QuantA&M Book Club
Meeting 6

QuantA&M of Texas A&M University

New Notation

Large Summations

Large Summations

- Summations use a sigma symbol
- The top of the sigma is the end of the summation, and the bottom of the sigma is the beginning.

$$\sum_{n=1}^{3} n$$

Large Summations

- Summations are equal to plugging in ‘n’ with the beginning integer, plus plugging in the next integer, plus the next… until the final integer.

$$\sum_{n=1}^{3} n = 1 + 2 + 3$$

Large Summations

- Examples of sigma notation:

$$\sum_{n=1}^{3} n \qquad \sum_{n=1}^{3} x^n$$

$$= 1 + 2 + 3 \qquad = x^1 + x^2 + x^3$$

New Notation

Large Products

Large Products

- Products use a pi symbol
- The top of the pi is the end of the product, and the bottom of the pi is the beginning.

$$\prod_{n=1}^{3} n$$

Large Products

- Summations are equal to plugging in ‘n’ with the beginning integer, multiplied by plugging in the next integer, multiplied by the next… until the final integer.

$$\prod_{n=1}^{3} n \quad = x^1 * x^2 * x^3$$

Large Products

- Examples of pi notation:

$$\prod_{n=1}^{3} n = 1 * 2 * 3$$

$$\prod_{n=1}^{3} x^n = x^1 * x^2 * x^3$$

New Notation

Large Tensor Products

Large Tensor Products

- Using similar notation to the prior two, we can define a string of tensor products like so:

$$\bigotimes_{n=1}^{3} e^{i\pi * k}|1\rangle$$

$$= e^{i\pi*1}|1\rangle \otimes e^{i\pi*2}|1\rangle \otimes e^{i\pi*3}|1\rangle$$

New Notation

Binary to Decimal Summations

Binary and Decimal

- We can convert from binary to decimal using summations:

$$6_{base2} = 110$$

$$6_{base10} = 2^{3-1} * 1 + 2^{3-2} * 1 + 2^{3-3} * 0$$

$$= 4 + 2 + 0 = 6$$

Binary and Decimal

- In this example, the ‘index’ of each digit is subtracted from the total number of bits (3):

$$6_{base2} = 110$$

$$6_{base10} = 2^{3-1} * 1 + 2^{3-2} * 1 + 2^{3-3} * 0$$

$$= 4 + 2 + 0 = 6$$

Binary and Decimal

$$6_{base2} = 110$$

$$6_{base10} = 2^{3-1} * 1 + 2^{3-2} * 1 + 2^{3-3} * 0$$

$$= 4 + 2 + 0 = 6$$

$$y_{base2} = y_1 y_2 \cdots y_n$$

$$y_{base10} = y_1^{n-1} * y_1 + 2^{n-2} * y_2 + \ldots + 2^{n-n} * y_n$$

Binary and Decimal

- Using this notation, we can index each 1 and 0 in a binary string and use those values in our summation to get the decimal value of y.

$$y_{base2} = y_1 y_2 \ldots y_n$$

$$y_{base10} = 2^{n-1} * y_1 + 2^{n-2} * y_2 + \ldots + 2^{n-n} * y_n$$

Binary and Decimal

- The notation for a string of binary and its decimal form can be shown as follows (b is short for base):

$$y_{b2} = y_1 y_2 \cdots y_n$$

$$y_{b10} = \sum_{k=1}^{n} 2^{n-k} * y_k$$

The Quantum Fourier Transform (QFT)

The Question:

- Imagine we have a qubit in the state |0000>, and this qubit increments by 1 up to the state |1111> and then resets back to zero.
  - |0000>, |0001>, |0010>, |0011>, |0100>, |0101>, |0110>, |0111>, |1000>, |1001>, |1010>, |1011>, |1100>, |1101>, |1110>, |1111>, …

The Question:

- Could we apply a function to these qubit values to convert the binary values to a rotation on four Bloch spheres?
- In other words, the more you count, the more you rotate around the equator of the Bloch spheres.

