Linear Algebra

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CHAPTER-1.

1) 2 eqⁿ – 2 unknown:

$$2x - y = 0$$
$$-x + 2y = 3$$

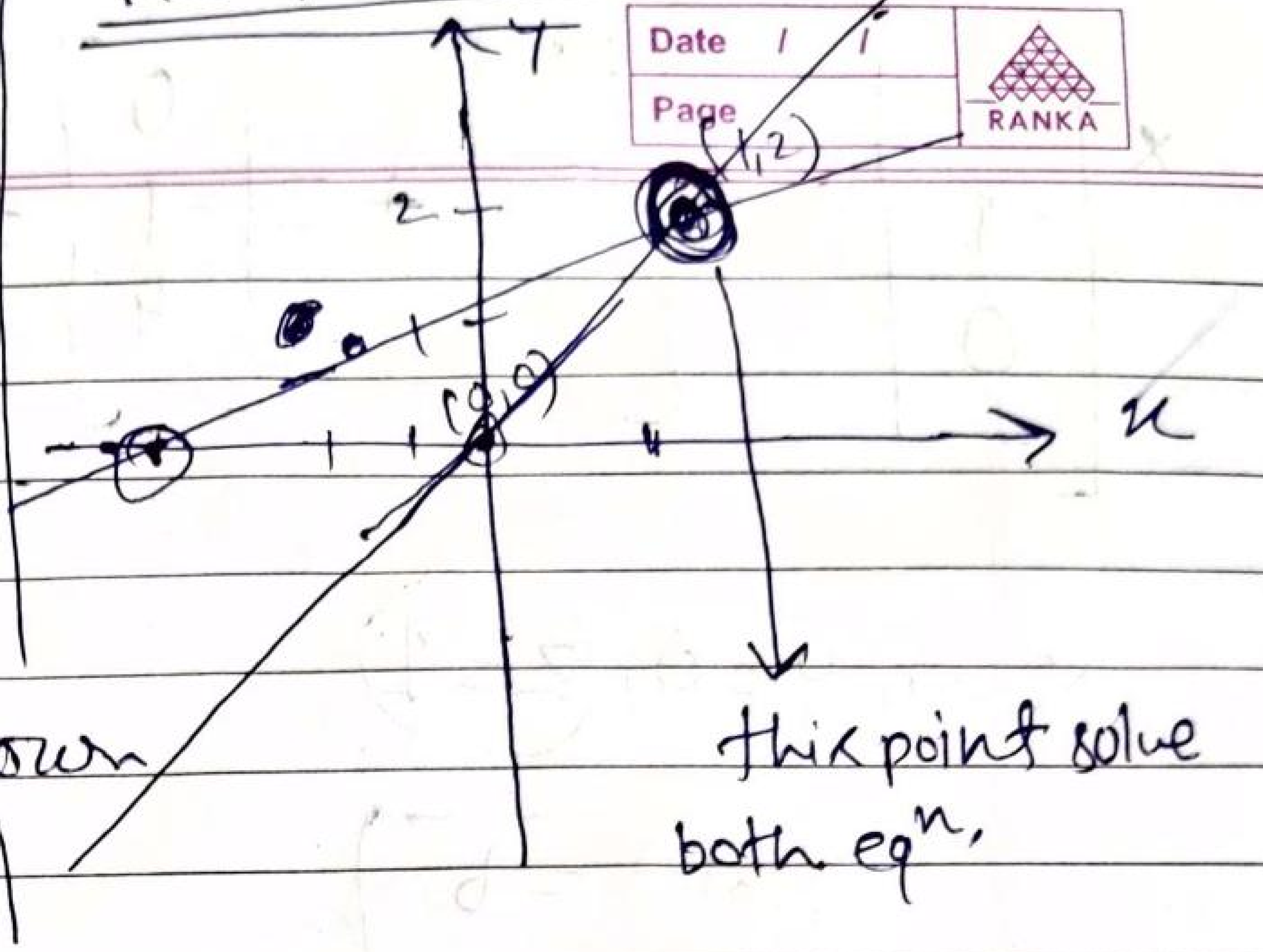
$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\cancel{X}} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{b}$$

A → coefficient matrix

X → vector of unknown

Linear eq: $AX = b$

Row picture



this point solve both eqⁿ.

