

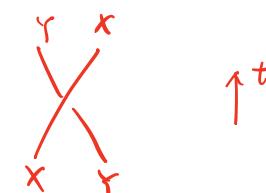
## Ch. Modular tensor categories, Drinfeld center and anyon models.

### 4.1. Braided monoidal categories

Def. A braided monoidal cat. consists of

- a monoidal cat.  $\mathcal{C}$
- a natural isomorphism call braiding that assigns to every pair of objects  $X, Y \in \mathcal{C}$  an iso.

$$b_{X,Y} : X \otimes Y \rightarrow Y \otimes X$$



such that the hexagon eq hold:

$$\begin{array}{ccccc}
 X \otimes (Y \otimes Z) & \xrightarrow{a_{X,Y,Z}^{-1}} & (X \otimes Y) \otimes Z & \xrightarrow{b_{X,Y \otimes Z}} & (Y \otimes X) \otimes Z \\
 \downarrow b_{X,Y \otimes Z} & & \downarrow & & \downarrow a_{Y,X,Z} \\
 (Y \otimes Z) \otimes X & \xleftarrow{a_{Y,Z,X}^{-1}} & Y \otimes (Z \otimes X) & \xleftarrow{id_Y \otimes b_{X,Z}} & Y \otimes (X \otimes Z) \\
 & & & \downarrow & \\
 (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{id_X \otimes b_{Y,Z}} & X \otimes (Z \otimes Y) \\
 \downarrow b_{X \otimes Y, Z} & & \downarrow & & \downarrow a_{X,Z,Y} \\
 Z \otimes (X \otimes Y) & \xleftarrow{a_{Z,X,Y}} & (Z \otimes X) \otimes Y & \xleftarrow{b_{X,Z} \otimes id_Y} & (X \otimes Z) \otimes Y
 \end{array}$$

Rem. 1.

$$b_{X,Y} = \begin{array}{c} \text{red diagram showing } b_{X,Y} \text{ with strands } X \text{ and } Y \text{ crossing} \\ \text{and labels } b_{Y,X} \text{ and } b_{Y,X}^{-1} \end{array}$$

$$b_{X,Y}^{-1} = \begin{array}{c} \text{red diagram showing } b_{X,Y}^{-1} \text{ with strands } X \text{ and } Y \text{ crossing} \\ \text{and labels } b_{X,Y} \text{ and } b_{X,Y}^{-1} \end{array}$$

$$b_{Y,X} \circ b_{Y,X}^{-1} = id_{X \otimes Y} = b_{X,Y}^{-1} \circ b_{X,Y}$$

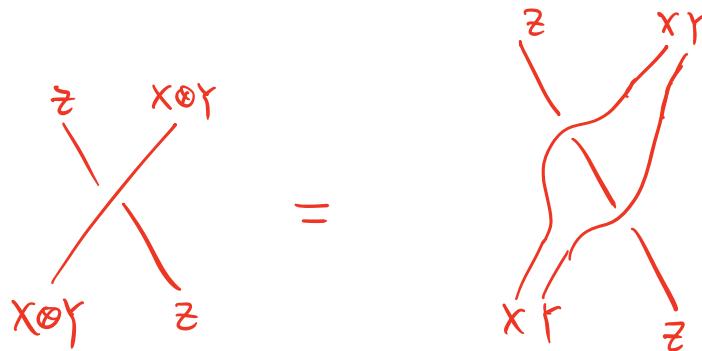
2. hexagon eq.  $\Leftrightarrow$

$$\begin{array}{c} \text{red diagram showing the hexagon equation with strands } X, Y, Z \text{ and labels } t^+, b, \text{ and } X \otimes (Y \otimes Z) \end{array}$$

$$\begin{array}{c} \text{red diagram showing the hexagon equation with strands } X, Y, Z \text{ and labels } a^{-1}, b, a, b, a^{-1} \end{array}$$

