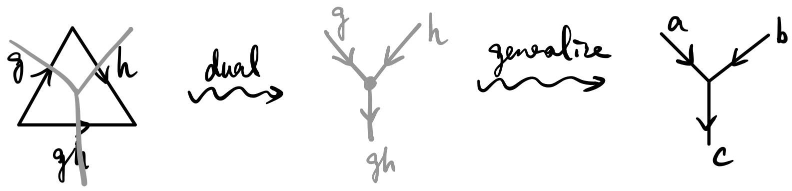
Fusion Categories and $\underbrace{\text{Turaev-Viro}}_{\text{spacetime}}$ - $\underbrace{\text{Levin-Wen}}_{\text{space}}$ Model

Motivation. Generalize TQDM of a group G.

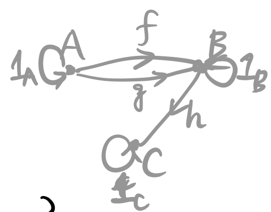
Hilbert space:



$g, h \in G$. $\qquad\qquad$ $a, b, c \in \mathcal{C}$

$$\mathcal{C} \begin{cases} \text{types: } a, b, c, \dots \\ \text{multiplication / tensor product} \\ \qquad\qquad\qquad\qquad a \times b \end{cases}$$

3.1. Categories



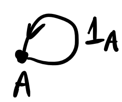
Def. A category $\mathcal{C}$ consists of

- A collection of objects $\mathrm{Obj}(\mathcal{C}) = \{A, B, \dots\}$
- A collection of morphisms $\mathrm{Hom}(A, B)$ for $\forall A, B \in \mathrm{Obj}(\mathcal{C})$
- A composition map for $\forall A, B, C \in \mathrm{Obj}(\mathcal{C}) = \mathcal{I}$

$$C_{A,B,C} : \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$
$$(f \;,\; g) \longmapsto g \circ f$$

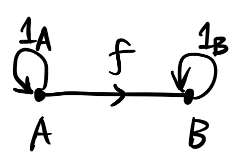
$$A \xrightarrow{f} B \xrightarrow{g} C \quad \longmapsto \quad A \xrightarrow{g \circ f} C$$

- An identity morphism $1_A = \mathrm{id}_A \in \mathrm{Hom}(A, A)$



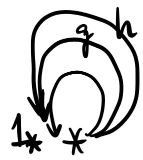
such that

- $(f \circ g) \circ h = f \circ (g \circ h)$ $\qquad$ associativity
- $f \circ 1_A = f = 1_B \circ f$ $\qquad$ identity



Examples (1) Set : set $A \xrightarrow{\text{map } f}$ set B

(2) a group G :



$Obj(\mathscr{C}) = \{*\}$

$Hom(*,*) = G$

$$* \xrightarrow{g} * \xrightarrow{h} * \quad = \quad * \xrightarrow{g\cdot h} *$$

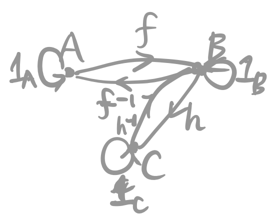
$$1_* = e \in G$$

Def: An isomorphism $f: A \to B$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

$$\begin{cases} g\circ f = 1_A \\ f\circ g = 1_B \end{cases}$$

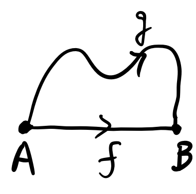
(3) groupoid : A cat $\mathscr{C}$ is a groupoid if every morphism is isomorphism.

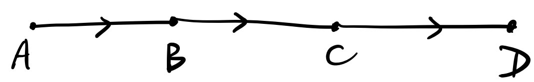


(4) fundamental groupoid of a topological space M :

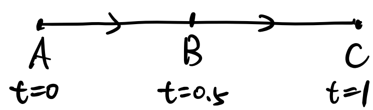
$obj = \{\text{points in } M\}$

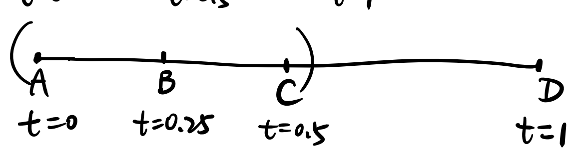
$Hom(A,B) = \{\text{Paths from } A \text{ to } B\} / \text{homotopy equivalence}$

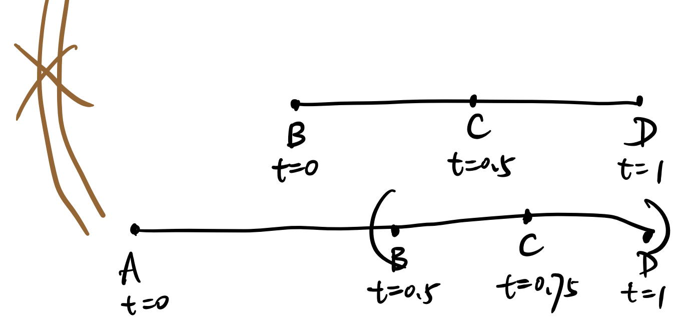




path: $I = [0,1] \to M$







(5) Grp : group $G \xrightarrow[\text{homomorphisms}]{\text{group}}$ group G'

(6) Vec : vector space $V \xrightarrow[\text{maps}]{\text{linear}}$ vector space W

(7) Top : topological space $M \xrightarrow[\text{maps}]{\text{continuous}}$ topological space N

Def A <u>functor</u> from categories $\mathscr{C}$ to $\mathscr{D}$ is a map sending

- any object $X \in \mathscr{C}$ to an object Y in $\mathscr{D}$.
- any morphism $f: X \to Y$ in $\mathscr{C}$ to a morphism
  $F(f) : F(X) \to F(Y)$ in $\mathscr{D}$.

such that

- F preserves identity : $F(1_X) = 1_{F(X)}$
- F preserves composition : $F(gh) = F(g)F(h)$

$$\underset{\mathscr{C}}{X \xrightarrow{f} Y} \quad \xRightarrow{F} \quad \underset{\mathscr{D}}{F(X) \xrightarrow{F(f)} F(Y)}$$

<u>Examples.</u> (1)

$$* \circlearrowleft g \in G \quad \xRightarrow{F} \quad V \circlearrowleft \rho(g)$$

$$G \xRightarrow{F} \mathrm{Vec}$$

representation of G.

