

Topological Quantum Matters (TQM)

Time:

Every Mon. & Wed. 13:30-15:05, from 2021-9-13 to 12-3 (weeks 1-12 of the fall semester in Tsinghua)

Venue:

It will be a combination of offline (宁斋W11, Ning Zhai W11) and online (腾讯会议tencent meeting: 5772849861, password: 654321)

Description:

In this course, we will use topology to understand some exotic quantum phases of matter. The topics will include topological insulators, topological orders, symmetry-protected topological phases, etc. The course will cover both condensed matter models in physics and general mathematical descriptions (such as group cohomology theory and modular tensor categories of knots).

Prerequisites:

Basic topology and quantum mechanics. We will try to make a compromise between mathematics and physics by introducing relevant concepts self-consistently, as there are audience from both sides.

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other courses :

- | | | |
|---|---|---|
| YMSC | { | Reshetikhin : Inv. of Knots and 3-mfd. |
| Haghighat : Topological Quantum Computation. | | |
| Zheng : Category theory | | |
| BIMSA | { | Papadimitriou : Quantum field theory anomalies and applications |
| Palcoux : On fusion categories | | |
| Yilong Wang : Modular categories and Reshetikhin-Turaev TQFTs | | |

Introduction to TQM.

(1) Why important to real life?

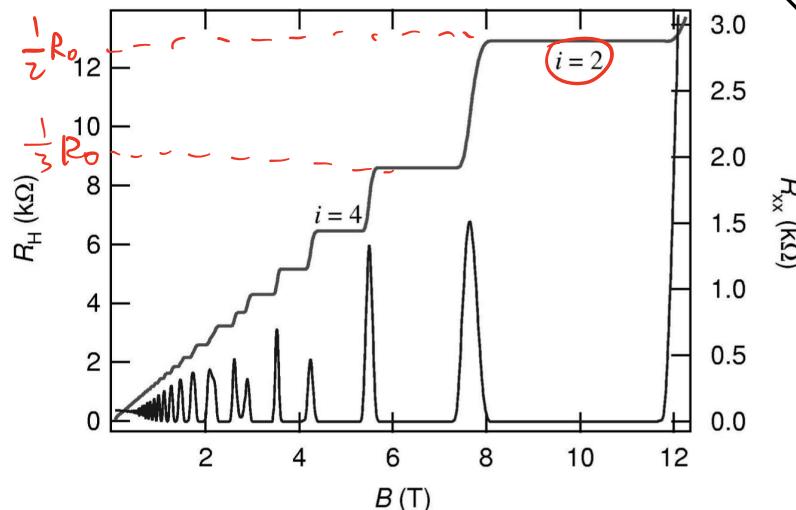
One example:

Q: how to measure $\left\{ \begin{array}{l} \text{electric charge } e \\ \text{Planck constant } h \end{array} \right.$ accurately?

① \checkmark Integer quantum Hall effect

Nobel 1985

$$R_0 = \frac{h}{e^2}$$

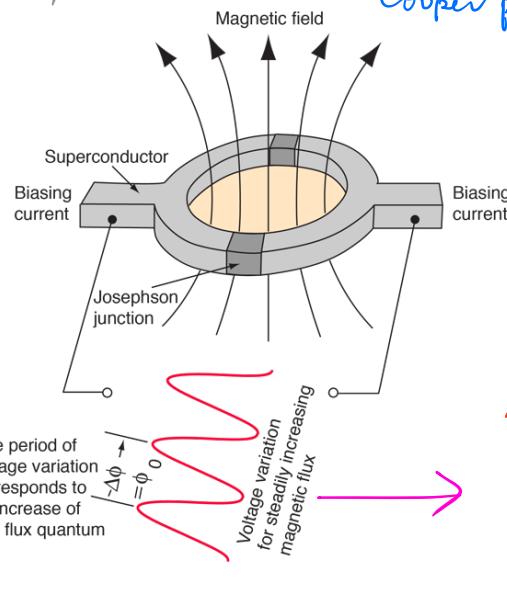


② Josephson effect

Nobel 1973

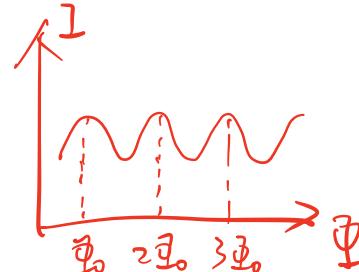
$$\Phi_0 = \frac{h}{2e}$$

Cooper pair



International System of Units (before 2018)

[After 2018, e, h are fixed by definition]



using many-body system to measure properties of a single electron!
accurately

(2) Why exotic?

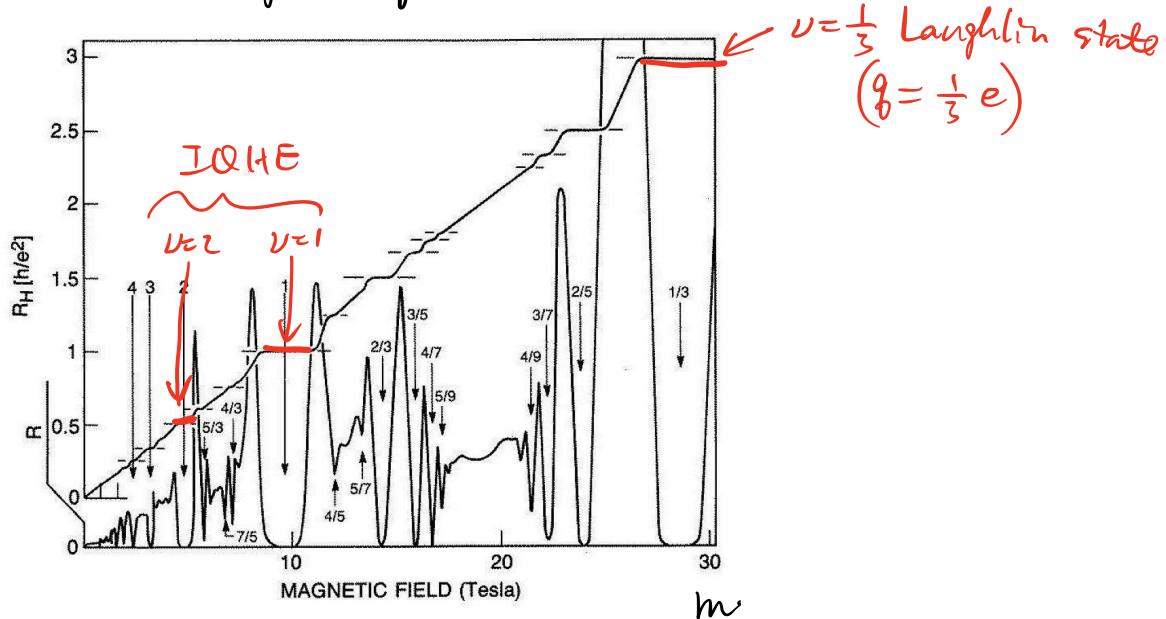
fundamental charge e = charge of one single electron/proton/...

