$\boxed{\text{Eigenvalue power method}}$ → used to find largest eigenvalue and it's eigenvector

Suppose A is an nxn matrix with n real distinct eigenvalues.

now, order eigenvalues by magnitude (with their eigenvector)

$|\lambda_1| > |\lambda_2| > |\lambda_3| > \dots |\lambda_n|$ → λ_1 is largest (dominant)

$e_1 \quad e_2 \quad e_3 \dots e_n$ — eigenvector (linearly independant)

Start with any initial vector x_0 (can be any).

any vector can be written as a combination of the eigenvectors: $x_0 = c_1e_1 + c_2e_2 + \dots c_ne_n$

$$\therefore x_0 = \sum_{i=1}^{n} c_ie_i = c_1e_1 + c_2e_2 + c_3e_3 + \dots c_ne_n$$

Now multiply by A: $x_1 = Ax_0$

$$= A\sum_{i=1}^{n} c_ie_i = \sum_{i=1}^{n}(c_iAe_i)$$

recall $AX = \lambda X$ where X is an eigenvector

$$\therefore \sum_{i=1}^{n} c_iAe_i$$

$$= \sum_{i=1}^{n} c_i\lambda e_i$$

now, if we multiply x_0 by A^p we have:

$$x_p = A^px_0 = \sum_{i=1}^{n} c_iA^pe_i = \sum_{i=1}^{n} c_i\lambda_i^pe_i$$

$\therefore$ since λ_1 is the largest eigenvalue, so λ_1^p is the largest number.

Factoring out λ_1 term:

$$x_p = c_1\lambda_1^pe_1 + \sum_{i=2}^{n} c_i\lambda_i e_i$$

$$\text{or, } x_p = \lambda_1^p\left[c_1e_1 + \sum_{i=2}^{n} c_i\left(\frac{\lambda_i}{\lambda_1}\right)e_i\right]$$

These terms tend to 0 as λ_1 in denominator is largest eigenvalue

$$\therefore x_p \simeq \lambda_1^pc_1e_1$$

multiply this by A again to get:

$$x_{p+1} = \lambda_1^{p+1}c_1e_1$$

$$\text{or } \lambda_1 = \frac{x_p^T x_{p+1}}{x_p^T \cdot x_p}$$

This is to get rid of c_1 and e_1 and get back the dominant λ_1.

$e_1 = x_{p+1}$ → because they are the same basic vectors (this x_p is normalized)

Note - overflow can happen $\lambda_1 > 1$ and is multiplied too many times by itself and becomes larger than the largest machine number.
OR there can be underflow when $\lambda_1 < 0$ and too many multiplications by itself can cause underflow (number is lower than real min).
We can avoid this by normalizing x_p:
x_p is saved with each iteration as $x_p = \dfrac{x_p}{(x_p^T x_p)^{1/2}}$

Power method example

$$A = \begin{bmatrix} 6 & 5 \\ 4 & 5 \end{bmatrix} \longrightarrow \lambda_1 = 10,\ v_1 = \begin{bmatrix} 5/4 \\ 1 \end{bmatrix} \oplus \lambda_2 = 1,\ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

First we need an initial vector to multiply by A

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore x_1 = Ax_0$

$$\text{or, } x_1 = \begin{bmatrix} 6 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

now continue to multiply the resultant vectors by A:

$x_2 = Ax_1$

$$\text{or, } x_2 = \begin{bmatrix} 6 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 56 \\ 44 \end{bmatrix}$$

$x_3 = Ax_2$

$$x_3 = \begin{bmatrix} 6 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 56 \\ 44 \end{bmatrix} = \begin{bmatrix} 556 \\ 444 \end{bmatrix}$$

$$x_4 = Ax_3 = \begin{bmatrix} 6 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 556 \\ 444 \end{bmatrix} = \begin{bmatrix} 5556 \\ 4444 \end{bmatrix}$$

$$x_5 = Ax_4 = \begin{bmatrix} 6 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 5556 \\ 4444 \end{bmatrix} = \begin{bmatrix} 55556 \\ 44444 \end{bmatrix}$$

$$\therefore \lambda_1 = \frac{x_4^T x_5}{x_4^T x_4} = \frac{506{,}178{,}271}{50{,}618{,}272} \simeq 10$$

$\therefore$ Eigenvector $e_1 = x_p$

$$\text{or, } e_1 = x_5 = \begin{bmatrix} 55556/44444 \\ 1 \end{bmatrix} \simeq \begin{bmatrix} 5/4 \\ 1 \end{bmatrix}$$

Another example: Use the power method to find dominant eigenvalue of: $A = \begin{bmatrix} -5 & 6 \\ 5 & -4 \end{bmatrix}$

Solution: start with an initial vector $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore x_1 = Ax_0 = \begin{bmatrix} -5 & 6 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

$$x_2 = Ax_1 = \begin{bmatrix} -5 & 6 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 55 \\ -45 \end{bmatrix}$$

$$x_3 = Ax_2 = \begin{bmatrix} -545 \\ 455 \end{bmatrix}$$

$$x_4 = Ax_3 = \begin{bmatrix} 5455 \\ -4545 \end{bmatrix}$$

$$x_5 = \begin{bmatrix} -54545 \\ 45455 \end{bmatrix}$$

$$\therefore \lambda_1 = \frac{x_4^T x_5}{x_4^T x_4}$$

$$\text{or, } \lambda_1 = \frac{\begin{bmatrix} -54545 & 45455 \end{bmatrix} \begin{bmatrix} -54545 \\ 45455 \end{bmatrix}}{\begin{bmatrix} -54545 & 45455 \end{bmatrix} \begin{bmatrix} 5455 \\ -4545 \end{bmatrix}}$$

$$\text{or, } \lambda_1 = \frac{5041314050}{-504135950} \simeq -10$$

$\therefore e_1 = x_5$ (normalized)

$$\therefore e_1 = \begin{bmatrix} -1.2 \\ 1 \end{bmatrix}$$