Quantum Double Models (QDM)

Kitaev, Annals of phys. 303, 2 (2003)

arXiv 1997

QDM : generalization of $\mathbb{Z}_2$ toric code
to lattice G gauge theory.
↳ finite (may be non-Abelian) group.

$$D(G) = \underset{\substack{\downarrow \\ \text{flux}}}{G} \times \underset{\substack{\downarrow \\ \text{charge}}}{\hat{G}} \quad \text{for Abelian } G.$$

(double)

2.1. The model based on a group algebra.

G.

$\mathcal{H} := \mathbb{C}[G] = \left\{ \sum_g a_g \cdot g \;\middle|\; g \in G,\ a_g \in \mathbb{C} \right\}$ be the group algebra.

↓ Hilbert space with orthonormal basis $|g\rangle$

$\dim \mathcal{H} = |G|$

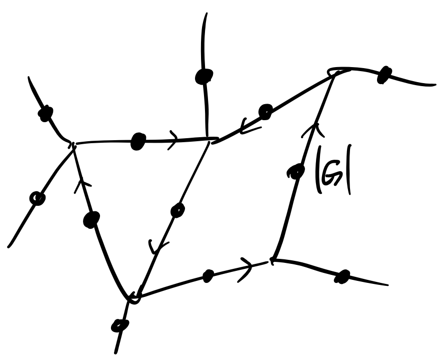
Def : $L_+^g |z\rangle = |gz\rangle \qquad T_+^h |z\rangle = \delta_{h,z} |z\rangle$

$L_-^g |z\rangle = |zg^{-1}\rangle \qquad T_-^h |z\rangle = \delta_{h^{-1},z} |z\rangle$

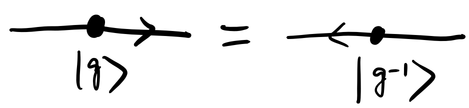
Satisfy : $L_+^g T_+^h = T_+^{gh} L_+^g, \quad L_+^g L_-^h = T_-^{hg^{-1}} L_+^g$

$L_-^g T_+^h = T_+^{hg^{-1}} L_-^g, \quad L_-^g T_-^h = T_-^{gh} L_-^g$

Hibert space :



|G|-dim vector space on each link of ∀ 2D lattice.

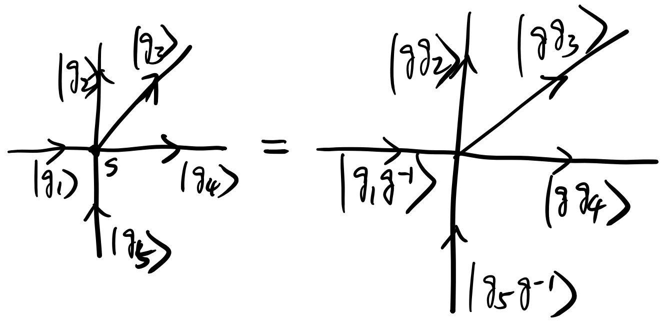


$, \; g \in G.$

QDM : $H = -\sum_s A_s - \sum_p B_p$

A_s

A_s^g

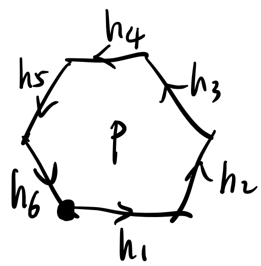


$$A_s^g := \prod_{\substack{\text{edge } l \\ \text{of } s}} L_\pm^g \,, \qquad A_s := \frac{1}{|G|}\sum_{g\in G} A_s^g$$

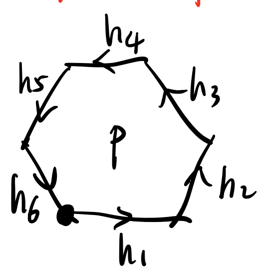
↳ gauge transf.

B_p

B_p^h



$$= \underbrace{\delta_{h_1 h_2 \cdots h_6, h}}_{\text{flux is } h}$$



$$B_p^h = \sum_{h_1 \cdots h_6 = h} \prod_{m=1}^{6} T_\pm^{h_m} \,, \qquad B_p := B_p^{h=1}$$

↳ zero flux condition for p.

relation

$$\left.\begin{aligned} A_s^2 &= A_s \\ B_p^2 &= B_p \end{aligned}\right\} \text{ projectors}$$

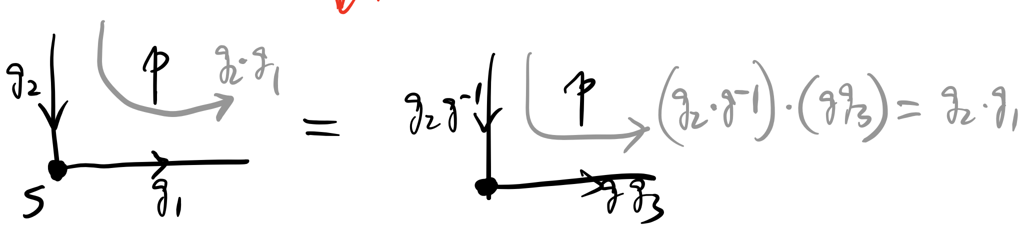
$$[A_s, A_{s'}] = 0$$

$$[B_p, B_{p'}] = 0$$

$$\boxed{[A_s, B_p] = 0}$$

proof

A_s



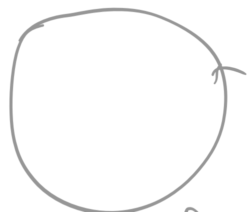
Ground state : $A_s|\Psi\rangle = B_p|\Psi\rangle = |\Psi\rangle\ ,\ \forall s, p$

A_s: gauge transf. B_p: flat connection condition

$\Rightarrow |\Psi\rangle$: flat connections up to gauge equivalence.

