

Quantum Double Models (QDM)

Kitaev, Annals of phys. 303, 2 (2003)

arXiv 1997

QDM : generalization of \mathbb{Z}_2 toric code

to lattice G gauge theory.

↳ finite (may be non-Abelian) group.

$$D(G) = G \times \hat{G} \text{ for Abelian } G.$$

↓ ↓
 flux charge
 (double)

2.1. The model based on a group algebra.

G .

$\mathcal{H} := \mathbb{C}[G] = \left\{ \sum_g a_g \cdot g \mid g \in G, a_g \in \mathbb{C} \right\}$ be the group algebra.

Hilbert space with orthonormal basis $|g\rangle$

$$\dim \mathcal{H} = |G|$$

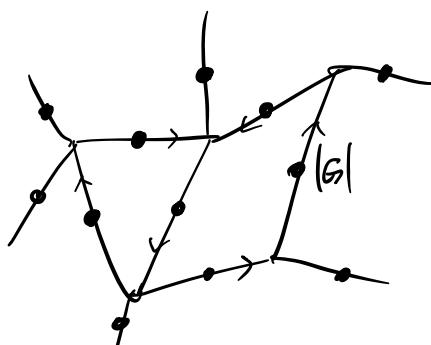
$$\text{Def: } L_+^g |z\rangle = |gz\rangle \quad T_+^h |z\rangle = \delta_{h,z} |z\rangle$$

$$L_-^g |z\rangle = |zg^{-1}\rangle \quad T_-^h |z\rangle = \delta_{h^{-1},z} |z\rangle$$

$$\text{Satisfy: } L_+^g T_+^h = T_+^{gh} L_+^g, \quad L_+^g L_-^h = T_-^{hg^{-1}} L_+^g$$

$$L_-^g T_+^h = T_+^{hg^{-1}} L_-^g, \quad L_-^g T_-^h = T_-^{gh} L_-^g$$

Hilbert space :



$|G|$ -dim vector space on each link of A 2D lattice.

$$\begin{array}{c} \nearrow \\ |g\rangle \end{array} = \begin{array}{c} \searrow \\ |g^{-1}\rangle \end{array}, \quad g \in G.$$

SDM : $H = - \sum_s A_s - \sum_p B_p$

$$A_s \left\{ \begin{array}{l} A_s^g \\ \text{---} \\ |g_1\rangle \quad |g_2\rangle \\ |g_3\rangle \quad |g_4\rangle \\ |g_5\rangle \end{array} \right. = \begin{array}{c} |gg_2\rangle \quad |gg_3\rangle \\ \text{---} \\ |g_1g^{-1}\rangle \quad |g_2g^{-1}\rangle \\ |g_5g^{-1}\rangle \end{array}$$

$$A_s^g := \prod_{\substack{\text{edge } l \\ \text{of } s}} L_\pm^g, \quad A_s := \frac{1}{|G|} \sum_{g \in G} A_s^g$$

↳ gauge transf.

$$B_p^h \left\{ \begin{array}{l} \text{---} \\ h_1 \quad h_2 \quad h_3 \quad h_4 \quad h_5 \quad h_6 \\ \text{---} \\ p \end{array} \right. = \underbrace{\delta_{h_1 h_2 \dots h_6, h}}_{\text{flux is } h}$$

$$B_p^h = \sum_{h_1 \dots h_6 = h} \prod_{m=1}^6 T_\pm^{h_m}, \quad B_p := B_p^{h=1}$$

↳ zero flux condition for p.

