Lecture: # 25

Date / /
Page

* Symmetric matrices.
  Eigenvalues / Eigen vectors.

* Start :- Positive Definite Matrices.

→ $A = A^T$

(1) The eigen values are Real.

(2) The eigenvectors are Perpendicular.
can be chosen (incase of repeated eigen value)

S ↓ eigen vector matrix

Usual case $\boxed{A = S \Lambda S^{-1}}$

symmetric case $\boxed{A = Q \Lambda Q^{-1}}$

Orthonormal eigenvectors
↳ columns of Q.

Note. matrix with orthonormal columns has $\boxed{Q^{-1} = Q^T}$

$\boxed{A = Q \Lambda Q^T}$ ✓ for symmetric matrix.

(#) Why real eigenvalues?

$\boxed{Ax = \lambda x}$ $\xrightarrow{\text{always}}$ $\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

↓ transpose

$\cancel{\bar{x}^T A \bar{x} = \lambda \bar{x}^T x}$ $\bar{x}^T A^T = \bar{x}^T \bar{\lambda}$

A if its symmetric then $A^T = A$

Date / /
Page

✱ $(a-ib)(a+ib) = a^2+b^2$
conjugate

↳ imaginary part is gone./

\# Good matrices
Real λ's
Perpendicular x's

→ $A = A^T$ (if real)

if A is complex & $\boxed{A = \bar{A}^T}$ ← good matrix./
not just transpose, but conjugate it./

\# $A = A^T$ (Symmetric)

$$A = Q \Lambda Q^T$$

$$= \begin{bmatrix} | & | & | & | \\ q_1 & q_2 & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ - & q_n^T & - \end{bmatrix}$$

ortho eig-vector e-value

$$= \lambda_1 \underbrace{q_1 q_1^T}_{\text{Projection matrix}} + \lambda_2 q_2 q_2^T + \dots$$

Note: Every Symm matrix is a combination of mutually ⊥r projection matrices./ (Spectral theorem)

\# Now, eigenvalues are real for symm matrix.
What about the positive & negative e-values?

Symmetric

Signs of pivots are same as sign of λ's.

- no of +ve pivots = # positive λ's
- $\Rightarrow$ product of pivot = product of eigen values
  if no row exchange.

Positive definite matrix
(symmetric)

fact
- all e-values are positive
- all the pivots are positive

ex:

product of pivot = det(A)

$A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow$ pivots are, 5, ?

product = det(A)

$15 - 4 = 11$

so 2nd pivot = $5/11$

pivots = 5, $\frac{5}{11}$

$\begin{bmatrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} \Rightarrow \lambda^2 - 8\lambda + 11 = 0$

E-value :- $\lambda = 4 \pm \sqrt{5}$

fact: determinant of positive definite matrix
all sub determinants are positive.

Problem

Explain why each of the following is true 'ε'

ⓐ Every +ve definite matrix is invertible.

ⓑ The only positive definite projection matrix is $P = I$

ⓒ D is diagonal with +ve entries is +ve definite

ⓓ S symmetric with det $S > 0$ might not be +ve definite.

ⓐ if A is invertible $\Rightarrow \det A = \overbrace{\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n}^{\text{E. Value}}$

if A is +ve definite $\Rightarrow$

$$\boxed{\lambda_1, \lambda_2, \cdots \lambda_n > 0}$$

$\hookrightarrow$ so $\det A = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n > 0 \neq 0$

Hence A must be invertible.

ⓑ if P is projection : $\Rightarrow$ E. values of P is either 0 or 1.

- if P is +ve definite

$\Rightarrow$ E. values > 0;

Conclusion

eign value of P must all equal 1.

which matrix has eign value 1 & symmetric.

$\hookrightarrow$ is Identity matrix.

if P diagonalizable

$$P = SIS^{-1} \Rightarrow SS^{-1} = I$$

$$\boxed{P = I}$$

(c) $D = \text{diag}\,(\underbrace{d_1\ d_2 \dots d_n}_{\text{items along diagonal}})$

if D is +ve definite

show for any vector x; $x \neq 0$

$$\boxed{x^T D x > 0}$$

$x^T = (x_1\ x_2 \dots x_n)$

$x^T D x = d_1 x_1^2 + d_2 x_2^2 + \dots + d_n x_n^2$

> 0

(d) $S = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} \rightarrow$ -ve ; → not positive definite

$\det S = 6 - 1 = 5$

$x^T S x \rightarrow x = (1\ 0)^T$

$x^T S x = -3$

↳ ~~Hence cannot~~

since element is -ve; this shows me that ~~the~~ ~~direc~~ along the direction [1, 0], the product $x^T S x$ is also -ve.

Hence, not +ve definite.