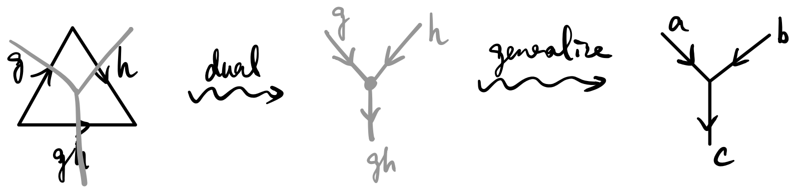
Fusion Categories and $\underbrace{\text{Turaev-Viro}}_{\text{spacetime}}$ - $\underbrace{\text{Levin-Wen}}_{\text{space}}$ Model

Motivation. Generalize TQDM of a group G.

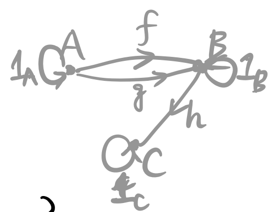
Hilbert space:



$g, h \in G$. $\qquad a, b, c \in \mathcal{C}$

$$\mathcal{C} \begin{cases} \text{types: } a, b, c, \ldots \\ \text{multiplication / tensor product} \\ \qquad\qquad a \times b \end{cases}$$

3.1. Categories



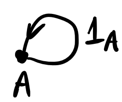
Def. A <u>category</u> $\mathcal{C}$ consists of

- A collection of objects $Obj(\mathcal{C}) = \{A, B, \cdots\}$
- A collection of morphisms $Hom(A, B)$ for $\forall A, B \in Obj(\mathcal{C})$
- A composition map for $\forall A, B, C \in Obj(\mathcal{C}) = \mathcal{I}$

$$C_{A,B,C} : \quad Hom(A, B) \times Hom(B, C) \longrightarrow Hom(A, C)$$
$$(f \;,\; g) \quad \longmapsto \quad g \circ f$$

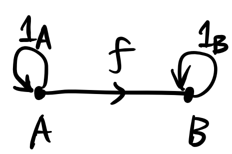
$$A \xrightarrow{f} B \xrightarrow{g} C \quad \longmapsto \quad A \xrightarrow{g \circ f} C$$

- An identity morphism $1_A = id_A \in Hom(A, A)$



such that

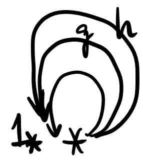
- $(f \circ g) \circ h = f \circ (g \circ h)$ associativity
- $f \circ 1_A = f = 1_B \circ f$ identity



Examples

(1) Set : set $A \xrightarrow{\text{map } f}$ set B

(2) a group G :



$Obj(\mathscr{C}) = \{*\}$

$Hom(*,*) = G$

$$* \xrightarrow{g} * \xrightarrow{h} * \;=\; * \xrightarrow{g\cdot h} *$$

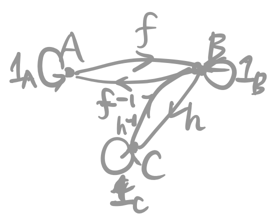
$$1_* = e \in G$$

Def: An isomorphism $f: A \to B$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

$$\begin{cases} g\circ f = 1_A \\ f\circ g = 1_B \end{cases}$$

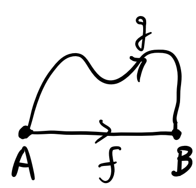
(3) groupoid : A cat $\mathscr{C}$ is a groupoid if every morphism is isomorphism.

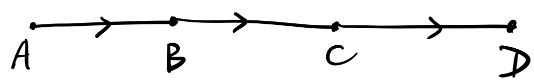


(4) fundamental groupoid of a topological space M :

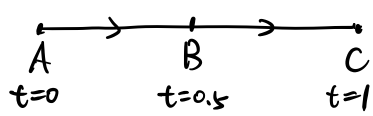
obj = {points in M}

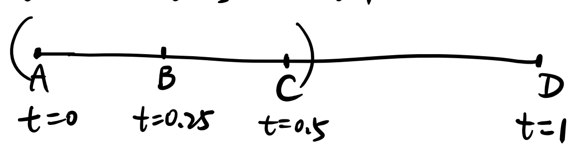
$Hom(A,B) = \{\text{Paths from } A \text{ to } B\} / \text{homotopy equivalence}$

