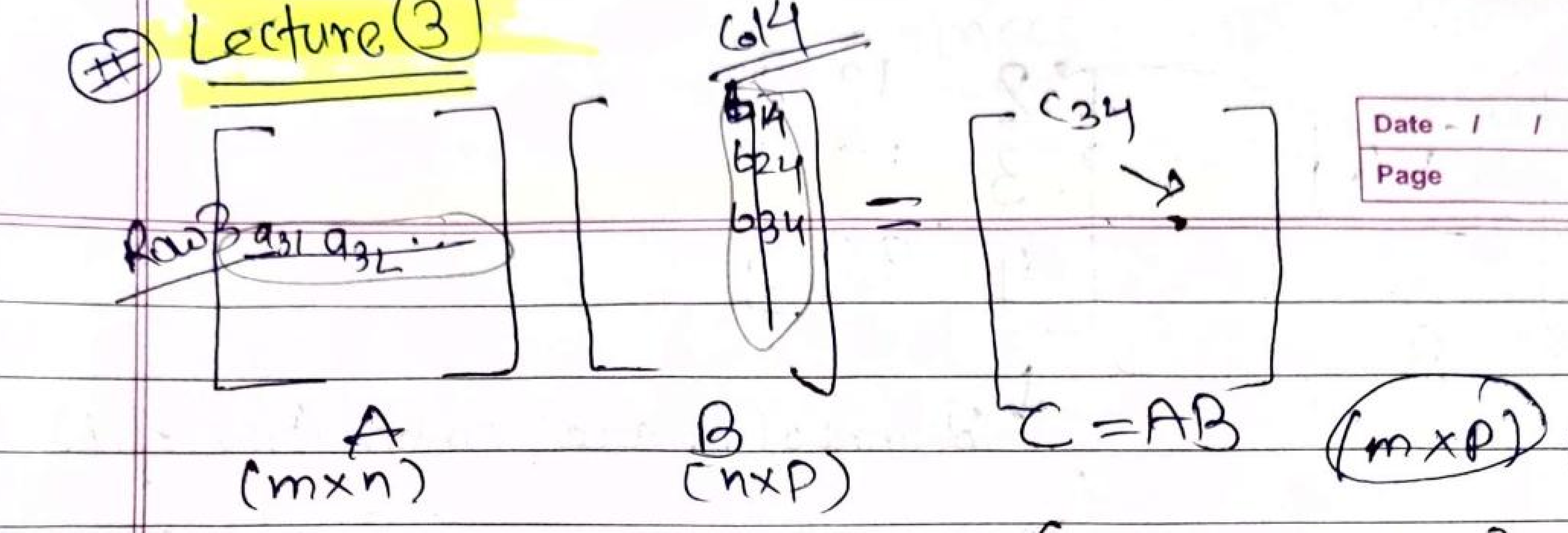
# Lecture 3

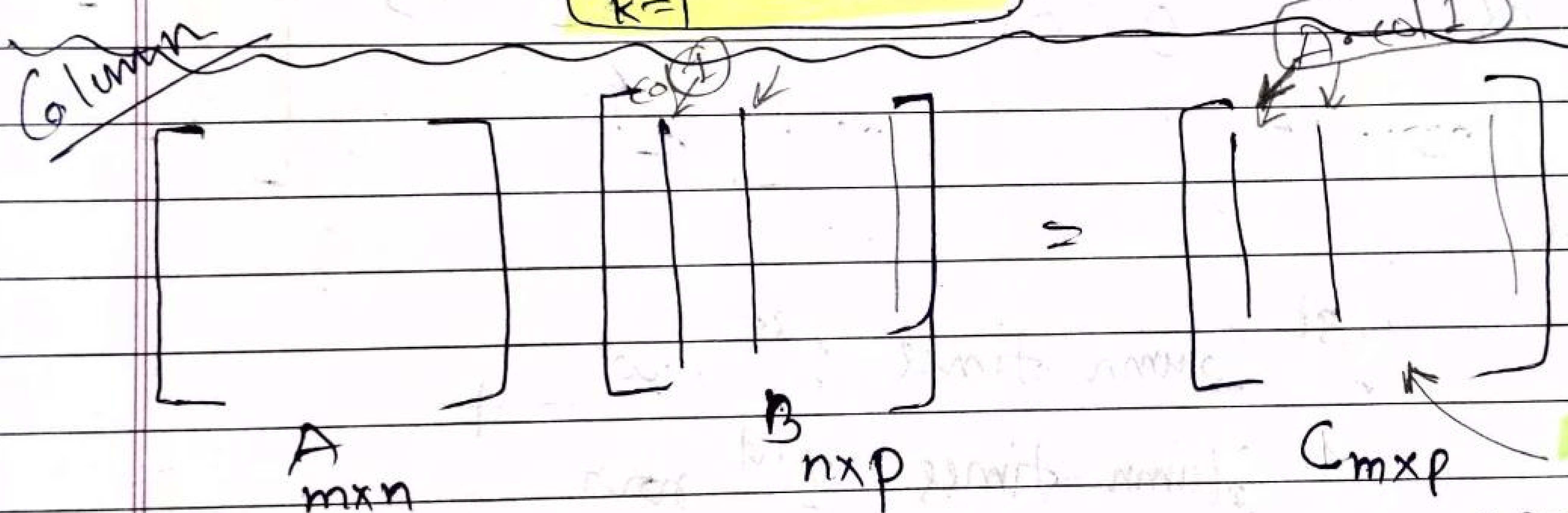



$$c_{34} = (\text{row 3 of } A) \cdot (\text{Column 4 of } B)$$

$$= a_{31} b_{14} + a_{32} b_{24} + \dots$$

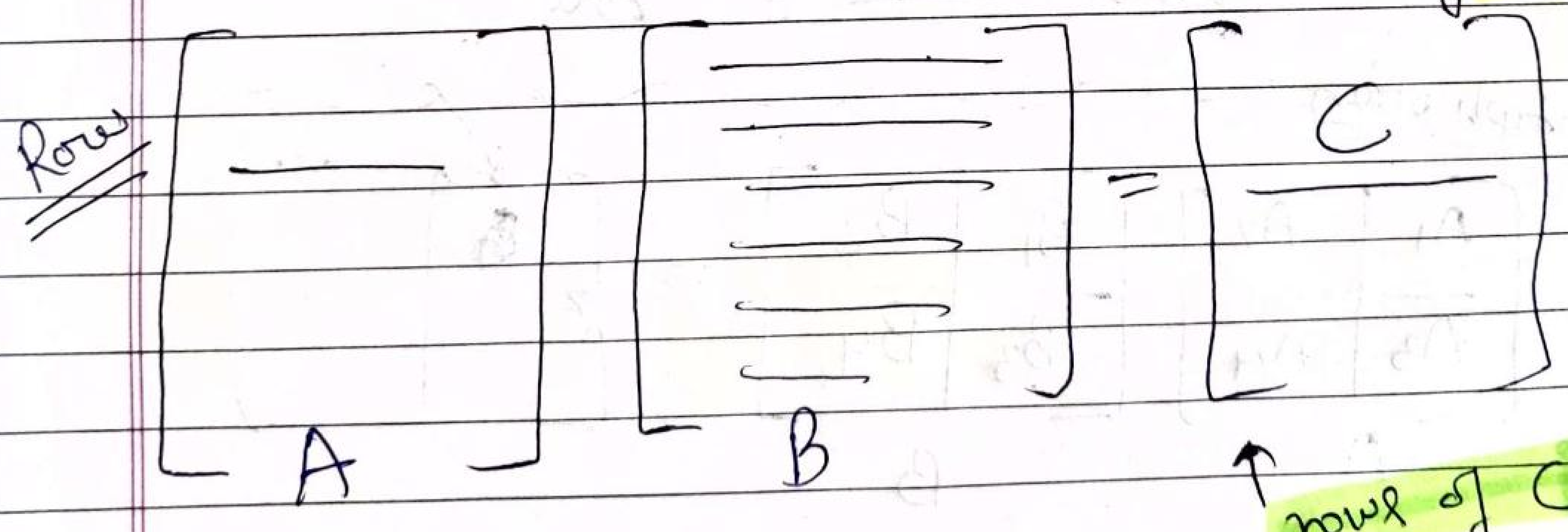
$$= \boxed{\sum_{k=1}^{4} a_{3k} b_{k4}}$$

## Column




columns of C are combinations of columns of A.

## Rows




rows of C are combinations of rows of B.

eg

$$\underset{A}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 6 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}}$$

Columns of C are combination of A

rows of C are multiple of row 1 B

4th way

**AB = Sum of (Cols of A) × (Rows of B).**

$$\underset{m \times n}{\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 4 \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

1st Column times 1st row +

2nd Column times 2nd row +

3rd Column — 3rd row

Block multiplication

$$\underset{A}{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}} = \begin{bmatrix} \ast & \\ & \end{bmatrix}$$

$A_1 B_1 + A_2 B_3$

# Inverses – (Square matrices)

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Not all matrices have inverse.

