QuantA&M Book Club
Meeting 2

QuantA&M of Texas A&M University

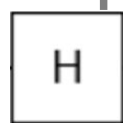
A Summary of Meeting 1

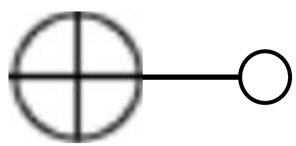
Qubits

- As discussed in the first book club, a qubit can be interpreted as a probability of measuring 0 or 1.

$$|0\rangle = 1 * |0\rangle + 0 * |1\rangle$$
$$|1\rangle = 0 * |0\rangle + 1 * |1\rangle$$

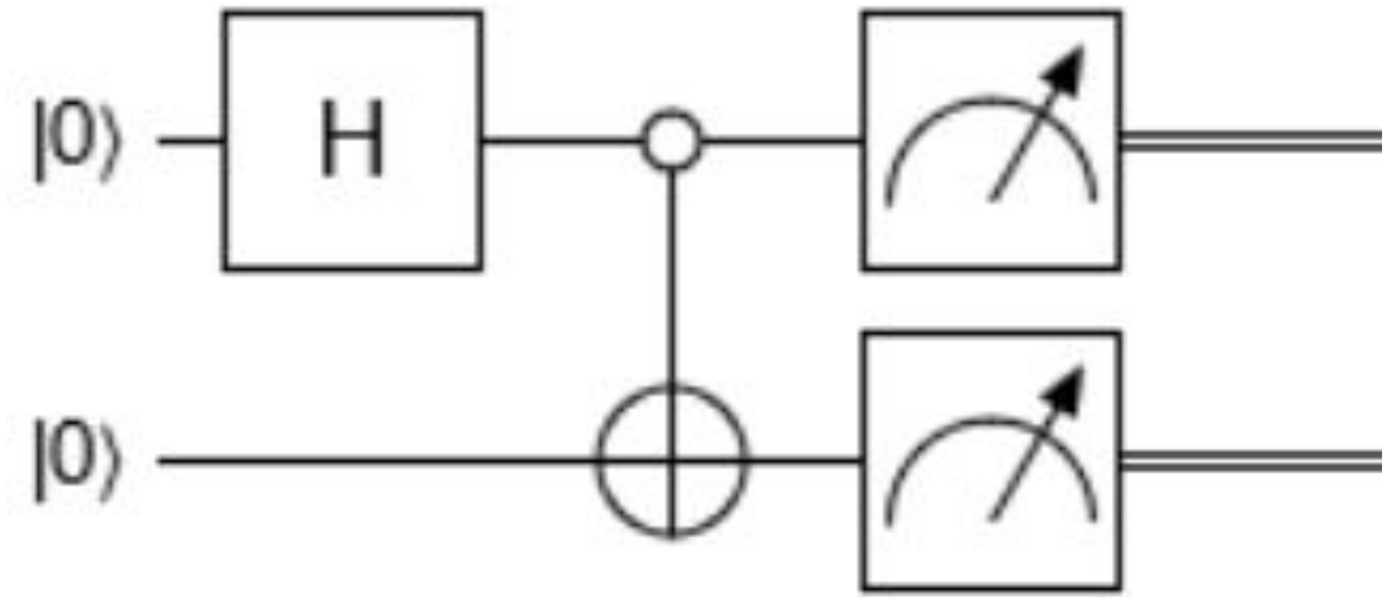
Covered Gates

- These qubits can be manipulated using gates. For example:
  - Hadamard Gate – 

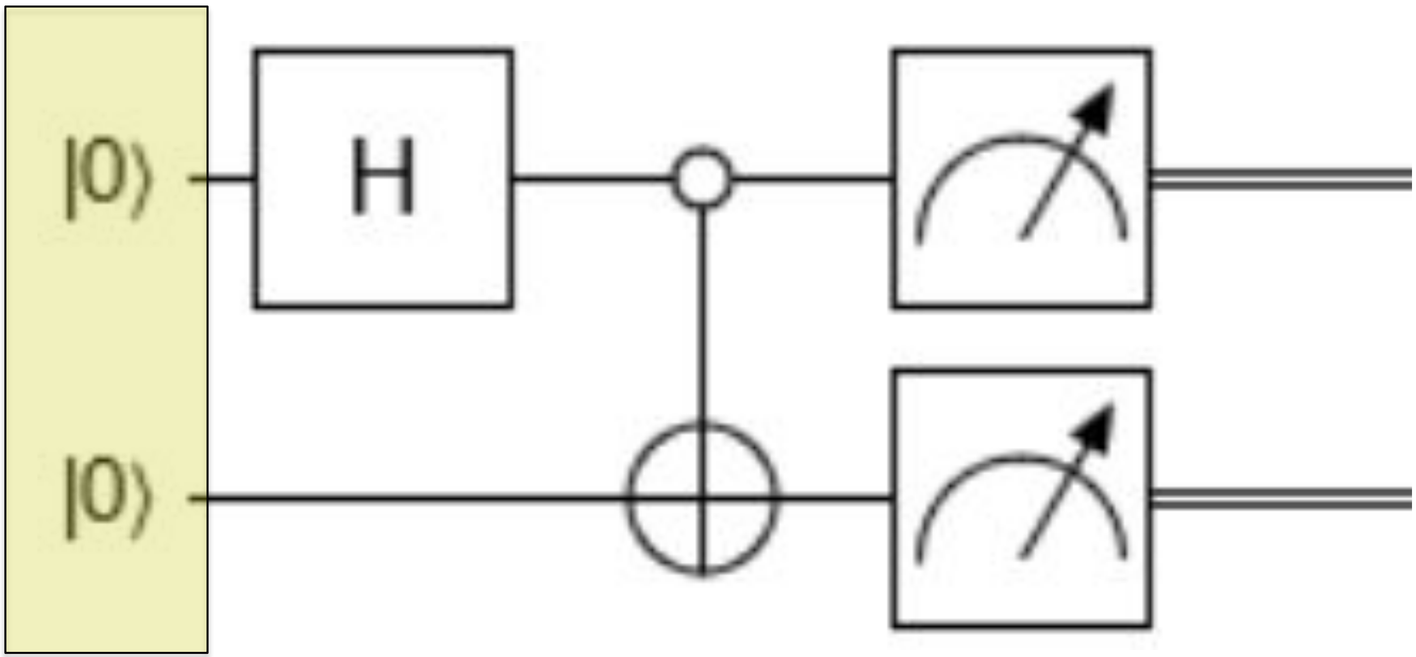
  - Not/X Gate – 
  - C-Not Gate – 

A Basic Quantum Circuit

- We also ran through a quantum circuit, implementing the bell states.

