QuantA&M Book Club
Meeting 3

QuantA&M of Texas A&M University

A Summary of Meeting 2

Some Useful Notation

$$\langle \Psi || \Phi \rangle = \langle \Psi \Phi \rangle \quad \text{"Inner Product"}$$

$$|\Psi\rangle\langle\Phi| = |\Psi\Phi| \quad \text{"Outer Product"}$$

$$|\Psi\rangle|\Phi\rangle = |\Psi\Phi\rangle \quad \text{"Tensor Product"}$$

$$AB = A \times B \quad \text{"Matrix Multiplication"}$$

Single-Qubits As Vectors

- Qubits can also be represented as vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Each element in the vector is an amplitude of a measurement.

Multi-Qubits As Vectors

- Multiple qubits can be represented as a single tensor product of qubits.

$$|ba\rangle = |b\rangle \otimes |a\rangle = \begin{bmatrix} b_0 \times \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \\ \\ b_1 \times \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{bmatrix}$$

- For every n-sized qubit, the size of the vector is 2^n.

Example Matrix Operation: Hadamard Gate

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} * \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} * \begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Example Matrix Operation: X/Not Gate

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example Matrix Operation: Identity Gate

$$I|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$I|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} * \begin{bmatrix} 6 & 5 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} -19 & -9 \\ -28 & 4 \end{bmatrix}$$

Combining Gates

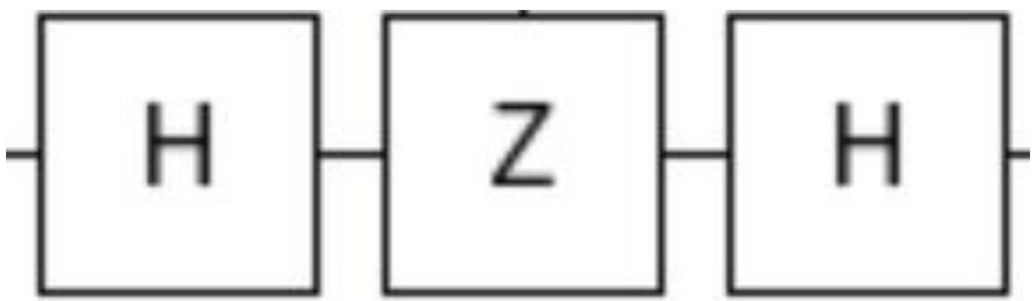
- Quantum logic gates can be combined in order to create more circuits.

$$HZH = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = X$$

- This is an example of a two Hadamards and a Z-Gate making an X-Gate.

Combining Gates

The original circuit for reference:



Tensor Product for Matrices

- Like with qubits, matrices affecting multiple qubits can also be represented as a larger matrix.
- If matrix A is effects Q0, and matrix B effects Q1, then the matrix for both qubits is simply the tensor product between matrix A and B.

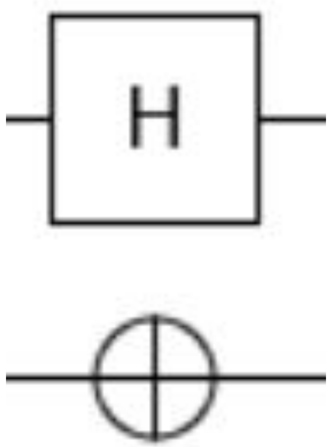
$$X \otimes H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Tensor Product for Matrices

- Once again, the circuit for reference:



