Lecture :- 23

- Differential Equations $\frac{du}{dt} = Au$.

- Exponential e^{At} of a matrix.

example :- $\frac{du_1}{dt} = -u_1 + 2u_2$; $\frac{du_2}{dt} = u_1 - 2u_2$

$$\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

suppose at time '0' everything is in u_1

$\Rightarrow$ $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$; eigenvalues = ?

$\rightarrow |A| = 0$.

↳ $\because$ its a singular matrix. (2nd column is (-2) times (1st column)

so one of the eigenvalue is $\lambda = 0$

↳ from trace : other eigen value :- $\lambda = -3$;

$\Rightarrow$ $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix}$

$= -\lambda^2 + 3\lambda = 0$

$\lambda(\lambda + 3)$

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

$\rightarrow$ eigen vector :- $\lambda_1 = 0$; , $\lambda_2 = -3$

$Ax_1 = 0x_1$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $x_2 =$ subtract through the diagonal.

$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

$(A + 3I) \rightarrow$ should be singular

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

$(A - \lambda_k I) = 0$

$\boxed{Ax_2 = -3x_2}$

sol^n to diff. eq^n

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Solution:

$$\boxed{u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2}$$

two eigen value ; Hence two special sol^n

= Two pure exponential sol^n

Check: $\boxed{\frac{du}{dt} = Au}$; Plug in $e^{\lambda_1 t} x_1$

$\frac{d}{dt}$

~~$\lambda_1 e^{\lambda_1 t} x_1$~~

$$\lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda_1 t} x_1$$

$$= c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So how do we get c1 & c2 ?

$\hookrightarrow u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{Sc = u(0)}$

At $t=0$;

$$\Rightarrow c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution: $c_1 = 1/3$; $c_2 = 1/3$

$$\Rightarrow \boxed{\underbrace{\frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\text{steady state}} + \underbrace{\frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{if } t \text{ goes to } \infty \text{ disappears}}}$$

$$\boxed{u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

① So; when do we get stability?

$u(t) \to 0$

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⇒ negative eigenvalues

suppose eigenvalues are complex numbers;

we need all these $e^{\lambda t} \to 0$ & that ask us

to have lambda negative. ($\lambda < 0$);

suppose λ is complex number; then what's

the test?

ex: $\left| e^{(-3+bi)t} \right|$ → how big is it? $= e^{-3t}$

because $|e^{bit}| = 1$

then real part of λ has to be negative.

② Steady State: always in the same direction?

(when $\lambda_1 = 0$ & other eigenvalue - real part negative.

③ Blow up: if any Real part of $\lambda > 0$;

Suppose I change all the sign of matrix A? What it will do to eigen values & vectors? → change signs

2x2 stability: looking for $Re\lambda_1 < 0$, $Re\lambda_2 < 0$;

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ How can I tell by looking at matrix?

; trace $a+d = \lambda_1 + \lambda_2 < 0$ is it enough? to make stable.

cond ①

so now i looking for an example where trace is negative; but still it will blow up. ?

ex. $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 1, \lambda_2 = -2$

so e^{+t} & it will blow up if it has any second component at all.

I need another condition

$$\boxed{\det > 0} \text{ — cond-(2)}$$

$(\lambda_1 \lambda_2)$

$\frac{du}{dt} = Au$ → coupled matrix. → diagonalising → decoupling

Set $u = Sv$ → eigenvector matrix

$$S\frac{dv}{dt} = ASv$$

$$\frac{dv}{dt} = \underbrace{S^{-1}AS}_{\text{diagonal matrix}(\Lambda)} v$$

$$\boxed{\frac{dv}{dt} = \Lambda v}$$

so if I'm using the eigen vectors as my basis, then my system of equations is just diagonal.

→ there's no coupling anymore

$\frac{dv_1}{dt} = \lambda_1 v_1$ } system of eqn but not connected, so they are easy to solve

⋮

solution $v_1 = c_1 e^{\lambda_1 t}$

$$\boxed{v(t) = e^{\Lambda t} v(0)}$$

$$\boxed{u(t) = S e^{\Lambda t} S^{-1} u(0)}$$

$$= e^{At} u(0)$$

$$\boxed{e^{At} = S e^{\Lambda t} S^{-1}}$$

what it mean?

exponential of a matrix; why the formula correct?