Fractional quantum Hall effect (FQHE) Nobel 1998



$$R = \frac{h}{ne^2} \quad \left\{ \begin{array}{ll} \nu \in \mathbb{Z} : & \text{integer QHE} \\ \nu \in \mathbb{Q} : & \text{FQHE} \end{array} \right.$$

Fractional charge $q = \nu \cdot e$



Fractional charge appears in many-body systems!

(3) Why interesting to many people?

related to many other topics:

{ math: knot theory, Jones polynomial, Turaev-Viro inv,
 Reshetikhin-Turaev inv., modular tensor categories,
 2-knot, higher categories ...
 lecture → Zheng's lecture

phys.: IQHE, FQHE, topological insulator, ...
 (real materials)

quantum information: topological quantum computation
 (by Freedman, Kitaev)

fault-tolerant quantum computation

(4) Why hard / deep ?

∞ number of degrees of freedom

∞ dimensional space .

foundations of quantum field theory (= many-body quantum mechanics)

emergence of spacetime/gravity (?)

Syllabus (tentative):

It may vary depending on the actual speed of the course. By weeks (4*45min/week, 12 weeks):

(1) introduction to TQM, different classes of TQM (bosonic/fermionic, long/short range entangled, with/without symmetry),
1st example: Kitaev's toric code model (homology enters),

2nd example: Haldane's honeycomb model (homotopy enters) Qikun Xue : quantum anomalous Hall effect.
1988 2013

Part I. Bosonic topological orders

(2) quantum double model, twisted quantum double model = Dijkgraaf-Witten gauge theory

(3) introduction to fusion categories, Levin-Wen model = Turaev-Viro model

(4) introduction to modular tensor categories, general description of anyon models by Kitaev

(5) 3+1D Walker-Wang model = Crane-Yetter model

Part II. Topological insulators (fermionic symmetry-protected topological phases without interactions)

(6) introduction to band theory, integer quantum Hall effect, Thouless–Kohmoto–Nightingale–den Nijs number, Chern insulator

TKNN : Hall conductance = Chern number

(7) examples: Kitaev's Majorana chain, Su-Schrieffer-Heeger model, 2+1D and 3+1D topological insulators, edge theories

(8) symmetries in free fermion system, 10-fold way classification → topological K theory, Clifford algebra

Part III. Symmetry-protected topological phases

(9) introduction to symmetry-protected topological (SPT) phases, Haldane chain

(10) introduction to projective representation, tensor product state, classification of 1+1D bosonic SPT

(11) Levin-Gu model, introduction to group cohomology, bosonic SPT model from group cohomology

(12) introduction to fermionic SPT phases, other related topics

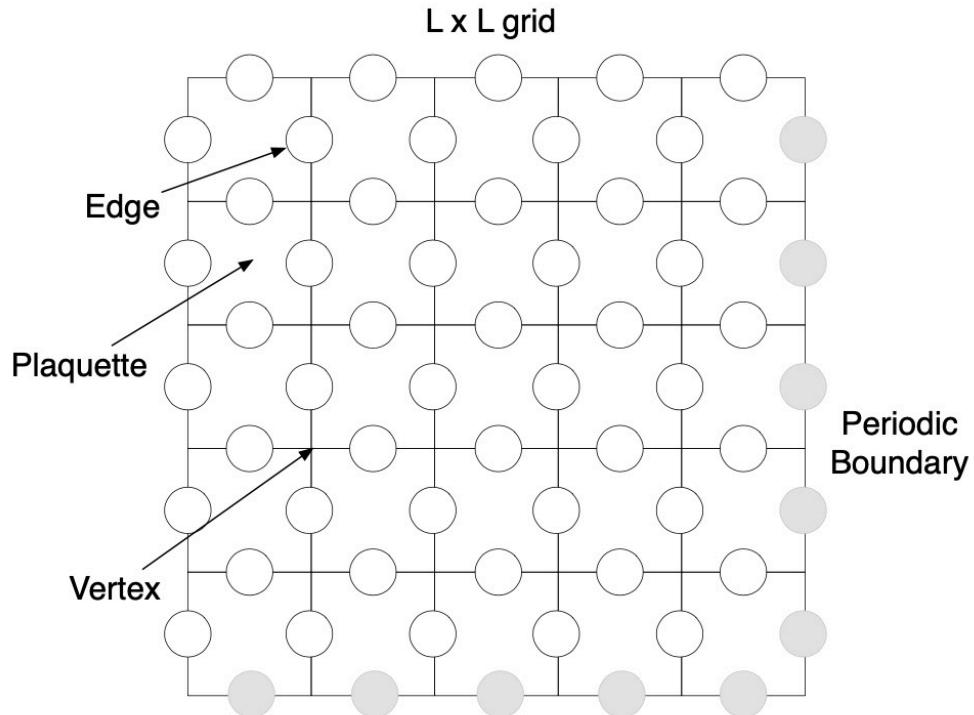
1st example of TQI:

Toric code model (= \mathbb{Z}_2 gauge theory)
(by Kitaev 1997) on torus

Hilbert space on torus

$L \times L$ square lattice, \mathbb{C}^2 (spin 1/2) on each link

$$\mathcal{H} = \bigotimes_{\text{link } j} \mathbb{C}^2$$



\mathcal{H} : Hilbert space

Hamiltonian $H : \mathcal{H} \rightarrow \mathcal{H}$

eigenvalues of H are energies ϵ_n of the system

Ground state is the eigenvector with the lowest energy ϵ_0 .

