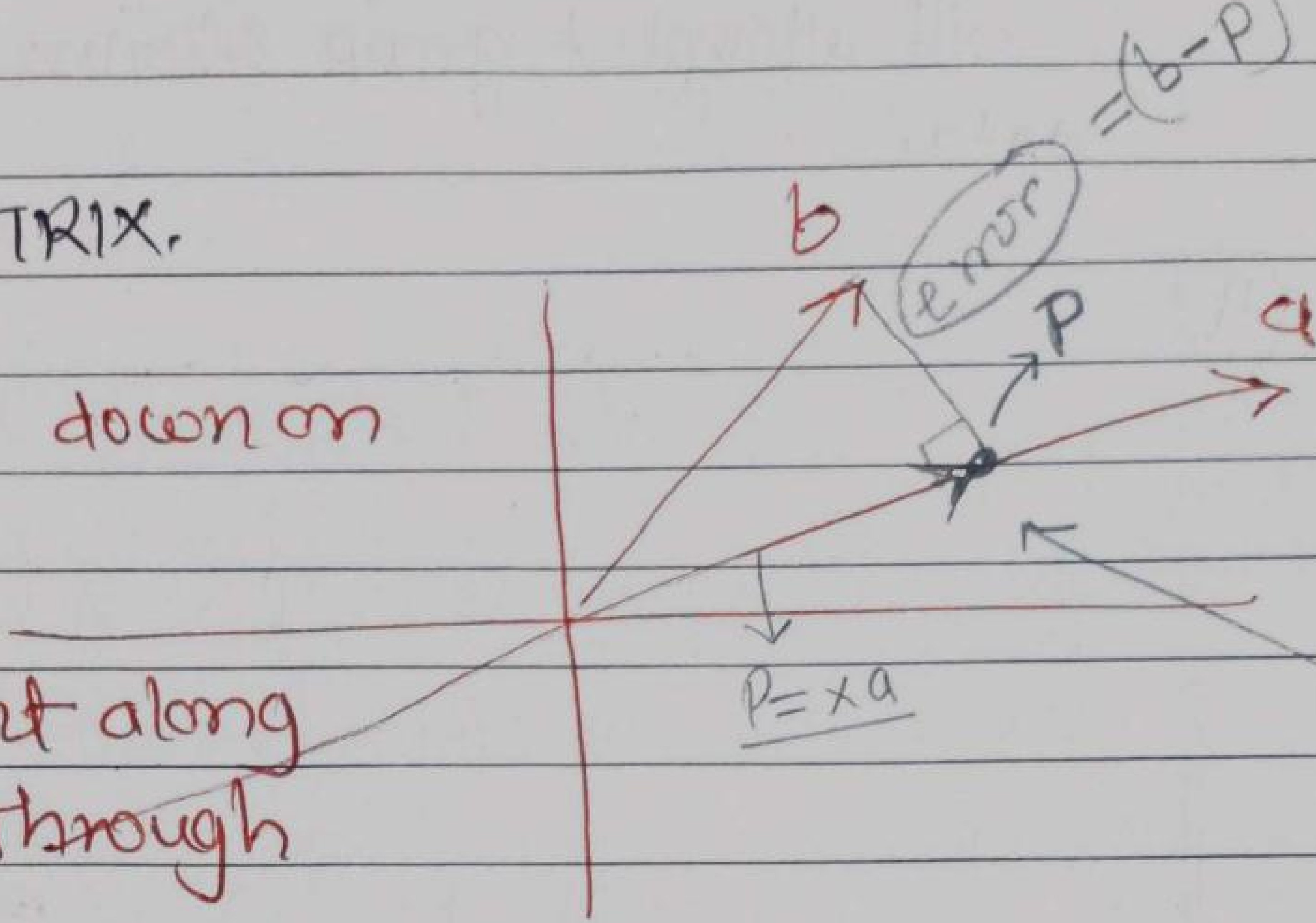
Lecture :- 15

Projections onto Subspaces :

→ Projections
— Least squares
— Projection MATRIX.