Circuit Identities

- Using matrix multiplication, we can find the following useful formulas and relationships between our gates:

$$HZH = X$$

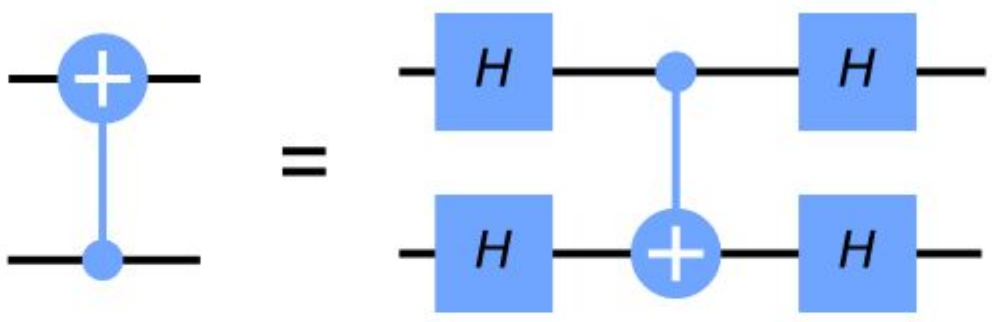
$$HXH = Z$$

$$HYH = -Y$$

$$H \otimes H * CNOT[Q1, Q0] * H \otimes H = CNOT[Q0, Q1]$$

Circuit Identities

- The last identity looks confusing but is actually quite intuitive. Here is the circuit diagram if done on qiskit:



The T-Gate

Phase Kickback: The Controlled T-Gate

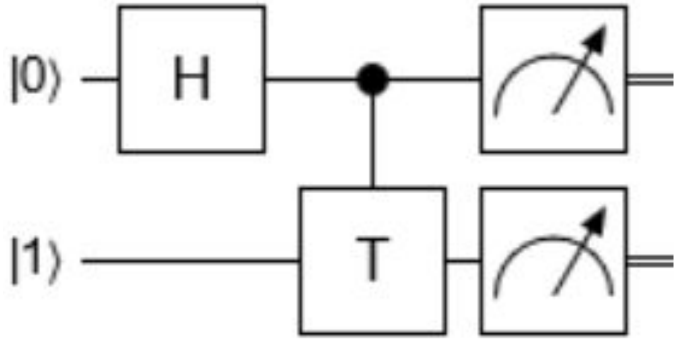
- An interesting gate can be created that changes the amplitude of our control bit. This is called a Controlled T-gate.

$$\text{Controlled-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}(1+i) \end{bmatrix}$$

$$T|1\rangle = e^{i\pi/4}|1\rangle$$

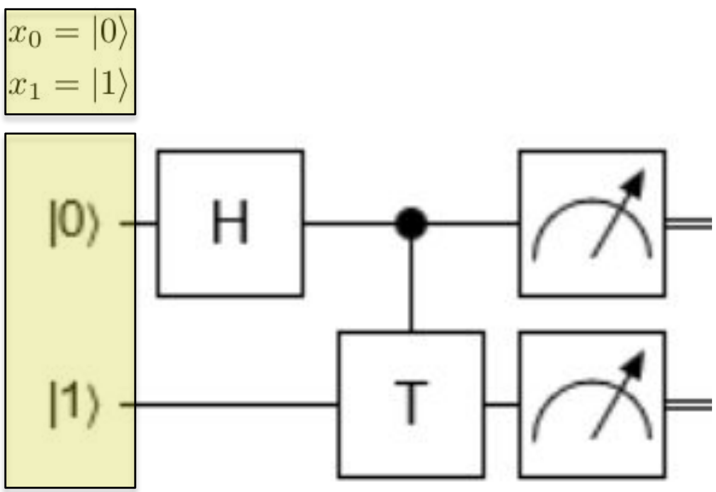
Phase Kickback: A Controlled T-Gate's Output

- We can of course use a controlled T-gate to manipulate a qubit.



Phase Kickback: A Controlled T-Gate's Output

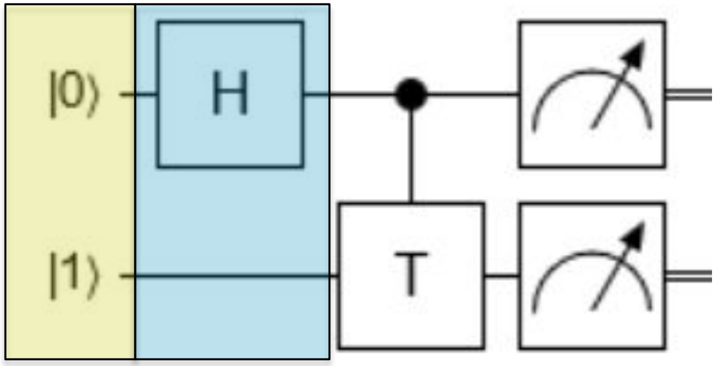
- The starting values are obviously zero and one.