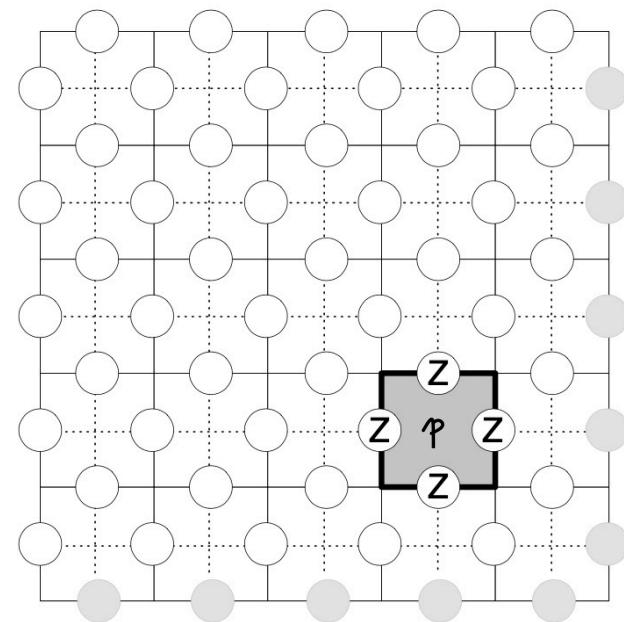
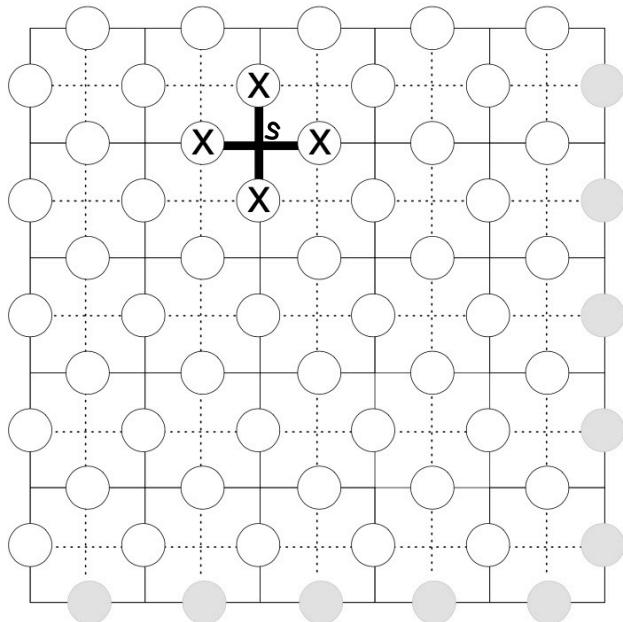
- Claim {
- The ground state of TC represents $H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$
 - robust ground states
 - topological quantum information

Hamiltonian

$$H = - \sum A_s - \sum B_p$$

sites

plaquettes



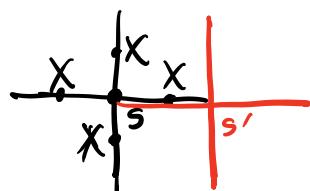
$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x \quad \text{Pauli operator}$$

$X = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

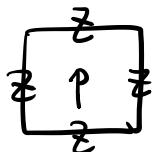
$$B_p = \prod_{j \in \partial p} \sigma_j^z$$

$Z = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^6$

Algebraic relations of A_s , B_p .

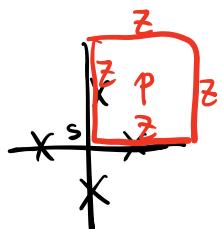


$$\left\{ \begin{array}{l} A_s^+ = A_s \\ (A_s)^2 = 1 \\ [A_s, A_{s'}] = A_s A_{s'} - A_{s'} A_s = 0 \end{array} \right. \quad \rightsquigarrow A_s \text{ has eigenvalues } \pm 1$$



$$\left\{ \begin{array}{l} B_p^+ = B_p \\ (B_p)^2 = 1 \\ [B_p, B_{p'}] = 0 \end{array} \right. \quad \rightsquigarrow B_p \text{ --- } \pm 1.$$

$$[A_s, B_p] = A_s B_p - B_p A_s = 0.$$



$$XZ = -ZX \Rightarrow A_s B_p = B_p A_s$$

$$H = - \sum_s A_s - \sum_p B_p$$

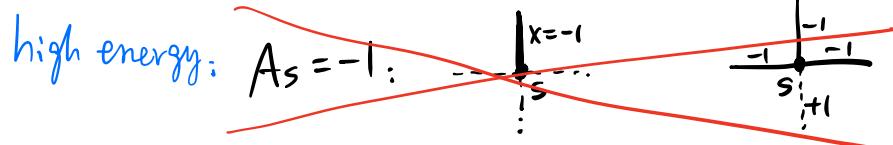
Ground state $| \Psi \rangle$ of TC:

$$A_s | \Psi \rangle = B_p | \Psi \rangle = +1 | \Psi \rangle, \forall s, p.$$

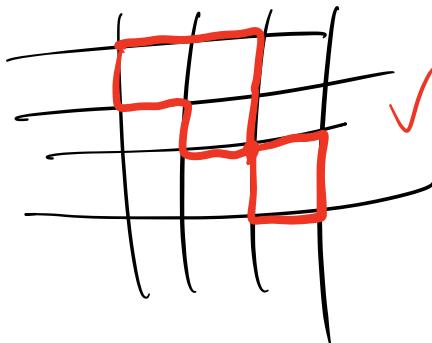
String representation. $A_s = \pi \sigma^x$. $\sigma^x: \pm 1$

$$\text{def: } \left\{ \begin{array}{ll} \sigma_j^x = -1 & : \text{string} \\ \sigma_j^x = +1 & : \text{no string} \end{array} \right. \quad \begin{array}{c} \text{---} \\ \text{.....} \end{array}$$

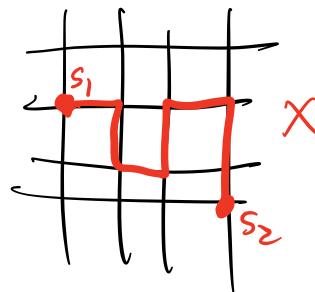
Low energy: $A_s = 1: \quad \begin{array}{c} x=+1 \\ \overline{x}=1 \\ x=1 \end{array} \quad \begin{array}{c} -1 \\ +1 \end{array} \quad \begin{array}{c} -1 \\ -1 \end{array} \quad \begin{array}{c} -1 \\ -1 \end{array} \quad \checkmark \end{array}$



ground state: $A_s |\Psi\rangle = |-\Psi\rangle \iff$ closed strings (loops)

