A K-matrix twisted $BF + BB$ field theory approach to the interplay of self-statistics, braiding statistics, shrinking rules and fusion rules in 3D topologically ordered phases

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While 2D topological orders, i.e., anyon systems, have been studied thoroughly for the past decades, it is fundamentally important to explore topological orders in all other dimensions, through, e.g., topological quantum field theory (TQFT). While previous TQFT attempts in 3D topological orders focus on either self-statistics, braiding statistics, shrinking rules, or fusion rules, it is yet to systematically put all such topological data together. Here, we construct the topological BF field theory of gauge group $G = \prod_{i=1}^{n} \mathbb{Z}_{N_i}$, in the presence of both twisted terms (e.g., $AAdA$ and AAB) and a K-matrix BB term. In this TQFT, in analogy to the K-matrix Chern-Simons theory of 2D topological orders, we are allowed to simultaneously explore the self-statistics of particles, particle-loop braiding, multi-loop braiding, Borromean Rings braiding, shrinking rules, and fusion rules. More concretely, we present general mathematical formulas and show how the K-matrix BB term confines either partially or completely G, and how self-statistics of deconfined particles is determined. In order to reach anomaly-free topological orders, we explore how the principle of gauge invariance affects the compatibility between braiding statistics and emergent fermions. For example, suppose the flux (loop excitation) and the charge (particle excitation) of gauge subgroup $\mathbb{Z}_{N_i}$ are denoted as ϕ_i and q_i respectively in a 3D bosonic topological order. The field-theoretical analysis simply tells us, within the present TQFTs, if ϕ_i nontrivially participates a multi-loop braiding or a Borromean-rings braiding, then fermionic q_i is prohibited; otherwise, gauge invariance cannot be guaranteed. Therefore, the possible ways of self-statistics assignment on particles are highly restricted once other topological data (e.g., braiding statistics) are given. Our analysis provides a field-theoretical "algorithm" for constructing anomaly-free topological orders in 3D. Together with the previous research, our work paves the way toward a more complete field-theoretical analysis of 3D topological orders.

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I. INTRODUCTION

The study of low-energy effective field theory of quantum many-body systems has a long history in condensed matter physics. For example, the Ginzburg-Landau theory is applied to symmetry-breaking phases and phase transitions, and nonlinear sigma models with θ term are applied to quantum spin chains [1]. Since the discovery of the fractional quantum Hall effect in the 1980s, the notion of topological order has been introduced as a route toward exotic phases of matter that cannot be characterized by the mechanism of symmetry-breaking. While there has been a broad consensus that the essence of topological order is deeply rooted in patterns of long-range entanglement that is robust against local unitaries of finite depth [2], the original definition of *topological* order really comes from the fact that the low energy effective field theory of the prototypical topological order–fractional quantum Hall states—is the Chern-Simons theory which is a *topological* quantum field theory (TQFT). Particularly, our expectation for topological phases of matter—topological robustness against any local perturbations—is achievable by simply noting the fact that correlation functions of all spatially-local operators in TQFTs vanish [3]. As the most general Abelian formulation, the K-matrix Chern-Simons theory (KCS) [4, 5] whose action is written in terms of $\sim \int \frac{K_{IJ}}{4\pi} A^I dA^J$, serves as the standard TQFT framework of 2D Abelian topological orders, providing a highly efficient algorithm for computing topological data, such as anyon types, self-statistics, mutual statistics, fusion algebra, chiral central charge, and ground state degeneracy. Besides topological orders, the KCS has also been successfully applied to the study of symmetry-enriched topological phases (SET) [6, 7] and symmetry-protected 'topological' phases [8–12] where global symmetry is nontrivially imposed.

While the KCS works very well in 2D topological phases of matter, it is no longer applicable to 3D and higher where exotic spatially extended excitations (e.g., loops in 3D and higher, membranes in 4D and higher) induce very rich and unique emergent phenomena. Instead, if particles and loops respectively carry gauge charges and gauge fluxes of a discrete Abelian gauge group $G = \prod_{i=1}^{n} \mathbb{Z}_{N_i}$, one may apply the twisted "BF field theory" [13, 14] by properly including "twisted terms" [15–19]. This series of TQFTs have been proved to efficiently describe various types of nontrivial braiding statistics, such as, particle-loop braiding [14, 20–24], multi-loop braiding [25] (with the twisted terms $\sim AAdA$ and $\sim AAAA$), and particle-loop-loop braiding (i.e., Borromean-Rings braiding with the twisted term $\sim AAB$) [19]. Recently, the untwisted / twisted BF theory has also been successfully applied to SET [26–30] and SPT [17, 31–34] in 3D.

In the twisted BF theory symbolically denoted as "$BF +$ twists", each type of braiding statistics is associated with a particular formulation of TQFT actions, which doesn't mean that all types of braiding statistics are mutually compatible and can thereby coexist in an anomaly-free topological order. To further examine whether two different types of braiding processes are allowed to coexist in the same topological order, Ref. [35] exhausted all combinations of twisted terms and found that TQFTs of some combinations inevitably violate the principle of gauge invariance. Thus, among all combinations, only a part of combinations are legitimate such that braiding processes can coexist. After TQFTs with mutually compatible braiding processes are obtained, fusion rules and shrinking rules in the TQFTs are further investigated, from which quantum dimensions of both particles and loops are computed [36]. Recently, the ideas of Ref. [35] and Ref. [36] have been subsequently extended to 4D real space [37, 38] where membrane excitations are allowed and hierarchy of shrinking rules is definable.

On the other hand, in the untwisted BF theory with the inclusion of the BB term [39] symbolically denoted as "$BF + BB$", the boson-fermion statistical transmutation of self-statistics (i.e., exchange statistics) of particles in 3D [16] has been studied through equations of motion, where the scenario of Dirac-string-attachment implied by the equations of motion mimics, to some extent, the physics of dyons studied intensively in other contexts [30, 40–43]. Along this line, it has become clear that all particles with anyonic statistics (neither fermionic nor bosonic) are exactly confined and thus disappear in the low energy spectrum, which is consistent with the well-known fact that anyons are not possible in 3D and higher [44–46]. Thanks to the statistical transmutation induced by the BB term, we may realize both *emergent fermions* (defined as topologically nontrivial particles that are fermionic) and transparent fermions (defined as topologically trivial particles that are fermionic). By definition, the latter case corresponds to fermionic topological order. In addition to boson-fermion transmutation, the single-component BB term further provides a novel "Higgs" mechanism that confines either partially or completely the gauge group G set by the coefficient of the BF term [47]. Besides, the multi-component BB term was successfully applied to 3D bosonic topological insulators (bosonic SPTs with particle number conservation and time-reversal symmetry) where bulk topological order is trivial but boundary admits anomalous surface topological orders [31].

Logically, once we have understood (*i*) how to obtain compatible braiding processes via legitimate combinations of twisted terms in the twisted BF theory ($BF+$twists) and (*ii*) how to assign self-statistics on particles via the boson-fermion transmutation in the untwisted BF theory with the BB term ($BF + BB$), it becomes urgent to make a step forward by examining whether braiding statistics is compatible with the assignment of self-statistics on particles in the twisted BF theory with the BB term ($BF +$ twists $+ BB$). In this work, we are motivated to study such compatibility in order to finally achieve a more complete field-theoretical description of topological data encoded in 3D topological orders, in analogy to the spirit of the KCS theory of 2D topological orders.

For this purpose, we begin Sec. II by systematically formulating the untwisted BF theory with the K-matrix BB term (i.e., $\sim \frac{K_{ij}}{4\pi} B^i B^j$ with a symmetric integer matrix K). In the presence of the K-matrix BB term, we present general mathematical formulas that can be applied to efficiently determine (*i*) excitation contents, i.e., deconfined particles and deconfined loops, and (*ii*) self-statistics assignment on particles. In particular, the previous results in the presence of the single-component BB term are naturally included by reducing K to an integer. And the self-statistics of particles is rigorously derived by computing the expectation values of *framed* Wilson loops. In Table I, we collect physically useful properties of untwisted BF theory with the single-component BB term. To determine whether a topological order is fermionic or bosonic, we may calculate the self-statistics of trivial particles. To determine whether a topological order supports emergent fermions, we may calculate the self-statistics of particles that carry nontrivial gauge charges of G. In Sec. **??** xxxx

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This paper is organized as follows. In Sec. **??**, we study the BF theory with an n-component $B \wedge B$ whose coefficients are given by a K-matrix. We show that how a $B \wedge B$ term alters the properties of a $\prod_{i=1}^{n} \mathbb{Z}_{N_i}$ gauge theory. Next, in Sec. III we study how a $B \wedge B$ term affect the braiding phases and fusion rules of an Abelian topological order. In Sec. IV we consider flux attachment in the Borromean rings topological order that is featured by its non-Abelian fusion rules. We investigate how an additional $B \wedge B$ influence the properties of BR topological order.

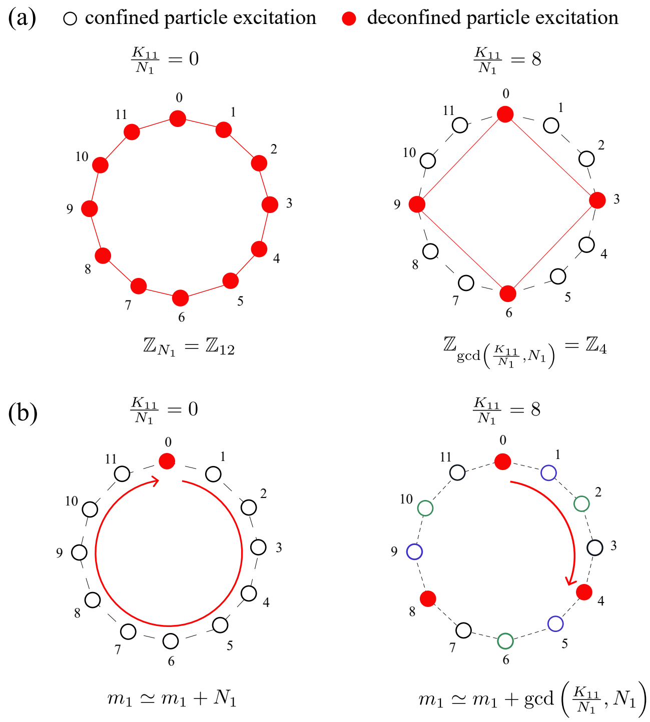

FIG. 1. (a) When $\frac{K_{11}}{N_1} = 0$, the charges of deconfined particle excitations of action (37) are labeled by $\mathbb{Z}_{N_1} = \mathbb{Z}_{12}$. Once a $B \wedge B$ term with $\frac{K_{11}}{N_1}$ is added, some particle excitations become confined. Those deconfined are labeled by the unbroken gauge group $\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)} = \mathbb{Z}_4$. (b) Period of fluxes for different values of $\frac{K_{11}}{N_1}$. When $\frac{K_{11}}{N_1} = 0$, fluxes has a period of N_1. When $\frac{K_{11}}{N_1} = 8$, some fluxes are actually equivalent as shown in the same color. In this case the minimal period of flux is $\gcd\left(\frac{K_{11}}{N_1}, N_1\right) = 4$.

II. THE UNTWISTED BF THEORY WITH THE K-MATRIX BB TERM

In this section, we introduce the $(3+1)$D TQFT consisting of a BF term and a $B \wedge B$ term, where B is a 2-form gauge field. As shown in the follows, the physical picture of $B \wedge B$ term is to attach a flux string to a particle excitation. Due to the tension on the flux string, some of such composed particles become confined. According to the correspondence between particle excitations and loop excitations, only part of loop excitations are deconfined. On the other hand, the attached flux may alternate the exchange statistics of particle excitations. Furthermore, taking all of these effects into consideration, we study the interplay of $B \wedge B$ term with other topological terms in $(3+1)$D. Through this way, we reveal the changes of braiding and fusion data in $(3+1)$D topological order due to attaching flux to particle.