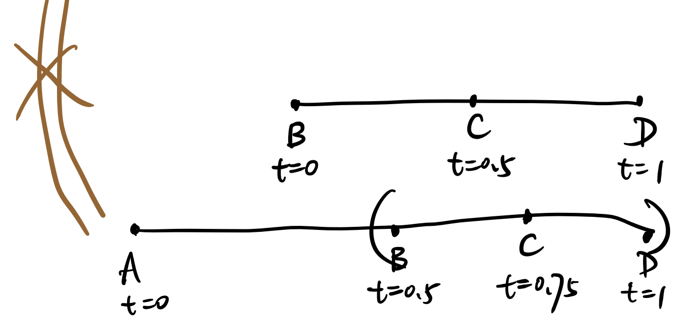




path: $I = [0,1] \to M$







(5) Grp : group $G \xrightarrow[\text{homomorphisms}]{\text{group}}$ group G'

(6) Vec : vector space $V \xrightarrow[\text{maps}]{\text{linear}}$ vector space W

(7) Top : topological space $M \xrightarrow[\text{maps}]{\text{continuous}}$ topological space N

Def A functor from categories $\mathcal{C}$ to $\mathcal{D}$ is a map sending

- any object $X \in \mathcal{C}$ to an object Y in $\mathcal{D}$.
- any morphism $f: X \to Y$ in $\mathcal{C}$ to a morphism

  $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$.

such that

- F preserves identity : $F(1_X) = 1_{F(X)}$
- F preserves composition : $F(gh) = F(g)F(h)$

$$\underset{\mathcal{C}}{X \xrightarrow{f} Y} \quad \xRightarrow{F} \quad \underset{\mathcal{D}}{F(X) \xrightarrow{F(f)} F(Y)}$$

Examples. (1) $* \circlearrowleft g \in G \xRightarrow{F} V \circlearrowleft \rho(g)$

$$G \xRightarrow{F} \mathrm{Vec}$$

representation of G.

(2) $H_n : \mathbf{Top} \longrightarrow \mathbf{Abel}$

$$\begin{array}{c} M \\ f\downarrow \\ N \end{array} \quad \overset{F}{\Longrightarrow} \quad \begin{array}{c} H_n(M) \\ \downarrow f_* \\ H_n(N) \end{array}$$

<u>Def</u>. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha : F \Rightarrow G$ assigns to every object X in $\mathcal{C}$ a morphism $\alpha_X : F(X) \to G(X)$ in $\mathcal{D}$, s.t.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

$$X \xrightarrow{f} Y \quad \overset{F}{\underset{G}{\rightrightarrows}} \quad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

<u>Example</u>. a group G, given two rep (functors) $\rho : G \to V$
$\rho' : G \to V'$

a natural transformation (intertwiner) is a map $f : V \to V'$. s.t. $f \circ \rho(g) = \rho'(g) \circ f$ for $g \in G$.

$$* \xrightarrow{g} * \quad \overset{\rho}{\underset{\rho'}{\rightrightarrows}} \quad \begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "f = \alpha_*"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "f"] \\ V' \arrow[r, "\rho'(g)"'] & V' \end{tikzcd}$$

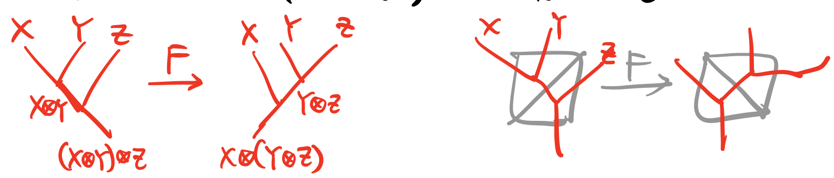
3.2. Fusion categories

Def. A <u>monoidal category</u> consists of

- a category $\mathcal{C}$
- a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a unit object $\mathbb{1} \in \mathcal{C}$
- a natural isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \quad \forall X, Y, Z \in \mathcal{C}$$

