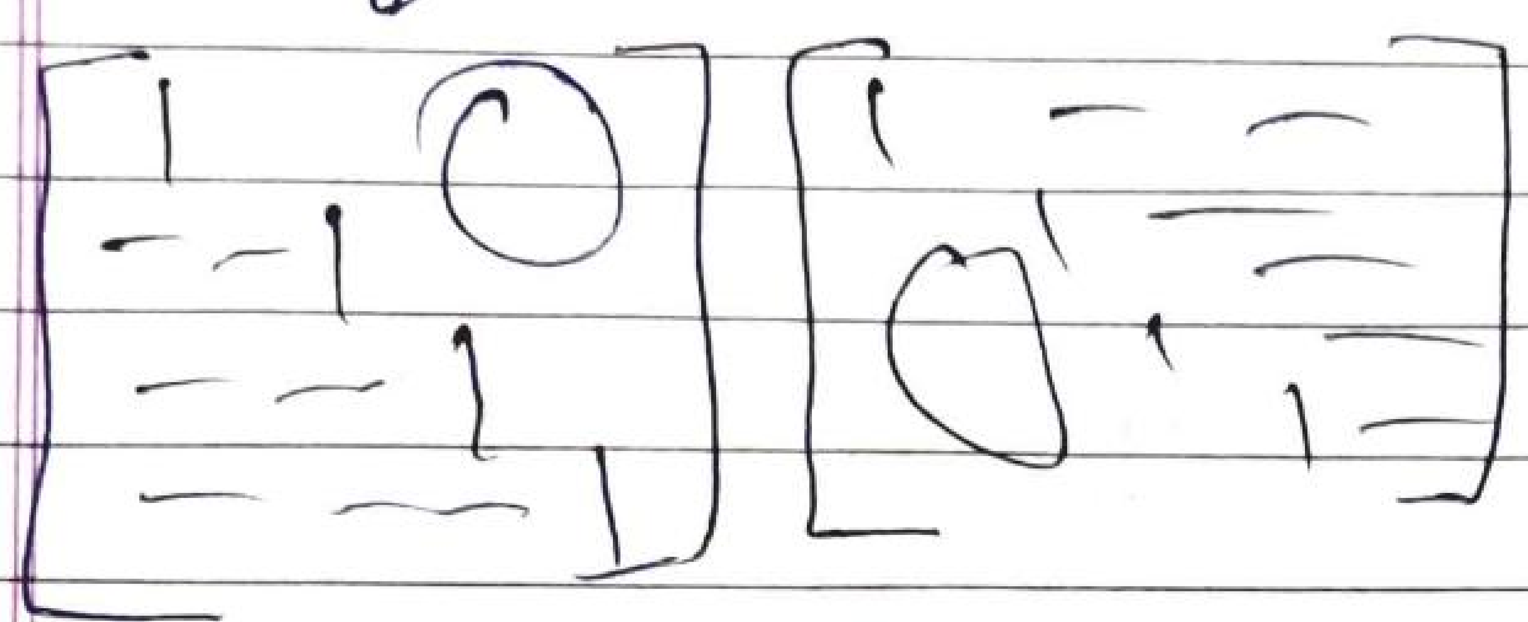
L5

Permutations P :- execute row exchanges.

$\boxed{A = LU}$ becomes $\boxed{PA = LU}$



P is the matrix which does the row exchange.

for any invertible A.

Todo

\# Elimination in matlab

P ⇒ only if A need row exchange

P = identity matrix with reordered rows.

how many possibility of P :- = $\boxed{n!}$

counts reorderings, counts all n×n permu

all these matrix are invertable, &

$$\boxed{P^{-1} = P^T}$$

$$\boxed{P^T P = I}$$

Ex

$(3\times 2) \rightarrow (2\times 3)$

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

R^T (rectangular) $\qquad R$

Transpose: $\boxed{(A^T)_{ij} = A_{ji}}$

what number in row i & column j in A^T

Symmetric matrices: $\boxed{A^T = A}$ exa: $\begin{bmatrix}3&1&7\\1&2&9\\7&9&4\end{bmatrix}$

(#) $R^T R$ is always symmetric

$$\begin{bmatrix}1&3\\2&3\\4&1\end{bmatrix}^T \begin{bmatrix}1&2&4\\3&3&1\end{bmatrix} = \begin{bmatrix}10&11&7\\11&-&-\\7&-&-\end{bmatrix}$$

WH? TAKE TRANSPOSE!