(2) $H_n : \mathbf{Top} \longrightarrow \mathbf{Abel}$

$$\begin{tikzcd} M \arrow[d, "f"'] \\ N \end{tikzcd} \overset{F}{\Longrightarrow} \begin{tikzcd} H_n(M) \arrow[d, "f_*"] \\ H_n(N) \end{tikzcd}$$

<u>Def</u>. Given two functors $F, G : \mathscr{C} \to \mathscr{D}$, a natural transformation $\alpha : F \Rightarrow G$ assigns to every object X in $\mathscr{C}$ a morphism $\alpha_X : F(X) \to G(X)$ in $\mathscr{D}$, s.t.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

$$X \xrightarrow{f} Y \quad \begin{matrix} \overset{F}{\Longrightarrow} \\ \underset{G}{\Longrightarrow} \end{matrix} \quad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

<u>Example</u>. a group G, given two rep (functors) $\rho : G \to V$
$\rho' : G \to V'$

a natural transformation (intertwiner) is a map $f : V \to V'$, s.t. $f \circ \rho(g) = \rho'(g) \circ f$ for $g \in G$.

$$* \xrightarrow{g} * \quad \begin{matrix} \overset{\rho}{\Longrightarrow} \\ \underset{\rho'}{\Longrightarrow} \end{matrix} \quad \begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "f = \alpha_*"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "f"] \\ V' \arrow[r, "\rho'(g)"'] & V' \end{tikzcd}$$

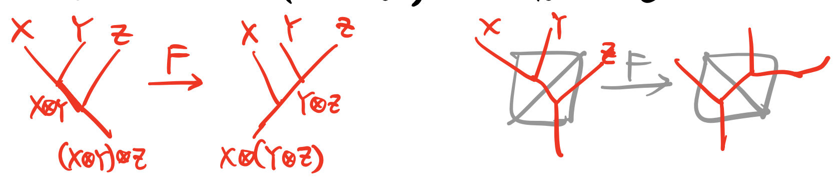
3.2. Fusion categories

Def. A monoidal category consists of

- a category $\mathcal{C}$
- a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a unit object $1 \in \mathcal{C}$
- a natural isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \quad \forall X, Y, Z \in \mathcal{C}$$

$\uparrow$
associator
F move



- natural isomorphism (left/right unitors) for $X \in \mathcal{C}$

$$l_X : 1 \otimes X \to X$$

$$r_X : X \otimes 1 \to X$$

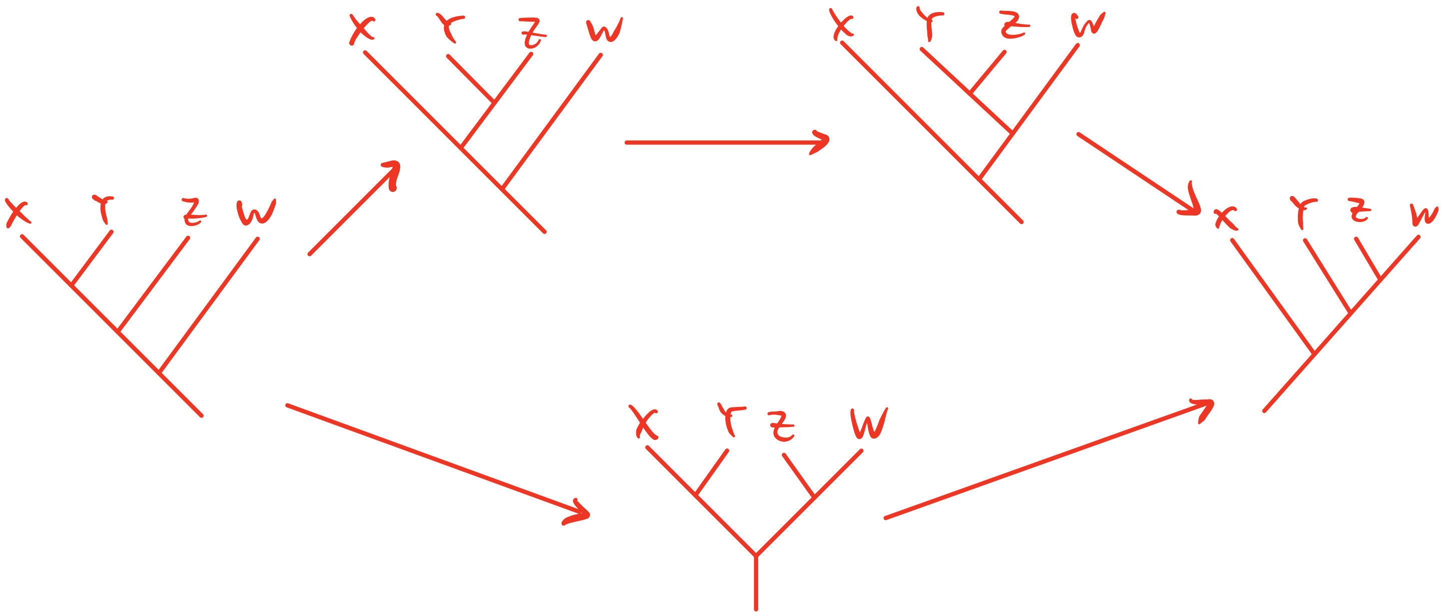
such that

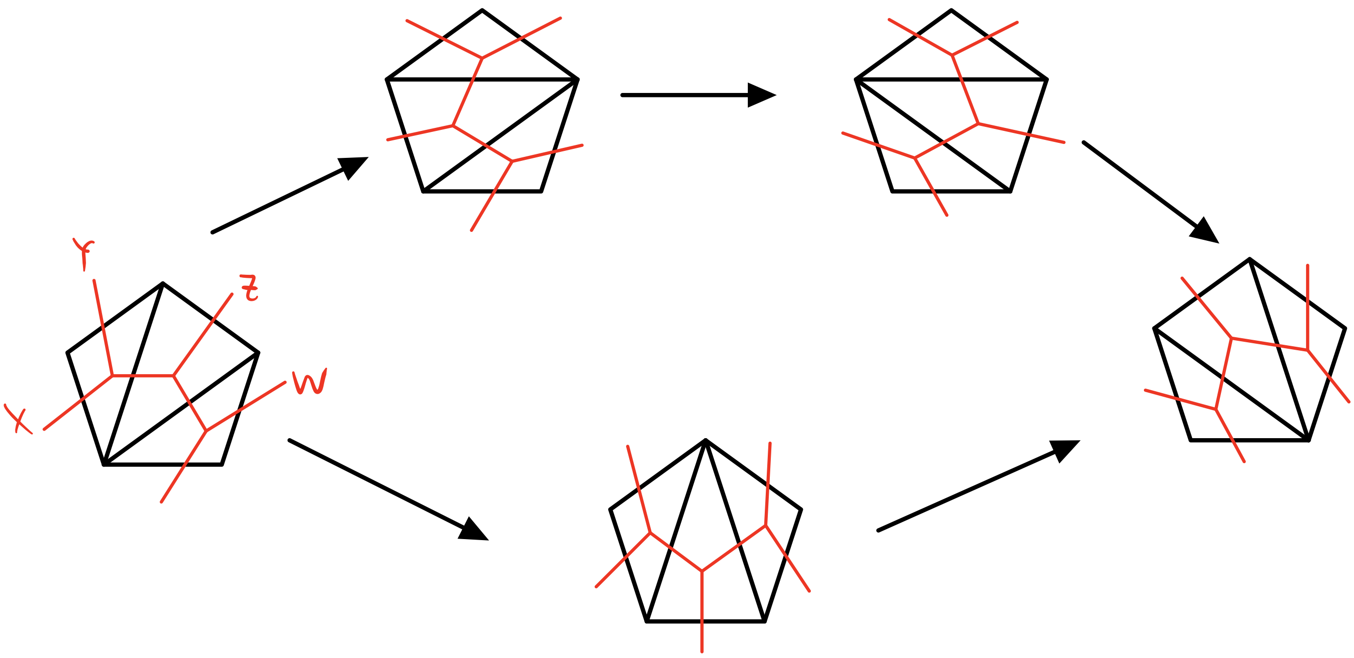
-

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X \otimes 1_Y"'] & \arrow[d, phantom, "\circlearrowleft"] & X \otimes (1 \otimes Y) \arrow[dl, "1_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