Phase Kickback: A Controlled T-Gate's Output

- Afterwards, we put the zero through the Hadamard, getting the half-and-half state.

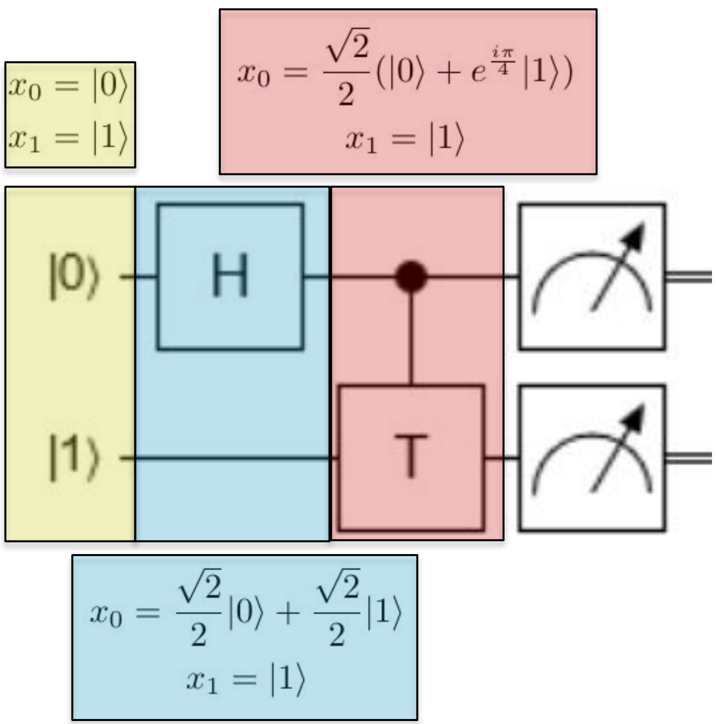
$$x_0 = |0\rangle$$
$$x_1 = |1\rangle$$



$$x_0 = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$x_1 = |1\rangle$$

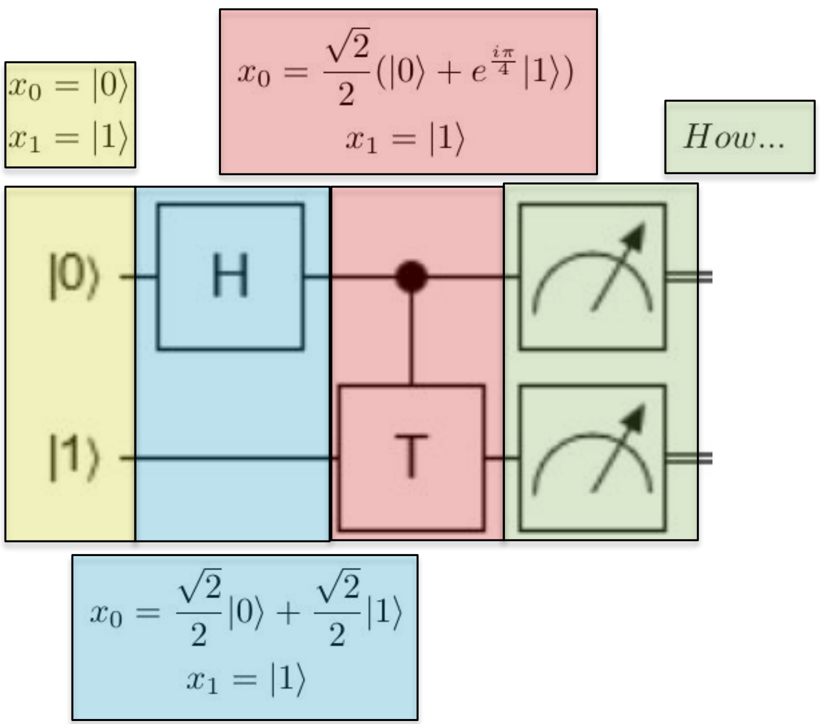
Phase Kickback: A Controlled T-Gate's Output

- Finally, we run the gate through the T-Gate!



