



# QuantA&M Book Club

## Meeting 6

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*QuantA&M of Texas A&M University*



# New Notation

## Large Summations



# Large Summations

- Summations use a sigma symbol
- The top of the sigma is the end of the summation, and the bottom of the sigma is the beginning.

$$\sum_{n=1}^3 n$$



# Large Summations

- Summations are equal to plugging in 'n' with the beginning integer, plus plugging in the next integer, plus the next... until the final integer.

$$\sum_{n=1}^3 n = 1 + 2 + 3$$

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# Large Summations

- Examples of sigma notation:

$$\sum_{n=1}^3 n$$

$$= 1 + 2 + 3$$

$$\sum_{n=1}^3 x^n$$

$$= x^1 + x^2 + x^3$$

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# New Notation

## Large Products



# Large Products

- Products use a pi symbol
- The top of the pi is the end of the product, and the bottom of the pi is the beginning.

$$\prod_{n=1}^3 n$$



# Large Products

- Summations are equal to plugging in 'n' with the beginning integer, multiplied by plugging in the next integer, multiplied by the next... until the final integer.

$$\prod_{n=1}^3 n = x^1 * x^2 * x^3$$

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# Large Products

- Examples of pi notation:

$$\prod_{n=1}^3 n$$
$$= 1 * 2 * 3$$

$$\prod_{n=1}^3 x^n$$
$$= x^1 * x^2 * x^3$$



# New Notation

## Large Tensor Products



# Large Tensor Products

- Using similar notation to the prior two, we can define a string of tensor products like so:

$$\bigotimes_{n=1}^3 e^{i\pi * k} |1\rangle$$
$$= e^{i\pi * 1} |1\rangle \otimes e^{i\pi * 2} |1\rangle \otimes e^{i\pi * 3} |1\rangle$$

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# New Notation

## Binary to Decimal Summations



# | Binary and Decimal

- We can convert from binary to decimal using summations:

$$6_{base2} = 110$$

$$\begin{aligned} 6_{base10} &= 2^{3-1} * 1 + 2^{3-2} * 1 + 2^{3-3} * 0 \\ &= 4 + 2 + 0 = 6 \end{aligned}$$



# Binary and Decimal

- In this example, the 'index' of each digit is subtracted from the total number of bits (3):

$$6_{base2} = 110$$

$$\begin{aligned} 6_{base10} &= 2^{3-1} * 1 + 2^{3-2} * 1 + 2^{3-3} * 0 \\ &= 4 + 2 + 0 = 6 \end{aligned}$$



# | Binary and Decimal

$$6_{base2} = 110$$

$$\begin{aligned} 6_{base10} &= 2^{3-1} * 1 + 2^{3-2} * 1 + 2^{3-3} * 0 \\ &= 4 + 2 + 0 = 6 \end{aligned}$$

$$y_{base2} = y_1 y_2 \dots y_n$$

$$y_{base10} = y_1^{n-1} * y_1 + 2^{n-2} * y_2 + \dots + 2^{n-n} * y_n$$

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# Binary and Decimal

- Using this notation, we can index each 1 and 0 in a binary string and use those values in our summation to get the decimal value of  $y$ .

$$y_{base2} = y_1 y_2 \dots y_n$$

$$y_{base10} = 2^{n-1} * y_1 + 2^{n-2} * y_2 + \dots + 2^{n-n} * y_n$$

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# Binary and Decimal

- The notation for a string of binary and its decimal form can be shown as follows (b is short for base):

$$y_{b2} = y_1 y_2 \dots y_n$$

$$y_{b10} = \sum_{k=1}^n 2^{n-k} * y_k$$



# The Quantum Fourier Transform (QFT)



# The Question:

- Imagine we have a qubit in the state  $|0000\rangle$ , and this qubit increments by 1 up to the state  $|1111\rangle$  and then resets back to zero.
    - $|0000\rangle, |0001\rangle, |0010\rangle, |0011\rangle, |0100\rangle, |0101\rangle, |0110\rangle, |0111\rangle, |1000\rangle, |1001\rangle, |1010\rangle, |1011\rangle, |1100\rangle, |1101\rangle, |1110\rangle, |1111\rangle, \dots$
-



# The Question:

- Could we apply a function to these qubit values to convert the binary values to a rotation on four Bloch spheres?
  - In other words, the more you count, the more you rotate around the equator of the Bloch spheres.
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# The Question:

- Let's say the first qubit rotates a total of 16 times before making a full rotation. Then the second rotates a total of 8 times. Then the third rotates a total of 4 times. Then the fourth rotates a total of 2 times.
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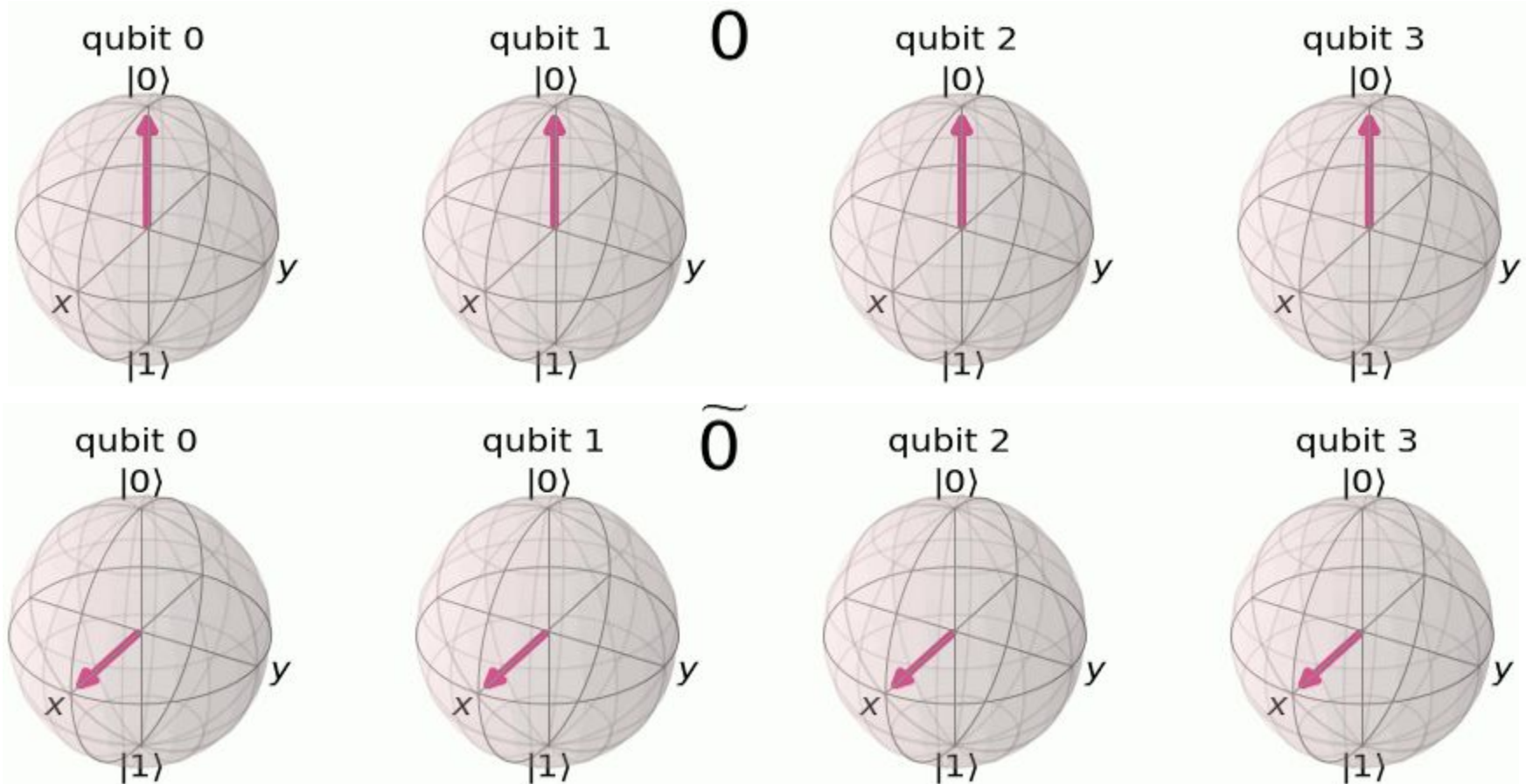
# The Quantum Fourier Transform (QFT)



# The Question



# The Question:







# The Question:

- This is what is known as the Quantum Fourier Transform. It turns binary valued qubits into a series of rotations along the x-y axis.
-



## The Question:

- These series of rotations are what is known as the 'Fourier Basis'
  - Values along the Fourier Basis are represented using  $|\tilde{x}\rangle$ , in which  $x$  is your usual integer.
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# Implementing a 1-bit QFT



## QFT Formula

The following formula gives us our function for the QFT:

Note:  $N = 2^n$

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x * y)}{N}} |y\rangle$$

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# QFT Formula

Look Familiar?

