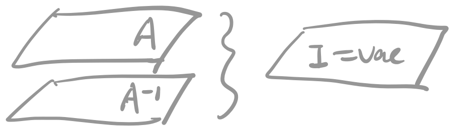
8. Introduction to symmetry-protected topological (SPT) phases

	short-range entangled (SRE)	long-range entangled (LRE)
without symmetry	invertible TO	intrinsic TO (MTC)
with symmetry	SPT	SET (G+MTC)

SET ← enriched

Note: Sometimes, are called invertible phases, in the sense that there exist inverse for these phase.



$$\begin{cases} \text{Topological phases} \longrightarrow \text{Abelian monoid under stacking} \\ \text{invertible phases} \longrightarrow \text{Abelian group under stacking.} \end{cases}$$

(monoid $\downarrow \exists$ inverse group)

invertible phase
$$\begin{cases} \text{invertible TO : Gapped states without anyons, but still can NOT be deformed into trivial product state.} \\ \text{G-SPT : Gapped states without anyons, but still can NOT be deformed into trivial product state while preserving the symmetry } G. \end{cases}$$

Known classification for invertible TO:

dim	0	1	2	3
bosonic	0	0	$\mathbb{Z}$	0
fermionic	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

bosonic $\mathbb{Z}$ (dim 2) → E_8 state

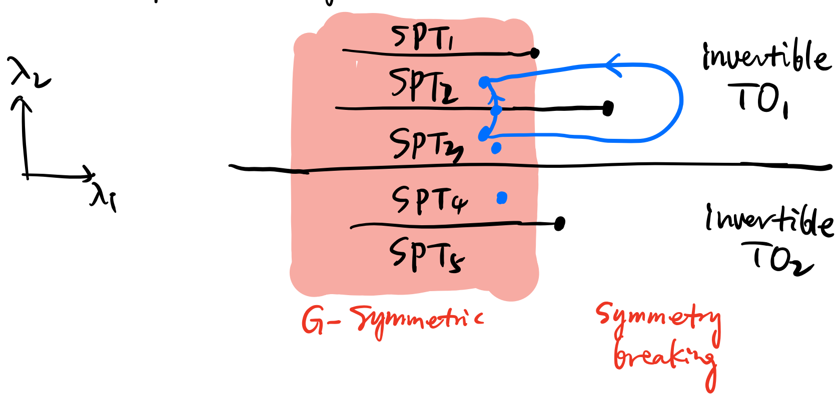
fermionic $\mathbb{Z}_2$ (dim 0) → bosonic / fermionic

fermionic $\mathbb{Z}_2$ (dim 1) → Majorana chain

fermionic $\mathbb{Z}$ (dim 2) → $p_x + i p_y$ SC

Note: fermionic iTO = fermion SPT protected by $G_f = \mathbb{Z}_2^f = \{1, (-1)^F\}$

Schematic phase diagram:



Depending on microscopic dof & (non) interacting:

SPT	noninteracting	interacting
bosonic	/	bSPT → this chapter
fermionic	TI/TSC ↓ last chapter	fSPT → last chapter?

Similar to TI/TSC, <u>nontrivial properties</u> of SPT is on the <u>edge.</u>

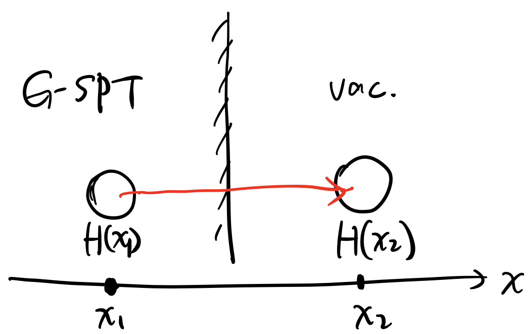
Possible edge states for SPT:

(1) G symmetry breaking

(2) anomalous SET (with edge GSD)

(3) gapless

~~(4)~~ unique gapped symmetric edge state.

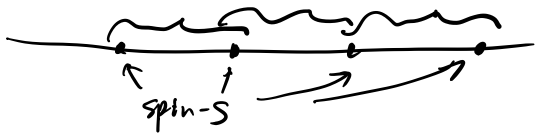


homotopy path $H(x)$: $H(x_1) \to H(x_2)$

8.1. Haldane chain (1983)

Consider 1D antiferromagnetic spin-S chain:

$$\hat{H} = \sum_i \hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1}$$



classical configuration: $\uparrow \downarrow \uparrow \downarrow \uparrow$

$$\vec{S}_i \approx (-1)^i S\, \hat{n}_i + \vec{l}_i$$

low E mode (for $\hat{n}_i$) high E mode (for $\vec{l}_i$)

$\int D\vec{l}_i \rightarrow$ involves $\hat{n}_i$

$$Z = \int \left(\prod_i D\vec{n}_i\right) e^{-S[\vec{n}_i]}$$

$$S = \int d^2x\, \underbrace{\frac{1}{2g^2}(\partial_\mu \hat{n})^2}_{\text{NL}\sigma\text{M}} + \underbrace{i\theta W[\vec{n}]}_{\theta\text{ term}} \qquad \begin{cases} \theta = 2\pi S \\ W = \int d^2x \frac{1}{4\pi} \hat{n}\cdot(\partial_t \hat{n} \times \partial_x \hat{n}) \end{cases}$$

Haldane conjecture:

$$H \text{ is } \begin{cases} \text{gapped} \\ \text{gapless} \end{cases} \quad \text{if } S \in \begin{cases} \mathbb{Z} \\ \mathbb{Z}+\frac{1}{2} \end{cases}$$

$$\begin{cases} \text{If } S \in \mathbb{Z}, \quad Z = \int D\hat{n}\, e^{-S_{\text{NL}\sigma\text{M}}} \longrightarrow \text{gapped} \\ \text{If } S \in \mathbb{Z}+\frac{1}{2}, \quad \begin{cases} ① \ S=\frac{1}{2}, \text{ Bethe ansatz} \\ ② \ S \in \mathbb{Z}+\frac{1}{2}, \text{ Lieb-Shultz-Mattis thm} \\ ③ \ \int D\hat{n}\, (-1)^{W[\vec{n}]} e^{-S_{\text{NL}\sigma\text{M}}} \end{cases} \end{cases}$$

Derivation of θ term.