$\uparrow$
associator
F move



- natural isomorphism (left/right unitors) for $X \in \mathcal{C}$

$$l_X : \mathbb{1} \otimes X \to X$$

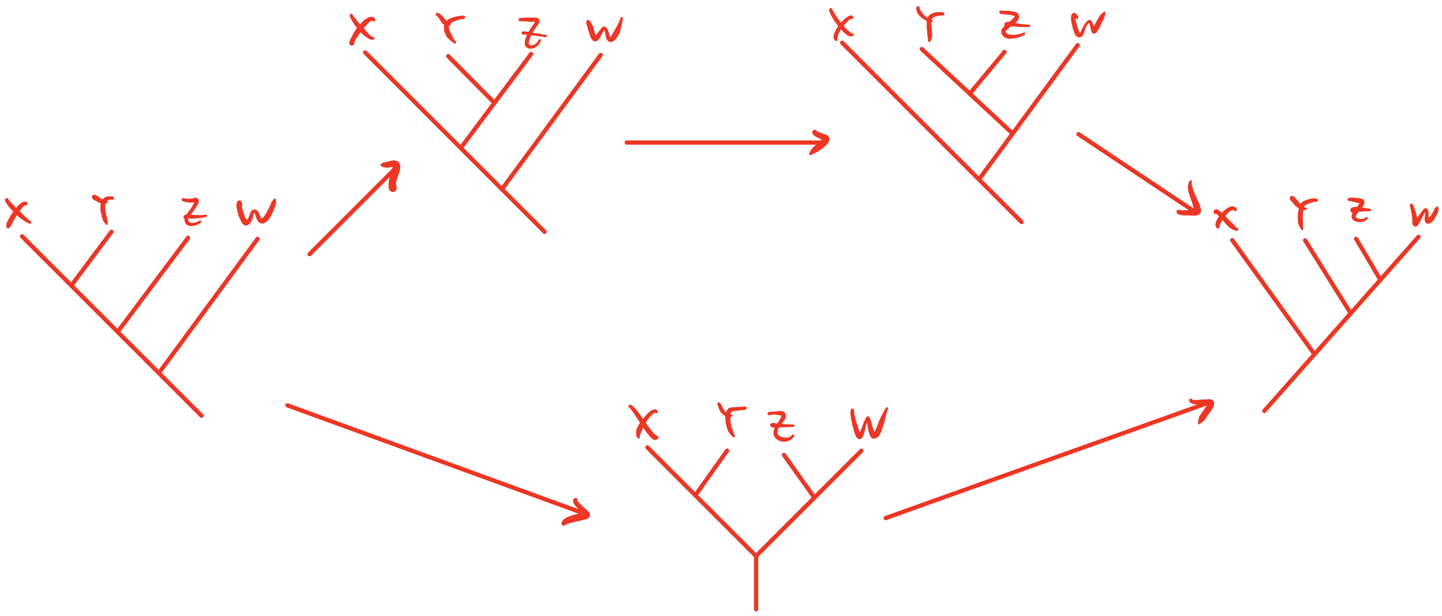
$$r_X : X \otimes \mathbb{1} \to X$$

such that

- $$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes 1_Y"'] & \arrow[d, phantom, "\circlearrowleft"] & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "1_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

- pentagon equation:

$$\begin{tikzcd}
& (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "a_{X,Y\otimes Z,W}"] & X \otimes ((Y \otimes Z) \otimes W) \arrow[dr, "1_X \otimes a_{Y,Z,W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "a_{X,Y,Z} \otimes 1_W"] \arrow[drr, "a_{X\otimes Y,Z,W}"'] & & \circlearrowleft & X \otimes (Y \otimes (Z \otimes W)) \\
& & (X \otimes Y) \otimes (Z \otimes W) \arrow[ur, "a_{X,Y,Z\otimes W}"'] &
\end{tikzcd}$$


x y z w
x y z w
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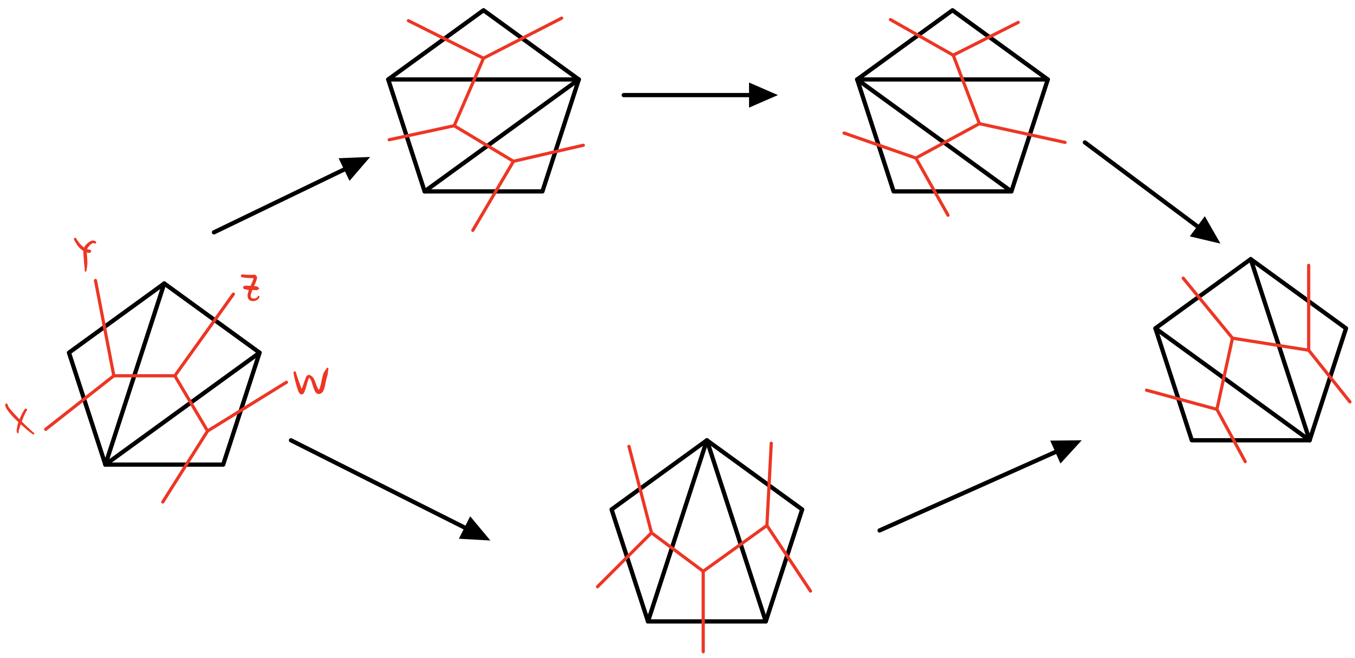