Matrix exponentials

$$e^{At} = \frac{I}{1} + \frac{At}{1} + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots \frac{(At)^n}{n!} + \cdots$$

convergence by div 1 $n!$

Taylor series

similar thing will be there:

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$$

if t is small then $\boxed{I + At}$

$$|\lambda(At)| < 1$$

eigenvalue < 1.

$$\boxed{e^x = \sum_0^\infty \frac{x^n}{n!}}$$

another one

$$\frac{1}{1-x} = \sum_0^\infty x^n$$

geometric series

$$= I + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1}}{2} t^2 + \cdots$$

$\boxed{SS^{-1}}$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$\Lambda + At + \frac{\Lambda^2 t^2}{2} + \cdots$$

Here, assumption are :- A can be diagonalized;

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

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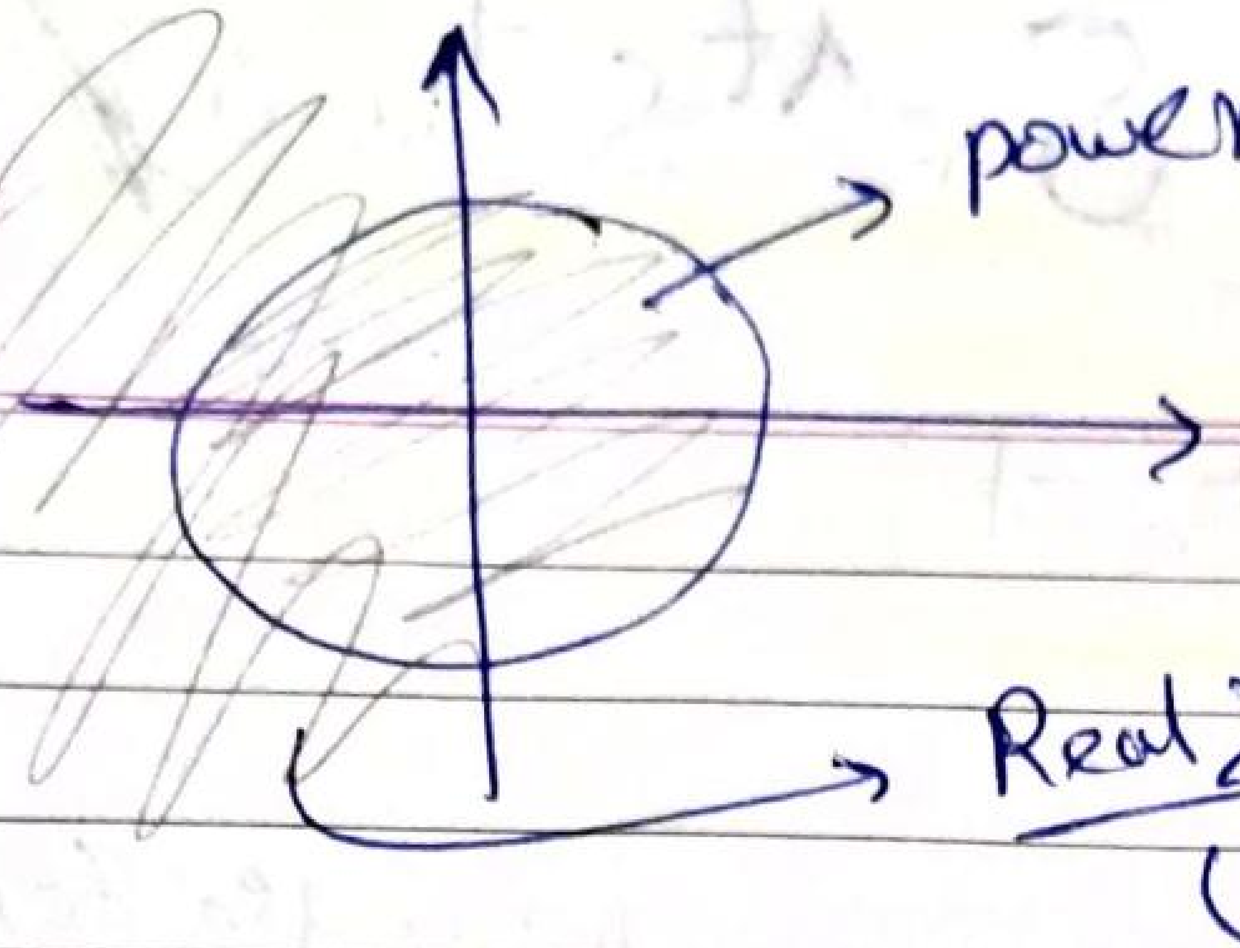

