Prof. M. Magill **Econ 659 - Problem 4** Fall 2022

given: Sunday November 6: due: Thursday November 17.

Subject: Inequalities on Option Prices and Two-period Option Pricing

Questions of the kind that follow will appear on the Midterm Exam: they require the use of simple no-arbitrage arguments which the problems that follow will teach you to use. You will find MM Lectures 9 and 10 useful. We also solve for a contingent market equilibrium of an economy with log utility functions.

1. Consider an interval of time $[0, T]$ and the current date $t \in [0, T]$: we think of $[0, T]$ as a relatively small interval of time in the subset $[0, \infty)$ during which we assume there are no dividend payments on the equity. For $t \in [0, T]$ let t denote the current time and let Q_t^e denote the known current price of equity: we let $\widetilde{Q}_T^e$ denote the random price of equity at date T. Let's think of $\widetilde{V}_T^e = \widetilde{Q}_T^e$ as the random payoff of equity at date T.

The other securities traded are the riskless bond and options on the stock expiring at date T. Let Q_t^b denote the price of the bond at date t and let V_T^b denote its payoff at date T with $V_T^b = 1$. We assume the interest rate is constant so that Q_t^b is defined in terms of the interest rate r by the relation $Q_t^b = e^{-r(T-t)}$.

We consider put and call options with *exercise price* K, also called the *strike price*: a *European call option* is a contract which gives the right to buy the stock at date T at the price K; a *European put option* is a contract which gives the right to sell the stock at date T at the price K. An *American call option* is a contract which gives the right to buy the stock at any date at or before date T at the price K, while an *American put option* is a contract which gives the right to sell the stock at any date at or before date T at the price K. Let Q_t^c and Q_t^p denote the date t prices of the European call and the put options, and let Q_t^{ca} and Q_t^{pa} denote the date t prices of the American call and the put options, all with the common exercise price K.

(a) On a graph draw the payoffs of the call and put options as a function of the equity price realized at date T. On the same graph draw the payoff of the equity as a function of its date T price (i.e. the diagonal). It will be convenient to use this graph to give a geometric interpretation to questions b, c, d.

(b) Prove the *put-call parity relation* for European options. Give a geometric interpretation of this result using the figure in (a).

(c) Show that the following put-call inequality holds for American options

$$Q_t^e - K \leq Q_t^{ca} - Q_t^{pa} \leq Q_t^e - KQ_t^b$$

(d) Show that if $K_1 \leq K_2$ then $Q_t^{c1} \geq Q_t^{c2}$.

(e) Show that $Q_t^b(K_2 - K_1) \geq Q_t^{c1} - Q_t^{c2}$.

(f) Show that the price of a call option is a convex function of its striking price, i.e. if K_1 and K_2 are the striking prices of two different options and if a third option has a striking price $K_\lambda = \lambda K_1 + (1-\lambda)K_2$ then $Q_t^{c_\lambda} \leq \lambda Q_t^{c_1} + (1-\lambda)Q_t^{c_2}$ with obvious notation. [Hint: find a portfolio which gives a larger payoff than the option with striking price K_λ]. Explain the intuition for the result.

(g) Consider the special case where $T = 1$ and there are only two periods, $t = 0, 1$. If there are 3 or more states of nature at date 1 then given only the equity and the bond, the markets are incomplete and it is not possible to derive a unique price Q_0^c for a call option. Prove the following bounds for the price of the call option

$$\max\{Q_0^e - \frac{K}{1+r}, 0\} \leq Q_0^c \leq Q_0^e$$

(h) Note that the inequality in (g) may only imply rather weak bounds on the price of the call, as the following example shows. Suppose the price of equity is 100 at date 0, 120 in state 1, 110 in state 2, 100 in state 3, let the interest rate be 7 % and suppose the strike price is $K = 107$: find the interval in (g).

(i) Use the the more precise information implied by the equivalence of no-arbitrage and the existence of state prices to find a more precise *interval* in which the price of the option must lie. Show that the length of this interval is about 1/50 of the length of the interval given in (h). Explain.