Y
Z
W
X

Examples (1) Rep(G). obj = vector space

Hom = G - invariant linear maps.

$V \otimes W$

1 = trivial rep on $\mathbb{C}$.

$a_{x,y,z}$ is trivial

(2) Vec_G. obj = V_g for $\forall g \in G$

$$\mathrm{Hom}(V_g, V_h) = \begin{cases} \mathbb{C}, & g = h \\ 0, & g \neq h \end{cases}$$

$V_g \otimes V_h = V_{gh}$

a_{V_g, V_h, V_k} is trivial

$1 = V_e$, e identity element in G.

(3) $\mathrm{Vec}_G^{\nu_3}$: $a_{V_g, V_h, V_k} := \nu_3(g,h,k) \in U(1)$

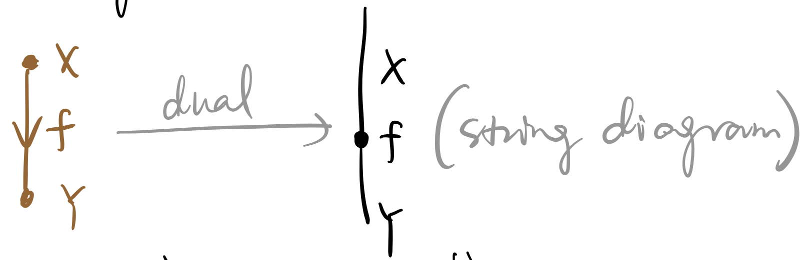
pentagon eq $\Leftrightarrow$ $d\nu_3 = 1$

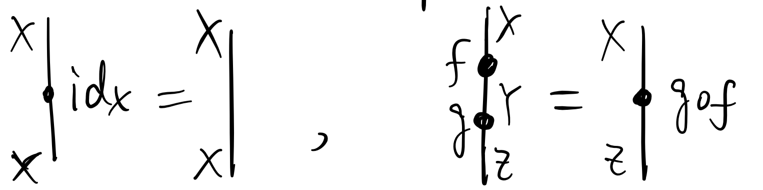
$\Leftrightarrow$ $\nu_3 \in Z^3(G, U(1))$

with dual a⊕b hom set is vector space.

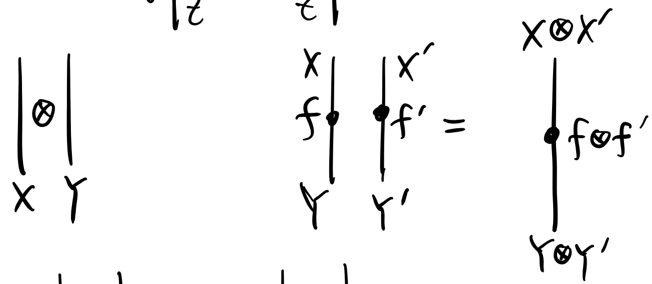
Def. A fusion category is a rigid, semisimple, linear monoidal category with finitly many isomorphism classes of simple objects such that $\mathrm{Hom}(1,1) = \mathbb{C}$.