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x * y)}{N}} |y\rangle$$

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## QFT Formula

If we use only a single qubit, we get the following:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i(x*y)}{N}} |y\rangle$$

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# QFT Formula

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i(x*y)}{N}} |y\rangle \\ & \frac{1}{\sqrt{2}} \sum_{y=0}^{2-1} e^{\frac{2\pi i(x*y)}{2}} |y\rangle \\ & = \frac{1}{\sqrt{2}} \sum_{y=0}^1 e^{\frac{2\pi i(x*y)}{2}} |y\rangle \end{aligned}$$

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## QFT Formula

$$\begin{aligned} &= \frac{1}{\sqrt{2}} \sum_{y=0}^1 e^{\frac{2\pi i(x*y)}{2}} |y\rangle \\ &= \frac{1}{\sqrt{2}} (e^{\frac{2\pi i(x*0)}{2}} |0\rangle + e^{\frac{2\pi i(x*1)}{2}} |1\rangle) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + e^{\frac{2\pi i(x*1)}{2}} |1\rangle) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + e^{\pi i x} |1\rangle) \end{aligned}$$

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# QFT Formula

Plugging in 0 and 1 for x:

$$|0\rangle \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + e^{\pi i 0}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + e^{\pi i 1}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



## QFT Formula

Turns out, this formula is exactly equal to the Hadamard for  $N = 2$ .

$$\frac{1}{\sqrt{2}} \sum_{y=0}^1 e^{\frac{2\pi i(x*y)}{2}} |y\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^1 e^{\pi i(x*y)} |y\rangle$$

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# Implementing an $n$ -bit QFT



## QFT Formula

While the previously given formula works, I rewrote the formula for clarity as the notation can be a bit confusing:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * y_{b10})}{N}} |y_{b2}\rangle$$

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## QFT Formula

This is to help differentiate between the base<sub>10</sub> and base<sub>2</sub> forms of x and y.

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * y_{b10})}{N}} |y_{b2}\rangle$$

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## QFT Formula

We can rewrite  $y_{b2}$  using the binary notation discussed prior:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * y_{b10})}{N}} |y_1 y_2 \dots y_n\rangle$$



## QFT Formula

We can rewrite  $y_{b10}$  using the binary-decimal notation discussed prior:

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * y_{b10})}{N}} |y_1 y_2 \dots y_n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * \sum_{k=1}^n \frac{2^k y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle \end{aligned}$$

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## QFT Formula

The  $2^n$  in the inner-most summation can be moved outside of the summation

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * \sum_{k=1}^n \frac{2^n y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} 2^n * \sum_{k=1}^n \frac{y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle \end{aligned}$$

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# QFT Formula

Since we defined  $N$  to be equal to  $2^n$ :

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} 2^n * \sum_{k=1}^n \frac{y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle$$
$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} N * \sum_{k=1}^n \frac{y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle$$

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## QFT Formula

The  $N$ s cancel out, leaving us with the following:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} \cancel{N} * \sum_{k=1}^n \frac{y_k}{2^k})}{\cancel{N}}} |y_1 y_2 \dots y_n\rangle$$
$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i (x_{b10} * \sum_{k=1}^n \frac{y_k}{2^k})} |y_1 y_2 \dots y_n\rangle$$

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# QFT Formula

Using the product rule for exponentials:

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i (x_{b10} * \sum_{k=1}^n \frac{y_k}{2^k})} |y_1 y_2 \dots y_n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{k=1}^n e^{\frac{2\pi i (x_{b10} * y_k)}{2^k}} |y_1 y_2 \dots y_n\rangle \end{aligned}$$

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# QFT Formula

We can redistribute the amplitudes here and rewrite this as a summation of tensor products:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{k=1}^n e^{\frac{2\pi i (x_{b10} * y_k)}{2^k}} |y_1 y_2 \dots y_n\rangle$$
$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * y_1)}{2^1}} |y_1\rangle \otimes e^{\frac{2\pi i (x_{b10} * y_2)}{2^2}} |y_2\rangle \otimes \dots \otimes e^{\frac{2\pi i (x_{b10} * y_n)}{2^n}} |y_n\rangle$$

