Lecture 31

- Change of Basics
- Compression of Images
- Tranefomation $\longleftrightarrow$ Matrix.

Standard basis -

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix};\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix};\begin{bmatrix}0\\1\\0\\0\\\vdots\\0\end{bmatrix}\cdots\begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix}$$

ith Pixel

512 $\times$ 512

$0 \leq x_i \leq 255$

$\boxed{8 \text{ bits}}$

Better basis -

$x \in R^n$

$n = (512)^2$

$$\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\\-1\\-1\\-1\end{bmatrix}\qquad\begin{bmatrix}1\\-1\\1\\-1\\1\end{bmatrix}$$

low fr

High freq

↳ why because vector with all one's by itself, one vector is able to completely give the information on a solid image.

→ image (half black & half white)

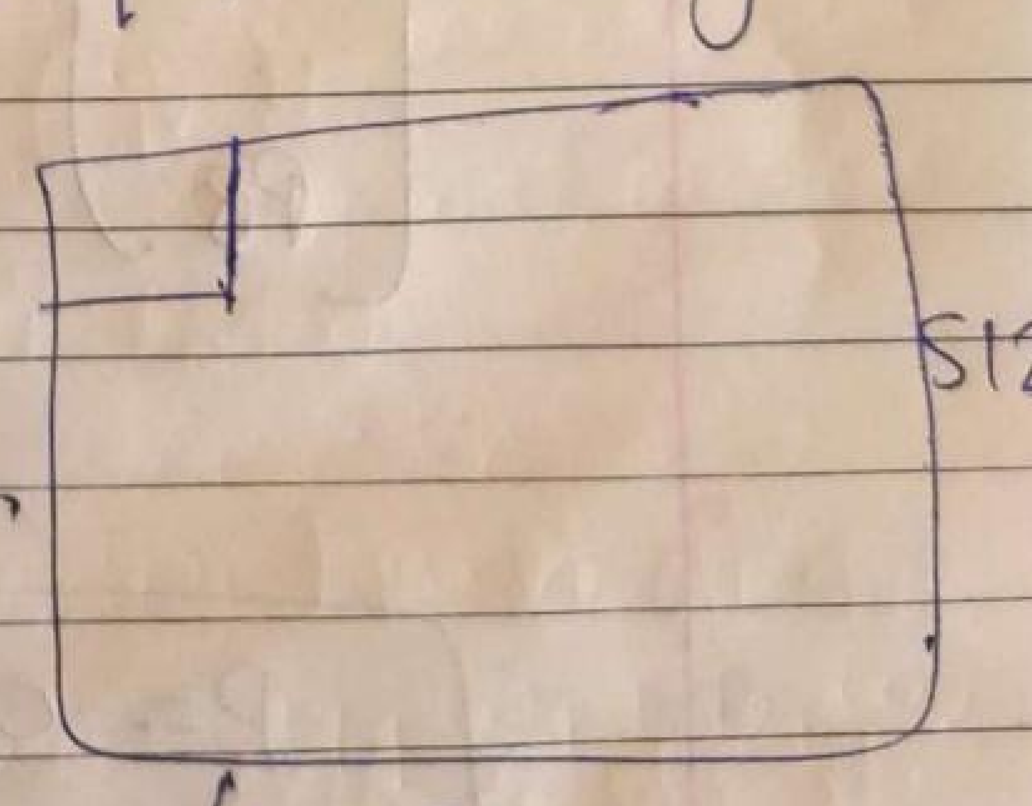
→ checkerboard vector
↓ ~~that~~
if the image was like a huge checkerboard of plus, minus. that vector would carry the whole signal.

Best known basis which JPEG uses -

fourier basis - (8x8)

1st way

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1\\w\\w^2\\w^3\\\vdots\\w^{n-1}\end{bmatrix}\cdots$$



512

512

↓ brake this image into (8x8)

Signal ~~x~~ (P) (64 pixels comes in)

lossless ↓ change basis

coff. c → 64 coeff. comes out

lossy ↓ compression

$\hat{C}$ (many zeros) ⟵ throws away small c (that's called thresholding.

reconstruct signal ↓

$\hat{x}$

$\hat{x} = \sum \hat{c}_i v_i$

Wavelets

2nd way

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix} \begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix} \begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix} \cdots$$

