7. Symmetries and classifications of free-fermion TI / TSC

7.1. Symmetries in quantum systems

Goal: Difference and relation of

① symmetries of operators acting on Hilbert space

② symmetries of physical states.

- Hilbert space $\neq$ physical states space

$\mathcal{H} \cong \mathbb{C}^N$, two states $|\psi\rangle$ and $|\psi'\rangle$ describes the same physical states iff $|\psi\rangle = z|\psi'\rangle$ for $0 \neq z \in \mathbb{C}$.

physical states space $P\mathcal{H} = \mathbb{C}P^{N-1} \cong (\mathcal{H} - \{0\})/\mathbb{C}^\times$.

Ex. spin $\frac{1}{2}$, $\mathcal{H} = \mathbb{C}^2 = \{a|\uparrow\rangle + b|\downarrow\rangle \mid a, b \in \mathbb{C}\} = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2\}$

$$\forall \begin{pmatrix} a \\ b \end{pmatrix} \begin{cases} ① \ a \neq 0: & \begin{pmatrix} a \\ b \end{pmatrix} \sim \dfrac{1}{\sqrt{|a|^2+|b|^2}} \begin{pmatrix} |a| \\ |a| \cdot b/a \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix} \\ ② \ a = 0: & \begin{pmatrix} 0 \\ b \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix} \text{ with } \begin{pmatrix} 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi \end{pmatrix}$$

$$\Rightarrow P\mathcal{H} \cong \mathbb{C}P^1 \cong S^2 \to \text{Bloch sphere.}$$

$$\dim_{\mathbb{R}} \mathcal{H} = 4 \underbrace{\quad , \quad}_{\substack{\text{normalization } -1 \\ U(1) \text{ phase } -1}} \dim_{\mathbb{R}} P\mathcal{H} = 2$$

- Symmetries of Hilbert space $\neq$ Symmetries of physical states.

$\mathrm{Aut}_{qtm}(P\mathcal{H})$: set of automorphisms of quantum systems.

transformation should preserve probability

$$|\langle\psi|\varphi\rangle|^2 = |\langle U\psi | U\varphi\rangle|^2$$

$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$: set of unitary and antiunitary transformations on $\mathcal{H}$.

$$U: |\psi\rangle \mapsto U|\psi\rangle$$

U is $\mathbb{R}$-linear

$$\begin{cases} \text{unitary} : & U(a|\psi\rangle + b|\varphi\rangle) = a\,U|\psi\rangle + b\,U|\varphi\rangle \\ \text{anti-unitary} : & U(a|\psi\rangle + b|\varphi\rangle) = a^*\,U|\psi\rangle + b^*\,U|\varphi\rangle \end{cases}$$

Ex. spin $\frac{1}{2}$. $P\mathcal{H} = S^2$, $\mathcal{H} = \mathbb{C}^2$

$$\mathrm{Aut}_{qtm}(P\mathcal{H}) = \mathrm{Aut}_{qtm}(S^2) \cong O(3) \cong \mathbb{Z}_2 \times SO(3)$$

$$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^2) = \{\text{unitary/antiunitary transf. actioning on } \mathbb{C}^2\}$$

$$= U(2) \oplus U(2) = \mathbb{Z}_2 \times U(2)$$

Relation :

$$0 \to U(1) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}_{qtm}(P\mathcal{H}) \to 0$$

Wigner thm : Every quantum automorphism in $\mathrm{Aut}_{qtm}(P\mathcal{H})$ is induced by a unitary or anti-unitary operator in $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ on Hilbert space $\mathcal{H}$.

$$1 \to U(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\phi} \mathbb{Z}_2 \longrightarrow 0$$

$$\phi(s) = \begin{cases} +1, & \text{if } s \text{ is unitary} \\ -1, & \text{if } s \text{ is antiunitary.} \end{cases}$$

- Physical symm and twisted symm.

$\mathrm{Aut}_{qtm}(P\mathcal{H})$: all symmetries of a q system.

Add Hamiltonian $\hat{H}$: smaller symmetry G.

$$\rho : G \to \mathrm{Aut}_{qtm}(P\mathcal{H})$$

twisted extension G^{tw} of G :

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & G^{tw} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r] & \mathrm{Aut}_{qtm}(P\mathcal{H}) \arrow[r] & 1 \end{tikzcd}$$

$$\begin{cases} G: & \text{physical symmetry acting on physical states} \\ G^{tw}: & \text{virtual symmetry acting on Hilbert space.} \end{cases}$$

Ex. $G = \mathbb{Z}_2^T = \{1, T\}$ is time reversal symmetry.

$$\begin{cases} \phi(1) = 1 \\ \phi(T) = -1 \end{cases}$$

$$1 \to U(1) \to G^{tw} \to \mathbb{Z}_2^T \to 1$$

G^{tw} is classified by $H^2(\mathbb{Z}_2^T, U(1)) = \mathbb{Z}_2 \ni \omega_2$

$$\begin{cases} (1) \ G^{tw} \cong \{ zT \mid zT = Tz^{-1},\ z \in U(1),\ T^2 = 1 \} \cong U(1) \rtimes \mathbb{Z}_2^T \\ (2) \ G^{tw} \cong \{ zT \mid zT = Tz^{-1},\ z \in U(1),\ T^2 = -1 \} \cong U(1) \rtimes_{\omega_2} \mathbb{Z}_2^T \end{cases}$$

7.2. 10-fold way of TI/TSC.

