7. Symmetries and classifications of free-fermion TI/TSC

7.1. Symmetries in quantum systems

Goal: Difference and relation of

① symmetries of operators acting on Hilbert space

② symmetries of physical states.

- Hilbert space $\neq$ physical states space

$\mathcal{H} \cong \mathbb{C}^N$, two states $|\psi\rangle$ and $|\psi'\rangle$ describes the same physical states iff $|\psi\rangle = z|\psi'\rangle$ for $0 \neq z \in \mathbb{C}$.

physical states space $P\mathcal{H} = \mathbb{C}P^{N-1} \cong (\mathcal{H} - \{0\})/\mathbb{C}^\times$.

Ex. spin $\frac{1}{2}$, $\mathcal{H} = \mathbb{C}^2 = \{a|\uparrow\rangle + b|\downarrow\rangle \mid a, b \in \mathbb{C}\} = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2\}$

$$\forall \begin{pmatrix} a \\ b \end{pmatrix} \begin{cases} \nearrow ① \ a \neq 0: \ \begin{pmatrix} a \\ b \end{pmatrix} \sim \frac{1}{\sqrt{|a|^2+|b|^2}} \begin{pmatrix} |a| \\ |a| \cdot b/a \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\ \searrow ② \ a = 0: \ \begin{pmatrix} 0 \\ b \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \text{ with } \begin{pmatrix} 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi \end{pmatrix}$$

$$\Rightarrow P\mathcal{H} \cong \mathbb{C}P^1 \cong S^2 \rightarrow \text{Bloch sphere.}$$

$$\dim_{\mathbb{R}} \mathcal{H} = 4 \ , \ \dim_{\mathbb{R}} P\mathcal{H} = 2$$

normalization -1

$U(1)$ phase -1

- Symmetries of Hilbert space $\neq$ Symmetries of physical states.

$\mathrm{Aut}_{qtm}(P\mathcal{H})$: set of automorphisms of quantum systems.

transformation should preserve probability

$$|\langle\psi|\varphi\rangle|^2 = |\langle U\psi|U\varphi\rangle|^2$$

$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$: set of unitary and antiunitary transformations on $\mathcal{H}$.

$$U: |\psi\rangle \mapsto U|\psi\rangle$$

U is $\mathbb{R}$-linear

$$\begin{cases} \text{unitary} : & U\big(a|\psi\rangle + b|\varphi\rangle\big) = a\,U|\psi\rangle + b\,U|\varphi\rangle \\ \text{anti-unitary} : & U\big(a|\psi\rangle + b|\varphi\rangle\big) = a^*\,U|\psi\rangle + b^*\,U|\varphi\rangle \end{cases}$$

Ex. spin $\frac{1}{2}$. $P\mathcal{H} = S^2$, $\mathcal{H} = \mathbb{C}^2$

$$\mathrm{Aut}_{qtm}(P\mathcal{H}) = \mathrm{Aut}_{qtm}(S^2) \cong O(3) \cong \mathbb{Z}_2 \times SO(3)$$

$$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^2) = \{\text{unitary/antiunitary transf. actioning on } \mathbb{C}^2\}$$

$$= U(2) \oplus U(2) = \mathbb{Z}_2 \times U(2)$$

Relation:

$$0 \to U(1) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}_{qtm}(P\mathcal{H}) \to 0$$

Wigner thm: Every quantum automorphism in $\mathrm{Aut}_{qtm}(P\mathcal{H})$ is induced by a unitary or anti-unitary operator in $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ on Hillbert space $\mathcal{H}$.

$$1 \to U(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\phi} \mathbb{Z}_2 \longrightarrow 0$$

$$\phi(s) = \begin{cases} +1, & \text{if } s \text{ is unitary} \\ -1, & \text{-- -- antiunitary.} \end{cases}$$

- Physical symm and twisted symm.

$\mathrm{Aut}_{qtm}(P\mathcal{H})$: all symmetries of a q system.

