Quantum Double Models (QDM)

Kitaev, Annals of phys. 303, 2 (2003)

arXiv 1997

QDM : generalization of $\mathbb{Z}_2$ toric code
to lattice G gauge theory.
↳ finite (may be non-Abelian) group.

$$D(G) = \underset{\substack{\downarrow \\ \text{flux}}}{G} \times \underset{\substack{\downarrow \\ \text{charge}}}{\hat{G}} \quad \text{for Abelian } G.$$

(double)

2.1. The model based on a group algebra.

G.

$\mathcal{H} := \mathbb{C}[G] = \{ \sum_g a_g \cdot g \mid g \in G,\ a_g \in \mathbb{C} \}$ be the group algebra.

↓ Hilbert space with orthonormal basis $|g\rangle$

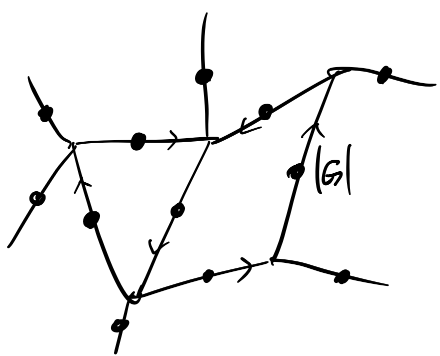
$\dim \mathcal{H} = |G|$

Def : $L_+^g |z\rangle = |gz\rangle$ $\quad\quad T_+^h |z\rangle = \delta_{h,z} |z\rangle$

$L_-^g |z\rangle = |z g^{-1}\rangle$ $\quad\quad T_-^h |z\rangle = \delta_{h^{-1},z} |z\rangle$

Satisfy : $L_+^g T_+^h = T_+^{gh} L_+^g$, $\quad L_+^g L_-^h = T_-^{hg^{-1}} L_+^g$

$L_-^g T_+^h = T_+^{hg^{-1}} L_-^g$, $\quad L_-^g T_-^h = T_-^{gh} L_-^g$

Hibert space : 

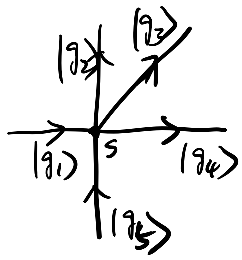
$|G|$-dim vector space on each link of $\forall$ 2D lattice.

$$, \quad g \in G.$$

QDM : $H = -\sum_s A_s - \sum_p B_p$

A_s

A_s^g



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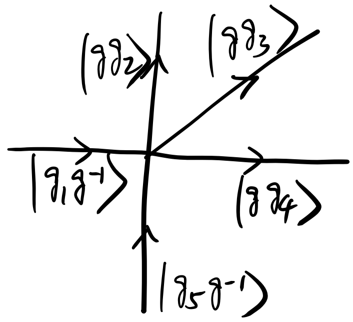
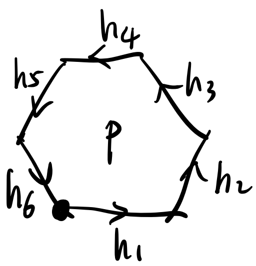


$$A_s^g := \prod_{\substack{\text{edge } l \\ \text{of } s}} L_\pm^g \,, \qquad A_s := \frac{1}{|G|}\sum_{g\in G} A_s^g$$

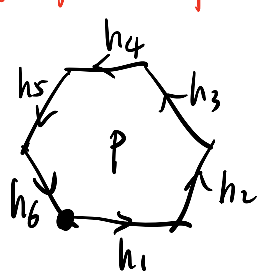
↳ gauge transf.

B_p

B_p^h



$$= \underbrace{\delta_{h_1 h_2 \cdots h_6 ,\, h}}_{\text{flux is } h}$$



$$B_p^h = \sum_{h_1 \cdots h_6 = h} \prod_{m=1}^{6} T_\pm^{h_m} \,, \qquad B_p := B_p^{h=1}$$

↳ zero flux condition for p.