TC. $|\Psi\rangle =$  $+ \cdots$

$g = e^{iA} \in U(1)$

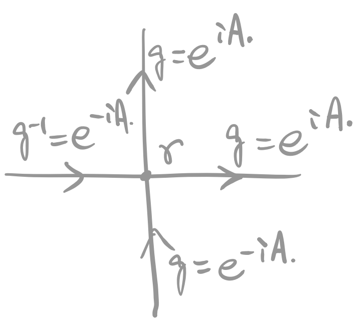


flat : $\Phi_C := \int_C A_\mu dx^\mu = 0$

$\downarrow$

$\prod g_1 \cdots g_N = 1 \Leftrightarrow B_p^{h=1}$

$\boxed{g_{ij} = e^{iA_{ij}}}$ (A_{ij} → continuous, g_{ij} → discrete)



Gauss law

$$\nabla \cdot \vec{A} = \rho$$

$$\| $$

$$A(\vec{r}+\hat{x}) - A(\vec{r}) + A(\vec{r}+\hat{y}) - A(\vec{r})$$

$$\downarrow$$

$$g_{\vec{r}+\hat{x}}\, g_{\vec{r}}^{-1}\, g_{\vec{r}+\hat{y}}\, g_{\vec{r}}^{-1}$$

Important notice:

The courses of week 3 (on 2021-09-27 and 2021-09-29) will be cancelled. And the ending date of the course will be postponed to week 13 of the fall semester.

From 2021-10-04 on, we will use another Tencent Meeting Room [9952830954] (password 654321) of the university account, such that there are more storage space for the recordings.

Recall. QDM.

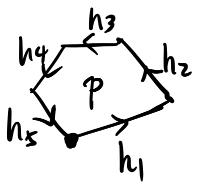
$$H = -\sum_s A_s - \sum_p B_p$$

$$[A_s, B_p] = 0$$

① $B_p |\Psi\rangle = |\Psi\rangle, \quad \forall p$

$\xrightarrow{h} = \xleftarrow{h^{-1}}$

$\Rightarrow$



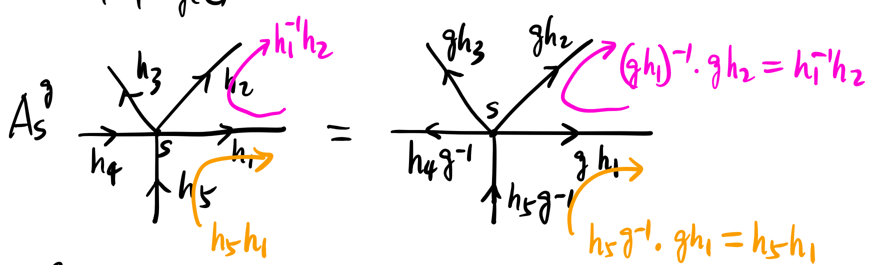
$h_1 h_2 \cdots h_5 = 1 \in G.$

(flat connection condition
zero flux condition)

② $A_s |\Psi\rangle = |\Psi\rangle, \quad \forall s$

$$A_s := \frac{1}{|G|} \sum_{g\in G} A_s^g$$

A_s^g : gauge transformation.



A_s^g fluctuate $\{h_{ij}\}$ within the subspace $B_p = 1, \forall p$.

$\Rightarrow$ GS $|\Psi\rangle$ is an equal weight superposition of all flat connections that are gauge equivalent.

continuous: U(1) gauge theory

$$A_\mu(r) \longrightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

$$\Phi' = \oint_C A'_\mu dx^\mu = \oint_C (A_\mu - \partial_\mu \chi) dx^\mu = \oint_C A_\mu dx^\mu = \Phi$$

is gauge inv.

discrete: $i \xrightarrow{g_{ij}} j$

$$\boxed{\begin{aligned} g_{ij} &= e^{i\int_{\text{site } i}^{\text{site } j} A_\mu dx^\mu} \\ g_i &= e^{i\chi(r_i)} \end{aligned}}$$

g_{ij} → gauge field

g_i → gauge transf. para.

gauge transf

$$g'_{ij} = e^{i\int_i^j (A_\mu - \partial_\mu \chi) dx^\mu} = e^{i\int_i^j A_\mu dx^\mu - \chi(r_j) + \chi(r_i)}$$

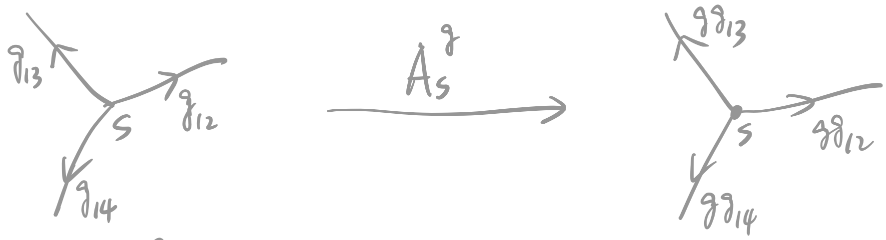
$$= e^{i\chi(r_i)} e^{i\int_i^j A_\mu dx^\mu} e^{-i\chi(r_j)}$$

