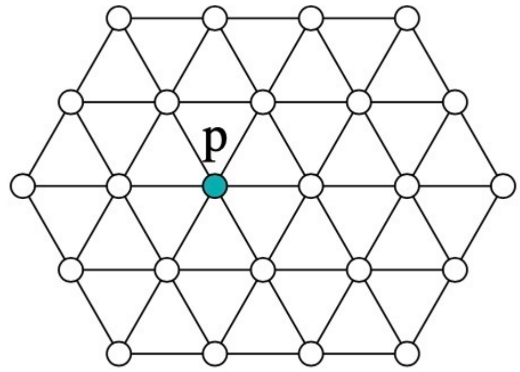
9.1. Levin-Gu model (2012)

2D SPT protected by $G=\mathbb{Z}_2$.

- Ising paramagnet.



$$H_0 = -\sum_p \sigma_p^x$$

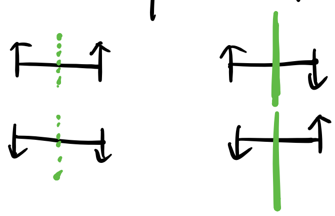
$$|\Psi_0\rangle = \bigotimes_p |\sigma_p^x = 1\rangle$$

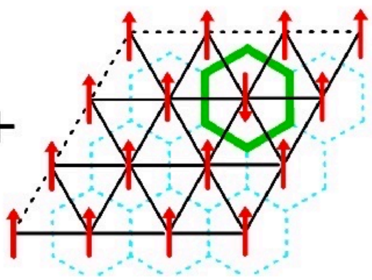
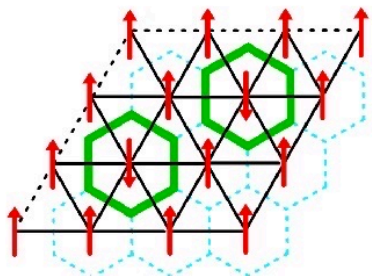
$$= \bigotimes_p \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_p + |\downarrow\rangle_p\right)$$

$$\propto \sum_{\{\sigma_p^z=\pm 1\}} |\{\sigma_p^z\}\rangle$$

$$\propto \sum_{\substack{\text{DW conf.}\\ (\text{even},\,\text{even})}} |\text{DW conf.}\rangle$$

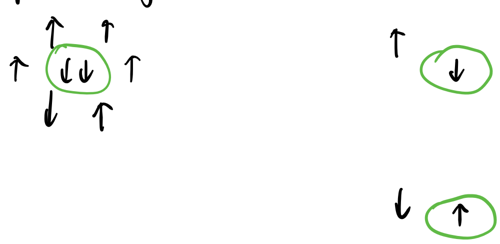
Domain wall picture:



$|\Psi_0\rangle = +$  $+$  $+ \dots$

On the plane:

spin conf. $\xrightarrow{2:1}$ DW conf.



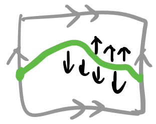
On the torus:

spin conf. $\xrightarrow{2:1}$ DW conf. (even, even)

(odd, even)

2:1

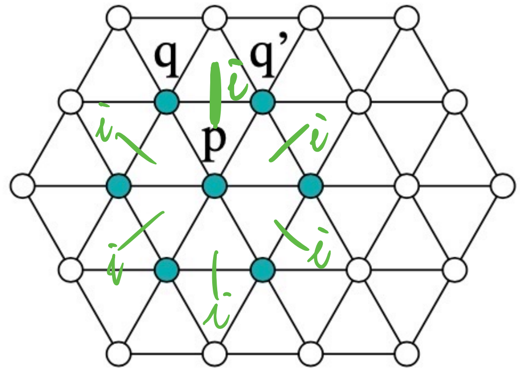
Spin conf. $\longrightarrow$ DW conf. (even, odd) (odd, odd)

? $\longmapsto$ 

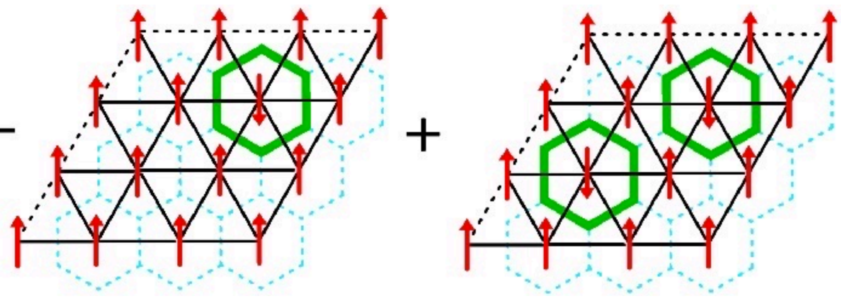
- Levin-Gu state.