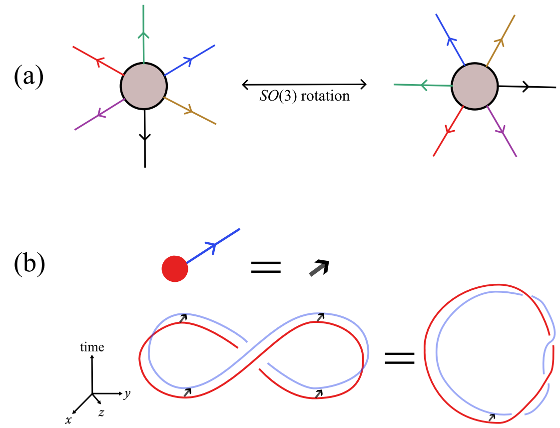

FIG. 2. (a) The physical picture of $B \wedge B$ term is to bind fluxes string (lines with arrow) to particle excitations (red solid circle), see the Wilson operator of a particle excitation (23). The bound state of particle excitation and attached flux strings is no longer isotropic in 3D space: the particle excitation itself is isotropic, but the flux strings has an orientation. Different orientations can be connected by a $SO(3)$ rotation. (b) Consider a particle excitation attached by a flux string, e.g., Eq. (38), exchanging such two particle excitations can be viewed as a self 2π rotation of a particle.

A. Topological action, gauge transformations, coefficient quantization and periods

The BF theory with a general n-component $B \wedge B$ term and an Abelian gauge group $G = \prod_{i=1}^{n} \mathbb{Z}_{N_i}$ is ($\wedge$ is omitted)

$$S = \int \sum_{i=1}^{n} \frac{N_i}{2\pi} B^i dA^i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{K_{ij}}{4\pi} B^i B^j, \tag{1}$$

whose gauge transformations are

$$A^i \to A^i + d\chi^i - \sum_{j=1}^{n} \frac{K_{ij}}{N_i} V^j, \tag{2}$$

$$B^i \to B^i + dV^i. \tag{3}$$

First we need to find out the quantization and period of elements of the K-matrix. After gauge transformation,

$$\begin{aligned} S_{BB} \to S'_{BB} =& \int \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{K_{ij}}{4\pi} \left(B^i + dV^i\right)\left(B^j + dV^j\right) \\ =& \int \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{K_{ij}}{4\pi} B^i B^j \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{K_{ij}}{4\pi} B^i dV^j + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{K_{ij}}{4\pi} B^j dV^i \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{K_{ij}}{4\pi} dV^i dV^j \\ =& S_{BB} + \Delta S_{BB}^{(1)} + \Delta S_{BB}^{(2)} \end{aligned} \tag{4}$$

where

$$\Delta S_{BB}^{(1)}=\int\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}B^{i}dV^{j}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}B^{j}dV^{i} \quad (5)$$

$$\Delta S_{BB}^{(2)}=\int\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}dV^{i}dV^{j}. \quad (6)$$

$\Delta S_{BB}^{(1)}$ is evaluated as

$$\Delta S_{BB}^{(1)}=\int\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}\frac{2\pi n_i}{N_i}2\pi m_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}\frac{2\pi n_j}{N_j}2\pi m_i \quad (7)$$

where n_i, n_j, m_i, m_j are integers. Demanding $\Delta S_{BB}^{(1)}\in 2\pi\mathbb{Z}$, we find the following constraints:

$$\frac{K_{ij}}{N_i}\in\mathbb{Z}, \quad (8)$$

$$\frac{K_{ij}}{N_j}\in\mathbb{Z}. \quad (9)$$

Specially, for $i\neq j$, we have $K_{ij}\in\operatorname{lcm}(N_i,N_j)\mathbb{Z}$ where $\operatorname{lcm}(N_i,N_j)$ is the least common multiplier of N_i and N_j.

For the calculation of $\Delta S_{BB}^{(2)}$, we need to consider whether a spin structure is taken into account. On a non-spin manifold, $\frac{1}{4\pi^2}\int dV^i dV^i$ is quantized to $\mathbb{Z}$; while on a spin manifold, it is quantized to $2\mathbb{Z}$. For $\frac{1}{4\pi^2}\int dV^i dV^j$ with $i\neq j$, it is quantized to $\mathbb{Z}$ no matter on a spin or non-spin manifold. In order to keep $\Delta S_{BB}^{(2)}\in 2\pi\mathbb{Z}$ for gauge invariance, we have

$$\text{non-spin manifold:} K_{ii}\in 2\mathbb{Z}, K_{ij}\in\mathbb{Z}, i\neq j \quad (10)$$

$$\text{spin manifold:} K_{ii}\in\mathbb{Z}, K_{ij}\in\mathbb{Z}, i\neq j. \quad (11)$$

Only on a spin manifold the diagonal elements K_{ii} can be an odd integer. As shown in the following main text, the parity of K_{ii} controls the exchange statistics of trivial particle excitations of $\mathbb{Z}_{N_i}$ gauge subgroup. As long as one of the diagonal elements K_{ii} is odd, there exists a fermionic trivial particle excitation thus by definition the theory (1) is a fermionic one. This is coincident with tha fact that a fermionic theory can only be defined on a spin manifold. On the other hand, when all K_{ii}'s are even, this theory (1) is a bosonic one.

For the period of K-matrix's elements, we conside

$$S_{BB}=\int\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}B^iB^j \in\sum_{i=1}^{n}\frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i}\mathbb{Z}+\sum_{i\neq j}^{n}2\cdot\frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j}\mathbb{Z} \quad (12)$$

$\exp(\mathrm{i}S_{BB})$ should be invariant if we shift $\frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i}$ or $2\cdot\frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j}$ by 2π which indicates identification relations (we use $\simeq$ to denote such identification relation):

$$\frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i}\simeq\frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i}+2\pi, \quad (13)$$

$$2\cdot\frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j}\simeq 2\cdot\frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j}+2\pi, \quad (14)$$

They in fact gives

$$K_{ii}\simeq K_{ii}+2(N_i)^2, \quad (15)$$

$$K_{ij}\simeq K_{ij}+N_iN_j, i\neq j. \quad (16)$$

In conclusion, for each elements in the K-matrix, its quantization and period are given by

$$\frac{K_{ij}}{N_i}\in\mathbb{Z},, \frac{K_{ij}}{N_j}\in\mathbb{Z}; \quad (17)$$

$$K_{ij}\in\mathbb{Z}, i\neq j; \quad (18)$$

$$K_{ii}\simeq K_{ii}+2(N_i)^2; \quad (19)$$

$$K_{ij}\simeq K_{ij}+N_iN_j, i\neq j; \quad (20)$$

$$\text{bosonic theory:} K_{ii}\in 2\mathbb{Z}; \quad (21)$$

$$\text{fermionic theory :at least one of } K_{ii}\text{'s is odd.} \quad (22)$$

B. Confinement and deconfinement of excitation contents

A particle excitation carrying e_i units of $\mathbb{Z}_{N_i}$ gauge charges can be labeled by a charge vector $\mathbf{l}=(e_1,e_2,\cdots,e_n)^T$ with $e_i\in\mathbb{Z}_{N_i}$, whose Wilson operator is

$$W(\mathbf{l},\gamma)=\exp\left(\int_\gamma \mathrm{i}\sum_{i=1}^{n}e_iA^i+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\mathrm{i}e_iK_{ij}}{N_i}\int_{\Sigma_i^j}B^j\right) \quad (23)$$

where Σ_i^j's are Seifert surfaces of γ. The physical picture of (23) is a particle excitation being attached by flux strings. Due to the tension on strings, a particle excitation may be confined. Only those attached by 2π fluxes are deconfined, i.e.,

$$\frac{e_iK_{ij}}{N_i}\int_{\Sigma_i^j}B^j=\frac{e_iK_{ij}}{N_i}\frac{2\pi n_j}{N_j}\in 2\pi\mathbb{Z} \quad (24)$$

where n_j is an integer. If a particle excitation labeled by $\mathbf{l}$ is deconfined, it is required that

$$\frac{e_iK_{ij}}{N_iN_j}\in\mathbb{Z}, \forall i,j\in\{1,\cdots,n\}. \quad (25)$$

For example, the constraints on e_1 are

$$\frac{e_1K_{11}}{N_1N_1}\in\mathbb{Z}, \frac{e_1K_{12}}{N_1N_2}\in\mathbb{Z},\cdots,\frac{e_1K_{1n}}{N_1N_n}\in\mathbb{Z}, \quad (26)$$

which demand

$$e_1\in\frac{N_1}{\gcd\left(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},\cdots,\frac{K_{1n}}{N_n},N_1\mathbb{Z}\right)}, \quad (27)$$

where $\gcd(a, b, \cdots)$ is the greatest common divisor of $a, b, \cdots$. In other words, for a deconfined particle excitation carrying $\mathbb{Z}_{N_1}$ gauge charges, the minimal amount of $\mathbb{Z}_{N_1}$ gauge charges is

$$e_{1\min} = \frac{N_1}{\gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, \cdots, \frac{K_{1n}}{N_n}, N_1\right)}. \tag{28}$$

Since e_1 is equivalent to $e_1 + N_1$, the number of nonequivalent values of e_1 is $\gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, \cdots, \frac{K_{1n}}{N_n}, N_1\right)$, i.e., e_1 is labeled by $\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, \cdots, \frac{K_{1n}}{N_n}, N_1\right)}$. Similarly, in order to make this particle deconfined, other e_i's need to satisfy

$$e_{i\min} = \frac{N_i}{\gcd\left(\frac{K_{i1}}{N_1}, \frac{K_{i2}}{N_2}, \cdots, \frac{K_{in}}{N_n}, N_i\right)}. \tag{29}$$

Any particle excitation carrying e_i units of $\mathbb{Z}_{N_i}$ gauge charges with $e_i \notin e_{i\min}\mathbb{Z}$ is confined. For an illustration, see Fig. 1 in which we consider a BF theory with a single component $B \wedge B$ term with $\mathbb{Z}_{N_1} = \mathbb{Z}_{12}$ and $\frac{K_{11}}{N_1} = 8$.

The confinement on $\mathbb{Z}_{N_i}$ gauge charge would also alternates the period of $\mathbb{Z}_{N_i}$ gauge fluxes. Since the $\mathbb{Z}_{N_i}$ gauge fluxes carried by a loop excitation can be detected by braiding a particle excitation around this loop excitation, we can consider the following particle-loop braiding phase:

$$\begin{aligned}\Theta_{\mathrm{PL}}(e_{i\min}, m_i) &= \exp\left[-\frac{\mathrm{i}2\pi e_{i\min} m_i}{N_i}\right] \\ &= \exp\left[-\frac{\mathrm{i}2\pi m_i}{\gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, \cdots, \frac{K_{1n}}{N_n}, N_1\right)}\right].\end{aligned} \tag{30}$$

One can see that

$$m_i \simeq m_i + \gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, \cdots, \frac{K_{1n}}{N_n}, N_1\right) \tag{31}$$

in the sense that $\Theta_{\mathrm{PL}}(e_{i\min}, m_i)$ differs by an integral multiple of 2π. An illustration for the smaller period of m_i is presented in Fig. 1 where a BF theory with a single component $B \wedge B$ term with $\mathbb{Z}_{N_1} = \mathbb{Z}_{12}$ and $\frac{K_{11}}{N_1} = 8$ is considered.

