Lecture :- #29

Linear Algebra

Singular Value Decomposition = SVD

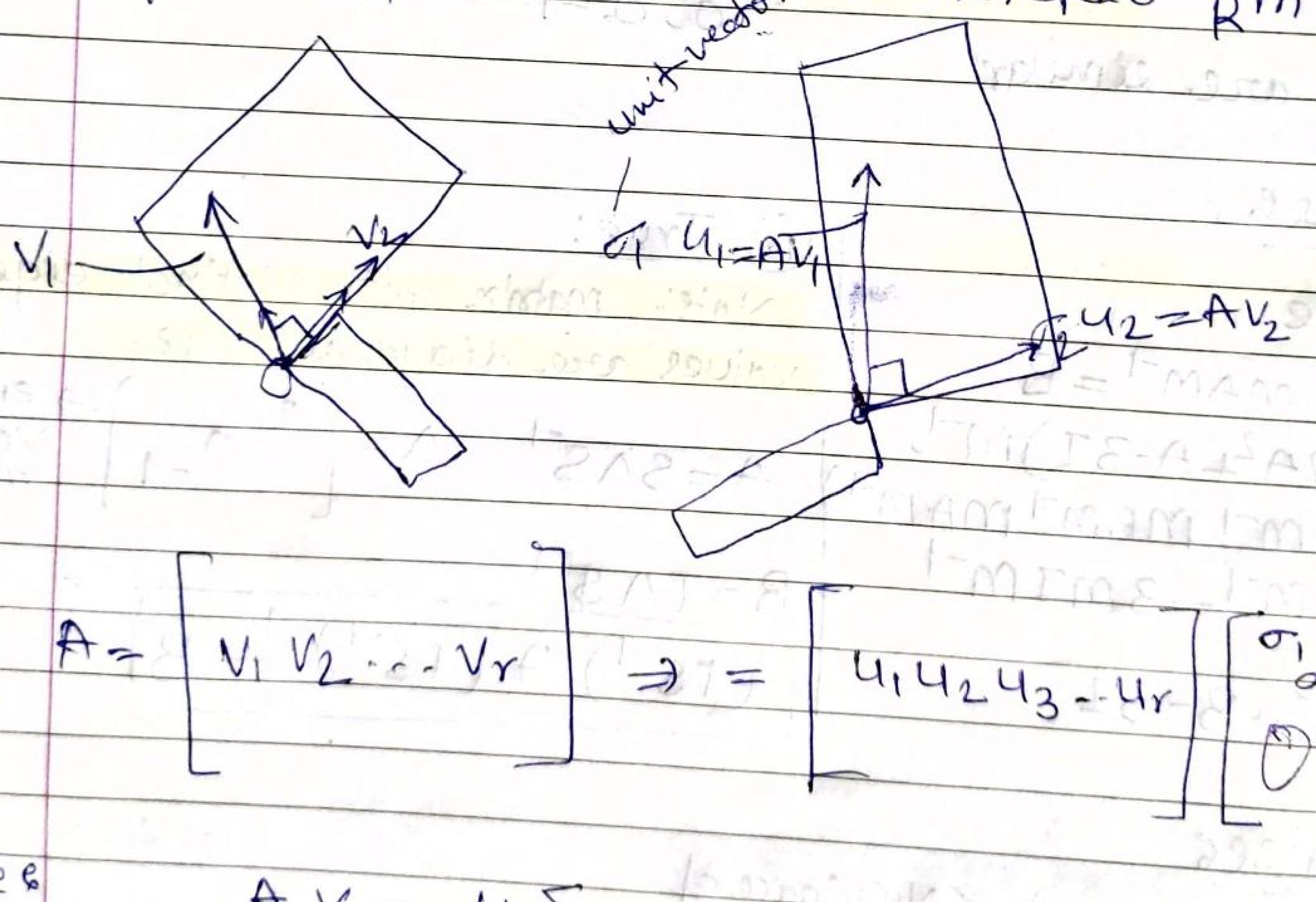
$A = U \Sigma V^T$ // Σ diagonal
// U, V orthogonal

Symm pos. def:

$$A = Q \Lambda Q^T$$

$$A = S \Lambda S^{-1}$$

R^n row space Column Space R^m



$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} \Rightarrow = \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \; \sigma_r \end{bmatrix}$$

Note: $AV = U\Sigma$

→ orthonormal basis in column space

→ find orthonormal basis in row space

so that I have sort of diagonalized the matrix

⇒ Matrix (A) is, like getting converted to this diagonal matrix sigma.