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# QFT Formula

Which gives us the following formula:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * y_1)}{2^1}} |y_1\rangle \otimes e^{\frac{2\pi i(x_{b10} * y_2)}{2^2}} |y_2\rangle \otimes \dots \otimes e^{\frac{2\pi i(x_{b10} * y_n)}{2^n}} |y_n\rangle$$
$$= \frac{1}{\sqrt{N}} \bigotimes_{k=1}^n (|0\rangle + e^{\frac{2\pi i x_{b10}}{2^k}} |1\rangle)$$



# QFT Formula

Plugging in  $x=6$  provides the following answer:

$$\begin{aligned} & \frac{1}{\sqrt{8}} \bigotimes_{k=1}^n (|0\rangle + e^{\frac{2\pi i 6}{2^k}} |1\rangle) \\ &= \frac{1}{\sqrt{8}} [(|0\rangle + e^{\frac{2\pi i 6}{2^1}} |1\rangle) \otimes (|0\rangle + e^{\frac{2\pi i 6}{2^2}} |1\rangle) \otimes (|0\rangle + e^{\frac{2\pi i 6}{2^3}} |1\rangle)] \end{aligned}$$

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# Building the Circuit



# Building the Circuit

To start building the circuit, we need to figure out the pattern behind our formula.

$ x_1x_2\dots x_{n-2}x_{n-1}x_n\rangle$	$x_n amp. =$
$ 00\dots 000\rangle$	$\frac{1}{\sqrt{N}}$
$ 00\dots 001\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^n})}}{\sqrt{N}}$
$ 00\dots 010\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-1}})}}{\sqrt{N}}$
$ 00\dots 100\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-2}})}}{\sqrt{N}}$
...	...
$ 11\dots 111\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^1} + \frac{x_{b10}}{2^2} + \dots + \frac{x_{b10}}{2^n})}}{\sqrt{N}}$





## Building the Circuit

We notice that each '1' in our binary string results in our first x-value having an amplitude of  $\exp(2\pi i/2^{n-k})$ .

Also, k is a specific power of 2 in x.

$ x_1x_2\dots x_{n-2}x_{n-1}x_n\rangle$	$x_n amp. =$
$ 00\dots000\rangle$	$\frac{1}{\sqrt{N}}$
$ 00\dots001\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^n})}}{\sqrt{N}}$
$ 00\dots010\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-1}})}}{\sqrt{N}}$
$ 00\dots100\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-2}})}}{\sqrt{N}}$
...	...
$ 11\dots111\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^1} + \frac{x_{b10}}{2^2} + \dots + \frac{x_{b10}}{2^n})}}{\sqrt{N}}$



# Building the Circuit

We can also notice that if multiple '1's exist, the exponentials of each respective  $k$  get multiplied together.

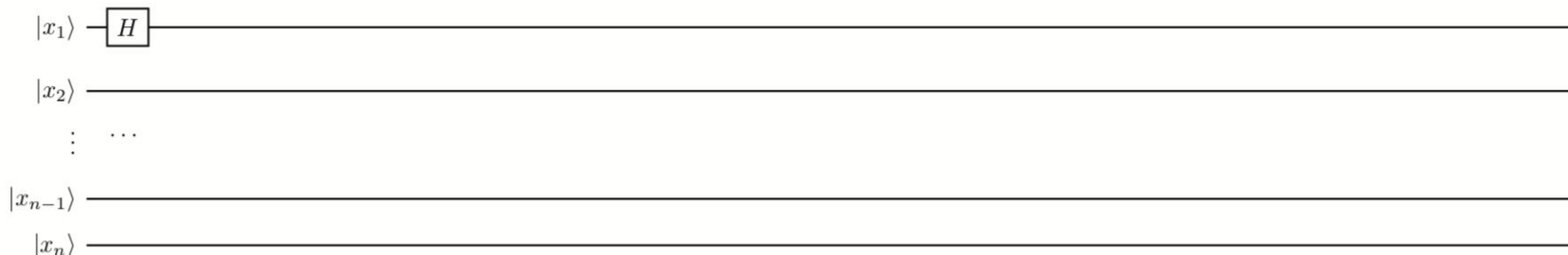
$ x_1x_2...x_{n-2}x_{n-1}x_n\rangle$	$x_n amp. =$
$ 00...000\rangle$	$\frac{1}{\sqrt{N}}$
$ 00...001\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^n})}}{\sqrt{N}}$
$ 00...010\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-1}})}}{\sqrt{N}}$
$ 00...100\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-2}})}}{\sqrt{N}}$
...	...
$ 11...111\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^1} + \frac{x_{b10}}{2^2} + ... + \frac{x_{b10}}{2^n})}}{\sqrt{N}}$



# Building the Circuit

Base case:

We know that the basic implementation of a 1-qubit QFT is the Hadamard.





