Topological Quantum Matters (TQM)

Time:

Every Mon. & Wed. 13:30-15:05, from 2021-9-13 to 12-3 (weeks 1-12 of the fall semester in Tsinghua)

Venue:

It will be a combination of offline (宁斋W11, Ning Zhai W11) and online (腾讯会议tencent meeting: 5772849861, password: 654321)

Description:

In this course, we will use topology to understand some exotic quantum phases of matter. The topics will include topological insulators, topological orders, symmetry-protected topological phases, etc. The course will cover both condensed matter models in physics and general mathematical descriptions (such as group cohomology theory and modular tensor categories of knots).

Prerequisites:

Basic topology and quantum mechanics. We will try to make a compromise between mathematics and physics by introducing relevant concepts self-consistently, as there are audience from both sides.

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other courses:

Reshetikhin : Inv. of knots and 3-mfd.

Haghighat : Topological Quantum Computation.

Zheng : Category theory

Introduction to TQM.

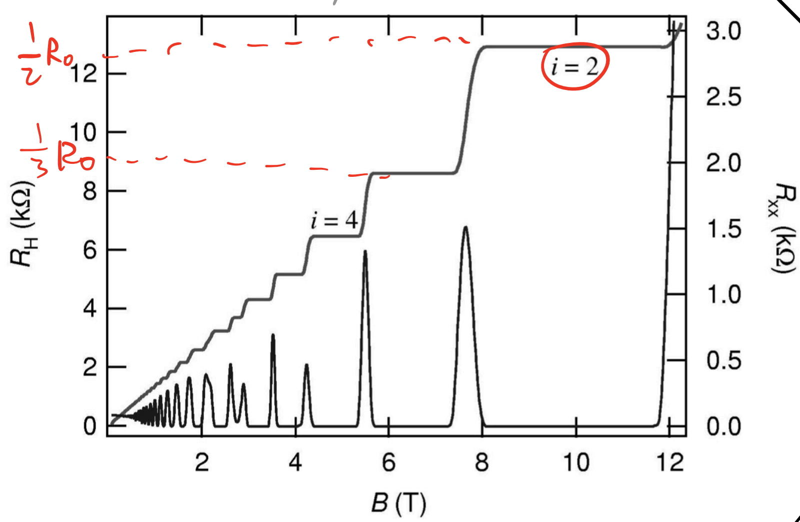
(1) Why important to real life?

One example:

Q: how to measure $\begin{cases} \text{electric charge } e \\ \text{Planck constant } h \end{cases}$ accurately?

① Integer quantum Hall effect $R_0 = \frac{h}{e^2}$

Nobel 1985



$$\begin{cases} e = \frac{2\Phi_0}{R_0} \\ h = \frac{4\Phi_0^2}{R_0} \end{cases}$$

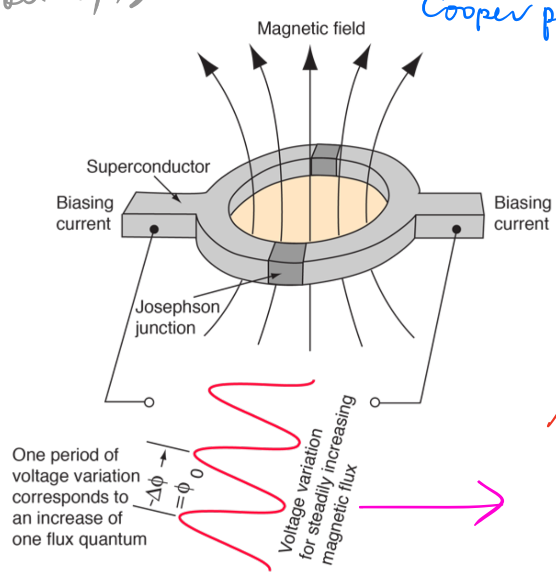
$\Downarrow$

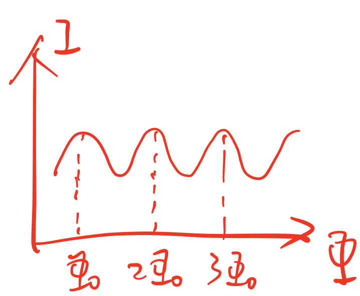
International System of Units (before 2018)

[After 2018, e, h are fixed by definition]

② Josephson effect $\Phi_0 = \frac{h}{2e}$ Cooper pair

Nobel 1973





using many-body system to measure accurately properties of a single electron!