C. Self-statistics of particle excitations via framing

Integrating out A^i results in $\langle W(\mathbf{l}, \gamma)\rangle$ results in

$$B^i = -\frac{2\pi q_i}{N_i}\delta^{\perp}(\Sigma_i) \tag{32}$$

with $\partial\Sigma_i = \gamma$. Plugging this solution back to path integral, we have

$$\begin{aligned}\langle W(\mathbf{l}, \gamma)\rangle =& \exp\left[\mathrm{i}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{ij}}{4\pi}\frac{2\pi q_i}{N_i}\frac{2\pi q_j}{N_j}\#(\Sigma_i \cap \Sigma_j)\right] \\ &\times \exp\left[-\mathrm{i}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\mathrm{i}q_i K_{ij}}{N_i}\frac{2\pi q_j}{N_j}\cdot\#\left(\Sigma_i^j \cap \Sigma_j\right)\right]\end{aligned} \tag{33}$$

Using $\#\left(\left(\Sigma_i - \Sigma_i^j\right) \cap \left(\Sigma_j - \Sigma_j^i\right)\right) = 0$, one has

$$\begin{aligned}&\#(\Sigma_i \cap \Sigma_j) - \#\left(\Sigma_i^j \cap \Sigma_j\right) - \#\left(\Sigma_i \cap \Sigma_j^i\right) \\ &= -\#\left(\Sigma_i^j \cap \Sigma_j^i\right)\end{aligned} \tag{34}$$

Therefore we have

$$\langle W(\mathbf{l}, \gamma)\rangle = \exp\left[-\mathrm{i}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\pi K_{ij} q_i q_j}{N_i N_j}\#\left(\Sigma_i^j \cap \Sigma_j^i\right)\right]. \tag{35}$$

$\#\left(\Sigma_i^j \cap \Sigma_i^j\right) = 1$ if a nontrivial framing is introduced. The exchange statistics of a particle with e_i charge is $\exp\left(-\mathrm{i}\frac{\pi K_{ii} e_i e_i}{N_i N_i}\right)$. Off-diagonal terms $\exp\left(-\mathrm{i}\frac{2\pi K_{ij} e_i e_j}{N_i N_j}\right)$ with $i \neq j$ is the mutual statistics of two particle excitations with e_i units of $\mathbb{Z}_{N_i}$ gauge charges and e_j units of $\mathbb{Z}_{N_j}$ gauge charges. Remember that for a deconfined particle excitation, $e_i \in e_{i\min}\mathbb{Z} \mod N_i$, such e_i's guarantee that the exchange statistics of two particles is ± 1 and the mutual statistics of two particles are always trivial. In conclusion, for a particle labeled by $\mathbf{l} = (e_1, e_2, \cdots, e_n)^T$, the statistics angle is given by

$$\Theta = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\pi K_{ij} e_i e_j}{N_i N_j} = -\pi \mathbf{l}^T \widetilde{K}\mathbf{l} \tag{36}$$

where $\left(\widetilde{K}\right)_{ij} = \frac{K_{ij}}{N_i N_j}$. So far, we have seen that how an n-component $B \wedge B$ term dramatically changes the number of deconfined operators and exchange statistics of a $\prod_{i=1}^{n}\mathbb{Z}_{N_i}$ gauge theory.

The exchange and mutual statistics can be better explained by the examples of the BF theory with a single (two-) component $B \wedge B$ term. In the single component case, the action is

$$S = \int \frac{N_1}{2\pi}B^1 dA^1 + \frac{K_{11}}{4\pi}B^1 B^1, \tag{37}$$

and the Wilson operator of a particle excitation carrying e_1 units of $\mathbb{Z}_{N_1}$ gauge charges is

$$W(e_1, \gamma) = \exp\left(\mathrm{i}e_1\int_{\gamma_1} A^1 + \frac{\mathrm{i}e_1 K_{11}}{N_1}\int_{\Sigma_1} B^1\right), \tag{38}$$

which describe a particle excitation with one attached flux string. For a deconfined particle excitation, it is required that

$$e_{1\min} = \frac{N_1}{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)}. \tag{39}$$

To calculate exchange statistics of this particle excitation, we can make use of spin-statistics theorem, see Fig. 2(b). Its expectation value is

$$\langle W(e_1, \gamma)\rangle = \exp\left[-\frac{\mathrm{i}\pi K_{11} e_1 e_1}{N_1 N_1}\cdot\#(\Sigma_1 \cap \Sigma_1)\right]. \tag{40}$$

The value of $\#\left(\Sigma_1 \cap \Sigma_1\right)$ depends on whether the framing of γ is nontrivial or not. A framing of γ can be understood as assigning a vector on each point along γ. Actually, we are now considering a particle attached with a flux string. In regularization, the charge and the flux cannot be placed on the same lattice site, i.e., the charge-flux composite is not isotropic. It is reasonable to use a vector to indicate the shape of the charge-flux composite. Such vectors along the world line of particle consistitute the framing. In $(2+1)$D, there are different ways to equip a vector to each point along the world line. The number of ways to equip is $\pi_1\left(SO\left(2\right)\right)=\mathbb{Z}$ that counts nonequivalent mappings from S_1 (the world line) to $SO\left(2\right)$ (2D rotation of vector on each point). $\pi_1\left(SO\left(2\right)\right)=\mathbb{Z}$ means that in $(2+1)$D there can be anyonic statistics. In $(3+1)$D, the 3D rotation of vector on each point is captured by $SO\left(3\right)$ and $\pi_1\left(SO\left(3\right)\right)=\mathbb{Z}_2$ means that there are only two kinds of statistics in $(3+1)$D.

For a nontrivial framing of γ, $\#\left(\Sigma_1\cap\Sigma_1\right)=1$, which also indicates a 2π-rotation of this particle excitation that induces a phase

$$\Theta_{\text{ex}}\left(e_1,e_1\right)=\exp\left(-\frac{\mathrm{i}\pi K_{11}e_1e_1}{N_1N_1}\right). \tag{41}$$

According to spin-statistics theorem, $\Theta=\exp\left[-\frac{\mathrm{i}\pi K_{11}e_1e_1}{N_1N_1}\right]$ is the exchange statistics of particle excitation with e_1 units of gauge charge. Notice that $e_1\in\frac{N_1}{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}\mathbb{Z}$, we find $\Theta_{\text{ex}}\left(e_{1\,\text{min}},e_{1\,\text{min}}\right)=\exp\left[-\frac{\mathrm{i}\pi\,\text{lcm}\left(\frac{K_{11}}{N_1},N_1\right)}{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}\right]=\pm1$ corresponding to bosonic or fermionic statistics.

For a trivial particle excitation, i.e., that with $e_1=0 \mod N_1$, its exchange statistics is given by

$$\Theta_{\text{trivial}}=\exp\left(-\mathrm{i}\pi K_{11}\right). \tag{42}$$

When $K_{11}$is odd, $\Theta_{\text{tri}}=-1$ meaning the trivial particle excitation is fermionic which tells us the theory (37) is a fermionic theory. Notice that an odd K_{11} can only happen when the theory is defined on a spin manifold. When K_{11} is even, $\Theta=1$ indicating that the trivial particle excitation is a boson, i.e., the theory (37) is a bosonic one.

For nontrivial particle excitations, i.e., those with $e_1\neq0 \mod N_1$, their exchange statistics depends on the values of N_1, $\frac{K_{11}}{N_1}$, and n. Among all possible combinations, it is possible that some particle excitations with $e_1\neq0 \mod N_1$ are fermionic while the trivial one is bosonic. We call such particle excitations *emergent fermions* in the sense that they exhibit fermionic statistics in a bosonic theory. Below we summary the properties of theory (37) for different N_1 and K_{11} in Table I.

The second example is a two-component $B\wedge B$ term with the action is

$$S=\int\sum_{i=1}^{2}\frac{N_i}{2\pi}B^idA^i+\sum_{i,j=1}^{2}\frac{K_{ij}}{4\pi}B^iB^j. \tag{43}$$

Consider a particle excitation carrying two types of gauge charges, denoted by $\mathbf{l}=\left(e_1,e_2\right)^T$, its exchange statistics is given by

$$\begin{aligned}\left\langle W\left(\mathbf{l},\gamma\right)\right\rangle=&\frac{1}{\mathcal{Z}}\int\mathcal{D}\left[A^i\right]\mathcal{D}\left[B^i\right]\exp\left(\mathrm{i}S\right)\\ &\times\exp\left(\oint_\gamma\mathrm{i}\sum_{i=1}^{2}e_iA^i+\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\mathrm{i}e_iK_{ij}}{N_i}\int_\Sigma B^j\right)\\ =&\exp\left[-\mathrm{i}\sum_{i,j=1}^{2}\frac{\pi K_{ij}e_ie_j}{N_iN_j}\#\left(\Sigma\cap\Sigma\right)\right].\end{aligned} \tag{44}$$

By choosing a nontrivial framing of γ, i.e., $\#\left(\Sigma\cap\Sigma\right)=1$, we find the exchange statistics of a particle excitation labeled by $\mathbf{l}=\left(e_1,e_2\right)^T$ is

$$\Theta_{\text{ex}}\left(\mathbf{l},\mathbf{l}\right)=\exp\left(-\frac{\mathrm{i}\pi K_{11}e_1e_1}{N_1N_1}-\frac{\mathrm{i}\pi K_{22}e_2e_2}{N_2N_2}-\frac{\mathrm{i}2\pi K_{12}e_1e_2}{N_1N_2}\right), \tag{45}$$

where $\exp\left(-\frac{\mathrm{i}2\pi K_{12}e_1e_2}{N_1N_2}\right)$ is the mutual statistics of $\mathbb{Z}_{N_1}$ charges and $\mathbb{Z}_{N_2}$ charges. Keep in mind that for a deconfined particle excitation, $q_i\in q_{i\,\text{min}}\mathbb{Z} \mod N_i$ where $q_{i\,\text{min}}=\frac{N_i}{\gcd\left(\frac{K_{i1}}{N_1},\frac{K_{i2}}{N_2},N_i\right)}$. Such q_i's guarantee that the exchange statistics of two particles is ±1 and the mutual statistics of two particles are always trivial. To see this, we consider

$$\Theta_{\text{mutual}}\left(e_{1\,\text{min}},e_{2\,\text{min}}\right)=\exp\left(-\frac{\mathrm{i}2\pi K_{12}e_{1\,\text{min}}e_{2\,\text{min}}}{N_1N_2}\right). \tag{46}$$

We can see that

$$\begin{aligned}\frac{K_{12}e_{1\,\text{min}}e_{2\,\text{min}}}{N_1N_2}=&\frac{K_{12}}{N_1N_2}\frac{N_1}{\gcd\left(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},N_1\right)}\frac{N_2}{\gcd\left(\frac{K_{21}}{N_1},\frac{K_{22}}{N_2},N_2\right)}\\ \in&\frac{K_{12}}{N_1N_2}\frac{N_1}{\gcd\left(\frac{K_{12}}{N_2},N_1\right)}\frac{N_2}{\gcd\left(\frac{K_{21}}{N_1},N_2\right)}\mathbb{Z}\end{aligned} \tag{47}$$

since $\gcd\left(\frac{K_{12}}{N_2},N_1\right)\in\gcd\left(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},N_1\right)\mathbb{Z}$ and $\gcd\left(\frac{K_{21}}{N_1},N_2\right)\in\gcd\left(\frac{K_{21}}{N_1},\frac{K_{22}}{N_2},N_2\right)\mathbb{Z}$. Furthermore, using

$$\begin{aligned}&\frac{K_{12}}{N_1N_2}\frac{N_1}{\gcd\left(\frac{K_{12}}{N_2},N_1\right)}\frac{N_2}{\gcd\left(\frac{K_{21}}{N_1},N_2\right)}\\ =&\frac{K_{12}N_1N_2}{\gcd\left(K_{12},N_1N_2\right)\gcd\left(K_{21},N_1N_2\right)}\\ =&\frac{\text{lcm}\left(K_{12},N_1N_2\right)}{\gcd\left(K_{12},N_1N_2\right)}\end{aligned} \tag{48}$$

we can see that $\Theta_{\text{mutual}}\left(e_{1\,\text{min}},e_{2\,\text{min}}\right)=\exp\left(-\mathrm{i}2\pi\mathbb{Z}\right)=1$. This is consistent with the fact that in 3D space the mutual statistics of two particles is trivial. These two examples shows the two effects of $B\wedge B$ term as Refs. [15**?** , 16] did.