## Building the Circuit

Next step:

Let us define a gate called URot-Gate:

The URot-gate rotates an amount depending on the value of  $k$ .

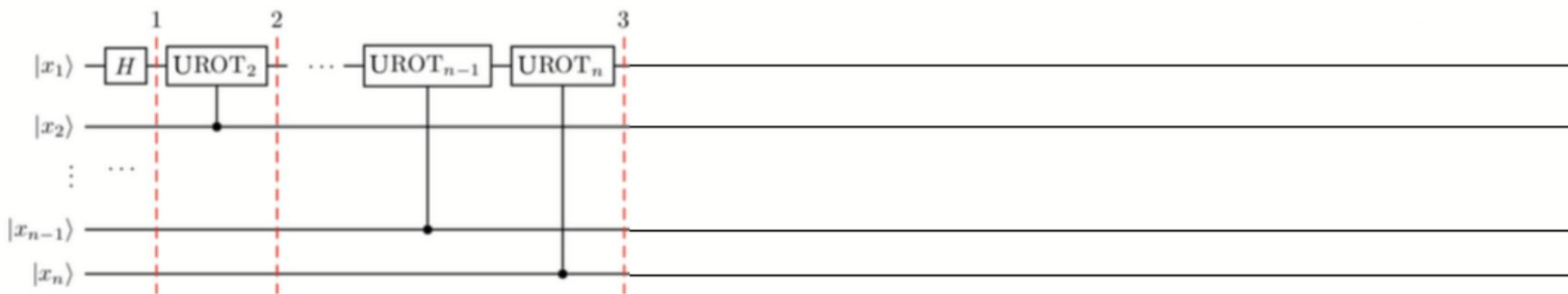
$$UROT_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{2^k}\right) \end{bmatrix}$$



# Building the Circuit

Next step:

We attach these gates to a control bit. The  $k$ th bit attached to the control bit is equal to the  $k$ -distance from the UROT gate.

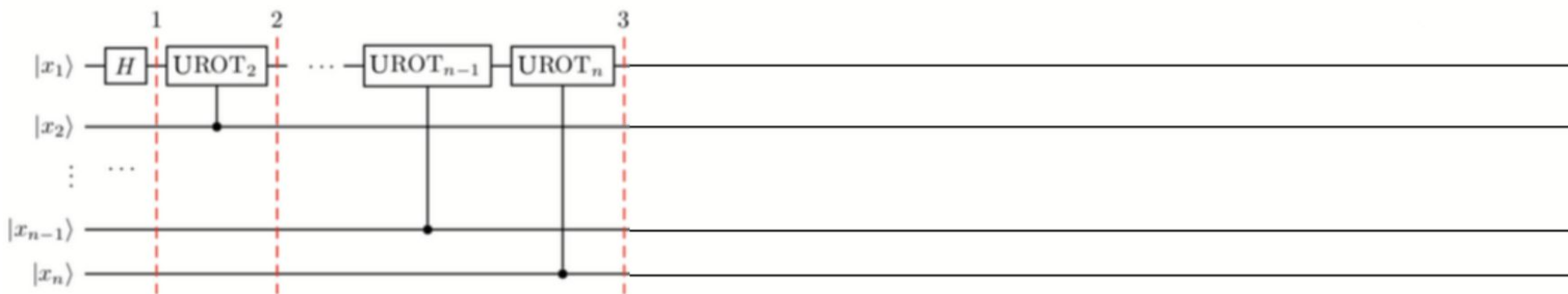




# Building the Circuit

After the 1<sup>st</sup> URot:

$$\frac{1}{\sqrt{2}} \left[ |0\rangle + \exp\left(\frac{2\pi i}{2} x_1\right) |1\rangle \right] \otimes |x_2 x_3 \dots x_n\rangle$$

