Solving $Ax=b$: Row Reduced Form R.
Lecture : 8

- Complete solution of $\boxed{Ax=b}$
- Rank r
- $r=m$: Solution exists $r=n$: Solution is unique.

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

sum of row 1 + row 2.

- So, what elimination is going to discover about the right-hand sides.
- what's - there is a condition on b1, b2, b3 for this system to have a solution.

example: if I take 1, 5 and 17, there would not be a solution.

- I need $b1 + b2 = b3$
- Let's just see how elimination discovers that.

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccccc} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right]$$

pivot

pivot column

→ Elimination with the 1st column completed.

→ $\boxed{1}$, $\boxed{2}$ are pivot; Hence Col 1 & Col 2 are pivot column.

row3 - row2

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

→ Last row: $[0 = b_3 - b_2 - b_1]$ ✓

that's the condition for solvability.

- suppose :- $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$; ~~Bqsed~~ then

matrix will be $\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

so this 'b' is OK ⇒ allows the solution.

Solvability Condition on b :-

→ $Ax = b$ solvable if
when b is in $C(A)$ [column space of A].

Another way to answer:
→ If a combination of rows of A gives zero row; then the same combination of the entries of b must give ~~zero~~. 0.

Steps to find the solution for $Ax = b$:-

(a) $X_{particular\ solution}$: Set all free variable to zero. then solve $Ax=b$ for pivot variables

- free variables in matrix are
x_2 (column 2) & x_4 (column 4)

so: $x_2 = 0$; $x_4 = 0$;

~~pivot~~:

- so $x_1 + 2x_3 = 1 \Rightarrow x_1 = -2$

$2x_3 = 3 \Rightarrow x_3 = 3/2$

- $X_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$. one particular solution.
- we can plug into original system

- Now, I am looking for all solutions :-

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(b) Add on X; ~~out of~~ anything out of the null space

$X_{nullspace}$
then add (a) + (b)

- $X = X_p + X_n$

correct RHS

- $A x_p = b$
$A x_n = 0$ ← as the RHS is 0. Hence on adding it doesn't affect RHS. & you will get correct RHS (b)

$A(\underbrace{x_p}_{\text{one particular}} + \underbrace{x_n}_{\text{whole subspace}}) = b$

(c) $X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

free variable

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$x_2 = 2$, $x_4 = 0$

to make it zero; $x_1 = -2$, $x_3 = 1$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$x_2 = 0$, $x_4 = 1$

$x_1 + 0 + 2x_3 + 2 = 0$
↳ $x_1 = 2$; $x_3 = -2$

These are the special solutions.

Complete solution?
→ Multiple null space by $\boxed{C}$ ✓

- X_n ⇒ all combination of special solutions.

$$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

→ There were two special solution; because there were two special solution.

m by n matrix A of rank r:

current def of rank = # of pivot.

- What relation b/w r and m?

$$\underline{r \leq m},$$

- r and n?

$\boxed{r \leq n}$ ✓ because column can't have more than one pivot.

Full column rank: means $r = n$:

- What does that tell us about the null space?
- What does that tell us about complete solution?

→ means:
- → pivot in every column.
- → pivot variable $= n$
- → How many free variables are there? $= 0$
  - ↳ No free variable:

→ N(A) got only the zero vector,

$$N(A) = \begin{Bmatrix} \text{zero} \\ \text{vector} \end{Bmatrix}$$

Solution to $Ax = b$:

$\boxed{X_{complete} = X_p}$ → only one sol^n if it exists (unique sol^n)

solution.

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Matrix that has full column Rank :-

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

what rref = ?

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ — Independent

Full row rank means $\boxed{r = m}$:-

- pivot = m;
- every row has a pivot
- solvability ?

• Can solve $Ax = b$ for which RHS.
for every b. Exists ✓

• free variable ⇒ $(n-r)$ free variable
$\hookrightarrow (n-m)$

$$A = \begin{bmatrix} \boxed{1} & 2 & 6 & 5 \\ 3 & \boxed{1} & 1 & 1 \end{bmatrix}$$; rank = ? ⇒ $\boxed{2.}$

rref $R = \begin{bmatrix} 1 & 0 & F \\ 0 & 1 & \end{bmatrix}$

pivot position

$r = m = n$; (square matrix) Full Rank

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \Rightarrow \text{invertible matrix}$$

rref; $R = I$

N = zero vector.

what condition $b_1, b_2 \Rightarrow$ No ✓

$r = m = n$;	$n < m$, $r = n$	$r = m$, $m < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$
1 solution to $Ax = b$	(0 or 1) solution	$= [\cancel{I}\ F]$
		always a solution. (1 or ∞ solution)

$r < m$, $r < n$;

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

((0, or ∞ solution)

Note: The rank tells you everything about the number of solutions.

