Lecture-13

Review for Exam 1

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① u, v, w non-zero vector in R^7. What are possible dimension span by u, v, w ?
ⓐ 1 ⓑ 2 ⓒ 3

② $5\times3 \rightarrow u$ (echelon form) ; 3 pivot
(i) NullSpace(u)

Sol:

$$\begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix}$$

rank = 3.
- 3 column independent
- No combination of column will be zero vector except zero vector.

$$N(u) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(ii) ~~10×3~~ $B = \begin{bmatrix} u \\ 2u \end{bmatrix}$ $\boxed{10\times3}$; what is the echelon form ? rank ?

$$B = \begin{bmatrix} u \\ 2u \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} u \\ 0 \end{bmatrix}$$ rank = 3 ✓

(iii) $C = \begin{bmatrix} u & u \\ u & 0 \end{bmatrix}$; echelon form ? $\boxed{10\times6}$

$$C = \begin{bmatrix} u & u \\ u & 0 \end{bmatrix} \xrightarrow{row2-row1} \begin{bmatrix} u & u \\ 0 & -u \end{bmatrix} \xrightarrow{row1-row2} \begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}$$

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \xleftarrow{+1 row2} \begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}$$

Rank(C) ⟶ 6

given u has rank 3:-

(iv) $N(C^T)$:- $= m - r$
dimension
6×10
$= 10 - 6 = 4$ ✓

(3) Eqn :- $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$; (3×1) complete Soln $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ //
(3×1)

dim (Row Space (A)) = ?

Soln :- size of (A) = $\boxed{3\times3}$ ✓

rank = 1 //

$m \times n$

dim N(A) = two vectors & are independent
= 2.

$m - r = 2$
$3 - r = 2$
$r = -2 + 3$
$= 1$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

the matrix that has $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in its null space

tells me that last column of the matrix is zero.

this is in null space. So what's the second column?

means $A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

means take 1st column of A + 2nd column of A = 0

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$A\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & - \\ 2 & - \\ 1 & - \end{bmatrix}$

- 2 of 1 column = $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ ✓

(11) $Ax=b$ can be solved if b has the form

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad \boxed{b = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

Don't forget other cases; $r=m$; $r=n$;

(4) $n \times n \rightarrow$ square

independent cols. $\rightarrow$ means = rank = n

$Ax=b$ always Solvable? YES ✓

(5) $$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{m \times n} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{n \times p}$$

F

$m \times p$

(a) basis for NullSpace? ($Bx=0$) $\underline{3 \times 4}$

$B = 3 \times 4$

$N(B) \subseteq R^4$

~~Is it invertible → YES.~~

~~$\hookrightarrow N(CD) = N(D)$~~ ✓

~~if C is invertible~~

(b) basis for $N(B)$:— looking for 2 special solution

have **2** pivot Column

will comes from column 3 & 4.

to get zero $\leftarrow \left\{ \right.$ $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\}$ ← free

free variable →

$\boxed{\begin{matrix} -F \\ I \end{matrix}}$ ✓

Invertible Matrix:

Note: ~~Do~~

- Do Gaussian Elimination. Then if you are left with a matrix with all zeros in (a) row, your matrix is not <u>invertible</u>

(b) Solve ~~Bx~~ Complete Solution6 — Particular / Null part

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$$Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

$$X_p + X_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \underbrace{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} + d \begin{bmatrix} \; \\ \; \\ \; \\ \; \end{bmatrix}}_{\text{Null Space}}$$

1st column matched with (B) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

any soln can be particular soln

(6) Do the matrices A and −A have the same column space, same null space, same row space.

⇒ YES

(7) If A & B have the same four subspaces; then A is a multiple of B. ? $A = cB$

⇒ false

~~ex:~~ A, B any invertible 6X6

what could you say ~~wh~~ about these matrices?

(8) :- If I exchange two rows of A which subspaces stay the same?

⇒ row space does stay the same & null space stay the space.

(9) Why can vector one, two, three not be a row and also in the null space?

