# Lecture : 24

- Markov matrices
- Fourier series of Projections

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix} \rightarrow \text{Markov matrix.}$$

<u>properties</u>: ① every entry is greater equal zero / $\geq 0$

→ On powers of matrix : this property will exist /

② All columns add to 1.

→ ~~sam~~ True for power matrix.

<u>Note</u>: $\lambda = 0$; lead to steady state; when matrix A.

what's in the case of power matrix ? → $\boxed{\lambda = 1}$ & its eigen vector.

§ Matrix A has eigenvalue of 1. the property ② turns out

↳ guarantees that one is an eigenvalue. /

↳ so eigenvalue of Markov matrix ~~is com~~ can be computed without computing $|A - \lambda I|$

<u>Keypoints</u> :-

① :- $\lambda = 1$ is an eigenvalue.

② :- All other eigenvalues $|\lambda_i| < 1$;

$$u_K = A^K u_0 = c_1 \underset{=1}{\lambda_1^K} x_1 + c_2 \underset{<1}{\lambda_2^K} x_2 + \cdots \cdots$$

Here $\lambda_1 = 1$ & over the time it will be zero

$$\boxed{u_K = A^K u_0 = c_1 x_1} \checkmark$$

→ x part of $u_0$ (Steady State). /

# Why one is an eigenvalue?

$\Rightarrow$

# special about eigenvector?

$x_1 \Rightarrow$ all its components are +ve

$\boxed{x_1 \geq 0} \leftarrow$ steady state is +ve

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

$=1 \quad =1 \quad =1$

$\because$ 1 is eigen value then

$$A - 1I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -0.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$$

shifted M-matrix by 1

if $A-1I$ is singular; that tell me $\lambda = 1$.

Note: Eigenvalue are the numbers that I subtract off (shift); (The number that I subtract from the diagonal) to make it singular.

Now: Why that matrix is singular?
(we want to see the reason which works for all M-matrix)

All columns add to zero of $(A-I)$ $\longrightarrow$ this means $A-I$ is singular.

reason
- How do I see those 3 rows are dependent?
  what comb$^n$ of those rows gives the zero row?
- because rows are dependent;
  because $(1,1,1)$ its not in the null space of matrix
  but it's in the null space of the transpose.
  $(1,1,1)$ is in $N(A^T)$

Now, what combinations of column gives zero?

then $x_1$ is in $n(A)$ //

(11) eigen value of $A$ are the same.
eigen value of $A^T$.
Why?

$\lambda$ of $A$ = $\det(A-\lambda I)=0$.

property (10) $\det(A) = \det(A^T)$.

$$\Downarrow$$

$$\Rightarrow \det(A^T - \lambda I) = 0$$

Application of M-matrices (Markov-Matrices)

Problem

$u_{k+1} = A u_k$    $A$ is Markov

stay at cal at t=k    moved

$$\begin{bmatrix} u_{cal} \\ u_{mall} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mall} \end{bmatrix}_{t=k}$$

moved from cal to mall.

stay in mall

$$\begin{bmatrix} u_{cal} \\ u_{mall} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} \begin{matrix} \text{cal} \\ \\ \end{matrix}$$

at time zero    population of mall

at time 1

$$\begin{bmatrix} u_{cal} \\ u_{mall} \end{bmatrix}_1 = 0 \times \begin{bmatrix} .9 \\ .1 \end{bmatrix} + 1000 \begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

after $K=1$; 200 people are there in cal.

(12) Eigen value & Eigen Vector

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \Rightarrow \lambda_1 = 1 \qquad \lambda_2 = (1.7\,(\text{trace}) - \lambda_1) = \underline{0.7}$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad x_2$$

Problem: what's the population at infinity?