- all the information except the exact entries in the solutions.

Lecture : 9

Independence, Basis and Dimension.

→ Linear Independence
→ Spanning a Space
→ BASIS & dimension.

Suppose A is m by n with $m < n$.
Then there are nonzero solutions to $Ax = 0$

→ More unknown x-s than equation.

Reason: there will be free variable; atleast (1)

Independence:
When x_1, x_2, x_n are independent if

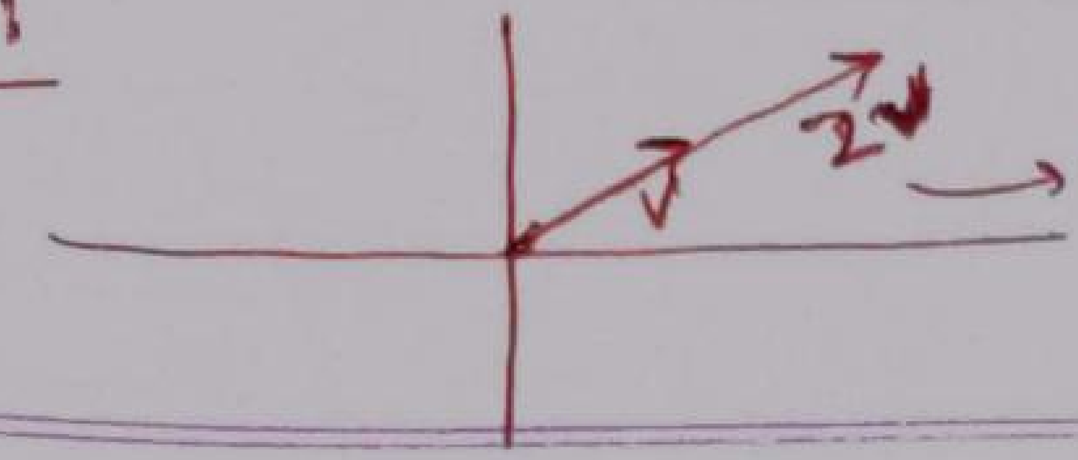
- Do any combinations gives zero?
  ⇒ if some combinations of those vectors gives zero vector, other than combination of all zeros, then they are $\boxed{\text{dependent}}$.

- if No combinations give the zero vector (except the zero ~~vector~~ combⁿ, all $c_i = 0$)
  ↳ $\boxed{\text{Independent}}$

$$c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n \neq 0$$

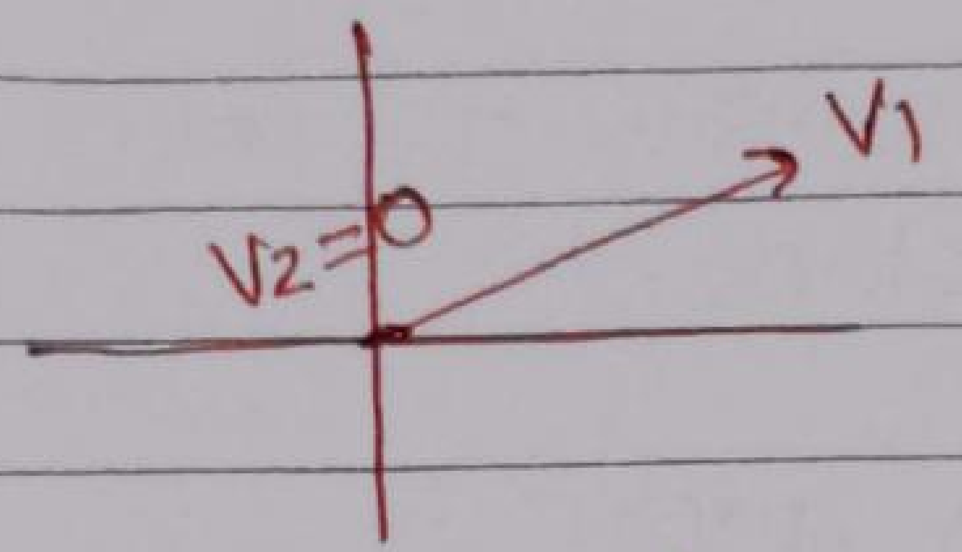
example:

(i)



