The theory

A logistic map is a 1-D map
start with a initial value and plug into some function $f(x_n)$ to get a sequential value
$x_0, x_1, x_2 \ldots x_n$
and $x_{n+1} = f(x_n)$

There may be special 'fixed points' that do not change
$x_* = f(x_*)$ (no change)

We want to determine stability of x_* (see how distance from fixed point changes)
start with point that is ϵ distance away from fixed point $\longrightarrow$ $x_n = x_* + \epsilon_n$
$x_{n+1} = x_* + \epsilon_{n+1}$

if ϵ_{n+1} causes x_{n+1} to approach x_* it is stable (through iteration)
if ϵ_{n+1} causes x_{n+1} to diverge from x_* it is unstable (through iteration)

plug x_n into $f(x_n)$:
$\therefore x_* + \epsilon_{n+1} = f(x_* + \epsilon_n)$
do taylor series about x_*:
$\therefore x_* + \epsilon_{n+1} = f(x_*) + \epsilon_n f'(x_*) + \ldots$ higher terms neglected
or, $x_* + \epsilon_{n+1} = x_* + \epsilon_n f'(x_*)$
or, $\dfrac{\epsilon_{n+1}}{\epsilon_n} = f'(x_*)$

to see if fixed point is stable, take magnitude:
$\therefore \left|\dfrac{\epsilon_{n+1}}{\epsilon_n}\right| = |f'(x_*)| \xrightarrow{\text{stable}}$ if $|f'(x_*)| < 1$
$\xrightarrow{\text{unstable}}$ if $|f'(x_*)| > 1$

Logistic map equation: $x_{n+1} = \mu x_n(1 - x_n)$
$0 < \mu < 4$

[for assignment need to plot x vs. μ]

To find out fixed points: $x_* = \mu x_*(1 - x_*)$
or, $x_*(1 - \mu(1 - x_*)) = 0$
$\therefore x_* = 0$ | $1 - \mu(1 - x_*) = 0$
or, $1 - x_* = \dfrac{1}{\mu}$
or, $x_* = 1 - \dfrac{1}{\mu}$

Also derivative $\rightarrow$ $x_* = \mu x_*(1 - x_*)$
$f'(x) = \mu - 2\mu x_*$

if $x_* = 0$, $f'(x) = \mu$ and fixed point will be stable if $|f'(x)| < 1$, ie
$0 < \mu < 1$

if $x_* = 1 - \dfrac{1}{\mu} \longrightarrow f'(1 - \frac{1}{\mu}) = \mu - 2\mu(1 - \frac{1}{\mu})$
or, $f'(1 - \frac{1}{\mu}) = 2 - \mu$

To be stable $|f'(1 - \frac{1}{\mu})| < 1 \Rightarrow |2 - \mu| < 1$
$-(2 - \mu) < 1$, $2 - \mu < 1$
or, $\mu < 3$, $\mu > 1 \longrightarrow 1 < \mu < 3$

if $\mu > 4$ then the logistic map will show 2 more points and x_* will not be fixed anymore (to be seen in matlab).

Period-2 cycle map with $x_2 = f(x_1)$ and $x_1 = f(x_2)$

∴ $x_1 = f(f(x_1))$ with $f(x) = rx(1-x)$ and $x = 0, 1-\frac{1}{\mu}$ are roots

Find the other roots:

$$x = \mu(\mu x(1-x))(1-\mu x(1-x))$$
$$= \mu(\mu x - \mu x^2)(1-\mu x+\mu x^2)$$
$$= \mu(\mu x - \mu^2x^2 + \mu^2 x^3 - \mu x^2 + \mu^2 x^3 - \mu^2 x^4)$$
$$= -\mu^3x^4 + 2\mu^3x^3 - \mu^2(\mu x^2 + x^2) + \mu^2 x \longrightarrow f(f(x))$$
$$\therefore \mu^3x^4 - 2\mu^3x^3 + \mu^2(\mu+1)x^2 + (1-\mu^2)x = 0$$

since $x = 1-\frac{1}{\mu}$ is a root:

use long-division:

$$\mu^2x^3 - (\mu+\mu^2)x^2 + (1+\mu)x$$
$$\mu x - \mu + 1 \,\big)\, \mu^3x^4 - 2\mu^3x^3 + \mu^2(\mu+1)x^2 + (1-\mu^2)x = 0$$
$$\mu^3x^4 - \mu^3x^3 + \mu^2x^3$$
$$-(\mu^2+\mu^3)x^3 + \mu^2(\mu+1)x^2 + (1-\mu^2)x$$
$$-(\mu^2+\mu^3)x^3 + (\mu^2+\mu^3)x^2 - (\mu+\mu^2)x^2$$
$$(\mu+\mu^2)x^2 + (1-\mu^2)x$$
$$(\mu+\mu^2)x^2 - (\mu+\mu^2)x + (1+\mu)x$$
$$0$$

$$[\mu^2x^3 - (\mu+\mu^2)x^2 + (\mu+1)x](\mu x - \mu + 1) = 0$$

since $x = 0$ is a factor:

$$\therefore [\mu^2x^2 - \mu(1+\mu)x + (\mu+1)][\mu x - \mu + 1][x] = 0$$
$$\therefore \mu^2x^2 - \mu(1+\mu)x + \mu + 1 = 0$$
$$\therefore x = \frac{\mu(1+\mu) \pm \sqrt{\mu^2(1+\mu)^2 - 4(\mu^2)(\mu+1)}}{2\mu}$$

or, $x = \dfrac{\mu+\mu^2 \pm \sqrt{(\mu+1)[\mu^2(\mu+1) - 4\mu^2]}}{2\mu}$

or, $x = \dfrac{1+\mu}{2} \pm \dfrac{\sqrt{(\mu+1)(\mu-3)\mu^2}}{2\mu}$

or, $x = \dfrac{1+\mu}{2} \pm \dfrac{\sqrt{(\mu+1)(\mu-3)}}{2}$

Now to test stability do $f'[f(x)]$

∴ $f'[f(x)] = -4\mu^3x^3 + 6\mu^3x^2 - 2\mu^2(\mu+1)x + \mu^2$

∴ for point to be stable, $|f'(x)|$ evaluated at $\dfrac{1+\mu}{2} \pm \dfrac{\sqrt{(\mu+1)(\mu-3)}}{2}$ must be less than 1.

Logistic map algorithm

pseudocode :

loop1 : Start at $\mu = 2.4$, end at $\mu = 4$ [calculates $x_{n+1} = \mu x_n(1-x_n)$ for range of μ. Index used for μ is i]

set $x = x_0$ [use $\frac{1}{2}$ as starting value]

loop 1.1 : keep solving for x_{n+1} to get to a stabilized x_{n+1} (fixed point) from initial x_0 (this is stability analysis)

$$x = \mu(i).x(1-x_n)$$

loop 1.1 end [saves the stable point in x]

loop 1.2 : calculate the x_{n+1} per $\mu(i)$ and put x_{n+1} into array called x_data (k, i) [k indicates number of times to calculate x_{n+1} and i indicates the μ value used in the loop]

(This loop utilizes the stabilized fixed point found in loop 1.1 to find period cycles)

loop 1.2 end

Notes — if we run the code, x_data array has same values in each column (i). This is because x stays the same in the kth iteration and a single full column denotes constant μ.

Parameters to be set in code

1) number of μ points
2) number of iterations for loop 1.1 to get fixed point from initial value.
3) data size for x_data.
4) resolution of graph.