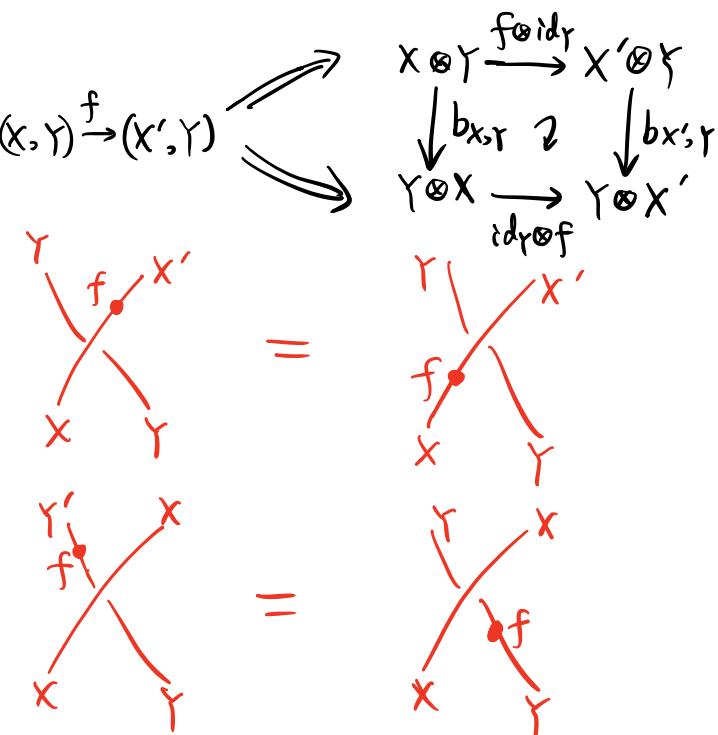
$$b = a^{-1} b a b a^{-1}$$

$$\Leftrightarrow aba = bab$$

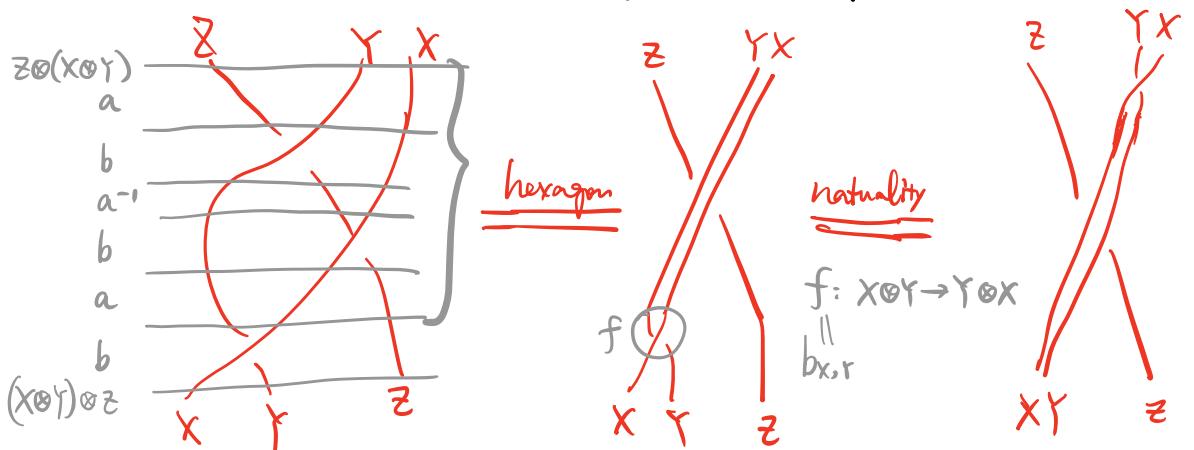


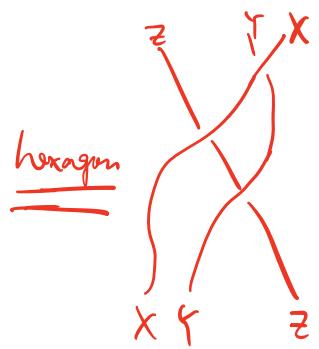
3. naturality of braiding:

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

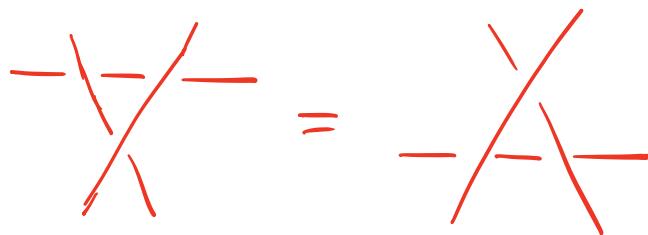


4. naturality + hexagon eq.  $\Rightarrow$  Yang-Baxter eq.





Reidemeister move III for knot  $\Leftrightarrow$  Yang-Baxter eq.  
 (geometry) (algebra)



5.  $\mathcal{C}$  is called symmetric monoidal category if

$$b_{x,y}^{-1} = b_{y,x} \iff b_{yx} \circ b_{xy} = \text{id}_{x \otimes y}$$

6. String diagram.

Split  $a \otimes b = \sum_c N_c^{ab} c$  in the simple obj. basis.



braiding  $\rightarrow$  a trivalent graph in 3D.

