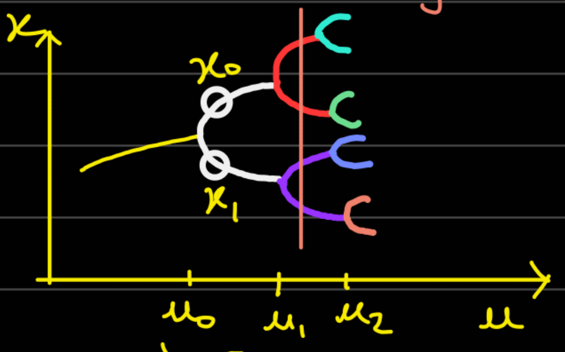
The Figenbaum delta

Recall the logistic map — $x_{i+1} = \mu x_i(1-x_i)$
with the bifurication map (x vs μ)

alternating fixed points along this line



values of μ where bifurcation occurs

To interpret the logistic map:

(i) Period 1: these are single fixed points of x_{i+1} and $x_i = x_{i+1}$ for a certain range of μ.

(ii) Period 2: these are values of x_{i+1} and here x_{i+1} alternates between 1 of 2 possible fixed points (x_0 and x_1 circled in diagram) for a certain range of μ.

(iii) Period 3 — same as section II but has 4 fixed points that alternate

Same situation for other sections

Figenbaum delta — as we get to higher and higher μ's the width's get smaller

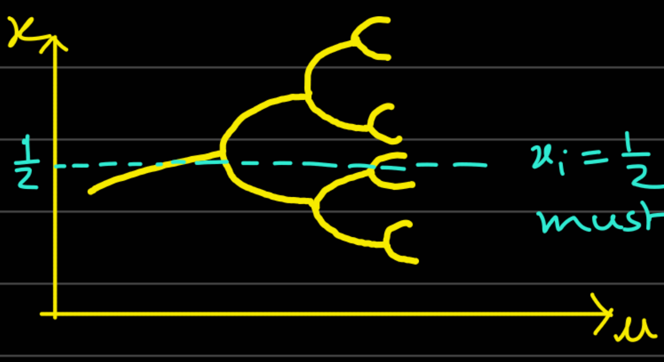
$$\delta = \lim_{n\to\infty} \frac{\mu_{n-1} - \mu_{n-2}}{\mu_n - \mu_{n-1}}$$ the n is the period number

Recall to get to fixed points we had to eliminate transients through iteration. We want to get rid of these transients by using <u>superstable</u> method.

The superstable method

$\boxed{N = 2^n}$ Orbit: $x_0, x_1, x_2 \ldots x_{n-1}, x_0$ (aka the fixed points) after the transients. These points repeat again cyclically after final point in orbit (x_{n-1}) is reached.

In the superstable cycle we have a point $x = \frac{1}{2}$ which is a point that all the cycles of fixed points of x_{i+1} must go through in this case $x = \frac{1}{2}$ and is shown in diagram below:



$x_i = \frac{1}{2}$ is a point that all the cycles must go through. This point reduces time for computation of transients

So to reduce computation to find δ, we find μ's where the orbit contains $x_0 = \frac{1}{2}$

Find first 2 μ's in the orbit that contains $x_0 = \frac{1}{2}$:

Define m_n to be value of μ in period 2^n

m_0 : $\frac{1}{2} = \mu \frac{1}{2}(1-\frac{1}{2})$ [$\frac{1}{2}$ is a fixed point]

or $\mu = 2$ (period 1) $\longrightarrow m_0 = 2$

m_1 : $x_1 = \mu x_0(1-x_0)$

$x_0 = \mu x_1(1-x_1)$ [period 2]

eliminating x_1 : $x_0 = \mu \cdot \mu x_0(1-x_0)[1-\mu x_0(1-x_0)]$

now since this cycle also goes through the point $x_0 = \frac{1}{2}$ (super stable cycle) we can find μ at $x_0 = \frac{1}{2}$:

$\therefore \frac{1}{2} = \mu^2 \cdot \frac{1}{2}(1-\frac{1}{2})[1-\frac{1}{2}\mu + \frac{1}{4}\mu]$

or $\frac{1}{4}\mu^2[1-\frac{1}{4}\mu] = \frac{1}{2}$

or $\mu^2 - \frac{\mu^3}{4} = 2$

or $\mu^3 - 4\mu^2 + 8 = 0 \longrightarrow$ let $f(\mu) = \mu^3 - 4\mu^2 + 8 = 0$

$\therefore f(2) = 0 \Rightarrow \mu - 2$ is a factor

$$
\begin{array}{r|l}
 & \mu^2 - 2\mu - 4 \\
\mu - 2 & \mu^3 - 4\mu^2 + 8 \\
 & \mu^3 - 2\mu^2 \\
\hline
 & -2\mu^2 + 8 \\
 & -2\mu^2 + 4\mu \\
\hline
 & -4\mu + 8 \\
 & -4\mu + 8 \\
\hline
 & \times
\end{array}
$$

$\longrightarrow \mu^2 - 2\mu - 4 = 0$

$\mu = \dfrac{2 \pm \sqrt{4 - 4(1)(-4)}}{2}$

or, $\mu = 1 \pm \sqrt{5} \rightarrow$ eliminate $-ve$ root because μ is only $+ve$.