→ this is giving steady state

$\lambda_2 = .7 ; \; x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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population after 100 times steps –

$$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cancel{\frac{2000}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

↳ disappearing

## # projection with orthonormal basis –

↓ expansion $q_1, \ldots q_n$

any vector

$$v = x_1 q_1 + x_2 q_2 + \ldots x_n q_n$$

⇒ expanding vector in the basis, & basis is orthonormal

$$q_1^T v = x_1 \underbrace{q_1^T q_1}_{1} + 0 \ldots + 0$$

$$\boxed{q_1^T v = x_1}$$

↳ matrix language:

$$\begin{bmatrix} | & & | \\ q_1 & — & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = v$$

$$\boxed{Qx = v}$$

orthonormal

$$\boxed{x = Q^{-1} v} \; \boxed{= Q^T v}$$

$$\boxed{x_1 = q_1^T v}$$

Fourier Series :-



# $f(x) = a_0 + a_1 \cos(x) + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$

~~diff~~ infinite but key property of things being orthogonal is still true for sines & cosines so it's the property makes fourier series work. so that's called fourier series.

Instead of vector $q_1, q_2, q_3 \cdots q_n$; I can have orthogonal ~~vectors~~ functions.

# ~~Inner product of $f^n$.~~ (In vector $v^T w = v_1 w_1 + \cdots v_n w_n$

$$f^T g = \int_0^{2\pi} f(x)\, g(x)\, dx$$

what you have for adding in continuous function.

$f(x) = f(x+2\pi)$ PERIODIC FUNC

$$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} (\sin x)^2 \Big|_0^{2\pi} = 0$$

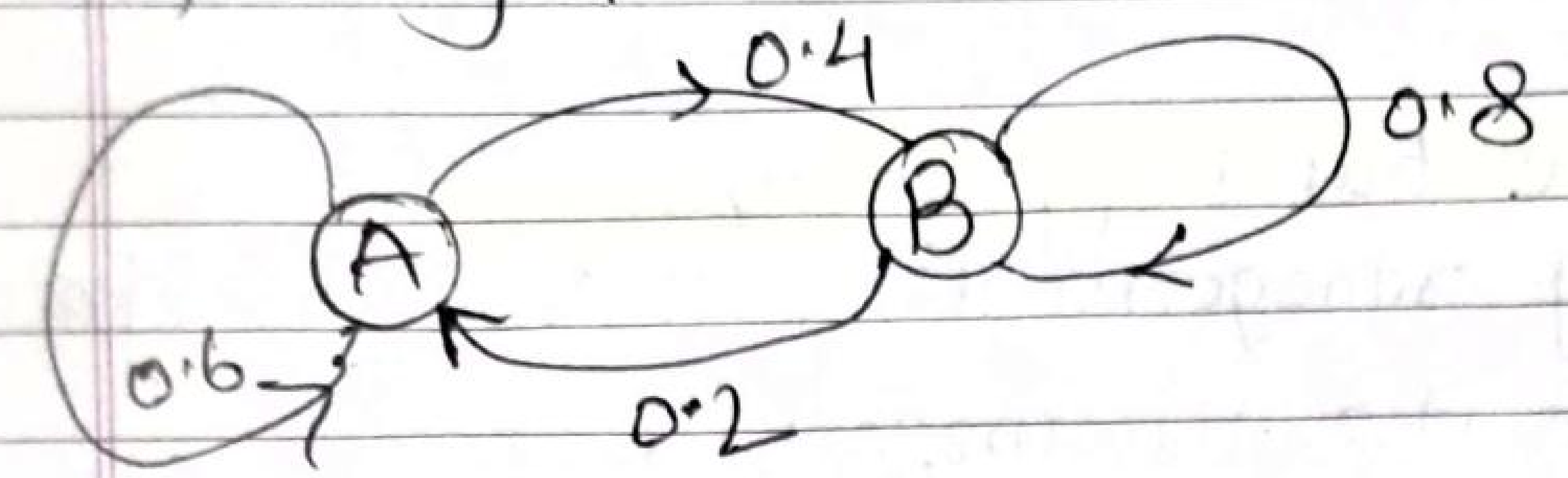
$a_1 = ?$ first f. coff'
(take inner product with $\cos x$)

$$\int_0^{2\pi} f(x) \cos x \, dx = a_1 \underbrace{\int_0^{2\pi} (\cos x)^2 dx}_{=\pi}$$

$$\boxed{a_1 = 1/\pi \int_0^{2\pi} f(x) \cos x \, dx}$$

Problems

A particle jumps b/w positions A & B with the following possibilities




If it starts at (A); what is the probability it is at (A) & (B) after

(i) 1 step (ii) n steps (iii) $\infty$ steps.