- unitary symmetries are NOT important in the classification of TI/TSC.

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^+ \mathcal{H}_{A,B} \hat{\psi}_B \qquad \mathcal{H}^T \sim \mathcal{H}^*$$

$$\{\psi_A, \psi_B^+\} = \delta_{AB}, \quad \{\psi_A, \psi_B\} = \{\psi_A^+, \psi_B^+\} = 0.$$

If we have a unitary symmetry:

$$\begin{cases} \hat{U} \hat{\psi}_A \hat{U}^+ = \sum_B U_{A,B}^+ \hat{\psi}_B \\ \hat{U} \psi_A^+ \hat{U}^+ = \sum_B \psi_B^+ U_{B,A} \end{cases} \qquad U \text{ is unitary matrix.}$$

$$\hat{U} \hat{H} \hat{U}^{-1} = \hat{H} \iff U \mathcal{H} U^+ = \mathcal{H}$$

We can block-diagonalize $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda$$

For each λ, we have several copies of a fixed irrep. of G.

$$V_\lambda = \underset{\substack{\uparrow \\ \text{Hamiltonian} \\ \text{acting on}}}{V_\lambda^{(H)}} \otimes \underset{\substack{\uparrow \\ \text{Symmetry} \\ \text{acting on}}}{V_\lambda^{(G)}}$$

Within $V_\lambda^{(H)}$, there is No constraints for $\mathcal{H}$.

Ex. $G = SO(3)$.

$$V = \mathbb{C}^6 = \underset{\lambda=\text{spin-}1}{\overset{SO(3)\circlearrowleft}{\mathbb{C}^3_\lambda}} \oplus \underset{\lambda=\text{spin-}1}{\overset{SO(3)\circlearrowleft}{\mathbb{C}^3_\lambda}} = \mathbb{C}^2 \otimes \mathbb{C}^3$$

$$[\mathcal{H}, SO(3)] = 0 \Rightarrow \mathcal{H} = \mathcal{H}_{2\times 2} \otimes I_{3\times 3}$$

$$SO(3) \ni U(g) = I_{2\times 2} \otimes U_\lambda(g)$$

- Antiunitary symmetries and 10-fold way.

(1) time reversal

$$\begin{cases} \hat{\mathcal{T}} \hat{\psi}_A \hat{\mathcal{T}}^{-1} = \sum_B (U_T^\dagger)_{A,B} \hat{\psi}_B \\ \hat{\mathcal{T}} \hat{\psi}_A^\dagger \hat{\mathcal{T}}^{-1} = \sum_B \hat{\psi}_B^\dagger (U_T)_{B,A} \\ \hat{\mathcal{T}}\, i\, \hat{\mathcal{T}}^{-1} = -i \end{cases}$$

$$\hat{\mathcal{T}} \hat{H} \hat{\mathcal{T}}^{-1} = \hat{H} \Longleftrightarrow \boxed{U_T \mathcal{H}^* U_T^\dagger = \mathcal{H}}$$

$$\hat{\mathcal{T}}^2 = \pm 1 \Leftrightarrow U_T U_T^* = \pm 1$$

$$T = \begin{cases} 0, & \text{if no } \mathcal{T} \text{ symm.} \\ +1, & \text{if } \mathcal{T}^2 = +1 \\ -1, & \text{if } \mathcal{T}^2 = -1 \end{cases} \quad \to 3$$

(2) charge conjugation (particle-hole) symmetry.

$$\begin{cases} \hat{C} \hat{\psi}_A \hat{C}^{-1} = \sum_B (U_C^{*\dagger})_{AB} \hat{\psi}_B^\dagger \\ \hat{C} \hat{\psi}_A^\dagger \hat{C}^{-1} = \sum_B \hat{\psi}_B (U_C^*)_{BA} \\ \hat{C}\, i\, \hat{C}^{-1} = i \end{cases}$$

$$\hat{C} \hat{H} \hat{C}^{-1} = \hat{H} \Longleftrightarrow \boxed{U_C \mathcal{H}^{\overset{T}{*}} U_C^\dagger = -\mathcal{H}}$$

$$C = \begin{cases} 0, & \text{if no } C \text{ symm.} \\ +1, & \text{if } C^2 = +1 \\ -1, & \text{if } C^2 = -1 \end{cases} \quad \to 3$$

(3) chiral (sublattice) symmetry.

$$\hat{S} = \hat{\mathcal{T}} \cdot \hat{C}$$

$$S = U_S = U_T \cdot U_C^*$$

$$\hat{S} \hat{H} \hat{S}^{-1} = \hat{H} \Leftrightarrow \boxed{U_S \mathcal{H} U_S^{+} = -\mathcal{H}}$$

$$S = \begin{cases} 0, & \text{if no } S \text{ symm.} \\ 1, & \text{if } S \text{ symm.} \end{cases}$$

if $T = C = 0$, $\quad S = T \cdot C = \begin{cases} 0 \\ 1 \end{cases}$

In total, $3 \times 3 + 1 = 10$ classes

TABLE - “Ten Fold Way” [`CARTAN Classes’] — Examples

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall Z2-Top.Ins. Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

(Ludwig 2015)