$(R^T R)^T =$ symmetric

$(R^T R^{TT}) = (R^T R)$ ✓

$R^{TT} \to R$

CHAPTER 3 VECTOR SPACES

Exa R^2 = all 2-dim-real-vectors $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}$

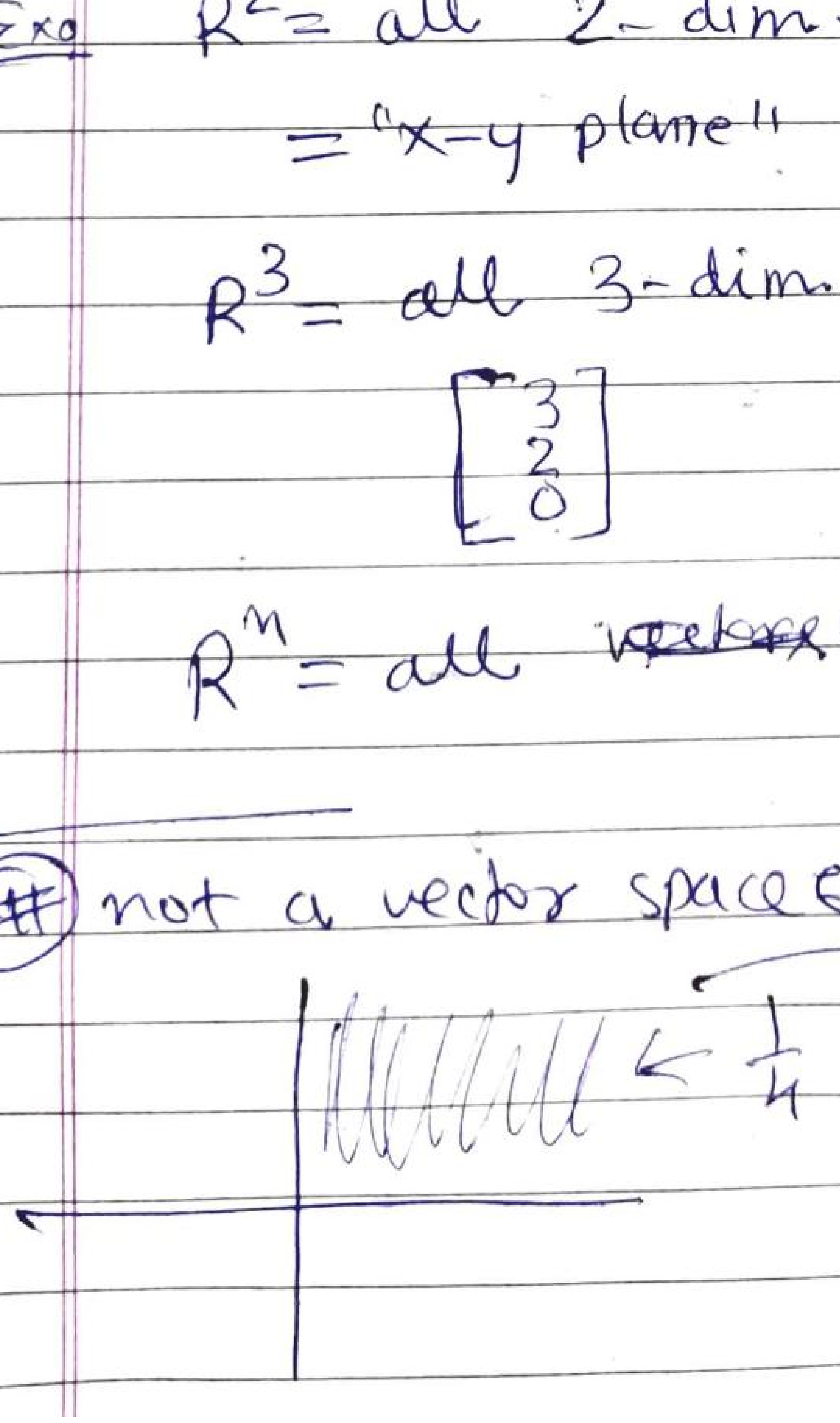
= "x-y plane"