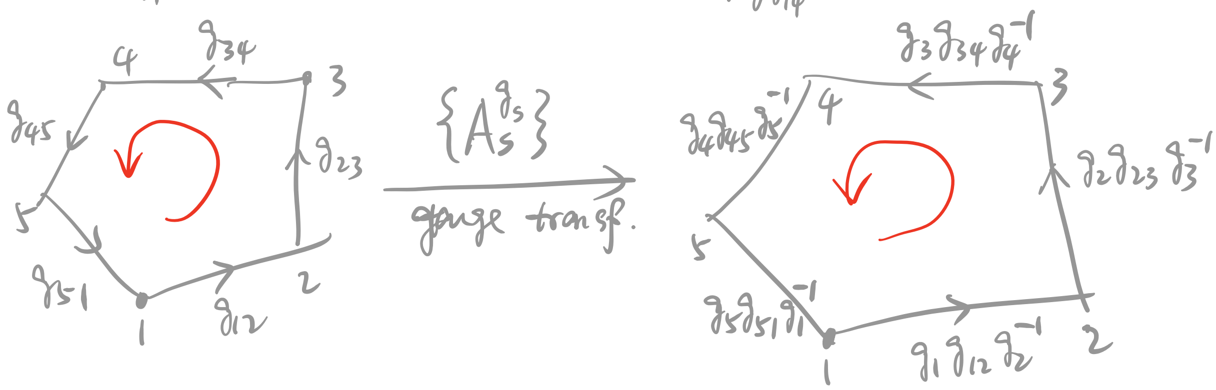
$$= g_i \cdot g_{ij} \cdot g_j^{-1}$$

$i \longrightarrow j$

$$g'_{ij} = g_i g_{ij} g_j^{-1}$$

eg:





$$\Phi = g_{12} \cdot g_{23} \cdots g_{51}$$

$$= e^{i\int_1^1 A_\mu dx^\mu}$$

$$\Phi' = g_1 g_{12} g_2^{-1} \cdot g_2 g_{23} g_3^{-1} \cdot \ldots \cdot g_5 g_{51} g_1^{-1}$$

$$= g_1 (g_{12} \cdots g_{51}) g_1^{-1}$$

$$= g_1 \Phi g_1^{-1}$$

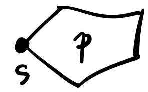
$$\mathrm{Tr}\,\Phi' = \mathrm{Tr}\,\Phi$$

2.2. Anyon excitations and ribbon operators

$$H = -\sum_s \underbrace{\frac{1}{|G|}\sum_{g\in G} A_s^g}_{\text{proj. to gauge inv. subspace}} - \sum_p \underbrace{B_p^{h=1}}_{\text{proj. to } h=1 \text{ zero flux}}$$

Excitations: ① non gauge invu: $\frac{1}{|G|}\sum_g A_s^g \neq 1$

② non zero flux: $B_p^{h\neq 1}$

To understand the excitations of QDM, we first try to understand the algebra of A_s^g, B_p^h for 

$$\begin{cases} A^g A^{g'} = A^{gg'} \\ (A^g)^+ = A^{g^{-1}} \\ B^h B^{h'} = \delta_{hh'} B^h \\ (B^h)^+ = B^h \\ A^g B^h = B^{ghg^{-1}} A^g \end{cases}$$

note: (A_s^g, B_p^h) for different (s,p) commute with each other and are isomorphic.

Def $D_{(h,g)} =$  $:= B_h \cdot A_g$

$$D_{(h_1,g_1)} \cdot D_{(h_2,g_2)} = \delta_{h_1, g_1 h_2 g_1^{-1}} D_{(h_1, g_1 g_2)}$$

The algebra generated by $D_{(h,g)}$ is called the Drinfeld quantum double $D(G)$ of G.

① Co-algebra structure $\Delta: D(G) \longrightarrow D(G) \times D(G)$

how to act on two anyons by one $D(G)$ element

act on anyon b

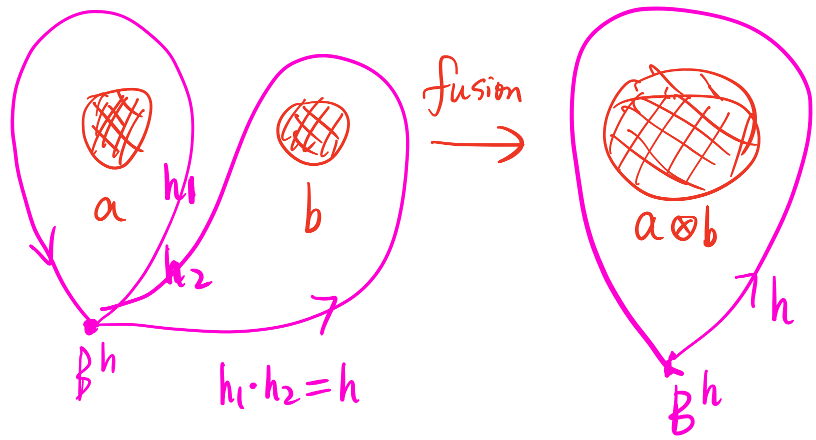
act on anyon a

$$\Delta(A^g) = A^g \otimes A^g$$

$$\Delta(B^h) = \sum_{h_1 h_2 = h} B^{h_1} \otimes B^{h_2}$$

$\Rightarrow$ Rep $D(G)$ is a tensor/monoidal category.

$a \otimes b \to c$



② $D(G)$ is quasi-triangular: $R = \sum_{g \in G} B^g \otimes A^g$

braiding

$\Rightarrow$ Rep $D(G)$ is a braided tensor category.

$D(G)$ is a quasi-triangular Hopf algebra.

Excitation space supports a representation of $D(G)$.