Phase Kickback: A Controlled T-Gate's Output

- Wait a second… Why did X_0 change when we clearly ran T through X_1?



Phase Kickback

Phase Kickback: A Controlled T-Gate's Output

- Let's run through this again, but with the math we learned from the last lecture.

- Maybe we can get a better understanding on what happened.

Phase Kickback: A Controlled T-Gate's Output

- Notice how the modified qubit is the control bit.

$$|1\rangle \otimes \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$$
$$= \frac{\sqrt{2}}{2}(|10\rangle + |11\rangle)$$
$$\implies T|1\rangle \otimes \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$$
$$= T\frac{\sqrt{2}}{2}(|10\rangle + |11\rangle)$$
$$= \frac{\sqrt{2}}{2}(|10\rangle + e^{\frac{i\pi}{4}}|11\rangle)$$
$$= |1\rangle\frac{\sqrt{2}}{2}(|0\rangle + e^{\frac{i\pi}{4}}|1\rangle)$$

Phase Kickback: A Controlled T-Gate's Output

$$T|1\rangle \otimes |1\rangle$$
$$= |1\rangle \otimes e^{\frac{i\pi}{4}}|1\rangle$$

$$T|1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + |1\rangle)$$
$$= |1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + e^{\frac{i\pi}{4}}|1\rangle)$$

Phase Kickback: Why is This Important?

- This is one of the magic tricks behind some famous quantum computing algorithms. It is called “phase kickback”.

- You can indirectly affect the amplitude of a qubit using phase kickback.

Phase Kickback: Why is This Important?

- The magnitude of the amplitude that gets shifted is still the same, it is a rotation.
- Take the following example:

$$\begin{matrix} 1|1\rangle \\ -1|1\rangle \\ i|1\rangle \\ -i|1\rangle \end{matrix} \implies \begin{matrix} \|1\| = 1 \\ \|-1\| = 1 \\ \|i\| = 1 \\ \|-i\| = 1 \end{matrix}$$

Phase Kickback: Why is This Important?

- The magnitude of the amplitude that gets shifted is still the same, it is a rotation.
- Take the following example:

$$\begin{matrix} 1|1\rangle \\ -1|1\rangle \\ i|1\rangle \\ -i|1\rangle \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \|1\| = 1 \\ \|-1\| = 1 \\ \|i\| = 1 \\ \|-i\| = 1 \end{matrix}$$

Phase Kickback: Why is This Important?

- On a classical computer, this is meaningless, as all that really matters is the magnitude of the amplitude.
- With superpositions however, a rotation of one of the superpositions can have a measurable impact on computations.

Phase Kickback: Why is This Important?