Sol:

$$A = \begin{array}{cc} A & B \\ \left(\begin{array}{cc} 0.6 & 0.2 \\ 0.4 & 0.8 \end{array}\right) & \begin{array}{l} \leftarrow A \\ \leftarrow B \end{array} \end{array} \rightarrow M\text{-matrix}$$

$$P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow A \\ \leftarrow B \end{array}$$
at time 0

(i) 1 steps

$$P_1 = A P_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

(ii) n steps

$$P_1 = A p_0 ; \; P_2 = A p_1 = A^2 p_0$$

$$\boxed{P_n = A^n P_0}$$ so how to take n-power of Matrix?

Recall eigenvalue $\Rightarrow$ $\lambda = 1$; $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = 0.4$; $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\boxed{A = S \Lambda S^{-1}}$$

$$S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$$

$$S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P_n = A^n P_0 = S \Lambda^n S^{-1} P_0$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.4^n \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2(0.4)^n + 1 \\ -2(0.4)^n + 2 \end{pmatrix}$$

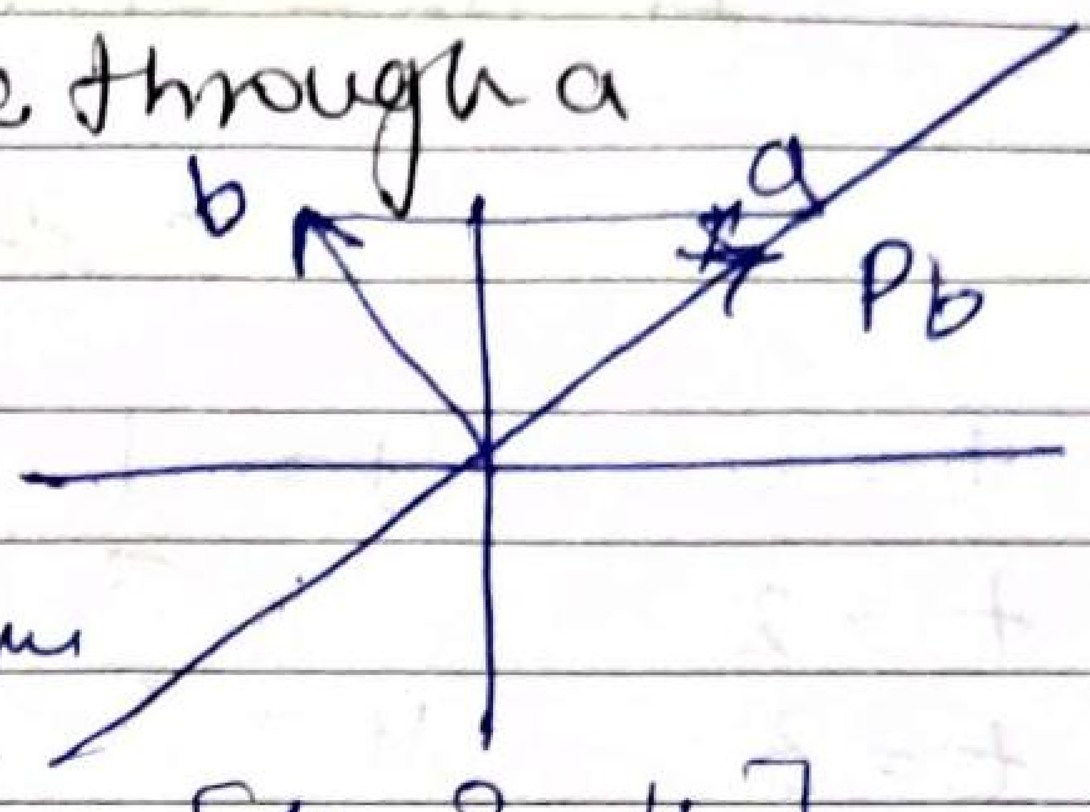
(iii)

$$P_\infty = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

# Lecture: 24b

(1) $a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ ; find projection matrix (P)

that projects onto line through a




Sol $\boxed{P = A(A^TA^{-1})A^T}$

$\Rightarrow \frac{aa^T}{a^Ta}$ (for 1 column matrix.

$= \frac{1}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \Rightarrow \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$

ColumnSpace is a line through (2,1,2) & rank 1

& eigenvalue ?