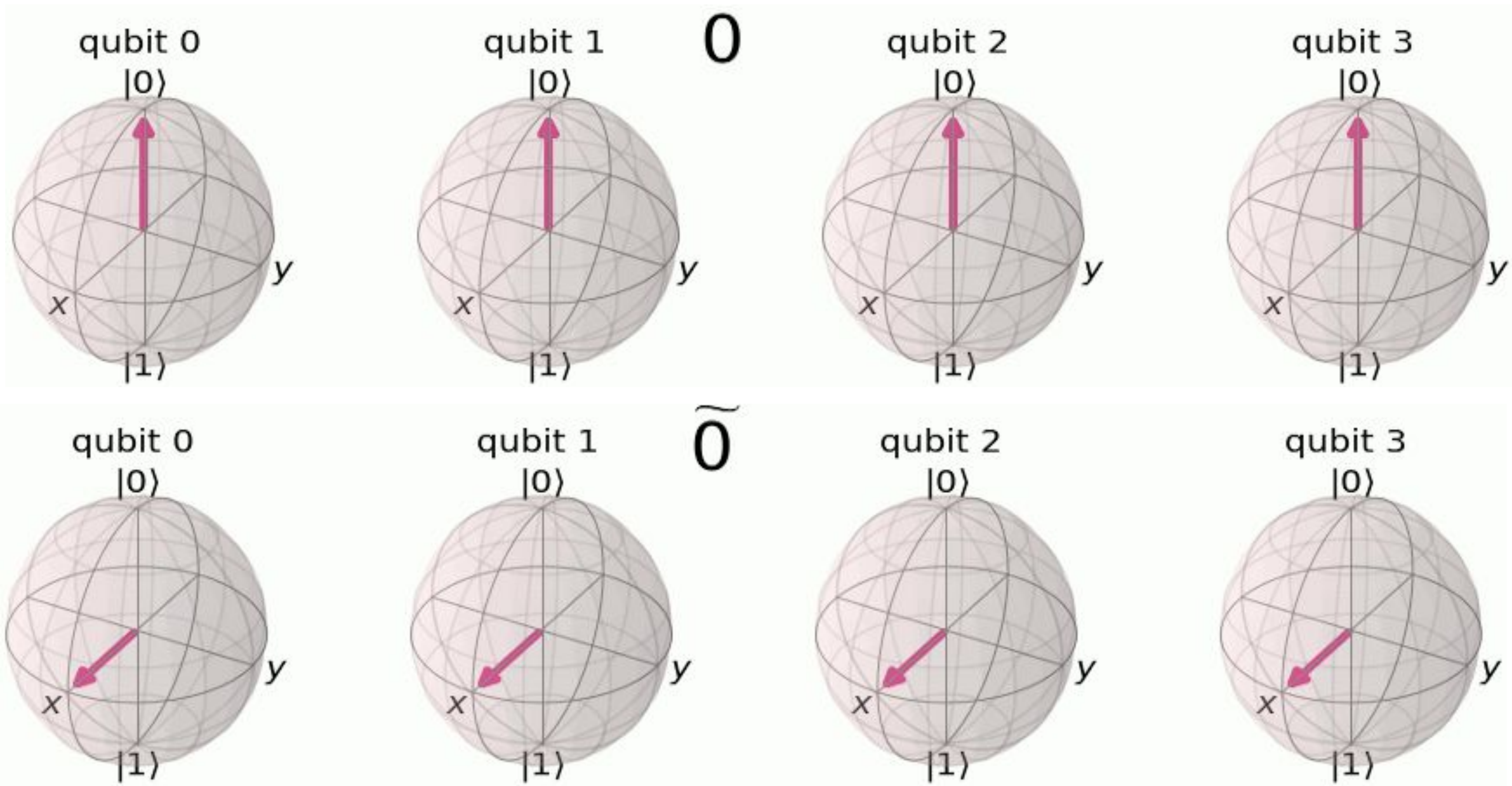
The Question:

- Let's say the first qubit rotates a total of 16 times before making a full rotation. Then the second rotates a total of 8 times. Then the third rotates a total of 4 times. Then the fourth rotates a total of 2 times.

The Quantum Fourier Transform (QFT)

The Question

The Question:



The Question:

- This is what is known as the Quantum Fourier Transform. It turns binary valued qubits into a series of rotations along the x-y axis.

The Question:

- These series of rotations are what is known as the ‘Fourier Basis’
- Values along the Fourier Basis are represented using $|\tilde{x}\rangle$, in which x is your usual integer.

Implementing a 1-bit QFT

QFT Formula

The following formula gives us our function for the QFT:

Note: $N = 2^n$

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x*y)}{N}} |y\rangle$$

QFT Formula

Look Familiar?