TABLE I. Properties of $S = \int \frac{N_1}{2\pi} B^1 \wedge dA^1 + \frac{K_{11}}{4\pi} B^1 \wedge B^1$ with different values of N_1 and K_{11}. Θ trivial is the exchange statistics of two trivial particle excitations, i.e., those carrying $0 \mod N$ unit of gauge charge. $\Theta_{\text{trivial}} = 1\,(-1)$ means that the trivial particle excitation is a boson (fermion), or equivalently, the theory is a bosonic (fermionic) one. $\Theta_{e_1 \min}$ is the exchange statistics of two particle excitations with $e_1 \min$ units of gauge charge where $e_1 \min = N_1/\gcd\left(\frac{K_11}{N_1}, N_1\right)$. $e_1 \min$ is the minimal units of gauge charge carried by a deconfined particle excitation in theory $S = \int \frac{N_1}{2\pi} B^1 \wedge dA^1 + \frac{K_{11}}{4\pi} B^1 \wedge B^1$. An emergent fermion appears when $\Theta_{\text{trivial}} = 1$ while $\Theta_{e_1 \min} = -1$. In other words, there exists nontrivial fermionic particle excitations in a bosonic theory.

N_1	K_{11}	Θ_{trivial}	$\Theta_{e_1 \min}$	bosonic/fermionic theory?	emergent fermion?
odd	even	1	1	bosonic	No
odd	odd	-1	-1	fermionic	-
even	odd	1	1	bosonic	No
even	even and $\frac{\text{lcm}\left(\frac{K_{11}}{N_1}, N_1\right)}{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)} \in 2\mathbb{Z}$	1	1	bosonic	No
even	even and $\frac{\text{lcm}\left(\frac{K_{11}}{N_1}, N_1\right)}{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)} \in 2\mathbb{Z}+1$	1	-1	bosonic	Yes

III. BRAIDING AND FUSION DATA IN THE PRESENCE OF FERMIONS

In this section, we study the situation when emergent fermions take part in braiding and fusion processes in 3D Abelian bosonic topological orders. There are three root braiding processes in 3D Abelian bosonic topological orders: particle-loop braiding, multi-loop braiding, and Borromean rings (BR) braiding and the combinations of them classify 3D Abelian bosonic topological orders [35]. Each root braiding process coresponds to a topological term. For each combination of braiding process, we can write a TQFT action from which one can calculate braiding phases and fusion rules. Each topological excitation e can be represented by a gauge invariant Wilson operator $\mathcal{O}_{\mathsf{e}}$. Using path integral, we can extract fusion rules $\mathsf{a} \otimes \mathsf{b} = \oplus_i N^{\mathsf{ab}}_{\mathsf{e}_i} \mathsf{e}_i$ from [?]

$$\begin{aligned}\langle \mathsf{a} \otimes \mathsf{b} \rangle =& \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A^i, B^i\right] \exp\left(\mathrm{i}S\right) \times \left(\mathcal{O}_{\mathsf{a}} \times \mathcal{O}_{\mathsf{b}}\right) \\ =& \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A^i, B^i\right] \exp\left(\mathrm{i}S\right) \times \left(\sum_i N^{\mathsf{ab}}_{\mathsf{e}_i} \mathcal{O}_{\mathsf{e}_i}\right) \\ =& \left\langle \oplus_i N^{\mathsf{ab}}_{\mathsf{e}_i} \mathsf{e}_i \right\rangle. \end{aligned} \quad (49)$$

Since emergent fermion can be induced by a proper $B \wedge B$ term, we can couple a $B \wedge B$ term to other topological terms such that we can study whether and how emergent fermion would influence braiding statistics and fusion rules.

A. $BF + BB$: interplay of particle-loop braiding statistics and fusion in BF theory with fermion

A pure BF theory describes the particle-loop braiding. A BF term is compatible with a $B \wedge B$ term to form a legitimate TQFT action: ($\wedge$ is omitted)

$$S = \int \frac{N_1}{2\pi} B^1 dA^1 + \frac{K_{11}}{4\pi} B^1 B^1. \quad (50)$$

This is just the simplest BF theory with a single component $B \wedge B$ term. Emergent fermionic particle excitations are possible provided proper values of N_1 and K_{11}. To explicitly show how emergent fermion influences particle-loop braiding, we can consider the phase of particle-loop braiding given by

$$\begin{aligned}\Theta =& \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A^i\right] \mathcal{D}\left[B^i\right] \exp\left(\mathrm{i}S\right) \\ & \times \exp\left(\mathrm{i}e_1 \int_\gamma A^1 + \mathrm{i}\frac{e_1 K_{11}}{N_1} \int_\Sigma B^1\right) \exp\left(\mathrm{i}m_1 \int_\sigma B^3\right)\end{aligned} \quad (51)$$

where γ is a closed curve with $\partial\Sigma = \gamma$, σ is a closed surface, e_1 and m_1 are the numbers of charges and fluxes carried by the particle and the loop. γ and σ can be understood as the world line and world sheet of the particle and the loop. The phase of particle-loop braiding is

$$\Theta = \exp\left[-\frac{\mathrm{i}\pi K_{11} e_1 e_1}{N_1 N_1} \#\left(\Sigma \cap \Sigma\right) - \frac{\mathrm{i}2\pi e_1 m_1}{N_1} \#\left(\Sigma' \cap \sigma\right)\right] \quad (52)$$

where $\partial\Sigma' = \gamma$ and $\#\left(\Sigma' \cap \sigma\right)$ is the linking number of γ and σ. There are two contributions to this phase. $\exp\left[-\frac{\mathrm{i}2\pi e_1 m_1}{N_1} \#\left(\Sigma' \cap \sigma\right)\right]$ is the usual particle-loop braiding phase due to the particle traveling around the loop. $\exp\left[-\frac{\mathrm{i}\pi K_{11} e_1 e_1}{N_1 N_1} \#\left(\Sigma \cap \Sigma\right)\right]$ is just the exchange statistics of the particle excitation. If this particle is an emergent fermion, it may contribute an extra $\exp\left(\mathrm{i}\pi\right)$ phase to the usual particle-loop braiding phase. As discussed in previous section, the values of e_1 and m_1 are constrained by

$$e_1 = e_{1\min} \cdot p = \frac{N_1}{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)} \cdot p, p \in \mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)}; \quad (53)$$

$$m_1 \simeq m_1 + \gcd\left(\frac{K_{11}}{N_1}, N_1\right). \quad (54)$$

Consider a particle and a loop carrying minimal gauge charge and flux, the phase contributed by a particle-loop braiding is given by

$$\Theta_{\mathrm{PL}} = \exp\left(-\frac{\mathrm{i}2\pi e_{1\min} m_1}{N_1}\right) = \exp\left(-\frac{\mathrm{i}2\pi m_1}{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)}\right) \quad (55)$$

where $m_1 \in \mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}$. This means that the BF theory with a $B \wedge B$ term only labels fewer topologically ordered phases than a pure BF theory. This is because a $B \wedge B$ term would confine part of topological excitations, making the physical observable braiding phases fewer.

Consider a general topological excitation labeled by (e_1, m_1), when $m_1 = 0$ it is a pointlike particle excitation; when $e_1 = 0$, it is a pure loop excitation[1]; when $e_1, m_1 \neq 0$, it is a decorated loop excitation, i.e., the bound state of a particle and a pure loop. It is straightforward to see that the fusion rule of two topological excitations is given by

$$(e_1, m_1) \otimes (e_1', m_1') = (e_1 + e_1', m_1 + m_1') \,. \tag{56}$$

Since both e_1 and m_1 are labeled by $\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}$, the fusion rules can be captured by a $\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)} \times \mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}$ group. While for a pure BF theory $S = \int \frac{N_1}{2\pi} B^1 dA^1$, its fusion rules are captured by a $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_1}$ group [?].

In conclusion, the coefficient of $B \wedge B$ term K_{11} would break the gauge group structure of $\mathbb{Z}_{N_1} BF$ theory, leaving the unbroken gauge group to be $\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}$. Only $\gcd\left(\frac{K_{11}}{N_1}, N_1\right)$ of N_1 particle excitations are deconfined and the gauge fluxes are labeled by $\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}$. The cyclic structure of particle-loop braiding phase is described by the unbroken gauge group, so are the fusion rules. The fusion rules are still Abelian.

B. Three-loop braiding is not compatible with emergent fermion

A three-loop braiding is described by an $AAdA$ topological term. The simplest TQFT action consisting of BF term and $AAdA$ term is

$$S = \int \sum_{i=1}^{2} \frac{N_i}{2\pi} B^i dA^i + q_1 A^1 A^2 dA^2 + q_2 A^2 A^1 dA^1, \tag{57}$$

where q_1 and q_2 are proper coefficients and the gauge group is $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. We find that no $B \wedge B$ term can be added to action (57) because no well-defined gauge transformation could be written down [35]. To make this point more clear, then we prove that there is no proper gauge transformation for it. We can assume that the gauge transformations take the form of

$$A^i \to A^i + d\chi^i + X^i, \tag{59}$$
$$B^i \to B^i + dV^i + Y^i, \tag{60}$$

where χ^i and V^i are 0- and 1-form gauge parameters with $\int d\chi^i \in 2\pi\mathbb{Z}$ and $\int dV^i \in 2\pi\mathbb{Z}$ on closed manifolds. X^i and Y^i are so-called shift terms to be determined. Obviously X^i and Y^i cannot be written as total derivative, otherwise they can be absorbed into $d\chi^i$ and dV^i. The key point of this proof is to show that there exists no solution for X^i and Y^i[35].

Below we follow the spirit in [35] to prove that we could not find a proper gauge transformation for action (58). According to the above assumed gauge transformations, the action changes as

$$\begin{aligned} S \to S' = &\int \sum_{i=1}^{2} \frac{N_i}{2\pi} \left(B^i + dV^i + Y^i\right)\left(dA^i + dX^i\right) \\ &+ \sum_{i,j=1}^{2} q_i \left(A^i + d\chi^i + X^i\right)\left(A^j + d\chi^j + X^j\right)\left(dA^j + dX^j\right) \\ &+ \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} \left(B^i + dV^i + Y^i\right)\left(B^j + dV^j + Y^j\right) \\ = &S + \Delta S_{BF} + \Delta S_{AAdA} + \Delta S_{BB}, \end{aligned} \tag{61}$$

where

$$\Delta S_{BF} = \int \sum_{i=1}^{2} \frac{N_i}{2\pi} dV^i dA^i + \frac{N_i}{2\pi} Y^i dA^i + \frac{N_i}{2\pi} B^i dX^i + \frac{N_i}{2\pi} dV^i dX^i + \frac{N_i}{2\pi} Y^i dX^i, \tag{62}$$

we first write down the following action

$$\begin{aligned} S = &\int \sum_{i=1}^{2} \frac{N_i}{2\pi} B^i dA^i + q_1 A^1 A^2 dA^2 + q_2 A^2 A^1 dA^1 \\ &+ \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} B^i B^j, \end{aligned} \tag{58}$$

[1] Pure loop excitation is a loop excitation that has no particle attached on it.

$$\Delta S_{AAdA} = \int \sum_{i,j=1}^{2} q_i \left(A^i d\chi^j dA^j + A^i X^j dA^j + A^i A^j dX^j + A^i d\chi^j dX^j + A^i X^j dX^j\right)$$
$$+ \sum_{i,j=1}^{2} q_i \left(d\chi^i A^j dA^j + d\chi^i d\chi^j dA^j + d\chi^i X^j dA^j + d\chi^i A^j dX^j + d\chi^i d\chi^j dX^j + d\chi^i X^j dX^j\right)$$
$$+ \sum_{i,j=1}^{2} q_i \left(X^i A^j dA^j + X^i d\chi^j dA^j + X^i X^j dA^j + X^i A^j dX^j + X^i d\chi^j dX^j + X^i X^j dX^j\right), \quad (63)$$

$$\Delta S_{BB} = \int \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} \left(B^i dV^j + B^i Y^j\right)$$
$$+ \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} \left(dV^i B^j + dV^i dV^j + dV^i Y^j\right)$$
$$+ \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} \left(Y^i B^j + Y^i dV^j + Y^i Y^j\right) \quad (64)$$

Notice that in this $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ gauge theory, the $\mathbb{Z}_{N_i}$ ($i = 1, 2$) cyclic group structure should be encoded in the Wilson operators of gauge fields. In other words, the $\mathbb{Z}_{N_i}$ cyclic group structure is encoded in $\oint A^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$ or $\oiint \oint B^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$ ($i = 1, 2$).