$\therefore m_0 = \mu_1 = 2$
$m_1 = \mu_2 = 1+\sqrt{5}$ } Period 1 cycle

$\boxed{\text{The Algorithm}}$ Till now we have computed $m_0 = 2$, $m_1 = 1+\sqrt{5}$ and the rest $m_2, m_3 \ldots m_n$ for $x_0 = \frac{1}{2}$ which is in the orbit

recall: $\delta = \lim\limits_{n\to\infty} \dfrac{m_{n-1} - m_{n-2}}{m_n - m_{n-1}}$

we need to calculate these m's by using root finding method (specifically newton's method).

The orbit of the period cycles are: $x_0, x_1, x_2, \ldots x_{N-1}, x_N$

[$x_0 = x_N = \frac{1}{2}$ and cycle repeats again]

To convert this to a root finding problem we define a function of μ where the root is $x_0 = x_N = \frac{1}{2}$:

$\therefore g(\mu) = x_N - \frac{1}{2} = 0 \longrightarrow$ Newton's method will give a root for μ from here.

$\therefore$ For newton's method: $g'(\mu) = x'_N$

where $x_{i+1} = \mu x_i(1-x_i)$ is our logistic map

$$\boxed{\begin{array}{l}\text{Recall Newton's method eq:}\\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}\end{array}}$$

$\therefore x'_N = x'_{i+1}$ w.r.t μ:

$$x'_{i+1} = x_i(1-x_i) + \mu x_i' - 2\mu x_i x_i'$$

or, $$x'_{i+1} = x_i(1-x_i) + \mu x_i'(1-2x_i)$$

$$\therefore \left.\begin{array}{l} x_{i+1} = \mu x_i(1-x_i) \\ x'_{i+1} = x_i(1-x_i) + \mu x_i'(1-2x_i) \end{array}\right\} \text{coupled difference equations.}$$

Now, since we are working with $x_0 = \frac{1}{2}$ for all starting values we can say that $x_0' = 0$. These starting values will be used by the 2 equations above.

$\therefore$ The newton's method eq for finding μ's is then:

$$\mu^{J+1} = \mu^J - \frac{x_N - \frac{1}{2}}{x_N'} \quad \text{at } J^{th} \text{ iteration.}$$

notice $\mu^{J^{th}}$ term. This needs an initial estimate which is m_{n-1}.

To get this initial guess we can use the feigenbaum delta equation:

$$\delta = \lim_{n\to\infty} \frac{m_{n-1} - m_{n-2}}{m_n - m_{n-1}} \quad \text{—— (11)}$$

or $$m_n = m_{n-1} + \frac{m_{n-1} - m_{n-2}}{\delta_{n-1}} \longrightarrow$$ For this we have $m_0 = 2$ (m_{n-2}), $m_1 = 1+\sqrt{5}$ (m_{n-1}) and use an initial guess $\delta = 5$

This m_n is then used as μ^J to calculate for μ^{J+1}. This μ^{J+1} then is used in eq (11) as m_n to calculate for best δ estimate. Cyclic view:

Final algo:

loop 1 : Start at period 2^n, $n=2$

Increment n with each iteration

compute initial guess for m_n using $m_{n-1}, m_{n-2}, \delta_{n-1}=5$

loop 2 : Iterate Newton's method

$x_0 = \frac{1}{2}$, $x_0' = 0$

loop 3 : Iterate logistic map 2^n times

compute x_i, x_i'

(end 3)

Newton's method for m_n

(end 2)

save m_n, compute δ_n

(end 1)

$$m_n = m_{n-1} + \frac{m_{n-1} - m_{n-2}}{\delta_{n-1}}$$

first guess is $\delta_0 = 5$, this δ_{n-1} is updated later

m_n is used as μ^J

compute x_{i+1} and x'_{i+1} using μ^J

use x_i and x_i' as x_N and x_N'

$$\mu^{J+1} = \mu^J - \frac{x_N - \frac{1}{2}}{x_N'}$$

μ^{J+1} is used as m_n. Old m_{n-1} becomes m_{n-2} here

$$\delta = \lim_{n \to \infty} \frac{m_{n-1} - m_{n-2}}{m_n - m_{n-1}}$$

new δ estimate

δ is used as δ_{n-1}. m_n is now m_{n-1} and so on.