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x*y)}{N}} |y\rangle$$

QFT Formula

If we use only a single qubit, we get the following:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x * y)}{N}} |y\rangle$$

QFT Formula

$$\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi i(x*y)}{N}}|y\rangle$$

$$\frac{1}{\sqrt{2}}\sum_{y=0}^{2-1} e^{\frac{2\pi i(x*y)}{2}}|y\rangle$$

$$=\frac{1}{\sqrt{2}}\sum_{y=0}^{1} e^{\frac{2\pi i(x*y)}{2}}|y\rangle$$

QFT Formula

$$= \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{\frac{2\pi i (x*y)}{2}} |y\rangle$$

$$= \frac{1}{\sqrt{2}} (e^{\frac{2\pi i (x*0)}{2}} |0\rangle + e^{\frac{2\pi i (x*1)}{2}} |1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{\frac{2\pi i (x*1)}{2}} |1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{\pi i x} |1\rangle)$$

QFT Formula

Plugging in 0 and 1 for x:

$$|0\rangle \implies \frac{1}{\sqrt{2}}(|0\rangle + e^{\pi i 0}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \implies \frac{1}{\sqrt{2}}(|0\rangle + e^{\pi i 1}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

QFT Formula

Turns out, this formula is exactly equal to the Hadamard for N = 2.

$$\frac{1}{\sqrt{2}}\sum_{y=0}^{1} e^{\frac{2\pi i(x*y)}{2}}|y\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{1} e^{\pi i(x*y)}|y\rangle$$

Implementing an n-bit QFT

QFT Formula

While the previously given formula works, I rewrote the formula for clarity as the notation can be a bit confusing:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * y_{b10})}{N}} |y_{b2}\rangle$$

QFT Formula

This is to help differentiate between the $base_{10}$ and $base_2$ forms of x and y.

$$\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * y_{b10})}{N}} |y_{b2}\rangle$$

QFT Formula

We can rewrite y_{b2} using the binary notation discussed prior:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * y_{b10})}{N}} |y_1 y_2 \ldots y_n\rangle$$

QFT Formula

We can rewrite y_{b10} using the binary-decimal notation discussed prior:

$$\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * y_{b10})}{N}} |y_1 y_2 \ldots y_n\rangle$$

$$= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * \sum_{k=1}^{n} \frac{2^n y_k}{2^k})}{N}} |y_1 y_2 \ldots y_n\rangle$$

QFT Formula

The 2^n in the inner-most summation can be moved outside of the summation

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * \sum_{k=1}^{n} \frac{2^n y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} 2^n * \sum_{k=1}^{n} \frac{y_k}{2^k})}{N}} |y_1 y_2 \dots y_n\rangle$$

QFT Formula

Since we defined N to be equal to 2^n:

$$\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10}2^n * \sum_{k=1}^{n}\frac{y_k}{2^k})}{N}}|y_1 y_2 \ldots y_n\rangle$$

$$= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10}N * \sum_{k=1}^{n}\frac{y_k}{2^k})}{N}}|y_1 y_2 \ldots y_n\rangle$$

QFT Formula

The Ns cancel out, leaving us with the following:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} \cancel{N} * \sum_{k=1}^{n} \frac{y_k}{2^k})}{\cancel{N}}} |y_1 y_2 \ldots y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i (x_{b10} * \sum_{k=1}^{n} \frac{y_k}{2^k})} |y_1 y_2 \ldots y_n\rangle$$

QFT Formula

Using the product rule for exponentials:

$$\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i(x_{b10} * \sum_{k=1}^{n} \frac{y_k}{2^k})} |y_1 y_2 \ldots y_n\rangle$$

$$= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} \prod_{k=1}^{n} e^{\frac{2\pi i(x_{b10} * y_k)}{2^k}} |y_1 y_2 \ldots y_n\rangle$$

QFT Formula

We can redistribute the amplitudes here and rewrite this as a summation of tensor products:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{k=1}^{n} e^{\frac{2\pi i (x_{b10} * y_k)}{2^k}} |y_1 y_2 ... y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i (x_{b10} * y_1)}{2^1}} |y_1\rangle \otimes e^{\frac{2\pi i (x_{b10} * y_2)}{2^2}} |y_2\rangle \otimes ... \otimes e^{\frac{2\pi i (x_{b10} * y_n)}{2^n}} |y_n\rangle$$

