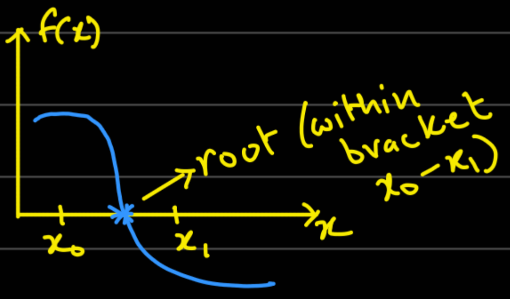
Bisection method

slow but always finds the root

Steps:

(i) Start with two values x_0, x_1 that bracket the root of $f(x)$ →

Evaluate $f(x_0)$ and $f(x_1)$



(ii) Bisect the range x_0 — x_1 to get x_2

∴ To bisect → $x_2 = \frac{x_0 + x_1}{2} = x_1 - \frac{(x_1 - x_0)}{2}$

(iii) Evaluate $f(x_2)$ and compare with $f(x_1)$ and $f(x_0)$

if $f(x_2)$ and $f(x_1)$ have opposite signs → root is between x_1 and x_2. x_1 and x_2 are our new brackets.

if $f(x_2)$ and $f(x_0)$ have opposite signs → root is between x_1 and x_0. x_0 and x_2 are our new brackets.

(iv) Repeat steps (i) to (iii) with the brackets found in step (iii)

(v) Iterate till $f(x_n) \simeq 0$ [x_n is new x value after nth iteration].

Estimate $\sqrt{3}$, use $x_0 = 1$, $x_1 = 2$

∴ $f(x) = 0$

or, $x^2 - 3 = 0$

(i) $f(x_0) = 1^2 - 3 = -2$, $f(x_1) = 2^2 - 3 = 1$ → root lies in this range

(ii) $x_2 = \frac{x_0 + x_1}{2} = \frac{1+2}{2} = 1.5$ → Bisecting range

(iii) $f(x_2) = 1.5^2 - 3 = -0.75$ → there is a sign change between $f(x_1)$ and $f(x_2)$. The root lies between x_1 and x_2.

(iv) Iterating using x_1 and x_2 as starting values:

$x_3 = \frac{x_1 + x_2}{2} = \frac{2 + 1.5}{2} = 1.75$

$f(x_3) = 1.75^2 - 3 = 0.0625$ [root inbetween x_2 and x_3]

∴ $x_4 = \frac{x_2 + x_3}{2} = \frac{1.75 + 1.5}{2} = 1.625$

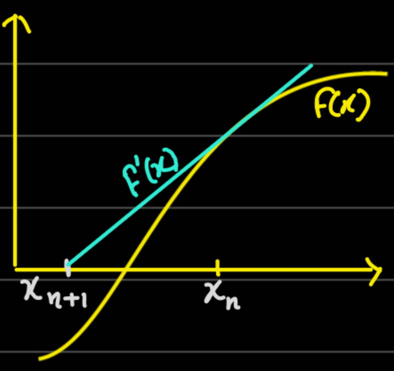
$f(x_4) = 1.625^2 - 3 = -0.359375$

∴ Since $f(x_3) \simeq 0$, $x_3 = 1.75$ is the closest solution

∴ $\sqrt{3} \simeq 1.75$

Newton's Method

Using calculus to find roots. Equation is approximated using straight line.



Equation

$\therefore y - y_n = f'(x_n)(x - x_n)$

if $x = x_{n+1}$ and $y = y_{n+1} = 0$ $[y_n = f(x)]$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Start with an estimate for x_n and keep iterating till a steady state value for x_{n+1} emerges [Solution converges].

Example: estimate $\sqrt{3}$, start with $x_0 = 1$

$\therefore f(x) = x^2 - 3$

$f'(x) = 2x$

Newton's method equation: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-2)}{2} = 2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{1}{4} = 1.75$$

$$x_3 = 1.75 - \frac{1/16}{7/2} = 1.732$$

$\rightarrow$ similar; convergence achieved.