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$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \to$ invertable $\to$ rank 2

look for v_1, v_2 in row space R^2, $\to$ orthonormal

u_1, u_2 in column space R^2 orthonormal

$\sigma_1 > 0 \quad \sigma_2 > 0 \to$ scaling factor

$Av_1 = \sigma_1 u_1 \; ; \; Av_2 = \sigma_2 u_2 \} \to$

$$\boxed{\begin{aligned} \Rightarrow \quad AV &= U\Sigma \\ V &= U\Sigma V^{-1} = U\Sigma V^T \\ A^TA &= V\Sigma U^T U \Sigma V \\ A^TA &= V\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_r^2 \end{bmatrix} V^T \end{aligned}}$$

$V \to$ eigen vector matrix

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

eigen vector: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

normalized form (divide by length) $= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

multiply A^TA with e-vector

$$\Rightarrow \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 32 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad 18 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$A^TA = V\Sigma V^T$ (I)

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{u} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Find u's & u_1, u_2

$$(AA^T) = u\Sigma u^T \overset{I}{V^TV}\Sigma^T u^T$$

tranen

$$= u\Sigma\Sigma^T u^T$$

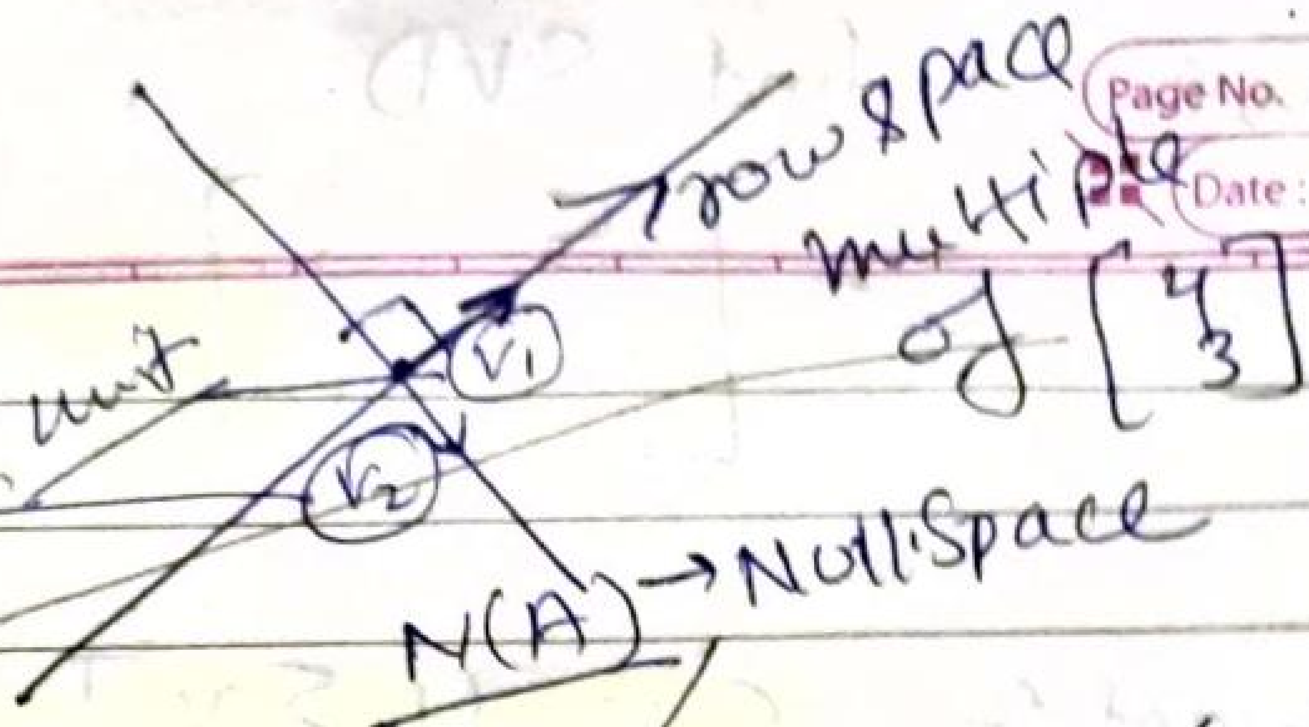
$$\boxed{AA^T = u\Sigma\Sigma^T u^T}$$

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

↳ eigen vecto $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

example: $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

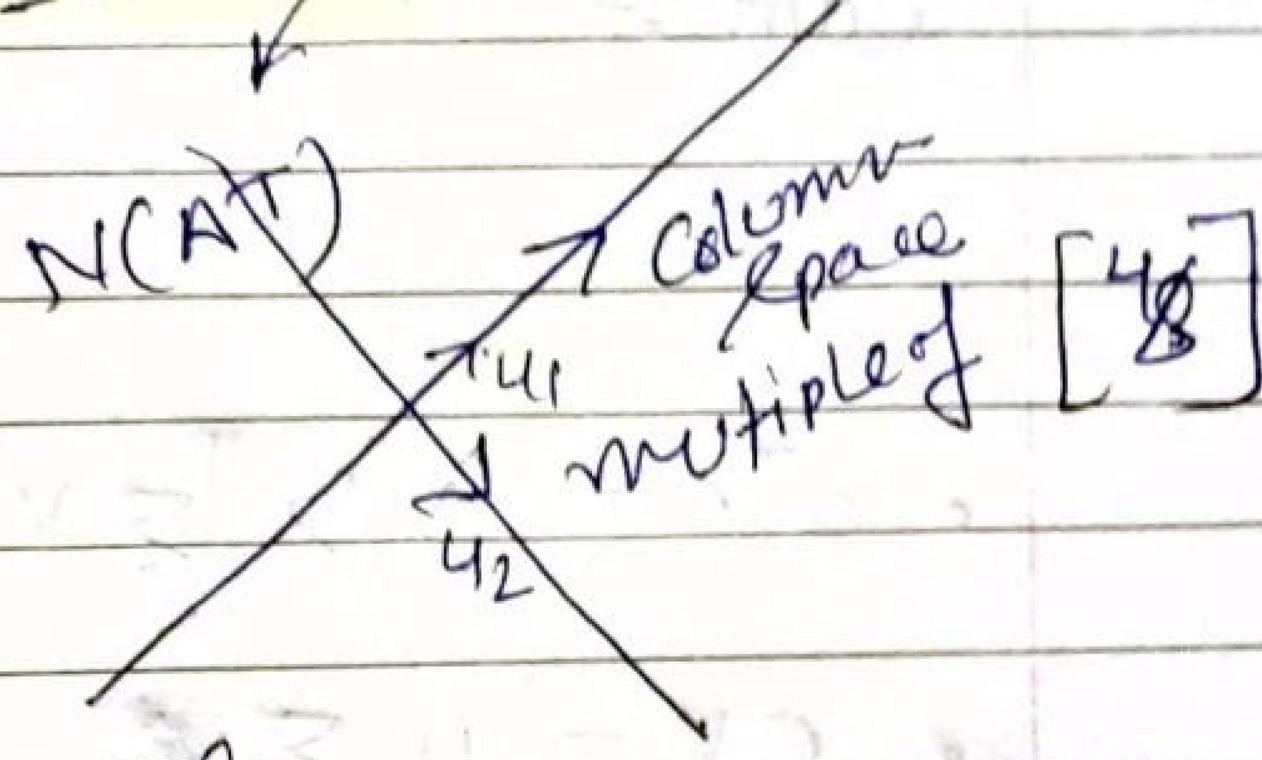


rank 1

v_1 = unit vector of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$v_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



so one

$$\underbrace{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}}_{A} = \underbrace{\begin{array}{cc} u_1 & u_2 \\ \hline \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{array}}_{U} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{cc} v_1 & v_2 \\ \hline .8 & .6 \\ .6 & -.8 \end{array}$$