$\boxed{\begin{array}{l} \text{Irreps of quantum double } D(G) \\ \xleftrightarrow{1:1} \text{ anyon excitation types} \end{array}}$

described by braided fusion cat.

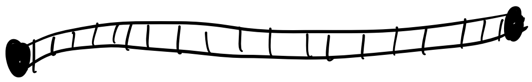
Irreps of $D(G)$.

Let $u \in G$, $C = \{ g u g^{-1} \mid g \in G \}$ the conjugacy class,

$E = \{ g \in G \mid gu = ug \}$ the centralizer.

Claim. There is one irreducible rep (C, χ) (flux: C, charge: χ)
for each conjugacy class C and each irrep χ of the centralizer group E.

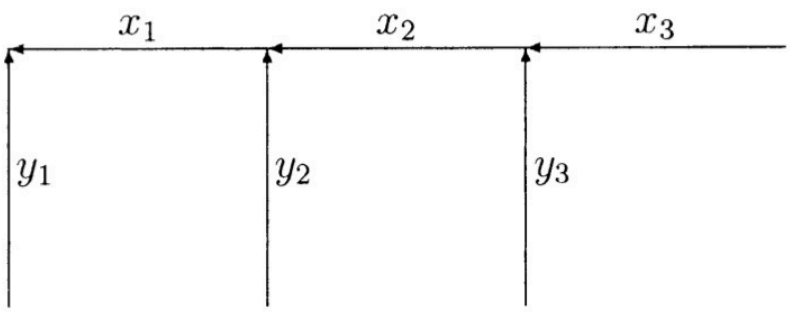
Ribbon operators



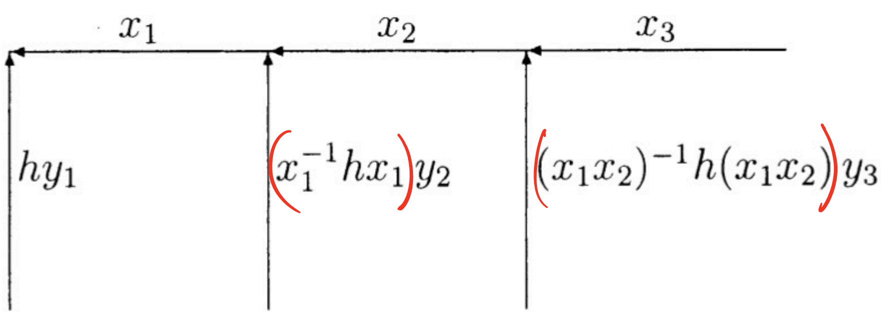
Creat a pair of anyonic excitations at the ends of the ribbon operators.

↳ finite width (framing)

$F^{(h,g)}(t)$



$= \quad \delta_{g,x_1x_2x_3}$

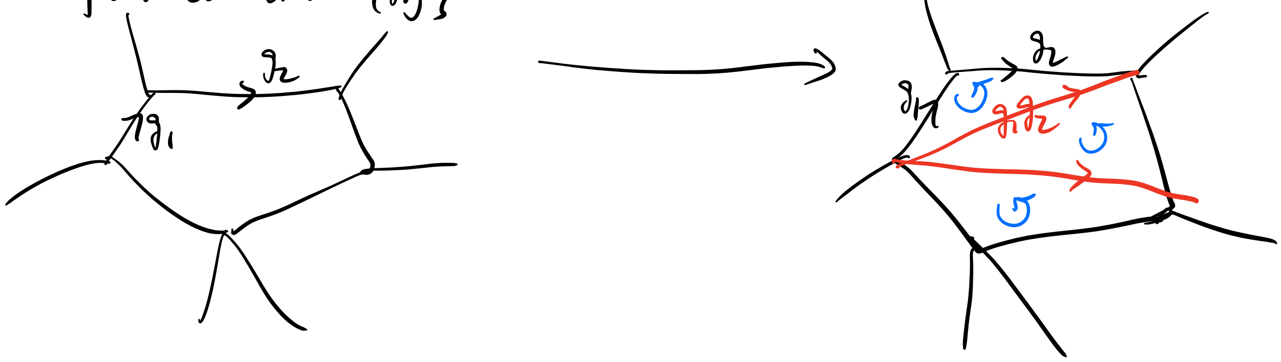


$$[F^{(h,g)}(t),\ B_p] = 0$$
$$[F^{(h,g)}(t),\ A_s] = 0$$
for $s, p \neq \partial t$

F gives us fusion, braiding of anyons.
(see Kitaev's original paper)

2.3. Spacetime Path Integral Picture

∀ 2D lattice flat connection $\{g_{ij}\}$ → 2D triangulation flat connection

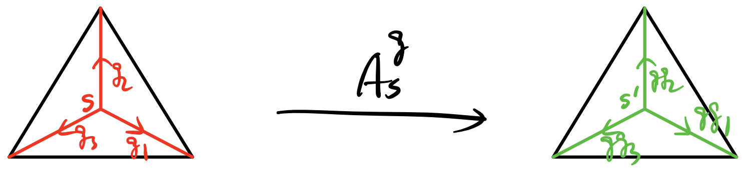


B_p operator: defined for ∀ triangle. flat connection for all spacial triangles

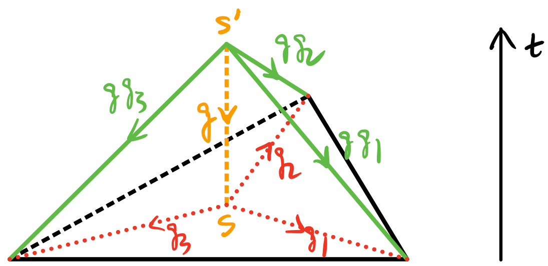
A_s^g operator:

$\partial(\text{spacetime}) = \text{space}(t_2) - \text{space}(t_1)$

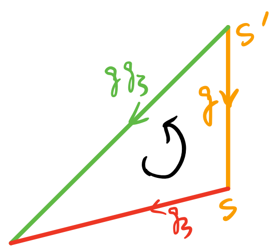
space:



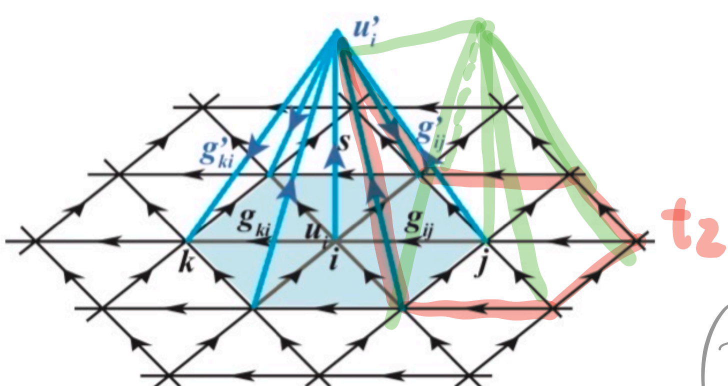
spacetime:



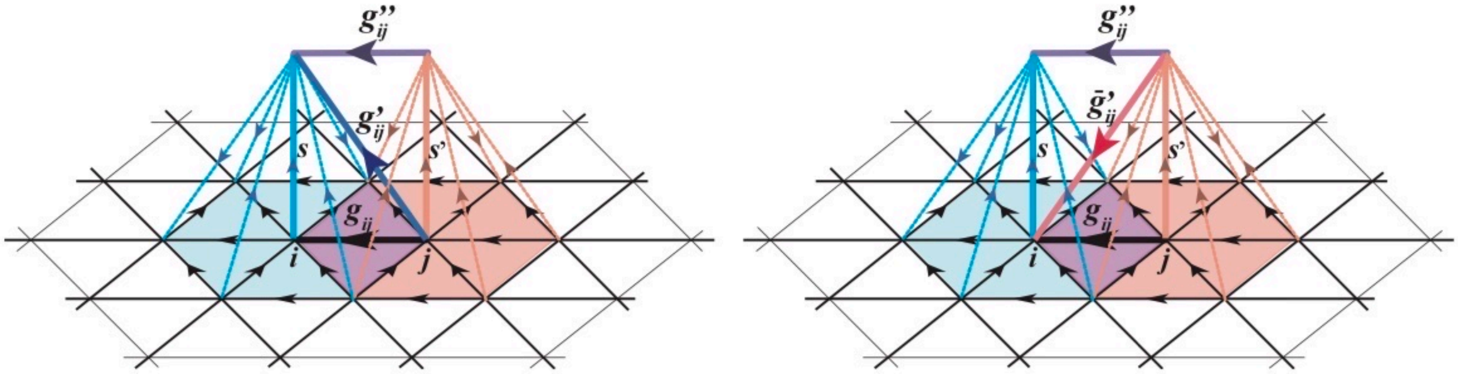
new link g along t direction



flat connection for all spacetime triangles.



(figs from Mesaros, Ran arXiv:1212.0835)

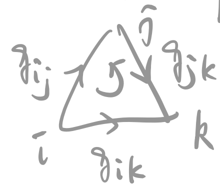


$$\underset{\uparrow t_3}{\,} A_j^{g'} \underset{\uparrow t_2}{\,} A_i^{g} \underset{\uparrow t_1}{\,} \underset{\text{exercise}}{=} \underset{\uparrow t_3}{\,} A_i^{g} \underset{\uparrow t_2}{\,} A_j^{g'} \underset{\uparrow t_1}{\,}$$

- Partition function

discrete : $Z_{QDM} = \frac{1}{\mathcal{N}} \sum_{\{g_{ij}\}} \underbrace{\prod_{\langle ijk\rangle} \delta_{g_{ij} g_{jk}, g_{ik}}}_{0 \text{ or } 1}$

flat connection for triangle $\langle ijk \rangle$

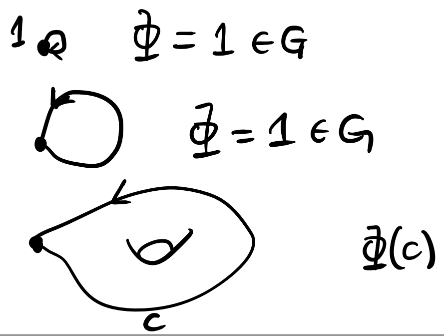


continuum : $Z_{QDM}(M_3) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M_3), G)} 1 \longrightarrow e^{-E(\{g_{ij}\})}$

$E(\{g_{ij}\}) = \begin{cases} 0, & \text{flat} \\ \infty, & \text{con-flat} \end{cases}$

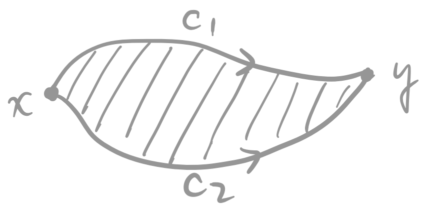
where $\gamma : \pi_1(M_3) \longrightarrow G = \pi_1(BG)$

loop $c \longmapsto \underset{\text{flux}}{\Phi(c)} = e^{i\oint_c A} = \prod_{\langle ij\rangle \in c} g_{ij}$

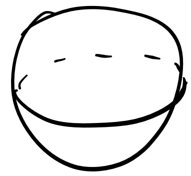


note : loops $c_1 \overset{\text{homotopy}}{\sim} c_2$

$\Rightarrow \Phi(c_1) = \Phi(c_2)$ for flat connections.