QFT Formula

Which gives us the following formula:

$$\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i(x_{b10} * y_1)}{2^1}} |y_1\rangle \otimes e^{\frac{2\pi i(x_{b10} * y_2)}{2^2}} |y_2\rangle \otimes \ldots \otimes e^{\frac{2\pi i(x_{b10} * y_n)}{2^n}} |y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \bigotimes_{k=1}^{n} (|0\rangle + e^{\frac{2\pi i x_{b10}}{2^k}} |1\rangle)$$

QFT Formula

Plugging in x=6 provides the following answer:

$$\frac{1}{\sqrt{8}}\bigotimes_{k=1}^{n}(|0\rangle + e^{\frac{2\pi i6}{2^k}}|1\rangle)$$

$$= \frac{1}{\sqrt{8}}[(|0\rangle + e^{\frac{2\pi i6}{2^1}}|1\rangle) \otimes (|0\rangle + e^{\frac{2\pi i6}{2^2}}|1\rangle) \otimes (|0\rangle + e^{\frac{2\pi i6}{2^3}}|1\rangle)]$$

Building the Circuit

Building the Circuit

To start building the circuit, we need to figure out the pattern behind our formula.

$\lvert x_1 x_2 ... x_{n-2} x_{n-1} x_n \rangle$	$x_n amp. =$
$\lvert 00...000\rangle$	$\frac{1}{\sqrt{N}}$
$\lvert 00..001\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^n})}}{\sqrt{N}}$
$\lvert 00..010\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-1}})}}{\sqrt{N}}$
$\lvert 00..100\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-2}})}}{\sqrt{N}}$
$\cdots$	$\cdots$
$\lvert 11...111\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^1}+\frac{x_{b10}}{2^2}+\ldots+\frac{x_{b10}}{2^n})}}{\sqrt{N}}$

Building the Circuit

We notice that each '1' in our binary string results in our first x-value having an amplitude of $\exp(2\pi i/2^{n-k})$.

Also, k is a specific power of 2 in x.

$\|x_1 x_2 \ldots x_{n-2} x_{n-1} x_n\rangle$	$x_n amp. =$
$\|00\ldots000\rangle$	$\frac{1}{\sqrt{N}}$
$\|00..001\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^n})}}{\sqrt{N}}$
$\|00..010\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-1}})}}{\sqrt{N}}$
$\|00..100\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-2}})}}{\sqrt{N}}$
$\ldots$	$\ldots$
$\|11\ldots111\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^1}+\frac{x_{b10}}{2^2}+\ldots+\frac{x_{b10}}{2^n})}}{\sqrt{N}}$

Building the Circuit

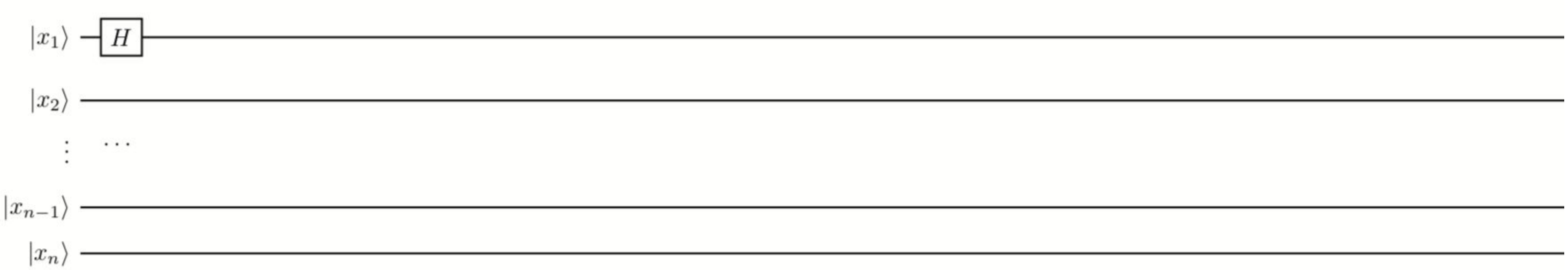
We can also notice that if multiple ‘1’s exist, the exponentials of each respective k get multiplied together.