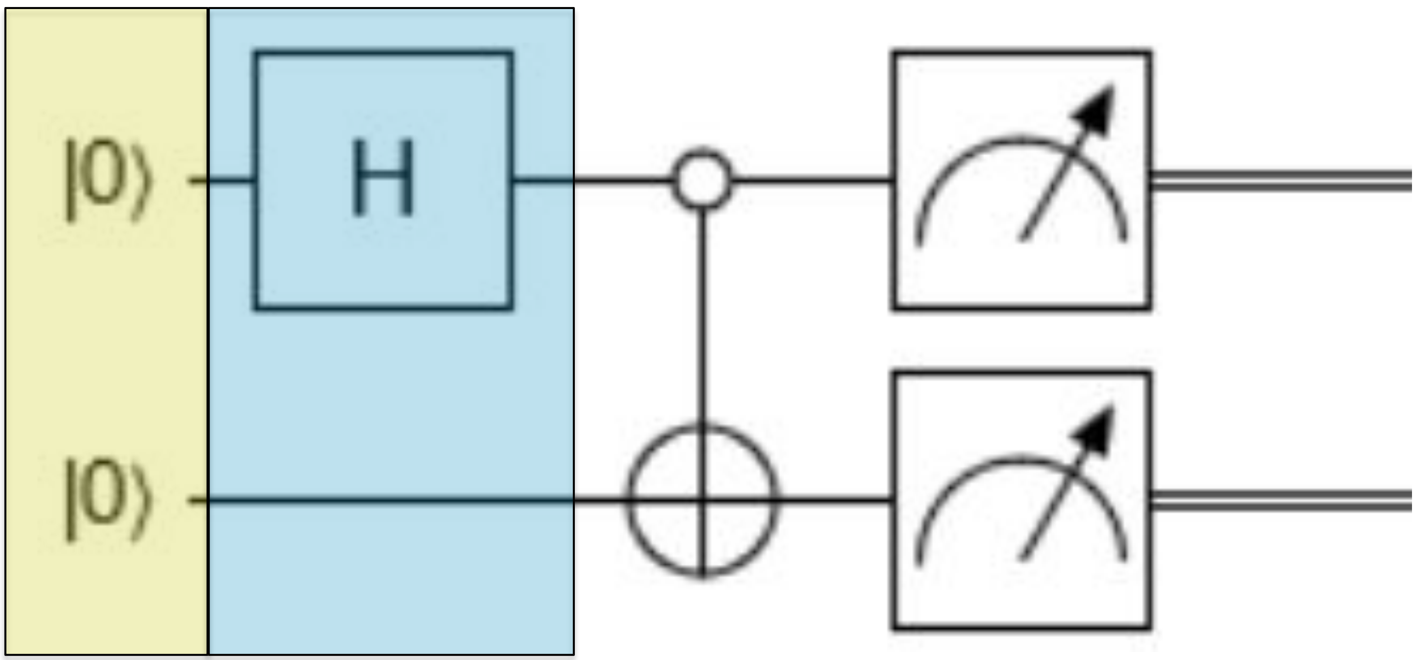


A Basic Quantum Circuit



$\langle 0|$
$\langle 0|$

A Basic Quantum Circuit



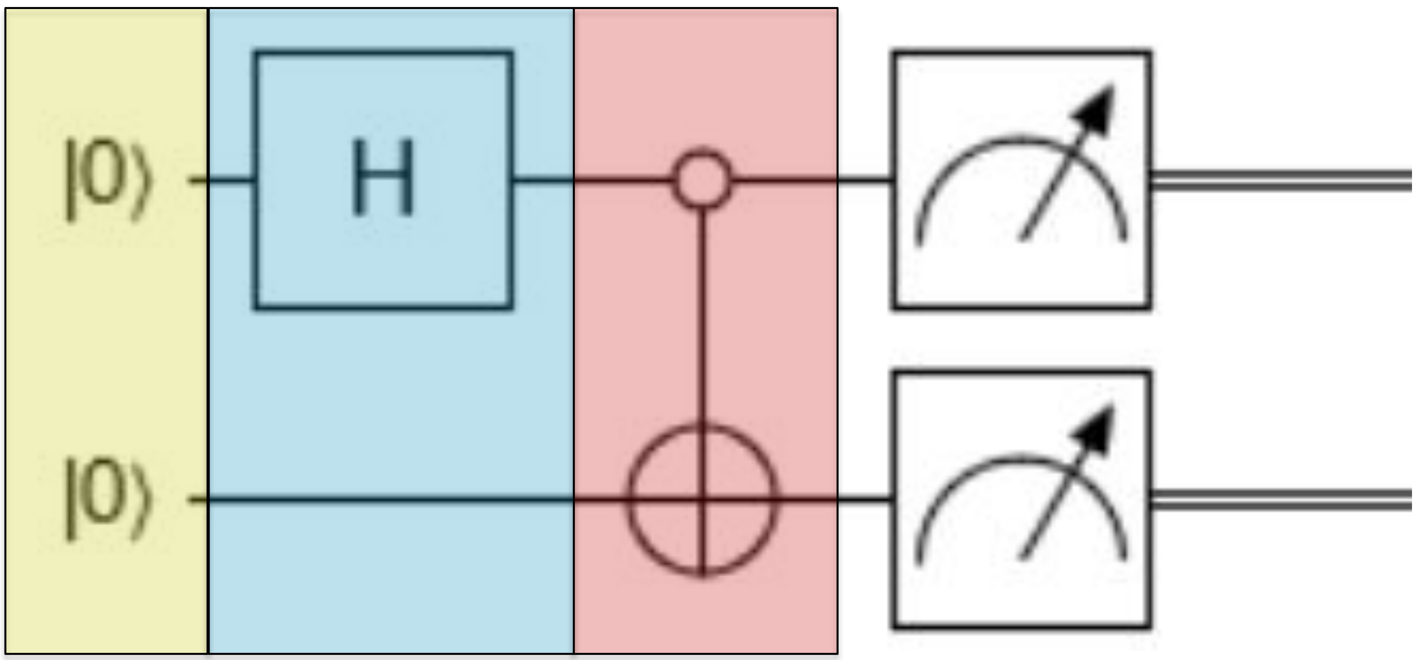
$$\langle 0| \qquad \langle 0|$$

$$\langle 0| \implies \frac{\sqrt{2}}{2}\langle 0| + \frac{\sqrt{2}}{2}\langle 1|$$
$$\langle 0| \implies \langle 0|$$