relation

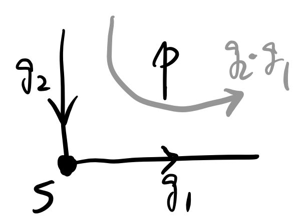
$$\left.\begin{array}{l} A_s^2 = A_s \\ B_p^2 = B_p \end{array}\right\} \text{projectors}$$

$$[A_s, A_{s'}] = 0$$

$$[B_p, B_{p'}] = 0$$

$$\boxed{[A_s, B_p] = 0}$$

A_s



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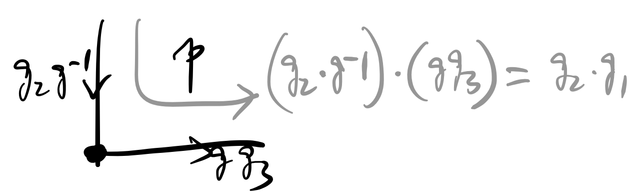
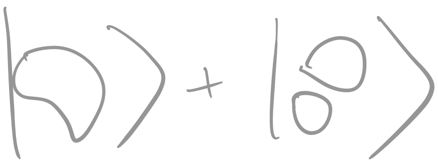


Ground state : $A_s|\Psi\rangle = B_p|\Psi\rangle = |\Psi\rangle, \ \forall s, p$

gauge transf.   flat connection condition

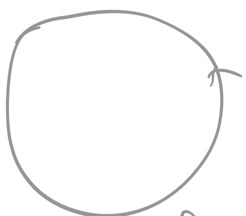
$\Rightarrow |\Psi\rangle$: flat connections up to gauge equivalence.

TC. $|\Psi\rangle =$



$+ \cdots$

$g = e^{iA} \in U(1)$



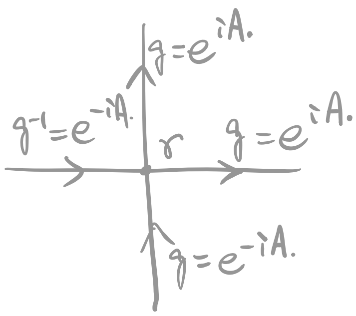
flat : $\Phi_c := \int_c A_\mu dx^\mu = 0$

$\downarrow$

$\prod g_1 \cdots g_N = 1 \Leftrightarrow B_p^{h=1}$

$\boxed{g_{ij} = e^{iA_{ij}}}$ → continuous

→ discrete



Gauss law

$\nabla \cdot \vec{A} = \rho$

$\|$

$A(\vec{r}+\hat{x}) - A(\vec{r})$

$+ A(\vec{r}+\hat{y}) - A(\vec{r})$

$\downarrow$

$g_{\vec{r}+\hat{x}} \, g_{\vec{r}}^{-1} \, g_{\vec{r}+\hat{y}} \, g_{\vec{r}}^{-1}$

Important notice:

The courses of week 3 (on 2021-09-27 and 2021-09-29) will be cancelled. And the ending date of the course will be postponed to week 13 of the fall semester.

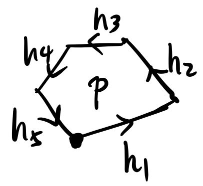
From 2021-10-04 on, we will use another Tencent Meeting Room 9952830954 (password 654321) of the university account, such that there are more storage space for the recordings.

Recall. QDM.

$$H = -\sum_s A_s - \sum_p B_p$$

$$[A_s, B_p] = 0$$

① $B_p |\Psi\rangle = |\Psi\rangle, \quad \forall p$ $\qquad \xrightarrow{h} \; = \; \xleftarrow{h^{-1}}$

$\Rightarrow$ 

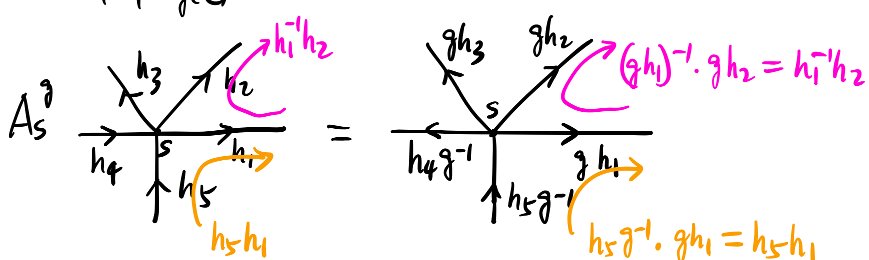
$h_1 h_2 \cdots h_5 = 1 \in G.$

(flat connection condition
zero flux condition)

② $A_s |\Psi\rangle = |\Psi\rangle, \quad \forall s$

$$A_s := \frac{1}{|G|} \sum_{g\in G} A_s^g$$

A_s^g : gauge transformation.



