Lecture :- 21

- Eigenvalues – Eigenvectors.
- $\det[A - \lambda I] = 0$
- $TRACE = \lambda_1 + \lambda_2 + \cdots + \lambda n.$

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eigenvectors:

Ax

in goes x & outcomes vector Ax

& here i am interested in ~~those~~ they went in & come out in the same direction. x

But there are certain vectors where Ax comes out parallel to x. those are eigen vectors./

Sol:- Ax parallel to x./ (Eigen Vector)//

equation form

$$\Rightarrow \boxed{Ax = \lambda x}$$

(6, -6 or 0)

eigen value

⇒ What are the eigenvectors with eigenvalue zero?

⇒ there are in nullspace $Ax = 0$

⇒ If A is singular ; $\lambda = 0$ is eigen value./

⇒ What singular means: it takes some vector x into ze

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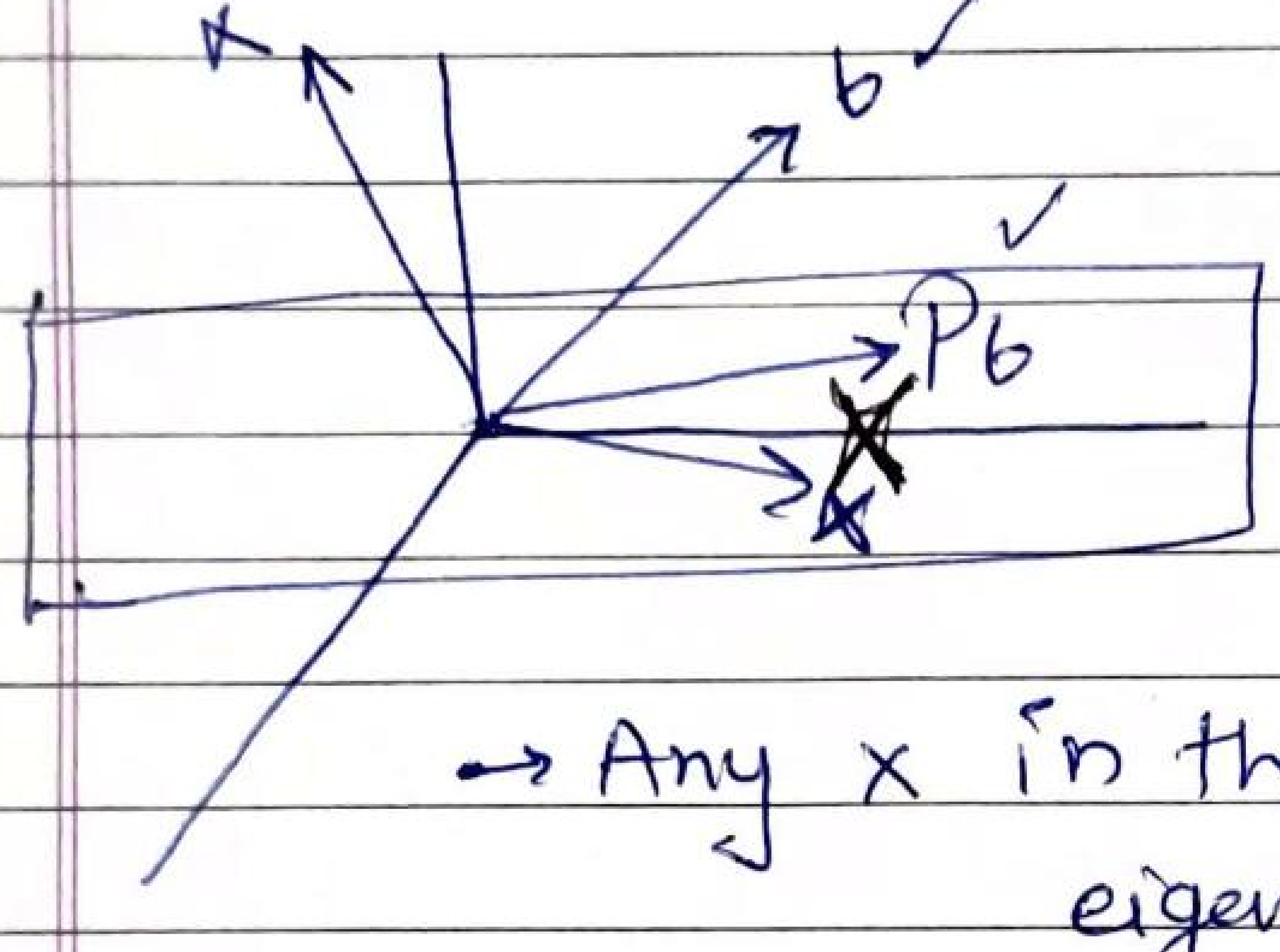
(#) What are x's & λ's for projection matrix?