Add Hamiltonian $\hat{H}$: smaller symmetry G.

$$\rho : G \to \mathrm{Aut}_{qtm}(P\mathcal{H})$$

twisted extension G^{tw} of G :

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & G^{tw} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r] & \mathrm{Aut}_{qtm}(P\mathcal{H}) \arrow[r] & 1 \end{tikzcd}$$

$$\begin{cases} G: \text{ physical symmetry acting on physical states} \\ G^{tw}: \text{ virtual symmetry acting on Hilbert space.} \end{cases}$$

Ex. $G = \mathbb{Z}_2^T = \{1, T\}$ is time reversal symmetry.

$$\begin{cases} \phi(1) = 1 \\ \phi(T) = -1 \end{cases}$$

$$1 \to U(1) \to G^{tw} \to \mathbb{Z}_2^T \to 1$$

G^{tw} is classified by $H^2(\mathbb{Z}_2^T, U(1)) = \mathbb{Z}_2 \ni \omega_2$

$$\begin{cases} (1)\ G^{tw} \cong \{ zT \mid zT = Tz^{-1},\ z \in U(1),\ T^2 = 1 \} \cong U(1) \rtimes \mathbb{Z}_2^T \\ (2)\ G^{tw} \cong \{ zT \mid zT = Tz^{-1},\ z \in U(1),\ T^2 = -1 \} \cong U(1) \rtimes_{\omega_2} \mathbb{Z}_2^T \end{cases}$$

7.2. 10-fold way of TI/TSC.

- unitary symmetries are NOT important in the classification of TI/TSC.

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^\dagger \mathcal{H}_{A,B} \hat{\psi}_B \qquad \mathcal{H}^T \sim \mathcal{H}^*$$

$$\{\psi_A, \psi_B^\dagger\} = \delta_{AB}, \quad \{\psi_A, \psi_B\} = \{\psi_A^\dagger, \psi_B^\dagger\} = 0.$$

If we have a unitary symmetry:

$$\begin{cases} \hat{U} \hat{\psi}_A \hat{U}^\dagger = \sum_B U_{A,B}^\dagger \hat{\psi}_B \\ \hat{U} \psi_A^\dagger \hat{U}^\dagger = \sum_B \psi_B^\dagger U_{B,A} \end{cases} \qquad U \text{ is unitary matrix.}$$

$$\hat{U} \hat{H} \hat{U}^{-1} = \hat{H} \iff U \mathcal{H} U^\dagger = \mathcal{H}$$

We can block-diagonalize $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda$$

For each λ, we have several copies of a fixed irrep. of G.

$$V_\lambda = \underset{\substack{\uparrow \\ \text{Hamiltonian} \\ \text{acting on}}}{V_\lambda^{(H)}} \otimes \underset{\substack{\uparrow \\ \text{Symmetry} \\ \text{acting on}}}{V_\lambda^{(G)}}$$

Within $V_\lambda^{(H)}$, there is No constraints for $\mathcal{H}$.

Ex. $G = SO(3)$. $\quad V = \mathbb{C}^6 = \underset{\lambda=\text{spin-1}}{\mathbb{C}^3_\lambda} \oplus \underset{\lambda=\text{spin-1}}{\mathbb{C}^3_\lambda} = \mathbb{C}^2 \otimes \mathbb{C}^3$

(each $\mathbb{C}^3_\lambda$ acted on by $SO(3)$)

$$[\mathcal{H}, SO(3)] = 0 \Rightarrow \left.\begin{array}{l} \mathcal{H} = \mathcal{H}_{2\times2} \otimes I_{3\times3} \\ U(g) = I_{2\times2} \otimes U_\lambda(g) \end{array}\right\}$$

- Antiunitary symmetries and 10-fold way.