- pentagon equation:

$$\begin{tikzcd}
& (X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "a_{X, Y \otimes Z, W}"] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[dr, "1_X \otimes a_{Y,Z,W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "a_{X,Y,Z} \otimes 1_W"] \arrow[drr, "a_{X \otimes Y, Z, W}"'] & & \circlearrowleft & & X \otimes (Y \otimes (Z \otimes W)) \\
& & (X \otimes Y) \otimes (Z \otimes W) \arrow[urr, "a_{X,Y,Z \otimes W}"'] & &
\end{tikzcd}$$


x
y
z
w


x
y
z
w

Examples (1) $\mathrm{Rep}(G)$. obj = vector space

Hom = G-invariant linear maps.

$V \otimes W$

1 = trivial rep on $\mathbb{C}$.

$a_{x,y,z}$ is trivial

(2) Vec_G. obj = V_g for $\forall g \in G$

$$\mathrm{Hom}(V_g, V_h) = \begin{cases} \mathbb{C}, & g = h \\ 0, & g \neq h \end{cases}$$

$$V_g \otimes V_h = V_{gh}$$

a_{V_g, V_h, V_k} is trivial

$1 = V_e$, e identity element in G.

(3) $\mathrm{Vec}_G^{\nu_3}$: $a_{V_g, V_h, V_k} := \nu_3(g,h,k) \in U(1)$

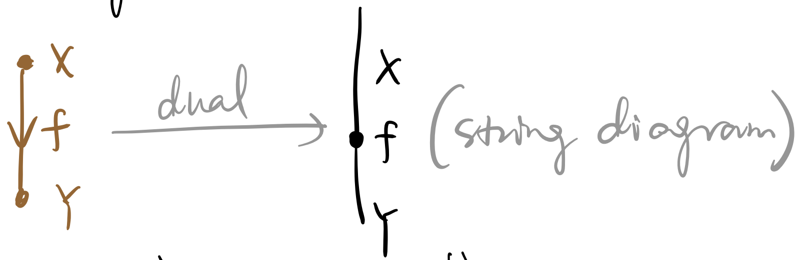
pentagon eq $\Leftrightarrow$ $d\nu_3 = 1$

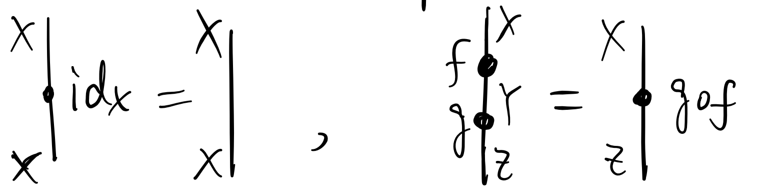
$\Leftrightarrow$ $\nu_3 \in Z^3(G, U(1))$

with dual a⊕b hom set is vector space

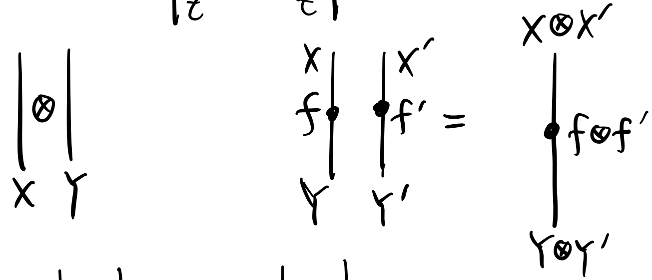
Def. A fusion category is a rigid, semisimple, linear monoidal category with finitely many isomorphism classes of simple objects such that $\mathrm{Hom}(1, 1) = \mathbb{C}$.

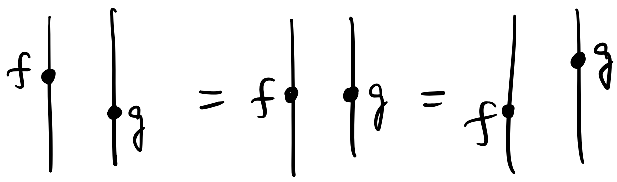
3.3. String diagram

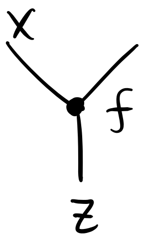




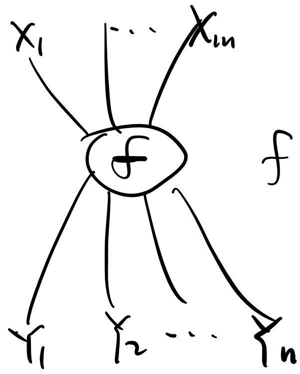
tensor: $X \otimes Y$





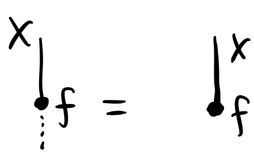


$f \in Hom(X \otimes Y, Z)$

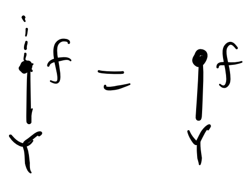


$f \in Hom(X_1 \otimes \cdots \otimes X_m, Y_1 \otimes \cdots \otimes Y_n)$

identity 1 :

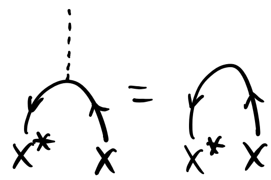


$f \in Hom(X, 1)$

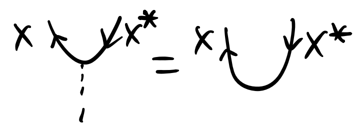


$f \in Hom(1, Y)$

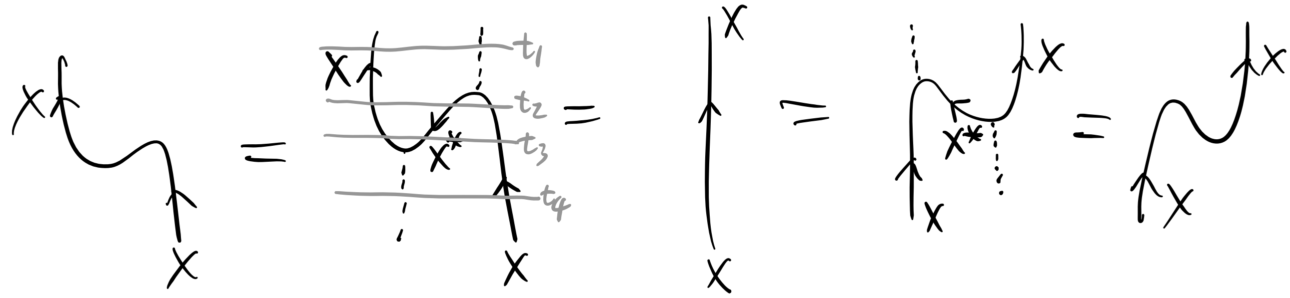
duals : unit $i_X : 1 \longrightarrow X^* \otimes X$



counit $e_X : X \otimes X^* \longrightarrow 1$



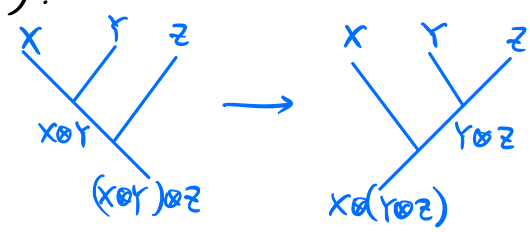
s.t.