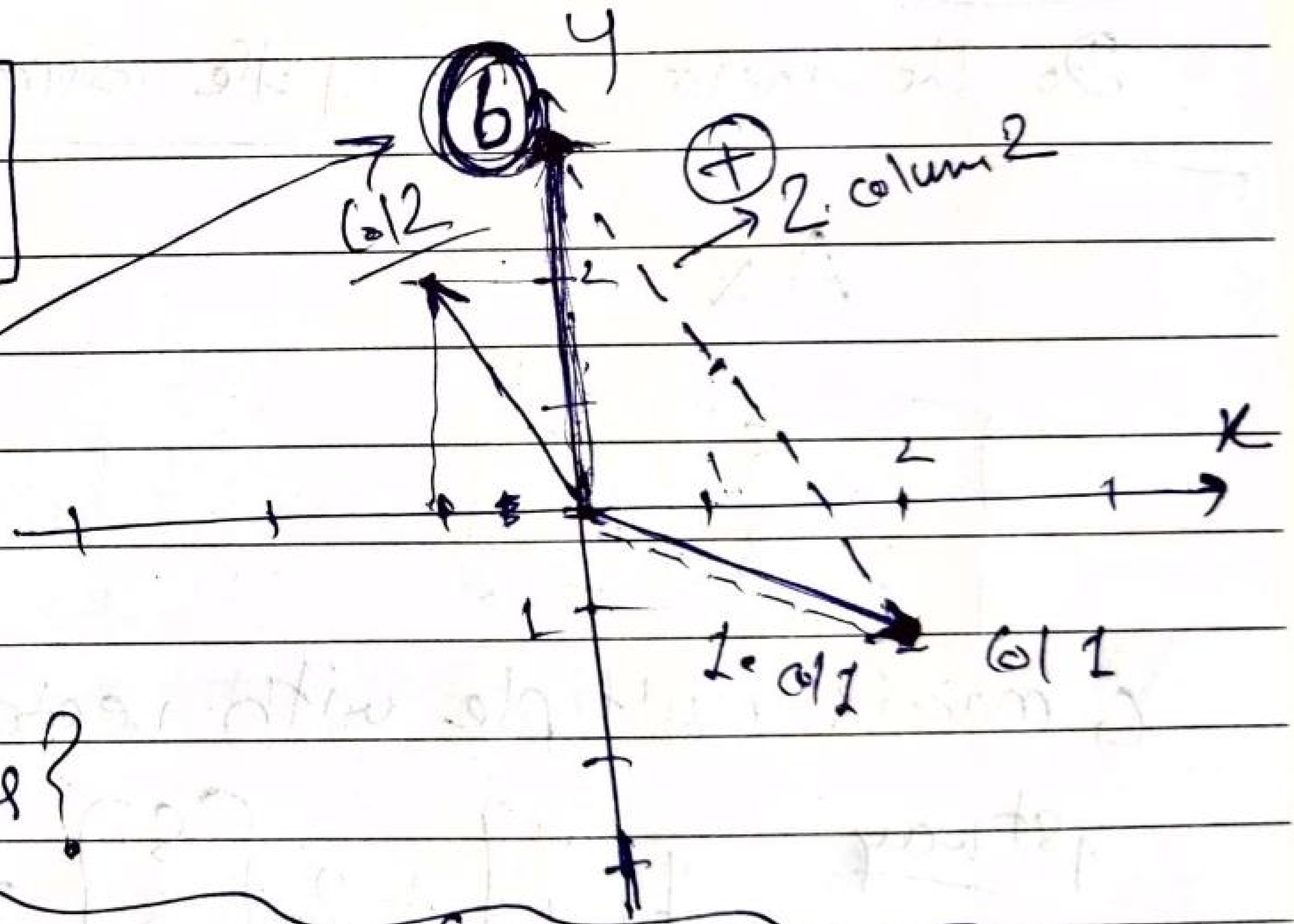
Column picture:

find linear combⁿ to find $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

right picture



What are the all combinations?