R^3 = all 3-dim. vectors with 3-real components.

$\begin{bmatrix}3\\2\\0\end{bmatrix}$

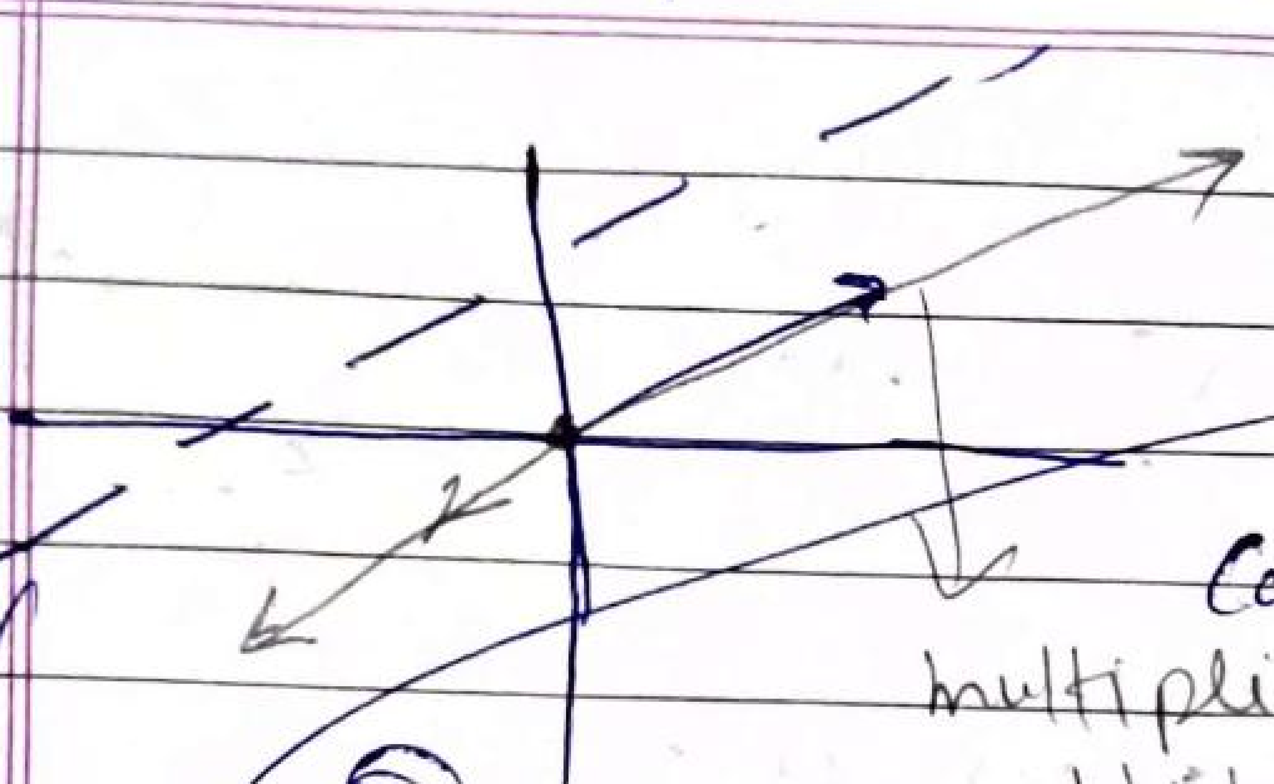
R^n = all ~~vectors~~ column vectors with n-real components.