A Basic Quantum Circuit

$$\frac{\sqrt{2}}{2}\langle 0| + \frac{\sqrt{2}}{2}\langle 1| \implies \frac{\sqrt{2}}{2}\langle 00| + \frac{\sqrt{2}}{2}\langle 11|$$

$$\langle 0| \implies \frac{\sqrt{2}}{2}\langle 00| + \frac{\sqrt{2}}{2}\langle 11|$$



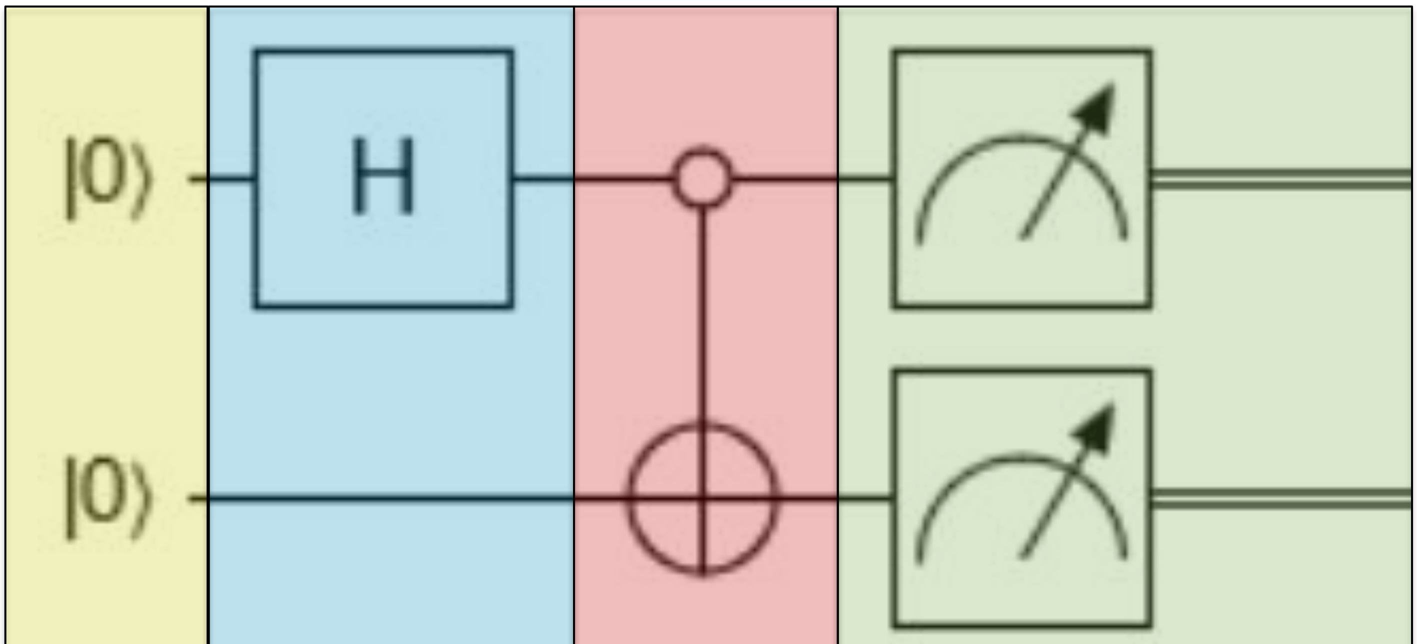
$$\langle 0|$$
$$\langle 0|$$

$$\langle 0| \implies \frac{\sqrt{2}}{2}\langle 0| + \frac{\sqrt{2}}{2}\langle 1|$$

$$\langle 0| \implies \langle 0|$$

A Basic Quantum Circuit

$$\frac{\sqrt{2}}{2}\langle 0| + \frac{\sqrt{2}}{2}\langle 1| \implies \frac{\sqrt{2}}{2}\langle 00| + \frac{\sqrt{2}}{2}\langle 11|$$
$$\langle 0| \implies \frac{\sqrt{2}}{2}\langle 00| + \frac{\sqrt{2}}{2}\langle 11|$$



$$\langle 0|$$
$$\langle 0|$$

$$\langle 0| \implies \frac{\sqrt{2}}{2}\langle 0| + \frac{\sqrt{2}}{2}\langle 1|$$
$$\langle 0| \implies \langle 0|$$

Classical Bit Measurement

Qubit Math

Useful Notation

Some Useful Notation

$$\langle\Psi||\Phi\rangle = \langle\Psi\Phi\rangle \quad \text{"Inner Product"}$$
$$|\Psi\rangle\langle\Phi| = |\Psi\Phi| \quad \text{"Outer Product"}$$
$$|\Psi\rangle|\Phi\rangle = |\Psi\Phi\rangle \quad \text{"Tensor Product"}$$
$$AB = A \times B \quad \text{"Matrix Multiplication"}$$

Qubit Math

Qubits as Vectors

Single-Qubits As Vectors

- Qubits can also be represented as vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Each element in the vector is an amplitude of a measurement.