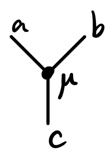
$$\begin{tikzcd}
\underset{t_1}{X \otimes 1} \arrow[r, "id_X \otimes i_X"] \arrow[drr, "r_X"'] & \underset{t_2}{X \otimes (X^* \otimes X)} \arrow[r, "a^{-1}_{X,X^*,X}"] \arrow[d, phantom, "\circlearrowright"] & \underset{t_3}{(X \otimes X^*) \otimes X} \arrow[r, "e_X \otimes id_X"] & \underset{t_4}{1 \otimes X} \arrow[dl, "l_X"] \\
 & & X &
\end{tikzcd}$$

associator (F move):

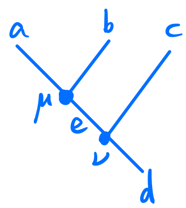
$a_{X,Y,Z}$:



In simple object basis :

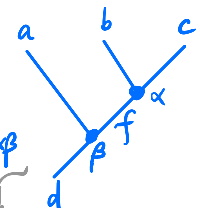
$V^{ab}_c := Hom(a \otimes b, c) \ni$  $\longleftrightarrow |a,b;c,\mu\rangle$

$N^{ab}_c := \dim V^{ab}_c$



$$= \sum_{f,\alpha,\beta} \left(F^{abc}_d\right)_{\underbrace{e\mu\nu}_{\text{row}},\,\underbrace{f\alpha\beta}_{\text{col}}}$$

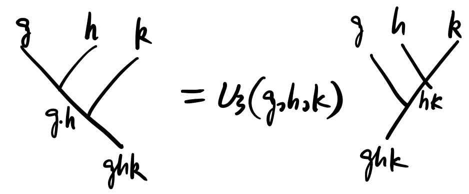
parameters



$$\sum_e N^{ab}_e N^{ec}_d = \sum_f N^{bc}_f N^{af}_d$$

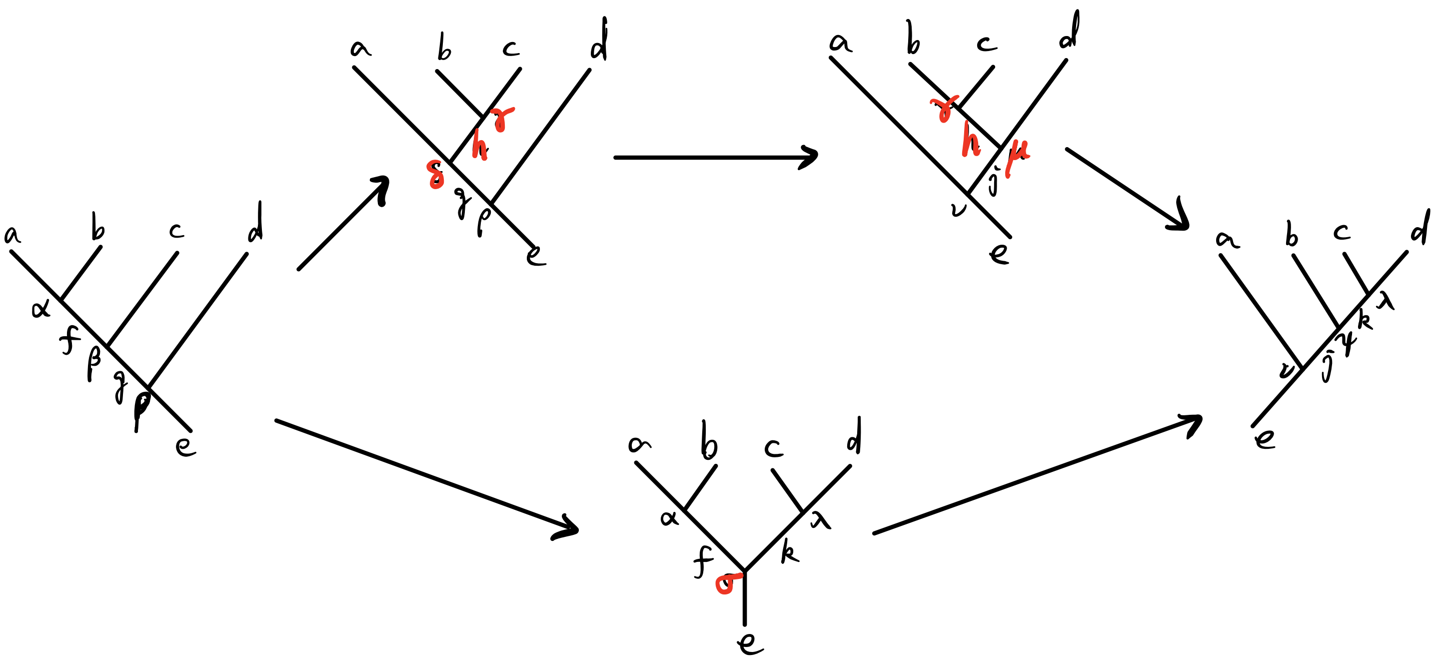
unitary fusion category : F is unitary.

$\mathcal{C} = \mathrm{Vec}_G^{\omega_3}$:



$\omega_3(g,h,k) := \left(F^{g,h,k}_{ghk}\right)_{1,1}$

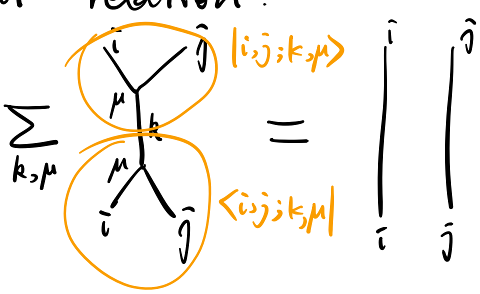
pentagon eq:



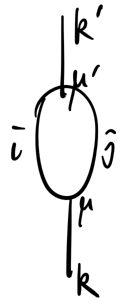
$$\sum_{h;\gamma,\delta,\mu} \left(F^{abc}_{g}\right)_{f\alpha\beta,h\gamma\delta} \left(F^{ahd}_{e}\right)_{g\delta\rho,j\mu\nu} \left(F^{bcd}_{j}\right)_{h\gamma\mu,k\lambda\psi}$$

$$= \sum_{\sigma} \left(F^{fcd}_{e}\right)_{g\beta\rho,k\lambda\sigma} \left(F^{abk}_{e}\right)_{f\alpha\sigma,j\psi\nu}$$

important relation :