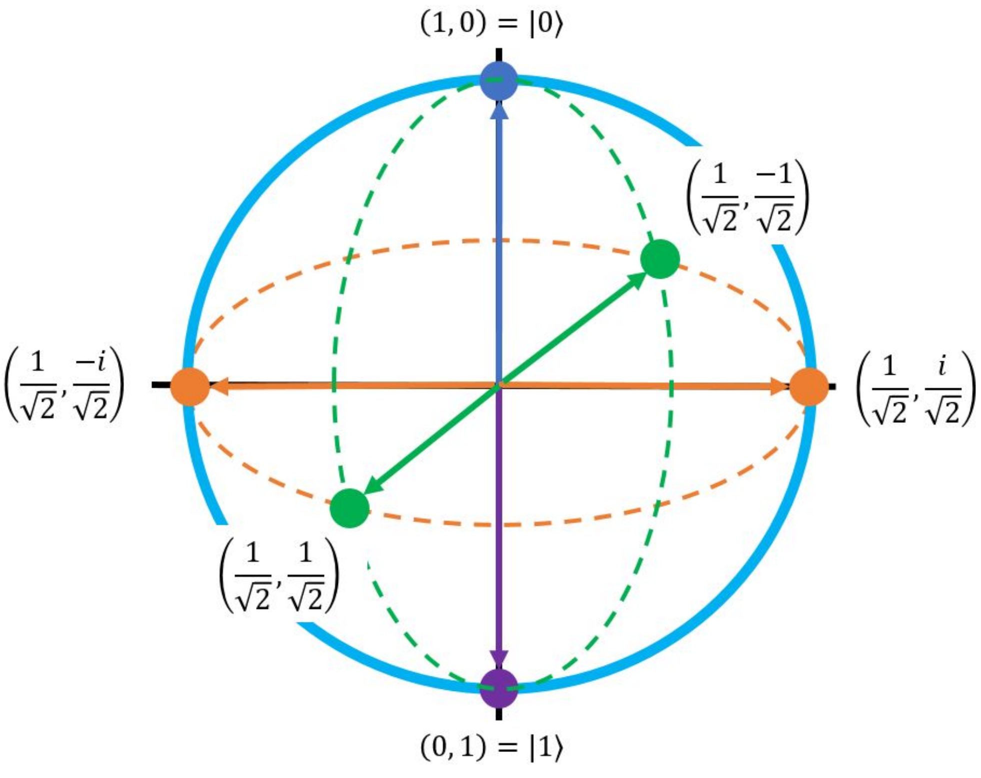
- The importance of superposition and rotation can be demonstrated graphically, but how?
- What do quantum computer scientists use to visualize qubits in superposition?

Bloch Sphere

The Bloch Sphere:

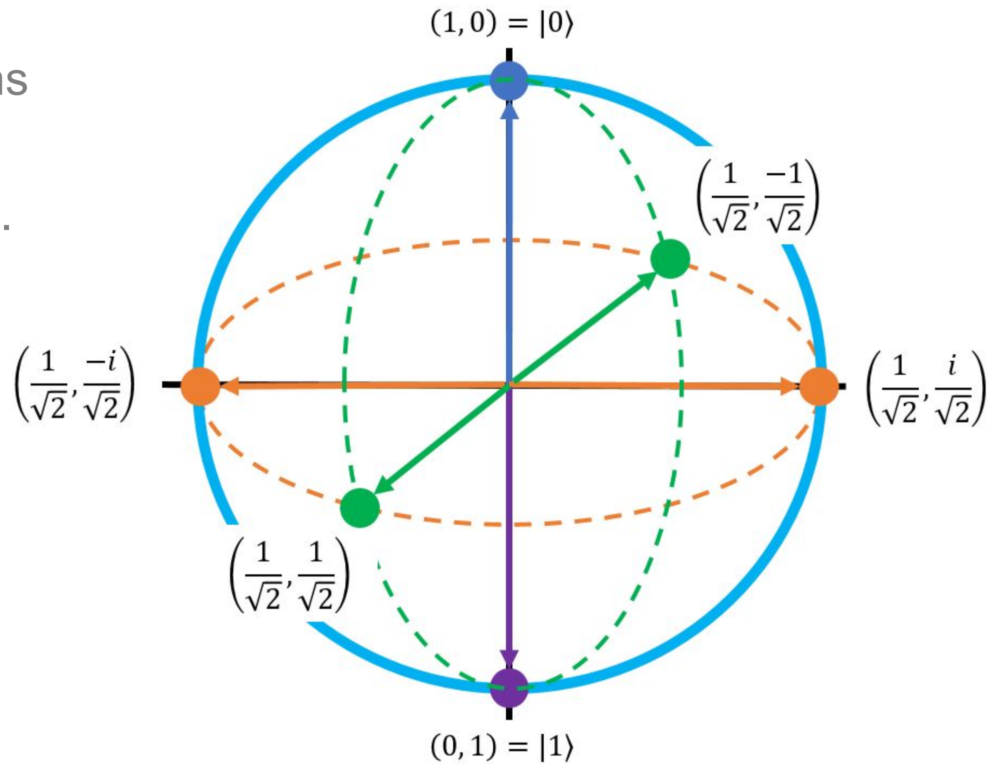
- The Bloch sphere is a visual representation of our qubit vector.
- It is a sphere with a radius of 1. Every point in the sphere is a possible amplitude value.