power of A goes to zero $|\lambda| < 1$

Real $\lambda < 0$

↳ ~~diff eq~~^{n}

expon of $A = 0$

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$y'' + by' + ky = 0$ (1 eq^n, 2nd order)

↓ change into

$u = \begin{bmatrix} y' \\ y \end{bmatrix}$ → vector unknown

A.u

→ (2×2) 1st order

$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \underline{\begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} y' \\ y \end{bmatrix}$ (just I did in fibonacci.

↳ for 5th order

coeff.

$$\begin{bmatrix} - & - & - & - \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Problem: App: Higher order linear diff eq^n

eq^n :- $y''' + 2y'' - y' - 2y = 0$

for the general solution. What is the matrix A?

find the first column of $\exp(A_t)$ -

$y''' = -2y'' + y' + 2y$

Sol:

$$\begin{bmatrix} y''' \\ y'' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y'' \\ y' \\ y \end{bmatrix}$$

$u'(t) = A \cdot u(t)$

→ transformed in 1st order of $u(t)$ diff eq^n

Although u(t) is vector, but if we can solve this equation for u, we have all the info we need for y.

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$\boxed{u'(t) = A\,u(t)}$ Solve this eq^n

→ we need λ & x's of A

$$\det(A-\lambda I) = \det\begin{pmatrix} -2-\lambda & 1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= (1-\lambda)(1+\lambda)(2+\lambda)$$

three roots

$\lambda_1 = 1;\ \lambda_2 = -1,\ \lambda_3 = -2;$

$\lambda_1 \to x_1$

$$(A-I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (0)$$

$$\begin{pmatrix} -3 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$\boxed{a = b = c}$

$x_1 = (1,1,1)^T$

$\lambda_2 \to x_2$

$x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_3 \to x_3 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}.$

→ general sol^n for u(t):

$$u(t) = c_1 e^{t} x_1 + c_2 e^{-t} x_2 + c_3 e^{-2t} x_3;$$

y?:- last item in $x_1, x_2, x_3 = 1$ so

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t}}$$

$$\exp(At) = S e^{\Lambda t} S^{-1}$$

$$S = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{t} & & 0 \\ & e^{-t} & \\ 0 & & e^{-2t} \end{bmatrix}$$

we want first column of result

$$\exp(At) = \underbrace{S e^{\Lambda t}}_{\text{multiplying column of } S \text{ by diagonal of } e^{\Lambda t}} S^{-1}$$

$$= \begin{bmatrix} e^{t}x_1 & e^{-t}x_2 & e^{-2t}x_3 \end{bmatrix} \underbrace{\begin{bmatrix} 1/6 & - & - \\ -1/2 & - & - \\ 1/2 & - & - \end{bmatrix}}_{\text{inverse}}$$

first column of $S^{-1} = ?$

$$= \frac{1}{\det(S)} C^{T} = \frac{1}{6} \begin{bmatrix} 1 & - & - \\ -3 & - & - \\ 2 & - & - \end{bmatrix}$$

cofactor of S_{11} S_{12}

S_{13}

$$\exp(At) = \begin{bmatrix} \frac{e^{t}}{6}x_1 - \frac{e^{-t}}{2}x_2 + \frac{e^{-2t}}{3}x_3 & \\ & \end{bmatrix}$$