(#) not a vector space:



Can I add these safely?

$\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}5\\6\end{bmatrix}$ ✓ in same quarter a vector space inside R^3?



SUBSPACE of R^2

Conditions to be a subspace//
multiplication result will be on line
~~addition also~~

$\circledast$ line in R^2 must go through zero vector

for this line multiple by 0. then its not in a subspace of (- -)

Subspaces of R^2

(i) all of R^2 → Plane

(ii) any line through $\begin{bmatrix}0\\0\end{bmatrix}$ → L

(iii) zero vector alone // (Z)

$$A=\begin{bmatrix}1 & 3\\ 2 & 3\\ 4 & 1\end{bmatrix}$$

$\times x \quad \times y$

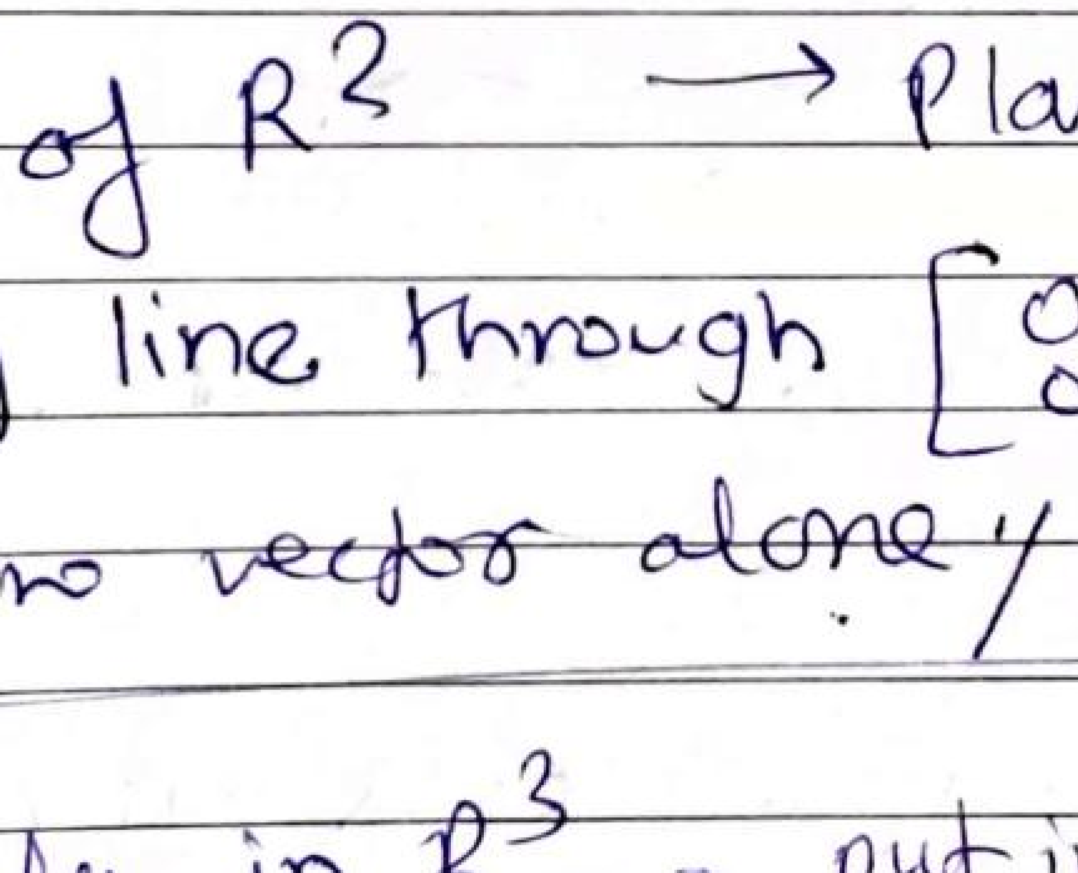
Cols in R^3 → put in subspace

all their linear comb^n form a subspace.

called column space // C(A) /

$\boxed{C(A)}$

How to draw whole column ?