relation

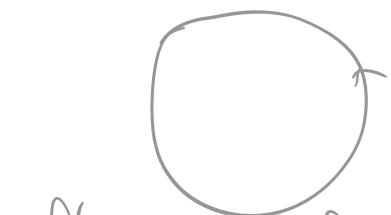
$$\left\{ \begin{array}{l} A_s^2 = A_s \\ B_p^2 = B_p \\ [A_s, A_{s'}] = 0 \\ [B_p, B_{p'}] = 0 \\ \boxed{[A_s, B_p] = 0} \end{array} \right\} \text{ projectors}$$

$$A_s \quad \begin{array}{c} \downarrow g_2 \\ \text{---} \\ s \end{array} \quad \begin{array}{c} \curvearrowright \varphi \\ \nearrow g_2 \cdot g_1 \\ \text{---} \end{array} = \begin{array}{c} \downarrow g_2 g^{-1} \\ \text{---} \\ s \end{array} \quad \begin{array}{c} \curvearrowright \varphi \\ (\varphi \cdot \varphi^{-1}) \cdot (g g_3) = g \cdot g_1 \end{array}$$

Ground state : $A_s | \Psi \rangle = B_p | \Psi \rangle = | \Psi \rangle$, Hs, p
 ↓
 gauge transf. ↓
 flat connection condition

$\Rightarrow | \Psi \rangle$: flat connections up to gauge equivalence.

Tc. $| \Psi \rangle = | \Psi \rangle + | \emptyset \rangle + \dots$
 $g = e^{iA} \in U(1)$



flat: $\oint_C A_\mu dx^\mu = 0$

$g_{ij} = e^{iA_{ij}}$ → continuous
 ↪ discrete

$\prod g_1 \dots g_N = 1 \Leftrightarrow B_p^{h=1}$

\vec{r}

$g = e^{iA}$
 $g^{-1} = e^{-iA}$
 $g = e^{iA}$
 $g = e^{-iA}$

Gauss law

$\nabla \cdot \vec{A} = \rho$

$A(\vec{r} + \hat{x}) - A(\vec{r})$

$+ A(\vec{r} + \hat{y}) - A(\vec{r})$



$g_{\vec{r} + \hat{x}} \quad g_{\vec{r}}^{-1} \quad g_{\vec{r} + \hat{y}} \quad g_{\vec{r}}^{-1}$

Important notice:

The courses of week 3 (on 2021-09-27 and 2021-09-29) will be cancelled. And the ending date of the course will be postponed to week 13 of the fall semester.

From 2021-10-04 on, we will use another Tencent Meeting Room [9952830954](#) (password 654321) of the university account, such that there are more storage space for the recordings.

Recall: QDM.

$$H = - \sum_s A_s - \sum_p B_p$$

$$[A_s, B_p] = 0$$

$$\textcircled{1} \quad B_p |\Psi\rangle = |\Psi\rangle, \quad \forall p \quad \rightarrow = \frac{h}{h^{-1}}$$

$$\Rightarrow \begin{array}{c} h_3 \\ \swarrow \quad \searrow \\ h_4 \quad h_5 \\ \downarrow \quad \uparrow \\ h_2 \quad h_1 \\ \text{flat connection condition} \\ \text{zero flux condition} \end{array} \quad h_1 h_2 \dots h_5 = 1 \in G.$$

(flat connection condition)
(zero flux condition)

$$\textcircled{2} \quad A_s |\Psi\rangle = |\Psi\rangle, \quad \forall s$$

$$A_s := \frac{1}{|G|} \sum_{g \in G} A_s^g \quad A_s^g: \text{gauge transformation.}$$

$$A_s^g = \begin{array}{c} h_3 \quad h_2 \\ \nearrow \quad \searrow \\ h_4 \quad h_5 \\ \uparrow \quad \downarrow \\ h_1 \end{array} = \begin{array}{c} gh_3 \quad gh_2 \\ \nearrow \quad \searrow \\ h_4 g^{-1} \quad h_5 g^{-1} \\ \uparrow \quad \downarrow \\ g h_1 \end{array} \quad (gh)^{-1} \cdot gh = h^{-1}h \quad h_5 g^{-1} \cdot gh_1 = h_5 h_1$$

A_s^g fluctuate $\{h_{ij}\}$ within the subspace $B_p=1, \forall p$.