→ dependent because $\boxed{2v_1 - v_2 = 0}$

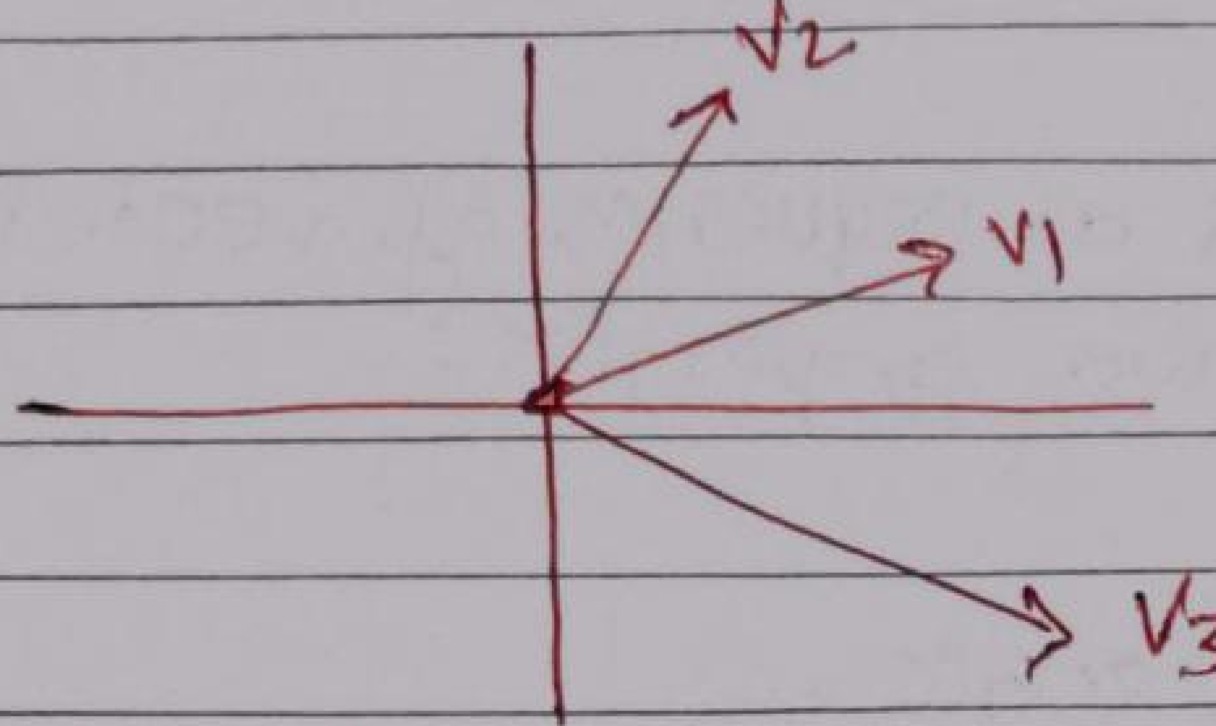
(ii)



⇒ dependent

$$0\,v_1 + 6\,v_2 = 0$$

(iii)



$n = 3$.

⇒

3-vector in a plane has to be dependent

$$A = \begin{array}{ccc} \left[\begin{matrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{matrix}\right] \\ v_1 \quad v_2 \quad v_3 \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Columns are dependent if there is something in the null space

Repeat when $v_1 \ldots v_n$ are columns of A.

(*) They are independent if null space of A is only {zero vector}

(*) Dependent: if something in null space.

$$\boxed{Ac = 0}$$

Free Columns tells that they are combination of other column

Independent means ⇒ all are pivot column ⇒ $\boxed{\text{rank} = n}$ ⇓ $N(A) = \{\text{zero vector}\}$

(*) Dependent ⇒ $\boxed{\text{rank} < n}$ ✓ ⇒ Yes free variable

Spanning a Space:

Vectors $v_1, --- v_l$ span a space means:- The space consists of all combinations of those vectors./

⇒ Columns of a matrix span column space.

Basis for a space is a sequence of vectors $v_1, v_2 --- v_d$ with two properties:-

(a) Independent

(b) they span the space./

ex:.

Space is $\mathbb{R}^3$

One basis is:- (list of vector) ⇒ ?

$$\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{Standard basis}} \quad \Bigg| \quad \underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}}$$

✓ (a) are they independent? YES

✗ (b) Do they span R^3?

→ No, because there are some vector in R^3 that are not combinations of those.

↳ $\begin{bmatrix}3\\3\\7\end{bmatrix}$ ⇒ we cann't take this, ∵ it are combination of $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\8\end{bmatrix}$ & it lies in same space./

so next $\begin{bmatrix}3\\3\\8\end{bmatrix}$

error Lecture 10: When I look at the matrix A that has those three columns: $\begin{bmatrix}1&2&3\\1&2&3\\2&5&8\end{bmatrix}$ → 2 rows are identical

& they are ~~dependent~~ hence column $\begin{bmatrix}3\\3\\8\end{bmatrix}$ ~~are~~ is dependent.