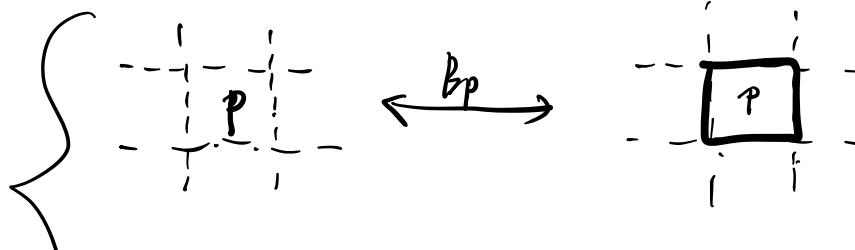


$$A_s = +1 \\ (\text{S})$$

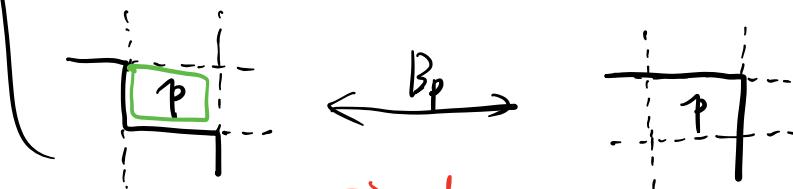


$$A_{s_1} = A_{s_2} = -1 \\ (\text{high energy})$$

$$B_p = \pi \mathbb{Z}: \quad \text{flip } \sigma^x = +1 \leftrightarrow \sigma^x = -1$$



create
and/or
small loops.



change shapes
of closed loops.

$$B_p |c\rangle = |c + \partial p\rangle \mod \mathbb{Z}$$

\forall closed loop configuration

$$H = -\sum A_s - \sum B_p$$

\downarrow \downarrow

closed loop. change shapes.

Assume $|\Psi\rangle = \sum_c a_c |c\rangle$, $a_c \in \mathbb{C}$ is the ground state

$$\Rightarrow B_p |\Psi\rangle = \sum_c a_c |c + \partial p\rangle = \sum_c a_{c+\partial p} |c\rangle$$

$$= |\Psi\rangle = \sum_c a_c |c\rangle$$

$$\Rightarrow a_c = a_{c+\partial p} \text{ for } \forall p$$

The ground state coefficients of $|c\rangle$ and $|c+\partial_p\rangle$ should be the same.
or Loops that are homologous to each other have the same coefficient.

- Summary
- $A_S = +1$: enforce closed-loop constraints
conf. $|c\rangle$ such that there are only closed loops in c .
 - $B_p = +1$: $|c\rangle$ and $|c + \partial_p \pmod{2}$ have the same coefficient.
mod out homologous loops.

Ground state subspace of TC

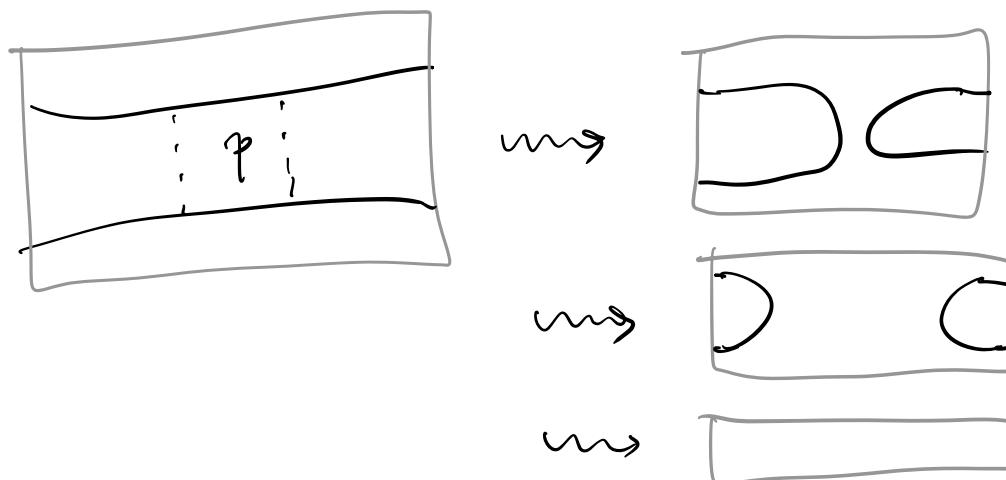
$$= \{\text{closed loops}\} / \{\text{boundaries of 2D surfaces on torus}\}$$

$$= H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$|\Psi\rangle = |\rightarrow\rangle + |0^\circ\rangle + |\text{O}\text{O}\rangle + |\leftarrow\rangle + \dots$$

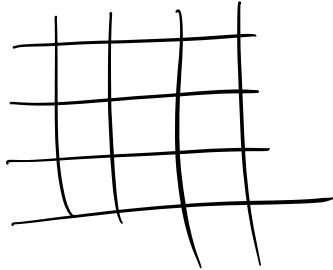
4-fold degeneracy of TC on torus:

$$\begin{cases} |\Psi_{00}\rangle = |\rightarrow\rangle + |0^\circ\rangle + |00\rangle + \dots \\ |\Psi_{10}\rangle = |\leftarrow\rangle + |\frac{0^\circ}{\circ}\rangle + |\text{O}\text{O}\rangle + \dots \\ |\Psi_{01}\rangle = |\text{O}\text{I}\rangle + |\text{O}\text{I}\text{O}\rangle + |\text{O}\text{I}^\circ\rangle + \dots \\ |\Psi_{11}\rangle = |\text{I}\text{O}\rangle + |\text{I}\text{O}\rangle + |\text{O}\text{O}\rangle + \dots \end{cases}$$



Last time:

TC:



torus
square lattice

$$H = - \sum_{\text{sites } s} A_s - \sum_{\text{plaquette } p} B_p$$

$$A_s = \begin{array}{c} x \\ \bullet \\ x \end{array}$$

$$B_p = \begin{array}{c} z \\ | \\ p \\ z \end{array}$$

$$[A_s, B_p] = 0$$

$$(A_s)^2 = 1$$

$$\left(\frac{1+A_s}{2}\right)^2 = \underbrace{\frac{1+A_s}{2}}$$

projector
to $A_s=1$ states,

commuting - projector Hamiltonian.