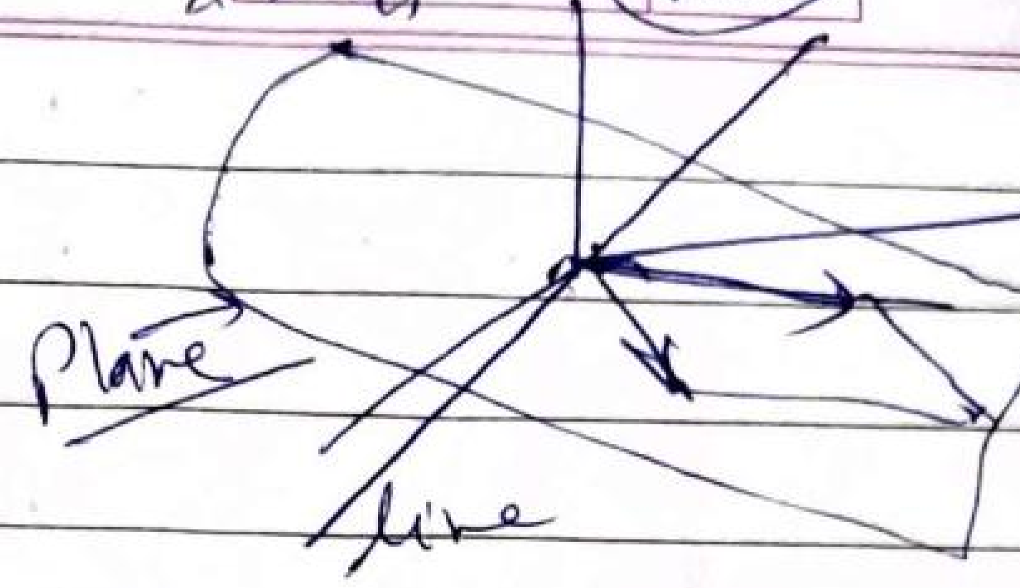
$\begin{bmatrix}1\\2\\4\end{bmatrix}$ $\begin{bmatrix}3\\3\\1\end{bmatrix}$ take all combinations

Plane through the origin/

Subspace cond^n ← ① multiplication (cv) – stay in space
② $v+w$ "
③ $\boxed{cv+dw}$ "

Lecture 6

Plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is subspace ✓
line is " "



---

2 subspaces P and L

$P \cup L$ (union) = all vectors in P or L or both
This ~~is~~ (is not) a subspace ✓ — take one vector from P, one vector from L, add, result will not be in space

$P \cap L$ = all vectors in both P & L.

$\boxed{\text{Subspaces } S \,\&\, T \text{ intersection } S \cap T \text{ is a subspace}}$ → v, w in intersection (both in S & T)
$(v+w)$ in intersection in ~~in~~
$(v+w)$ in intersect space

Column Space of $A_{(4\times3)}$ is subspace of R^4. $\boxed{C(A)}$ ✓

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} = \text{all linear Comb}^n \text{ of the columns.}$$

R_4

Does $Ax = b$ have a solution for every b? → No

$\boxed{4 \text{ eq}^n \text{ only 3 unknown}}$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cancel{x_4} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Which b's allow this system to be solved??

$\cancel{A}$ $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Date / /
Page

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

I can solve $Ax=b$ when b is a combination of the columns; when its' in the column space.

I can solve $Ax=b$ exactly when b is in $C(A)$.//
column space//

Can I throw away ~~some~~ any column & have the same space?
which one I can throw? (3) $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

what's so bad about three?
it's a sum of Col 1 & Col 2
Col 1 & Col 2 → pivot columns.