(1) time reversal

$$\begin{cases} \hat{\mathcal{T}}\, \hat{\psi}_A\, \hat{\mathcal{T}}^{-1} = \sum_B (U_T^\dagger)_{A,B}\, \hat{\psi}_B \\ \hat{\mathcal{T}}\, \hat{\psi}_A^\dagger\, \hat{\mathcal{T}}^{-1} = \sum_B \hat{\psi}_B^\dagger (U_T)_{B,A} \\ \hat{\mathcal{T}}\, i\, \hat{\mathcal{T}}^{-1} = -i \end{cases} \qquad \mathcal{T} = U_T \cdot K$$

$$\hat{\mathcal{T}}\hat{H}\hat{\mathcal{T}}^{-1} = \hat{H} \Leftrightarrow \boxed{U_T \mathcal{H}^* U_T^\dagger = \mathcal{H}}$$

$$\hat{\mathcal{T}}^2 = \pm 1 \Leftrightarrow U_T U_T^* = \pm 1$$

$$T = \begin{cases} 0, & \text{if no } \mathcal{T} \text{ symm.} \\ +1, & \text{if } \mathcal{T}^2 = +1 \\ -1, & \text{if } \mathcal{T}^2 = -1 \end{cases} \quad \rightarrow 3$$

(2) charge conjugation (particle-hole) symmetry.

$$\begin{cases} \hat{C}\, \hat{\psi}_A\, \hat{C}^{-1} = \sum_B (U_C^{*\dagger})_{AB}\, \hat{\psi}_B^\dagger \\ \hat{C}\, \hat{\psi}_A^\dagger\, \hat{C}^{-1} = \sum_B \hat{\psi}_B\, (U_C^*)_{BA} \\ \hat{C}\, i\, \hat{C}^{-1} = i \end{cases}$$

Comes from T (transpose)

$$\hat{C}\hat{H}\hat{C}^{-1} = \hat{H} \Leftrightarrow \boxed{U_C \mathcal{H}^* U_C^\dagger = -\mathcal{H}}$$

$$C = \begin{cases} 0, & \text{if no } C \text{ symm.} \\ +1, & \text{if } C^2 = +1 \\ -1, & \text{if } C^2 = -1 \end{cases} \quad \rightarrow 3$$

(3) chiral (sublattice) symmetry.

$$\hat{S} = \hat{\mathcal{T}} \cdot \hat{C}$$

$$S = U_S = U_T \cdot U_C^*$$

$$\hat{S}\hat{H}\hat{S}^{-1} = \hat{H} \Leftrightarrow \boxed{U_S \mathcal{H} U_S^{+} = - \mathcal{H}}$$

$$S = \begin{cases} 0, & \text{if no } S \text{ symm.} \\ 1, & \text{if } S \text{ symm.} \end{cases}$$

if $T = C = 0$, $\quad S = T \cdot C = \begin{cases} 0 \\ 1 \end{cases}$

$T = 0, \pm \quad C = 0, \pm \quad \begin{cases} T = C = 0 \\ S = 1 \end{cases}$

In total, $3 \times 3 + 1 = 10$ classes

TABLE - "Ten Fold Way" [`CARTAN Classes'] Examples

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall Z2-Top.Ins. Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

(Ludwig 2015)

7.3. Examples of TI/TSC classification

2 complex classes
$$\begin{cases} A & : \quad TCS = 000 \\ A\mathrm{III} & : \quad TCS = 001 \end{cases}$$

8 real classes

Class	TCS	
AI	$+00$	insulators, $G_f = U(1)_f$
AII	-00	
D	$0+0$	Superconductors, $G_f = \mathbb{Z}_2^f$
BDI	$++1$	
DIII	$-+1$	
C	$0-0$	superconductors with $SU(2)_s$ symm.
CI	$+-1$	
CII	$--1$	

- classification of 2D class A TI.