$R_c^{ab}$  is assumed to be unitary.

$$R^{ab} = \sum_{\mu,\nu} (R_c^{ab})_{\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}} = \sum_{c,p} \sqrt{\frac{d_c}{d_a d_b}} (R_c^{ab})_{\mu,\nu} \sum_{\mu,\nu} (R_c^{ab})_{\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}}$$

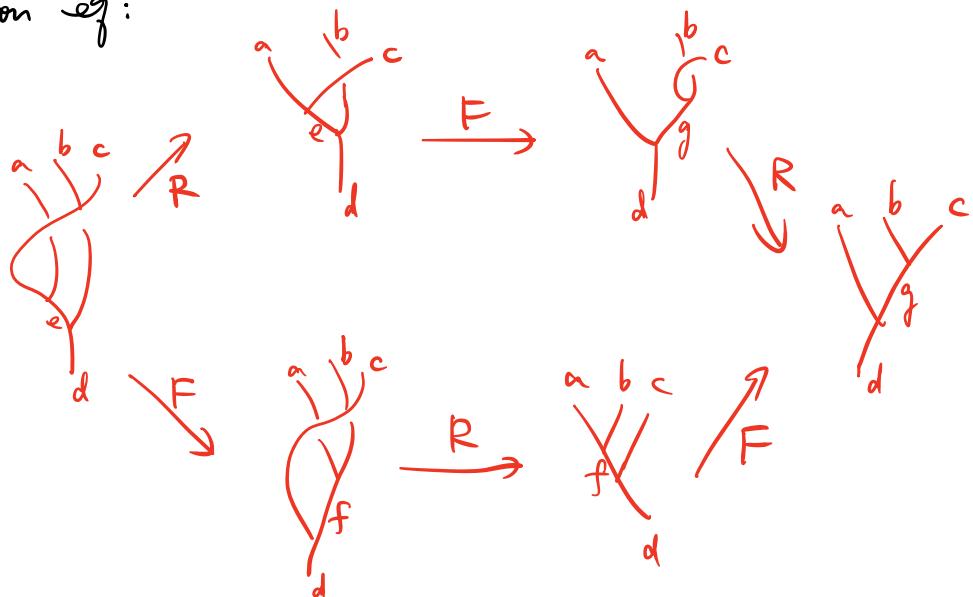
(3D diagram)

(2D diagram)

naturality :

$$\cancel{X} = X \quad \cancel{X} = X$$

hexagon eq:



$$\sum (R^{\cdot\cdot})..(F^{\cdot\cdot})..(R^{\cdot\cdot}).. = \sum F R F$$

A 3D trivalent graph  $\xrightarrow{R, F}$  2D trivalent graph  $\xrightarrow{F} \emptyset$

7. modular data.

$$\theta_a = \frac{1}{da} \text{GO}_a = \sum_{c\mu} \frac{dc}{da} (R_c^{aa})_{\mu\mu}$$

$$\begin{cases} T_{ab} = \theta_a \delta_{a,b} \\ S_{ab} := \frac{1}{D} \text{GO}_a \text{GO}_b = \frac{1}{D} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} dc \end{cases}$$

$\ell$  is called modular if  $S, T$  are non-singular.

$S, T$  generate rep of  $PSL_2(\mathbb{Z})$ .

8. 2+1D anyon models are described by a unitary modular tensor category (UMTC)

anyon	$a$	object
anti-particle	$a^*$	dual object
vacuum	$1$	identity obj.
fusion of anyons	$a \otimes b$	tensor product
anyon braiding	$R_{a,b}$	braiding
anyon worldline in 2+1 D		string diagram in 3D
transition amplitude		linear map: $a_1 \otimes a_2 \otimes \dots \rightarrow b_1 \otimes b_2 \otimes \dots$

Partition function  linear map:  $1 \rightarrow 1$

## 4.2. Drinfeld center construction.

Let  $\mathcal{C}$  be a monoidal category, a half braiding  $\beta_X$  for  $X \in \mathcal{C}$  is a family  $\{\beta_X(Y) \in \text{Hom}_{\mathcal{C}}(X \otimes Y, Y \otimes X) \mid Y \in \mathcal{C}\}$  of iso,

natural w.r.t.  $Y$ , satisfying  $\beta_X(1) = \text{id}_X$  and  $\beta_X(Y \otimes Z) = [\text{id}_Y \otimes \beta_X(Z)] \circ [\beta_X(Y) \otimes \text{id}_Z]$

$$\cancel{\begin{array}{c} Y \\ \times \\ X \end{array}}_f = f \cancel{\begin{array}{c} X \\ \times \\ Y \end{array}}$$

$$\beta_X(Y \otimes Z) = \cancel{\begin{array}{c} \text{id}_Y \otimes \beta_X(Z) \\ \beta_X(Y) \otimes \text{id}_Z \end{array}} = \cancel{\begin{array}{c} \beta_X(Y) \\ \times \\ \beta_X(Z) \end{array}}$$

The Drinfeld center  $\mathcal{Z}(\mathcal{C})$  of  $\mathcal{C}$  has obj.  $(X, \beta_X)$ , where  $X \in \mathcal{C}$  and  $\beta_X$  is a half braiding for  $X$ .