$\therefore \sqrt{3} \simeq 1.732$

Secants method

used when analytical derivation of function is not possible. Need to have 2 initial values: x_n and x_{n-1} to find x_{n+1}

the equation to get x_{n+1} is similar to Newton's method $\longrightarrow$ $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

but, $f'(x_n)$ is evaluated numerically

$$\therefore f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Substituting into x_{n+1} eq we get: $x_{n+1} = \dfrac{x_n x_{n-1} + 2}{x_n + x_{n-1}}$

we solve this iteratively to get a convergence

Order of convergence of Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The error at the nth iteration is $\epsilon_n = r - x_n$

The error at the n+1 th iteration is $\epsilon_{n+1} = r - x_{n+1}$

$$\therefore \epsilon_{n+1} = r - x_n + \frac{f(x_n)}{f'(x_n)}$$

$^* x_n = r - \epsilon_n$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \frac{f(x_n)}{f'(x_n)} \longrightarrow \epsilon_{n+1} = \epsilon_n + \frac{f(r - \epsilon_n)}{f'(r - \epsilon_n)}$$

use taylor series to expand $f'(r - \epsilon_n)$ and $f(r - \epsilon_n)$

$$\therefore \epsilon_{n+1} = \epsilon_n + \frac{\cancelto{0}{f(r)} - \epsilon_n f'(r) + \frac{\epsilon_n^2}{2}.f''(r) + \ldots}{f'(r) - \epsilon_n f''(r) + \frac{\epsilon_n^2}{2}.f'''(r) + \ldots}$$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \frac{-\epsilon_n f'(r) + \frac{\epsilon_n^2}{2}.f''(r) + \ldots}{f'(r) - \epsilon_n f''(r) + \frac{\epsilon_n^2}{2}.f'''(r) + \ldots}$$

divide by $f'(r)$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \frac{-\epsilon_n + \frac{\epsilon_n^2}{2}.\frac{f''(r)}{f'(r)} + \ldots}{1 - \epsilon_n \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2}.\frac{f'''(r)}{f'(r)} + \ldots}$$

$$\frac{1}{1-\Delta}$$

$$\frac{1}{1 - \epsilon_n \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2}.\frac{f'''(r)}{f'(r)} + \ldots}$$

$$= 1 + \epsilon_n \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2}.\frac{f'''(r)}{f'(r)} + \ldots$$

$$= 1 + \Delta + \Delta^2 + \Delta^3 + \ldots$$

can be thought of as $\frac{1}{1-\Delta}$ where Δ is a very small number. Thus we can write $\frac{1}{1-\Delta}$ as a taylor series expansion

(Here $\Delta = \epsilon_n \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2}.\frac{f'''(r)}{f'(r)} + \ldots$)

$$\therefore \epsilon_{n+1} = \epsilon_n + \left(-\epsilon_n + \frac{\epsilon_n^2}{2}.\frac{f''(r)}{f'(r)} + \ldots\right)\left(1 + \epsilon_n \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2}.\frac{f'''(r)}{f'(r)} + \ldots\right)$$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \left(-\epsilon_n - \epsilon_n^2 \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2}.\frac{f''(r)}{f'(r)} + \ldots\right)$$

$$\text{or, } \epsilon_{n+1} = \frac{-\epsilon_n^2}{2}.\frac{f''(r)}{f'(r)}$$

$$\text{or, } \epsilon_{n+1} = -\frac{f''(r)}{f'(r)}\epsilon_n^2 \xrightarrow{\text{same form as}} \epsilon_{n+1} = k|\epsilon_n|^p$$

$\therefore p=2$ and the method is order 2 convergence

Order of convergence of Secant method

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})} \quad \text{——— (1)}$$

$\therefore \epsilon_n = r - x_n, \quad \epsilon_{n+1} = r - x_{n+1}$

subtract eq (1) from r:

$$\epsilon_{n+1} = \epsilon_n + \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \frac{(r - \epsilon_n - r + \epsilon_{n-1})f(r - \epsilon_n)}{f(r-\epsilon_n) - f(r - \epsilon_{n-1})}$$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \frac{(\epsilon_{n-1} - \epsilon_n)f(r - \epsilon_n)}{f(r-\epsilon_n) - f(r - \epsilon_{n-1})}$$

we know $f(r)=0$ and ϵ's are small, therefore taylor series expand $f(r-\epsilon_n)$ and $f(r-\epsilon_{n-1})$

$$\text{or, } \epsilon_{n+1} = \epsilon_n + \frac{(\epsilon_{n-1} - \epsilon_n)(f(r) - \epsilon_n f'(r) + \frac{1}{2}f''(r)\epsilon_n^2 + \ldots)}{(f(r) - \epsilon_n f'(r) + \frac{1}{2}f''(r)\epsilon_n^2 + \ldots) - (f(r) - \epsilon_{n-1}f'(r) + \frac{1}{2}f''(r)\epsilon_{n-1}^2 + \ldots)}$$