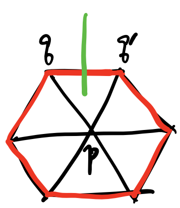
$$H_1 = -\sum_p B_p$$

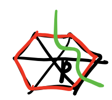
$$B_p = -\sigma_p^x \prod_{\langle pqq'\rangle} i^{\frac{1-\sigma_q^z\sigma_{q'}^z}{2}}$$



$$|\Psi_1\rangle = \sum_{\substack{\{DW\ conf\} \\ (even,\ even)}} (-1)^{\#(DW)} |DW\ conf\rangle$$

$|\Psi_1\rangle = -$  $+ \dots$



$$-\prod_{\langle pqq'\rangle} \underbrace{i^{\frac{1-\sigma_q^z\sigma_{q'}^z}{2}}}_{} = -i^{\#(DW)\text{ crossing boundary of}}$$ 

$\sigma_q^z\sigma_{q'}^z = \pm 1$

$$\begin{cases} 1, & \text{if } \sigma_q^z\sigma_{q'}^z = +1 \Leftrightarrow \text{No DW for } \langle qq'\rangle \\ i, & \text{if } \sigma_q^z\sigma_{q'}^z = -1 \Leftrightarrow \text{DW for } \langle qq'\rangle \end{cases}$$

$$= \begin{cases} -i^2 = 1, & \text{if } \#(DW) \text{ crossing } \hexagon = 2 \mod 4 \\ -i^0 = -1, & \text{if } \quad \dots \quad = 0 \mod 4 \end{cases}$$

$$= (-1)^{\text{changes of } \#(\text{total DW}) \text{ under } B_p}$$

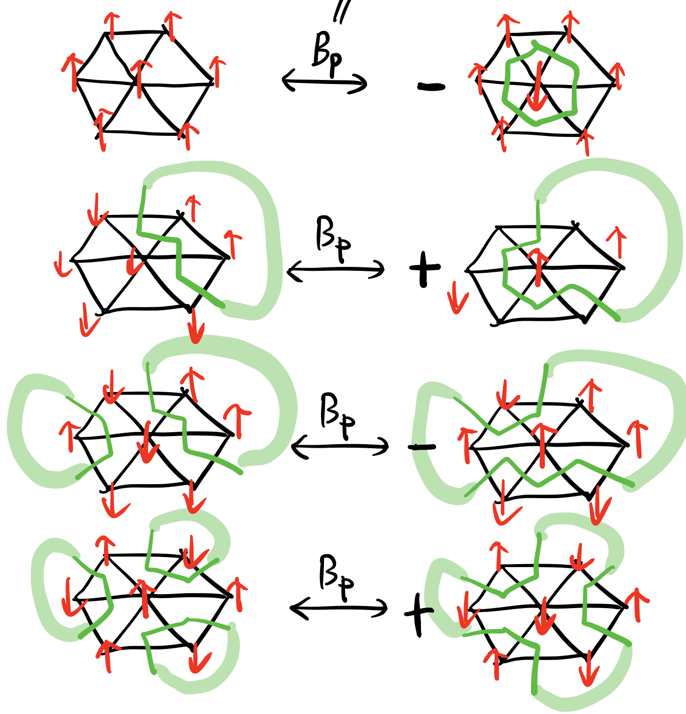
$(\pm)\cdot\sigma_p^x$

#(DW) = 0:   #(total DW): $0 \to 1$

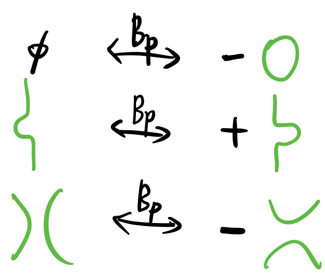
#(DW) = 2: $1 \to 1$

#(DW) = 4: $2 \to 1$

#(DW) = 6: $3 \to 1$



In summary (in the continuum):