The morphisms are

$$\text{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \beta_X), (Y, \beta_Y)) := \{f \in \text{Hom}_{\mathcal{C}}(X, Y) \mid [(\text{id}_Z \otimes f)] \circ \beta_X(z) = \beta_Y(z) \circ [f \otimes \text{id}_X], \forall z \in \mathcal{C}\}$$

$$\cancel{\begin{array}{c} z \\ \times \\ X \end{array}}_f = \cancel{\begin{array}{c} z \\ \times \\ Y \end{array}}_z$$

The tensor product in  $\mathcal{Z}(\mathcal{C})$  is given by

$$(X, \beta_X) \otimes (Y, \beta_Y) = (X \otimes Y, \beta_{X \otimes Y})$$

where  $\beta_{X \otimes Y}(z) := [\beta_X(z) \otimes \text{id}_Y] \circ [\text{id}_X \otimes \beta_Y(z)]$

$$\cancel{\begin{array}{c} X \otimes Y \\ \times \\ z \end{array}} = \cancel{\begin{array}{c} \beta_X(z) \\ \times \\ Y \end{array}}_z$$

The tensor unit is  $(1, \beta_1)$  where  $\beta_1(X) := \text{id}_X$ .

$$\cancel{\begin{array}{c} 1 \\ \times \\ X \end{array}} = \cancel{|}$$

The composition and tensor product of morphisms in  $\mathcal{Z}(\mathcal{C})$  are inherited from  $\mathcal{C}$ .

The braiding in  $\mathcal{Z}(\mathcal{C})$  is given by

$$b_{(X, \beta_X), (Y, \beta_Y)} := \beta_X(Y)$$

$$\begin{array}{c} \times \\ (x, \beta_x) \quad (\gamma, \beta_\gamma) \end{array}$$

$\Rightarrow \mathcal{Z}(\mathcal{C})$  is a braided monoidal cat.

Rem. 1.  $\mathcal{C}$  is monoidal  $\xrightarrow{\mathcal{Z}(\cdot)}$   $\mathcal{Z}(\mathcal{C})$  is braided monoidal

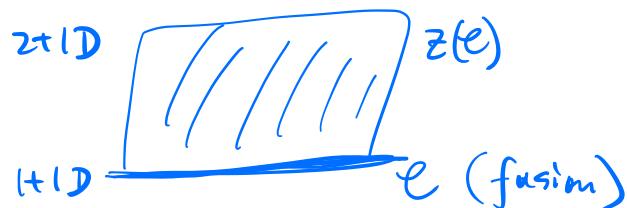
2. If  $\mathcal{C}$  is fusion, then  $\mathcal{Z}(\mathcal{C})$  is modular,

and  $\dim \mathcal{Z}(\mathcal{C}) = (\dim \mathcal{C})^2$ , where  $\dim \mathcal{C} := \sum_a d_a^2$

3. If  $\mathcal{C}$  is modular tensor cat, then

$$\mathcal{Z}(\mathcal{C}) = \mathcal{C} \otimes \mathcal{C}^{op}$$

↓  
same as  $\mathcal{C}$  with inverse braiding



4. String diagram in simple object basis.

$$\begin{array}{c} \times \\ (a, \beta_a) \quad b \end{array} = \beta_a(b) \quad \begin{array}{c} b \\ \backslash \quad / \\ c \quad a \\ \backslash \quad / \\ a \quad b \end{array} \quad (N_c^{ab} = \delta_{ab})$$

$a \otimes b = c = ab$  is simple.