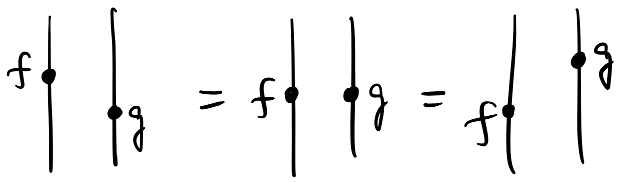
3.3. String diagram

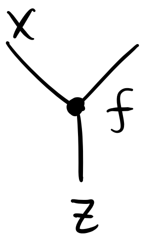




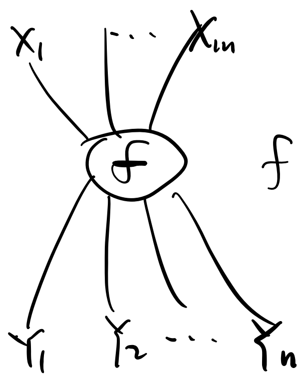
tensor: $X \otimes Y$





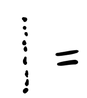


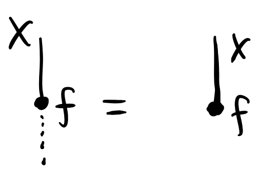
$f \in Hom(X\otimes Y, Z)$



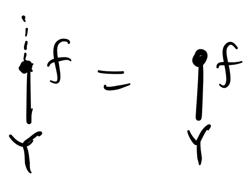
$f \in Hom(X_1\otimes\cdots\otimes X_m, Y_1\otimes\cdots\otimes Y_n)$

identity 1 :



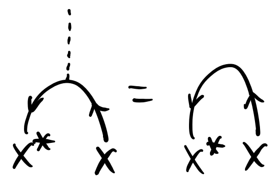


$f \in Hom(X, 1)$

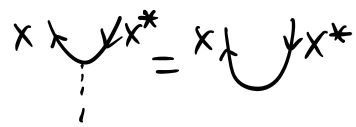


$f \in Hom(1, Y)$

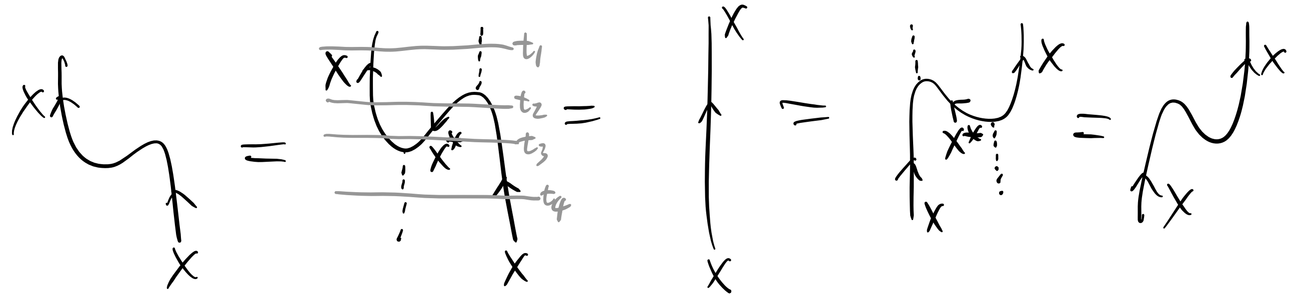
duals : unit $i_X : 1 \longrightarrow X^* \otimes X$



counit $e_X : X \otimes X^* \longrightarrow 1$



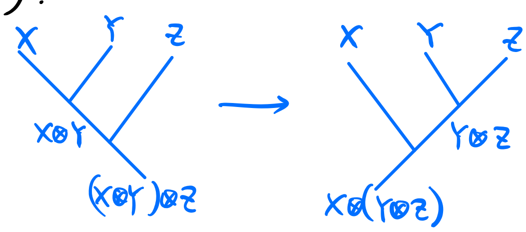
s.t.



$$\begin{tikzcd}
\underset{t_1}{X \otimes 1} \arrow[r, "id_X \otimes i_X"] \arrow[drr, "r_X"'] & \underset{t_2}{X \otimes (X^* \otimes X)} \arrow[r, "a^{-1}_{X,X^*,X}"] \arrow[d, phantom, "\circlearrowright"] & \underset{t_3}{(X \otimes X^*) \otimes X} \arrow[r, "e_X \otimes id_X"] & \underset{t_4}{1 \otimes X} \arrow[dl, "l_X"] \\
 & & X &
\end{tikzcd}$$

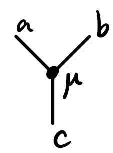
associator (F move):

$a_{X,Y,Z}$:



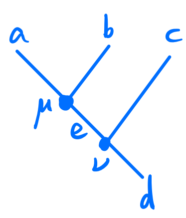
In simple object basis :