→ Any x in the plane: $Px = X$
eigen value ⇒ $\lambda = 1$

(#) What is the right eigenvector that's not in the plane?
⇒ good one is ~~that of~~ one that's ⊥r to the plane

What's projection → $Px = 0x$; eigen value $\boxed{\lambda = 0}$

(#) What about permutation matrix?

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; tell me vector x?

What vector can I multiply & end up in the same direction? (eigenvector)

→ that's the matrix, that switches the component of x.

How could the vector with its $x_2\ x_1$ with permuted turn out to be a multiple of $x_1\ x_2$,

eigen Vector with eigen value (1) → so if I permute it doesn't change.

$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\lambda = 1}$

I want a vector, when I multiply by A, which reverses the two components. I want the thing to come out minus the original

2nd eigen vector with eigen value -1,

$X = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$; $Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\boxed{\lambda = -1}$ $(Ax = -X)$

Facts: eigen values — $n \times n$ matrices will have 'n' eigen values.

Facts: Sum of eigen value = sum of diagonal values of matrix A.
$(a_{11} + a_{22} + \cdots a_{33})$

How to solve $Ax = \lambda x$?

Rewrite: $\underbrace{(A - \lambda I)}_{\text{SINGULAR}} x = 0$

∴ like the matrix A with shifted by (lambda. I) that it has to be singular.

$\therefore \; |(A - \lambda I)| = 0$

1st job FIND λ FIRST (N- ~~diff~~ λ)

2nd Job Now find the null space.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \underset{\substack{\downarrow \\ \lambda \text{ along diagonal}}}{(A - \lambda I)} = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$

$-3I$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A'$$

$$= \lambda^2 - 6\lambda + 8$$

$(3+3)$ determinant

linear combinations

$$= \left.\begin{matrix} (\lambda - 4) \\ (\lambda - 2) \end{matrix}\right\} \text{factors}$$

Now
$\lambda_1 = 1 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = -1 \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenvalue $\lambda_1 = 4; \quad \lambda_2 = 2$

$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ → singular; ∴ x is in nullspace.

$Ax_1 = \lambda_1 x_1$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 4$

$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ - singular

$Ax_2 = \lambda_2 x_2$

$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \; \lambda_2 = 2$

eigenvalue of $A' = A - 3$ & vector will be same

$Ax = \lambda x$

then $(A+3I)x = \lambda x + 3x = (\lambda + 3)x$

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If

$Ax = \lambda x$; B has eigen value α_1

$Bx = \alpha x$ if $B = 3I$ but $By = \alpha y$

$+$

$\cancel{(A+B)x = (\lambda+\alpha)x}$

B is multiple of I, then $(A+B)x = (\lambda+\alpha)x$
but if B is some general matrix, then for
$A+B \Rightarrow$ you have to solve eigenvalue problem

Rotation matrix 6 (Q) (90° rotation)

Every vector by 90

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\rightarrow$ orthogonal matrix

$\rightarrow$ trace

find eigen values?