2. *Option pricing with 2 states and 2 periods.* Consider the model above in which there are $S = 2$ states of nature at date 1. We think of these as primitive shocks which affect the price of the stock at date 1: the first is "up", the second is "down" (good news and bad news). Let $R = 1 + r$ denote the return on the riskless bond with payoff $V^b = (1, 1)$ at date 1 and let u and d denote the returns on equity in the "up" and "down" states respectively, with $0 < d < u$. Let the date 0 price and the two prices at date 1 for the equity contract be given by

$$Q_0^e = Q, \quad Q_u^e = uQ, \quad Q_d^e = dQ$$

Since no dividends are paid on the equity, the payoff matrix at date 1 for the bond and the stock is given by

$$V = [V^b \; V^e] = \begin{bmatrix} 1 & uQ \\ 1 & dQ \end{bmatrix}$$

(a) Show that there are no arbitrage opportunities if and only if $d < R < u$. Interpret this condition.

(b) Exhibit an arbitrage opportunity when (a) is violated.

(c) In what follows assume that (a) holds. Find the vector of state prices $\pi = (\pi_u, \pi_d)$ for the two states at date 1.

(d) Find the price Q^c at date 0 of the call option on the stock with exercise price K. Explain the idea underlying the method you are using: try to be as clear and thorough as you can!

(e) Show that the formula in (d) can be written as

$$Q^c = \mu_u \frac{V_u^c}{R} + \mu_d \frac{V_d^c}{R}, \quad \text{with } \mu_u > 0,\ \mu_d > 0, \text{ and } \mu_u + \mu_d = 1$$

How do you interpret this expression?

(f) Here's another way of pricing the call. Find the portfolio (Δ, B) of the stock and the bond which replicates the date 1 payoff of the option. Use this to deduce the price Q^c: check that it coincides with what you found in (d).

(g) Here's yet another way of pricing the call ! Show that there is a portfolio which consists of buying Δ units of stock and going short one unit on the call option which generates a riskless income stream at date 1: use this hedge to price the call. Give the intuition underlying the hedge.

3. Suppose the equity price is $Q_0^e = 40$ and is expected to go up by 10% or down by 10% for each of the next two three-month periods. Suppose the interest rate is known to be 12% per annum with continuous compounding for each period. Find the value of a six-month European put option with strike price $K = 42$, and the value of a six-month American put option with the same strike price.

4. *In question 5 below we will calculate a contingent market equilibrium: to this end we begin by deriving a preliminary result on demand functions of agents with log utility functions* v^i. There is a simple piece of gymnastics that all of you should know and that only needs to be done once: for ever after life is simple. Consider a one period economy $\mathcal{E}(\mathbb{R}^L, u, \omega)$ with L goods and with spot markets for these goods. Suppose that an agent has a so-called *Cobb Douglas utility function* for bundles of the L goods i.e. $v(x_1, \ldots, x_L) = \gamma_1 \log(x_1) + \ldots + \gamma_L \log(x_L)$. Suppose also that the agent has a vector of initial endowments of the L goods, $\omega = (\omega_1, \ldots, \omega_L) \in \mathbb{R}_+^L$ and faces spot

prices for the goods $p = (p_1, \ldots, p_L) \gg 0$. Show that the agent's *demand function* (i.e. the solution of his utility maximizing problem over his budget set—it should be clear what his budget set is in this context) is given by $f(p, p\omega) = (f_1(p, p\omega), \ldots, f_L(p, p\omega))$ with

$$f_\ell(p, p\omega) = \left(\frac{\gamma_\ell}{\sum_{\ell=1}^{L} \gamma_\ell} \right) \frac{p\omega}{p_\ell}, \quad \ell = 1, \ldots, L$$

Give a simple interpretation of this result and explain the proportion of his income that the agent spends on each good.