$V^{ab}_c := \mathrm{Hom}(a \otimes b, c) \ni$



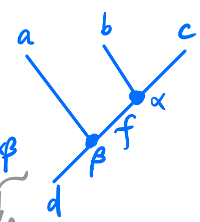
$\longleftrightarrow |a, b; c, \mu\rangle$

$N^{ab}_c := \dim V^{ab}_c$



$$= \sum_{f,\alpha,\beta} \Big(F^{abc}_d\Big)_{\underbrace{e\mu\nu}_{\text{row}},\, \underbrace{f\alpha\beta}_{\text{col}}}$$

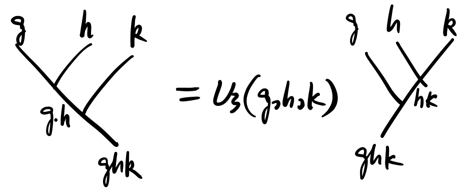
parameters



$$\sum_e N^{ab}_e N^{ec}_d = \sum_f N^{bc}_f N^{af}_d$$

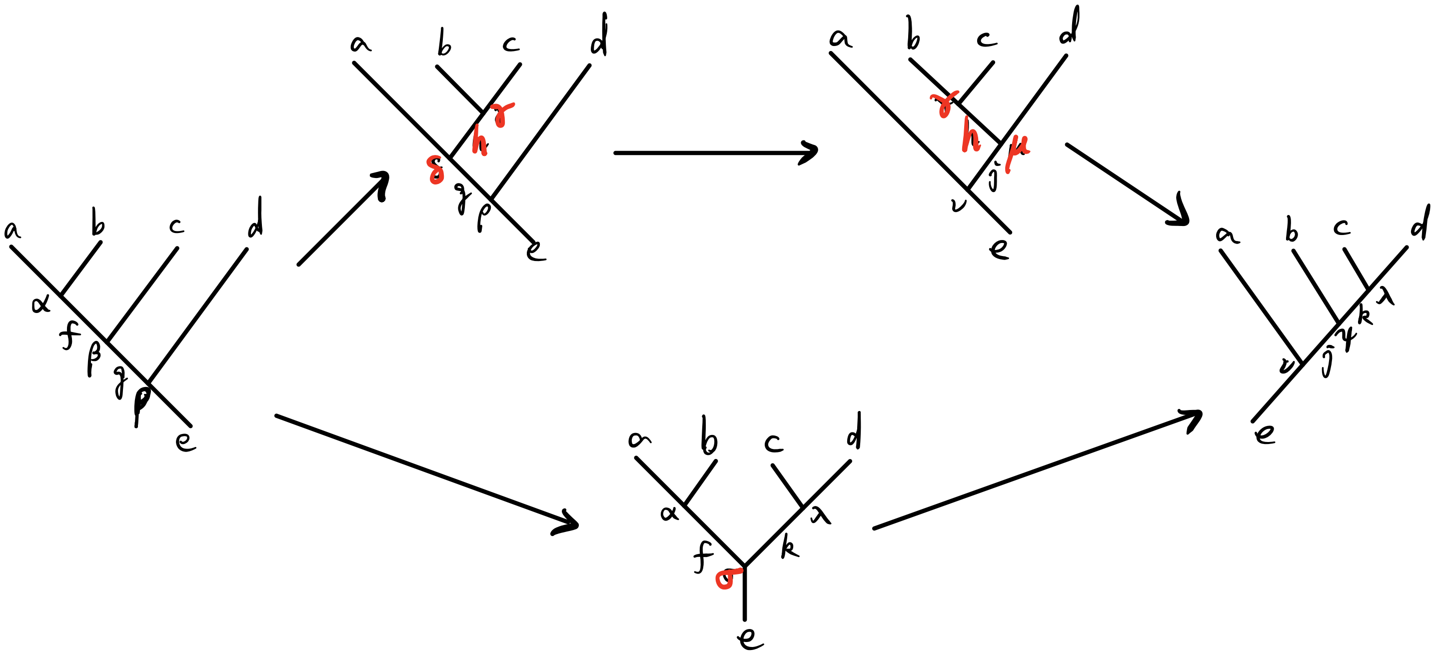
unitary fusion category: F is unitary.