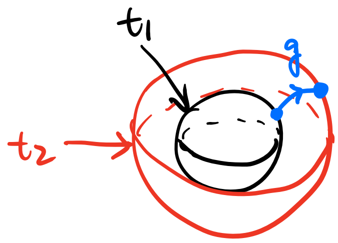


Example: (1) $M_3 = S^2 \times S^1$



space $= S^2$

$\pi_1(S^2) = 0 \Rightarrow$ All flat connections are gauge equivalent.



$\gamma : \pi_1(S^2 \times S^1) = \mathbb{Z} \rightarrow G$

$$\left| \{\text{flat conn.}\} \right| = |G|$$

$$\Rightarrow Z_{QDM}(S^2 \times S^1) = \frac{1}{|G|} |G| = 1$$

$$= \underset{GSD}{\text{ground state degeneracy}}(S^2)$$

(2) $M_3 = T^2 \times S^1 = T^3$, $G = \mathbb{Z}_2 \rightarrow$ toric code model.

$\gamma : \pi_1(T^3) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \longrightarrow G = \mathbb{Z}_2$

$$\left| \{\text{flat conn.}\} \right| = 2^3 = 8$$

$$Z_{QDM}(T^3) = \frac{1}{|G|} \left| \{\text{flat conn.}\} \right| = \frac{1}{2} \times 8 = 4$$

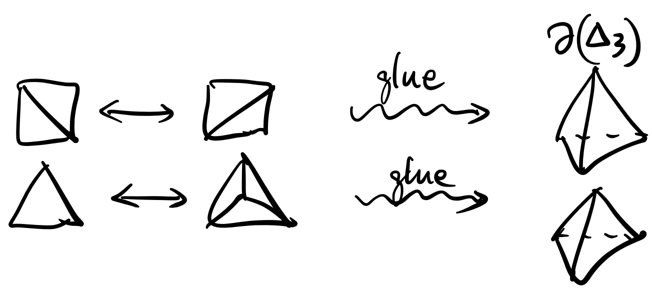
$$= GSD(T^2)$$

- Retriangulation invariance of Z_{QDM}.

<u>Pachner moves.</u>

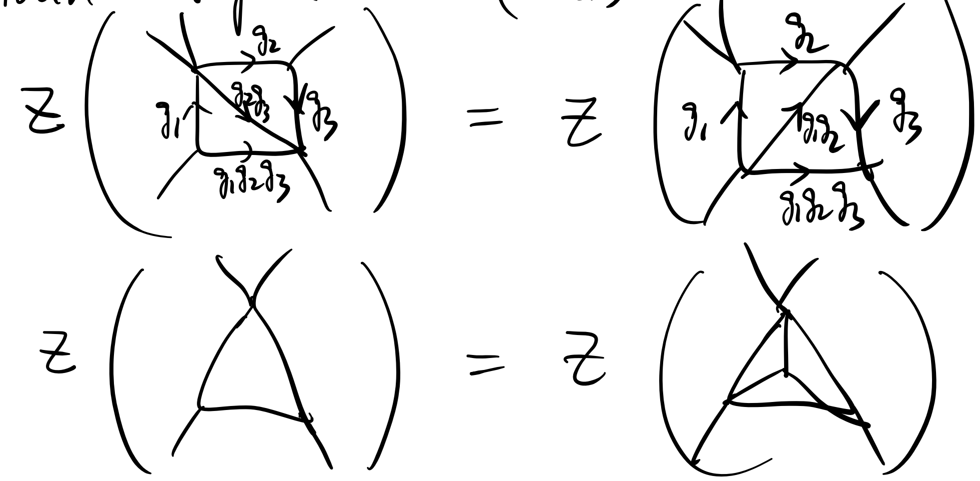
eg: 2D 2-2 move

1-3 move



Any two triangulations of a piecewise linear manifold can be related by a finite sequence of Pachner moves.

(1) Path integral (1+1D)



$$\text{flat connection} \xrightarrow[\text{local patch change}]{\text{Pachner move}} \text{flat connection}$$

$$\boxed{Z_{QDM}(M, T; G) = Z_{QDM}(M, T'; G)}$$

spacetime manifold

triangulation

(2) wave function

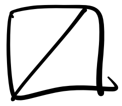
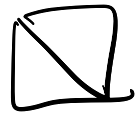
(2+1)D : $\displaystyle |\Psi_{GS}\rangle = \sum_{\substack{\text{flat conn.} \\ \{g_{ij}\}}} \Psi(\{g_{ij}\}) \, |\{g_{ij}\}\rangle$

$\Psi(\{g_{ij}\}) = \Psi(\{g'_{ij}\})$ if $\{g_{ij}\} \underset{\text{equiv.}}{\overset{\text{gauge}}{\sim}} \{g'_{ij}\}$

equal amplitute superposition of gauge equiv. flat connections.

generalization from one fix lattice to arbitrary lattice :

$$|\Psi_{GS}\rangle = \sum_{\substack{\text{lattice } T \\ (\text{triangulation})}} \frac{1}{N_T} \sum_{\text{flat conn.} \{g_{ij}\}} \Psi(\{g_{ij}\}) \, |\{g_{ij}\}\rangle$$