5. *Now lets calculate the contingent market equilibrium of a stochastic economy in which agents have log preferences.* Consider a one-good two period-economy $\mathcal{E}(\mathbb{R}^{S+1}, u, \omega)$ in which agents have log Bernouilli utility functions

$$u^i(x^i) = \log(\alpha_i + x_0^i) + \delta \sum_{s=1}^{S} \rho_s \log(\alpha_i + x_s^i), \quad \alpha_i \in \mathbb{R}, \quad i = 1, \ldots, I$$

where δ is the common discount factor of the agents, and $0 < \delta \leq 1$. Suppose agents can buy and sell on *contingent markets*: let $\pi = (1, \pi_1, \ldots, \pi_S)$ denote the vector of present-value prices. Define $\tilde{\rho} = (\tilde{\rho}_0, \tilde{\rho}_1, \ldots, \tilde{\rho}_S)$ by $\tilde{\rho}_0 = 1, \tilde{\rho}_s = \delta \rho_s,\ s = 1, \ldots, S$, and $\Delta = \sum_{s=0}^{S} \tilde{\rho}_s$. Let $\tilde{\mathbf{1}} = (1, 1, \ldots, 1) \in \mathbb{R}^{S+1}$. In this problem we will be a bit sloppy about the non-negativity constraints for consumption, and assume that the admissible consumption for agent i in any state is the set of $\xi \in \mathbb{R}$ such that $\alpha_i + \xi > 0$.

(a) Using the change of variable $X_s^i = \alpha_i + x_s^i,\ s = 0, \ldots, S$ and question (4), show that agent i's demand function is given by

$$x_s^i = f_s^i(\pi, \pi\omega^i) = \left(\frac{\tilde{\rho}_s}{\Delta} \right) \frac{\pi(\omega^i + \alpha_i \tilde{\mathbf{1}})}{\pi_s} - \alpha_i, \quad s = 0, 1, \ldots, S, \quad i = 1, \ldots, I$$

Interpret.

(b) Derive the *aggregate excess demand function* $Z : \mathbb{R}_{++}^{S+1} \to \mathbb{R}^{S+1}$ defined by

$$Z(\pi, \omega) = \sum_{i=1}^{I} (f^i(\pi, \pi\omega^i) - \omega^i)$$

(c) Let $S = 3$ (three states at date 1). Explain why we only need to solve the equations $Z_s(\pi, \omega) = 0$, for $s = 1, 2, 3$, to find the equilibrium present-value prices. [Hint: *use the fact that every agent satisfies his budget equation: add them and then notice that this implies that the aggregate excess demands across the states are not independent. This is a very important property to understand.*]

(d) Find the equilibrium prices for the case (c) and show that they can be expressed as simple functions of the aggregate output $w = \sum_{i=1}^{I} \omega^i$, the aggregate coefficient $\alpha = \sum_{i=1}^{I} \alpha_i$, the discount factor δ, and the vector of probabilities ρ.

(e) Show that the equilibrium consumption of agent i is of the form

$$\bar{x}^i = \bar{b}_i w + \bar{a}_i \tilde{\mathbf{1}} \quad \text{where} \quad \bar{b}_i = \frac{\bar{\pi}(\omega^i + \alpha_i \tilde{\mathbf{1}})}{\bar{\pi}(w + \alpha \tilde{\mathbf{1}})}$$

Find $\bar{a}_i$. Clearly $\sum_i \bar{b}_i = 1$. Check that $\sum_i \bar{a}_i = 0$.

(f) Let $T^i(\xi)$ denote the *risk tolerance* of agent i defined by

$$T^i(\xi) = \frac{1}{A^i(\xi)}$$

where $A^i(\xi)$ is the risk aversion of the agent (defined in Problem Set#1). Show that for the log Bernouilli functions

$$T^i(\xi) = \alpha_i + \xi$$

What does this imply about the way agents can differ in their risk tolerance, for this class of utility functions? Show that if $i \neq j$ are two agents with $\omega^i = \omega^j$ and $\alpha^i < \alpha^j$, then $\bar{b}_i < \bar{b}_j$ and $\bar{a}^i > \bar{a}^j$ in (e). Interpret.