→ orthogonal basis/ not orthonormal/

↳ once convert to orthonormal

w^{-1} is w^T

so its a fast way to calculate w^{-1} in $c = w^{-1}p$

~~Pixel~~ Standard basis

$$\begin{bmatrix}P_1\\P_2\\\vdots\\P_8\end{bmatrix} = c_1 \overset{w_1}{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}} + c_2 \overset{w_2}{\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}} + \cdots$$

$$\boxed{P = c_1 w_1 + c_2 w_2 + \cdots c_8 w_8}$$

← transformation step.

$$P = \begin{bmatrix} | & | & & \\ | & | & W & \\ | & | & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_8 \end{bmatrix}$$

$$\boxed{P = Wc}$$

$$\boxed{c = W^{-1}P}$$

GOOD BASIS :- ① Fast FFT, FWT

~~by~~ these basis (fourier basis) ~~&~~ ~~using~~ FFT to ~~calculate~~ ~~F^{-1}~~ change basis.

② few basis vector should come close to the signal.

⇒ few vector to reproduce the image.

Change of basis :-

columns of W = new basis vectors.

$$[x] \longrightarrow [c] \qquad \boxed{x = Wc} \checkmark$$

x: In old basis; c: new basis

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linear transformation (T) with respect to v_1 to v_8. It has matrix A

with respect to $w_1 \ldots w_8$ it has matrix B.

If I have a same T & I'm compute on its matrix in one basis, and ~~&~~ then I compute it in another basis, those two matrices are similar meene :

SIMILAR : $\boxed{B = M^{-1} A M}$

M could be same as W.

⇒ What is A? Using basis $v_1 \dots v_8$, i/p

Know T completely from $T(v_1), T(v_2) \dots T(v_8)$ ← o/p

Because every $x = c_1 v_1 + c_2 v_2 + \dots + c_8 v_8$

then $\boxed{T(x) = c_1 T(v_1) + c_2 T(v_2) + \dots c_8 T(v_8)}$

⇒ write $T(v_1) = a_{11} v_1 + a_{21} v_2 + \dots a$

$T(v_2) = a_{12} v_1 + a_{22} v_2 + \dots$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & \\ | & | & \dots \\ a_{18} & a_{28} & \end{bmatrix}$$

Suppose basis are eigenvector basis.

$T(v_i) = \lambda_i v_i$

what is A?

$$A = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

perfect basis

1st i/p is v1 → col 1

its o/p is $\lambda_1 v_1$

2nd i/p → v2

o/p → $\lambda_2 v_2$

(but to find eigen vector in ~~image processing~~ pixel matrix is too expensive)

Problem :-

The vector space of all polynomials in x of degree $x \le 2$ has a basis $1, x, x^2$.

Let $\omega_1, \omega_2, \omega_3$ be a different basis of polynomials whose values at $x = -1, 0, 1$ are given by :-

x	ω_1	ω_2	ω_3
-1	1	0	0
0	0	1	0
1	1	0	1

y	1	x	x^2
6	1	-1	1
5	1	0	0
4	1	1	1

(a) Express $y(x) = -x+5$ in this basis!

(b) find the change of basis matrices $(1, x, x^2) \rightleftarrows (\omega_1, \omega_2, \omega_3)$

(c) find the matrix of "taking derivatives" in both bases!

(a) $y(x) = \alpha \cdot \omega_1(x) + \beta \cdot \omega_2(x) + \gamma \omega_3(x)$ / $x=-1$, $x=0$, $x=1$

$$\begin{cases} y(-1) = \alpha \cdot \omega_1(-1) + \beta \cdot \omega_2(-1) + \gamma \cdot \omega_3(-1) \\ \cdots(0) = \cdots \\ \cdots(1) = \cdots \end{cases}$$

(c) $$D_x = \begin{array}{cc} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

$$\boxed{D_\omega = A \cdot D \cdot A^{-1}}$$

$$= \begin{bmatrix} -3/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & -2 & 3/2 \end{bmatrix} \checkmark$$

$y = 6\omega_1 + 5\omega_2 + 4\omega_3$

(b) $1 = \omega_1 + \omega_2 + \omega_3$
$x = -\omega_1 + \omega_3$
$x^2 = \omega_1 + \omega_3$

$$A \begin{array}{cc} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ & \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \end{array}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

$(1, x, x^2) \longrightarrow (\omega_1, \omega_2, \omega_3)$

$(\omega_1, \omega_2, \omega_3) \rightleftarrows (1, x, x^2)$