$\Rightarrow Gs |\Psi\rangle$ is an equal weight superposition of all flat connections that are gauge equivalent.

continuous: U(1) gauge theory

$$A_\mu(r) \rightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

$$\Phi' = \oint_c A'_\mu dx^\mu = \oint_c (A_\mu - \partial_\mu \chi) dx^\mu = \oint_c A_\mu dx^\mu = \Phi$$

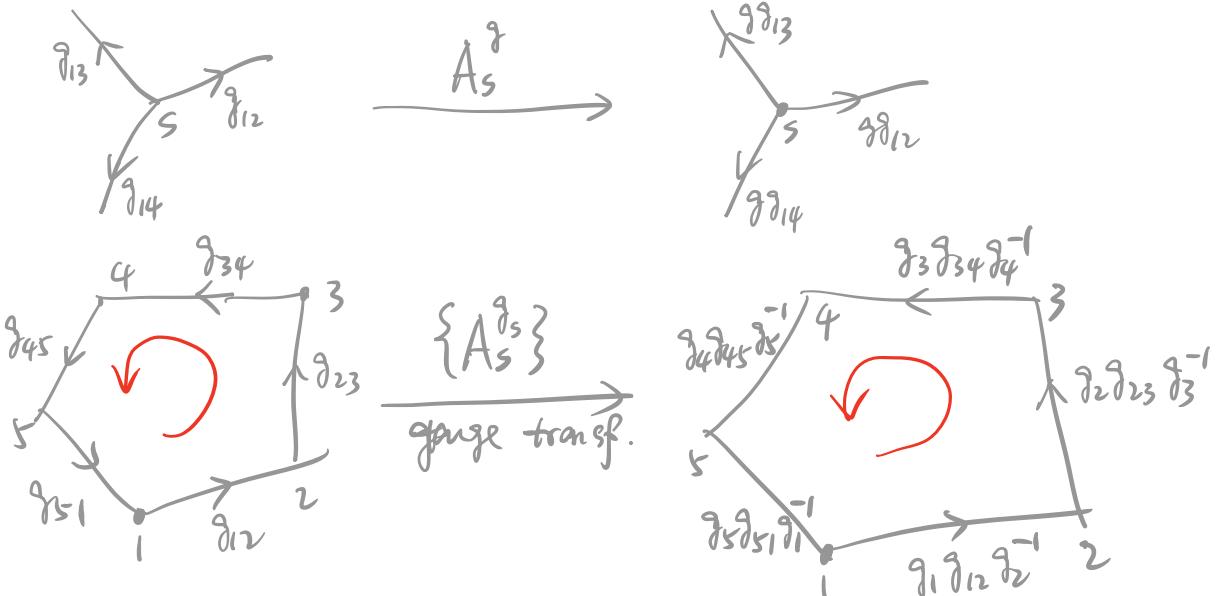
is gauge inv.

discrete:

$$\begin{aligned} g_{ij} &= e^{i \int_{\text{site } i}^{\text{site } j} A_\mu dx^\mu} && \rightarrow \text{gauge field} \\ g_i &= e^{i \chi(r_i)} && \rightarrow \text{gauge transf. para.} \\ g'_{ij} &= e^{i \int_i^j (A_\mu - \partial_\mu \chi) dx^\mu} = e^{i \int_i^j A_\mu dx^\mu - \chi(r_j) + \chi(r_i)} \\ &= e^{i \chi(r_i)} e^{i \int_i^j A_\mu dx^\mu} e^{-i \chi(r_j)} \\ &= g_i \cdot g_{ij} \cdot g_j^{-1} \end{aligned}$$

$$g'_i = g_i \cdot g_j \cdot g_j^{-1}$$

e.g.:



$$\begin{aligned} \Phi &= g_{12} \cdot g_{23} \cdots g_{51} \\ &= e^{i \int_1^5 A_\mu dx^\mu} \end{aligned}$$