naturality :

$$\begin{array}{c} a \quad b \\ \backslash \quad / \\ c \quad a \otimes b \\ \parallel \\ ab \end{array} = \begin{array}{c} a \quad b \\ \backslash \quad / \\ c \quad ab \end{array}$$

$$LHS = \beta_c(a \otimes b)$$

$$\begin{array}{c} a \quad b \\ \backslash \quad / \\ c \quad ab \\ \backslash \quad / \\ a \quad b \\ \backslash \quad / \\ c \quad ab \end{array}$$

$$RHS = \beta_c(a) \beta_c(b)$$

$$\begin{array}{c} a \\ \diagdown \\ c \\ \diagup \\ b^* \end{array} = (?) \cdot \begin{array}{c} a \\ \diagup \\ c \\ \diagdown \\ b^* \end{array}$$

$$\text{Hom}(c \otimes b^*, a) \cong \text{Hom}(c, a \otimes b)$$

$$\begin{array}{c} a \\ \diagup \\ c \\ \diamond \\ \diagdown \\ a \end{array} = (?) \cdot \begin{array}{c} a \\ \diagup \\ c \\ \square \\ \diagdown \\ a \end{array}$$

$\parallel$

$$\sqrt{\frac{dc \otimes b}{da}} \quad \begin{array}{c} a \\ \diagup \\ F_a^{a,b,b^*} \\ \square \\ \diagdown \\ a \end{array}$$

$\parallel$

$$F_a^{a,b,b^*} \quad \begin{array}{c} a \\ \diagup \\ b \\ \square \\ \diagdown \\ a \end{array}$$

$\parallel$

$$F_a^{a,b,b^*} \quad \begin{array}{c} a \\ \diagup \\ b \\ \square \\ \diagdown \\ a \end{array}$$

assume  
1

$$\text{LHS} = \beta_c(a \otimes b)$$

$\parallel$

Naturality

$$\text{RHS} = \beta_c(a) \beta_c(b)$$
$$= \beta_c(a) \beta_c(b)$$
$$= \beta_c(a) \beta_c(b) (F_{abc b^*}^{a,b,c,b^*})^{-1}$$

$\lambda_b \rightarrow \begin{array}{c} b^* \\ \diagup \\ b \end{array}$

$$\begin{array}{c} ab \\ \diagup \\ a \\ \square \\ \diagdown \\ ab \end{array} = \sqrt{\frac{da \otimes b}{da \otimes b}} \quad \begin{array}{c} ab \\ \diagup \\ ab \end{array} = \begin{array}{c} ab \\ \diagup \\ ab \end{array}$$

$$da \otimes b = \begin{array}{c} ab \\ \diagup \\ a \\ \square \\ \diagdown \\ b \end{array} = da \cdot db$$

$$= \beta_c(a) \beta_c(b) (F_{abc b^*}^{a,b,c,b^*})^{-1} (F_{abc}^{a,b,c})^{-1}$$

$$F_{abc b^* a^*}^{a,b,c,b^*,a^*}$$

$$= \beta_c(a) \beta_c(b) (F_{abc b^*}^{a, bc, b^*})^{-1} (F_{abc}^{a, b, c})^{-1} F_{abc b^* a^*}^{a, b, c, b^*, a^*}$$



$$\Rightarrow \beta_c(a \otimes b) = \beta_c(a) \beta_c(b) (F_{abc b^*}^{a, bc, b^*})^{-1} (F_{abc}^{a, b, c})^{-1} F_{abc b^* a^*}^{a, b, c, b^*, a^*}$$

↓

$\beta_c(-)$  half braiding of  $c$ .

5. Example of  $Z(\text{Vec}_{\mathbb{Z}_2})$ : toric code.  
 $\mathbb{Z}_2$  gauge theory.

$$\mathcal{C} = \text{Vec}_{\mathbb{Z}_2} \quad \text{obj.} = \{1, a\}, \quad a \otimes a = 1$$



$$\beta_1(1), \beta_1(a); \quad \beta_a(1), \beta_a(a)$$

||

$$\left\{ \begin{array}{l} \beta_1(a): \\ \beta_a(1): \end{array} \right. \quad \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \text{---} \end{array} \neq \quad \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \text{---} \end{array} =$$

①  $\beta_1(a):$

$$\cancel{\text{---}} = \cancel{\text{---}} = \text{---}$$

|| naturality

$$\cancel{\text{---}} = [\beta_1(a)]^2 \quad \cancel{\text{---}} = [\beta_1(a)]^2 \quad \text{---} \quad \left. \right\} \Rightarrow \beta_1(a) = \pm 1$$

②  $\beta_a(a):$

$$\cancel{\text{---}} = \cancel{\text{---}} = \text{---} = F_a^{aaa} \text{---} \quad \left. \right\} \begin{array}{l} \text{for } \nu_3 \in H^3(\mathbb{Z}_2, U_1) \\ (\text{semion}) \end{array}$$

|| naturality

$$\cancel{\text{---}} = [\beta_a(a)]^2 \quad \cancel{\text{---}} = [\beta_a(a)]^2 \quad \text{---} = [\beta_a(a)]^2 \quad \text{---} \quad \left. \right\} \begin{array}{l} \beta_a(a) = \pm i \\ \text{for } \nu_3 \in H^3 \end{array}$$