$\|x_1 x_2 \ldots x_{n-2} x_{n-1} x_n\rangle$	$x_n amp. =$
$\|00\ldots000\rangle$	$\frac{1}{\sqrt{N}}$
$\|00..001\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^n})}}{\sqrt{N}}$
$\|00..010\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-1}})}}{\sqrt{N}}$
$\|00..100\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^{n-2}})}}{\sqrt{N}}$
$\ldots$	$\ldots$
$\|11\ldots111\rangle$	$\frac{e^{2\pi i(\frac{x_{b10}}{2^1}+\frac{x_{b10}}{2^2}+\ldots+\frac{x_{b10}}{2^n})}}{\sqrt{N}}$

Building the Circuit

Base case:

We know that the basic implementation of a 1-qubit QFT is the Hadamard.



Building the Circuit

Next step:

Let us define a gate called URot-Gate:

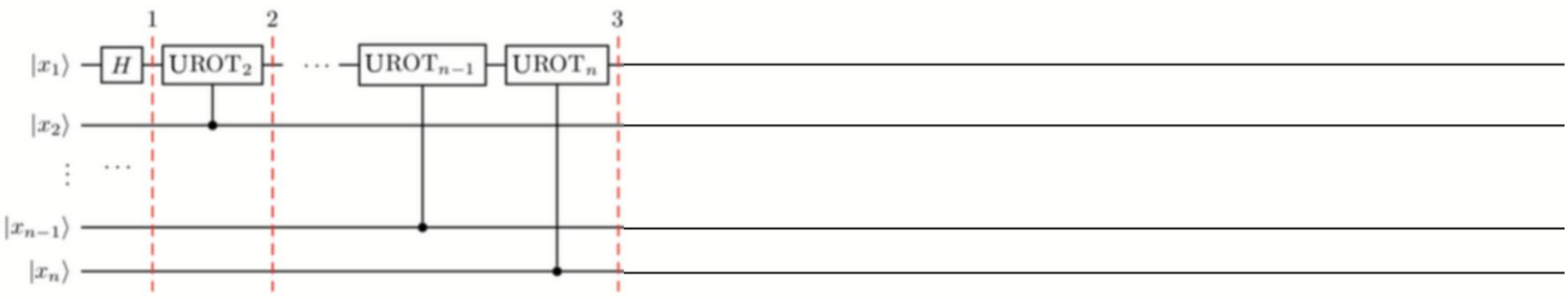
The URot-gate rotates an amount depending on the value of k.

$$UROT_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{2^k}\right) \end{bmatrix}$$

Building the Circuit

Next step:

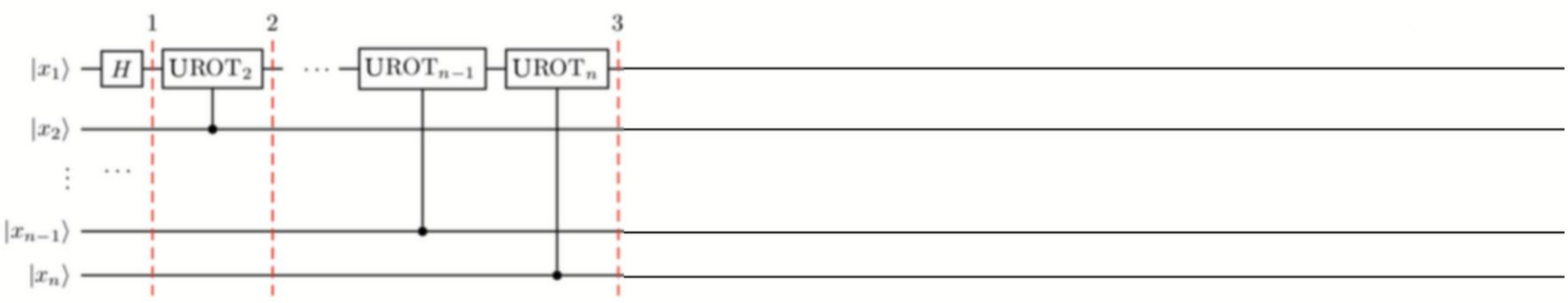
We attach these gates to a control bit. The kth bit attached to the control bit is equal to the k-distance from the UROT gate.