$$\begin{aligned} \Phi' &= g_1 g_{12} g_{12}^{-1} \cdot g_2 g_{23} g_{23}^{-1} \cdots \\ &\quad \cdot g_5 g_{51} g_{51}^{-1} \\ &= g_1 (g_{12} \cdots g_{51}) g_1^{-1} \\ &= g_1 \Phi g_1^{-1} \end{aligned}$$

$$\text{Tr } \Phi' = \text{Tr } \Phi$$

2.2. Anyon excitations and ribbon operators

$$H = - \sum_s \underbrace{\frac{1}{|G|} \sum_{g \in G} A_s^g}_{\text{proj. to gauge inv. subspace}} - \sum_p \underbrace{B_p^{h=1}}_{\substack{\text{proj. to } h=1 \\ \text{zero flux}}}$$

Excitations:

- ① non gauge inv.: $\frac{1}{|G|} \sum_g A_s^g \neq 1$
- ② non zero flux: $B_p^{h \neq 1}$

To understand the excitations of QDM, we first try to understand the algebra of A_s^g , B_p^h for 

$$\left\{ \begin{array}{l} A_s^g A_s^{g'} = A_s^{gg'} \\ (A_s^g)^+ = A_s^{g^{-1}} \\ B_p^h B_p^{h'} = \delta_{hh'} B_p^h \\ (B_p^h)^+ = B_p^h \\ A_s^g B_p^h = B_p^{hg^{-1}} A_s^g \end{array} \right.$$

Def $D_{(h,g)} = \underbrace{h}_{g} := B_h \cdot A_g$

$$D_{(h_1,g_1)} \cdot D_{(h_2,g_2)} = \delta_{h_1, g_1 h_2 g_2^{-1}} D_{(h_1, g_1 g_2)}$$

The algebra generated by $D_{(h,g)}$ is called the Drinfeld quantum double $D(G)$ of G .

① Co-algebra structure how to act on two anyons by one $D(G)$ element

$$\Delta: D(G) \rightarrow D(G) \times D(G)$$

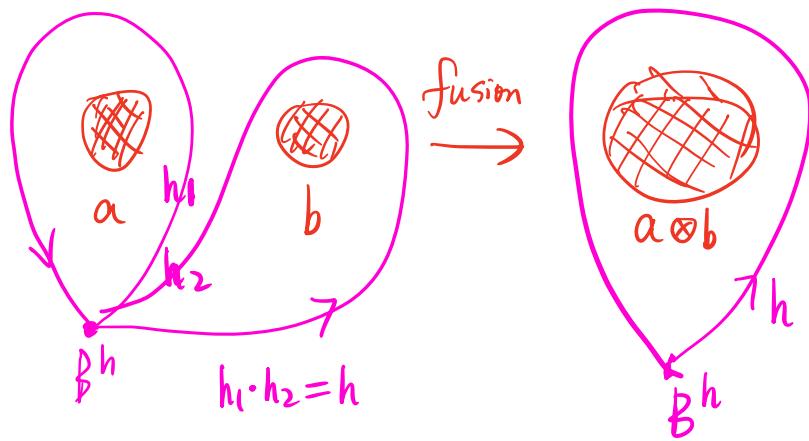
$$\Delta(A_s^g) = A_s^g \otimes A_s^g$$

$$\Delta(B_p^h) = \sum_{h_1 h_2 = h} B_p^{h_1} \otimes B_p^{h_2}$$

act on anyon b act on anyon a

$\Rightarrow \text{Rep } D(G)$ is a tensor/monoidal category.

$$a \otimes b \rightarrow c$$



② $D(G)$ is quasi-triangular: $R = \sum_{g \in G} B^g \otimes A^g$
 braiding
 $\Rightarrow \text{Rep } D(G)$ is a braided tensor category.

$D(G)$ is a quasi-triangular Hopf algebra.