$\Psi\Big($  $\Big) = 1 \times \Psi\Big($  $\Big)$

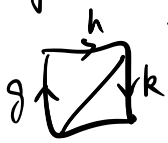
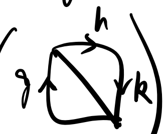
↳ QDM

$\nu_3(g,h,k)$

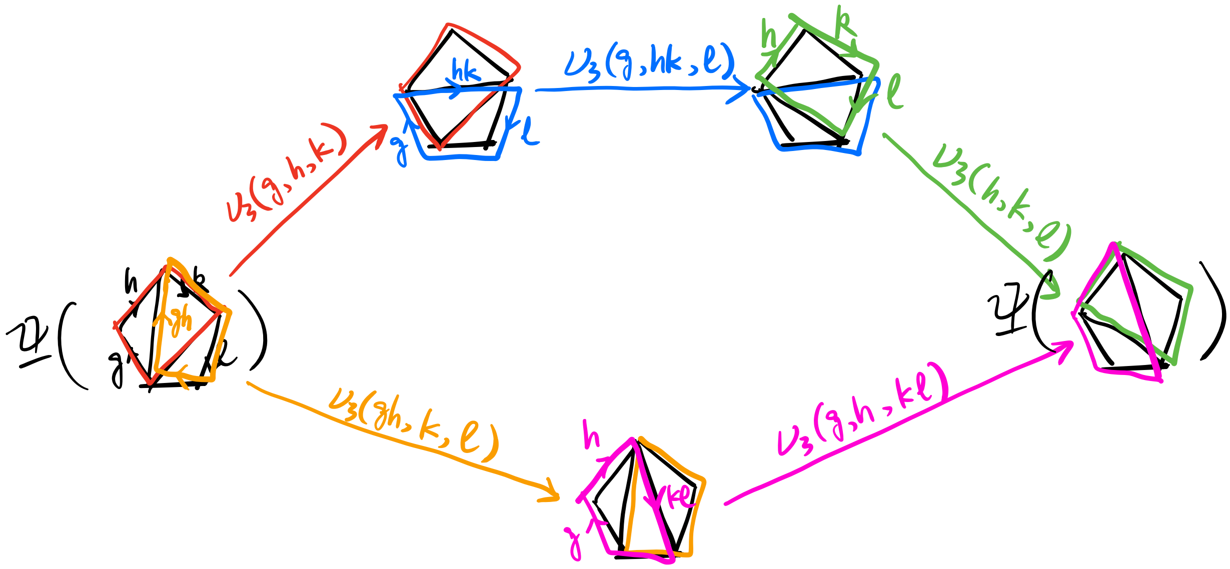
↳ TQDM

2.4. Dijkgraaf-Witten (1990) gauge theory as twisted QDM.

- Introduce d-cocycle for wavefunction retriangulation

$\Psi\Big($  $\Big) = \underbrace{\nu_3(g,h,k)}_{\text{function } \nu_3 :\ G^3 \to U(1)} \Psi\Big($  $\Big)$

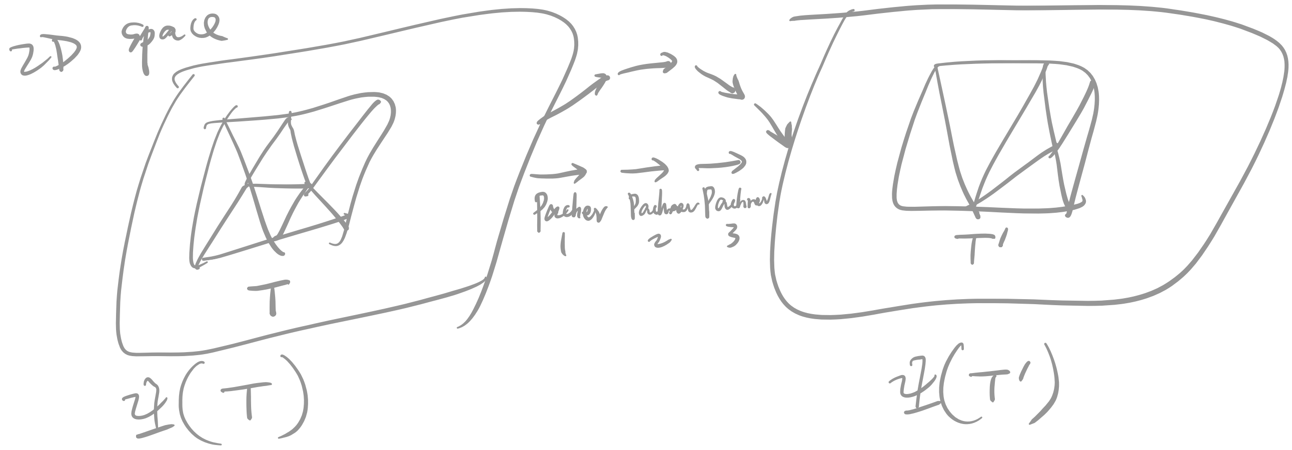
Pentagon equation as consistency condition:



$$\nu_3(g,h,k)\,\nu_3(g,hk,l)\,\nu_3(h,k,l) = \nu_3(gh,k,l)\,\nu_3(g,h,kl)$$

$$\Leftrightarrow \quad (d\nu_3)(g,h,k,l) := \frac{\nu_3(h,k,l)\,\nu_3(g,hk,l)\,\nu_3(g,h,k)}{\nu_3(gh,k,l)\ \ \nu_3(g,h,kl)} = 1$$

$\Leftrightarrow$ $\nu_3 \in Z^3(G, U(1))$ is a 3-cocycle.