(2) Why exotic?

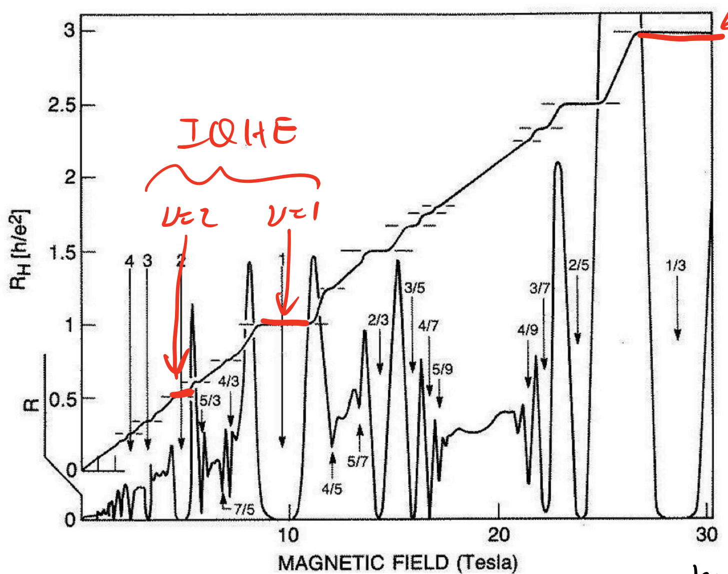
fundamental charge e = charge of one single electron/proton/...

Fractional quantum Hall effect (FQHE) Nobel 1998

$$R = \frac{h}{\nu e^2} \qquad \begin{cases} \nu \in \mathbb{Z}: & \text{integer QHE} \\ \nu \in \mathbb{Q}: & \text{FQHE} \end{cases}$$

$\downarrow$

Fractional charge $q = \nu \cdot e$



$\nu = \frac{1}{3}$ Laughlin state $\left(q = \frac{1}{3} e\right)$

Fractional charge appears in many-body systems!

(3) Why interesting to many people?

related to many other topics:

- math: knot theory, Jones polynomial, Turaev-Viro inv, Reshetikhin-Turaev inv., modular tensor categories, (Zheng's lecture) 2-knot, higher categories ... (lecture)
- physics: IQHE, FQHE, topological insulator, ... (real materials)
- quantum information: topological quantum computation (by Freedman, Kitaev)

fault-tolerant quantum computation

(4) Why hard/deep ?

∞ number of degrees of freedom

∞ dimensional space.

fundations of quantum field theory (= many-body quantum mechanics)

emergence of spacetime/gravity (?)

Syllabus (tentative):

It may vary depending on the actual speed of the course. By weeks (4*45min/week, 12 weeks):

(1) introduction to TQM, different classes of TQM (bosonic/fermionic, long/short range entangled, with/without symmetry),
1st example: Kitaev's toric code model (homology enters),
2nd example: Haldane's honeycomb model (homotopy enters) ⇝ Qikun Xue : quantum anomalous Hall effect.
1988 2013

Part I. Bosonic topological orders

(2) quantum double model, twisted quantum double model = Dijkgraaf-Witten gauge theory

(3) introduction to fusion categories, Levin-Wen model = Turaev-Viro model

(4) introduction to modular tensor categories, general description of anyon models by Kitaev

(5) 3+1D Walker-Wang model = Crane-Yatter model

Part II. Topological insulators (fermionic symmetry-protected topological phases without interactions)

(6) introduction to band theory, integer quantum Hall effect, Thouless–Kohmoto–Nightingale–den Nijs number, Chern insulator
TKNN : Hall conductance = Chern number