Excitation space supports a representation of $D(G)$.

Irreps of quantum double $D(G)$
 $\xleftrightarrow{1:1}$ anyon excitation types

(braiding, fusion)

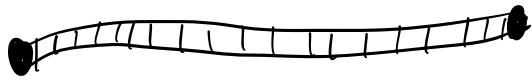
Irreps of $D(G)$.

Let $u \in G$, $C = \{gu g^{-1} \mid g \in G\}$ the conjugacy class,
 $E = \{g \in G \mid gu = ug\}$ the centralizer.

Claim: There is one irreducible rep (C, χ)
 for each conjugacy class C and each
 irrep χ of the centralizer group E .

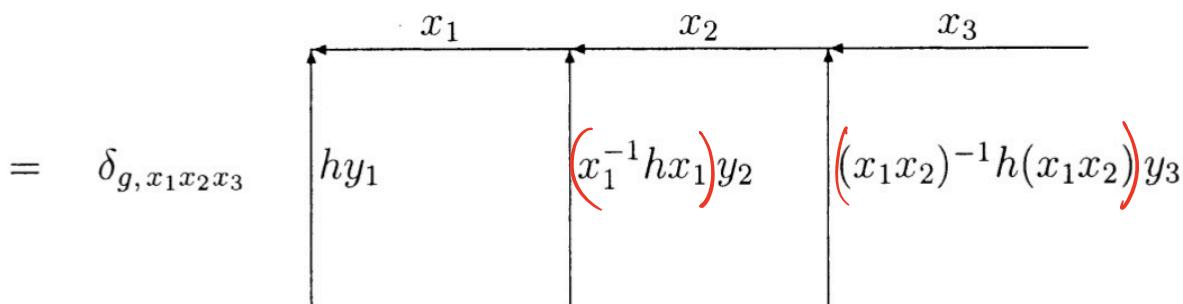
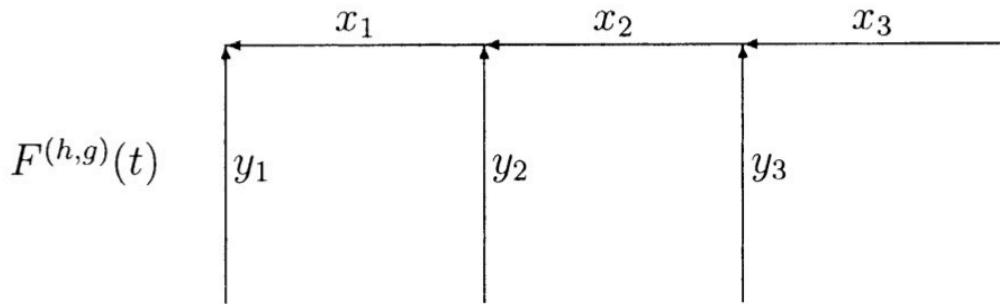
flux
 ↑
 charge

Ribbon operators



Create a pair of anyonic excitations at the ends of the ribbon operators.

↳ finite width (framing)



$$\begin{aligned} [F^{(h,g)}(t), B_p] &= 0 && \text{for } s, p \neq \partial t \\ [F^{(h,g)}(t), A_s] &= 0 \end{aligned}$$

F gives us fusion, braiding of anyons.

2.3. Spacetime Path Integral Picture

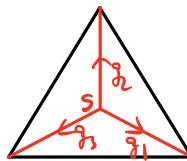
A hand-drawn diagram illustrating a 2D lattice structure. The lattice is represented by a network of black lines forming a grid-like pattern. Two specific edges are highlighted with arrows: one horizontal edge pointing to the right, labeled β_2 , and one vertical edge pointing upwards, labeled β_1 . The vertices of the lattice are also drawn with small circles.