$\therefore A^T A = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$

$\lambda_1 = 0, 125$

→ v_2 is in null space

⊥r to v_1 so

$v_2 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$

$v_1 \ldots v_r$ orthonormal basis for row space

$u_1 \ldots u_r$ orthonormal " " column

$v_{r+1} \ldots v_n$ " " " nullspace

$u_{r+1} \ldots u_m$ " " " $n(A^T)$

and

$$\boxed{A v_i = \sigma_i u_i}$$ //

Probleme find SVD-

$$C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$$

Sole-

Want :- $C = U \Sigma V^T$

U, V — orthoriogon

Σ — diagonal (non -ve value)

- $C^T C = V \Sigma^T \Sigma V^T$
- $CV = U \Sigma$ — (2)

$$C^T C = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}$$

λ

$$\det(C^T C - \lambda I) = \det \begin{bmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{bmatrix}$$

$$= \lambda^2 - 100\lambda + 1,600$$

$$= (\lambda - 20)(\lambda - 80)$$

eigen value 20, 80

x

$$C^T C - 20I = \begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix} ; \quad V_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

unit vector

$$C^T C - 80I = \begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \qquad V_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

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$$V = \frac{1}{\sqrt{10}}\begin{bmatrix} \cancel{3} & 1 \\ 1 & 3 \end{bmatrix} ; \quad \Sigma = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$$

square root of λ

Now I need to find (u) use eq-(2)

$$CV = \overset{C}{\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}} \overset{V}{\begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}} = \begin{bmatrix} -\sqrt{10} & 2\sqrt{10} \\ 2\sqrt{10} & 2\sqrt{10} \end{bmatrix}$$

this is u times Σ
so I ~~un~~ need to make it ~~unit~~ length

$\Rightarrow$

$$= \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{bmatrix} 2\sqrt{5} & \\ & 4\sqrt{5} \end{bmatrix}$$

$$u = \frac{1}{\sqrt{2}}\begin{bmatrix} +1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$