(7) examples: Kitaev's Majorana chain, Su-Schrieffer-Heeger model, 2+1D and 3+1D topological insulators, edge theories

(8) symmetries in free fermion system, 10-fold way classification → topological K theory, Clifford algebra

Part III. Symmetry-protected topological phases

(9) introduction to symmetry-protected topological (SPT) phases, Haldane chain

(10) introduction to projective representation, tensor product state, classification of 1+1D bosonic SPT

(11) Levin-Gu model, introduction to group cohomology, bosonic SPT model from group cohomology

(12) introduction to fermionic SPT phases, other related topics

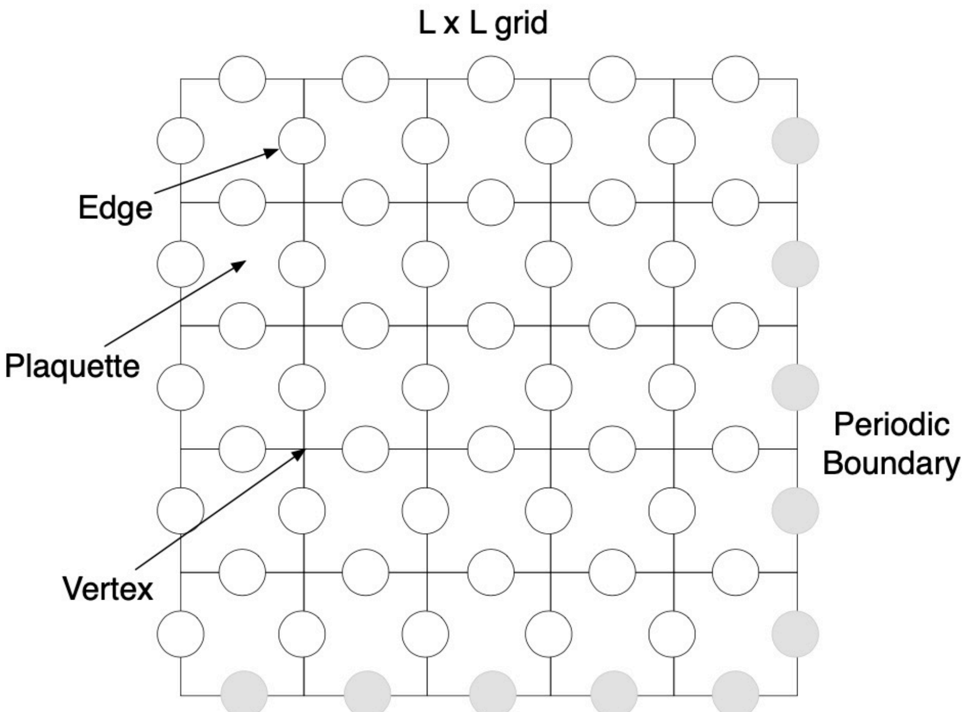
1st example of TQM:

Toric code model ($= \mathbb{Z}_2$ gauge theory on torus)
(by Kitaev 1997)