A_s^g fluctuate $\{h_{ij}\}$ within the subspace $B_p = 1, \forall p$.

$\Rightarrow$ GS $|\Psi\rangle$ is an equal weight superposition of all flat connections that are gauge equivalent.

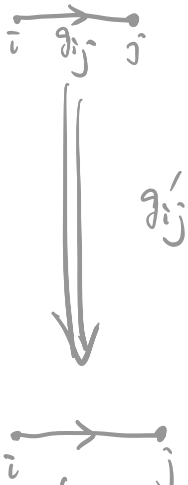
continuous: U(1) gauge theory

$$A_\mu(r) \longrightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

$$\Phi' = \oint_C A'_\mu dx^\mu = \oint_C (A_\mu - \partial_\mu \chi) dx^\mu = \oint_C A_\mu dx^\mu = \Phi$$

is gauge inv.

discrete :



$$\boxed{\begin{aligned} g_{ij} &= e^{i\int_{\text{site } i}^{\text{site } j} A_\mu dx^\mu} \\ g_i &= e^{i\chi(r_i)} \end{aligned}}$$

$\rightarrow$ gauge field

$\rightarrow$ gauge transf. para.

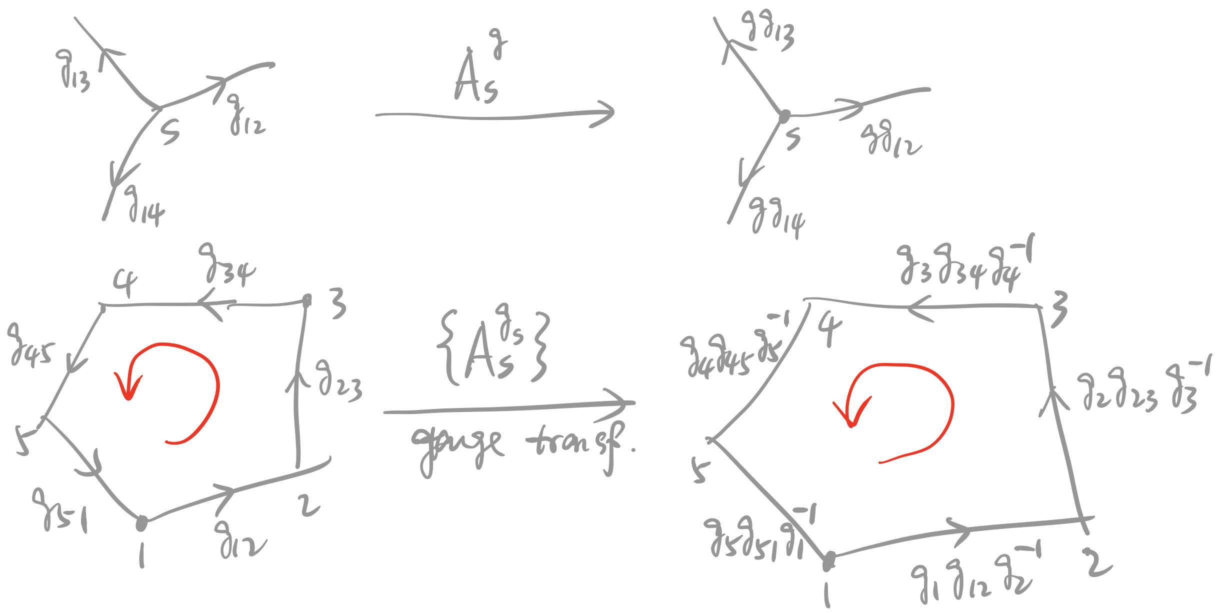
$$g'_{ij} = e^{i\int_i^j (A_\mu - \partial_\mu \chi) dx^\mu} = e^{i\int_i^j A_\mu dx^\mu - \chi(r_j) + \chi(r_i)}$$

$$= e^{i\chi(r_i)} e^{i\int_i^j A_\mu dx^\mu} e^{-i\chi(r_j)}$$

$$= g_i \cdot g_{ij} \cdot g_j^{-1}$$

$$g'_{ij} = g_i g_{ij} g_j^{-1}$$

eg:



$$\Phi = g_{12} \cdot g_{23} \cdots g_{51}$$

$$= e^{i\oint_1^1 A_\mu dx^\mu}$$

$$\Phi' = g_1 g_{12} g_2^{-1} \cdot g_2 g_{23} g_3^{-1} \cdot \ldots \cdot g_5 g_{51} g_1^{-1}$$

$$= g_1 (g_{12} \cdots g_{51}) g_1^{-1}$$

$$= g_1 \Phi g_1^{-1}$$

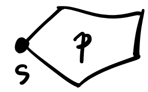
$$\mathrm{Tr}\, \Phi' = \mathrm{Tr}\, \Phi$$