We first assume that the $\mathbb{Z}_{N_i}$ cyclic group structure is encoded in $\oint A^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$, i.e., $X^i = 0$. We recognize that the $B^i dV^j$ term cannot be eliminated by subtraction or absorbed into a total derivative term. If we want to subtract $B^i dV^j$ in ΔS, we require that

$$B^i dV^j + Y^i dV^j = \left(B^i - B^i + \cdots\right) dV^j, \quad (65)$$

i.e.,

$$Y^i = -B^i + \cdots, \quad (66)$$

but this means that the B^i transforms as $B^i \to dV^i + \cdots$ which is not a well-defined gauge transformation for B^i. If we want to absorb $B^i dV^j$ into a total derivative term, ΔS should contain (yet not) a $dB^i V^j$ term since

$$d\left(B^i V^j\right) = dB^i V^j + B^i dV^j. \quad (67)$$

We can see that the $B^i dV^j$ term is an obstruction for ΔS being a total derivative term to be dropped out. This fact indicates that when $X^i = 0$ we cannot find a proper gauge transformation for B^i such that ΔS can be dropped out.

Next, we turn to assume that $\mathbb{Z}_{N_i}$ cyclic group structure is encoded in $\oint B^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$, i.e., $Y^i = 0$. Under this assumption, we find that the $A^i d\chi^j dA^j$ term is a stubborn term that cannot be eliminated. If we want to subtract it in ΔS, we require that

$$A^i d\chi^j dA^j + X^i d\chi^j dA^j = \left(A^i - A^i + \cdots\right) d\chi^j dA^j, \quad (68)$$

i.e.,

$$X^i = -A^i + \cdots, \quad (69)$$

but this means no well-defined gauge transformation for A^i. If we hope to absorbed $A^i d\chi^j dA^j$ into a total derivative term, ΔS should contain (yet not, again) a $dA^i \chi^j dA^j$ term since

$$d\left(A^i \chi^j dA^j\right) = dA^i \chi^j dA^j + A^i d\chi^j dA^j. \quad (70)$$

This $A^i d\chi^j dA^j$ term plays as an obstacle for ΔS to be a total derivative. We have to meet the fact that there is no proper gauge transformation for A^i when $Y^i = 0$.

Above discussion shows that we can never find well defined gauge transformations for A^i and B^i no matter the $\mathbb{Z}_{N_i}$ cyclic group structure is encoded in $\oint A^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$ or $\oiint \oint B^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$. The absence of gauge transformations in turn reveals the illegitimacy of action (58).

An alternate way to see the illegitimacy of action (58) is as follows. In action (57), i.e., setting $K_{ij} = 0$ in action (58), B^1 and B^2 are Lagrange multipliers to enforce the flat connection conditions $dA^1 = 0$ and $dA^2 = 0$. This means that A^1 and A^2 transform as $A^{1,2} \to A^{1,2} + d\chi^{1,2}$, i.e., $\oint A^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$. On the other hand, for $S = \int \sum_{i=1}^{2} \frac{N_i}{2\pi} B^i dA^i + \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} B^i B^j$, i.e., setting $q_1 = q_2 = 0$ in action(58), A^1 and A^2 serve as Lagrange multipliers to enforce $dB^1 = 0$ and $dB^2 = 0$, which means that $B^{1,2} \to B^{1,2} + dV^{1,2}$ and $\oiint B^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$.. Back to action (58) with nonzero q_1, q_2, and K_{ij}'s, either B^i or A^i can be the Lagrange multiplier. Therefore, one cannot write down a proper gauge transformation for action (58) without violating the $\mathbb{Z}_{N_i}$ cyclic group structure.

The illegitimacy of action (58) reveals that a $B \wedge B$ term is not compatible with an $AAdA$ term. In addition, a $B \wedge B$ term is not compatible with an $AAAA$ term (responsible for four-loop braiding) or an AAB term (responsible for Borromean rings braiding) as long as B and A correspond to the same gauge subgroup. On the other hand, an absence of $B \wedge B$ term means that all particle excitations are bosonic, i.e., no emergent fermion can be induced. Let us state this result as a theorem.

Theorem 1. *Consider a bosonic topological order with an Abelian gauge group $G=\prod_i \mathbb{Z}_{N_i}$, the flux ϕ_i and the charge q_i of gauge subgroup $\mathbb{Z}_{N_i}$. If ϕ_i is involved in a three-loop braiding, four-loop braiding, or a Borromean rings braiding, then the charge of the same gauge subgroup, q_i, must be carried by a particle excitation with bosonic exchange statistics.*

IV. BORROMEAN RINGS TOPOLOGICAL ORDER WITH FERMION

A Borromean rings braiding is described by an AAB topological term [19]. A system is equipped with Borromean rings topological order if it supports a Borromean rings braiding. A Borromean rings topological order is featured by non-Abelian fusion rules and loop shrinking rules [**?**]. Unlike $AAdA$ term, an AAB term can be compatible with $B\wedge B$ term, so we can consider the following TQFT action:

$$S=\int\sum_{i=1}^{3}\frac{N_i}{2\pi}B^i dA^i+qA^1A^2B^3+\frac{K_{33}}{4\pi}B^3B^3 \tag{71}$$

where $q=\frac{pN_1N_2N_3}{N_{123}}$ with $N_{123}=\gcd(N_1,N_2,N_3)$, $p\in\mathbb{Z}_{N_{123}}$, and the gauge group is $G=\prod_{i=1}^{3}\mathbb{Z}_{N_i}$. The gauge transformations are

$$A^1\to A^1+d\chi^1, \tag{72}$$

$$A^2\to A^2+d\chi^2, \tag{73}$$

$$\begin{aligned}A^3\to &A^3+d\chi^3-\frac{K_{33}}{N_3}V^3\\&-\frac{2\pi q}{N_3}\left(\chi^1A^2+\frac{1}{2}\chi^1d\chi^2\right)\\&+\frac{2\pi q}{N_3}\left(\chi^2A^1+\frac{1}{2}\chi^2d\chi^1\right),\end{aligned} \tag{74}$$

$$\begin{aligned}B^1\to &B^1+dV^1\\&-\frac{2\pi q}{N_1}\left(\chi^2B^3-A^2V^3+\chi^2dV^3\right),\end{aligned} \tag{75}$$

$$\begin{aligned}B^2\to &B^2+dV^2\\&+\frac{2\pi q}{N_2}\left(\chi^1B^3-A^1V^3+\chi^1dV^3\right),\end{aligned} \tag{76}$$

$$B^3\to B^3+dV^3. \tag{77}$$

The compatibility of AAB term and BB term indicates that emergent fermion is possible in Borromean rings topological order.

A. Borromean rings braiding phase

We use $\mathsf{P}_{e_1e_2e_3}$ to denote a particle excitation carrying e_i units of $\mathbb{Z}_{N_i}$ gauge charges and $\mathsf{L}_{m_1m_2m_3}$ to denote a pure loop excitation carrying m_i units of $\mathbb{Z}_{N_i}$ gauge fluxes ($i=1,2,3$). A decorated loop (formed by attaching a particle excitation to a pure loop excitation) is denoted by $\mathsf{L}^{e_1e_2e_3}_{m_1m_2m_3}$. Consdier a Borromean rings braiding involving L_{m_100}, L_{0m_20}, and P_{00e_3}, the phase is

$$\begin{aligned}\Theta(m_1,m_2,e_3)=&\exp\left[-\frac{\mathrm{i}2\pi pm_1m_2e_3}{N_{123}}\cdot\mathrm{Tlk}\right]\\&\times\exp\left[-\frac{\mathrm{i}\pi K_{33}e_3e_3}{N_3N_3}\#(\Sigma\cap\Sigma)\right]\end{aligned} \tag{78}$$

where Tlk is the Milnor's triple linking number of the link formed by the two loops' world sheets σ_1,σ_2 and the particle's world line γ, Σ is a Seifert surface of γ. The first term is the phase of Borromean rings braiding Θ_{BR} [19] while the second term is due to the possible self 2π rotation of P_{00e_3} during the braiding process. Since the self 2π rotation of P_{00e_3} would introduce an extra phase of ±1, depending on its own exchange statistics (spin-statistics theorem), we can just ignore it. Notice that the existence of $B\wedge B$ term will confine some particle excitation carrying $\mathbb{Z}_{N_3}$ gauge charges, the value of e_3 is given by

$$e_3\in\frac{N_3}{\gcd\left(\frac{K_{33}}{N_3},N_3\right)}\mathbb{Z}. \tag{79}$$

When $K_{33}=0$, i.e., no $B\wedge B$ term considered, e_3 is labeled by $\mathbb{Z}_{N_3}$ and the minimal e_3 is 1. The Borromean rings braiding phase

$$\Theta_{\mathrm{BR}}(m_1,m_2,1)=\exp\left(-\frac{\mathrm{i}2\pi pm_1m_2}{N_{123}}\right) \tag{80}$$

is labeled by $p\in\mathbb{Z}_{N_{123}}$. In the case of $K_{33}\neq0$, the minimal e_3 cannot be 1 any longer since it would be confined. The minimal value of e_3 is $e_{3\,\min}=\frac{N_3}{\gcd\left(\frac{K_{33}}{N_3},N_3\right)}$. The Borromean rings braiding phase is

$$\begin{aligned}\Theta_{\mathrm{BR}}(m_1,m_2,e_{3\,\min})&=\exp\left(-\frac{\mathrm{i}2\pi pm_1m_2e_{3\,\min}}{N_{123}}\right)\\&=\exp\left(-\frac{\mathrm{i}2\pi pm_1m_2N_3}{N_{123}\gcd\left(\frac{K_{33}}{N_3},N_3\right)}\right).\end{aligned} \tag{81}$$

Since $\frac{N_3}{N_{123}}\in\mathbb{Z}$, we can see that p is identified with $p+\gcd\left(\frac{K_{33}}{N_3},N_3\right)$. Combined with $p\in\mathbb{Z}_{N_{123}}$, we find that p is actually labeled by $\mathbb{Z}_{\gcd\left(N_1,N_2,N_3,\frac{K_{33}}{N_3}\right)}$. We can see that adding a $B\wedge B$ term to $S=\int\sum_{i=1}^{3}\frac{N_i}{2\pi}B^idA^i+qA^1A^2B^3$ may reduce the number of different Borromean rings braiding phases. This result is reasonable since some particle excitations are confined by $B\wedge B$ term hence cannnot contribute to an observable Borromean rings braiding phases .

B. Fusion rules and an example of $G=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6$

Each topological excitation e_i can be represented by a gauge invariant Wilson operator $\mathcal{O}_i$. Using path integral, we can extract fusion rules $\mathsf{e}_i \otimes \mathsf{e}_j = \oplus_k N_k^{ij} \mathsf{e}_k$ from [?]:

$$\begin{aligned}\langle \mathsf{e}_i \otimes \mathsf{e}_j \rangle =& \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A^i, B^i\right] \exp\left(\mathrm{i}S\right) \times \left(\mathcal{O}_i \times \mathcal{O}_j\right) \\ =& \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A^i, B^i\right] \exp\left(\mathrm{i}S\right) \times \left(\sum_i N_k^{ij} \mathcal{O}_k\right) \\ =& \left\langle \oplus_i N_k^{ij} \mathsf{e}_i \right\rangle . \end{aligned} \tag{82}$$

The algebraic fusion rules of topological excitations can be extracted from arithmetic calculation of corresponding Wilson operators. Since emergent fermion can be induced by a proper $B \wedge B$ term, we can couple a $B \wedge B$ term to other topological terms such that we can study whether and how emergent fermion would influence fusion rules.