Hilbert space on torus

$L \times L$ square lattice, $\mathbb{C}^2$ (spin 1/2) on each link

$$\mathscr{H} = \bigotimes_{\text{link } j} \mathbb{C}^2$$



$\mathcal{H}$: Hilbert space

Hamiltonian $H : \mathcal{H} \to \mathcal{H}$

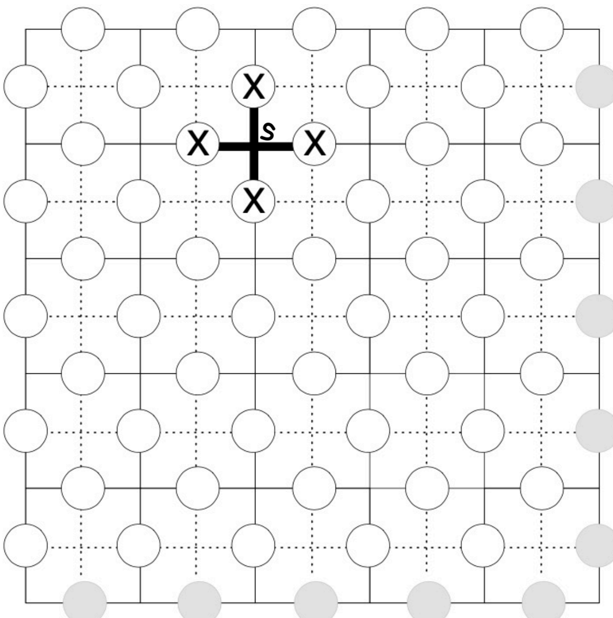
eigenvalues of H are energies ε_n of the system

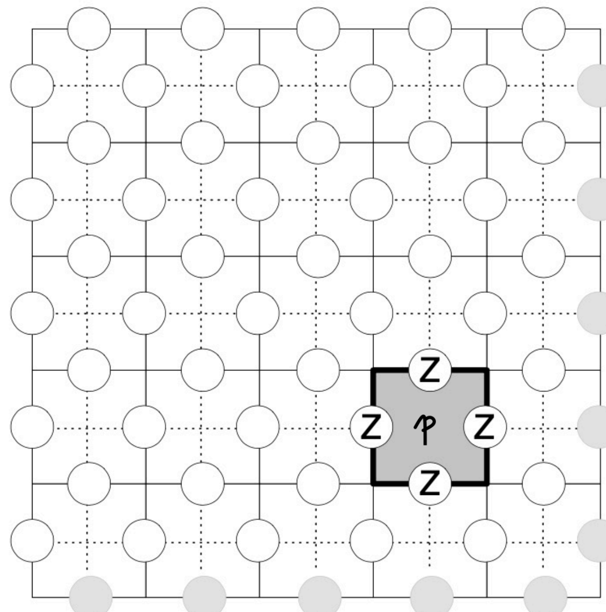
Ground state is the eigenvector with the lowest energy ε_0.

Claim
- The ground state of TC represents $H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$
- robust ground states
- Topological quantum information

Hamiltonian

$$H = -\sum_{\text{site } s} A_s - \sum_{\text{plaquette } p} B_p$$





$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x$$

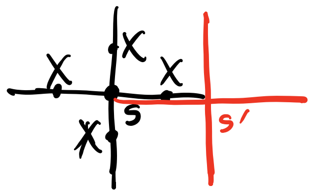
Pauli operator

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

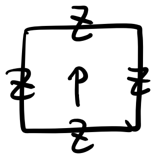
$$B_p = \prod_{j \in \partial p} \sigma_j^z$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Algebraic relations of A_s / B_p.

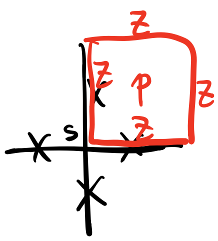


$$\begin{cases} A_s^{+} = A_s \\ (A_s)^2 = 1 & \rightsquigarrow A_s \text{ has eigenvalues } \pm 1 \\ [A_s, A_{s'}] = A_s A_{s'} - A_{s'} A_s = 0 \end{cases}$$



$$\begin{cases} B_p^{+} = B_p \\ (B_p)^2 = 1 & \rightsquigarrow B_p \ \cdots \ \pm 1. \\ [B_p, B_{p'}] = 0 \end{cases}$$

$$[A_s, B_p] = A_s B_p - B_p A_s = 0.$$



$$XZ = -ZX \Rightarrow A_s B_p = B_p A_s$$

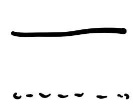
$$H = -\sum_s A_s - \sum_p B_p$$

Ground state $|\Psi\rangle$ of TC :

$$A_s |\Psi\rangle = B_p |\Psi\rangle = +1 \, |\Psi\rangle \ , \ \forall s, p.$$

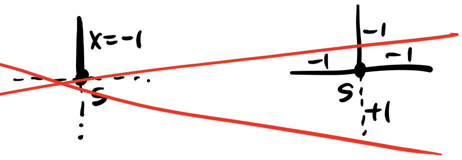
<u>String representation</u>. $\quad A_s = \prod \sigma^x$. $\quad \sigma^x : \pm 1$

def: $\begin{cases} \sigma_j^x = -1 : \text{ string} \\ \sigma_j^x = +1 : \text{ no string} \end{cases}$

