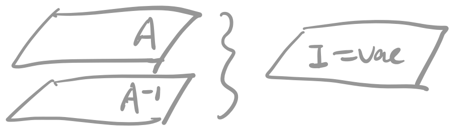
8. Introduction to symmetry-protected topological (SPT) phases

	short-range entangled (SRE)	long-range entangled (LRE)
without symmetry	invertible TO	intrinsic TO (MTC)
with symmetry	SPT	SET (G+MTC)

SET ← enriched

Note: Sometimes, are called <u>invertible</u> phases, in the sense that there exist inverse for these phase.



$$\begin{cases} \text{Topological phases} \longrightarrow \text{Abelian monoid under stacking} \\ \text{invertible phases} \longrightarrow \text{Abelian group under stacking.} \end{cases}$$

$\downarrow$ ∃ inverse

invertible phase
- invertible TO: Gapped states without anyons, but still can NOT be deformed into trivial product state.
- G-SPT: Gapped states without anyons, but still can NOT be deformed into trivial product state while preserving the symmetry G.

Known classification for invertible TO:

dim	0	1	2	3
bosonic	0	0	$\mathbb{Z}$	0
fermionic	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

bosonic $\mathbb{Z}$ (dim 2) → E_8 state

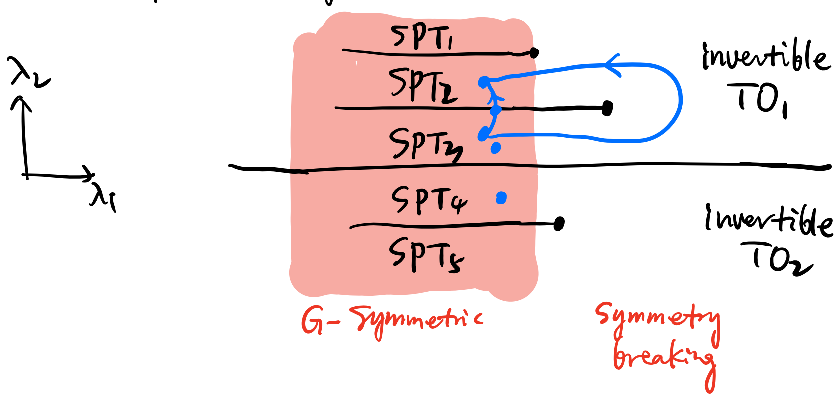
fermionic $\mathbb{Z}_2$ (dim 0) → bosonic/fermionic

fermionic $\mathbb{Z}_2$ (dim 1) → Majorana chain

fermionic $\mathbb{Z}$ (dim 2) → p_x+ip_y SC

Note: fermionic iTO = fermion SPT protected by $G_f = \mathbb{Z}_2^f = \{1, (-1)^F\}$

Schematic phase diagram:



Depending on microscopic dof & (non) interacting:

SPT	noninteracting	interacting
bosonic	/	bSPT → this chapter
fermionic	TI/TSC → last chapter	fSPT → last chapter?

Similar to TI/TSC, <u>nontrivial properties</u> of SPT is on the <u>edge</u>.

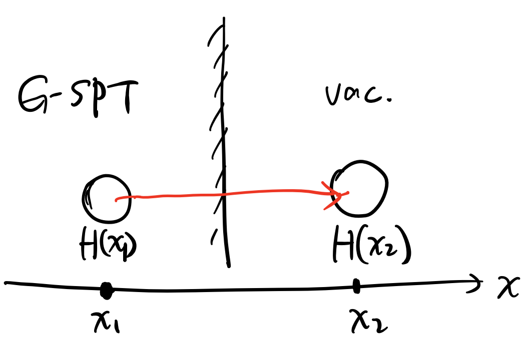
Possible edge states for SPT:

(1) G symmetry breaking

(2) anomalous SET (with edge GSD)

(3) gapless

~~(4)~~ unique gapped symmetric edge state.