Trace $= 0+0 = \lambda_1 + \lambda_2$ $\rightarrow$ say $1, -1$ but
determinant $= 1 = \lambda_1 \lambda_2$ $\rightarrow$ product is -1

? What vector will come out parallel to itself after rotation?

but there's a way out

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 + 1 = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}}$$

Complex Number

If matrix is symmetric
or close to symmetric
their eign value will be real.

Anti-symm ($Q^T = -Q$)
because
when you fill across
diagonal, you change the sign.

(#) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow$ triangular

eigenvalue = 3, 3

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RANKA

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)$$

$\lambda_1 = 3 ; \lambda_2 = 3$ ##

eigenvector : $(A - \lambda I)x \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

independent eigen ~~vect~~ vector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; x_2 = [$

(shortage of independent eigen vector)

Linear Algebra Lecture :- 22

* Diagonalizing a matrix $S^{-1}AS = \Lambda$
* Powers of A / equation $u_{k+1} = Au_k$

$A - \lambda I$ Singular

$Ax = \lambda x$

(#) Suppose n linearly independent eigen vector of A ✓

Put them in columns of S ✓

$$AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = [Ax_1 \;\cdots\; Ax_n)$$

$\boxed{\because Ax = \lambda x}$

$$= \begin{bmatrix} \lambda_1 x_1 & \cdots\cdots & \lambda_n x_n \end{bmatrix} \xleftarrow{\text{so}}$$

factor it out

$$= \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & 0 \\ 0 & & \vdots \\ 0 & & \lambda_n \end{bmatrix}$$

diagonal eigenvalue matrix. (Λ)

$AS = S\Lambda$ ✓

$S^{-1}AS = \Lambda \leftarrow$ diagonilization ✓

$\boxed{A = S\Lambda S^{-1}}$ ✓

(#) eigen value & eigen V of A^2 ?

sol: If $Ax = \lambda x$ then multiply A both side

One way

$A^2 x = \lambda \underbrace{Ax}_{\lambda x}$

$\boxed{A^2 x = \lambda^2 x}$ ⇒ eigenvalue are squared

⇒ eigen V are same

Second way

$A^2 = S\Lambda \overbrace{S^{-1} S}^{I} \Lambda S^{-1}$

$= S\Lambda^2 S^{-1}$ telling me that S is the same (EV) but Eigenvalue are squared ✓

$\boxed{A^K = S\Lambda^K S^{-1}}$

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eigen vectors are same
eigen value are kth power.

\# If I take k^{th} power of a matrix,
the pivots are all over the place

examples: when does the powers of matrix go to zero?
⇒ Stable matrix.

Theorem: $A^K \to 0$ as $K \to \infty$
if all $|\lambda_i| < 1$ //

\# A is sure to have n independent vectors
(and be diagonalizable)
if all the λ's are different
(no repeated eigen value λ's)

\# Repeated possibility
↳ eigen values, // may or may not have
have n indep. vectors /

\# Suppose A is triangular: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

eigen values? $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)^2$

$\boxed{\lambda = 2, 2}$

eigen vectors

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Nullspace → 1D (Not enough EigenVec)
↳ $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → eigen vector

Equation: $u_{k+1} = A u_k$

Start with given vector $\boxed{u_0}$

$\Rightarrow u_1 = A u_0; \quad u_2 = A^2 u_1; \quad \boxed{u_k = A^k u_0}$ ✓

To really solve: write u_0 as a combination of eigenvector

$$u_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = Sc$$

$$u_{100} = A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \cdots + c_n \lambda_n^{100} x_n \ //$$

$$= \Lambda^{100} S c$$

Fibonacci example: 0, 1, 1, 2, 3, 5, 8, ...

$F_{100} = ?$

How fast are they growing; is lies in the eigen values?

$F_{k+2} = F_{k+1} + F_k$; (Rule) — (1) ← single eqn not a system/ & its 2nd order (like 2nd order diff. eq with second derivatives) I want to get first derivative.

TRICK $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ ← unknown

so i am going to get a two by two system, first order instead of a one (instead of scalar system second order).