$TCS = 000 \;\to\;$ $\mathcal{T}$ broken insulator $\Longrightarrow$ Chern insulator $\mathbb{Z}$

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^{\dagger} H_{AB} \hat{\psi}_B$$

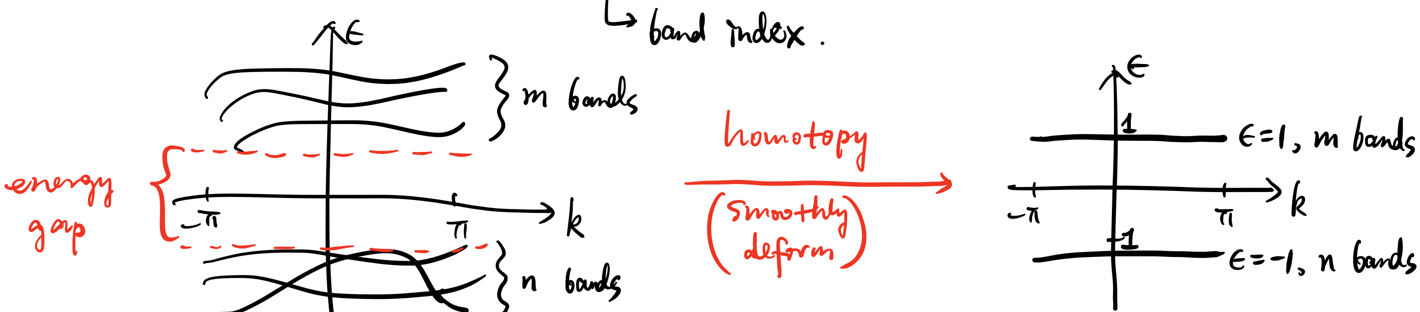
H_{AB}: hermitian matrix $\quad H^{\dagger} = H \Leftrightarrow H \in u(N) \Leftrightarrow e^{iH} \in U(N)$

$\{\text{class A Hamiltonians}\} = u(N)$

If we have translational symmetry,

$$\hat{H} = \sum_{\vec{k} \in BZ = T^2} \sum_{i=1}^{m+n} \epsilon_{ik} \psi_{ik}^{\dagger} \psi_{ik}$$

i ↳ band index.



$$H(k) = U(k)\begin{pmatrix} \epsilon_1(k) & & \\ & \ddots & \\ & & \epsilon_{m+n}(k) \end{pmatrix} U(k)^\dagger \longrightarrow \tilde{H}(k) = U(k)\begin{pmatrix} I_m & \\ & -I_n \end{pmatrix} U^\dagger(k)$$

wavefuntion unchanged
(U same)

$H \in u(m+n)$ simplified $\tilde{H} \in \dfrac{U(m+n)}{U(m)\times U(n)}$

For $U = \begin{pmatrix} U_m & \\ & U_n \end{pmatrix}$ with $U_m \in U(m)$, $U_n \in U(n)$,

$$\tilde{H} = U\begin{pmatrix} I_m & \\ & -I_n \end{pmatrix} U^\dagger = \begin{pmatrix} U_m U_m^\dagger & \\ & -U_n U_n^\dagger \end{pmatrix} = \begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}$$

$$\{\text{class A simplified Hamiltonians}\} = \frac{U(m+n)}{U(m)\times U(n)} = C_0$$

2D class A TI

$$\Longleftrightarrow \quad \tilde{H}: \ BZ = T^2 \longrightarrow C_0$$
$$\vec{k} \longmapsto \tilde{H}(\vec{k})$$

Classification of 2D class A strong TI

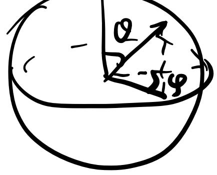
$$= (\tilde{H}: S^2 \to C_0)$$
$$= \pi_2(C_0) = \mathbb{Z}$$