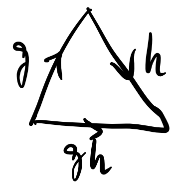


- Spacetime Path integral on the lattice

<u>1+1D :</u>

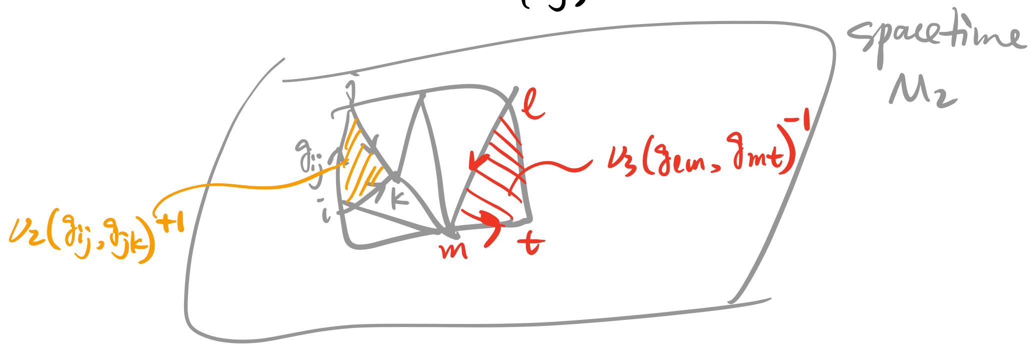
1-space : $\Psi(\bullet \xrightarrow{gh} \bullet) \overset{1\text{-}2}{\underset{\text{move}}{=}} \nu_2(g,h)\, \Psi(\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet)$

2-spacetime :

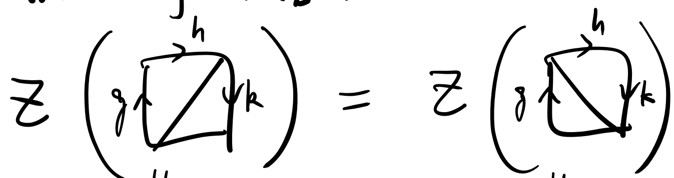


$= \nu_2(g,h)$

$$Z_{TQDM}(M_2) := \frac{1}{N} \sum_{\substack{\text{flat} \\ \text{conn.} \\ \{g_{ij}\}}} \prod_{\substack{\langle ijk \rangle \\ \in T}} \nu_2(g_{ij}, g_{jk})^{\underbrace{s(ijk)}_{\pm 1 \text{ for orientations}}}$$



retriangulation of M_2 :



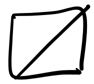
$$\nu_2(g,h)\,\nu_2(gh,k) = \nu_2(g,hk)\,\nu_2(h,k)$$

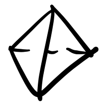
$$\Longleftrightarrow (d\nu_2)(g,h,k) = 1$$

$$\Longleftrightarrow \quad Z(\partial \Delta_3) = Z\Big($$ 

$$\Big) = Z(S^2) = 1$$

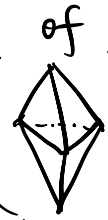
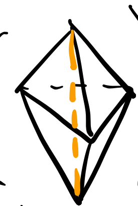
2+1 D:

2-space: $\Psi\Big($  $\Big) = \nu_3(g,h,k)\ \Psi\Big($  $\Big)$

3-spacetime:  $= \nu_3(g,h,k)$

$$Z_{TQDM} := \frac{1}{\mathcal{N}} \sum_{\substack{\text{flat} \\ \text{conn.} \\ \{g_{ij}\}}} \prod_{\substack{\langle ijkl \rangle \\ \in T}} \nu_3(g_{ij}, g_{jk}, g_{kl})^{s(ijkl)}$$

retriangulation of M_3:

$Z\Big($  $\Big) = Z\Big($  $\Big)$

$\qquad \| \qquad\qquad\qquad \|$

$\quad \nu_3 \cdot \nu_3 \qquad\qquad \nu_3 \nu_3 \nu_3$

$$\Longleftrightarrow \quad d\nu_3 = 1$$

$$\Longleftrightarrow \quad Z(\partial \Delta_4) = Z\Big($$ 

$$\Big) = Z(S^3) = 1$$

Generalize to dD:

d-spacetime: d-simplex $\Delta_d \longmapsto \nu_d(g_1, \cdots, g_d)$

consistency condition: $\nu_d(\partial \Delta_{d+1}) = \nu_d(S^d) = 1$

partition function:

$$\boxed{Z_{TQDM}(M_d, T; G) = \frac{1}{\mathcal{N}} \sum_{\substack{\text{flat conn.} \\ \{g_{ij}\}}} \prod_{\Delta_d \in T} \nu_d(\Delta_d)^{s(\Delta_d)}}$$

$$\boxed{Z_{TQDM}(M_d, T; G) = Z_{TQDM}(M_d, T'; G)}$$

- Spatime path integral in the continuum:
  (DW in 1990)

$$Z_{QDM} = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M_d), G)} 1$$

$$\Big\downarrow \text{twist by } \nu_d \in Z^d(G, U(1))$$

$$\boxed{Z_{TQDM} = Z_{DW} = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M_d), G)} \langle \gamma^* \nu_d, [M] \rangle}$$

$$\gamma^* \nu_d : \quad \pi_1(M_d) \xrightarrow{\gamma} G \xrightarrow{\nu_d} U(1)$$

$$\langle \gamma^* \nu_d, [M] \rangle \sim e^{i \int_{M_d} \tilde{\nu}_d} \sim \prod_{\Delta_d \in T} \nu_d(\Delta_d)^{s(\Delta_d)}$$

additive $\nu_d = e^{i\tilde{\nu}_d}$