⊙ projecting a vector b down on a vector 'a'

- I'd like to find the point along this line so that line through

a → is a one-dimensional subspace,

- I'd like to find the point on that line closest to b.

- So where's the point closest to 'b' that's on that line ?

→ P is some multiple of a. $\boxed{P = xa}$

perpendicular

$$a^T(b-p) = 0$$

$$a^T(b - xa) = 0$$

$$\boxed{x\,a^Ta = a^Tb} \rightarrow \boxed{x = \frac{a^Tb}{a^Ta}}$$

$$\boxed{P = ax}$$

$$P = a \cdot \frac{a^Tb}{a^Ta}$$

b to 2b
what happen to P ?
↳ double too.

a to 2a → P = ? → doesn't change.

Looks like projection is carried out by some matrix.
↳ projection matrix
→ in another words:-

$$\boxed{p = P \cdot b}$$

→ ?

$$\frac{a \cdot a^T}{a^Ta}$$

column times a row

a^Ta → just a number

length of square (a^2)

MATRIX

$$\boxed{P = \frac{a \cdot a^T}{a^T a}}$$ $a \cdot a^T$ → Column × a row; $a^T a$ → number

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$C(P) \Rightarrow$ P time b $\Rightarrow$ Column space ← Combination of Column Space.

= line through a. | $\frac{a \cdot a^T}{a^T a}$

rank(P) = 1

symmetric = YES ($\boxed{P^T = P}$)

↳ How? $\left(\frac{a a^T}{a^T a}\right)^T = \frac{a a^T}{a^T a}$ → $(a a^T)^T$ $= a^{TT} \cdot a^T$ $= a a^T$ ✓

($a^T a$ ↓ number.)

· What happens if I do the projection twice?

↳ projection is same point.//

(*) $\boxed{P^T = P \quad P^2 = P}$

Why project? Because $Ax = b$ may have no solution.