$\mathcal{C} = \mathrm{Vec}_G^{\nu_3}$:



$$\nu_3(g,h,k) := \left(F^{g,h,k}_{ghk}\right)_{1,1}$$

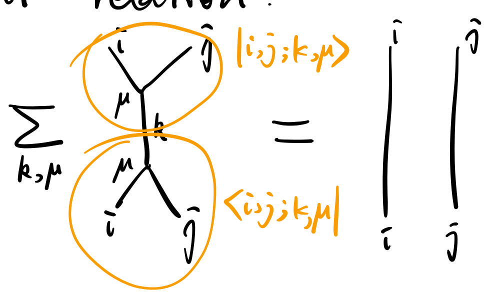
pentagon eq:



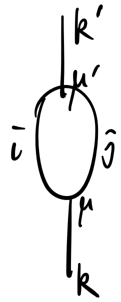
$$\sum_{h;\gamma,\delta,\mu} \left(F^{abc}_{g}\right)_{f\alpha\beta,\, h\gamma\delta} \left(F^{ahd}_{e}\right)_{g\delta\rho,\, j\mu\nu} \left(F^{bcd}_{j}\right)_{h\gamma\mu,\, k\lambda\psi}$$

$$= \sum_{\sigma} \left(F^{fcd}_{e}\right)_{g\beta\rho,\, k\lambda\sigma} \left(F^{abk}_{e}\right)_{f\alpha\sigma,\, j\psi\nu}$$

important relation :



$$\sum_{k,\mu} |ij;k\mu\rangle\langle ij;k\mu| = I$$

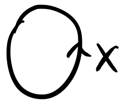


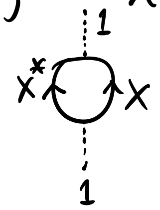
$= \delta_{kk'}\delta_{\mu\mu'}$


$\langle ij;k'\mu' | ij;k\mu\rangle = \delta_{kk'}\delta_{\mu\mu'}$

quantum dimension :

d_X of a simple object $X \in \mathcal{C}$ is

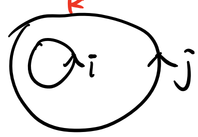
$d_X :=$ 

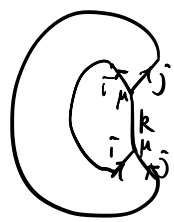
$=$ 

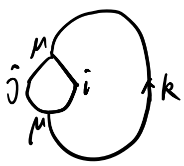
$\in \mathrm{Hom}(1,1) = \mathbb{C}$

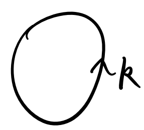
$d_X > 0$.

property : $d_i d_j = \sum_k N_k^{ij} d_k$

proof : $d_i d_j =$ 

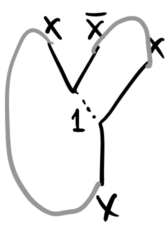
$= \sum_{k,\mu}$ 

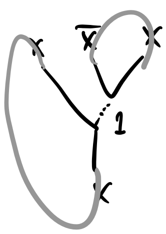
$= \sum_{k\mu}$ 

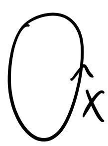
$= \sum_k N_k^{ij}$ 

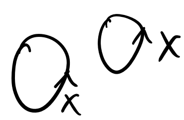
$= \sum_k N_k^{ij} d_k$

$d_X = \left[\left(F_X^{X,\bar{X},X}\right)_{1,1} \right]^{-1}$



$= \left(F_X^{X,\bar{X},X}\right)_{1,1}$




$= \left(F_X^{X,\bar{X},X}\right)_{1,1}$


$d_X = \left(F_X^{X,\bar{X},X}\right)_{1,1} (d_X)^2$

physical meaning of d_x :

$$N_k^{ij} d_k = d_i d_j$$

$$(N^i)_k^j d_k = d_i d_j$$

$$\begin{array}{c} \\ j \end{array}\begin{pmatrix} \\ N^i \\ \\ \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = d_i \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}\begin{array}{c} \\ j \end{array}$$

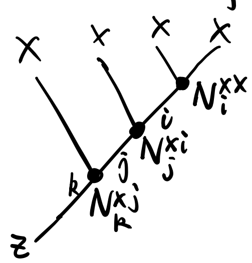
$\Rightarrow (d_1 \cdots d_n)^T$ is the eigenvector of N^i with eigenvalue d_i.

$N_k^{ij} \geq 0$, $d_i > 0$

$\Rightarrow$ d_i is the largest eigenvalue of N^i.

Q: what is dim of $(X)^{\otimes n}$, $n \to \infty$?

