Linear Algebra

Lecture : 18

→ Determinants. det A , |A|
→ Properties $\underline{1, 2, 3}$, 4-10
→ ± signs.

Determinants is number associated with every square matrix.

Invertible : when $|A| \neq 0$.

Singular : when $|A| = 0$.

Properties |A| :
① ⓐ $\det I = 1$.
② ⓑ What happens to a ~~detem~~ if you exchange the rows.
Exchange rows : reverse sign of det.

$$\det \underset{\text{permutation}}{P} = \begin{cases} 1 & \rightarrow \text{even } (\# \text{ of exchanges}) \\ -1 & \rightarrow \text{odd} \end{cases}$$

example

ⓐ $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$;

ⓑ $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$.

property

Linear Comb^n

③ ⓐ $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

ⓑ $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

formulas

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

~~det (A + B) = det A + det B~~

LINEARITY OF EACH ROW (3b)

~~por~~ property (4) 2 eq rows $\longrightarrow$ det = 0;

Two equal rows lead to det = 0

{ Exchange those rows $\longrightarrow$ same matrix. } ← property 2

property (5) Subtract $l \times$ row i from row k;

determinant doesn't change. (Elimination is OK (pivots on diagonal))

ex: $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \Rightarrow$ property (3(b))

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} \Rightarrow \text{use } 3(a)$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 \leftarrow \text{property (4)}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

property (6) Row of zeros $\longrightarrow$ det A = 0

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} \xrightarrow{3(a)} \begin{vmatrix} 5\times 0 & 5\times 0 \\ c & d \end{vmatrix} \Rightarrow 5 \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$$

Date / /
Page

property: 7 $\det U = \begin{bmatrix} d_1 & * & * & * & * \\ 0 & d_2 & * & * & * \\ 0 & 0 & d_3 & & \vdots \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$

$$\boxed{\det(U) = (d_1)(d_2)\cdots(d_n)}$$

= product of pivot // MATLAB // Elimination → rref ↓

property 8 $\det A = 0$
when A is singular.

invertible? or not: $A \to U \to$ rref. //

property 9 $\det AB = (\det A)(\det B)$

$\det A^{-1} = ?$ $A^{-1}A = I$

$(\det A^{-1})(\det A) = 1$

$\to 1/\det A \Rightarrow \boxed{\det A^{-1} = \frac{1}{\det A}}$

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

$\det A^2 = (\det A)^2$ — (a)

(3a)

$$\boxed{\det 2A = 2^n \det A}$$

property (10) $\boxed{\det A^T = \det A}$

Date / /
Page

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

if column is all zero :- $\det A = 0$ //

prove #10 $|A^T| = |A|$ //

$$|U^T L^T| = |LU|$$

using (9)

$$|U^T||L^T| = |L||U|$$

lower triangular → product of diagonal $|L|$

Lecture: 19

Date / /
Page

- formula for det(A) (n! terms)
- Cofactor formula
- Tridiagonal matrices.

simple properties

1. det I = 1
2. SIGN REVERSE WITH ROW EXCHANGE
3. det is linear in each row separately.

2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= \boxed{ad - bc}$$

3×3

survivor

$$\begin{vmatrix} \underline{a_{11}} & a_{12} & a_{13} \\ a_{21} & \underline{a_{22}} & a_{23} \\ a_{31} & a_{32} & \underline{a_{33}} \end{vmatrix} \Rightarrow \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} +$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} +$$

↳ row exchange   ↳ 2 exchange

$$= a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$

$$\cdots \pm \cdots$$

BIG FORMULA

$$\det A = \sum_{n!\ \text{terms}} \pm\, a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

$$(\alpha, \beta, \gamma, \cdots \omega) = \text{perm of } (1, 2, \cdots n).$$

example:

$$\det \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0$$

permutation $(4,3,2,1) \to +1$ $\quad \overbrace{(3,2,1,4)}^{\text{1 row exch}} \to -1$

COFACTORS:

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(\quad)$$

$$\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

cofactor

Cofactor of a_{ij} :- $= C_{ij}$

$$\pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \text{ erased} \\ \text{column } j \end{pmatrix}$$

$+$ if $i+j$: even

$-$ if $i+j$: odd

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

Cofactor formula: (along row 1)