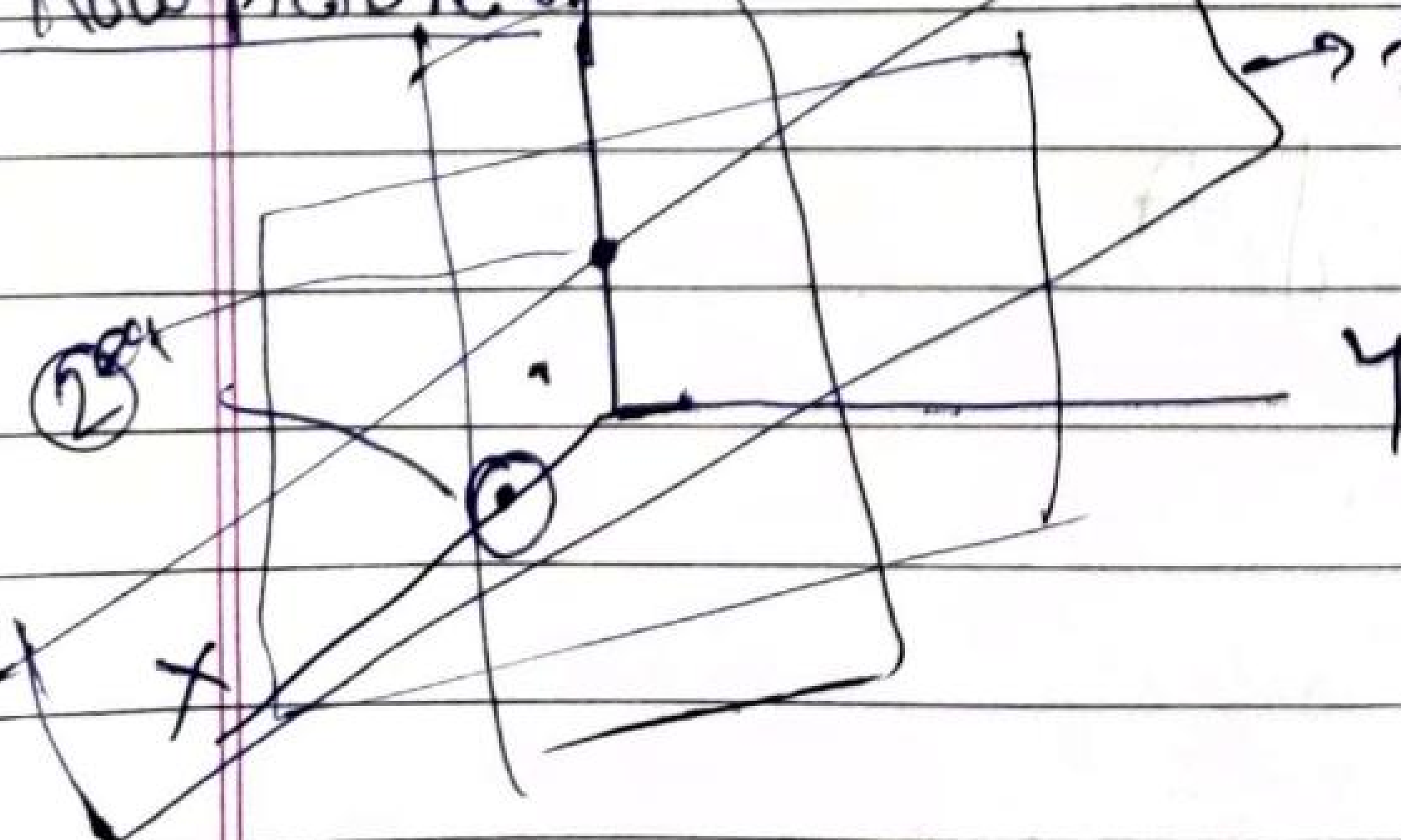
2) 3 eqⁿ – 3 unknown

$$2x - y = 0$$
$$-x + 2y - z = -1$$
$$-3y + 4z = 4$$

Matrix form

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

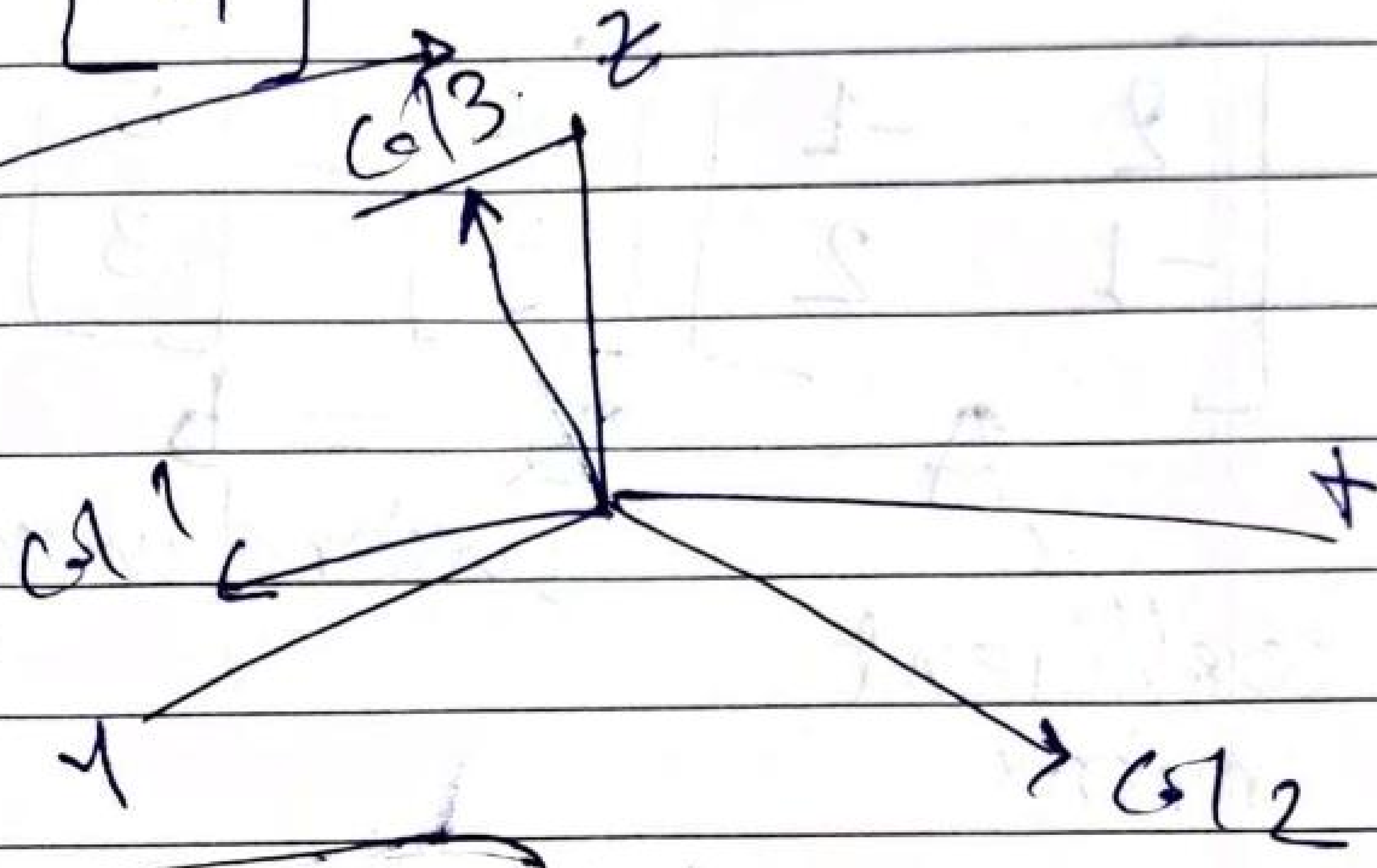
Row picture of



→ three planes meet a point is a solution.

column picture

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$



$x=0,\ y=0,\ z=1$

$\boxed{\text{col } 3 = b}$

Can I solve $Ax=b$ for every b?