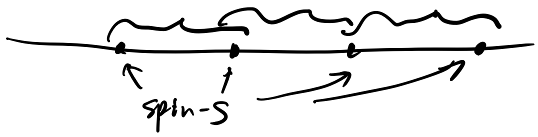


homotopy path $H(x)$: $H(x_1) \to H(x_2)$

8.1. Haldane chain (1983)

Consider 1D antiferromagnetic spin-S chain:

$$\hat{H} = \sum_i \hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1}$$



classical configuration: $\uparrow \downarrow \uparrow \downarrow \uparrow$

$$\vec{S}_i \approx (-1)^i S\, \hat{n}_i + \vec{l}_i$$

low E mode ($\hat{n}_i$) high E mode ($\vec{l}_i$)

$\int \mathcal{D}\vec{l}_i \rightarrow$ involves $\hat{n}_i$

$$Z = \int \left(\prod_i \mathcal{D}\vec{n}_i\right) e^{-S[\vec{n}_i]}$$

$$S = \underbrace{\int d^2x \frac{1}{2g^2} (\partial_\mu \hat{n})^2}_{\text{NL}\sigma\text{M}} + \underbrace{i\theta W[\vec{n}]}_{\theta \text{ term}} \qquad \begin{cases} \theta = 2\pi S \\ W = \int d^2x \frac{1}{4\pi} \hat{n} \cdot (\partial_t \hat{n} \times \partial_x \hat{n}) \end{cases}$$

Haldane conjecture:

H is $\begin{cases} \text{gapped} \\ \text{gapless} \end{cases}$ if $S \in \begin{cases} \mathbb{Z} \\ \mathbb{Z} + \frac{1}{2} \end{cases}$

$\begin{cases} \text{If } S \in \mathbb{Z}, \; Z = \int \mathcal{D}\hat{n}\; e^{-S_{\text{NL}\sigma\text{M}}} \longrightarrow \text{gapped} \\ \text{If } S \in \mathbb{Z} + \frac{1}{2}, \; \begin{cases} ① \; S = \frac{1}{2}, \text{ Bethe ansatz} \\ ② \; S \in \mathbb{Z} + \frac{1}{2}, \text{ Lieb-Shultz-Mattis thm} \\ ③ \; \int \mathcal{D}\hat{n}\, (-1)^{W[\vec{n}]} e^{-S_{\text{NL}\sigma\text{M}}} \end{cases} \end{cases}$

Derivation of θ term.

① Path int for a single spin

$$\hat{\vec{S}} = \frac{1}{2} \hat{\vec{\sigma}}$$

Consider $\hat{H} = -\vec{n} \cdot \vec{S} = -\frac{1}{2} \vec{n} \cdot \hat{\vec{\sigma}}$

$$\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

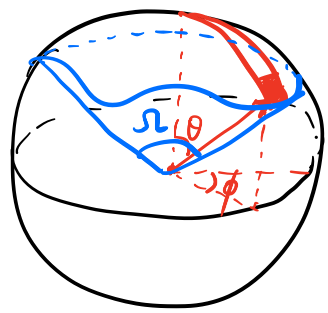
$E = \pm\frac{1}{2}$, ground state is

$$|\vec{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \Rightarrow \langle\vec{n}|\hat{\vec{\sigma}}_i|\vec{n}\rangle = n_i$$

For a time evolution $|\vec{n}(t)\rangle$, the Berry phase

$$\begin{aligned}
\gamma &= -i\int_0^T dt\ \langle\vec{n}(t)|\ \frac{d}{dt}\,|\vec{n}(t)\rangle \\
&= -i\int_0^T dt\ \left(\cos\frac{\theta(t)}{2}, e^{-i\phi(t)}\sin\frac{\theta(t)}{2}\right)\begin{pmatrix} \frac{d}{dt}\cos\frac{\theta(t)}{2} \\ \frac{d}{dt}\left[e^{i\phi(t)}\sin\frac{\theta(t)}{2}\right] \end{pmatrix} \\
&= -i\int_0^T dt\left[\cancel{-\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} + i\phi'\sin^2\frac{\theta(t)}{2} + \cancel{\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right] \\
&= \int_0^T dt\ \frac{1-\cos\theta(t)}{2}\ \frac{d\phi(t)}{dt} \\
&= \frac{1}{2}\int d\phi\,[1-\cos\theta] \\
&= \frac{1}{2}\int \sin\theta\ d\theta\, d\phi \\
&= \frac{1}{2}\ \Omega[\vec{n}(t)]
\end{aligned}$$