Consider the simpler case with $m=n=1$ (2 bands):

$$C_0 = \frac{U(2)}{U(1)\times U(1)} \cong S^2$$

$$U(2) \ni U = \begin{pmatrix} a & b \\ -e^{i\theta}b^* & e^{i\theta}a^* \end{pmatrix}, \quad a, b \in \mathbb{C}, \ |a|^2+|b|^2=1$$

$$U \sim \begin{pmatrix} \cos\frac{\theta}{2} & e^{i\varphi}\sin\frac{\theta}{2} \\ -e^{-i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \quad \begin{matrix} 0 \le \theta \le \pi \\ 0 \le \varphi < 2\pi \end{matrix}$$



$$\pi_2\left(\frac{U(2)}{U(1)\times U(1)}\right) = \pi_2(S^2) = \mathbb{Z}.$$

$$[T^2, S^2] = \mathbb{Z}$$

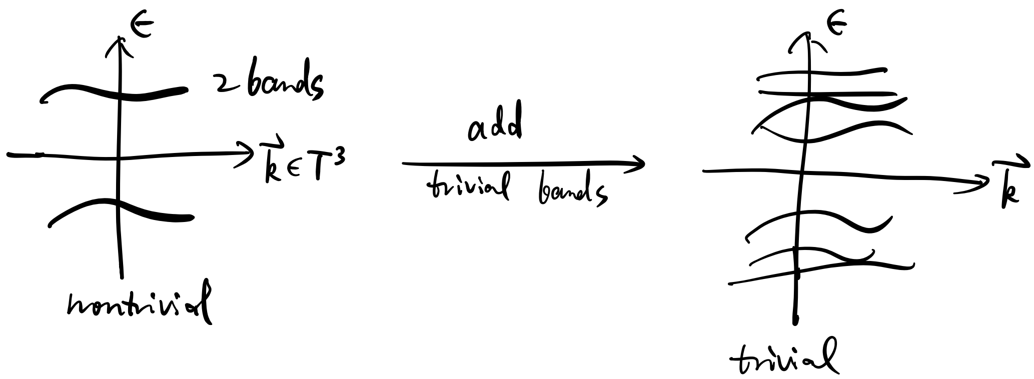
Remarks.

(1) Classification is obtained under stable condition

Example: 3D class A TI with 2 bands:

$$\tilde{H}: T^3 \to \frac{U(2)}{U(1)\times U(1)} = S^2$$

$$T^3 \to S^3 \xrightarrow[\mathbb{Z}]{\text{Hopf}} S^2$$



Adding bands.

$$H = H \oplus H_0$$

$$H_0 = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad X = U\begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}U^\dagger$$

$$X^2 = I$$

Claim. $H_0 \sim H_1 = \begin{pmatrix} & iI \\ -iI & \end{pmatrix}$

homotopy $H_t = \cos\left(\frac{\pi}{2}t\right) H_0 + \sin\left(\frac{\pi}{2}t\right) H_1$

$$\left.\begin{aligned} H_0^2 &= (\sigma_z\otimes X)^2 = I_2\otimes I = I \\ H_1^2 &= (-\sigma_y\otimes I)^2 = I \\ H_0H_1 &= -H_1H_0 \end{aligned}\right\} \Rightarrow H_t^2 = \left(\cos^2\frac{\pi}{2}t + \sin^2\frac{\pi}{2}t\right)I = I$$

$$\begin{pmatrix} X & \\ & -X \end{pmatrix} \underset{\text{gapped}}{\overset{\text{Symm.}}{\sim}} \begin{pmatrix} & iI \\ -iI & \end{pmatrix} \underset{\text{gapped}}{\overset{\text{Symm}}{\sim}} \begin{pmatrix} I & \\ & -I \end{pmatrix}$$

$\{X\}_{\text{Symm}}$ may have nontrivial topology,
but $\left\{\begin{pmatrix} X & \\ & -X \end{pmatrix}\right\}_{\text{Symm.}}$ is always contractible.

Equivalence of modes (with different number of bands)

- $H_n' \sim H_n''$ iff $H_n' \oplus Y_k \sim H_n'' \oplus Y_k$ for some Y_k.

