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A^TA is positive definite !

SIMILAR MATRICES A, B / JORDAN FORM

$B = M^{-1}AM$

Positive definite means

$x^TAx > 0$ (except for $x=0$)

if A, B are pos. def.; A+B

↳ then $x^T(A+B)x > 0$ ✓ ∵ $x^TAx > 0$ ∵ $x^TBx > 0$

Now A m×n (recantugal matrix)

A^TA → square, symmetric

→ Is it pos. def. ?

↳ look for x^TA^TAx] arrang parenth

$\Rightarrow (Ax)^T(Ax)$

$\Rightarrow |Ax|^2 > 0$

$|Ax|^2$ ~~should~~ will not zero till vector is not zero.

so if matrix A has no null space & rank 'n'

independent column

Note

With a positive definite matrix, you never have to do row exchanges.

Similar Matrices :

A and B are similar.
(nxn) (nxn)

means for some matrix M

$$\boxed{B = M^{-1} A M}$$

Examples: $S^{-1} A S = \Lambda$ → A is similar to Λ.
↳ eigenvalue matrix.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{M^{-1}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$$ (B)

What is common in these matrixs?
eigenvalue. $\lambda = 3, 1$
for A & B, Λ

Facts: # Similar matrices have same eigen-values !!

if λ is an eigen value of A.

$\boxed{Ax = \lambda x}$ — multiply by M^{-1} both side

$(M^{-1} A M) M^{-1} x = \lambda M^{-1} x$

…means: Matrix B also has the same eigen values.

…en Vector of B is M^{-1} (eigen vector of A)

$Ax = \lambda x \Rightarrow \boxed{BM^{-1}x = \lambda \boxed{M^{-1}x}}$

$\hookrightarrow x \Rightarrow$ e-vector of A

$\hookrightarrow$ e-vector of B.

…en two eigenvalues are same.

… CASES $\lambda_1 = \lambda_2$ then matrix might not be diagonalizable.

suppose $\lambda_1 = \lambda_2 = 4$;

ONE family with eigenvalue 4,4 :: has …mes I

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

~~sim~~ Family means similar

BIG FAMILY INCLUDES $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$\hookrightarrow$ one eigenvalue, Nondiagonal

Jordan-form

Jordan - found best looking matrix in each family.

…ne members of family:

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}; \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix} \quad \begin{bmatrix} a & n \\ n & 8-a \end{bmatrix};$$

$\rightarrow$ Trace = 8 & det = 18

$\hookrightarrow$ not diagonalible

trace = 8

det = 16

#
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \lambda = 0, 0, 0 0,$

matrix has 2 independent column

$4 - 2 = 2 \rightarrow$ rank

so! 2 eigen vectors. | 2 missing e-vector

so & dim $N(A) = 2$

#
$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ has similar eigenvalue & vectors but not similar

difference is block, A has 3×3 & 1×1, B has 2×2 & 2×2

these blocks are called Jordan blocks

Jordan block

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ 0 & & \lambda_i & 1 \\ & & & \ddots \\ & & & \lambda_i & 1 \end{bmatrix}$$

$\rightarrow$ has 1 eigen vector

In A & B; blocks are of diff. sizes hence A & B are not similar.

Jordan Theorem

Every square A is similar to a Jordan matrix J;

$$J = \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \ddots & \\ & & & \boxed{J_D} \end{bmatrix}$$

# of blocks = # eigenvectors.

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Problems True or false ? explain

(a) If A & B are similar matrices, then

$2A^2+A-3I$ & $2B^3+B-3I$

are similar.

(b) If A & B are 3x3 matrices with eigen values 1, 0, -1 then A & B are similar.

(c) The matrices $J=\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}$ & $J\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$

are similar

Solutions

(a) True

know:- $MAM^{-1}=B$

$M(2A^2+A-3I)M^{-1}$

$=2(MAM^{-1}MAM^{-1}MAM^{-1})$

$+MAM^{-1}-3MIM^{-1}$

$=2B^3+B-3I$

(b) True!

Note:- matrix with distinct eigen values are diagonalizable.

$A=S\Lambda S^{-1}$ $\Lambda=\begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix}$ → eigen value matrice

$B=T\Lambda \cancel{S}^{-1}$

$\boxed{(TS^{-1})A(TS^{-1})^{-1}=B}$

(c) False

$J_1+I=\begin{pmatrix}0 & 1 & \\ & 0 & 1\\ & & 0\end{pmatrix}$ ← Null Space of J_1+I is 1D

↳ because 1-e-vect. with e-value -1.

$J_2+I=\begin{pmatrix}0 & 1 & \\ & 0 & \\ & & 0\end{pmatrix}$ ← 2D

because 2 e-vector with e-value -1