↳ solid angle of trajectory of $\vec{n}(t)$ on S^2.



Another SU(2) invariant form of Ω is

$$\Omega = \int_0^1 d\rho \int_0^T dt\ \ \vec{n}\cdot(\partial_t\vec{n}\times\partial_\rho\vec{n})$$

$$\text{with}\quad \vec{n}(t,\rho) = \begin{cases} (0,0,1), & \text{if } \rho=0 \\ \vec{n}(t), & \text{if } \rho=1 \end{cases}$$

WZ term for one spin

For general spin S:

$$\gamma = S\cdot\Omega[\vec{n}(t)]$$

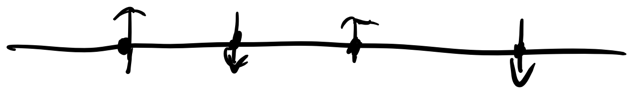
$$\gamma' = S\cdot(-\Omega') = -S(4\pi-\Omega) \underset{\substack{\uparrow \\ S\in\frac{1}{2}\mathbb{Z}}}{\overset{\text{mod } 2\pi}{=\!=\!=}} S\Omega = \gamma$$

single spin :

$$Z = \int D\vec{n}(t)\; e^{iS\,\Omega[\vec{n}(t)]} + \cdots$$

$\uparrow$ Wess-Zumino term for (0+1)D

② AF spin chain :



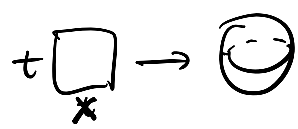
$$Z = \int \left(\prod_i D\vec{n}_i(t)\right) e^{iS \sum_i \Omega[\vec{n}_i(t)]} + \cdots$$

$\vec{n}_i(t) = (-1)^i \vec{m}_i(t)$, s.t. $\vec{m}(x,t) := \vec{m}_i(t)$ is smooth

$$\begin{aligned}
S \sum_i \Omega[\vec{n}_i(t)] &= S \sum_i \Omega[(-1)^i m_i(t)] \\
&= S \sum_i (-1)^i\, \Omega[m_i(t)] \\
&= S \sum_k \left(\Omega[m_{2k}(t)] - \Omega[m_{2k-1}(t)]\right) \\
&= \frac{S}{2}\int_0^L dx\, \frac{\partial}{\partial x}\Omega[m(x,t)] \\
&= \frac{S}{2}\int_0^L dx\, \frac{\delta\Omega}{\delta\vec{m}}\cdot\frac{\partial\vec{m}}{\partial x} \\
&= \frac{S}{2}\int dt\,dx\, (\vec{m}\times\partial_t\vec{m})\cdot\partial_x\vec{m} \\
&= 2\pi S\, \underbrace{\frac{1}{4\pi}\int dt\,dx\, \vec{m}\cdot(\partial_t\vec{m}\times\partial_x\vec{m})}_{\in W}
\end{aligned}$$

winding number $(T^2 \to S^2)$

$$= 2\pi W$$



$$Z = \int D\vec{n}(x,t)\; e^{i2\pi S\, W[\vec{n}(x,t)]} + \cdots$$

$\downarrow$

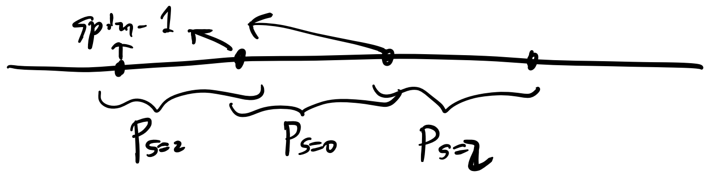
θ term for (1+1)D.