① Path int for a single spin

$$\hat{\vec{S}} = \frac{1}{2}\hat{\vec{\sigma}}$$

Consider $\hat{H} = -\vec{n}\cdot\vec{S} = -\frac{1}{2}\vec{n}\cdot\hat{\vec{\sigma}}$

$$\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$E = \pm\frac{1}{2}$, ground state is

$$|\vec{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \Rightarrow \langle\vec{n}|\hat{\vec{\sigma}}_i|\vec{n}\rangle = n_i$$

For a time evolution $|\vec{n}(t)\rangle$, the Berry phase

$$\gamma = -i\int_0^T dt\ \langle\vec{n}(t)|\ \frac{d}{dt}\,|\vec{n}(t)\rangle$$

$$= -i\int_0^T dt\ \left(\cos\frac{\theta(t)}{2}, e^{-i\phi(t)}\sin\frac{\theta(t)}{2}\right)\begin{pmatrix} \frac{d}{dt}\cos\frac{\theta(t)}{2} \\ \frac{d}{dt}\left[e^{i\phi(t)}\sin\frac{\theta(t)}{2}\right] \end{pmatrix}$$

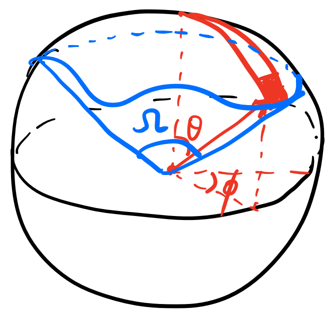
$$= -i\int_0^T dt\left[\cancel{-\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} + i\phi'\sin^2\frac{\theta(t)}{2} + \cancel{\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \int_0^T dt\ \frac{1-\cos\theta(t)}{2}\ \frac{d\phi(t)}{dt}$$

$$= \frac{1}{2}\int d\phi\,[1-\cos\theta]$$

$$= \frac{1}{2}\int \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\ \Omega[\vec{n}(t)]$$



↳ solid angle of trajectory of $\vec{n}(t)$ on S^2.

Another SU(2) invariant form of Ω is

$$\Omega = \int_0^1 d\rho\ \int_0^T dt\ \ \vec{n}\cdot(\partial_t\vec{n}\times\partial_\rho\vec{n})$$

with $\vec{n}(t,\rho) = \begin{cases} (0,0,1), & \text{if } \rho=0 \\ \vec{n}(t), & \text{if } \rho=1 \end{cases}$

WZ term for one spin

For general spin S:

$$\gamma = S\cdot\Omega[\vec{n}(t)]$$

$$\gamma' = S\cdot(-\Omega') = -S(4\pi-\Omega) \xlongequal{\text{mod } 2\pi} S\Omega = \gamma$$

$\uparrow$

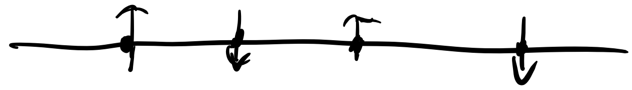
$S\in\frac{1}{2}\mathbb{Z}$

single spin:

$$Z = \int D\vec{n}(t)\, e^{iS\Omega[\vec{n}(t)]} + \cdots$$

$\uparrow$ Wess-Zumino term for (0+1)D

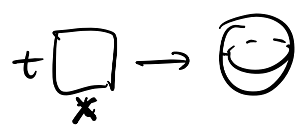
② AF spin chain:



$$Z = \int \left(\prod_i D\vec{n}_i(t)\right) e^{iS\sum_i \Omega[\vec{n}_i(t)]} + \cdots$$

$$\vec{n}_i(t) = (-1)^i \vec{m}_i(t), \quad \text{s.t. } \vec{m}(x,t) := \vec{m}_i(t) \text{ is smooth}$$

$$\begin{aligned}
S\sum_i \Omega[\vec{n}_i(t)] &= S\sum_i \Omega[(-1)^i m_i(t)] \\
&= S\sum_i (-1)^i \Omega[m_i(t)] \\
&= S\sum_k \left(\Omega[m_{2k}(t)] - \Omega[m_{2k-1}(t)]\right) \\
&= \frac{S}{2}\int_0^L dx\, \frac{\partial}{\partial x}\Omega[m(x,t)] \\
&= \frac{S}{2}\int_0^L dx\, \frac{\delta\Omega}{\delta\vec{m}}\cdot\frac{\partial\vec{m}}{\partial x} \\
&= \frac{S}{2}\int dt\, dx\, (\vec{m}\times\partial_t\vec{m})\cdot\partial_x\vec{m} \\
&= 2\pi S \underbrace{\frac{1}{4\pi}\int dt\, dx\, \vec{m}\cdot(\partial_t\vec{m}\times\partial_x\vec{m})}_{\in W \text{ winding number } (T^2\to S^2)} \\
&= 2\pi W
\end{aligned}$$



$$Z = \int D\vec{n}(x,t)\, e^{i2\pi S\, W[\vec{n}(x,t)]} + \cdots$$

$\downarrow$ θ term for (1+1)D.