"$\Leftarrow$": $H' \oplus Y \sim H'' \oplus Y \Rightarrow H' \oplus Y \oplus (-Y) \sim H'' \oplus \underbrace{Y \oplus (-Y)}$

$\Rightarrow H' \sim H''$

- diffence class $d(A,B) \to$ understood as $A \ominus B$

$(A_n', B_n') \sim (A_m'', B_m'')$ if $A_n' \oplus B_m'' \sim A_m'' \oplus B_n'$

$A' \ominus B' \sim A'' \ominus B''$ $\quad$ $A' \oplus B'' \sim A'' \oplus B'$

This equi. gives a notion of equiv. of matrices with different sizes.

$$(A_n, B_n) \sim (A_n \oplus (-B_n), B_n \oplus (-B_n)) = \left(A_n \oplus -B_n, \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}\right)$$

We can always choose $B_n = \begin{pmatrix} I_s & \\ & -I_s \end{pmatrix}$ with $n = 2s$.

Invariant of (A,B): $\quad k = k(A) - k(B) = k(A) - s$

\# negative energies of A

- Consider again class A

For each k, space of $\tilde{H}$ of class A is $\dfrac{U(2s)}{U(s+k) \times U(s-k)}$

$$\text{"} C_0 = \frac{U(m+n)}{U(m) \times U(n)} \text{"} \longrightarrow C_0 = \bigcup_{k \in \mathbb{Z}} \lim_{s \to \infty} \frac{U(2s)}{U(s+k) \times U(s-k)}$$

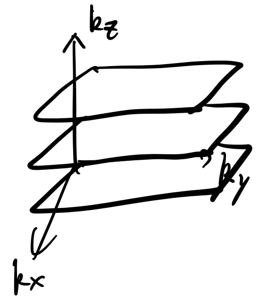
(2) strong / weak TI / TSc.

$BZ^d = T^d \neq S^d$

$$[T^d, C]_s \cong \pi_d(C) \oplus \bigoplus_{i=0}^{d-1} \binom{d}{i} \pi_i(C)$$

Strong TI

Weak TI
$\parallel$
stacking of strong TI of lower dims.



$$H_{3D}(k_x, k_y, \cancel{k_z}) = H_{2D}(k_x, k_y)$$

(3) We can use topological invariants to distinguish TI/TSC classes for each symm class and dim.

(4) There are nontrivial (gapless, symmetric) edge states for nontrivial TI/TSC.

(5) Space of Hamiltonians for each symm class:

symmetric
No gapped condition

$\left\{ \begin{array}{l} \text{① gapped} \\ \text{② flatten } \epsilon = \pm 1 \end{array} \right. \Leftrightarrow \tilde{H}^2 = 1$

	class	TCS	{Hamiltonian H}	{simplified Hamiltonian $\tilde{H}$}	
(complex)	A	0 0 0	$u(N)$	$U(m+n)/U(m)\times U(n)$	$= C_0$
	AIII	0 0 1	$u(M+N)/u(M)\times u(N)$	$u(N)$	$= C_1$
(real)	AI	+ 0 0	$u(N)/o(N)$	$O(m+n)/O(m)\times O(n)$	$= R_0$
	BDI	+ + 0	$o(M+N)/o(M)\times o(N)$	$O(n)$	$= R_1$
	D	0 + 0	$o(N)$	$O(2n)/U(n)$	$= R_2$
	DIII	− + 1	$so(2N)/u(N)$	$U(2n)/Sp(n)$	$= R_3$
	AII	− 0 0	$u(2N)/sp(N)$	$Sp(m+n)/Sp(m)\times Sp(n)$	$= R_4$
	CII	− − 1	$sp(M+N)/sp(M)\times sp(N)$	$Sp(n)$	$= R_5$
	C	0 − 0	$sp(N)$	$Sp(n)/U(n)$	$= R_6$
	CI	+ − 1	$sp(N)/u(N)$	$U(n)/O(n)$	$= R_7$

d-dim strong TI/TSC are classified by $\pi_d(C_i \text{ or } R_i)$.