$\left\{ \begin{array}{l} A_s = +1 \text{ for every site } s \text{ of the torus.} \\ \hookrightarrow \text{closed } \underline{\text{string}} \text{ property of the GS.} \\ (\sigma_j^x = -1 \text{ for link } j) \end{array} \right.$

$$B_p = +1 :$$

$$B_p \left| \text{loop } p \right\rangle = \left| \text{loop } p \text{ with red square} \right\rangle = \left| \text{loop } p \right\rangle$$

GS of TC: $A_s |\Psi\rangle = B_p |\Psi\rangle = |\Psi\rangle, \forall s, p.$

$$|\Psi\rangle = \sum_c a_c |c\rangle, a_c \in \mathbb{C}$$

$$\begin{cases} A_s |\Psi\rangle = |\Psi\rangle \Rightarrow c \text{ is closed loop conf.} \\ B_p |\Psi\rangle = |\Psi\rangle \Rightarrow B_p \sum_c a_c |c\rangle = \sum_c a_c \underbrace{|c+p\rangle}_{(\text{mod } 2)} \\ = \sum_c a_{c+p} |c\rangle \\ = \sum_c a_c |c\rangle \end{cases}$$

$$\Rightarrow a_{c+p} = a_c, \forall p$$

$$\Rightarrow a: \mathcal{H} \Big|_{A_S=1} \rightarrow \mathbb{C}$$

$$c \mapsto a_c$$

is a function of \mathbb{Z}_2 homology class of c .

Different classes of TQMs:

① bosonic / fermionic

{ Microscopic degrees of freedom.

{ Not emergent excitations.

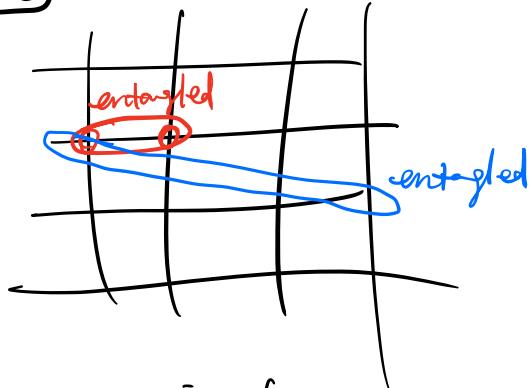
Example : bosonic Toric code has emergent fermions.

② long range entangled / short range entangled
(LRE) (SRE)

$$\text{entanglement } |\Psi\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$$

(in QM) ↓ ↓
 1^{st} 2^{nd} particle

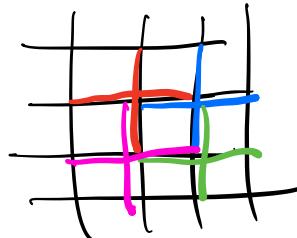
Add distance structure :



Example : T.C. is long range entangled.

$$A_S = \begin{array}{|c|c|} \hline * & * \\ \hline * & * \\ \hline \end{array}$$

$$\boxed{\prod_s A_s = 1.} \text{ on torus}$$



$$A_S = -1 \rightarrow A_i = -1 \text{ for } s' \in \bar{i}$$



measure LRE : topological entanglement entropy.

$$S = \frac{(\text{area law})}{L} + S_{\text{top.}} + \frac{\text{const.}}{L^0} + \dots + \frac{1}{L^{-1}}$$

③ with /without symmetries

Ising \mathbb{Z}_2 symm.

$SU(2)$, $U(1)$

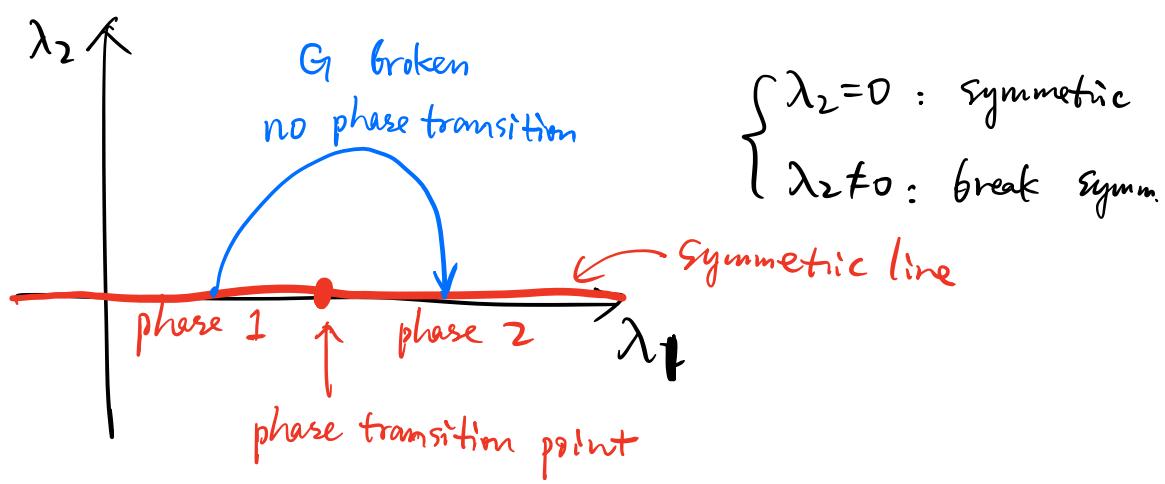
time reversal.

...

Given a symmetry group G , there may be many different topological phases. If we can break G , they may become the same phase.