Could I thrown away Col 1? Yes/

2D subspace of R^4

Null Space of A:
= all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $\boxed{Ax=0}$

$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

R^4 R^3

N(A)

What's the null space for this?

One Sol^n = zero vector $x \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Another Sol^n: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \ldots \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$

or $\boxed{c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$ — c could be 0 → ∞

→ Nullspace is line in R^3 //

Check that solutions to $Ax=0$ always give a subspace/

If $Ax=0$ & $Ax'=0$;

$Av=0$ $\quad Aw=0$; then $\boxed{A(v+w)=0}$

that's say v in the nullspace, w in the nullspace.

nullspace//

$\boxed{Av+Aw}$ ← distributive law.

then $A(12v)=0$

because $12(\underline{Av})=0$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Date / /
Page

do solutions form vector space?

$\boxed{\text{No}}$

reason:- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not a solⁿ:-

so solⁿ cann't be a vector space./

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

it says $\boxed{-\text{Col }2 + \text{Col }3}$

Lecture 7

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$ col 2 multiple of col 1

rectangular matrix

Elimination:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

row 2 − 2× row 1

row 3 − 3× row 1

1st stage of Elimination

I got 1st Column, Now, move on to second column. I look at position (2,2) I see a zero, I look below it, hoping for non-zero that I can do a row-exchange. But it's zero below

that's telling me that, the column is dependent on earlier column (col 1)

Now I go on to another column.

1st pivot

2nd pivot

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\downarrow$ matrix U — echelon form = stair case form

echelon $\rightarrow$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row 3 − row 2

No. of pivot = 2 $\Rightarrow$ Rank of matrix.

Pivot column

free column. (C2, C4 any number).

Sol^n to $Ux = 0$ $\sim$ $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

$x_4 = 0$

so $x_3 = 0$

$x_1 = -2$

$x_2 = 1$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

bring 1 at pivot position

R = Reduced Row echelon form.

zeros (above + below) pivots.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow (I) - (II)$$

pivot = 1

I can get zero above the pivot

I can divide eq (2) by the pivot (2)

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \rightarrow \text{rref}$$

reduced row echelon form

Note (2,2) Identity matrix sitting at pivot row & pivot column.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $Ax = 0$, $Ux = 0$, $Rx = 0$

Scanned by TapScanner

Date / /
Page

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{pivot col}} \quad \underbrace{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}}_{\text{free cols}} \rightarrow \text{free part} = F$$

I

$$\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}$$

rref form $\Rightarrow$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot rows}$$

I: r pivot cols

F - free cols

$$RN = 0$$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

nullspace matrix Columns = Special solutions

$$Rx = 0$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$$\boxed{X_{pivot} = -F X_{free}}$$

Another examples

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

col 3: sum of col 1 + col 2 → so only 2 pivot

col 1, col 2: pivot

col 3: dependent of col 1 & col 2 (free col)

$$R = ? \quad \downarrow \quad \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \begin{matrix} \text{row 2} - 2\,\text{row 1} \\ \text{row 3} - 2 \times r1 \\ \text{row 4} - 2 \times r1 \end{matrix}$$

do row exchange

Now perfect pivot

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad row4 - 2r2$$

form $\boxed{U}$.

Pivot free col.

$\boxed{rank = 2}$ ✓

How many special solⁿ for this matrix?

No. of pivot columns for A & A^T are same.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0$$

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

whole null space?

multiple by C

$$x = C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

it say $-1(\text{col }1) - 1(\text{col }2) + 1(\text{col }3)$ [A]
$= \text{Zero}()$
that's in the Null space.

One is to keep going to reduced matrix, R

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R \Leftarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I, F

$r_1 - r_2$

I want pivot to 1 so divide by 2

$$X = \begin{bmatrix} -F \\ I \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

N ⇒ space matrix.

Our null space matrix is the guy whose columns are the special solutions so their free variables have the special values (1) & pivot variable has $-F$.