Gaussian elimination without Pivoting

The basic gaussian elimination without pivoting (no row interchanges) is redundant due to roundoff error.

Example:

recall machine eps $\epsilon = 2^{-52}$

roundoff errors in machines:

$2+\epsilon = 2$ (in a machine next largest number after 2 is $2+2\epsilon$).

$4-\epsilon = 4$ (similarly, next largest number after 4 is $4+4\epsilon$).

say we have: $\epsilon x_1 + 2x_2 = 4 \rightarrow$ ϵx_1 is very small $\therefore x_2 = \frac{4}{2} = 2$

$x_1 - x_2 = 1 \rightarrow$ since $x_2 = 2 \rightarrow x_1 = 3$

sol 1

our 'theoretical' numerical simulation should give $\boxed{\text{this}}$ answer

augmented matrix $\rightarrow \begin{pmatrix} \epsilon & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{\epsilon}r1+r2} \begin{pmatrix} \epsilon & 2 & 4 \\ 0 & -\frac{2}{\epsilon}-1 & 1-\frac{4}{\epsilon} \end{pmatrix}$

$$\boxed{\begin{aligned} &\epsilon x_1 + 2x_2 = 4 \\ &\left(-\frac{2}{\epsilon}-1\right)x_2 = 1-\frac{4}{\epsilon} \end{aligned}}$$

$\therefore$ Multiply by $-\epsilon$:

$(2+\epsilon)x_2 = 4-\epsilon$

or, $2x_2 = 4 \rightarrow x_2 = 2$

$\epsilon x_1 + 2x_2 = 4$

put in $x_2 = 2$

$\therefore \epsilon x_1 = 0 \rightarrow x_1 = 0$

sol 2

solutions does not match up! this is because of roundoff errors that occurred when we said numbers with ϵ are equal to 0.

Example: Consider the equations

$$\left.\begin{aligned} 2\epsilon x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \end{aligned}\right] \text{———} \begin{bmatrix} 2\epsilon & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix} \text{ augmented matrix}$$

solve in matlab without pivoting $\rightarrow x_1 = 3, x_2 = 2$

solving analytically: $2\epsilon x_1 + 2x_2 = 4 \rightarrow 2\epsilon x_1$ is small

$\therefore x_2 = 2$

$\therefore x_1 = 1+2 = 3$

Partial pivoting

$\rightarrow$ row interchanges

$$\text{System: } \left.\begin{aligned} \epsilon x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \end{aligned}\right] \text{—} \begin{bmatrix} 2\epsilon & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

go down a column and look for number with largest magnitude. Here 1 is largest in column 1. Interchange this row

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2+\epsilon & 4-\epsilon \end{bmatrix} \xleftarrow{-\epsilon r_1 + r_2} \begin{bmatrix} 1 & -1 & 1 \\ \epsilon & 2 & 4 \end{bmatrix} \longleftarrow$$

$\therefore x_2 = \frac{4-\epsilon}{2+\epsilon} = 2$

$x_1 = 1 + \frac{4-\epsilon}{2+\epsilon} = 3$

solution with good pivoting

LU decomposition with partial pivots

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -8 & 4 \end{bmatrix} \xrightarrow[\text{row-wise pivot in 1st column}]{r1 \leftrightarrow r2} \underbrace{\begin{bmatrix} 6 & -6 & 7 \\ -2 & 2 & -1 \\ 3 & -8 & 4 \end{bmatrix}}_{P_{12}A} \xrightarrow[r3 - \frac{r1}{2}]{r2 + \frac{r1}{3}} \underbrace{\begin{bmatrix} 6 & -6 & 7 \\ 0 & 0 & \frac{4}{3} \\ 0 & -5 & \frac{1}{2} \end{bmatrix}}_{M_2M_1P_{12}A}$$

row-wise pivot in 2nd column

$$\begin{bmatrix} 6 & -6 & 7 \\ 0 & -5 & \frac{1}{2} \\ 0 & 0 & 4/3 \end{bmatrix}$$

$P_{23}M_2M_1P_{12}A =$ Upper triangular matrix

> * P_{12}, P_{23} are transformation matrices. Multiplication by P_{12}/P_{23} on right side to any matrix interchanges columns. Multiplication by P_{12}/P_{23} on left to any matrix interchanges rows

Now to compute L from the upper triangle, we use the LU decomposition formula:

In our case: $(P_{23}M_2M_1P_{23})P_{23}P_{12}A = U$

or, $P_{23}P_{12}A = (P_{23}M_2M_1P_{23})^{-1}U$

here, $(P_{23}M_2M_1P_{23})^{-1} = L$

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$$\boxed{\begin{aligned} P_{23} &= P_{23}^{-1} \\ P_{12} &= P_{12}^{-1} \end{aligned}}$$

where $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\underbrace{M_2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}}$

and $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\qquad M_2M_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$

$\therefore P_{23}M_2M_1P_{23}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 \\ \frac{1}{3} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$(P_{23}M_2M_1P_{23})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} = L$$

Matlab saves this L matrix as $P_{12}P_{23}L$ and calls it "a psychologically lower triangular matrix". All the decomposition algos completed till now is how Matlab handles matrices. It is faster to deduce LU decomposition of A in $AX = B$ and find solution than it is to find solution of $Ax = B$ using Gaussian elimination.