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}}$$

for 2x2:-

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\overset{\text{cofactor}}{d} + b\overset{\text{cofactor}}{(-c)}$$

$$= \boxed{ad - bc}$$

Ex 18

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

A_3

$|A_1| = 1$

$|A_2| = 0$

$|A_3| = -1.$

$|A_4| = 1 \cdot |A_3| - 1 |A_2|$

$$\boxed{|A_n| = |A_{n-1}| - |A_{n-2}|}$$

$|A_5| = 0; \ |A_6| = 1, \ |A_7| = 1;$

Lectures-20

Date / /
Page

- Formula for A^{-1}
- Cramers Rule for $x = A^{-1}b$
- $|\text{Det } A|$ = Volume of box.

a ~~fac~~ co factor

cofactor of (c)

– because (c) is at (2,1) positi (odd)

\# $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{cofactor Matrix}}$

\# $\boxed{A^{-1} = \dfrac{1}{\det A} C^T}$

C^T ← products n-1 entries

$\det A$ → products of n-entries

\# $\begin{bmatrix} a & b & c \\ d & e & f \\ g & i & h \end{bmatrix}^{-1}$

\# Check $AC^T = (\det A) I$ $\boxed{\text{prove}}$ ✓

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ c_{12} & & \vdots \\ \vdots & & \\ c_{1n} & & c_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & & 0 \\ & \det(A) & \\ 0 & & \det(A) \end{bmatrix}$$

row 1

2nd ~~Column~~ row of A * 2nd column C^T :-

$\because \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $-ab + ab$

similarly $A_S = \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \det(A) = ab - ab = 0$

$\therefore$ zero at lower triangle & upper triangle in

from Last Lecture :-

$i=1$

→ $a_{11}C_{11} + a_{12}C_{12} + \ldots$

Cofactor formula :-

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ //

Date / /
Page

$Ax=b$

$$\boxed{x = A^{-1}b = \frac{1}{\det A}\, C^T b}$$

$\downarrow\ c_{11}b_1 + c_{21} + b_2 + \dots$

CRAMER'S RULE

first component $x_1 = \dfrac{\det B_1}{\det A}$

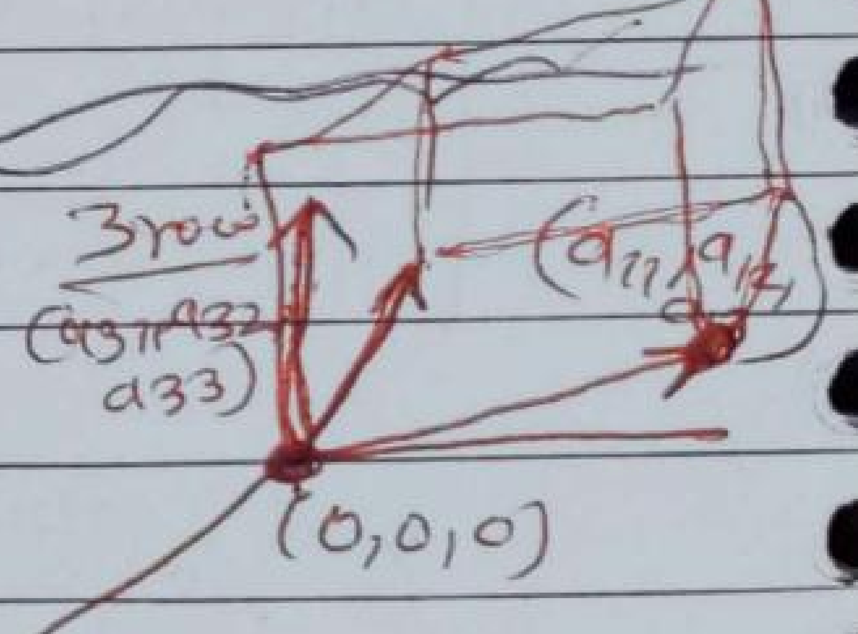
$x_2 = \dfrac{\det B_2}{\det A}$

$\Rightarrow B_1 = \begin{bmatrix} | & n-1 \text{ columns} \\ b & \text{of } A \\ | & \end{bmatrix}$; $B_j = \begin{bmatrix} A \text{ with column } j \text{ replaced by } b \end{bmatrix}$

A with column 1 replaced by b

$$\boxed{x_j = \frac{\det B_j}{\det A}}$$

$\underset{3\times 3}{|\det A|}$ = volume of box.



if $\cancel{A} = I$; Volume = $|\det A|$
(unit Cube)

if volume has det (first 3) property then its volume = $|\det A|$

$A = Q$ orthogonal Matrix.