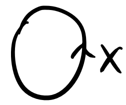
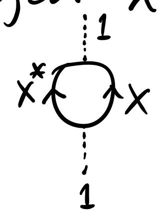
$$\sum_{k,\mu} |ij;k\mu\rangle\langle ij;k\mu| = I$$

 $= \delta_{kk'}\delta_{\mu\mu'}$ 

$\langle ij;k'\mu' | ij;k\mu\rangle = \delta_{kk'}\delta_{\mu\mu'}$

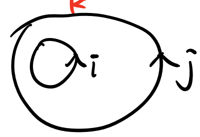
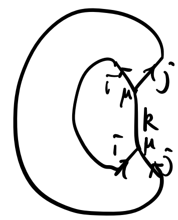
quantum dimension:

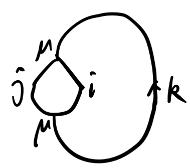
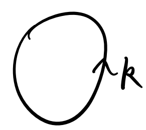
d_X of a simple object $X \in \mathcal{C}$ is

$d_X :=$  $=$  $\in \mathrm{Hom}(1,1) = \mathbb{C}$

$d_X > 0$.

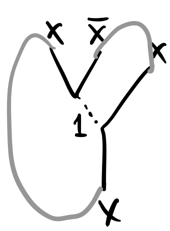
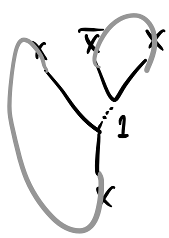
property: $d_i d_j = \sum_k N_k^{ij} d_k$

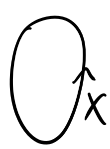
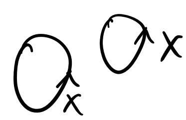
proof: $d_i d_j =$  $= \sum_{k,\mu}$ 

$= \sum_{k\mu}$  $= \sum_k N_k^{ij}$ 

$= \sum_k N_k^{ij} d_k$

$d_X = \left[\left(F_X^{X,\bar{X},X}\right)_{1,1}\right]^{-1}$

 $= \left(F_X^{X,\bar{X},X}\right)_{1,1}$ 

 $= \left(F_X^{X,\bar{X},X}\right)_{1,1}$ 

$d_X = \left(F_X^{X,\bar{X},X}\right)_{1,1} (d_X)^2$

physical meaning of d_x :

$$N_k^{ij}\, d_k = d_i\, d_j$$

$$(N^i)_k^{\,j}\, d_k = d_i\, d_j$$

$$j\left(\quad N^i \quad\right)\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = d_i \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} j$$

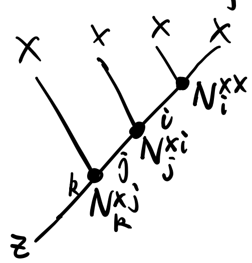
$\Rightarrow (d_1 \cdots d_n)^T$ is the eigenvector of N^i with eigenvalue d_i.

$N_k^{ij} \geq 0\,,\ d_i > 0$

$\Rightarrow$ d_i is the largest eigenvalue of N^i.

Q: what is dim of $(X)^{\otimes n}$, $n \to \infty$?

A:



$\dim X^{\otimes n} := \dim\left[\bigoplus_z \mathrm{Hom}(X^{\otimes n}, z)\right]$

$$\dim (X)^{\otimes n} = \sum_{i,j,k,\cdots} N_i^{XX}\, N_j^{Xi}\, N_k^{Xj} \cdots$$

$$= \sum_{i,j,k,\cdots} (N^X)_i^{\,X}\, (N^X)_j^{\,i}\, (N^X)_k^{\,j} \cdots$$

$$= \sum_z \left[(N^X)^n\right]_{X,z} \xrightarrow{n\to\infty} (d_X)^n$$

$\Rightarrow$ $\dim(X^{\otimes n}) \sim d_X^{\,n}$ for $n \to \infty$

quantum dimension of X.

eg: $\dim\left(\bullet_{\text{Majorana}} \quad \bullet_{\text{Majorana}}\right) = 2$

$\Rightarrow \dim\left(\bullet_{\text{Majorana}}\right) = \sqrt{2}$

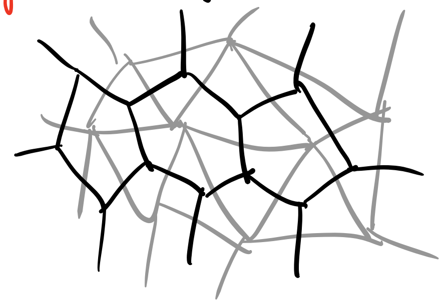
Finally, $\forall$ trivalent graph

label $\begin{cases} \text{edge by } X \in \mathrm{Obj}(\mathcal{C}) \\ \text{vertex by } \mathrm{Hom}(X \otimes Y, Z) \text{ or } \mathrm{Hom}(X, Y \otimes Z) \end{cases}$

$\updownarrow$ dual

triangulation of M_2



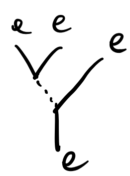
3.4. Examples.

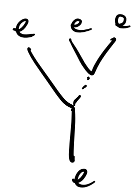
Classification of fusion cat ?

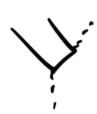
Very hard !!! $\mathcal{C} = \mathrm{Vec}_G \to$ classification of finite simple groups.
(already very hard !)

(1) $\mathrm{obj}(\mathcal{C}) = \{1, e\} = \mathbb{Z}_2$

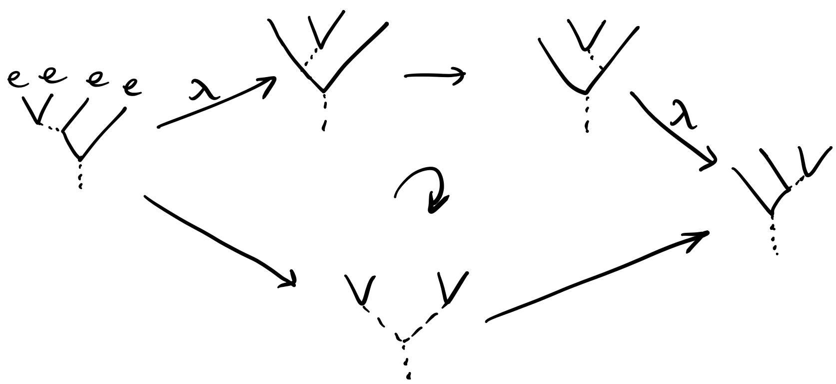
$e \otimes e = 1$, $e^* = e$

associator $a_{e,e,e}$: 

$= \lambda$ 

$a_{e,e,1}$:  $=$ 

pentagon:



$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \ni \omega_3 \begin{cases} ① \ \omega_3(a,b,c) = 1 \ (\forall a,b,c) : & \lambda = 1 \quad \text{toric code} \\ ② \ \omega_3(e,e,e) = -1 : & \lambda = -1 \quad \text{double semion.} \end{cases}$$