A:



$\dim X^{\otimes n} := \dim\left[\bigoplus_z \mathrm{Hom}(X^{\otimes n}, z)\right]$

$$\dim (X)^{\otimes n} = \sum_{i,j,k,\cdots} N_i^{XX} N_j^{Xi} N_k^{Xj} \cdots$$

$$= \sum_{i,j,k,\cdots} (N^X)_i^X (N^X)_j^i (N^X)_k^j \cdots$$

$$= \sum_z \left[(N^X)^n\right]_{X,z} \xrightarrow{n\to\infty} (d_X)^n$$

$\Rightarrow$ $\dim(X^{\otimes n}) \sim d_X^{\,n}$ for $n \to \infty$

quantum dimension of X.

eg: $\dim(\text{Majorana} \quad \text{Majorana}) = 2$

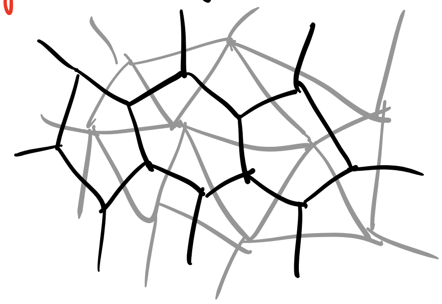
$\Rightarrow \dim(\text{Majorana}) = \sqrt{2}$

Finally, $\forall$ trivalent graph

label $\begin{cases} \text{edge by } X \in Obj(\mathcal{C}) \\ \text{vertex by } Hom(X\otimes Y, Z) \text{ or } Hom(X, Y\otimes Z) \end{cases}$

$\Updownarrow$ dual

triangulation of M_2



3.4. Examples.

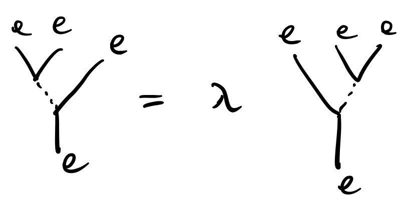
Classification of fusion cat ?

Very hard !!! $\mathcal{C} = Vec_G \to$ classification of finite simple groups.
(already very hard !)

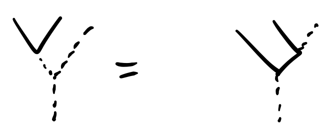
(1) $obj(\mathcal{C}) = \{1, e\} = \mathbb{Z}_2$

$e \otimes e = 1$, $e^* = e$

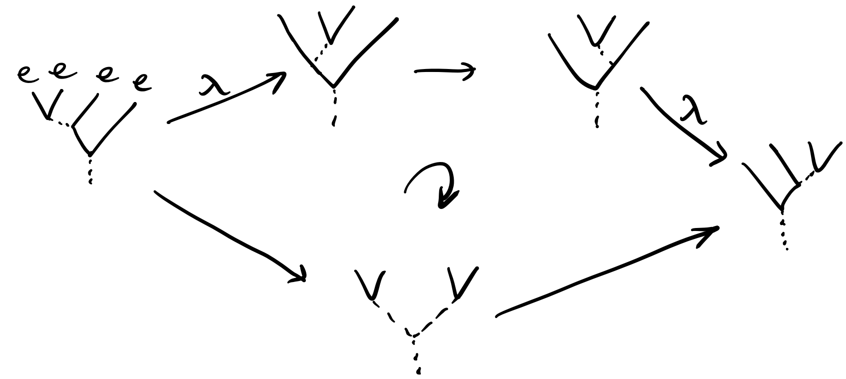
associator $a_{e,e,e}$:



$a_{e,e,1}$:



pentagon:



$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \ni \nu_3 \quad \begin{cases} ① \ \nu_3(a,b,c) = 1 \ (\forall a,b,c) : & \lambda = 1 \quad \text{toric code} \\ ② \ \nu_3(e,e,e) = -1 : & \lambda = -1 \quad \text{double semion.} \end{cases}$$

(2) Fibonacci ·

- $\mathrm{Obj}(\mathcal{C}) = \{1, \tau\}$
- $\tau \otimes \tau = 1 \oplus \tau \rightsquigarrow$ non-Abelian

$N_1^{\tau\tau} = N_\tau^{\tau\tau} = 1$

$\tau^{\otimes n} = F_{n-2} 1 \oplus F_{n-1} \tau$ where $F_n = 1, 1, 2, 3, 5, 8, \cdots$ is the Fibonacci number.

$$\Downarrow$$

$$\tau \otimes (\tau^{\otimes n}) = \tau \otimes (F_{n-2} 1 \oplus F_{n-1} \tau) = F_{n-2} \tau \oplus F_{n-1} (\tau \otimes \tau)$$

$$= F_{n-2} \tau \oplus F_{n-1} (1 \oplus \tau) = F_{n-1} 1 \oplus \underbrace{(F_{n-2} + F_{n-1})}_{F_n} \tau$$

$$= F_{n-1} 1 \oplus F_n \tau$$