First, let us find out Wilson operators for those topological excitation carrying only *one* kind of gauge charge or flux for the action

$$S = \int \sum_{i=1}^{3} \frac{N_i}{2\pi} B^i dA^i + q A^1 A^2 B^3 + \frac{K_{33}}{4\pi} B^3 B^3 \tag{83}$$

with $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$. The particle excitations carrying $\mathbb{Z}_{N_1}$ or $\mathbb{Z}_{N_2}$ gauge charges are represented by gauge invariant Wilson operators

$$\mathsf{P}_{e_1 00} = \mathcal{N}^{e_1 00} \exp\left(\mathrm{i} e_1 \int_\gamma A^1\right), \tag{84}$$

$$\mathsf{P}_{0 e_2 0} = \mathcal{N}^{0 e_2 0} \exp\left(\mathrm{i} e_2 \int_\gamma A^2\right), \tag{85}$$

and the pure loop excitations carrying $\mathbb{Z}_{N_3}$ gauge fluxes are represented by

$$\mathsf{L}_{00 m_3} = \mathcal{N}_{00 m_3} \exp\left(\mathrm{i} m_3 \int_\sigma B^3\right), \tag{86}$$

where the factors $\mathcal{N}$'s are to be determined. The operator for pure loop excitation carrying $\mathbb{Z}_{N_1}$ gauge fluxes is

$$\begin{aligned}\mathsf{L}_{m_1 00} =& \mathcal{N}_{m_1 00} \exp\left[\mathrm{i} m_1 \int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}\left(d^{-1}A^2 B^3 + d^{-1} B^3 A^2\right)\right] \\ & \times \delta\left(\int_\gamma A^2\right) \delta\left(\int_\sigma B^3\right), \end{aligned} \tag{87}$$

where the Kronecker delta function are $\delta\left(\int_\gamma A^2\right) = \begin{cases} 1, & \int_\gamma A^2 = 0 \mod 2\pi \\ 0, & \text{else}\end{cases}$ and $\delta\left(\int_\sigma B^3\right) = \begin{cases} 1, & \int_\sigma B^3 = 0 \mod 2\pi \\ 0, & \text{else}\end{cases}$ to ensure $d^{-1}A^2$ and $d^{-1}B^3$ are well-defined [15, 48?]. . Similarly, the operator for pure loop carrying $\mathbb{Z}_{N_2}$ gauge fluxes is

$$\begin{aligned}\mathsf{L}_{0 m_2 0} =& \mathcal{N}_{0 m_2 0} \exp\left[\mathrm{i} m_2 \int_\sigma B^2 - \frac{1}{2}\frac{2\pi q}{N_2}\left(d^{-1}B^3 A^1 + d^{-1} A^1 B^3\right)\right] \\ & \times \delta\left(\int_\gamma A^1\right) \delta\left(\int_\sigma B^3\right). \end{aligned} \tag{88}$$

The particle excitation carrying $\mathbb{Z}_{N_3}$ gauge charges is represented by

$$\begin{aligned}\mathsf{P}_{00 e_3} =& \mathcal{N}^{00 e_3} \exp\left[\mathrm{i} e_3 \int_\gamma A^3 + \mathrm{i} e_3 \int_\gamma \frac{1}{2}\frac{2\pi q}{N_3}\left(d^{-1}A^1 A^2 - d^{-1} A^2 A^1\right)\right. \\ & \left. + \mathrm{i}\frac{e_3 K_{33}}{N_3} \int_\Sigma B^3\right] \delta\left(\int_\gamma A^1\right) \delta\left(\int_\gamma A^2\right), \end{aligned} \tag{89}$$

These Kronecker delta functions can be expanded by summation of some exponentials, e.g., $\delta\left(\int_\gamma A^1\right) = \frac{1}{N_1}\sum_{k=1}^{N_1} \exp\left(\mathrm{i}k \int_\gamma A^1\right)$ [48?]. As mentioned in previous section, $\mathsf{P}_{00 e_3}$ is a particle excitation attached by a flux string and may be confined due to the tension on string. $\mathsf{P}_{00 e_3}$ is deconfined only when the flux on the string is a multiple of 2π. The minimal e_3 for deconfined $\mathsf{P}_{00 e_3}$ is

$$e_{3\min} = \frac{N_3}{\gcd\left(\frac{K_{33}}{N_3}, N_3\right)}. \tag{90}$$

Notice that the limitation of values of e_3 influences the period of $\mathbb{Z}_{N_3}$ gauge fluxes. Since a loop excitation is detected by a particle excitation, we consider the particle-loop braiding phase of $\mathsf{P}_{00 e_{3\min}}$ and $\mathsf{L}_{00 m_3}$:

$$\begin{aligned}\Theta_{\mathrm{PL}}\left(e_{3\min}, m_3\right) =& \exp\left[-\frac{2\pi e_{3\min} m_3}{N_3}\right] \\ =& \exp\left[-\frac{2\pi N_3 m_3}{\gcd\left(\frac{K_{33}}{N_3}, N_3\right)}\right]. \end{aligned} \tag{91}$$

We immediatley see that m_3 is equivalent with $m_3 + \gcd\left(\frac{K_{33}}{N_3}, N_3\right)$. In other words, m_3 has a period of $\gcd\left(\frac{K_{33}}{N_3}, N_3\right)$. This is important when we discuss the fusion rules in the following main text.

So far we have find out Wilson operators for topological excitation carrying only one kind of gauge charge or flux. Other excitation with multiple species of gauge charges or fluxes, e.g., a particle excitation with different $\mathbb{Z}_{N_i}$ gauge charges is defined by

$$\mathsf{P}_{e_1 00} \otimes \mathsf{P}_{0 e_2 0} \otimes \mathsf{P}_{00 e_3} \equiv \mathsf{P}_{e_1 e_2 e_3}, \tag{92}$$

or a decorated loop excitation with $\mathbb{Z}_{N_i}$ gauge fluxes and $\mathbb{Z}_{N_j}$ gauge charges is defined by

$$\mathsf{P}_{e_1 00} \otimes \mathsf{P}_{0 e_2 0} \otimes \mathsf{P}_{00 e_3} \otimes \mathsf{L}_{m_1 00} \otimes \mathsf{L}_{0 m_2 0} \otimes \mathsf{L}_{00 m_3} \equiv \mathsf{L}_{m_1 m_2 m_3}^{e_1 e_2 e_3}. \tag{93}$$

Next, we need to determine the factors $\mathcal{N}$ for each operators. For an illustration, we consider the loop excitation L_{100} which carries flux of $\mathbb{Z}_{N_1}$ gauge subgroup, its Wilson operator is

$$\begin{aligned}\mathsf{L}_{100} =& \mathcal{N}_{100} \exp\left[\mathrm{i} \int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}\left(d^{-1}A^2 B^3 + d^{-1} B^3 A^2\right)\right] \\ & \times \delta\left(\int_\gamma A^2\right) \delta\left(\int_\sigma B^3\right). \end{aligned} \tag{94}$$

Since L_{100} represent the element 1 in group $\mathbb{Z}_{N_1}$, according to the $\mathbb{Z}_{N_1}$ cyclic structure, it is natural to require

$$\underbrace{\mathsf{L}_{100} \otimes \mathsf{L}_{100} \otimes \cdots \otimes \mathsf{L}_{100}}_{N_1 \text{ terms}} = 1 + \cdots \tag{95}$$

where "$\cdots$" denotes other fusion channels if this fusion is non-Abelian. Here we have made an assumption: for an excitation with only kind of charge or flux, fusing it and its anti excitation would output *one* vacuum. This assumption is reansonable since a pair of particle and anti particle, or a pair of loop and anti loop, can be created from vacuum and then be annihilated to vacuum. For those with multiple kinds of non-Abelian charges or fluxes, fusion a pair of excitation and anti-excitation may output more than one vacuums [**?**]. In path integral, the fusion (95) is written as

$$\begin{aligned}
\left\langle (\mathsf{L}_{100})^{\otimes N_1} \right\rangle =& (\mathcal{N}_{100})^{N_1} \left\langle \exp\left[\mathrm{i} N_1 \int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}\left(d^{-1}A^2B^3 + d^{-1}B^3A^2\right)\right] \right.\\
&\left. \times \delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right)\right\rangle \\
=& (\mathcal{N}_{100})^{N_1}\left\langle \delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right)\right\rangle \\
=& (\mathcal{N}_{100})^{N_1}\left\langle \frac{1}{N_2}\sum_{e_2=1}^{N_2}\exp\left(\mathrm{i}e_2\int_\gamma A^2\right)\frac{1}{N_3}\sum_{m_3=1}^{N_3}\exp\left(\mathrm{i}m_3\int_\sigma B^3\right)\right\rangle \\
=& \frac{(\mathcal{N}_{100})^{N_1}}{N_2N_3}\left\langle 1 + \sum_{e_2=1}^{N_2-1}\exp\left(\mathrm{i}e_2\int_\gamma A^2\right) + \sum_{m_3=1}^{N_3-1}\exp\left(\mathrm{i}m_3\int_\sigma B^3\right)\right. \\
&\left. + \sum_{e_2=1}^{N_2-1}\sum_{m_3=1}^{N_3-1}\exp\left(\mathrm{i}e_2\int_\gamma A^2 + \mathrm{i}m_3\int_\sigma B^3\right)\right\rangle
\end{aligned} \tag{96}$$

where we have used

$$\left\langle \exp\left[\mathrm{i}N_1\int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}d^{-1}A^2B^3 + d^{-1}B^3A^2\right]\right\rangle = 1 \tag{97}$$

since B^1 is $\mathbb{Z}_{N_1}$ valued. Since the fusion coefficient of vacuum is 1, it is required that

$$\frac{(\mathcal{N}_{100})^{N_1}}{N_2N_3} = 1, \tag{98}$$

i.e., the factor of Wilson operator for L_{100} is

$$\mathcal{N}_{100} = \sqrt[N_1]{N_2N_3}. \tag{99}$$

Now we are going to shown that the factor $\mathcal{N}_{100}$ is exactly equal to the quantum dimension of L_{100}. Notice that the result in Eq. (96) tells us that the output of fusing N_1 L_{100}'s is

$$\begin{aligned}
(\mathsf{L}_{100})^{\otimes N_1} =& 1 \oplus \left(\oplus_{e_2=1}^{N_2-1}\mathsf{P}_{0e_20}\right) \oplus \left(\oplus_{m_3=1}^{N_3-1}\mathsf{L}_{00m_3}\right) \\
&\oplus\left(\oplus_{e_2=1}^{N_2-1}\oplus_{m_3=1}^{N_3-1}\mathsf{L}_{00m_3}^{0e_20}\right).
\end{aligned} \tag{100}$$

It is easy to see that P_{0e_20} is an Abelian particle excitation whose quantum dimension is 1. This is because

$$\begin{aligned}
\left\langle (\mathsf{P}_{010})^{\otimes N_2}\right\rangle =& \left\langle \left(\mathcal{N}^{010}\right)^{N_2}\exp\left(\mathrm{i}N_2\int_\gamma A^2\right)\right\rangle, \\
=& \left\langle \left(\mathcal{N}^{010}\right)^{N_2}\cdot 1\right\rangle \\
=& \left(\mathcal{N}^{010}\right)^{N_2}\cdot \mathsf{1}
\end{aligned} \tag{101}$$

where $\mathsf{1}$ denotes the vacuum. Our assumption above requires $\left(\mathcal{N}^{010}\right)^{N_2} = 1$ thus $\mathcal{N}^{0e_20} = 1, \forall e_2 \in \mathbb{Z}_{N_2}$. Similarly, we know that L_{00m_3}'s and $\mathsf{L}_{00m_3}^{0e_20}$'s are all Abelian excitations. For a fusion rule

$$\mathsf{e}_i \otimes \mathsf{e}_k = \oplus_m N_m^{ik}\mathsf{e}_m$$

where the quantum dimension of e_i is denoted as d_i, there is a relation of these quantum dimensions (the proof can be found in Appendix):

$$d_id_k = \sum_m N_m^{ik}d_m. \tag{102}$$

Let the quantum dimension of L_{100} be d_{100}. Applying Eq. (102) to fusion rule (100), we have

$$(d_{100})^{N_1} = \sum_m N_m^{ik}d_m = N_2N_3 \tag{103}$$

thus the quantum dimension of L_{100} is

$$d_{100} = \sqrt[N_1]{N_2 N_3}. \quad (104)$$

We can see that the quantum dimension of excitation L_{100} is just the factor of its Wilson operator.