↳ singular matrix so $\lambda = 0, 0, 1$ ← trace

rank is 1 so it will be 2D NullSpace

so ~~zero eigenvalue~~

eigen vector for $\lambda = 1$ means the vector which doesn't move

~~$A = I$~~

$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rightarrow$ is also eigen vector

because if I apply projection to a again I will get a

$\boxed{Pa = a}$

(2) Solve $u_{k+1} = Pu_k$ find $u_k$ $a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ given

$u_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$

Sol. $u_1 = Pu_0$ (it like $u_0$ is 'b' & projecting onto the line)

$u_1 = a\frac{a^Tu_0}{a^Ta} = a\frac{27}{9} = 3a = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$u_2 = Pu_1$ $u_k = P^ku_0 = Pu_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$P = P^2 = P^3$

if $P$ is not projection matrix

then $u_0 = \underline{c_1} x_1 + \underline{c_2} x_2 + \underline{c_3} x_3$ → find these

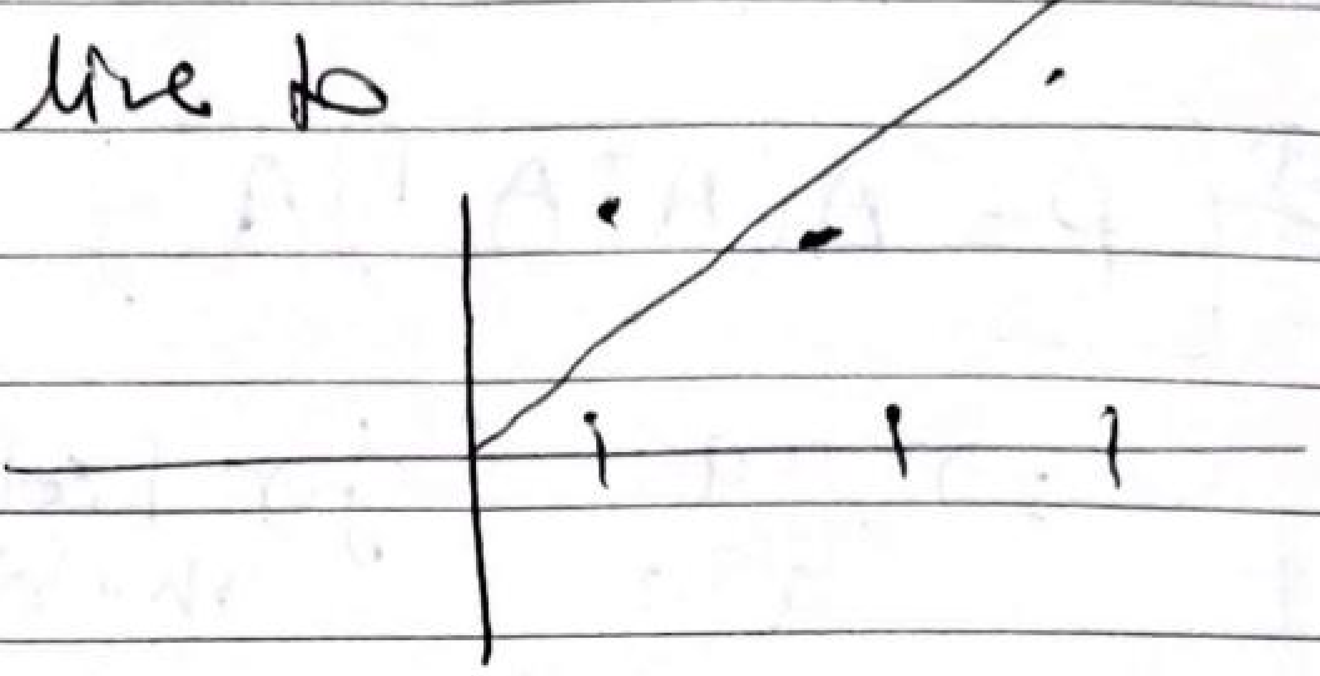
$$A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_2$$

(2): fitting a straight line to

$t=1$ $y=4$

$t=2$ $y=5$

$t=3$ $y=8$



$y = Dt$

Sol: 
$$\begin{bmatrix} 1D = 4 \\ 2D = 5 \\ 3D = 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

$\boxed{Ax = b}$

Best x $\boxed{A^T A \hat{D} = A^T b}$

$$14\hat{D} = 38$$

$$\boxed{\hat{D} = \frac{38}{14}}$$

(E) projecting b onto col space of A (a line)

Ex $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; find 2 orthogonal vectors?