$\Rightarrow$  There are 4 objects  $(X, \beta_X)$  in  $\mathcal{Z}(\text{Vec}_{\mathbb{Z}_2})$

$(1, \underbrace{\begin{array}{l} \beta_1(1)=1 \\ \beta_1(a)=+1 \end{array}}_{1 \in \mathcal{Z}(\text{Vec}_{\mathbb{Z}_2})}), (1, \underbrace{\begin{array}{l} \beta_1(1)=1 \\ \beta_1(a)=-1 \end{array}}_e), (a, \underbrace{\begin{array}{l} \beta_a(1)=1 \\ \beta_a(a)=+1 \end{array}}_m), (a, \underbrace{\begin{array}{l} \beta_a(1)=1 \\ \beta_a(a)=-1 \end{array}}_f)$

$\Rightarrow$  Toric code =  $\mathbb{Z}_2$  gauge theory has 4 excitations.

Statistics :



$$b_{e,m} = b_{\underbrace{(1, \beta_1(a)=-1)}_e, \underbrace{(a, \beta_a(a)=+1)}_m} = \beta_1(a) = -1$$

$$b_{(x, \beta_X), (Y, \beta_Y)} := \beta_X(Y)$$

$$b_{m,e} = b_{(a, \beta_a(a)=+1), (1, \beta_1(a)=-1)} = \beta_a(1) = 1$$

$$\begin{array}{c} \diagup \diagdown \\ e \quad m \end{array} = b_{me} \circ b_{em} \quad \begin{array}{c} || \\ em \end{array} = \beta_a(1) \cdot \beta_1(a) \quad \begin{array}{c} | \quad | \\ e \quad m \end{array} = - \quad \begin{array}{c} | \quad | \\ e \quad m \end{array}$$

4.3. Excitation in Levin-Wen model of fusion cat.  $\mathcal{C}$ .

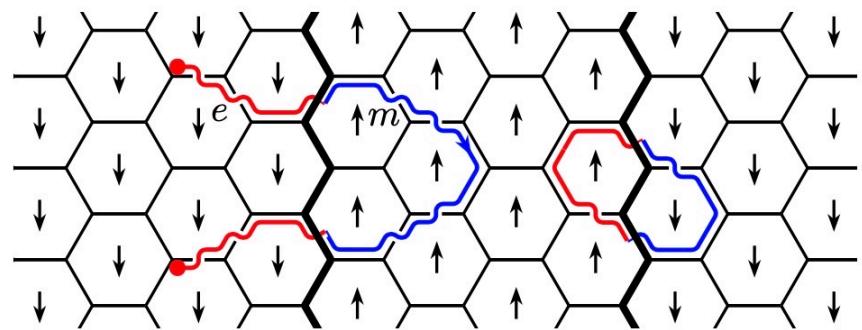
↳ described by  $\mathbb{Z}(\mathcal{C})$

(1) diagram for ribbon operators.

$$\left| \square \square^\alpha \right\rangle = \sum_i n_{\alpha,i} \left| \square \square^i \right\rangle$$

$$\left| \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle = \sum_{jst} (\Omega_{\alpha,sti}^j)_{\sigma\tau} \left| \begin{array}{c} i & j & s \\ t & i & \end{array} \right\rangle$$

$$\left| \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle = \sum_{jst} (\bar{\Omega}_{\alpha,sti}^j)_{\sigma\tau} \left| \begin{array}{c} t & j & i \\ i & s & \end{array} \right\rangle$$



ribbon op. are labelled by obj. in  $\mathcal{C}$ .

$$\begin{array}{c} \diagup \quad \diagdown \\ a \quad b \end{array} = \sum R \begin{array}{c} \diagup \quad \diagdown \\ a \quad b \end{array}$$

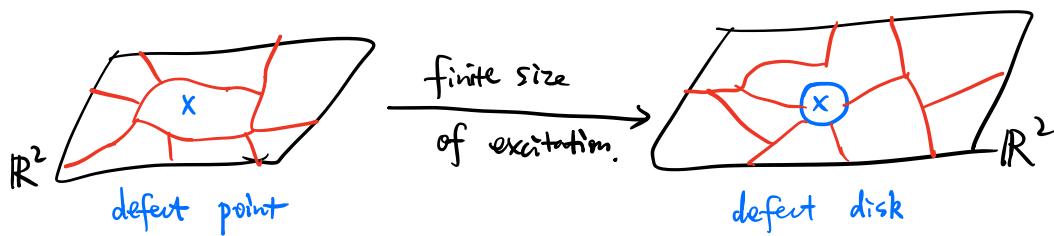
[ribbon op.,  $\frac{A_s}{B_p}$ ] = 0 if  $s, p \notin \text{ribbon}$ .