more eqn than unknown & I can't solve them

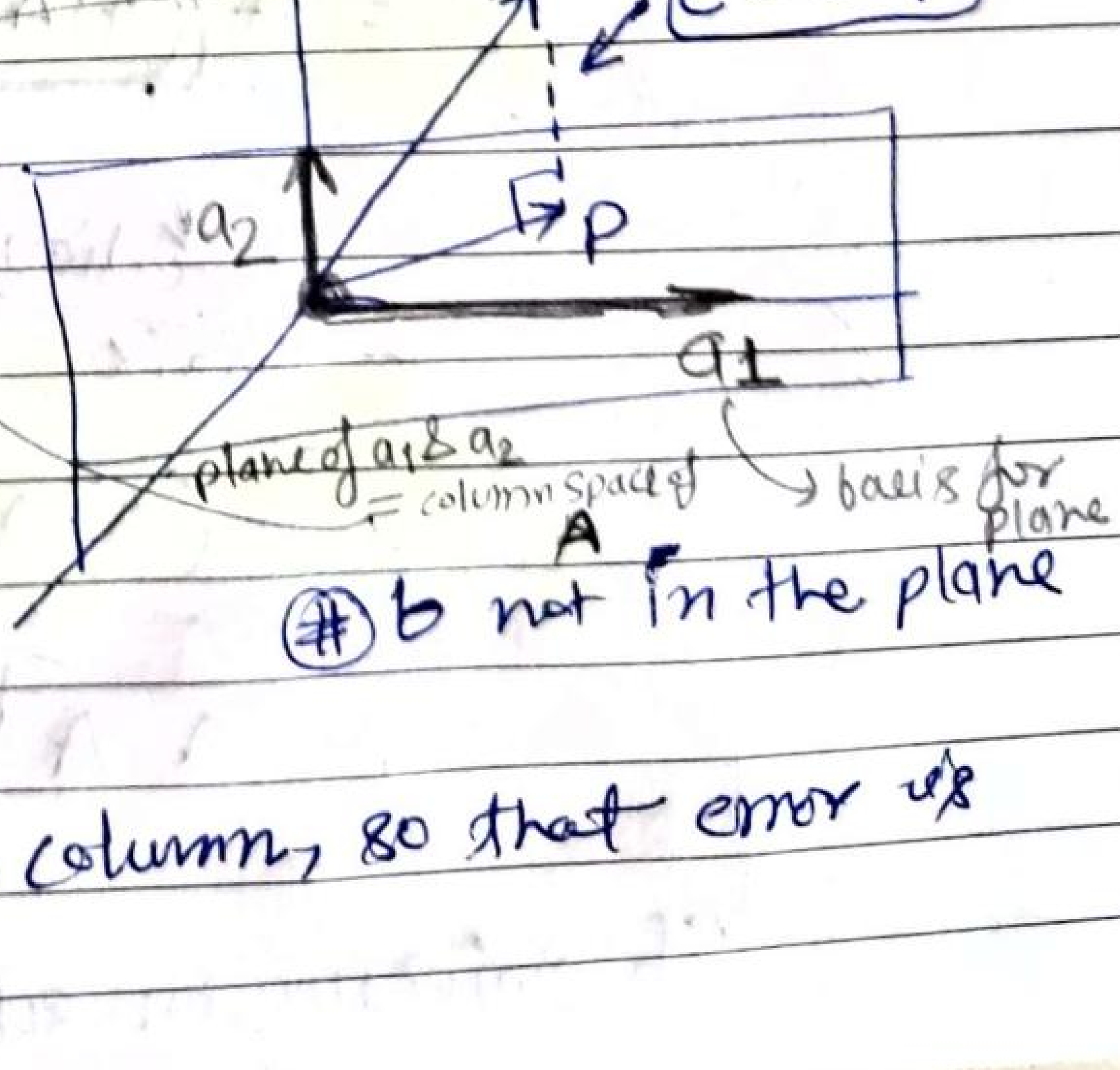
↳ So I solve the closest problem that I can solve.

And what's the closest one?

↳ Solve $\boxed{A\hat{x} = P}$ ↳ proj. of b onto space.

error in b → $\boxed{e = b - p}$

$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$;



↳ e is ⊥r to plane./

• p is some multiple of column.

$\boxed{P = \hat{x}_1 a_1 + \hat{x}_2 a_2}$

$\boxed{P = A\hat{x}}$

(#) b not in the plane

↳ find the right combination of the column, so that error is ⊥r to plane.

Problem: $P = A\hat{x}$ find $\hat{x}$:-

Key :- $b - A\hat{x}$ is $\perp r$ to plane.

eqⁿ: x_1 & x_2 (two eqⁿ) because I've got x_1 & x_2

- e is $\perp r$ plane.
  ↳ means $\perp r$ to a_1 & also to a_2 ✓

$$a_1^T(b - A\hat{x}) = 0; \quad a_2^T(b - A\hat{x}) = 0$$

matrix form

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

e ⇒ subspace ?