8.2. AKLT model

AKLT = Affleck, Lieb, Kennedy, Tasaki (1987)

- Exactly solvable model for Haldane phase.

$$H_{AKLT} = \sum_i P_{s=2}(\vec{S}_i + \vec{S}_{i+1})$$



spin: $1 \otimes 1 = 0 \oplus 1 \oplus 2$ $\qquad \vec{S}^2 = s(s+1)$

$$(\vec{S}_i + \vec{S}_{i+1})^2 = \vec{S}_i^{\,2} + \vec{S}_{i+1}^{\,2} + 2\vec{S}_i \cdot \vec{S}_{i+1}$$

$$= 1\times(1+1) + 1\times(1+1) + 2\vec{S}_i \cdot \vec{S}_{i+1}$$

$$= 4 + 2\vec{S}_i \cdot \vec{S}_{i+1}$$

$$(\vec{S}_i + \vec{S}_{i+1})^2 = S_{tot}(S_{tot}+1) = \begin{cases} 0, & S_{tot}=0 \\ 2, & S_{tot}=1 \\ 6, & S_{tot}=2 \end{cases}$$

$$P_{s=2} = \frac{1}{24}(\vec{S}_i + \vec{S}_{i+1})^2 \cdot \left[(\vec{S}_i + \vec{S}_{i+1})^2 - 2\right] = \begin{cases} 0, & \text{if } S_{tot}=0 \\ 0, & \text{if } S_{tot}=1 \\ 1, & \text{if } S_{tot}=2 \end{cases}$$

$$= \frac{1}{24}(4 + 2\vec{S}_i \cdot \vec{S}_{i+1})(4 + 2\vec{S}_i \vec{S}_{i+1} - 2)$$

$$= \underbrace{\frac{1}{2}\vec{S}_i \cdot \vec{S}_j}_{\text{Heisenberg}} + \underbrace{\frac{1}{6}(\vec{S}_i \cdot \vec{S}_j)^2 + \frac{1}{3}}_{\text{new terms}}$$

$$H = \sum_{\langle ij \rangle}\left[\cos\theta\, \vec{S}_i \cdot \vec{S}_j + \sin\theta\, (\vec{S}_i \cdot \vec{S}_j)^2\right]$$

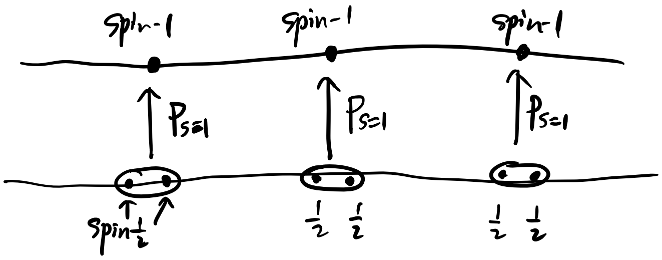
- AKLT state.

$H = \sum_{\langle ij \rangle} P_{s=2}(\vec{S}_i + \vec{S}_j)$ consists of projectors.

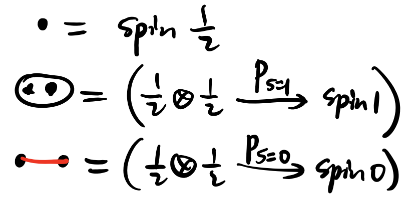
But $\left[P_{s=2}(\vec{S}_{i-1} + \vec{S}_i),\ P_{s=2}(\vec{S}_i + \vec{S}_{i+1})\right] \neq 0$

There is a state $|AKLT\rangle$, s.t. $P_{s=2}(\vec{S}_i + \vec{S}_{i+1})|AKLT\rangle = 0, \forall i$.