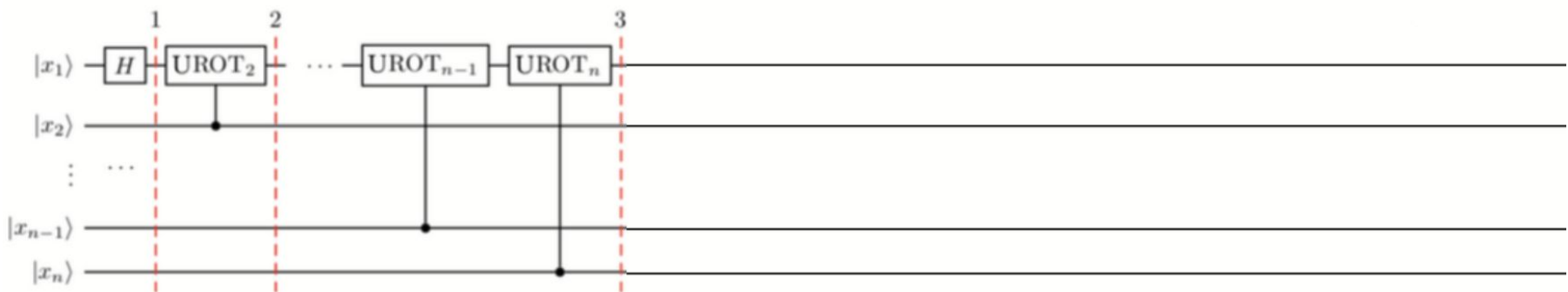




# Building the Circuit

After the 2<sup>nd</sup> URot:

$$\frac{1}{\sqrt{2}} \left[ |0\rangle + \exp \left( \frac{2\pi i}{2^2} x_2 + \frac{2\pi i}{2} x_1 \right) |1\rangle \right] \otimes |x_2 x_3 \dots x_n\rangle$$

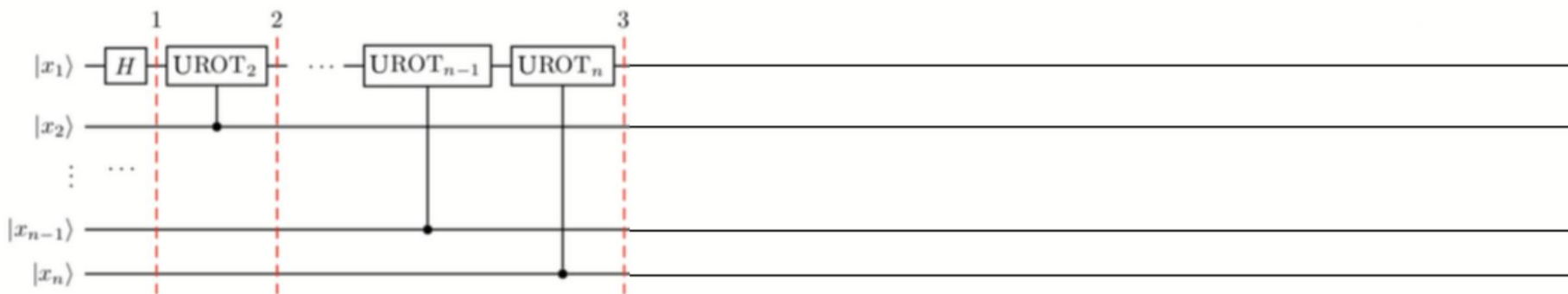




# Building the Circuit

After the  $n^{\text{th}}$  UROT:

$$\frac{1}{\sqrt{2}} \left[ |0\rangle + \exp \left( \frac{2\pi i}{2^n} x_n + \frac{2\pi i}{2^{n-1}} x_{n-1} + \dots + \frac{2\pi i}{2^2} x_2 + \frac{2\pi i}{2} x_1 \right) |1\rangle \right] \otimes |x_2 x_3 \dots x_n\rangle$$







# Building the Circuit

The Result:

- The resulting value:

$$\frac{1}{\sqrt{2}} \left[ |0\rangle + \exp \left( \frac{2\pi i}{2^n} x_n + \frac{2\pi i}{2^{n-1}} x_{n-1} + \dots + \frac{2\pi i}{2^2} x_2 + \frac{2\pi i}{2} x_1 \right) |1\rangle \right] \otimes |x_2 x_3 \dots x_n\rangle$$

- is the exact value of:

$$e^{\frac{2\pi i x_{b10}}{2^n}}$$



# Building the Circuit

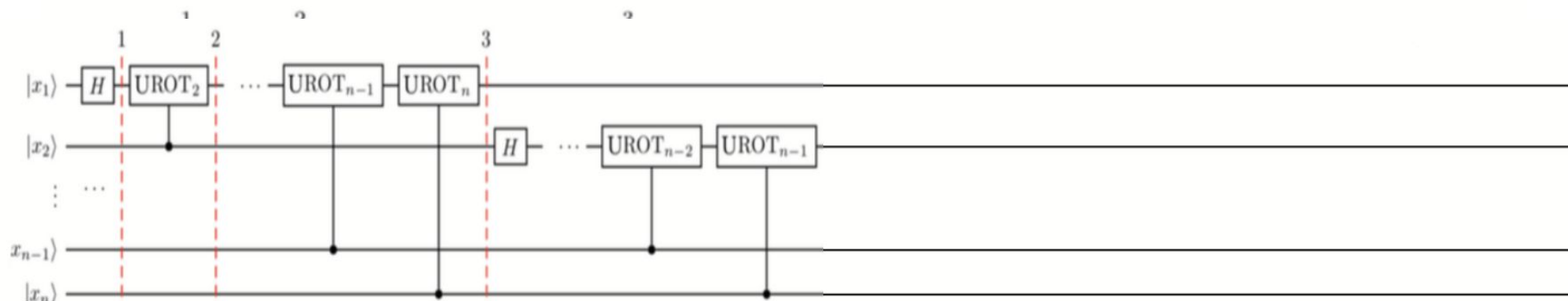
$$\begin{aligned}x_{b10} &= 2^{n-1} * x_1 + 2^{n-2} * x_2 + \dots + 2^{n-n} * x_n \\&\Rightarrow \frac{2\pi i}{2^n} x_n + \frac{2\pi i}{2^{n-1}} x_{n-1} + \dots + \frac{2\pi i}{2^2} x_2 + \frac{2\pi i}{2^1} x_1 \\&\Rightarrow \frac{1}{\cancel{2\pi i}} * \left( \cancel{\frac{2\pi i}{2^n}} x_n + \cancel{\frac{2\pi i}{2^{n-1}}} x_{n-1} + \dots + \cancel{\frac{2\pi i}{2^2}} x_2 + \cancel{\frac{2\pi i}{2^1}} x_1 \right) \\&= \frac{1}{2^n} x_n + \frac{1}{2^{n-1}} x_{n-1} + \dots + \frac{1}{2^2} x_2 + \frac{1}{2^1} x_1 \\&\Rightarrow 2^n * \left( \frac{1}{2^n} x_n + \frac{1}{2^{n-1}} x_{n-1} + \dots + \frac{1}{2^2} x_2 + \frac{1}{2^1} x_1 \right) \\&= 2^{n-n} x_n + 2^{n-n+1} x_{n-1} + \dots + 2^{n-2} x_2 + 2^{n-1} x_1 \\&= 2^{n-1} * x_1 + 2^{n-2} * x_2 + \dots + 2^{n-n} * x_n \\&= x_{b10} \\&\Rightarrow \frac{2\pi i}{2^n} x_{b10} = amp. |x\rangle\end{aligned}$$



# Building the Circuit

Next step:

We can repeat this process for the next value and get the second portion of the circuit.

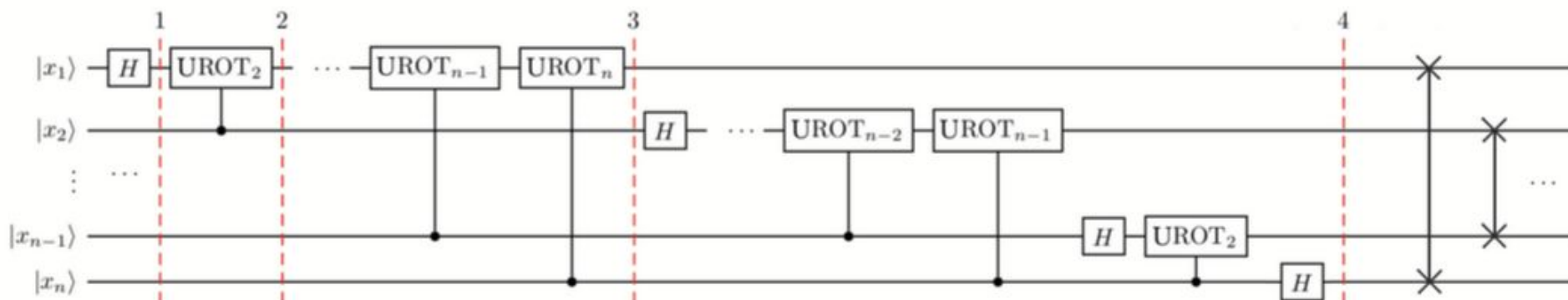




# Building the Circuit

Next step:

This can be repeated indefinitely until we get to the last qubit, which is by itself and therefore only requires a Hadamard gate.