(2) Fibonacci ·

- $Obj(\mathcal{C}) = \{1, \tau\}$
- $\tau \otimes \tau = 1 \oplus \tau$ $\rightsquigarrow$ non-Abelian

$$N_1^{\tau\tau} = N_\tau^{\tau\tau} = 1$$

$$\tau^{\otimes n} = F_{n-2}\, 1 \oplus F_{n-1}\, \tau$$ where $F_n = 1, 1, 2, 3, 5, 8, \cdots$ is the Fibonacci number.

$$\Downarrow$$

$$\tau \otimes (\tau^{\otimes n}) = \tau \otimes (F_{n-2}\, 1 \oplus F_{n-1}\, \tau) = F_{n-2}\, \tau \oplus F_{n-1} (\tau \otimes \tau)$$

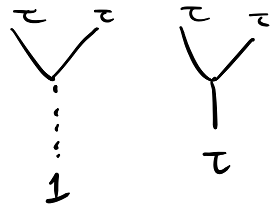
$$= F_{n-2}\, \tau \oplus F_{n-1} (1 \oplus \tau) = F_{n-1}\, 1 \oplus \underbrace{(F_{n-2} + F_{n-1})}_{F_n} \tau$$

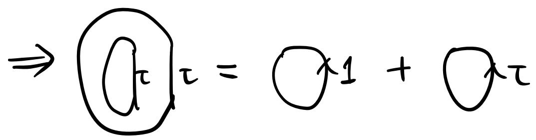
$$= F_{n-1}\, 1 \oplus F_n\, \tau$$

No class on 2021.10.20 and 2021.10.23 !

Prof. Liang Kong's talk

- $\tau \otimes \tau = 1 \oplus \tau$

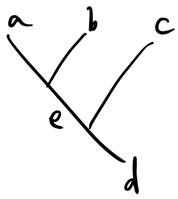




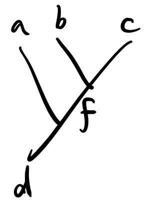
$\Rightarrow \quad (d_\tau)^2 = 1 + d_\tau$

$\Rightarrow \quad d_\tau = \frac{1+\sqrt{5}}{2} = \phi$ Golden ratio

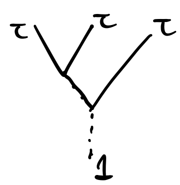
- F symbol.



$$= \sum_f \left(F^{abc}_d\right)_{e,f}$$



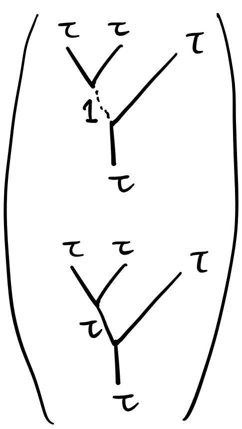
① $\tau \otimes \tau \otimes \tau \longrightarrow 1$



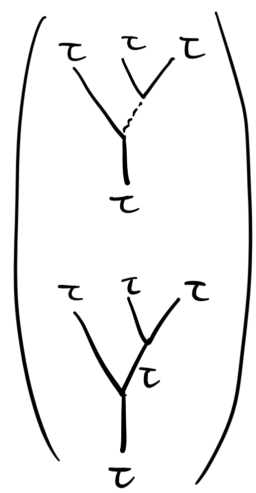
$$= \quad F^{\tau\tau\tau}_1$$

$F^{\tau\tau\tau}_1 = 1$

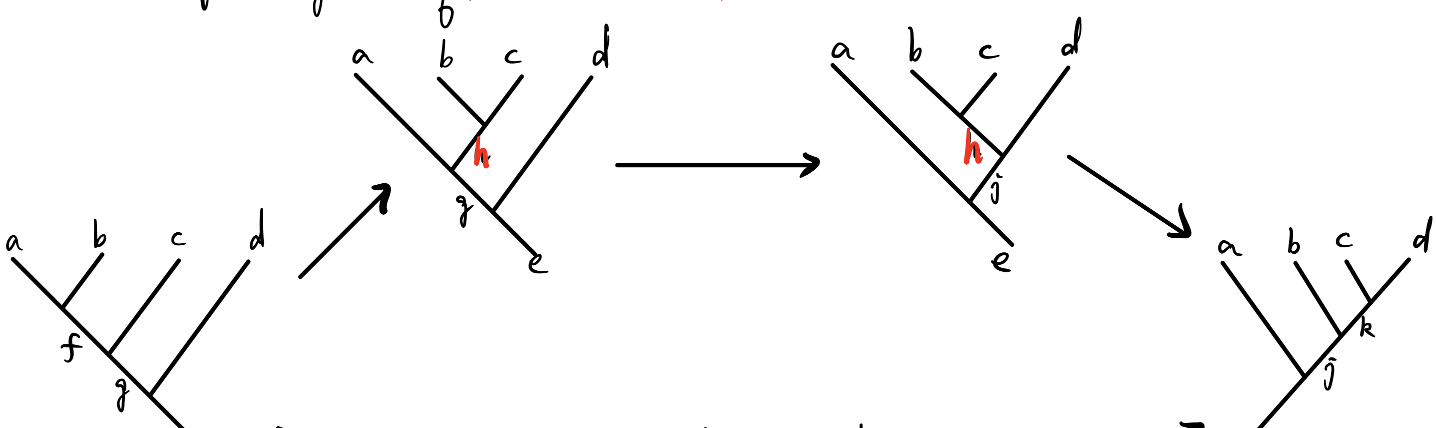
② $\tau \otimes \tau \otimes \tau \longrightarrow \tau$

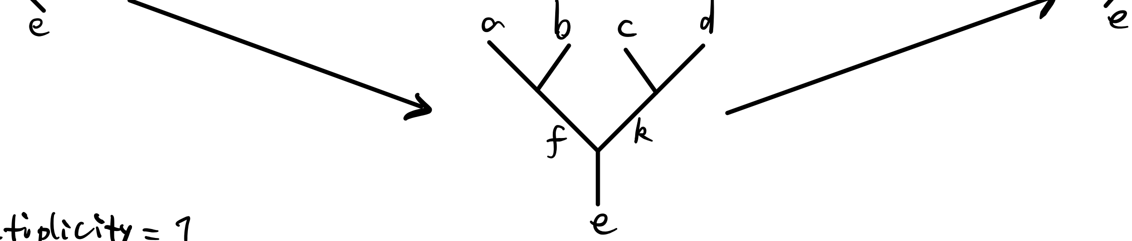


$$= \underbrace{\begin{pmatrix} \left(F^{\tau\tau\tau}_\tau\right)_{1,1} & \left(F^{\tau\tau\tau}_\tau\right)_{1,\tau} \\ \left(F^{\tau\tau\tau}_\tau\right)_{\tau,1} & \left(F^{\tau\tau\tau}_\tau\right)_{\tau,\tau} \end{pmatrix}}_{= \begin{pmatrix} \phi^{-1} & \sqrt{\phi^{-1}} \\ \sqrt{-\phi^{-1}} & -\phi^{-1} \end{pmatrix}}$$