spin-$\frac{1}{2}$ representation:



$|AKLT\rangle =$ 

where 

$$\underbrace{\tfrac{1}{2}\otimes\left(\tfrac{1}{2}\otimes\tfrac{1}{2}\right)\otimes\tfrac{1}{2}}_{\tfrac{1}{2}\otimes 0\otimes\tfrac{1}{2}\;=\;\tfrac{1}{2}\otimes\tfrac{1}{2}\;=\;0\oplus 1}$$

$$\Rightarrow P_{S=2}(\vec{S}_i+\vec{S}_{i+1})\,|AKLT\rangle = 0.$$

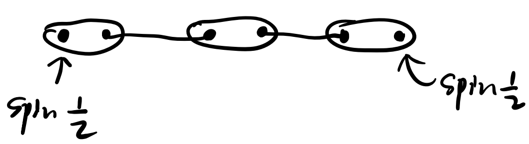
$\Rightarrow$ $|AKLT\rangle$ is a ground state of H_{AKLT}.

Properties:

(1) H_{AKLT} is gapped with GS $|AKLT\rangle$ (Haldane conjecture)

(2) H_{AKLT} and $|AKLT\rangle$ preserve $SO(3)_s$, time reversal, lattice inversion symmetries.

(3) $|AKLT\rangle$ has spin-$\frac{1}{2}$ edge state for open boundary condition.



GSD = 4

AKLT is an 1D SPT protected by $SO(3)_s$, time reversal, or lattice inversion symmetries.

- For general symm group G, we can construct AKLT-like state, s.t. the boundary is nontrivial preserving G.

projective rep:

$$U(g)\, U(h) = \omega_2(g,h)\; U(gh)\;, \quad \omega_2(g,h) \in U(1)$$

$$(g\cdot h)\cdot k = g\cdot(hk)$$

$$\Rightarrow [U(g)\cdot U(h)]\cdot U(k) = U(g)\cdot[U(h)\cdot U(k)]$$

$$\Rightarrow \omega_2(g,h)\, U(gh)\, U(k) = U(g)\cdot \omega_2(h,k)\, U(hk)$$

$$\Rightarrow \omega_2(g,h)\,\omega_2(gh,k)\, U(ghk) = \omega_2(h,k)\,\omega_2(g,hk)\; U(ghk)$$

$$\Rightarrow (d_2\omega_2)(g,h,k) = \frac{\omega_2(h,k)\;\omega_2(g,hk)}{\omega_2(gh,k)\;\omega_2(g,h)} = 1$$

ω_2 is called 2-cocycle of G.

If we do basis transf. $U'(g) := \mu_1(g)\, U(g)$, for $\mu_1 : G \to U(1)$.

$$\left.\begin{aligned} U'(g)\, U'(h) &= \omega_2'(g,h)\, U'(gh) \\ U(g)\; U(h) &= \omega_2(g,h)\, U(gh) \end{aligned}\right\}$$

$$\Rightarrow \omega_2'(g,h) = \omega_2(g,h)\cdot \underbrace{\frac{\mu_1(h)\,\mu_1(g)}{\mu_1(gh)}}_{=:\,(d_1\mu_1)(g,h)}$$

$$\Rightarrow \omega_2' = \omega_2\cdot(d_1\mu_1)$$

$\Rightarrow$ ω_2 and ω_2' give equivalent proj. rep.