$$|\Psi_1\rangle = \sum_{\{\sigma_p^z\}} (-1)^{\#(DW)} |\{\sigma_p^z\}\rangle$$
$$= \sum_{\text{DW conf } c} (-1)^{\#(DW \text{ in } c)} |c\rangle$$

$$B_p |c\rangle = (-1)^{\#(DW \text{ in } c+\partial p) - \#(DW \text{ in } c)} |\underset{(\text{mod } 2)}{c+\partial p}\rangle$$
$$\Rightarrow B_p (-1)^{\#(DW \text{ in } c)} |c\rangle = (-1)^{\#(DW \text{ in } c+\partial p)} |c+\partial p\rangle$$
$$\Rightarrow B_p \sum_c (-1)^{\#(DW \text{ in } c)} |c\rangle = \sum_c (-1)^{\#(DW \text{ in } c+\partial p)} |c+\partial p\rangle$$
$$= \sum_c (-1)^{\#(DW \text{ in } c)} |c\rangle$$

$$\Rightarrow B_p |\Psi_1\rangle = |\Psi_1\rangle$$

$\Rightarrow |\Psi_1\rangle$ is the GS of H_1.

$$\begin{cases} |\Psi_1\rangle = \sum_{\{\sigma_p^z\}} \underbrace{(-1)^{\#DW}}_{\text{nonlocal sign.}} |\{\sigma_p^z\}\rangle \\ H_1 = -\sum_p \underbrace{B_p}_{\text{local term}} \end{cases}$$

difference of nonlocal signs → local

9.2. Gauging a global symmetry.

System with onsite global symmetry G:

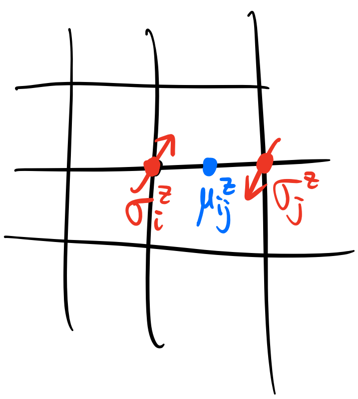
$$U(g) = \bigotimes_{\text{site } i} U_i(g) \quad \text{acting on} \quad \mathcal{H} = \bigotimes_i \mathcal{H}_i$$

$$[U(g), H] = 0, \quad \forall g \in G.$$

$$\Big\downarrow \text{gauge}$$

G gauge theory with local gauge symmetry / redundancy.

Gauge $\underbrace{\text{Ising paramagnet}}_{\text{(global } \mathbb{Z}_2 \text{ symm)}}$ to $\underbrace{\text{toric code}}_{(\mathbb{Z}_2 \text{ gauge symm})}$ on square lattice.



$$H_0 = -\sum_i \sigma_i^x, \quad |\Psi_0\rangle = \bigotimes_i |\sigma_i^x = 1\rangle$$

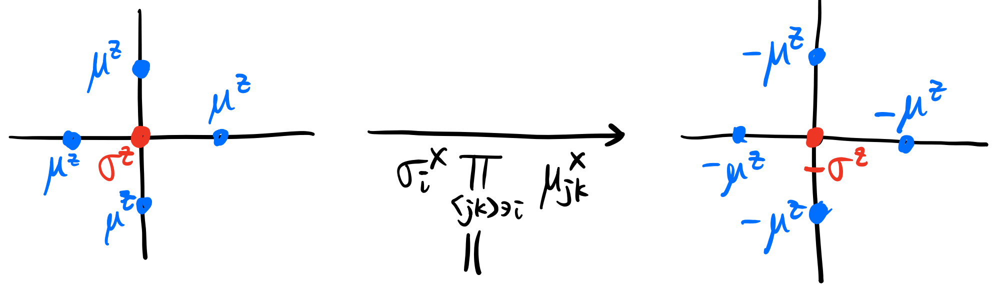
$$U = \bigotimes_i \sigma_i^x$$

$$[U, H_0] = 0$$

gauging procedure:

① Adding $\overset{U(1)}{\mathbb{Z}_2}$ gauge field $\overset{A_{ij}}{\mu_{ij}^z}$ on link $\langle ij \rangle$

② Enforcing local gauge symmetry (Gauss law) on the total Hilbert space.