G	3	4
phase 1		...
2		

phase diagram



① bosonic / fermion

	(2)	SRE	LRE
No symm.	invertible topo. order (iTO)	topo. order (TO)	
Symm.	Symmetry-protected topo. order (SPT)	Symmetry-enriched topo. order (SET)	

Statement: ① The collection of all topological phases forms an Abelian monoid (=group without inverse)
 ② The collection of all SRE phases has an Abelian group structure.



$$A * B = B * A$$

For
topo.
phases.
(monoid)

For
SPE
phases
(group)

phase B anyimb stacking operation

trivial phase
 0
= product state

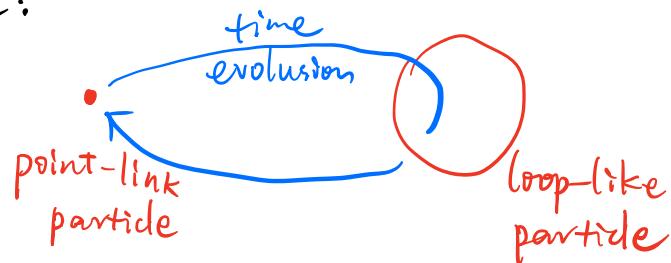
$\begin{matrix} 0 \\ A \end{matrix}$ = A identity
associativity obvious.

For
SRE
phases

$\begin{matrix} A \\ -A \end{matrix}$ } $\cong 0$

\exists inverse
for \mathcal{H} SRE phase

3+1D TC:



$$H = - \sum_s A_s - \sum_p B_p$$

$$A_s = \begin{array}{|c|c|} \hline \times & \times \\ \times & \times \\ \hline \end{array}$$

$$B_p = \boxed{\begin{array}{|c|c|} \hline z & z \\ z & z \\ \hline \end{array}}$$

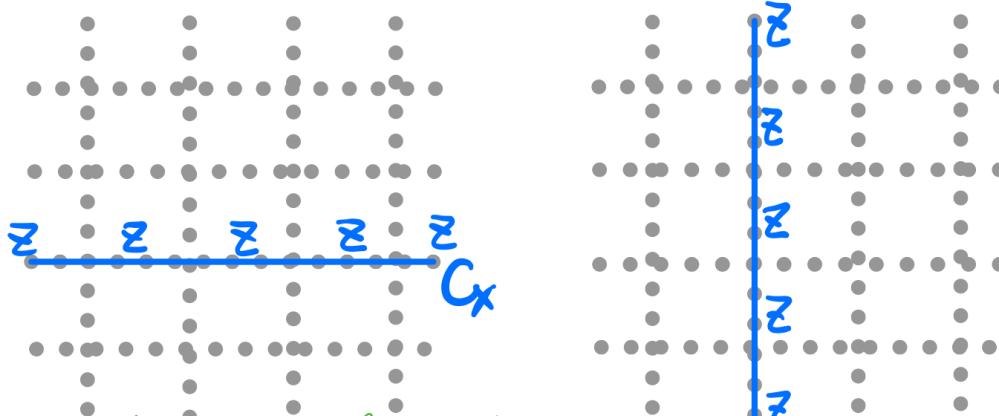
Wilson operators

U(1) gauge theory
 $W = e^{i \int_C A_\mu dx^\mu}$

For a loop C , define Wilson loop operator

$$W_C = \prod_{j \in C} \sigma_j^z, \text{ the ground states are related by}$$

$$W_{C_x}, W_{C_y}$$



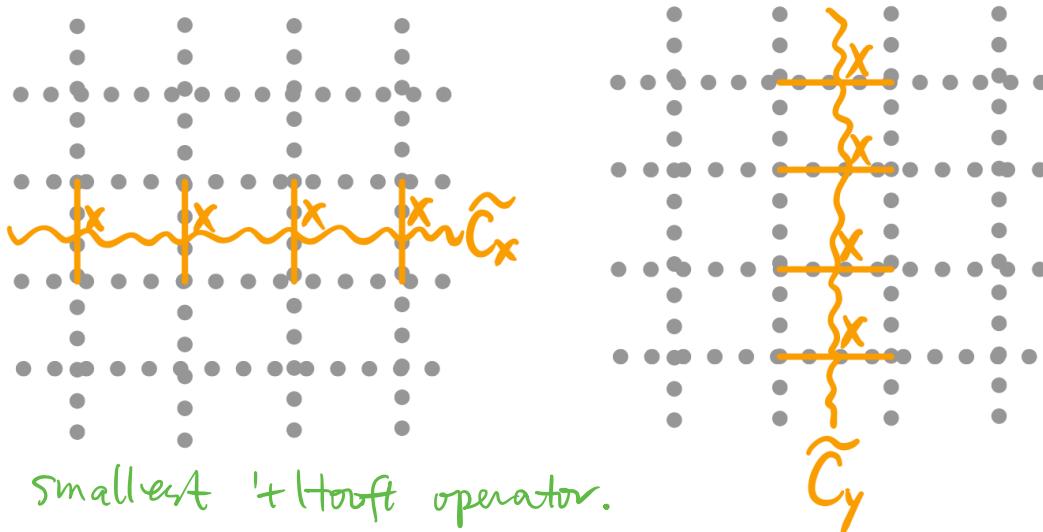
B_p is the smallest Wilson loop operator. C_y

't Hooft operators

For a dual loop C , define 't Hooft operator

$V_{\tilde{C}} = \prod_{j \perp \tilde{C}} \sigma_j^x$, the ground states are related by

$$V_{\tilde{C}_x}, V_{\tilde{C}_y}$$



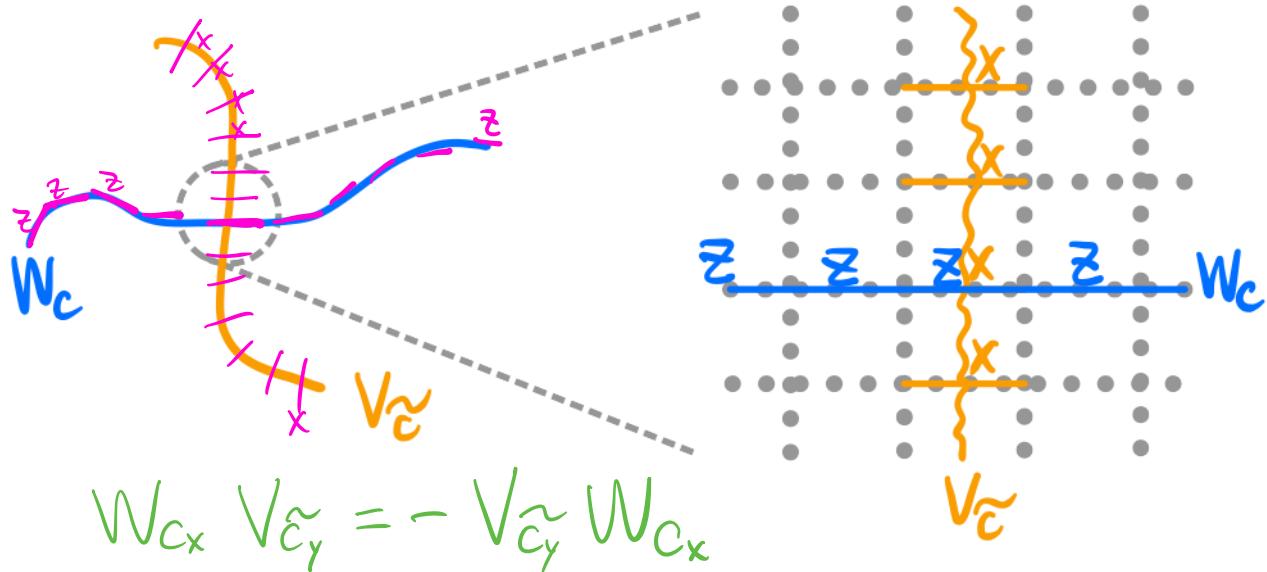
As is the smallest 't Hooft operator.