→ plane

Sol: (1) make $a_2$ orthogonal to $a_1$ & substract its projection

$B$ is $\perp r$ to $a_1$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\frac{a_1^T a_2}{a^T a_1} a_1}_{\text{subtract it projection}}$$

$$B = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{6}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$$
(⊥ to $a_1$)

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(3) $4\times4$; $\lambda_1, \lambda_2, \lambda_3, \lambda_4$;
What's condition on $\lambda$ so the matrix is invertible?

(a) Invertible ⇒ no zero eigen value

(b) $\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_4}\right)$

(c) trace of $\underbrace{(A+I)}_{\text{add eigen value } +1} = (\lambda_1+1) + (\lambda_2+1) + \dots$

$= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4.$

(4) $A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$D_n = \det A_n$.

(a) Use cofactors: $D_n = \underline{+}\, D_{n-1} + \underline{-}\, D_{n-2}$

for $a_{11} \to$ cofactor $(n-1)$

$\underline{\underline{a_{12}}} = -$

(b) Solution: $D_1 = 1;\ D_2 = 0$

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

?

(c) eigen val

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0$$

$\boxed{\lambda = \frac{1 \pm \sqrt{-3}}{2}}$

Complex

# eigen value of $A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

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$$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} =$$

det

$$= -\lambda^3 + 5\lambda = 0$$

$$\lambda(-\lambda^2 + 5)$$

$$\boxed{\begin{aligned} \lambda &= 0 \\ \lambda &= \sqrt{5} \\ &= -\sqrt{5} \end{aligned}}$$

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Problems ① Find all the non-zero terms in the big formula

$$\det A = \sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} a_{4\delta}$$

& compute det A.

② Find cofactors $C_{11}, C_{12}, C_{13}$ & $C_{14}$

③ Find column 1 of $A^{-1}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}$$

Sol:

①:

$$\det A = \sum \pm \underline{a_{1\alpha} a_{2\beta}} \underbrace{a_{3\gamma} a_{4\delta}}_{\text{choices}}$$

choices

$$= 1 \times 6 \times 9 \times 12 - 1 \times 6 \times 10 \times 11 - 2 \times 5 \times 9 \times 12$$

$a_{11}\ a_{12}\ a_{13}\ a_{14}$ — $(1\ 2\ 3\ 4)$ perfect alingm so +sign

$a_{11}\ a_{12}\ a_{34}\ a_{43}$ — $(1, 2, 4\ 3)$

column $\rightarrow \underbrace{(2, 1}, 3, 4)$ 1 exchange so −ve

$$+ 2 \times 5 \times 10 \times 11$$

$(\underbrace{2, 1}, \underbrace{4, 3})$ 2 row exchange

$$= 8 \checkmark$$

(2.) $c_{11} = \det \begin{pmatrix} 6 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{pmatrix}$

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$= 6 \times (9 \times 12 - 10 \times 11) = \underline{-12}$

$= 6 \times \text{cofactor}(a_{11})$ ✓

$\underset{i+j=3}{c_{12}} = -\det \begin{pmatrix} 5 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{pmatrix} = 10$ ✓

$c_{13} = \det \begin{pmatrix} 5 & 6 & 8 \\ 0 & 0 & 10 \\ 0 & 0 & 12 \end{pmatrix} = 0$

$c_{14} = \det \begin{pmatrix} 5 & 6 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 11 \end{pmatrix} = 0;$

$\det A = a_{11} c_{11} + a_{12} \cdot c_{12}$ ~~(because~~

$= 1(-12) + 2 \times 10 = 8$ //

→ dot product (row, its co-factor) ✓

(3) ✓ $A^{-1} = \dfrac{1}{\det A} C^T$

$1^{st}$ Column of $A^{-1} = \dfrac{1}{\det A} (1 \text{ row of } C)^T$

$= \dfrac{1}{8} \begin{bmatrix} -12 \\ 10 \\ 0 \\ 0 \end{bmatrix}$