- pentagon eq:



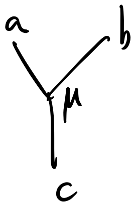


multiplicity = 1

$$\Downarrow$$

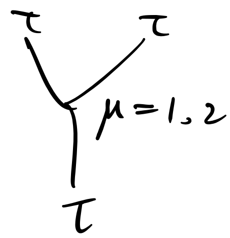
$$\sum_{h} \left(F^{abc}_{g}\right)_{f,h} \left(F^{ahd}_{e}\right)_{g,j} \left(F^{bcd}_{j}\right)_{h,k}$$

$$= \left(F^{fcd}_{e}\right)_{g,k} \left(F^{abk}_{e}\right)_{f,j}$$



$\mu \in \mathrm{Hom}(a \otimes b, c) = \mathbb{C}^{(\text{multiplicity})}$

If $\tau \otimes \tau = 1 \oplus \underset{\uparrow \atop \mu=1}{\tau} \oplus \underset{\uparrow \atop \mu=2}{\tau}$, then



3.5. Levin-Wen model

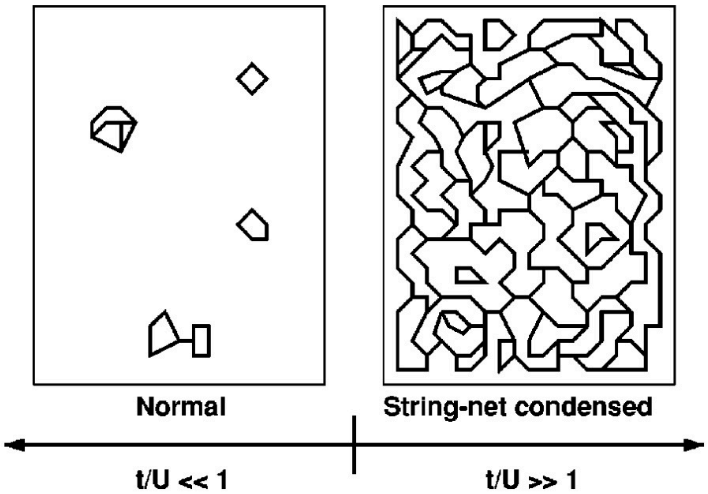
PRB 71, 045110 (2005)

- string-net condensation

string-net → trivalent graph with fusion category labels

condensation → $\langle b \rangle \neq 0$ $\quad |\Psi_{GS}\rangle = \sum_n \# (b^+)^n |0\rangle$

Σ (all possible string-net conf.)



Example : $H = -\sum_s \prod_{\text{star } s} X - t \sum_p \prod_{\partial p} Z - U \sum_{\text{links}} X$

$U \sum_{\text{links}} X$ → new string tension term

$\cdots\cdots$ $X=+1$

——— $X=-1$

① $t/U \ll 1$, $U \to +\infty$: $X=+1$ for all links

$|\Psi_{GS}\rangle = \bigotimes_{\text{link } \ell} |X=+1\rangle_\ell$ → product state.

② $t/U \gg 1$, $U \to 0$: TC,

$|\Psi\rangle = \Sigma$ (all closed loops) → string-net cond.

- string-net configuration :

color / label trivalent graph (V, E) by the fusion category $\mathcal{C}$.

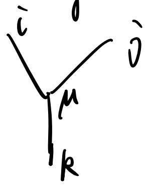
$E \to \mathrm{obj}(\mathcal{C})$

$V \to \mathrm{Hom}_{\mathcal{C}}(i \otimes j, k)$

(1) string type:

simple object $i = 0, 1, \cdots, N$ (N finite)

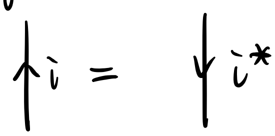
(2) branching rule:



$\mu \in \mathrm{Hom}(i \otimes j, k)$ $\overset{\text{assume}}{=\!=}$ $\begin{cases} 0, & \text{if } i, j \text{ can not fuse into } k. \\ \mathbb{C}, & \text{if } i, j \text{ can fuse into } k \end{cases}$

multiplicity $= 1$ $\Updownarrow$

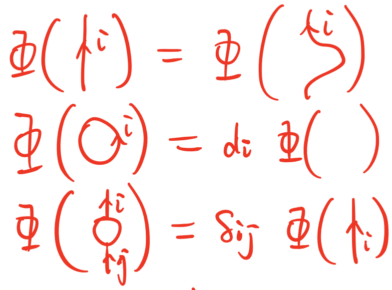
(3) string orientation:



- Ground state wave function

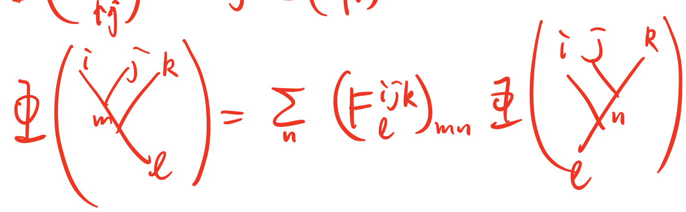
$$|\Psi_{GS}\rangle = \sum_{\substack{\text{string-net} \\ \text{conf. } X}} \Phi(X)\, |X\rangle$$

$\{\Phi(X)\}$ are related to each other under local move $X \to X'$:



invariant under ambient isotopy

scale invariance



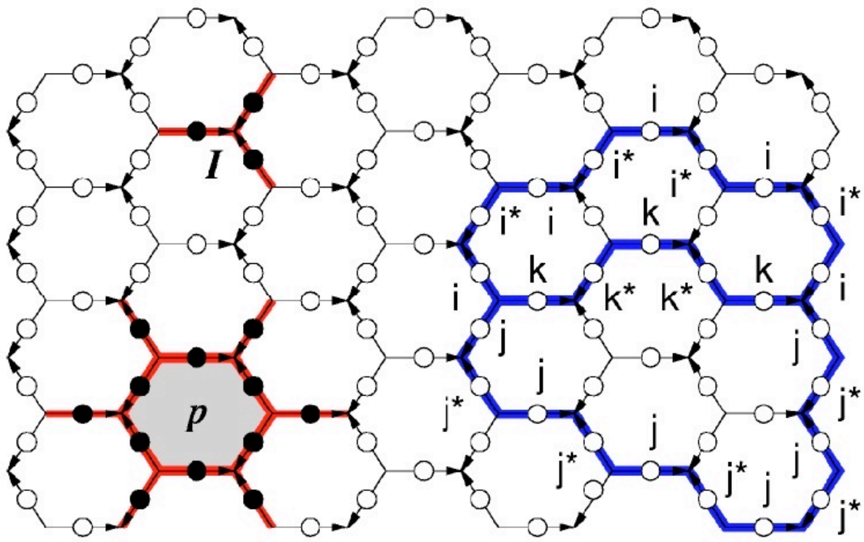
consistency condition: pentagon equation, rigid structure, ...