$$\boxed{A^T(b - A\hat{x}) = 0} \Rightarrow A^T \cdot e = 0$$

means e in $N(A^T)$

↓ saying

$\boxed{e \perp r\ C(A)}$

column

Nullspace

$$\boxed{A^T A\hat{x} = A^T b}$$

$\hat{x} = (A^T A)^{-1} A^T b$ put

$$\boxed{p = A\hat{x}}$$

$$\boxed{P = A\underline{((A^T A)^{-1} A^T} b)}$$

in 1D ↳ $\boxed{\frac{a \cdot a^T}{a^T \cdot a}}$

n-dimensions

projection matrix

$$\boxed{P = A(A^T A)^{-1} A^T}$$

→ is not a square matrix

Hence A^{-1} ⇒ doesn't exist

$$A A^{-1} (A^T)^{-1} A^T = I$$

↳ if projecting on whole space //

↳ if projecting on subspace. /

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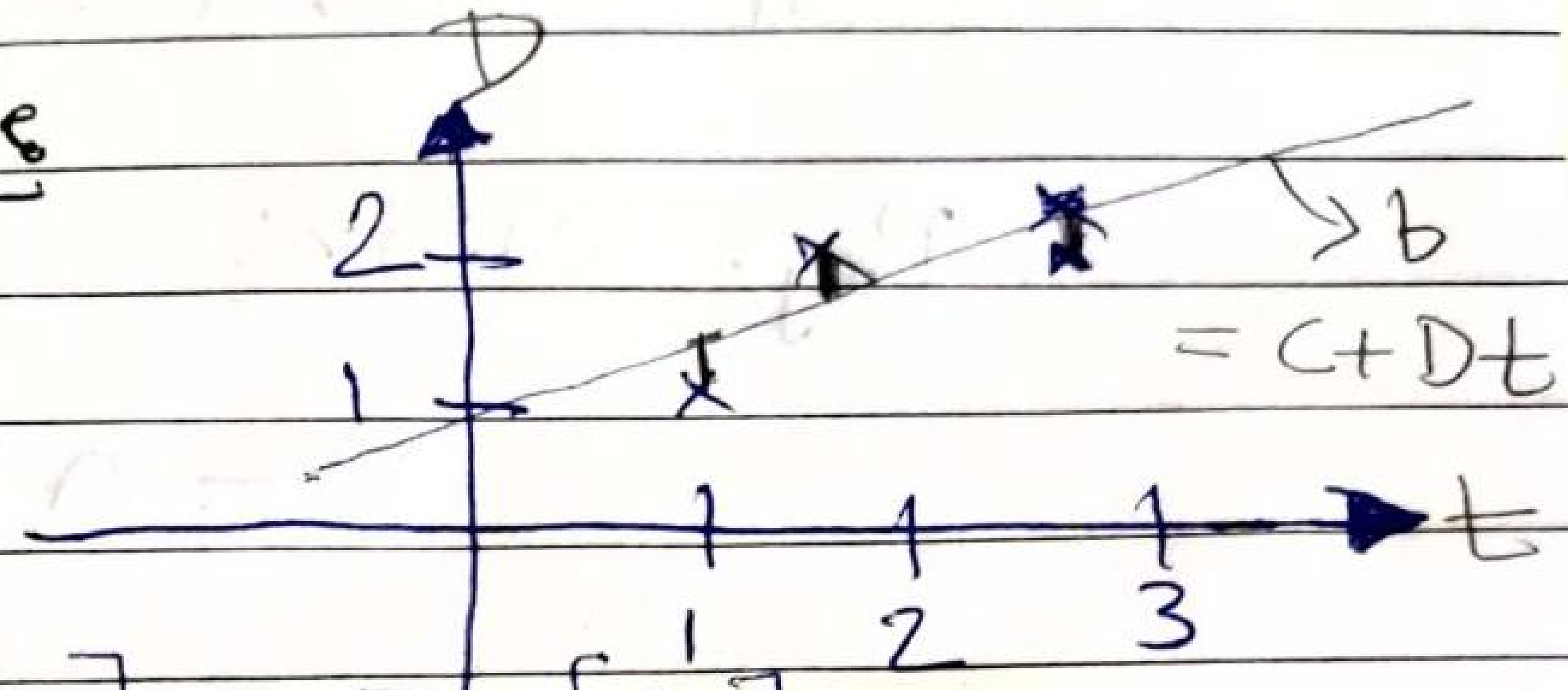
(*) $\boxed{P^T = P}$

(*) $\boxed{P^2 = P}$

Application: Least Squares

(*) fitting a line



$(1,1), (2,2), (3,2)$

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\boxed{A \quad x = b}$ → No solution.
still looking for best solution.//