Do the linear combⁿ of the columns fill 3-D space?

$$Ax = b$$

$$\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} \qquad \begin{bmatrix}2\\1\end{bmatrix}[1]+\begin{bmatrix}5\\3\end{bmatrix}[2]$$

↳ matrix multiple with vector.

1st way $\quad 1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}$

2nd way $\quad \begin{bmatrix}(2\times1+2\times5)\\(1\times1+2\times3)\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}$ Block method

Ax is a comb. of columns of A

Lecture #2

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

1st pivot

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{\boxed{A}} \xrightarrow[\substack{\text{wipe out} \\ (2,1)}]{\times 3 - (\text{i}) \quad (\text{ii}) - (\text{i})} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[(3,2)]{\times 2 \quad (\text{iii}) - (\text{ii})} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$

2nd pivot

pivot

$$\boxed{A} \rightarrow \cdots \rightarrow \cdots \rightarrow \boxed{U}$$

$\boxed{U}$

succeed

determinant = multiple the pivot ✓

Elimination

partial failure → $\boxed{2}$ has $\boxed{0}$ → position exchange → 0 at pivot position

complete failure — when $\boxed{5}$ has $\boxed{0}$ ✓

failure

if '0' is at the pivot position in last row(eq^n) ✓

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 0 & 4 & -4 \end{bmatrix} \Rightarrow$$ there wouldn't have been a third pivot //

The matrix would have not been invertible

(ii) Back-Substitution

$A \rightarrow U$

$B \rightarrow C$

$$\underbrace{\begin{matrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 2 \\ 12 \\ 2 \end{matrix}}_{b} \xrightarrow{(2,1)} \begin{matrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{matrix}$$

right hand side

lets call it vector (C)

$$\downarrow (3,2)$$

$$\underbrace{\begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}}_{U} \quad \underbrace{\begin{matrix} 2 \\ 6 \\ -10 \end{matrix}}_{C}$$

copy eq^n

$Ux = c$

$$x + 2y + z = 2 \Rightarrow x = 2$$
$$2y - 2z = 6 \Rightarrow y = 1 \quad — ② \text{ step}$$
$$5z = -10 \Rightarrow z = -2 \quad — ① \text{ solve this}$$

Matrics =
$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Subtract 3 time row1 from row2

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \leftarrow (2,3)$$

step ①

row1 not changing.

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ $-3 \times$ row1, $1 \times$ row2

→ row 3 not changing

E_{21} to fix (2,1) position

Note: multiplication

$$\begin{bmatrix} \; \end{bmatrix}_A \begin{bmatrix} \; \end{bmatrix}_x = \text{col1} \times \text{row1} + \text{col2} + \text{row2} + \ldots$$

$[\,|\,] \times [\text{row1}] + [\,|\,] + \text{row2}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 - 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1 + 3\times0 + 0, & 2 & 1 \\ -3+3 & -6+8 & \end{bmatrix}$$

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow 3 \times \text{col1} + 4 \times \text{col2} + 5 \times \text{col3}$$

matrix X column = column

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \Rightarrow 1 \times \text{row1} + 2 \times \text{row2} + 7 + \text{row3}$$

step ② Subtract 2 × row2 from row3

$$E_{32}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

no change

$E_{32}(E_{21}A) = U$ → move the parenthesis by associative law

$(E_{32}E_{21})A = U$

$(E_{32} \cdot E_{21})A = U$ ✓

Permutation

Exchange row1 & row2

$$\begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow P \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

exchange col1 & col2

$$\begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

NO

left side is only for row operation

column matrix

Hence $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

row operation multiplication

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

?

$0 \times (a+b) +$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Column operation multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

?

You can't exchange the order.

but move the parenthesis.

$$(E_{32} E_{21}) A = U$$

How to get $U \to A$

Inversese

$$\begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{add } 3 \times (\text{row}1)$$

$$\boxed{E^{-1} \cdot E = I}$$