Def. $H^2(G,U(1)) = \dfrac{Z^2(G,U(1))}{B^2(G,U(1))}$ 2nd group cohomology of G, where

$$Z^2(G,U(1)) := \{\omega_2 : G^{\times 2}\to U(1) \mid d_2\omega_2 = 1\} \;\to \text{2 cocycle}$$

$$B^2(G,U(1)) := \{d_1\mu_1 \mid \mu_1 : G\to U(1)\} \;\to \text{2 coboundary}$$

$H^2(G, U(1))$ is an Abelian group:

$$\left.\begin{array}{l} U(g)\,U(h) = \omega_2(g,h)\,U(gh) \\ U'(g) \quad \cdots \end{array}\right\}$$

$$\Rightarrow \left[U(g)\otimes U'(g)\right]\left[U(h)\otimes U'(h)\right] = \underbrace{\omega_2(g,h)\,\omega_2'(g,h)}\left[U(gh)\otimes U'(gh)\right]$$

Nontrivial proj. rep must be higher dim.

$$\left.\begin{array}{l} U(g)\,U(h) = \omega_2(g,h)\,U(gh) \\ U(g) \in U(1) \end{array}\right\} \Rightarrow \omega_2(g,h) = \frac{U(g)\,U(h)}{U(gh)} \in B^2(G, U(1))$$

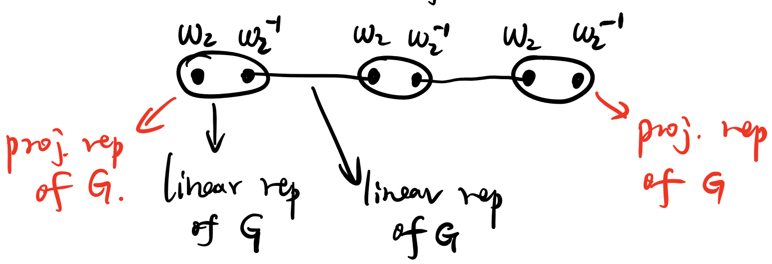
$$\Rightarrow [\omega_2] = [0].$$

Spin $\frac{1}{2}$: $\quad SO(3)$: $\quad \omega_2(\pi, \pi) = -1$

$\mathbb{Z}_2^T$: $\quad \omega_2(T, T) = -1$

projective reps.

AKLT-like state for G:



1D G-SPT is classified (at least) by $H^2(G, U(1))$.

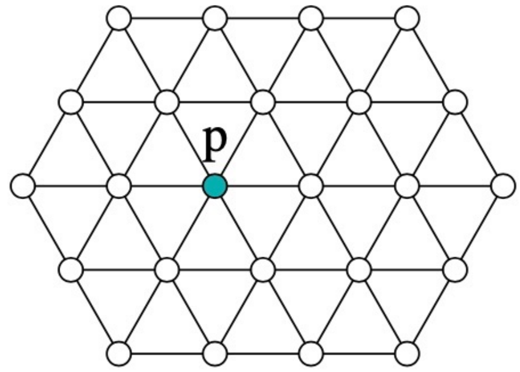
$\cdots$ (at most) $\cdots$

$\Rightarrow$ 1D G-SPT is classified by $H^2(G, U(1))$.

8.3. Levin-Gu model (2012)

2D SPT protected by $G = \mathbb{Z}_2$.

- Ising paramagnet.



$$H_0 = -\sum_p \sigma_p^x$$

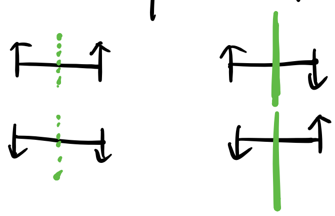
$$|\Psi_0\rangle = \bigotimes_p |\sigma_p^x = 1\rangle$$

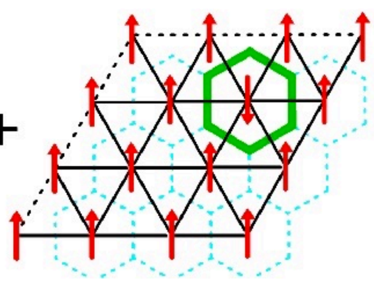
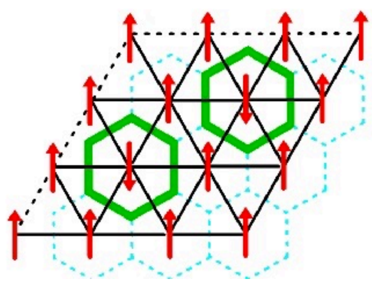
$$= \bigotimes_p \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_p + |\downarrow\rangle_p\right)$$

$$\propto \sum_{\{\sigma_p^z = \pm 1\}} |\{\sigma_p^z\}\rangle$$

$$\propto \sum_{\substack{\text{DW conf.} \\ (\text{even}, \text{even})}} |\text{DW conf.}\rangle$$