$\boxed{A^T A \hat{x} = A^T b}$ → Can sol^n.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}_{3\times 2} \hat{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1} \begin{bmatrix} 1 & 1 \end{bmatrix}_{1\times 2} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}[1] + \begin{bmatrix} 1 \\ 2 \end{bmatrix}[2] + \begin{bmatrix} 1 \\ 3 \end{bmatrix}[2]$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3.34 & 7.34 \\ 30 & 66 \end{bmatrix}$$

$$\begin{bmatrix} 1\times1 + 1\times1 + 1\times1 & 1\times1 + 2\times1 + 3\times1 \\ \cancel{1\times1 + 4 + 9} & 1 + 4 + 9 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$3C + 6D = 1 + 2 + 3$
$6C + 14D = 11$

$$\boxed{\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_{2\times 1}}$$

$$\begin{bmatrix} 7/6 & 4 \\ -1/2 & 2 \end{bmatrix}_{2\times 2}$$

$$\begin{bmatrix} \cancel{\tfrac{35}{6}} \; 5/2 & \tfrac{77}{6} - \tfrac{11}{2} \\ 30 & 66 \end{bmatrix}$$

$$\begin{bmatrix} 5\left(7/6 + -1/2\right) + 11\left(7/6 - 1/2\right) \\ 5(4 + 2) - 11(4 + 2) \end{bmatrix}$$

A^{-1}

$$\Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$6a - 6 = 1$ $\quad a = 7/6$

$\cancel{6}a + 12c = 1 \quad \times 2$
$- \quad 6a + 14c = 0$
$-2c = 1$
$c = -1/2$

$3a + 6c = 1 \qquad 6a + 14c = 0$

$2\times \quad 3b + 6d = 0 \qquad 6b + 14d = 1$

$6b + 12d = 0$
$- \quad 6b + 14d = 1$
$+2d = +1 \quad (d = 2)$

$2\times\frac{4}{6}$ $\quad 4$

Lecture :- 16

→ Projections

→ Least squares & best straight line.

Proj. matrix :-

$$P = A(A^TA)^{-1}A^T \quad — (1)$$

If b in column space $Pb = b$

If $b \perp$ column space $Pb = 0$

means :- b is another space

Q:- What vectors are $\perp$r to the column space ⇒ ?

⇒ Those are the guys in the null space of A^T

$\boxed{A^Tb = 0}$

$$p = Pb = A(A^TA)^{-1}A^T \cdot b = 0$$

$$\boxed{p = Pb = 0}$$

(*) if b is in the column space : then b is a combination of A's column.

means $b = Ax$.

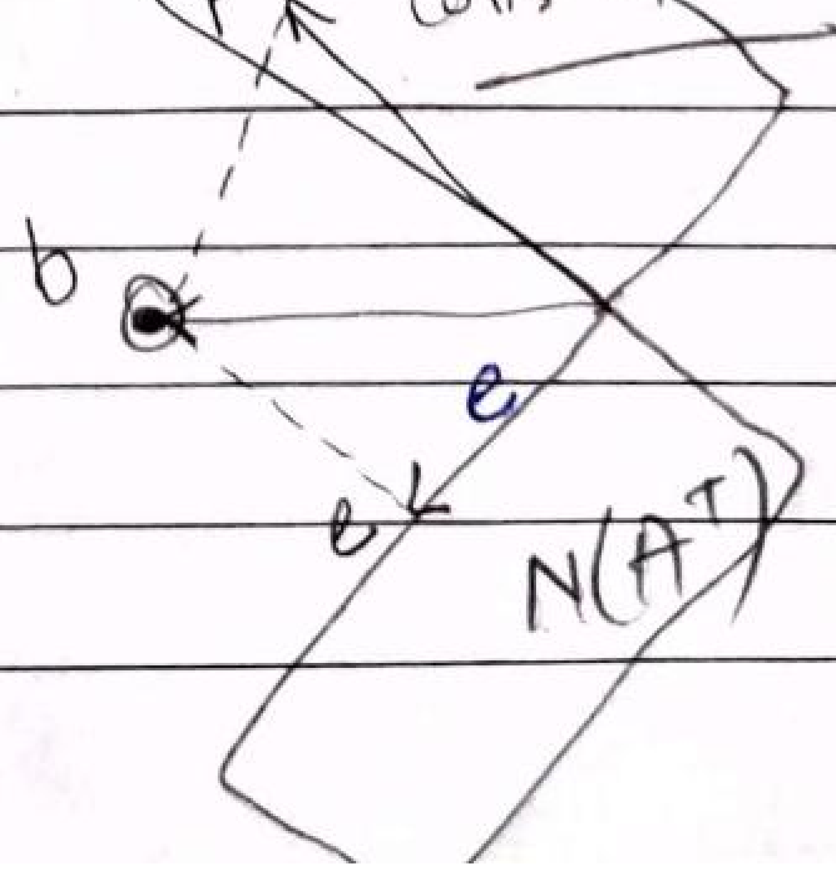
so (1)