Building the Circuit

After the 1st URot:

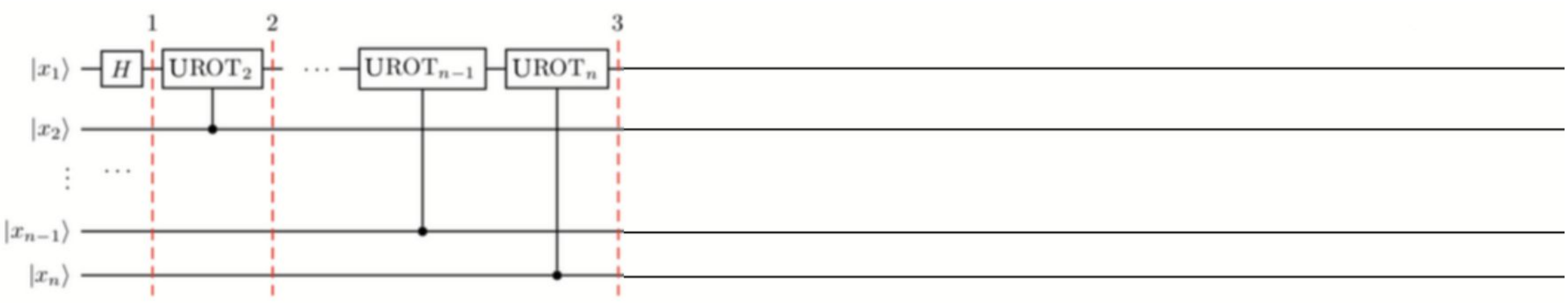
$$\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2}x_1\right)|1\rangle\right] \otimes |x_2 x_3 \ldots x_n\rangle$$



Building the Circuit

After the 2nd URot:

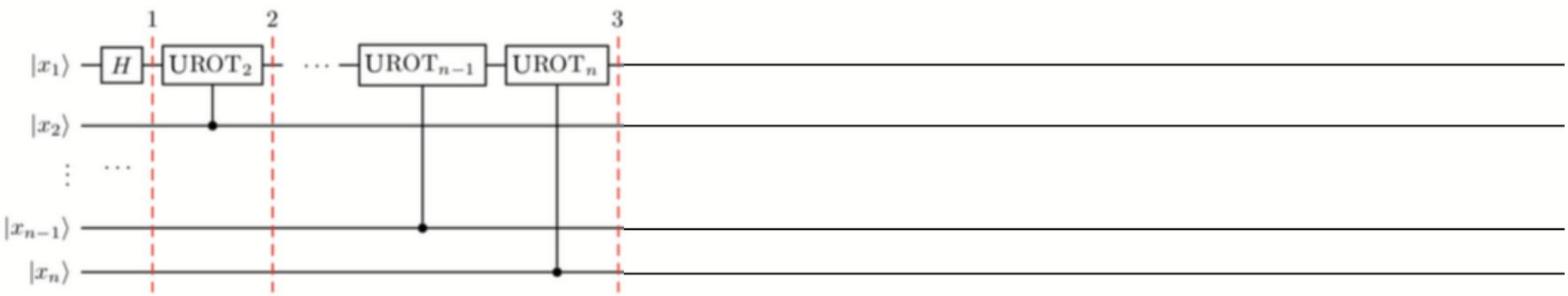
$$\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)|1\rangle\right] \otimes |x_2 x_3 \ldots x_n\rangle$$



Building the Circuit

After the n[th] UROT:

$$\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^n}x_n + \frac{2\pi i}{2^{n-1}}x_{n-1} + \ldots + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)|1\rangle\right] \otimes |x_2 x_3 \ldots x_n\rangle$$



Building the Circuit

The Result:

- The resulting value:

$$\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^n}x_n + \frac{2\pi i}{2^{n-1}}x_{n-1} + \ldots + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)|1\rangle\right] \otimes |x_2x_3\ldots x_n\rangle$$

- is the exact value of:

$$e^{\frac{2\pi i x_{b10}}{2^n}}$$

Building the Circuit

$$x_{b10} = 2^{n-1} * x_1 + 2^{n-2} * x_2 + ... + 2^{n-n} * x_n$$

$$\implies \frac{2\pi i}{2^n} x_n + \frac{2\pi i}{2^{n-1}} x_{n-1} + ... + \frac{2\pi i}{2^2} x_2 + \frac{2\pi i}{2^1} x_1$$

$$\implies \frac{1}{\cancel{2\pi i}} * (\frac{\cancel{2\pi i}}{2^n} x_n + \frac{\cancel{2\pi i}}{2^{n-1}} x_{n-1} + ... + \frac{\cancel{2\pi i}}{2^2} x_2 + \frac{\cancel{2\pi i}}{2^1} x_1)$$