The diagram shows a 2D triangulation of a surface. A curved arrow points from the left towards the triangulated area. The triangulation consists of several triangles defined by black lines. Red arrows indicate connections between vertices, labeled ∂_1 , ∂_2 , and ∂_{12} . The label "flat connection" is written above the diagram.

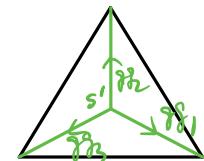
B_p operator: defined for Δ triangle.

As operator:

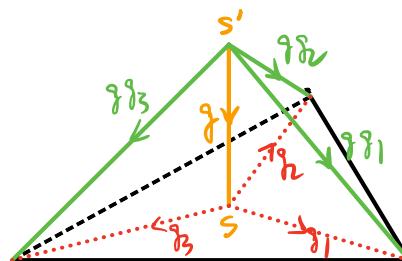
Space :



As^g

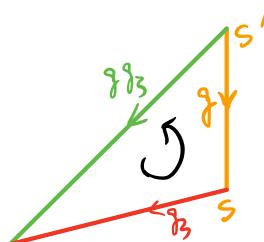


space time :

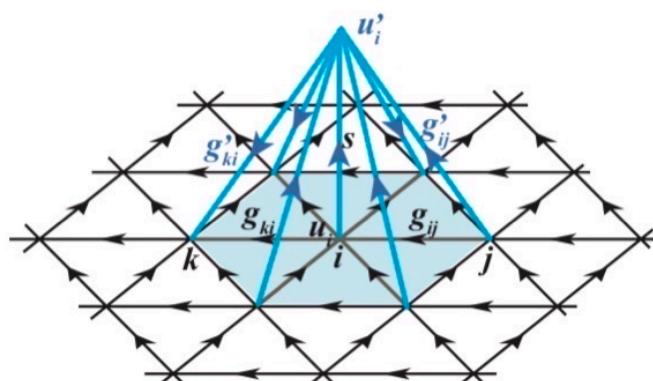


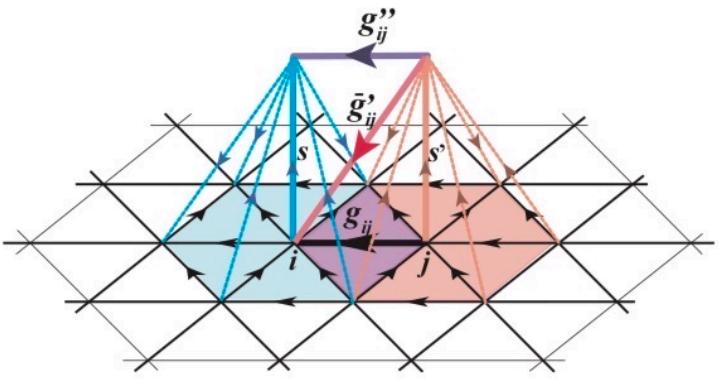
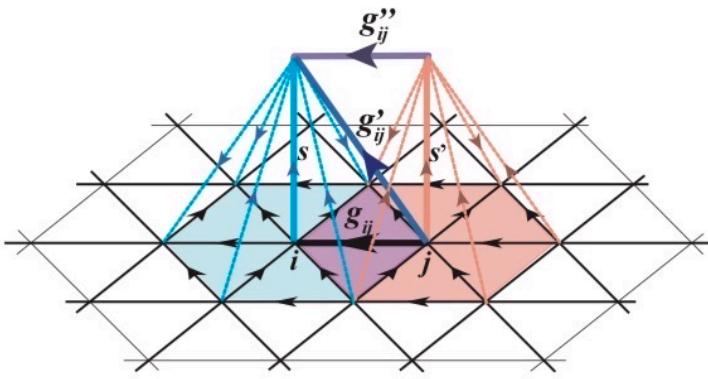
$\uparrow t$

new link along t direction



flat connection for new spacetime triangles.





$$A_j^{\vartheta'} A_i^{\vartheta} = A_i^{\vartheta} A_j^{\vartheta'}$$