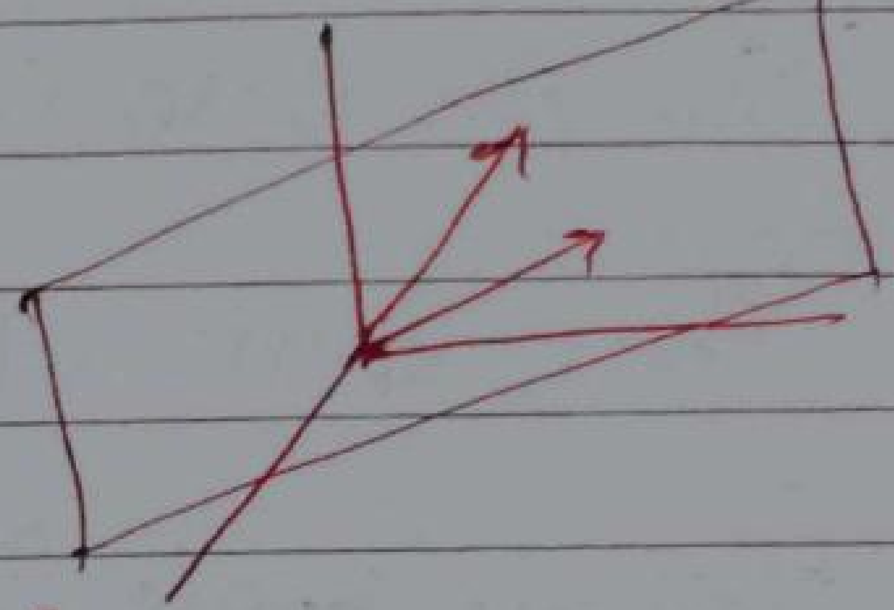
→ is invertible because 2 rows identical, → dependent

$R^n \Rightarrow$ n Vectors give a basis if the <u>nxn</u> matrix with those cols is $\boxed{\text{Invertible}}$.

$\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix} \Rightarrow$ Is there a space for which that's a basis?

~~Space~~ statisfy 1st Condition: (Independent)

Ans: The one that they span, (their combination)
It's a plane. ; inside $\mathbb{R}^3$



they're a basis for the plane, because they are independent,
if I stick in some third guy, like 3 3 7, which is in the plane
— suppose I put in, try to put in 3,3,7
then the three vectors would still span the plane.
But they wouldn't be a <u>basis</u> anymore because they're not independent anymore.

Given a space ($\mathbb{R}^3$, $\mathbb{R}^n$, NullSpace, ColumnSpace) :-
<u>Fact</u>: Every basis for the space has the <u>same number</u> of vectors.

$\mathbb{R}^3 \Rightarrow$ $\boxed{\text{3 vectors}}$ to have basis and so-on.

& that <u>number</u> is dimension (D) of space.

<u>ex</u>: Space is $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 1 \end{bmatrix}$$

Do they span the column space of that matrix?
Yes; By definition; that's what the column space.

- Are <u>they basis</u> for ~~colum~~ space? ; - Are they Independent?
↳ <u>No</u>;

⇒ Tell <u>me a vector</u> that's in the null space of that matrix

↳ So I'm <u>looking</u> for some vector that combines those columns & produces zero column.

or looking for $Ax=0$

$\Rightarrow$ vector in null space:- $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

means If i have $\longleftarrow$

$-v1 - v2 + v3 + 0.v4 = 0$

$-\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 0 =$

So; Are these vectors are independent? No

They span, but not independent

Tell me basis? for C(A):- $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

Column 1 & 2

so rank = ? $\Rightarrow$ 2

pivot

can't put in basis because Comb^n of col1 + col2

Rank(A) = # of pivot columns

rank of matrix

= dimension of column space, C(A)

not dimension of ~~matrix~~

it's dimension of space, subspace, the column space.

What's basis?

$\rightarrow$ First two columns

Another basis:

$\Rightarrow$ 1 & 3

$\rightarrow$ 2 & 3

$\rightarrow$ 2 & 4

another basis: $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$ sum of all col

Dimension of null space?

dimension of the column space is the rank

$$\boxed{\dim C(A) = r}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

free

these vectors in null space:- telling me combinations of columns that gives zero.

→ telling me in what way column are dependent

Are they basis of for null space?

→ In other words; does the null space consists of all combinations of those two guys?

→ Yes

→ Dimension of null space = # of free variable

$$= \boxed{n-r}$$ ✓