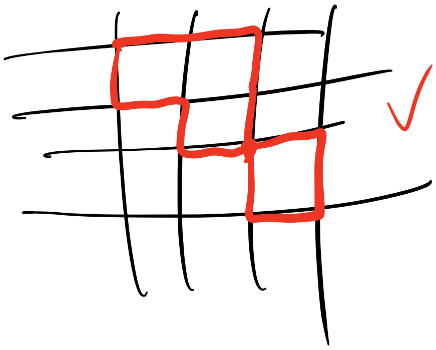


low energy: $A_s = 1$:

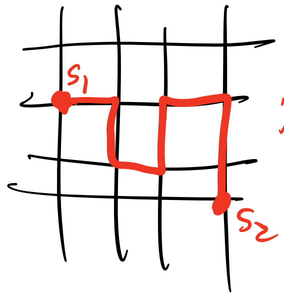
high energy: $A_s=-1$:



ground state : $A_s|\Psi\rangle = |\Psi\rangle \Longleftrightarrow$ closed strings (loops)

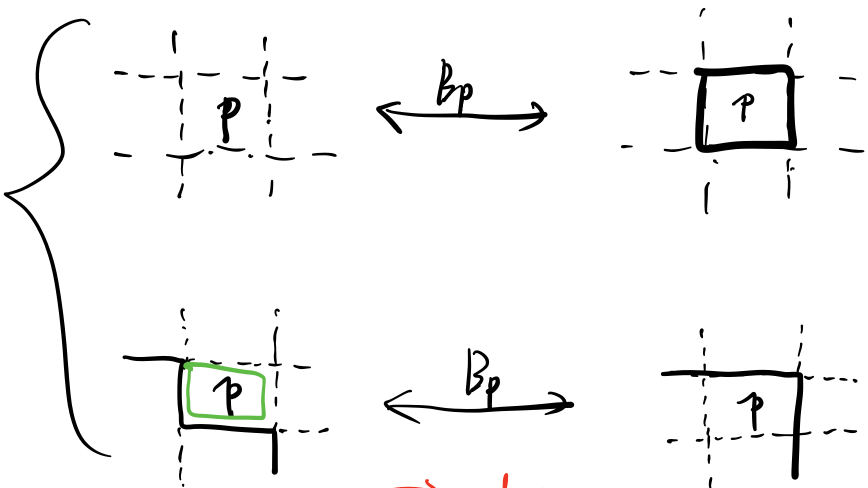


✓ $A_s=+1$ $(\forall s)$



X $A_{s_1}=A_{s_2}=-1$ (high energy)

$B_p = \prod Z$: flip $\sigma^x=+1 \leftrightarrow \sigma^x=-1$



create annihilation small loops.

change shapes of closed loops.

$$B_p\,|c\rangle = |c+\partial p\rangle \quad \text{mod } 2$$

$\forall$ closed loop configuration

$$H = -\sum A_s - \sum B_p$$

closed loop. change shapes.

Assume $|\Psi\rangle = \sum_{\text{closed loop } c} a_c |c\rangle$, $a_c \in \mathbb{C}$ is the ground state

$$\Rightarrow B_p|\Psi\rangle = \sum_c a_c\,|c+\partial p\rangle = \sum_c a_{c+\partial p}\,|c\rangle$$

$$= |\Psi\rangle = \sum_c a_c |c\rangle$$

$$\Rightarrow a_c = a_{c+\partial p} \quad \text{for } \forall p$$

The ground state coefficients of c and $c+\partial p$ should be the same.