Let us go through this line of thinking again: first we write the Wilson operator of L_{100} with an unknown factor $\mathcal{N}_{100}$. At this time we do not know any fusion rules of L_{100} yet. By demanding $\mathsf{L}_{100} \otimes \mathsf{L}_{(N_1-1)00} = 1 + \cdots$ from the $\mathbb{Z}_{N_i}$ cyclic structure, we obtain $\mathcal{N}_{100} = \sqrt[N_1]{N_2 N_3}$. Meanwhile, by expanding the Kronecker delta functions, we obtain the fusion rule (100) which tells us the channels are all Abelian excitations. Since the quantum dimension of Abelian excitation is 1, applying Eq. (102) we find the quantum dimension of L_{100} is $d_{100} = \sqrt[N_1]{N_2 N_3}$, same as its Wilson operator's factor. So far, we have seen that for topological excitation carrying only one species of charge or flux, its quantum dimension is same as the factor of its Wilson operator. For topological excitation carrying charges or fluxes from different $\mathbb{Z}_{N_i}$ subgroups, it is defined by fusion of those with only one kind of charge or flux, see Eqs. (92) and (93). Their quantum dimension can be obtained by Eq. (102) and the factor of their Wilson operator can obtained by path integral calculation according to Eqs. (92) and (93).

So far we are ready to discuss how the fusion rules of action (71) affected by the $B \wedge B$ term. We take an example of $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ and $\frac{K_{33}}{N_3} = 2$. We will compare the two situations of $K_{33} = 0$ and $\frac{K_{33}}{N_3} = 2$. The fusion rules of action (71) without $B \wedge B$ term in the case of $G = (\mathbb{Z}_2)^3$ are studied in Ref. [**?**].

We first take a look at the particle excitation P_{00e_3}:

$$\begin{aligned}\mathsf{P}_{00e_3} =& \mathcal{N}^{00e_3} \exp\left[\mathrm{i}e_3 \int_\gamma A^3 + \mathrm{i}e_3 \int_\gamma \frac{1}{2}\frac{2\pi q}{N_3}\left(d^{-1}A^1A^2 - d^{-1}A^2A^1\right)\right.\\ &\left.+\mathrm{i}\frac{e_3 K_{33}}{N_3}\int_\Sigma B^3\right]\delta\left(\int_\gamma A^1\right)\delta\left(\int_\gamma A^2\right). \quad (105)\end{aligned}$$

As shown in previous discussion, turning on the $\frac{K_{33}}{4\pi}B^3B^3$ term in action (71) would narrow the choices of e_3's. When $K_{33} = 0$, e_3 takes values from $\mathbb{Z}_{N_3} = \mathbb{Z}_6$. When $\frac{K_{33}}{N_3} = 2$, there exist a minimal value of e_3, $e_{3\,\min}$, and the charges of deconfined P_{00e_3} should satisfy $e_3 \in e_{3\,\min}\mathbb{Z}$ where

$$e_{3\,\min} = \frac{N_3}{\gcd\left(\frac{K_{33}}{N_3}, N_3\right)} = 3. \quad (106)$$

In the case of $K_{33} = 0$, the charges of P_{00e_3} are labeled by $\mathbb{Z}_6$. This $\mathbb{Z}_6$ cyclicity of e_3 indicates the following fusion rule

$$(\mathsf{P}_{001})^{\otimes 6} = 1 \oplus \mathsf{P}_{100} \oplus \mathsf{P}_{010} \oplus \mathsf{P}_{110}. \quad (107)$$

The quantum dimension of P_{001} is

$$\mathcal{N}^{001} = \sqrt[6]{1+1+1+1} = 2^{\frac{1}{3}}. \quad (108)$$

In the case of $\frac{K_{33}}{N_3} = 2$, the charges of deconfined P_{00e_3} are 3 and 6, labeled by $\mathbb{Z}_{\gcd\left(\frac{K_{33}}{N_3}, N_3\right)} = \mathbb{Z}_2$. By definition, $\mathsf{P}_{00e_{3\,\min}} = (\mathsf{P}_{001})^{\otimes 3} = \mathsf{P}_{003}$ and its operators is

$$\begin{aligned}\mathsf{P}_{00e_{3\,\min}} =& \mathcal{N}^{00e_{3\,\min}} \exp\left[\mathrm{i}e_{3\,\min}\int_\gamma A^3\right.\\ &+ \mathrm{i}e_{3\,\min}\int_\gamma \frac{1}{2}\frac{2\pi q}{N_3}\left(d^{-1}A^1A^2 - d^{-1}A^2A^1\right)\\ &\left.+\mathrm{i}\frac{e_{3\,\min}K_{33}}{N_3}\int_\Sigma B^3\right]\delta\left(\int_\gamma A^1\right)\delta\left(\int_\gamma A^2\right).\end{aligned} \quad (109)$$

Since P_{00e_3} is labeled by $\mathbb{Z}_2$ when $\frac{K_{33}}{N_3} = 2$, from $\langle \mathsf{P}_{00e_{3\,\min}} \otimes \mathsf{P}_{00e_{3\,\min}}\rangle$ and requiring the coefficient of vacuum is 1 we have

$$\mathsf{P}_{00e_{3\,\min}} \otimes \mathsf{P}_{00e_{3\,\min}} = 1 \oplus \mathsf{P}_{100} \oplus \mathsf{P}_{010} \oplus \mathsf{P}_{110}. \quad (110)$$

Compared to the case of $K_{33} = 0$, this is just the fusion rule of two P_{003}'s. Through this example, we see that one of the effect of $B \wedge B$ term is to confine some particle excitations, i.e., P_{00e_3} with $e_3 \neq 3\mathbb{Z}$. However, the fusion rules of deconfined particle excitations are unchanged. This result can be understood as that the flux attachment due ot $B \wedge B$ term does not change the particle excitation's internal degrees of freedom that correspond to fusion.

Next, we foucs on the loop excitation L_{00m_3}. As aforementioned, the $B \wedge B$ term makes m_3 has a smaller period than N_3: in the case of $\frac{K_{33}}{N_3} = 2$, m_3 is equivalent to $m_3 + 2$. In other words, m_3 is labeled by $\mathbb{Z}_2$: for $m_3 \in \{0, 2, 4\}$, L_{00m_3} is equivalent to the vacuum 1; for $m_3 \in \{1, 3, 5\}$, L_{00m_3} is equivalent to L_{001}. The corresponding fusion rules are:

$$\underbrace{\mathsf{L}_{001} \otimes \mathsf{L}_{001} \otimes \cdots \otimes \mathsf{L}_{001}}_{N_3=6\text{ terms}} = 1, K_{33} = 0; \quad (111)$$

$$\mathsf{L}_{001} \otimes \mathsf{L}_{001} = \mathsf{L}_{002}, K_{33} = 0; \quad (112)$$

$$\mathsf{L}_{001} \otimes \mathsf{L}_{001} = 1, \frac{K_{33}}{N_3} = 2. \quad (113)$$

The last example to show is the non-Abelian loop excitation L_{100}:

$$\begin{aligned}\mathsf{L}_{100} =& \mathcal{N}_{100}\exp\left[\mathrm{i}\int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}\left(d^{-1}A^2B^3 + d^{-1}B^3A^2\right)\right]\\ &\times\delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right), \quad (114)\end{aligned}$$

When $K_{33} = 0$, these two delta functions can be expanded as

$$\delta\left(\int_\gamma A^2\right) = \frac{1}{2}\left[1 + \exp\left(\mathrm{i}\int_\gamma A^2\right)\right], \quad (115)$$

$$\delta\left(\int_\sigma B^3\right) = \frac{1}{6}\sum_{m_3=1}^{6}\exp\left(\mathrm{i}m_3\int_\sigma B^3\right). \quad (116)$$

We can calculate the factor $\mathcal{N}_{100}$ from $\langle \mathsf{L}_{100} \otimes \mathsf{L}_{100}\rangle$:

$$\mathcal{N}_{100} = \sqrt{2 \times 6} = 2\sqrt{3}. \quad (117)$$

As shown in above discussion, $\mathcal{N}_{100}$ is also the quantum dimension of L_{100}. By setting $\frac{K_{33}}{N_3} = 2$ we turn on the $B \wedge B$ term. Due to the period of m_3, $m_3 \simeq m_3 + \gcd\left(\frac{K_{33}}{N_3}, N_3\right)$, the expansion of $\delta\left(\int_\sigma B^3\right)$ actually becomes (in the sense of correlation with other operators)

$$\delta\left(\int_\sigma B^3\right) = \frac{1}{2}\left[1 + \exp\left(\mathrm{i}\int_\sigma B^3\right)\right] \tag{118}$$

The factor $\mathcal{N}_{100}$ as well as the quantum dimension of L_{100} then becomes

$$\mathcal{N}_{100} = \sqrt{2 \times 2} = 2. \tag{119}$$

In summary, the influences of $B \wedge B$ term on fusion rules are as follows. First, $B \wedge B$ term would confine part of particle excitations. This in turn makes some loop excitations that used to distinguishable now become equivalent in the sense of correlation with other excitations. As in the above example, L_{00m_3} used to be labeled by $\mathbb{Z}_6$ but now labeled by $\mathbb{Z}_2$ due to the $B \wedge B$ term. Consequently, other topological excitations' quantum dimensions are changed. In the above example, the output of fusion two L_{100}'s used to be

$$\mathsf{L}_{100} \otimes \mathsf{L}_{100} = 1 \oplus \mathsf{P}_{010} \oplus \left(\oplus_{m_3=1}^{6} \mathsf{L}_{00m_3}\right) \oplus \left(\oplus_{m_3=1}^{6} \mathsf{L}_{00m_3}^{010}\right), \tag{120}$$

but due to $B \wedge B$ term, becomes

$$\mathsf{L}_{100} \otimes \mathsf{L}_{100} = 1 \oplus \mathsf{P}_{010} \oplus \mathsf{L}_{001} \oplus \mathsf{L}_{001}^{010}. \tag{121}$$

V. CONCLUSION AND OUTLOOK

In this paper, we study braiding phases and fusion rules in 3D topological orders with fermions where the fermions include trivial fermion and emergent fermion. A trivial fermion is a trivial particle excitation with fermionic statistics while an emergent fermion is a fermionic nontrivial particle excitation. To incorporate fermionic statistics in the TQFT description of 3D topological orders, we introduce a $B \wedge B$ term with proper coefficient to the action. The effects of a $B \wedge B$ term also includes that the corresponding $\mathbb{Z}_{N_i}$ gauge group would be broken to its subgroup. This in fact means that only part of particle excitations representing $\mathbb{Z}_{N_i}$ gauge group are deconfined. TQFT actions consisting of BF terms, $BwedgeB$ terms, and twisted terms with general Abelian gauge group $\prod_i \mathbb{Z}_{N_i}$ are studied. Specially, we illustrate how a general $B \wedge B$ term with a coefficient matrix K influences the exchange statistics of deconfined particle excitations. We find that the statistics angle of a particle excitation labeled by $\mathbf{l} = (e_1, e_2, \cdots, e_n)^T$ is given by $\Theta = \pi \mathbf{l}^T \widetilde{K} \mathbf{l}$ where $\widetilde{K}_{ij} = \frac{K_{ij}}{N_i N_j}$. The expression of this statistics angle is similar to that of a K-matrix Chern-Simons theory.

We study in what situation, respectively, trivial fermion and emergent fermion are possible and how they influence the braiding phase and fusion rules. We find that if 3-loop braiding and/or BR braiding are considered, the loops carry gauge fluxes from at least two different gauge subgroups. For those gauge subgroups whose gauge fluxes take part in 3-loop braiding or BR braiding, their gauge charges can only be carried by bosonic particle excitations. For example, when $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ and 3-loop braidings are presented, emergent fermion are forbidden. When $G = \prod_{i=1}^{3} \mathbb{Z}_{N_i}$ and a BR braiding is considered, emergent fermion is possible. Furthermore, we take BR topological order as an example to see how emergent fermion influences its fusion rules.