$$p = Pb = A(\underbrace{A^TA)^{-1}A^T}_{\text{cancels}}Ax$$

$$= Ax$$

$$= b.$$

Here, projecting b to P.
& same time finding otherpart i.e. e



$$\boxed{p + e = b^{\text{original}}}$$

$p \to Pb$, $e \to (I-P)b$

$\nearrow Pb$

$p + e \rightarrow (I-P)b$

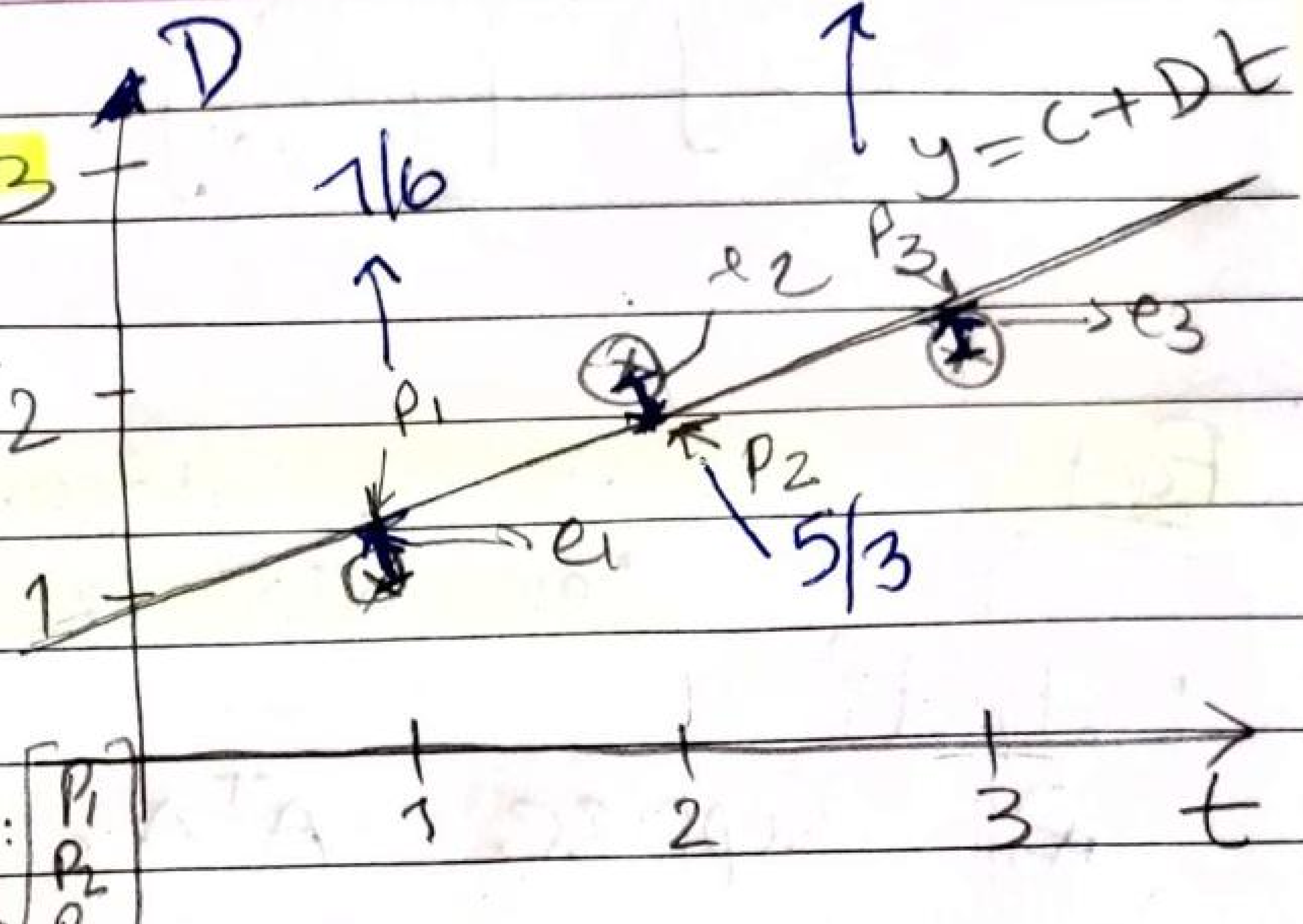
fitting a line (t) (D)