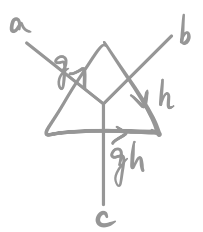
- Exactly solvable commuting-projector Hamiltonian

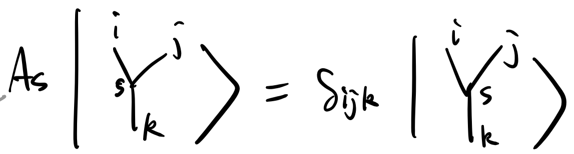
$$H = -\sum_s A_s - \sum_p B_p$$

A_s: constraint on conf.

B_p: "gauge transf"



(dual to QDM: )



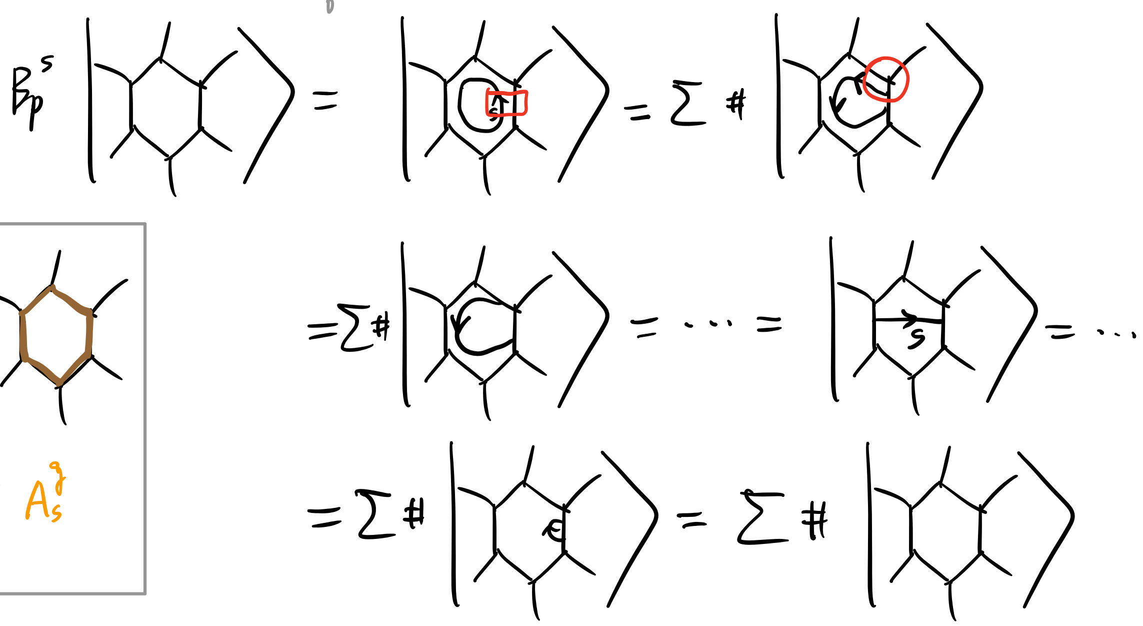
3-body interaction

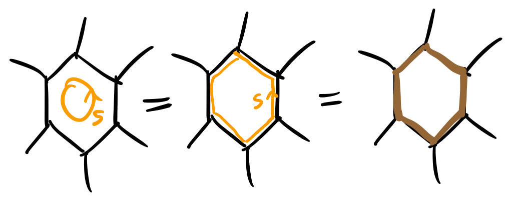
$\delta_{ijk} = \begin{cases} 1, & N_k^{ij} \neq 0 \\ 0, & N_k^{ij} = 0 \end{cases}$

$B_p = \frac{1}{D^2} \sum_{s \in obj(\mathcal{C})} d_s \cdot B_p^s$,

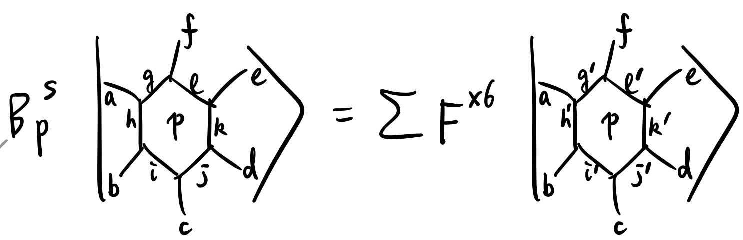
q dim

$D^2 := \sum_s d_s^2$ total q dim.



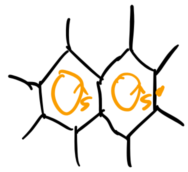


similar to $A_s = \frac{1}{|G|} \sum_g A_s^g$



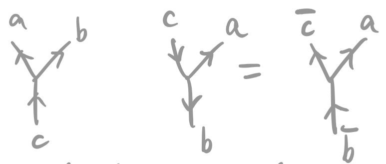
12-body interaction

properties :
$$\begin{cases} [A_s, A_{s'}] = 0 \\ [B_p, B_{p'}] = 0 \\ [A_s, B_p] = 0 \\ A_s^2 = A_s \\ B_p^2 = B_p \end{cases}$$



$$B_p^2 = \left(\frac{1}{D^2}\sum_s d_s \bigcirc_s\right)^2 = \frac{1}{D^4}\sum_{ss'} d_s d_{s'} \bigcirc_{s'}\bigcirc_s$$

$$= \frac{1}{D^4}\sum_{ss'}\sum_{t\mu} d_s d_{s'} \bigcirc_t = \frac{1}{D^4}\sum_{ss'}\sum_{t\mu} d_s d_{s'} \bigcirc_{s'} \bigcirc_t$$



$\mathrm{Hom}(a\otimes b, c) \cong \mathrm{Hom}(\bar{c}\otimes a, \bar{b})$

$N_c^{ab} = N_{\bar{b}}^{\bar{c}a}$

$$= \frac{1}{D^4}\sum_{ss't} d_s d_{s'} N_t^{s's} \bigcirc_t$$

$$= \frac{1}{D^4}\sum_{ss't} d_s d_{s'} N_{\bar{s}}^{\bar{t}s'} \bigcirc_t$$

$d_s = \bigcirc_s = \bigcirc_s = d_{\bar{s}}$

$$= \frac{1}{D^4}\sum_{s't} d_{s'}\left(\sum_{\bar{s}} N_{\bar{s}}^{\bar{t}s'} d_{\bar{s}}\right) \bigcirc_t$$

$d_i d_j = \sum_k N_k^{ij} d_k$

$$= \frac{1}{D^4}\sum_{s't} d_{s'}(d_{\bar{t}} d_{s'}) \bigcirc_t$$

$$= \frac{1}{D^2}\left[\frac{1}{D^2}\sum_{s'}(d_{s'})^2\right]\sum_t d_t \bigcirc_t$$

$$= \frac{1}{D^2}\sum_t d_t \bigcirc_t$$

$$= B_p$$

$\Rightarrow$ commuting-projector Hamiltonian

$$\Rightarrow \begin{cases} A_s|\Psi_{GS}\rangle = |\Psi_{GS}\rangle \\ B_p|\Psi_{GS}\rangle = |\Psi_{GS}\rangle \end{cases}$$

$B_p = \sum d_s \bigcirc_s$