Qubit Math

Tensor Products of Qubits

Multi-Qubits As Vectors

- Multiple qubits can be represented as a single tensor product of qubits.

$$|ba\rangle = |b\rangle \otimes |a\rangle = \begin{bmatrix} b_0 \times \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \\ \\ b_1 \times \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{bmatrix}$$

- For every n-sized qubit, the size of the vector is 2^n.

Qubit Math

Gates as Matrices

Gates as Matrices

- The previously mentioned gates can be interpreted as NxN Matrices.
- These gates will aways be NxN matrices. N being the number of qubits.

Example Matrix Operation: Hadamard Gate

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} * \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} * \begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Example Matrix Operation: X/Not Gate

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example Matrix Operation: Identity Gate

$$I|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$I|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Qubit Math

Matrix Multiplication

Matrix Multiplication

- Like numbers, matrices can be multiplied in order to create new matrices.
- There are subtle differences between how normal multiplication and matrix multiplication works.

Example Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} * \begin{bmatrix} 6 & 5 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} -19 & -9 \\ -28 & 4 \end{bmatrix}$$

Combining Gates

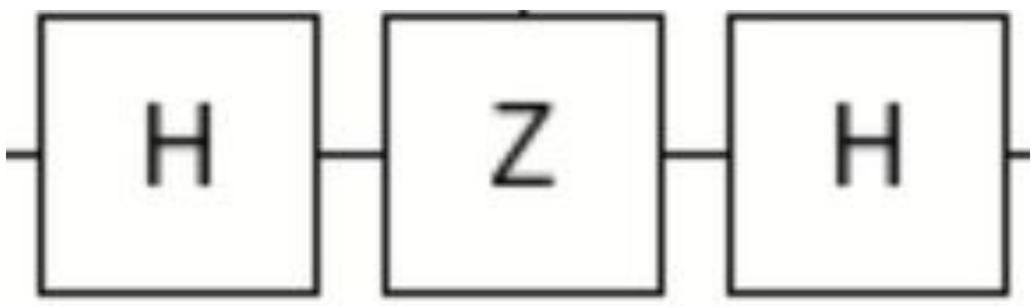
- Quantum logic gates can be combined in order to create more circuits.

$$HZH = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

- This is an example of a two Hadamards and a Z-Gate making an X-Gate.

Combining Gates

The original circuit for reference:



Qubit Math

Tensor Products of Gates

Tensor Product for Matrices

- Like with qubits, matrices affecting multiple qubits can also be represented as a larger matrix.
- If matrix A is effects Q0, and matrix B effects Q1, then the matrix for both qubits is simply the tensor product between matrix A and B.

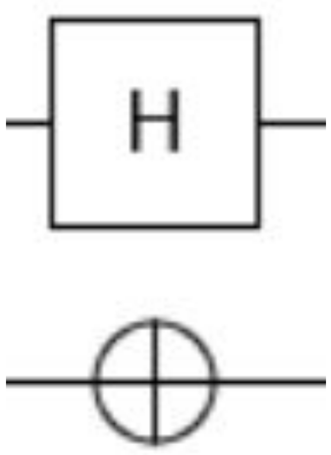
$$X \otimes H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Tensor Product for Matrices

- Once again, the circuit for reference:



Qubit Math

Circuit Identities

Circuit Identities

- Using matrix multiplication, we can find the following useful formulas and relationships between our gates:

$$HZH = X$$

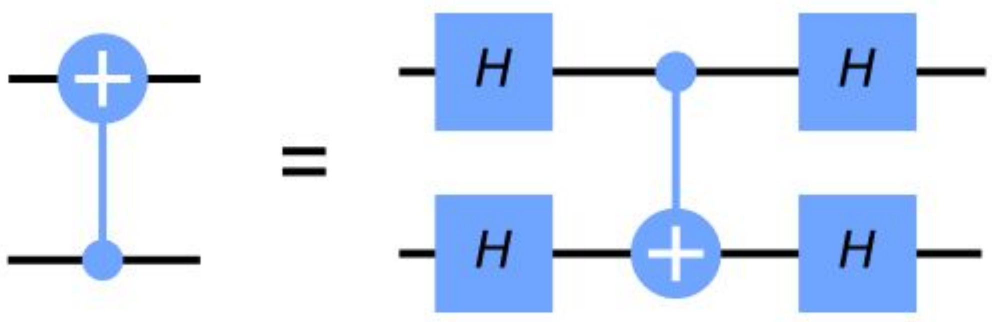
$$HXH = Z$$

$$HYH = -Y$$

$$H \otimes H * CNOT[Q1, Q0] * H \otimes H = CNOT[Q0, Q1]$$

Circuit Identities

- The last identity looks confusing but is actually quite intuitive. Here is the circuit diagram if done on qiskit:



Phase Kickback

Phase Kickback: The Controlled T-Gate

- An interesting gate can be created that changes the amplitude of our control bit. This is called a Controlled T-gate.

$$\text{Controlled-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}(1+i) \end{bmatrix}$$

$$T|1\rangle = e^{i\pi/4}|1\rangle$$

Phase Kickback: A Controlled T-Gate's Output

- We can of course use a controlled T-gate to manipulate a qubit. Notice how the modified qubit is the control bit.

$$|1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + |1\rangle)$$

$$= \frac{\sqrt{2}}{2} * (|10\rangle + |11\rangle)$$

$$\implies T|1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + |1\rangle)$$

$$= T\frac{\sqrt{2}}{2} * (|10\rangle + |11\rangle)$$

$$= T|1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + |1\rangle)$$

$$= |1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + e^{\frac{i\pi}{4}}|1\rangle)$$

Phase Kickback: Why is This Important?

- What was just observed was a concept known as phase kickback.
- Without superposition, a rotation of a qubit is meaningless, as all that really matters is the magnitude of the amplitude.
- With superposition however, a rotation of one of the superpositions can have a measurable impact on computations.
- The IBM textbook goes fairly in depth on the math behind this concept. I would recommend giving it a read.

Phase Kickback: Why is This Important?

$$T|1\rangle \otimes |1\rangle$$
$$= |1\rangle \otimes e^{\frac{i\pi}{4}}|1\rangle$$

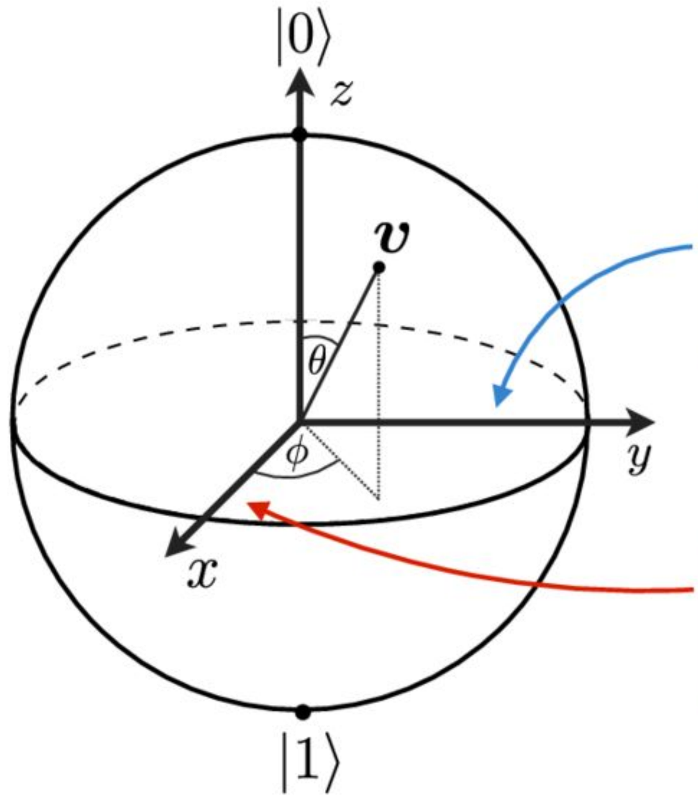
$$T|1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + |1\rangle)$$
$$= |1\rangle \otimes \frac{\sqrt{2}}{2} * (|0\rangle + e^{\frac{i\pi}{4}}|1\rangle)$$

Bloch Sphere

The Bloch Sphere:

- The Bloch sphere is a visual representation of our qubit vector.
- It is a sphere with a radius of 1. Every point in the sphere is a potential amplitude value.

Visualizing a Qubit:



Pole states:

$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Let’s Get Started