$$= \frac{1}{2^n} x_n + \frac{1}{2^{n-1}} x_{n-1} + ... + \frac{1}{2^2} x_2 + \frac{1}{2^1} x_1$$

$$\implies 2^n * (\frac{1}{2^n} x_n + \frac{1}{2^{n-1}} x_{n-1} + ... + \frac{1}{2^2} x_2 + \frac{1}{2^1} x_1)$$

$$= 2^{n-n} x_n + 2^{n-n+1} x_{n-1} + ... + 2^{n-2} x_2 + 2^{n-1} x_1$$

$$= 2^{n-1} * x_1 + 2^{n-2} * x_2 + ... + 2^{n-n} * x_n$$

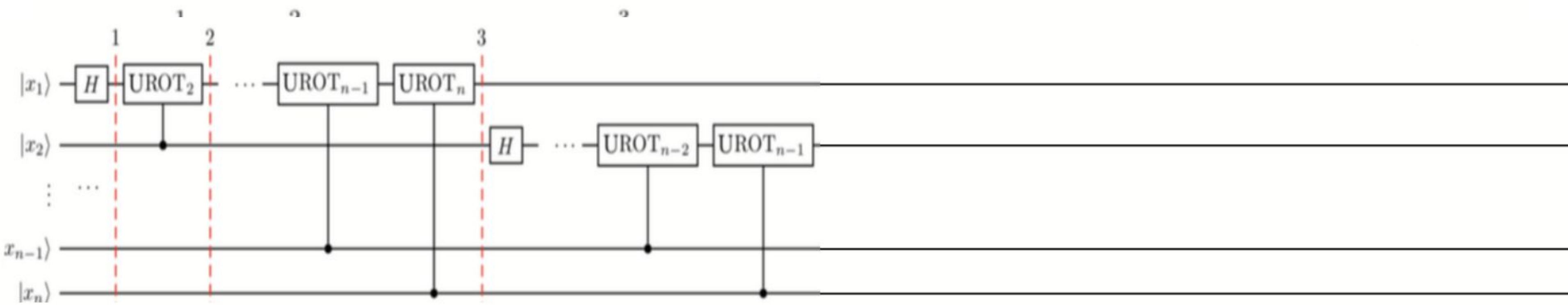
$$= x_{b10}$$

$$\implies \frac{2\pi i}{2^n} x_{b10} = amp.|x\rangle$$

Building the Circuit

Next step:

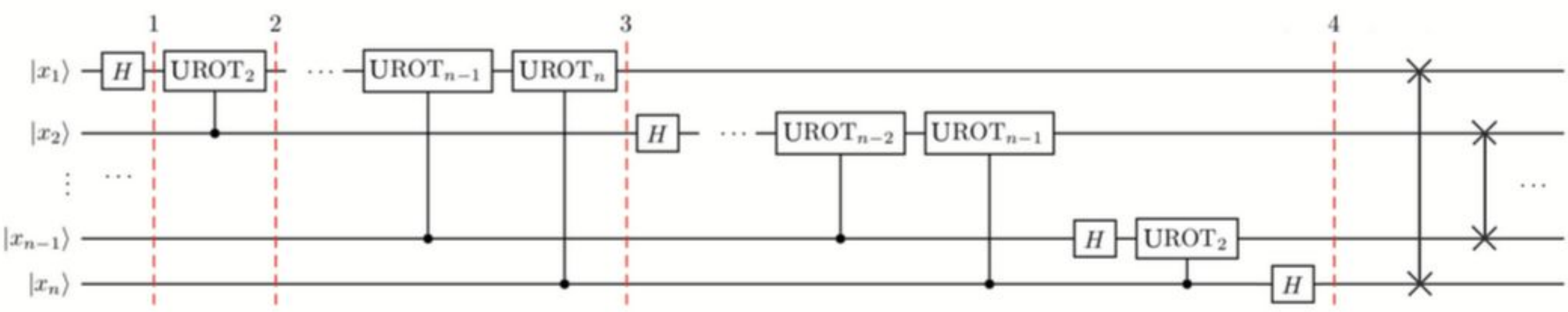
We can repeat this process for the next value and get the second portion of the circuit.



Building the Circuit

Next step:

This can be repeated indefinitely until we get to the last qubit, which is by itself and therefore only requires a Hadamard gate.