# Building the Circuit

After applying the circuit, we get the following state:

$$\frac{1}{\sqrt{2}} \left[ |0\rangle + \exp\left(\frac{2\pi i}{2^n} x\right) |1\rangle \right] \otimes \frac{1}{\sqrt{2}} \left[ |0\rangle + \exp\left(\frac{2\pi i}{2^{n-1}} x\right) |1\rangle \right] \otimes \dots \otimes \frac{1}{\sqrt{2}} \left[ |0\rangle + \exp\left(\frac{2\pi i}{2^2} x\right) |1\rangle \right] \otimes \frac{1}{\sqrt{2}} \left[ |0\rangle + \exp\left(\frac{2\pi i}{2^1} x\right) |1\rangle \right]$$

Which is equal to the function we defined earlier:

$$= \frac{1}{\sqrt{N}} \bigotimes_{k=1}^n (|0\rangle + e^{\frac{2\pi i x b_{10}}{2^k}} |1\rangle)$$

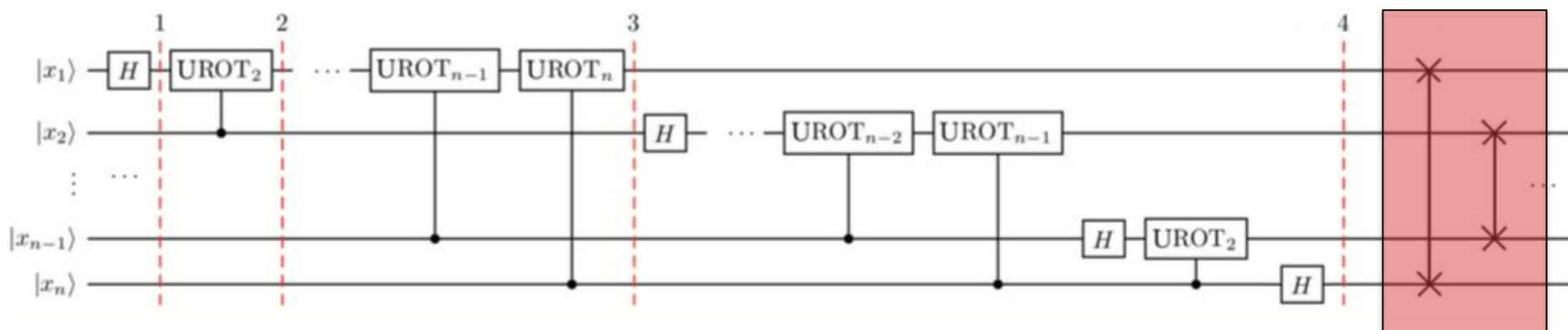
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# Building the Circuit

Final Step:

It is important to notice that our final output is upside down, our  $x_1$  gives the  $x_n$  amplitude!

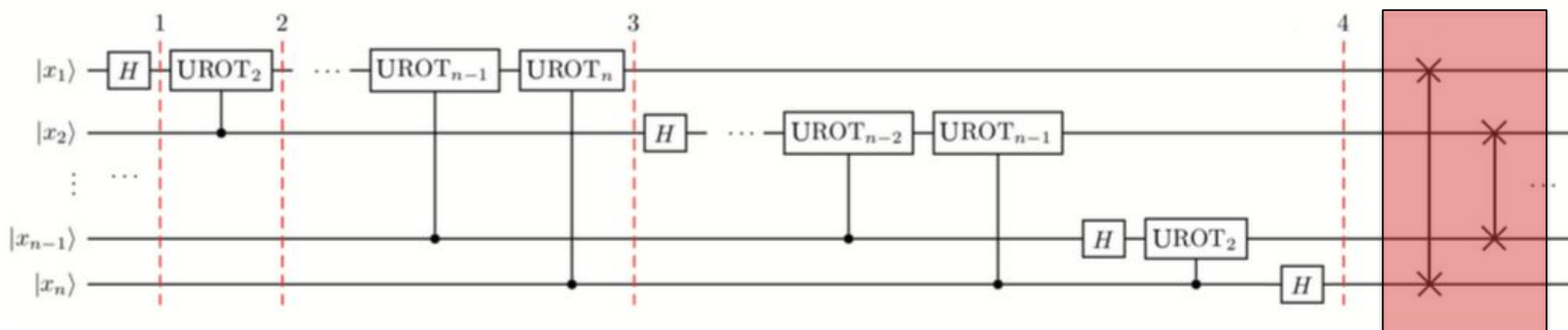




# Building the Circuit

Final Step:

To solve this, we simply add a swap gate that swaps the amplitudes.





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# Coding

Now, lets code this in Qiskit!

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Let's Get Started