Heisenberg algebra from W and V

For a loop C and dual loop \tilde{C} , we have

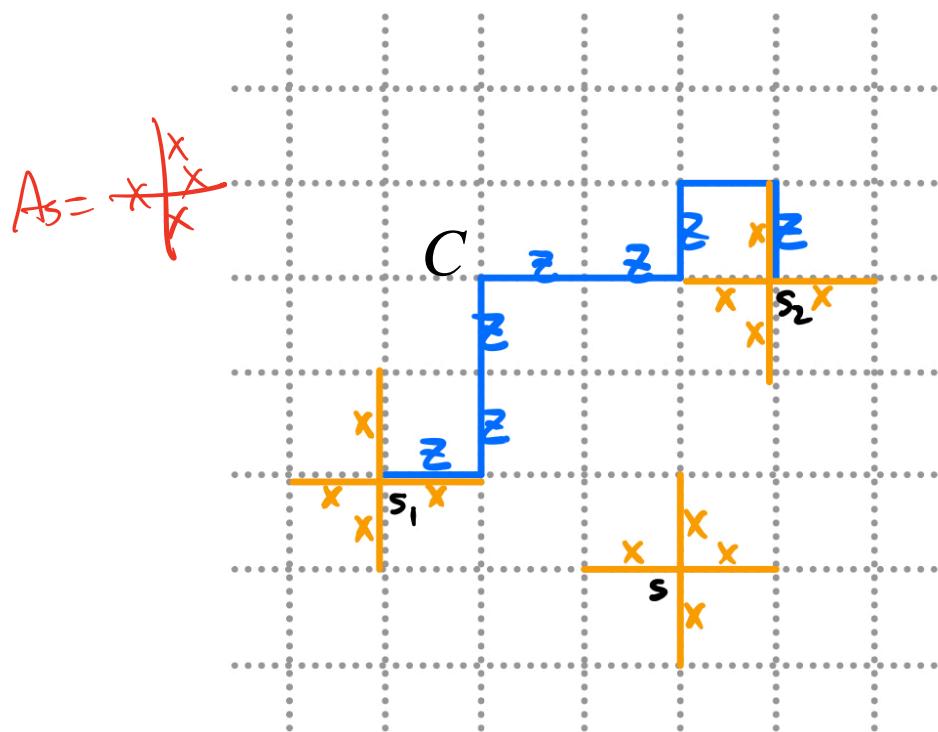
$$W_C V_{\tilde{C}} = (-1)^{\#C \cap \tilde{C}} V_{\tilde{C}} W_C$$

which has smallest representation $\dim 4$.



e excitations

Endpoints of Wilson operator W_C are electric charge excitations e .



$$A_s |GS\rangle = B_p |GS\rangle$$

$$|e_{s_1} e_{s_2}\rangle = W_C |GS\rangle$$

$$A_{s_2} W_C |GS\rangle = -W_C A_{s_2} |GS\rangle$$

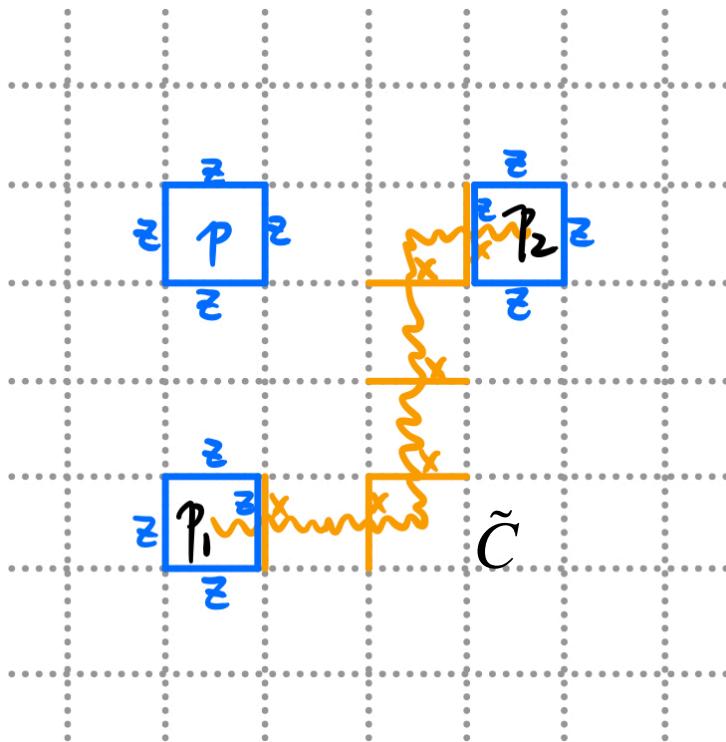
$$\hookrightarrow A_{s_1} = A_{s_2} = -1$$

$$A_{s \neq s_1, s_2} = 1$$

$$B_p = 1$$

m excitations

Endpoints of 't Hooft operator $V_{\tilde{C}}$ are magnetic charge excitations m .