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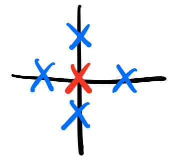
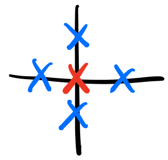
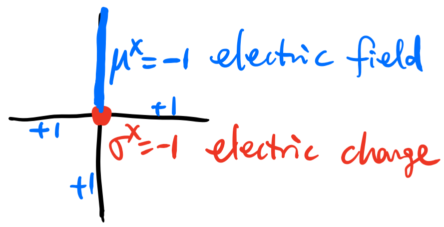


local gauge transformation

Gauss law:  $= \sigma_i^x \prod_{\langle jk\rangle \ni i} \mu_{jk}^x = 1$

$$\nabla \cdot \vec{E} = \rho$$

eg: 

$$\underbrace{\left(\sigma_i^x \prod_{\langle jk\rangle \ni i} \mu_{jk}^x\right)}_{\text{local gauge symm transf. at site } i} |\Psi_{phy}\rangle = |\Psi_{phy}\rangle$$

$$\prod_i \underbrace{\left(\sigma_i^x \prod_{\langle jk\rangle \ni i} \mu_{jk}^x\right)}_{\text{local symm}} = \prod_i \sigma_i^x = \underbrace{U}_{\text{global symm}}$$

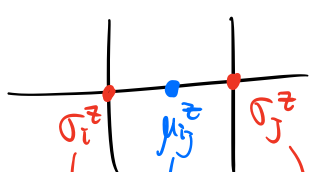
$(\mathbb{Z}_2)^N$ $\qquad\qquad$ $\mathbb{Z}_2$

infinite # of local symm/redundancies!

③ Minimal coupling

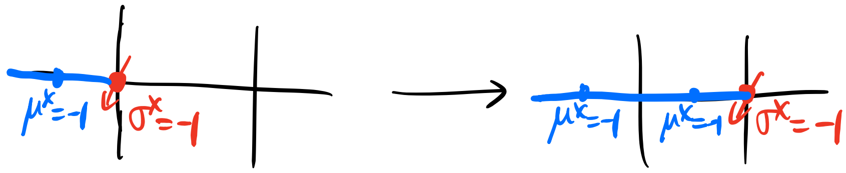
$$\sigma_i^z \sigma_j^z \longrightarrow \sigma_i^z \mu_{ij}^z \sigma_j^z$$

$$c_i^\dagger c_j \qquad\qquad c_i^\dagger e^{iA_{ij}} c_j$$

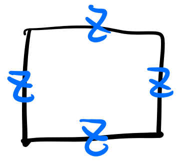


flip electric field

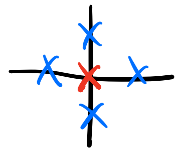
flip $\mathbb{Z}_2$ charge

eg. 

④ Adding zero flux condition and Gauss law to Hamiltonian.

$$-\sum_p \prod_{\langle ij\rangle\in\partial p} \mu_{ij}^z$$ 

$$\left(\oint_\square dx^\mu A_\mu\right)^2 = \vec{B}^2$$

$$-\sum_i \sigma_i^x \prod_{\langle jk\rangle\ni i} \mu_{jk}^x$$ 

For Ising paramagnet:

$$H_0 = -\sum_i \sigma_i^x \quad \left(-J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z\right)$$

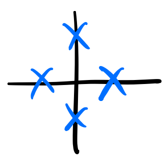
$\Big\downarrow$ gauging

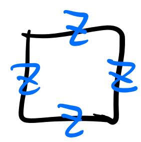
$$\widetilde{H}_0 = -\sum_i \sigma_i^x \quad \left(-J\sum_{\langle ij\rangle}\sigma_i^z\mu_{ij}^z\sigma_j^z\right)$$

$$-\sum_i \sigma_i^x \prod_{\langle jk\rangle\ni i}\mu_{jk}^x - \sum_p \prod_{\langle ij\rangle\in\partial p}\mu_{ij}^z$$

$\sigma_i^x=+1$ to minimize E

$$= -\sum_i \prod_{\langle jk\rangle\ni i}\mu_{jk}^x - \sum_p \prod_{\langle ij\rangle\in\partial p}\mu_{ij}^z$$

$$= -\sum_i$$ 

$$-\sum_p$$ 

= toric code model.