Left $\rightarrow$ $\boxed{A^{-1} A = I = AA^{-1}}$ ✓ ← for square

if this exists.
when does it exist?
how to find it?

these matrices are called invertible, nonsingular ✓

# Singular Case (No Inverse)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Why does this matrix have no inverse?

1st reason — C matrix will be having column of combination of A columns

& A columns $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ won't lead to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$AA^{-1} = A^{-1}A = I$

2nd reason

You can find a vector $x$ with A

$\boxed{Ax = 0}$ → $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓ if $x \neq 0$

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} [3] + \begin{bmatrix} 3 \\ 6 \end{bmatrix} [-1]$$

$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ✓

# ~~Singular~~ Non-Singular :-

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A $\quad$ $A^{-1}$

$A \times$ col j of $A^{-1}$ = Col j of I

$A \times$ column j of $A^{-1}$ = Column j of I

Gauss-Jordan (Solve 2 eqⁿ at once) :-

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{— (1)}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{— (2)}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{\times 2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} \text{(I)} \\ \text{(II)} \times 3 \end{array}$$

A $\quad$ I

eliminate

eliminate until this point is I

to remove 3

(I) − 2 × (II)

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

I $\quad$ $A^{-1}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} [7, -3] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [-2\ 1]$$

$$\cancel{\begin{bmatrix} 7 & -3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}}$$

~~-2 3~~

2 $\quad$ 1

-2

# Lecture 4

## Inverse of AB, $A^T$

$$\boxed{AA^{-1} = I = A^{-1}A}$$

$AB \rightarrow$ if A, B is invertible then AB's inverse?

what order do I multiply.

$$(AB)(B^{-1}A^{-1}) = I$$ In reverse order

$$B^{-1}A^{-1}AB = I$$

## What's the inverse of $A^T$?

$AA^{-1} = I$ Transpose

$(A^{-1})^T A^T = I$

this is a inverse of $\boxed{A^T}$

## $A = LU$

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$A = L\ U$ — L: lower triangle, U: upper triangle

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = E_{21}^{-1} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{d}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}$$

3x3 matrix:

$$\underbrace{E_{32}\,E_{31}\,E_{21}}_{\text{steps to eliminate}}\,A = U \quad \left(\begin{array}{c}\text{no row}\\ \text{exchange}\end{array}\right)$$

$$A = \underline{\quad ? \quad}\, U.$$ How do invert whole bunch?

$$= E_{21}^{-1}\,E_{31}^{-1}\,E_{32}^{-1}\,U$$

$$= L\,U.$$

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E\left(\begin{array}{c}\text{left}\\ \text{of}\\ A\end{array}\right)$$

$EA = U$

$+5$

inverses! (reverse order)

$$\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$\hookrightarrow L\left(\begin{array}{c}\text{left}\\ \text{of } U\end{array}\right)$

$$\boxed{A = LU}$$

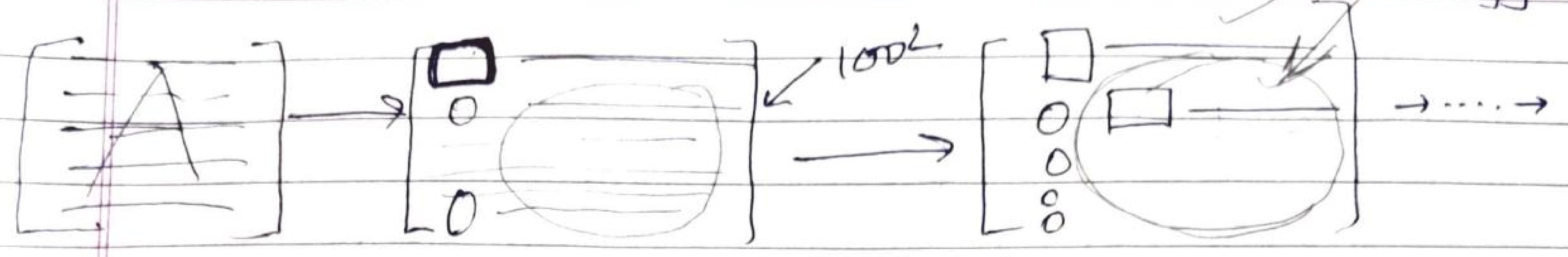
(*) $\boxed{A = LU}$

If no row exchanges

multipliers go directly into L. //

How many operations on $n \times n$ matrix $A$?

$n = 100$ [no zeros]  $n, n^2, n^3, n!$ ??




$100^2, 99^2, 98^2, \ldots\ldots 1^2$

$\Rightarrow n^2 + \ldots\ldots + 1^2 \rightarrow \int n^2 \rightarrow \boxed{\frac{1}{3} n^3}$

$\simeq \Rightarrow \boxed{\frac{1}{3} n^3}$ ✓ on $\boxed{A}$

what about extra column B ?

cost on right hand side $\Rightarrow \boxed{n^2}$ ✓

if row exchange ??

Permutation matrix

exchanges rows 1 & 2 · 1 & 3 · 2 & 3

no row exchange $= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$; $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

exchange $P_{12}$

exchange single pair of rows

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

6 P's

exchange all rows

inverse of each other

$\boxed{P^{-1} = P^T}$ about permutation matrices//

Q: if we multiply 2 of the above matrix = Answer?

inverse of (1)? own inverse

Q: 4x4; 24 P's.//