2.2. Anyon excitations and ribbon operators

$$H = -\sum_s \underbrace{\frac{1}{|G|}\sum_{g\in G} A_s^g}_{\text{proj. to gauge inv. subspace}} - \sum_p \underbrace{B_p^{h=1}}_{\text{proj. to } h=1 \text{ zero flux}}$$

Excitations: ① non gauge inv: $\frac{1}{|G|}\sum_g A_s^g \neq 1$

② non zero flux: $B_p^{h\neq 1}$

To understand the excitations of QDM, we first try to understand the algebra of A_s^g, B_p^h for 

$$\begin{cases} A^g A^{g'} = A^{gg'} \\ (A^g)^\dagger = A^{g^{-1}} \\ B^h B^{h'} = \delta_{hh'} B^h \\ (B^h)^\dagger = B^h \\ A^g B^h = B^{ghg^{-1}} A^g \end{cases}$$

Def $\quad D_{(h,g)} = {}_h\!\lfloor_g := B_h \cdot A_g$

$$D_{(h_1,g_1)} \cdot D_{(h_2,g_2)} = \delta_{h_1, g_1 h_2 g_1^{-1}} D_{(h_1, g_1 g_2)}$$

The algebra generated by $D_{(h,g)}$ is called the Drinfeld quantum double $D(G)$ of G.

① Co-algebra structure $\quad \Delta: D(G) \longrightarrow D(G) \times D(G)$

how to act on two anyons by one $D(G)$ element

act on anyon b

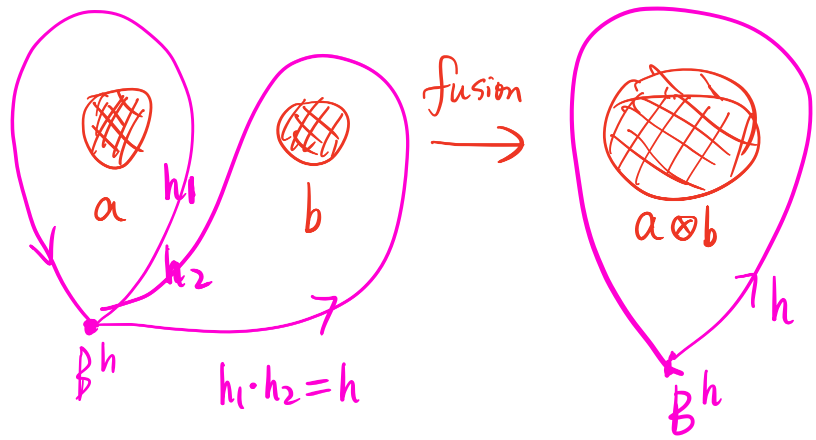
act on anyon a

$$\Delta(A^g) = A^g \otimes A^g$$

$$\Delta(B^h) = \sum_{h_1 h_2 = h} B^{h_1} \otimes B^{h_2}$$

$\Rightarrow$ Rep $D(G)$ is a tensor/monoidal category.

$a \otimes b \to c$



② $D(G)$ is quasi-triangular: $R = \sum_{g \in G} B^g \otimes A^g$

braiding

$\Rightarrow$ Rep $D(G)$ is a braided tensor category.

$D(G)$ is a quasi-triangular Hopf algebra.

Excitation space supports a representation of $D(G)$.

$\boxed{\text{Irreps of quantum double } D(G) \xleftrightarrow{1:1} \text{anyon excitation types}}$

(braiding, fusion)

Irreps of $D(G)$.