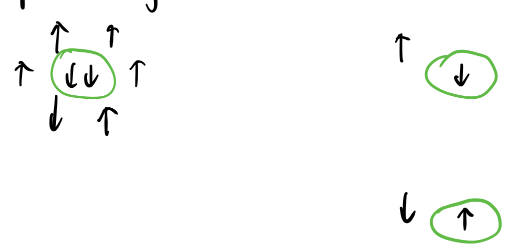
Domain wall picture:



$|\Psi_0\rangle = +$  $+$  $+ \cdots$

On the plane:

spin conf. $\xrightarrow{2:1}$ DW conf.

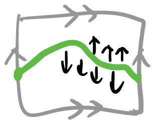


On the torus:

spin conf. $\xrightarrow{2:1}$ DW conf. (even, even)

(odd, even)

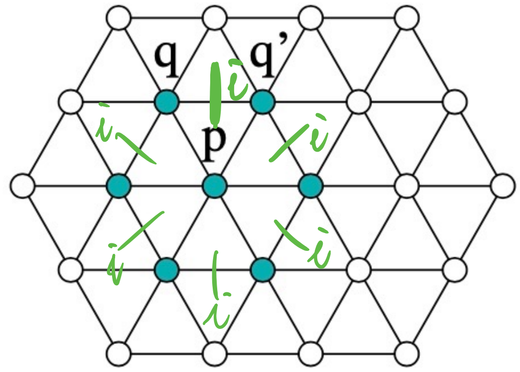
Spin conf. $\longrightarrow$ DW conf. (even, odd) (odd, odd)

? $\longmapsto$ 

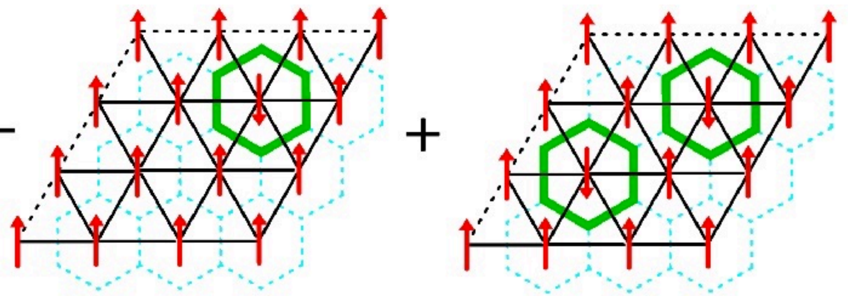
- Levin-Gu state.

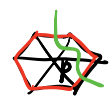
$$H_1 = -\sum_p B_p$$

$$B_p = -\sigma_p^x \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}}$$



$$|\Psi_1\rangle = \sum_{\substack{\{\text{DW conf}\} \\ (\text{even, even})}} (-1)^{\#(\text{DW})} \, |\text{DW conf}\rangle$$

$|\Psi_1\rangle = -$  $+ \dots$

$$-\prod_{\langle pqq' \rangle} \underbrace{i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}}}_{} = -\, i^{\#(\text{DW}) \text{ crossing boundary of}}$$ 

$\sigma_q^z \sigma_{q'}^z = \pm 1$

$$\begin{cases} 1, & \text{if } \sigma_q^z \sigma_{q'}^z = +1 \Leftrightarrow \text{No DW for } \langle qq' \rangle \\ i, & \text{if } \sigma_q^z \sigma_{q'}^z = -1 \Leftrightarrow \text{DW for } \langle qq' \rangle \end{cases}$$