$\boxed{f(r)=0}$

or, $\epsilon_{n+1} = \epsilon_n + \dfrac{(\epsilon_{n-1}-\epsilon_n)\left(-\epsilon_n f'(r) + \frac{1}{2} f''(r)\epsilon_n^2 + \cdots\right)}{(\epsilon_{n-1}-\epsilon_n) f'(r) - \frac{1}{2} f''(r)(\epsilon_{n-1}^2 - \epsilon_n^2) + \cdots)}$

write in $(a+b)(a-b)$ form

or, $\epsilon_{n+1} = \epsilon_n - \dfrac{\left(\epsilon_n f'(r) - \frac{1}{2} f''(r)\epsilon_n^2 + \cdots\right)}{f'(r) - \frac{1}{2} f''(r)(\epsilon_{n-1}+\epsilon_n) + \cdots}$

divide numerator and denominator by $f'(r)$:

$\therefore \epsilon_{n+1} = \epsilon_n - \dfrac{\left(\epsilon_n - \frac{1}{2}\epsilon_n^2 \frac{f''(r)}{f'(r)} + \cdots\right)}{1 - \frac{1}{2}\frac{f''(r)}{f'(r)}(\epsilon_{n-1}+\epsilon_n) + \cdots}$

write $\dfrac{1}{1 - \frac{1}{2}\frac{f''(r)}{f'(r)}(\epsilon_{n-1}+\epsilon_n) + \cdots}$ as $\dfrac{1}{1-\Delta}$ and taylor series expand it.

$\therefore \dfrac{1}{1-\Delta} = 1 + \Delta + \Delta^2 + \cdots$

or $\dfrac{1}{1 - \frac{1}{2}\frac{f''(r)}{f'(r)}(\epsilon_{n-1}+\epsilon_n) + \cdots} = 1 + \frac{1}{2}\frac{f''(r)}{f'(r)}(\epsilon_{n-1}+\epsilon_n) + \cdots$

$\therefore \epsilon_{n+1} = \epsilon_n - \left(\epsilon_n - \frac{1}{2}\epsilon_n^2 \frac{f''(r)}{f'(r)} + \cdots\right)\left(1 + \frac{1}{2}\frac{f''(r)}{f'(r)}(\epsilon_{n-1}+\epsilon_n) + \cdots\right)$

or, $\epsilon_{n+1} = -\frac{1}{2}\frac{f''(r)}{f'(r)}\epsilon_n(\epsilon_{n-1}+\epsilon_n) + \frac{1}{2}\epsilon_n^2 \frac{f''(r)}{f'(r)}$

or, $\epsilon_{n+1} = \frac{1}{2}\frac{f''(r)}{f'(r)}\left(\epsilon_n^2 - \epsilon_n(\epsilon_{n-1}+\epsilon_n)\right)$

or, $\epsilon_{n+1} = -\frac{1}{2}\frac{f''(r)}{f'(r)}(\epsilon_n\epsilon_{n-1})$

let $\epsilon_{n+1} = k|\epsilon_n|^p$ and $\epsilon_n = k|\epsilon_{n-1}|^p$

or, $k|\epsilon_n|^p = -\frac{1}{2}\frac{f''(r)}{f'(r)}(k\epsilon_{n-1}^{p+1})$

or, $k^2|\epsilon_{n-1}|^{p^2} = -\frac{1}{2}\frac{f''(r)}{f'(r)} \cdot k|\epsilon_{n-1}|^{p+1}$

$\therefore p^2 = p+1 \rightarrow p^2 - p - 1 = 0$

$\therefore p = \dfrac{1 \pm \sqrt{1-4(1)(-1)}}{2(1)} = 1.62, -0.62$

$\therefore p = 1.62$, secant method is order 1.62 convergence