$V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be in nullspace & ~~row space~~. Why not? be a row of A

Why $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ why is it impossible?

$$\begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

because this will be 14

Intersection of null space & ~~[illegible]~~ row space is only the zero vector.

ie. null space ⊥r row space.

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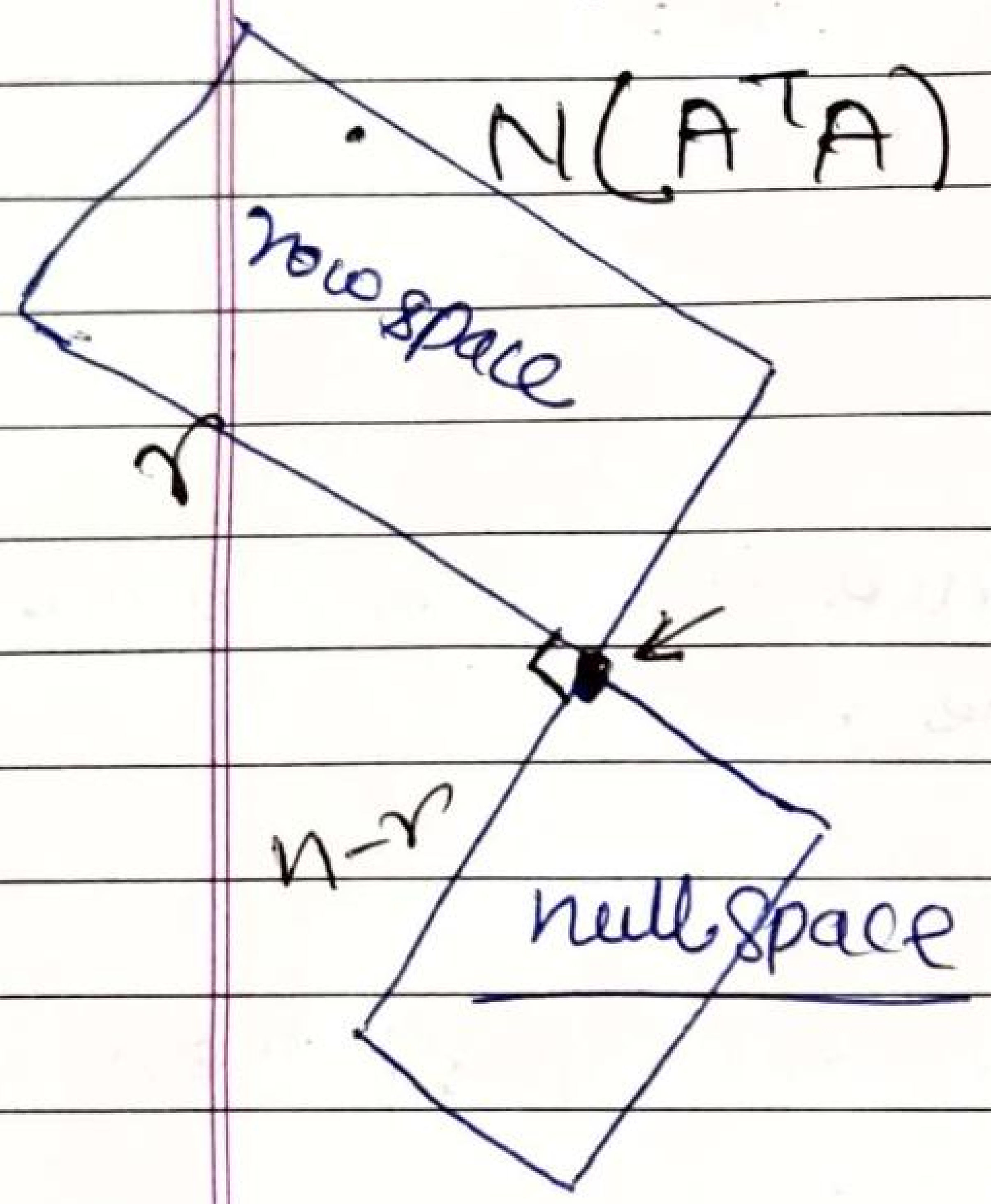
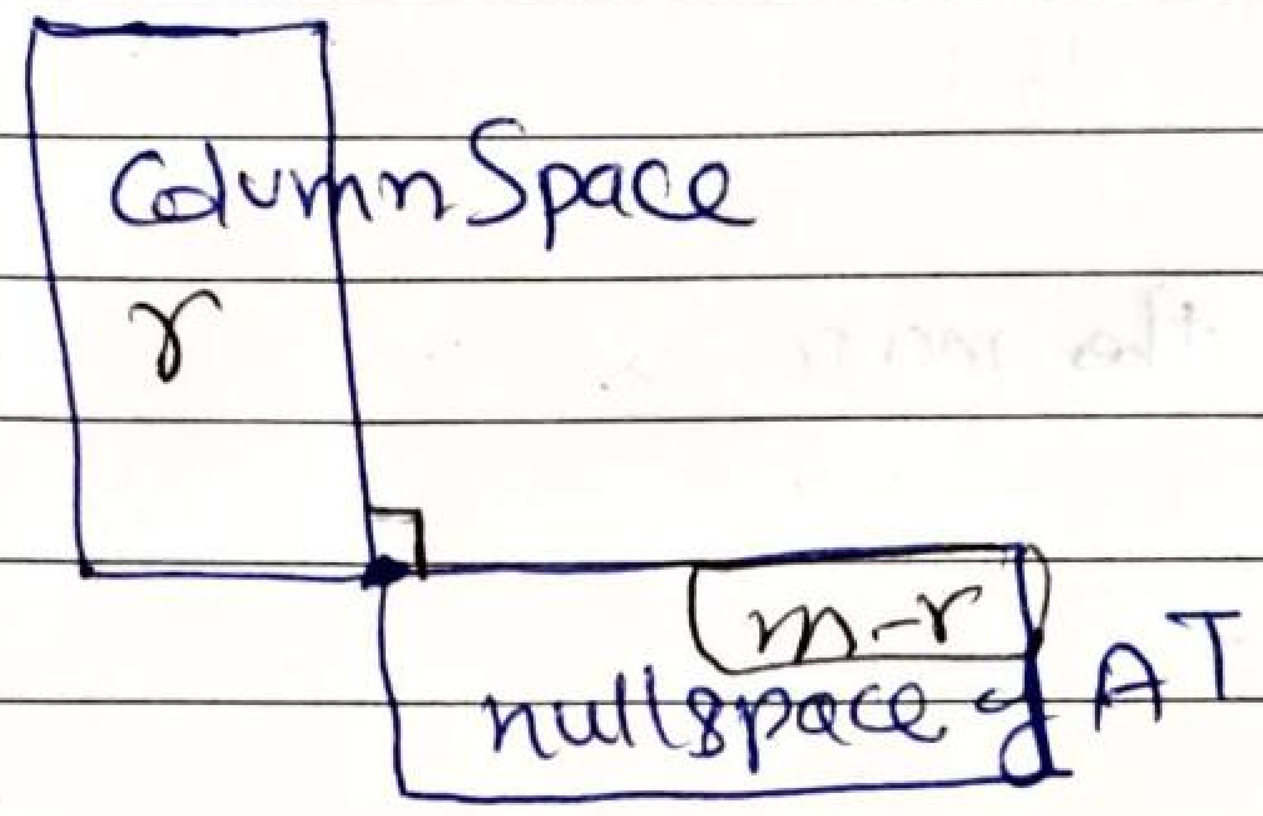
- orthogonal vectors ⊥ subspaces.
- nullspace ⊥ row space
- $N(A^TA) = N(A)$