$$\begin{array}{c} \diagup \quad \diagdown \\ a \quad b \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ a \quad b \end{array} \quad \text{naturality}$$

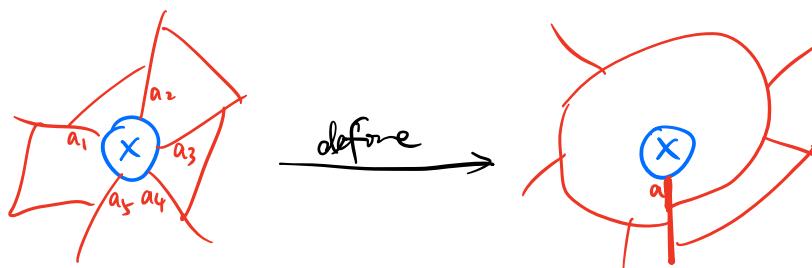
$\Rightarrow$  ribbon op. are labelled by  $\mathbb{Z}(\mathcal{C})$ .

## (2) Tube algebra approach

excitation = defect of the wavefunction or Hamiltonian.



String diagram on  $\mathbb{R}^2 \setminus D^2$

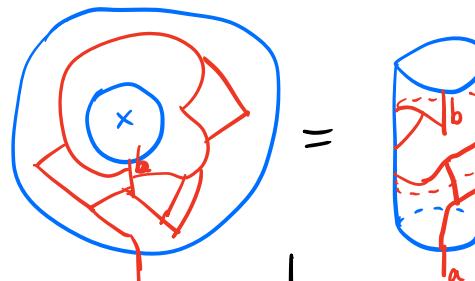


$\Rightarrow$  excitations are labelled by obj.  $a \in \mathcal{C}$ .

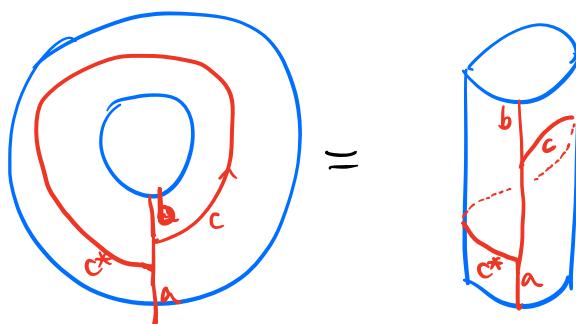
Q: What is the morphism between excitations?

$\leftrightarrow$  How to change one excitation into another?

A: Glue a cylinder  $S^1 \times I$  with string diagram.

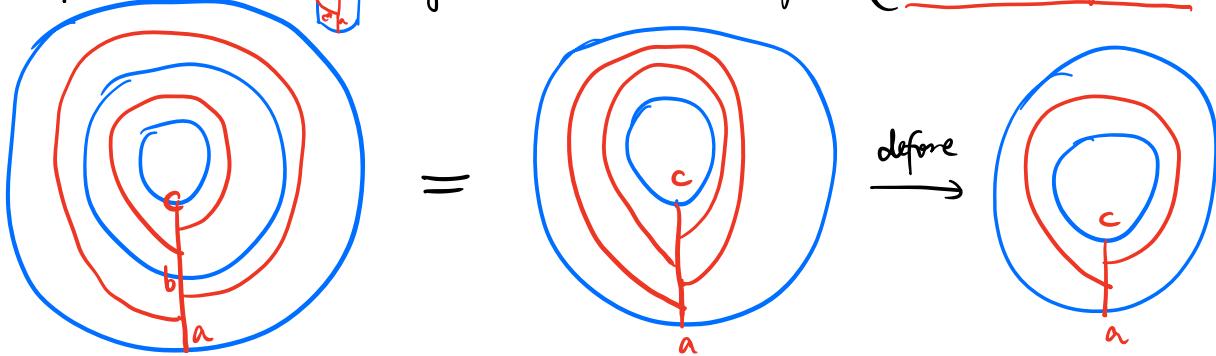


↓  
deform



Glue  $S^1 \times I$  with  $S^1$  gives another  $S^1$ .