Visualizing a Qubit:

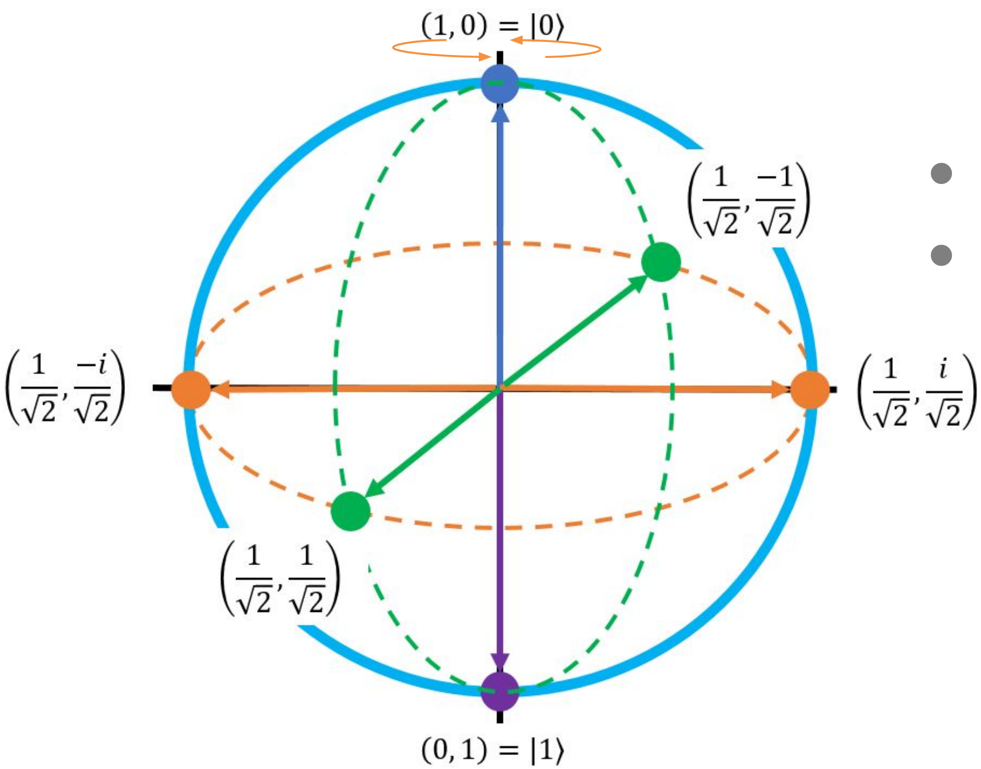


Visualizing a Qubit:

- Most calculations are done along the green axis.