Here, I want to get first difference/

$$\left.\begin{aligned} F_{k+1} &= F_{k+1} \quad (2) \\ F_{k+2} &= F_{k+1} + F_k \quad (1) \end{aligned}\right\} \rightarrow \text{system/}$$

What's my one step equation?

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k \qquad u_k \rightarrow \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

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$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

A — symmetric

eigen value → real, eigen vector → orthogonal

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} [F_{k+1}] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_k$$

$$= \begin{bmatrix} F_{k+1} \\ F_{k+1} \end{bmatrix} + \begin{bmatrix} F_k \\ 0 \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix}$$

Trace

$\lambda_1 + \lambda_2 = 1$ = sum(Trace)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2};$$

$$\boxed{\begin{aligned} \lambda_1 &= \tfrac{1}{2}(1+\sqrt{5}) \approx 1.6 \\ \lambda_2 &= \tfrac{1}{2}(1-\sqrt{5}) \approx -.618 \end{aligned}}$$

eigen value

Is the matrix diagonalizable :- Yes
because distinct eigen values

(How fast are those fibonacci numbers increasing?)

$$F_{100} \simeq c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

eigen vector

$x_1 =$

$x_2 =$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$u_k \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$u_0 = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}; = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_0$$

1st eigen vector, 2nd

Note: When things are evolving in time by first-order system starting from u_0; key is find e.vect & e.value of A

Problem: Find a formula for C^k where

$$C = \begin{pmatrix} 2b-a & a-b \\ 2b-2a & 2a-b \end{pmatrix}$$

Calculate C^{100} when $a = \cancel{b} = -1$

Sol:- eigen value & eigen vectors:

$$\det(C - \lambda I) = \det\begin{pmatrix} (2b-a)-\lambda & a-b \\ 2b-2a & (2a-b)-\lambda \end{pmatrix}$$

$$= \lambda^2 - (a+b)\lambda + ab$$

$$= (\lambda - a)(\lambda - b)$$

eigen value are a, b

eigen vector:

Null Space of $C - aI = \begin{pmatrix} 2b-2a & a-b \\ 2b-2a & a-b \end{pmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C - bI = \begin{pmatrix} b-a & a-b \\ 2b-2a & 2a-2b \end{pmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

-1 time of Colum 2

$C = S\Lambda S^{-1}$

$\Lambda \to$ matrix of eigen value

$S \to$ matrix of eigen vector

$$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$C^K = S\Lambda^K S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a^K & \\ & b^K \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} a^K & b^K \\ 2a^K & b^K \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2b^K - a^K & a^K - b^K \\ 2b^K - 2a^K & 2a^K - b^K \end{bmatrix}$$

Check

$K = 1$

80 $a = b = -1, \quad K = 100$

$$C^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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Probleme Given the invertible

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

find the eigenvalues & eigenvectors of A^2, $A^{-1} - I$.

Sol^n

$$Av = \lambda v$$

$$\begin{aligned} \cdot\ A^2 v &= A(Av) \\ &= A(\lambda v) \quad (\because \lambda \text{ is scalar} \\ &= \lambda(Av) \end{aligned}$$

$$\boxed{A^2 v = \lambda^2 v}$$

$$\cdot\ \underset{(\lambda \neq 0)}{A^{-1} v} = A^{-1} \cdot \frac{Av}{\lambda} = \underbrace{A^{-1}A}_{I} \frac{v}{\lambda} \Rightarrow \frac{1}{\lambda} v$$

$$\cdot\ (A^{-1} - I)v = (\lambda^{-1} - 1)v$$

$$\Rightarrow \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{bmatrix}$$

$$= (1-\lambda)(\lambda-2)(\lambda-3)$$

$$\boxed{\lambda = 1, 2, 3}$$

eigen vector $\quad (A - I)v = 0 = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} v$

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

	A	A^2	$A^{-1} - I$
e. value :	λ	λ^2	$\lambda^{-1} - 1$
vector	v	v	v