Building the Circuit

After applying the circuit, we get the following state:

$$\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^n}x\right)|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^{n-1}}x\right)|1\rangle\right] \otimes \ldots \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^2}x\right)|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^1}x\right)|1\rangle\right]$$

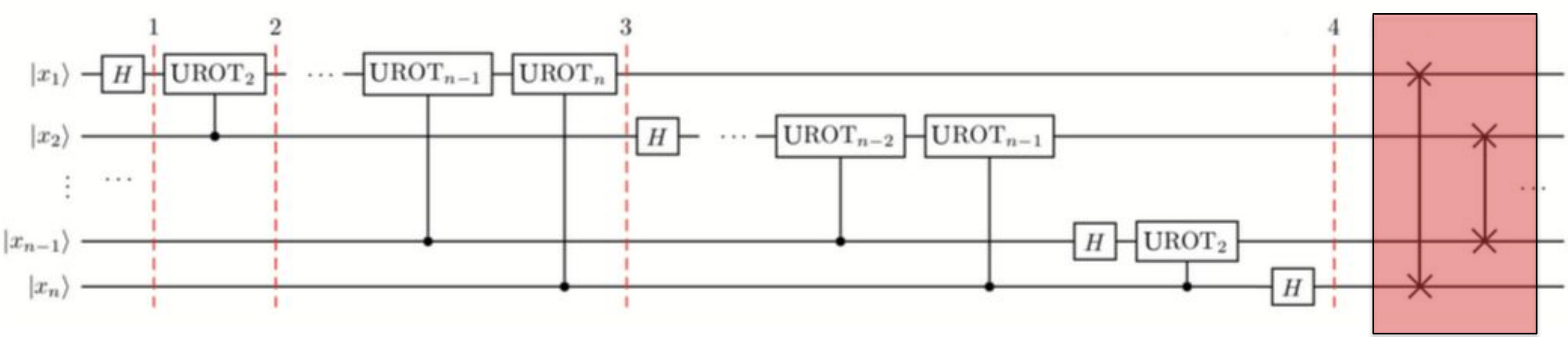
Which is equal to the function we defined earlier:

$$= \frac{1}{\sqrt{N}} \bigotimes_{k=1}^{n} \left(|0\rangle + e^{\frac{2\pi i x_{b10}}{2^k}} |1\rangle\right)$$

Building the Circuit

Final Step:

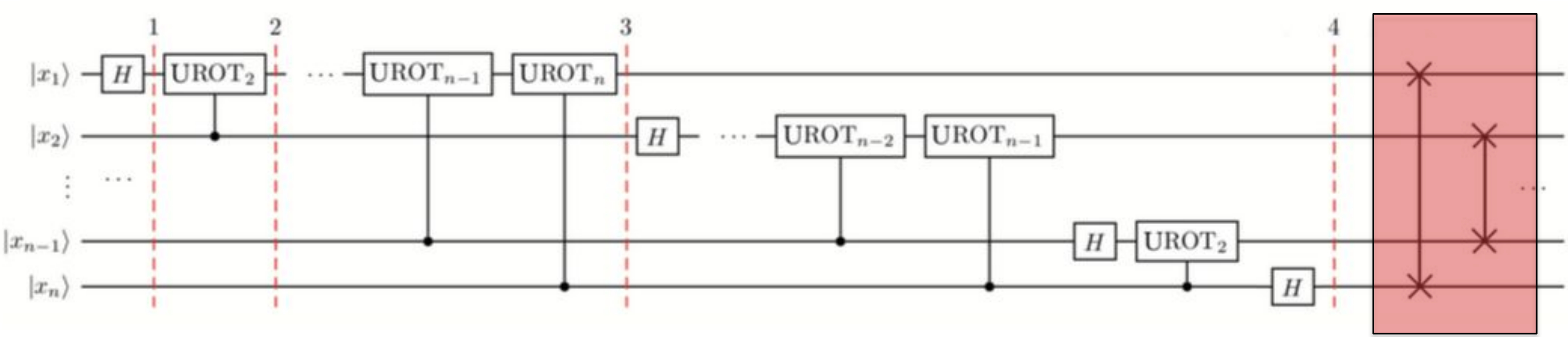
It is important to notice that our final output is upside down, our x_1 gives the x_n amplitude!



Building the Circuit

Final Step:

To solve this, we simply add a swap gate that swaps the amplitudes.



Coding

Now, lets code this in Qiskit!

Let’s Get Started