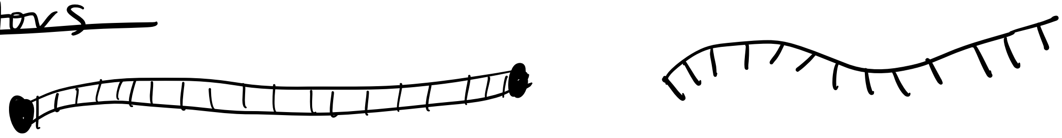
Let $u \in G$, $C = \{ g u g^{-1} \mid g \in G \}$ the conjugacy class,

$E = \{ g \in G \mid gu = ug \}$ the centralizer.

Claim. There is one irreducible rep (C, χ) for each conjugacy class C and each irrep χ of the centralizer group E.

flux (C), charge (χ)

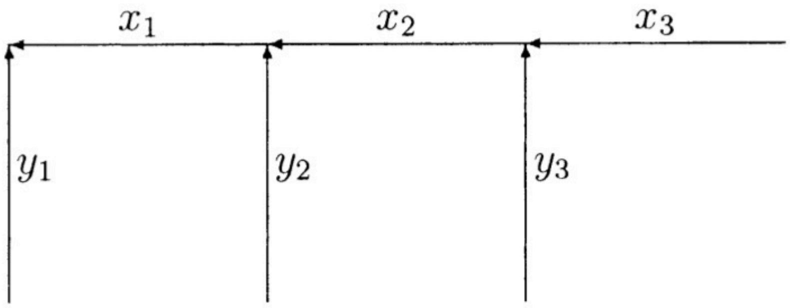
Ribbon operators



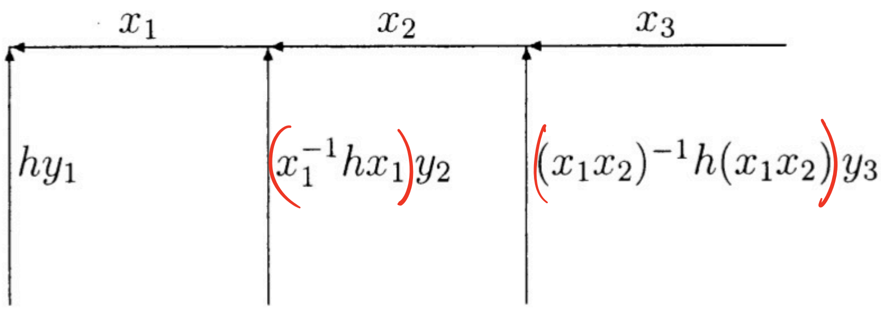
Creat a pair of anyonic excitations at the ends of the ribbon operators.

↳ finite width (framing)

$F^{(h,g)}(t)$



$= \quad \delta_{g,x_1x_2x_3}$



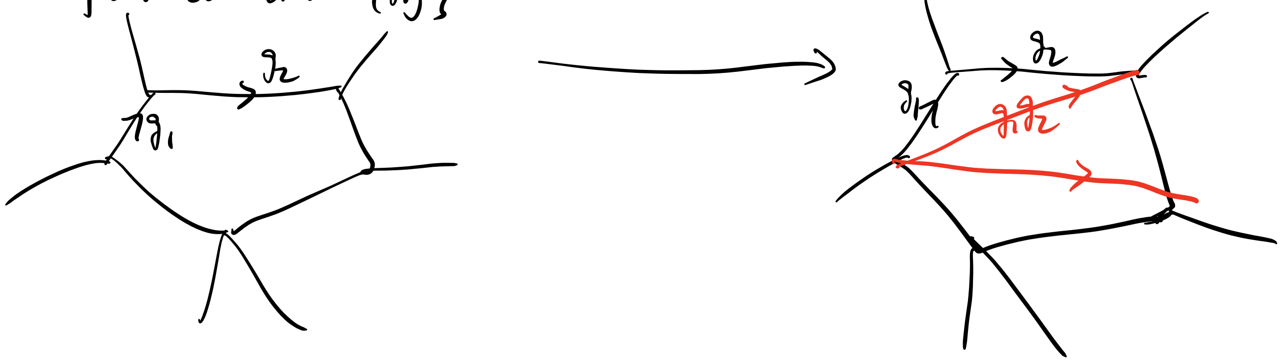
$$[F^{(h,g)}(t), B_p] = 0$$
$$[F^{(h,g)}(t), A_s] = 0 \qquad \text{for } s, p \neq \partial t$$

F gives us fusion, braiding of anyons.