$\Rightarrow$ what kind of box? (cube) (unit cube turned in space)

$\det Q = \pm 1$

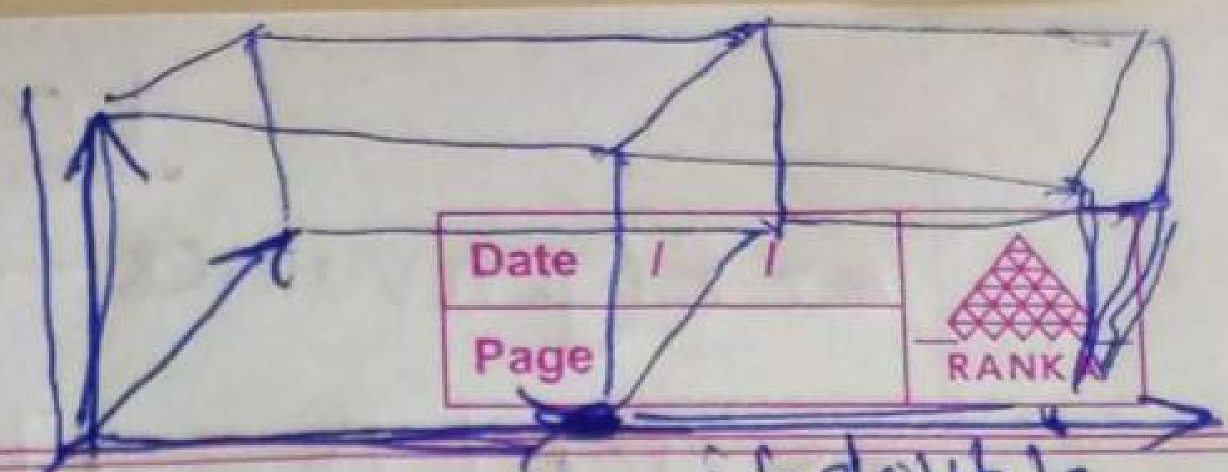
fact: $Q^T Q = I$ (take det both side)

$\cancel{\det}|Q^T|\,|Q| = 1$ rule (10)

$|Q|^2 = 1 \therefore \boxed{|Q| = \pm 1}$

= volume

⇒ recantuglar box:



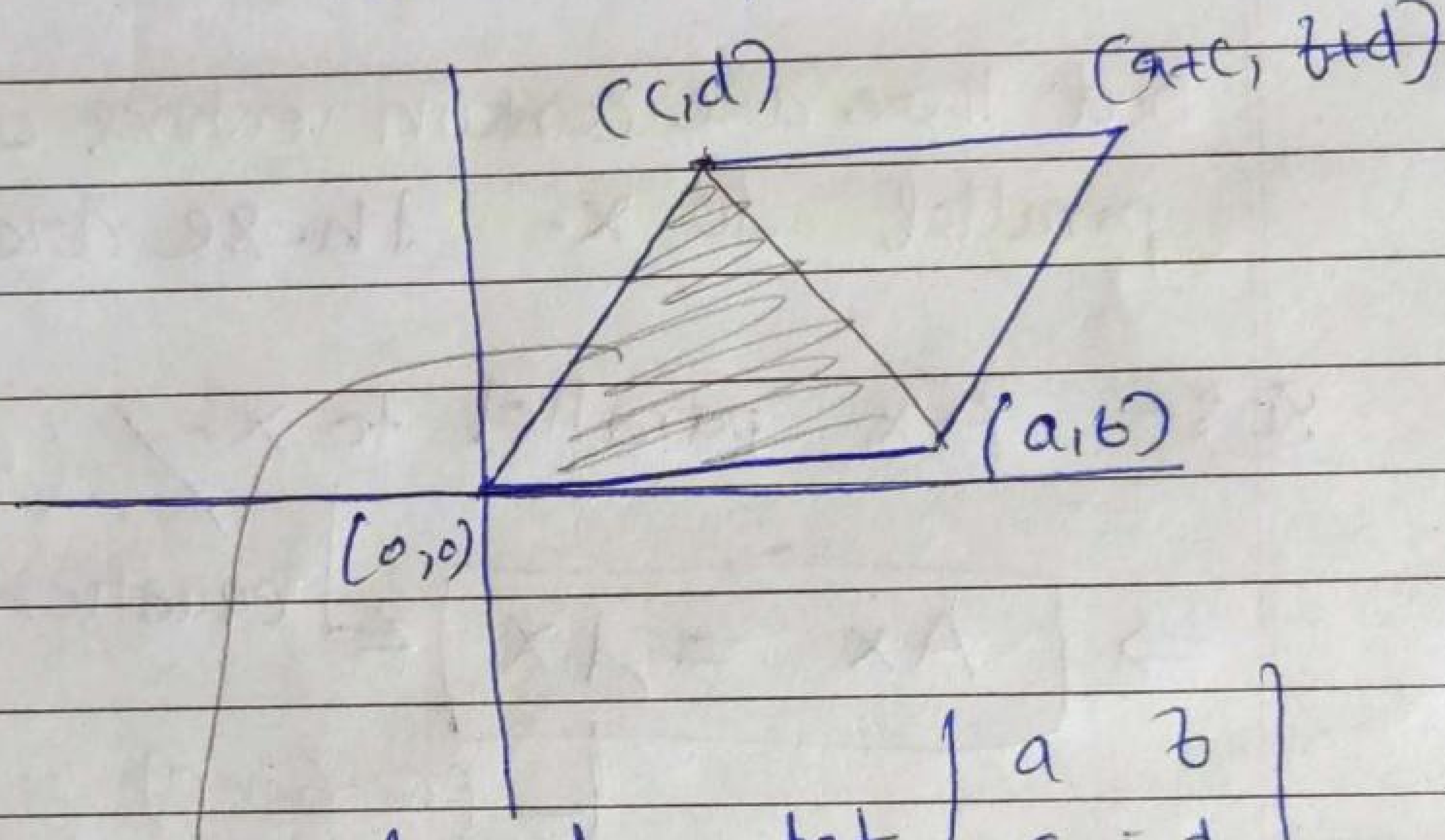
$|\det A|$ = Volume of box: → has property 1