$(1,1), (2,2), (3,2)$ → predicted point

$C + D = 1$ ✓ actual

$C + 2D = 2$ ✓

Basis $C + 3D = 2$ ✓



$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

closest sol: $\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$

$Ax = b \rightarrow$ No Sol^n

$$\boxed{A^T \cdot A\hat{x} = A^T b}$$

minimize this

error $\Rightarrow \boxed{Ax - b = e}$

$\Rightarrow \boxed{\|Ax - b\|^2 = \|e\|^2}$

$\Rightarrow e_1^2 + e_2^2 + e_3^2$

$A^T \cdot A =$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

symmetric, invertible

$A^T b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\left.\begin{aligned} 3C + 6D &= 5 \\ 6C + 14D &= 11 \end{aligned}\right\} \rightarrow \text{normal equations.}$$

$\rightarrow \frac{\partial}{\partial C} = 0$

$\frac{\partial}{\partial D} = 0$

→ Let's find through minimization

$$= e_1^2 + e_2^2 + e_3^2 \Rightarrow (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$$

using calculus $\frac{\partial}{\partial C} = 0; \; \frac{\partial}{\partial D} = 0;$

#
$3C + 6D = 5$

$6C + 14D = 11$

$$\boxed{y = \frac{2}{3} + \frac{1}{2}t}$$

#
$e_1 = -P_1 + \text{actual} = -1/6$

$e_2 = +2/6$

$e_3 = -1/6$

$$e = \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

$b = p + e$

they are $\perp r$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} \xrightarrow{\perp r \text{ to}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

dot product →

Fact: If A has independent columns (To prove) then A^TA is invertible.

If not?

Suppose $A^TAx = 0$; (To prove) x must be $\{[0]\}$

→ A is invertible, when its null space is $\{[0]\}$ only the zero vector

TRICK (IDEA) multiply x^T both side

$$x^T A^T A x = 0$$

square

$$(Ax)^T Ax = 0 \quad (y^T y = 0) \to \text{means } y \text{ vector has to be zero.}$$

$$\Rightarrow Ax = 0$$

hypothesis: if A is independent column;

$$\boxed{x = 0} \checkmark$$

this tell

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

because these column are independent — has to be invertible