2.3. Spacetime Path Integral Picture

$\forall$ 2D lattice — 2D triangulation

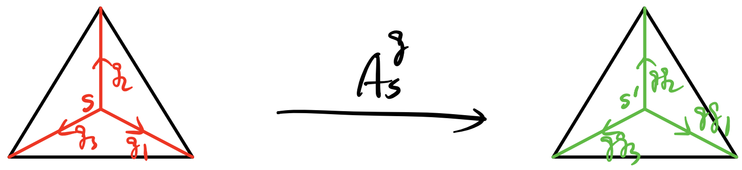
flat connection $\{g_{ij}\}$ — flat connection



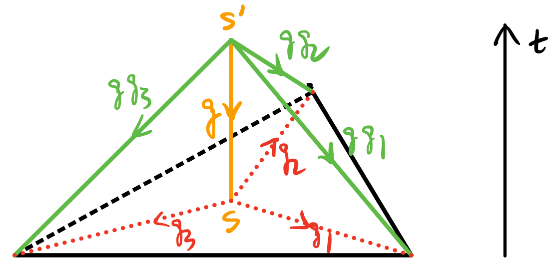
B_p operator: defined for $\forall$ triangle.

A_s^g operator:

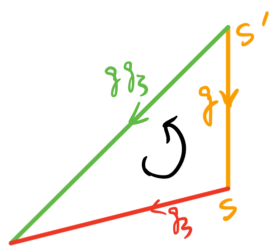
space :



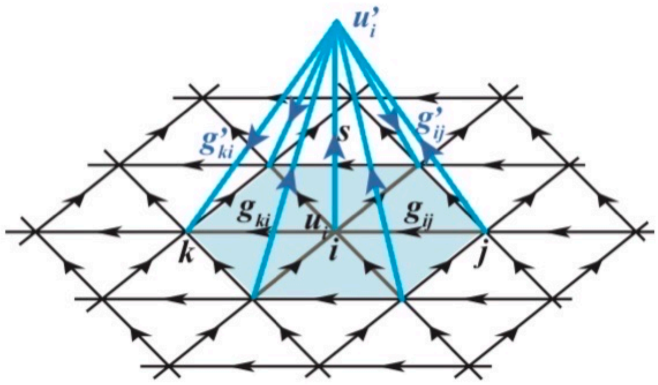
spacetime :

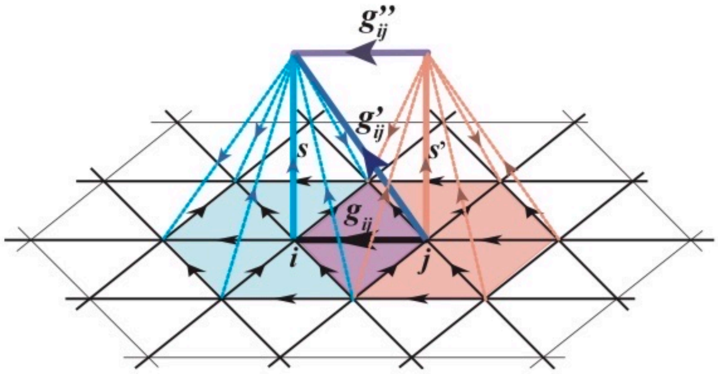


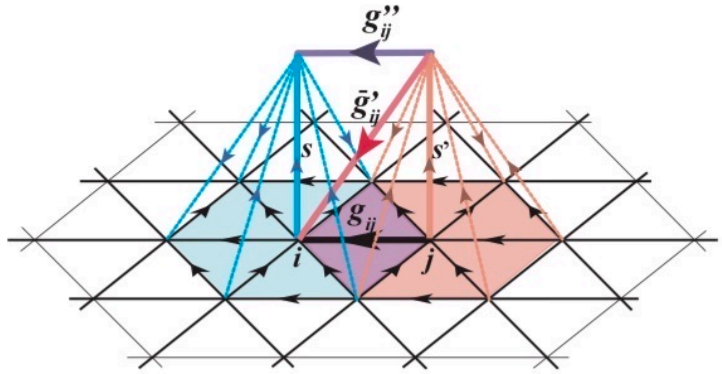
new link along t direction



flat connection for new spacetime triangles.







$$A_j^{g'} A_i^{g} = A_i^{g} A_j^{g'}$$