Summary $\begin{cases} A_s = +1 : & \text{conf. } |c\rangle \text{ such that there are only closed loops in } c. \\ B_p = +1 : & |c\rangle \text{ and } |c+\partial p\rangle_{(\text{mod } 2)} \text{ have the same coefficient.} \end{cases}$

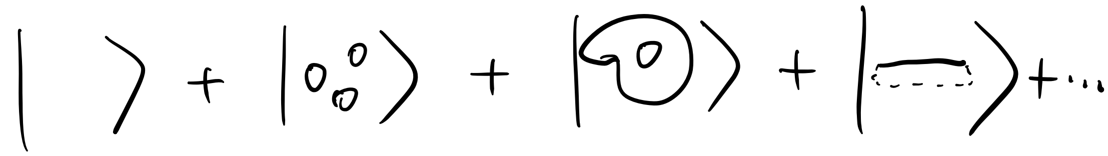
Ground state subspace of TC

$= \{\text{closed loops}\} \,/\, \{\text{boundaries of 2D plaquettes}\}$

$= H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$

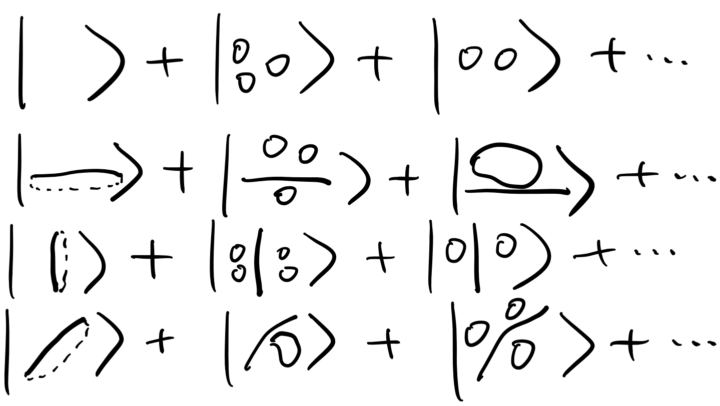
$\cup$

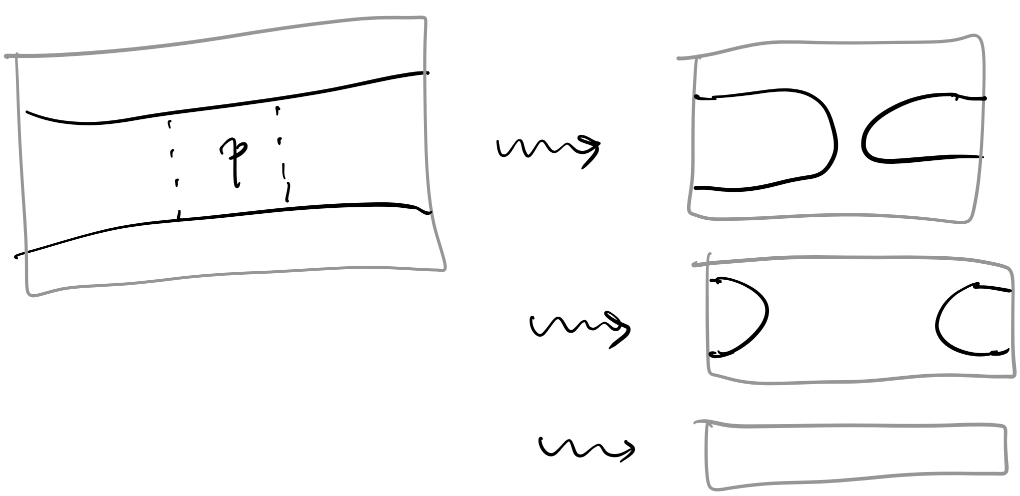
$|\Psi\rangle =$



4-fold degeneracy of TC on torus:

$|\Psi_{00}\rangle =$, $|\Psi_{10}\rangle =$, $|\Psi_{01}\rangle =$, $|\Psi_{11}\rangle =$





knot/link in (2+1)D spacetime

anyon worldlines in (2+1)D <—> knot/link in 3D