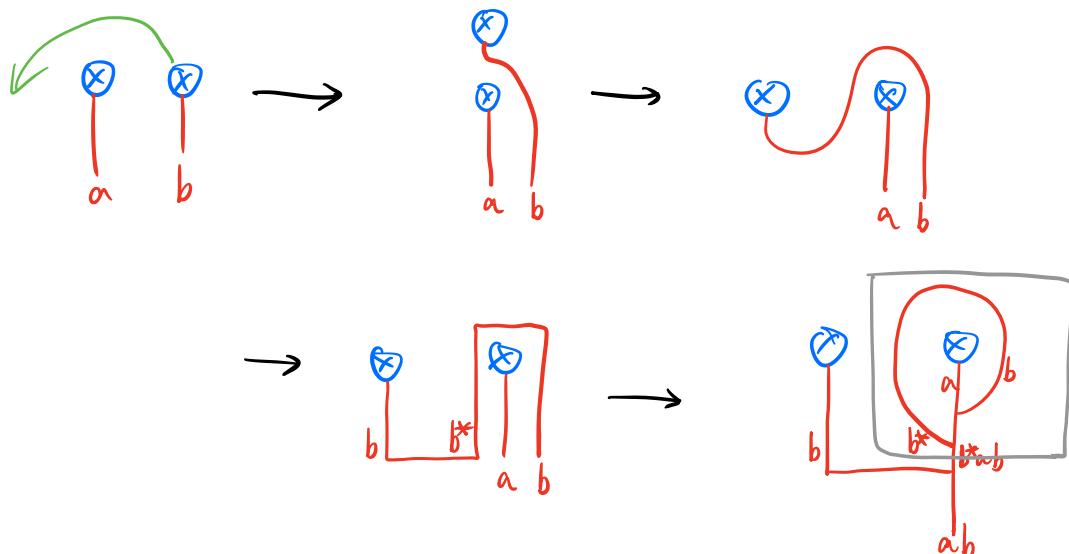
Composition of  gives us an algebra (Tube algebra)



$\Rightarrow$  Excitations can be acted by tube algebra

$\Rightarrow$  Excitations are reps of tube algebra }  $\xrightarrow{\text{excitations}}$   
 (act. of rep. of tube algebra =  $Z(\mathcal{C})$ ) }  $\in Z(\mathcal{C})$

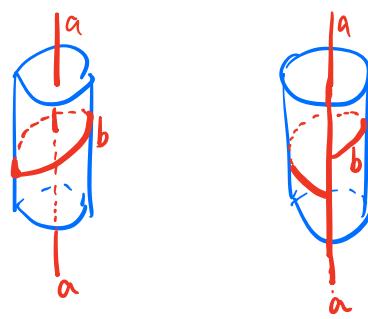
Braiding of excitations = defects :



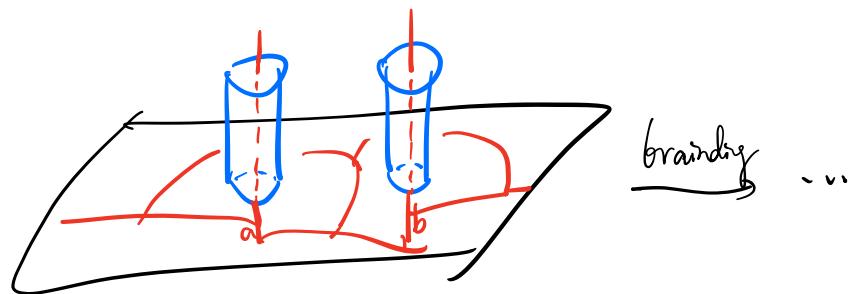
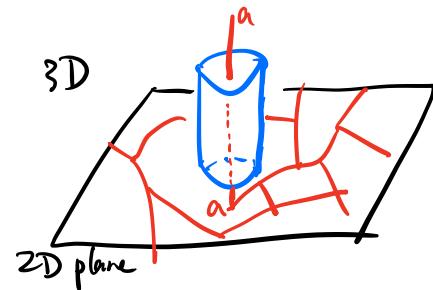
$$|a, b\rangle \xrightarrow{\text{braiding}} |b, a\rangle$$

$$\rightsquigarrow |(\alpha, \beta_a), (b, \beta_b)\rangle \xrightarrow{\text{braiding}} |(b, \beta_b), \beta_b(-)(\alpha, \beta_a)\rangle$$

$$\oint_{\gamma_b} = \beta_b(a) \quad (\oint_{\gamma_b})^*$$



Excitations can be understood as



$\Rightarrow$  Excitations should be labelled by  $(a, \beta_a) \in Z(\mathcal{C})$ .