→ reverse two rows (has Prop 2)
No change in box

→ 3a, → t

→ 3b →

$\det \begin{vmatrix} a & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

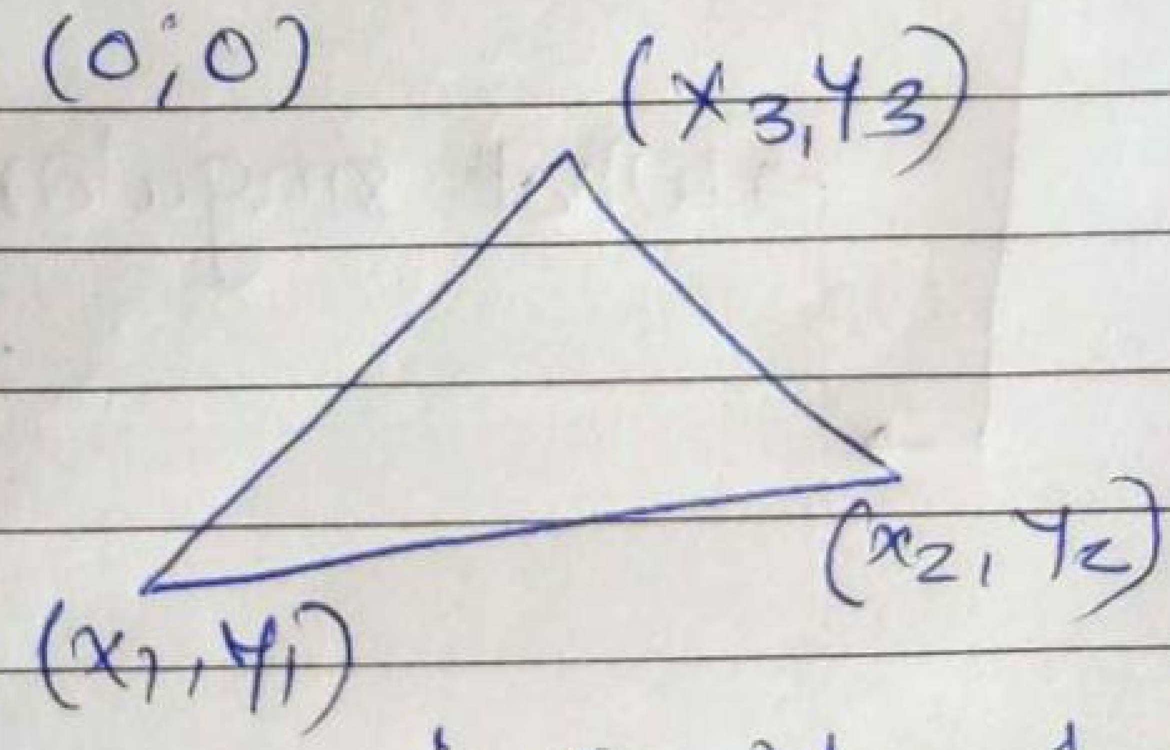


Area of parallelogram $= \det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

Area of $\Delta = \frac{1}{2}(ad - bc)$

what if it doesn't start at (0,0)



$$\text{area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$