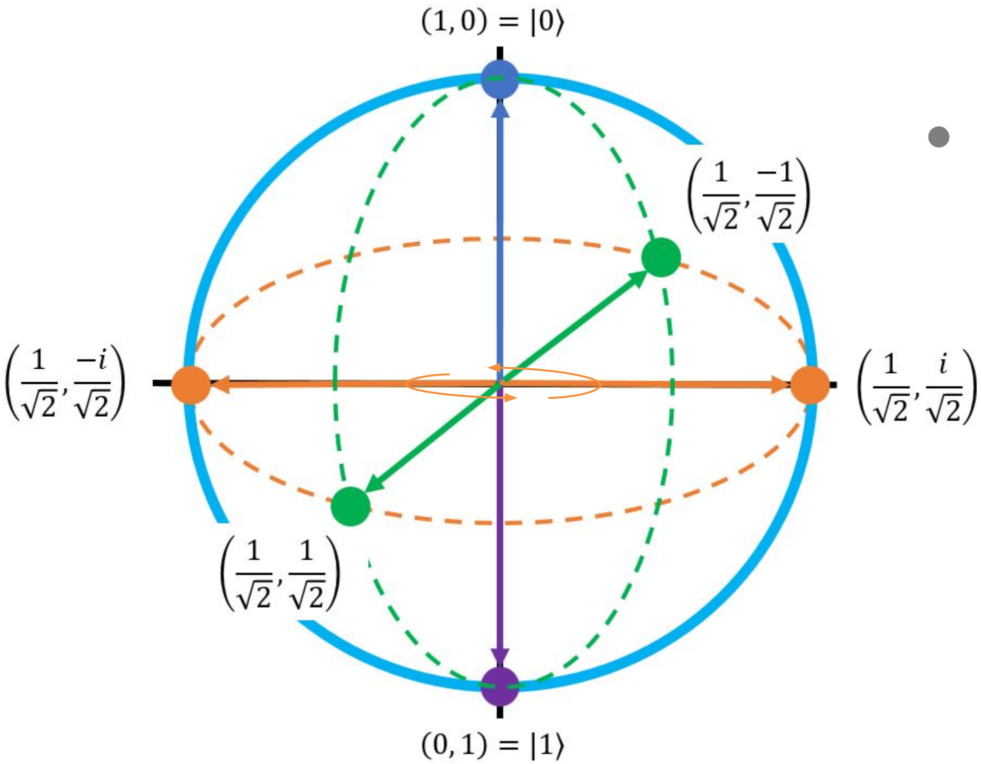
Visualizing a Qubit:



- Let assume our qubit is $|0\rangle$
- Rotating the qubit along the orange axis gives us:

$$1|0\rangle \implies -1|0\rangle \implies i|0\rangle \implies -i|0\rangle \implies \dots$$

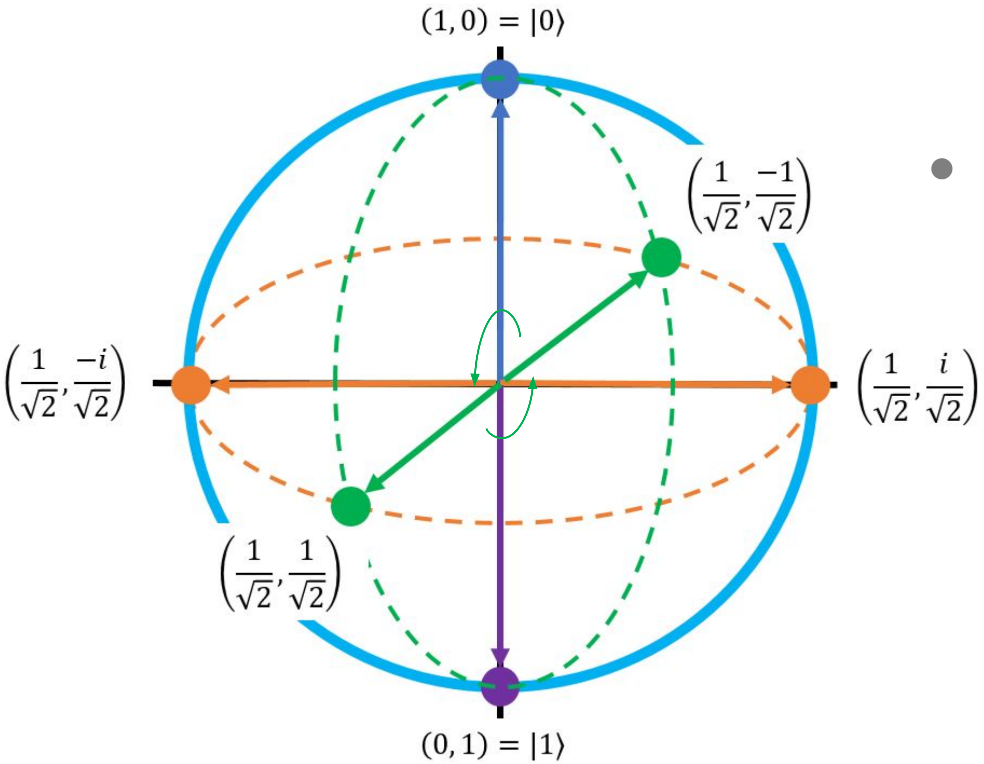
Visualizing a Qubit:



- Nothing really changes if we rotate it on the

$$1|0\rangle \implies -1|0\rangle \implies i|0\rangle \implies -i|0\rangle \implies \dots$$

Visualizing a Qubit:

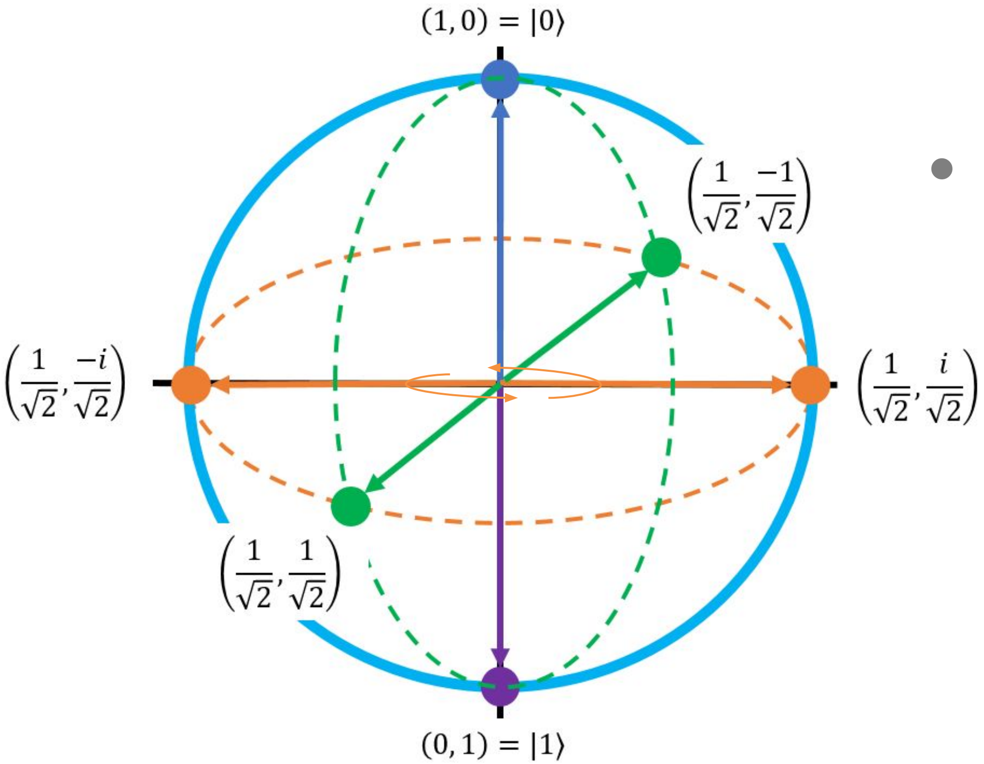


- Rotating the qubit along the green axis does the following:

$$1|0\rangle + 0|1\rangle \implies \frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$$

$$\implies 0|0\rangle + 1|1\rangle \implies \frac{\sqrt{2}}{2}|0\rangle - \frac{\sqrt{2}}{2}|1\rangle \implies \dots$$

Visualizing a Qubit:



- In contrast to the orange, the green magnitudes have clearly changed. Rotating along the green results in

$$1|0\rangle + 0|1\rangle \implies \frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$$
$$\implies 0|0\rangle + 1|1\rangle \implies \frac{\sqrt{2}}{2}|0\rangle - \frac{\sqrt{2}}{2}|1\rangle \implies \ldots$$

Let’s Get Started