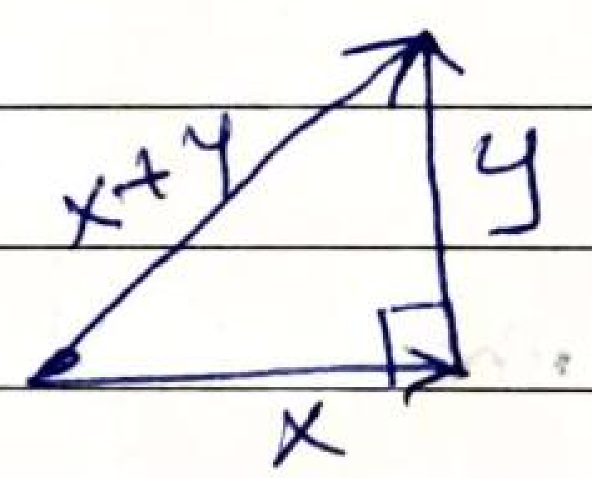




- What does it mean for subspaces to be orthogonal?

Orthogonal vectors:

How can I tell these are orthogonal vectors?



$\boxed{x^Ty = 0}$ ✓ connection?

Pythagoras

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$\cancel{x^Tx} + \cancel{y^Ty} = (x+y)^T(x+y)$$

$$= \cancel{x^Tx + y^Ty}$$

$$+ \underbrace{x^Ty + y^Tx}_{\text{no diff.}}$$

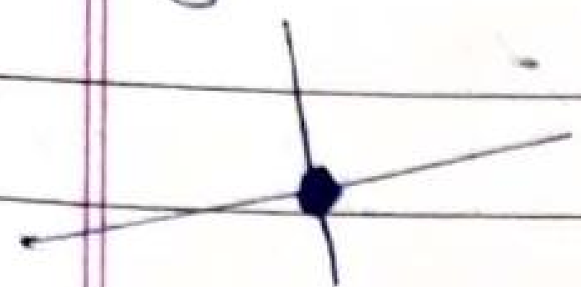
$$\boxed{x^Ty = 0}$$

Supspace S is orthogonal to subspace T. ?

means:

wall & floor — are they orthogonal?

- every vector in **S** is orthogonal to every vector in **T**



- So row space is orthogonal to nullspace. Why?

x in nullspace. $\underline{Ax=0}$

$$A = \begin{bmatrix} \text{row1} \\ \text{row 2} \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} \text{row1} \cdot \text{colum} = 0 \\ \text{2 colum} = 0 \end{array}\right\} \rightarrow \text{telling } x \text{ is orthogonal to all the rows}$$

then also x is $\perp r$ to a combination of the rows.

$$\left.\begin{array}{l} d_1 + c_1 (\text{row1})^T \cdot x = 0 \\ d_2 + c_2 (\text{row2})^T \cdot x = 0 \end{array}\right\} + (c_1 \text{row1} + c_2 \text{row2} \cdots)^T x = 0$$

- $A =$
$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbb{R}^3$

1-D row space so $D = 1$ Line

$n = 3$; rank $= 1$, $\boxed{\dim(N) = 2}$ → plane. What plane is it?

nullspace & rowspace are orthogonal complements in $\mathbb{R}^n$. plane $\perp r$ $[1\ 2\ 5]$

→ Null space contains all vectors $\perp$ row space.

$\underline{Ax = b}$; when there is no solution. (ie. b is not in column space)

good eq^n

$\Rightarrow A^T A \rightarrow$ Symmetric

$n \times m \quad m \times n$

$n \times n$

square

$(A^T A)^T$

$(A^T A)$

$\boxed{A^T A \hat{x} = A^T b}$ ✓ → this will have sol^n

When it is invertable?

→ take example →

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}; \quad m = 3, \quad n = 2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}}_{A^T \cdot A} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

Is it invertible?
it is because $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ is not multiple of $\begin{pmatrix} 8 \\ 30 \end{pmatrix}$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

A → has rank 1

invertible → No

Not always invertible

so product is not gonna have rank > 1.

so $\text{rank}(A^T A) = \text{rank}(A)$.

$N(A^T A) = N(A)$

$A^T A$ is invertible exactly if A has independent columns.