In the main text we states that charge of a gauge subgroup must be carried by a bosonic particle excitation when the flux of the same gauge subgroup is involved in a three-loop braiding. This rather interesting result is obtained from the fact that an $AAdA$ term is not compatible with a $B \wedge B$ term. This result further indicates that three-loop braiding and fermionic particle excitation is not compatible in an Abelian bosonic topologically ordered system. However, there seems no obvious reason to forbidden three-loop braiding in a fermionic topological order. Indeed, there are previous studies on multi-loop braiding in fermionic symmetry-protected topological (FSPT) phases, e.g., [**?**]. The exclusion of three-loop braiding and fermion may be due to the limit of our Abelian TQFT. One may need a non-Abelian theory to capture three-loop braiding in fermionic topological order.

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[1] Eduardo Fradkin, *Field theories of condensed matter physics* (Cambridge University Press, 2013).

[2] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, "Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order," Phys. Rev. B **82**, 155138 (2010).

[3] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma, "Non-abelian anyons and topological quantum computation," Rev. Mod. Phys. **80**, 1083–1159 (2008).

[4] B. Blok and X. G. Wen, "Effective theories of the fractional quantum hall effect: Hierarchy construction," Phys. Rev. B **42**, 8145–8156 (1990).

[5] X. G. Wen and A. Zee, "Classification of abelian quantum hall states and matrix formulation of topological fluids," Phys. Rev. B **46**, 2290–2301 (1992).

[6] Yuan-Ming Lu and Ashvin Vishwanath, "Classification and properties of symmetry-enriched topological phases: Chern-simons approach with applications to Z_2 spin liquids," Phys. Rev. B **93**, 155121 (2016).

[7] Ling-Yan Hung and Yidun Wan, "k matrix construction of symmetry-enriched phases of matter," Phys. Rev. B **87**, 195103 (2013).

[8] Yuan-Ming Lu and Ashvin Vishwanath, "Theory and classification of interacting integer topological phases in two dimensions: A chern-simons approach," Phys. Rev. B **86**, 125119 (2012).

[9] Peng Ye and Xiao-Gang Wen, "Projective construction of two-dimensional symmetry-protected topological phases with u(1), so(3), or su(2) symmetries," Phys. Rev. B **87**, 195128 (2013).

[10] Zheng-Cheng Gu, Juven C. Wang, and Xiao-Gang Wen, "Multikink topological terms and charge-binding domain-wall condensation induced symmetry-protected topological states: Beyond chern-simons/bf field theories," Phys. Rev. B **93**, 115136 (2016).

[11] Zheng-Xin Liu, Jia-Wei Mei, Peng Ye, and Xiao-Gang Wen, "$u(1) \times u(1)$ symmetry-protected topological order in gutzwiller wave functions," Phys. Rev. B **90**, 235146 (2014).

[12] Meng Cheng and Zheng-Cheng Gu, "Topological response theory of abelian symmetry-protected topological phases in two dimensions," Phys. Rev. Lett. **112**, 141602 (2014).

[13] Gary T. Horowitz and Mark Srednicki, "A quantum field theoretic description of linking numbers and their generalization," Communications in Mathematical Physics **130**, 83–94 (1990).

[14] T. H. Hansson, Vadim Oganesyan, and S. L. Sondhi, "Superconductors are topologically ordered," Annals of Physics **313**, 497–538 (2004).

[15] Pavel Putrov, Juven Wang, and Shing-Tung Yau, "Braiding statistics and link invariants of bosonic/fermionic topological quantum matter in 2+1 and 3+1 dimensions," Annals of Physics **384**, 254 – 287 (2017).

[16] Qing-Rui Wang, Meng Cheng, Chenjie Wang, and Zheng-Cheng Gu, "Topological quantum field theory for abelian topological phases and loop braiding statistics in $(3+1)$-dimensions," Phys. Rev. B **99**, 235137 (2019).

[17] Peng Ye and Zheng-Cheng Gu, "Topological quantum field theory of three-dimensional bosonic abelian-symmetry-protected topological phases," Phys. Rev. B **93**, 205157 (2016).

[18] Xueda Wen, Huan He, Apoorv Tiwari, Yunqin Zheng, and Peng Ye, "Entanglement entropy for (3+1)-dimensional topological order with excitations," Phys. Rev. B **97**, 085147 (2018).

[19] AtMa P. O. Chan, Peng Ye, and Shinsei Ryu, "Braiding with borromean rings in $(3+1)$-dimensional spacetime," Phys. Rev. Lett. **121**, 061601 (2018).

[20] Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in the quantum theory," Phys. Rev. **115**, 485–491 (1959).

[21] John Preskill and Lawrence M. Krauss, "Local discrete symmetry and quantum-mechanical hair," Nuclear Physics B **341**, 50 – 100 (1990).

[22] M. G. Alford and Frank Wilczek, "Aharonov-bohm interaction of cosmic strings with matter," Phys. Rev. Lett. **62**, 1071–1074 (1989).

[23] Lawrence M. Krauss and Frank Wilczek, "Discrete gauge symmetry in continuum theories," Phys. Rev. Lett. **62**, 1221–1223 (1989).

[24] Mark G. Alford, Kai-Ming Lee, John March-Russell, and John Preskill, "Quantum field theory of non-abelian strings and vortices," Nuclear Physics B **384**, 251 – 317 (1992).

[25] Chenjie Wang and Michael Levin, "Braiding statistics of loop excitations in three dimensions," Phys. Rev. Lett. **113**, 080403 (2014).

[26] Shang-Qiang Ning, Zheng-Xin Liu, and Peng Ye, "Fractionalizing global symmetry on looplike topological excitations," Phys. Rev. B **105**, 205137 (2022).

[27] Shang-Qiang Ning, Zheng-Xin Liu, and Peng Ye, "Symmetry enrichment in three-dimensional topological phases," Phys. Rev. B **94**, 245120 (2016).

[28] Peng Ye, "Three-dimensional anomalous twisted gauge theories with global symmetry: Implications for quantum spin liquids," Phys. Rev. B **97**, 125127 (2018).

[29] Peng Ye, Meng Cheng, and Eduardo Fradkin, "Fractional s-duality, classification of fractional topological insulators, and surface topological order," Phys. Rev. B **96**, 085125 (2017).

[30] Peng Ye, Taylor L. Hughes, Joseph Maciejko, and Eduardo Fradkin, "Composite particle theory of three-dimensional gapped fermionic phases: Fractional topological insulators and charge-loop excitation symmetry," Phys. Rev. B **94**, 115104 (2016).

[31] Peng Ye and Zheng-Cheng Gu, "Vortex-line condensation in three dimensions: A physical mechanism for bosonic topological insulators," Phys. Rev. X **5**, 021029 (2015).

[32] Juven C. Wang, Zheng-Cheng Gu, and Xiao-Gang Wen, "Field-theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology, and beyond," Phys. Rev. Lett. **114**, 031601 (2015).

[33] Bo Han, Huajia Wang, and Peng Ye, "Generalized wen-zee terms," Phys. Rev. B **99**, 205120 (2019).

[34] Peng Ye and Juven Wang, "Symmetry-protected topological phases with charge and spin symmetries: Response theory and dynamical gauge theory in two and three dimensions," Phys. Rev. B **88**, 235109 (2013).

[35] Zhi-Feng Zhang and Peng Ye, "Compatible braidings with hopf links, multiloop, and borromean rings in $(3+1)$-dimensional spacetime," Phys. Rev. Research **3**, 023132 (2021).

[36] Zhi-Feng Zhang, Qing-Rui Wang, and Peng Ye, "Non-abelian fusion, shrinking, and quantum dimensions of abelian gauge fluxes," Phys. Rev. B **107**, 165117 (2023).

[37] Zhi-Feng Zhang and Peng Ye, "Topological orders, braiding statistics, and mixture of two types of twisted BF theories in five dimensions," JHEP **04**, 138 (2022), arXiv:2104.07067 [hep-th].

[38] Yizhou Huang, Zhi-Feng Zhang, and Peng Ye, "Fusion rules and shrinking rules of topological orders in five dimensions," (2023), arXiv:2306.14611 [hep-th].

[39] Gary T Horowitz, "Exactly soluble diffeomorphism invariant theories," Commun. Math. Phys. **125**, 417–437 (1989).

[40] Peng Ye and Xiao-Gang Wen, "Constructing symmetric topological phases of bosons in three dimensions via fermionic projective construction and dyon condensation," Phys. Rev. B **89**, 045127 (2014).

[41] Edward Witten, "Dyons of charge e$\theta/2\pi$," Phys. Lett. B **86**, 283–287 (1979).

[42] Alfred S Goldhaber, R MacKenzie, and Frank Wilczek, "Field corrections to induced statistics," Mod. Phys. Lett. A **4**, 21–31 (1989).

[43] Alfred S. Goldhaber, "Connection of spin and statistics for charge-monopole composites," Phys. Rev. Lett. **36**, 1122–1125 (1976).

[44] J. M. Leinaas and J. Myrheim, "On the theory of identical particles," Il Nuovo Cimento B (1971-1996) **37**, 1–23 (1977).

[45] Yong-Shi Wu, "General theory for quantum statistics in two dimensions," Phys. Rev. Lett. **52**, 2103–2106 (1984).

[46] Frank Wilczek, *Fractional statistics and anyon superconductivity*, Vol. 5 (World Scientific, 1990).

[47] Anton Kapustin and Nathan Seiberg, "Coupling a QFT to a TQFT and Duality," JHEP **04**, 001 (2014), arXiv:1401.0740 [hep-th].

[48] Huan He, Yunqin Zheng, and Curt von Keyserlingk, "Field theories for gauged symmetry-protected topological phases: Non-abelian anyons with abelian gauge group $Z_2^{\otimes 3}$," Phys. Rev. B **95**, 035131 (2017).

Appendix A: Prrof of the relation of quantum dimensions in a fusion rule

Let the quantum dimension of e_i be d_i. Now we prove that for

$$\mathsf{e}_i \otimes \mathsf{e}_k = \oplus_m N_m^{ik} \mathsf{e}_m \tag{A1}$$

one can know

$$d_i d_k = \sum_m N_m^{ik} d_m. \tag{A2}$$

From the commutativity of fusion, we have

$$\mathsf{e}_i \otimes \mathsf{e}_j \otimes \mathsf{e}_k = \mathsf{e}_j \otimes \mathsf{e}_i \otimes \mathsf{e}_k. \tag{A3}$$

The left hand side can be written as $\oplus_m N_m^{ij} \mathsf{e}_m \otimes \mathsf{e}_k = \oplus_m N_m^{ij} \oplus_l N_l^{mk} \mathsf{e}_l$ and the right hand side can be written as $\mathsf{e}_j \otimes \left(\oplus_m N_m^{ik} \mathsf{e}_m\right) = \oplus_m N_m^{ik} \oplus_l N_l^{jm} \mathsf{e}_l$. The fusion coefficients N_k^{ij}'s can form a matrix N_i with $(N_i)_{kj} = N_k^{ij}$. Therefore we have

$$\oplus_m N_m^{ij} N_l^{mk} = \oplus_m N_m^{ik} N_l^{jm}. \tag{A4}$$

Notice that $\oplus_m N_m^{ij} N_l^{mk} = \sum_m N_m^{ij} N_l^{mk} = \sum_m (N_i)_{mj} (N_k)_{lm} = (N_i N_k)_{lj}$ and $\oplus_m N_m^{ik} N_l^{im} = \sum_m N_m^{ik} N_l^{jm} = \sum_m N_m^{ik} (N_m)_{lj}$, where we have used $N_c^{ab} = N_c^{ba}$. So we have a relation between matrices:

$$N_i N_k = \sum_m N_m^{ik} N_m. \tag{A5}$$

Since N_i's are commutative, their largest eigenvalues d_i's, i.e., quantum dimensions of corresponding topological excitations, satisfy

$$d_i d_k = \sum_m N_m^{ik} d_m. \tag{A6}$$