$$|m_{p_1} m_{p_2}\rangle = V_{\tilde{C}} |\text{GS}\rangle$$

$$B_{p_1} = B_{p_2} = -1$$

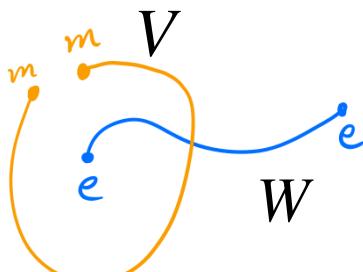
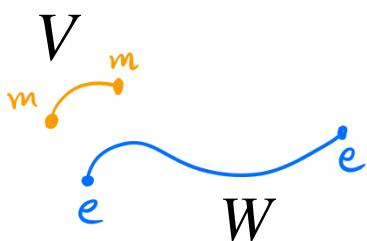
$$B_{p \neq p_1, p_2} = 1$$

$$A_s = 1$$

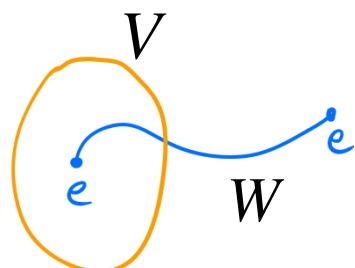
Statistics



- W_C and $V_{\tilde{C}}$ can be used to create / move e and m excitations
- Full braiding of e and m gives -1 sign



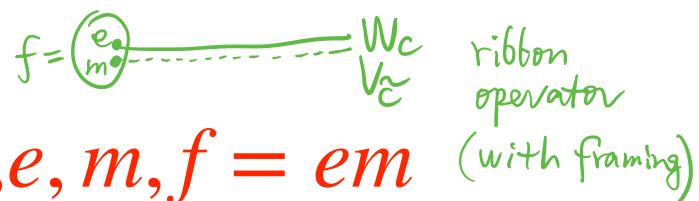
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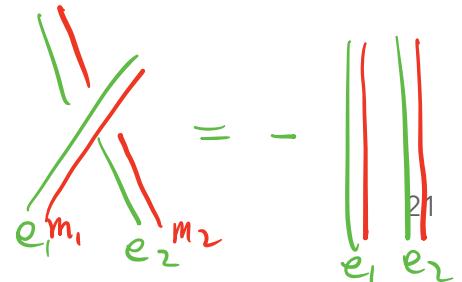
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Anyons

- In (2+1) dimensions, point-like excitations (**anyons**) can have any rational number statistical phases.
- Anyons in toric code: $1, e, m, f = em$ (with framing)
- Both e and m are bosons; f is fermion
- e, m, f have mutual braiding phase -1

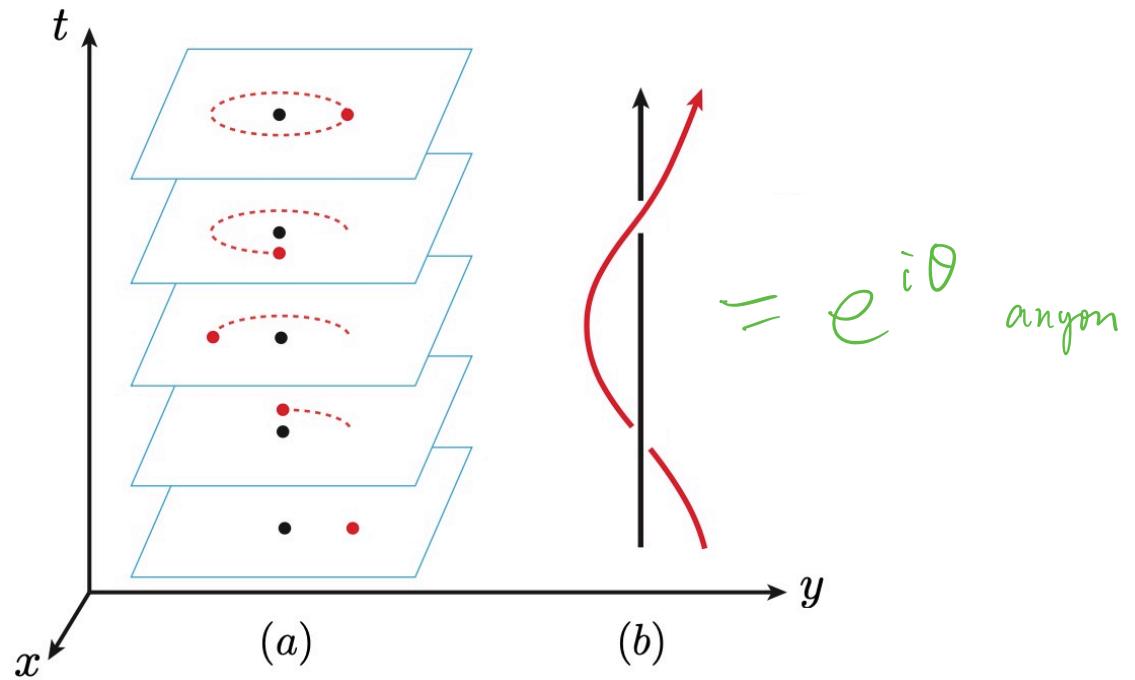


boson/fermion: $e_1 \xrightarrow{e_2} e_2 \Rightarrow \pm \dot{e}_2 \dot{e}_1$



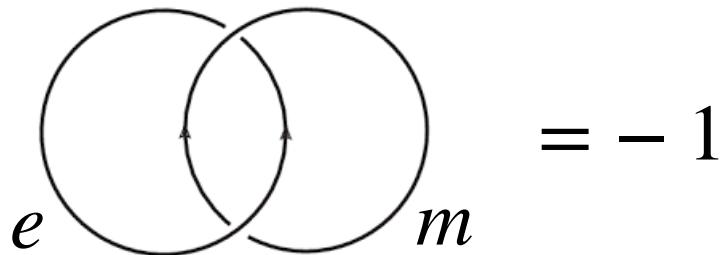
knot/link in (2+1)D spacetime

anyon worldlines in (2+1)D \longleftrightarrow knot/link in 3D



Hopf link

For the toric code model, mutual statistics of e and m :



In general, for a link of e and m worldlines, we have statistical phase $(-1)^{\text{linking \#}}$.

Anyon models -> knot/link invariants

Summary

- Toric code model is a commuting projector Hamiltonian on torus
- Ground state subspace $\cong H_1(T^2, \mathbb{Z}_2)$
- Anyons braiding statistics and knot/link invariants
- Generalizations of toric code model later...