Category Framework

Release 10.3

The Sage Development Team

Mar 20, 2024

CONTENTS

CHAPTER

ONE

INTRODUCTION

1.1 Sage categories quickref

- `sage.categories.primer?` a primer on Elements, Parents, and Categories
- `sage.categories.tutorial?` a tutorial on Elements, Parents, and Categories
- `Category?` technical background on categories
- `Sets()`, `Semigroups()`, `Algebras(QQ)` some categories
- `SemiGroups().example()??` sample implementation of a semigroup
- `Hom(A, B)`, `End(A, Algebras())` homomorphisms sets
- `tensor`, `cartesian_product` functorial constructions

Module layout:

- `sage.categories.basic` the basic categories
- *`sage.categories.all`* all categories
- *`sage.categories.semigroups`* the `Semigroups()` category
- *`sage.categories.examples.semigroups`* the example of `Semigroups()`
- *`sage.categories.homset`* morphisms, ...
- *`sage.categories.map`*
- *`sage.categories.morphism`*
- `sage.categories.functors`
- *`sage.categories.cartesian_product`* functorial constructions
- *`sage.categories.tensor`*
- *`sage.categories.dual`*

CHAPTER

TWO

THE SAGE CATEGORY FRAMEWORK

2.1 Elements, parents, and categories in Sage: a primer

Contents

2.1.1 Abstract

The purpose of categories in Sage is to translate the mathematical concept of categories (category of groups, of vector spaces, …) into a concrete software engineering design pattern for:

- organizing and promoting generic code
- fostering consistency across the Sage library (naming conventions, doc, tests)
- embedding more mathematical knowledge into the system

This design pattern is largely inspired from Axiom and its followers (Aldor, Fricas, MuPAD, …). It differs from those by:

- blending in the Magma inspired concept of Parent/Element
- being built on top of (and not into) the standard Python object oriented and class hierarchy mechanism. This did not require changing the language, and could in principle be implemented in any language supporting the creation of new classes dynamically.

The general philosophy is that *Building mathematical information into the system yields more expressive, more conceptual and, at the end, easier to maintain and faster code* (within a programming realm; this would not necessarily apply to specialized libraries like gmp!).

One line pitch for mathematicians

Categories in Sage provide a library of interrelated bookshelves, with each bookshelf containing algorithms, tests, documentation, or some mathematical facts about the objects of a given category (e.g. groups).

One line pitch for programmers

Categories in Sage provide a large hierarchy of abstract classes for mathematical objects. To keep it maintainable, the inheritance information between the classes is not hardcoded but instead reconstructed dynamically from duplication free semantic information.

2.1.2 Introduction: Sage as a library of objects and algorithms

The Sage library, with more than one million lines of code, documentation, and tests, implements:

- Thousands of different kinds of objects (classes):

  Integers, polynomials, matrices, groups, number fields, elliptic curves, permutations, morphisms, languages, … and a few racoons …

- Tens of thousands methods and functions:

  Arithmetic, integer and polynomial factorization, pattern matching on words, …

Some challenges

- How to organize this library?

  One needs some bookshelves to group together related objects and algorithms.

- How to ensure consistency?

  Similar objects should behave similarly:

```
sage: Permutations(5).cardinality()
120

sage: GL(2,2).cardinality()
↪# needs sage.modules
6

sage: A = random_matrix(ZZ, 6, 3, x=7)
↪# needs sage.modules
sage: L = LatticePolytope(A.rows())
↪# needs sage.geometry.polyhedron sage.modules
sage: L.npoints()                # oops!   # random
↪# needs sage.geometry.polyhedron sage.modules
37
```

- How to ensure robustness?

- How to reduce duplication?

  Example: binary powering:

  ```
  sage: m = 3
  sage: m^8 == m*m*m*m*m*m*m*m == ((m^2)^2)^2
  True
  ```

  ```
  sage: # needs sage.modules
  sage: m = random_matrix(QQ, 4, algorithm='echelonizable',
  ....:                   rank=3, upper_bound=60)
  sage: m^8 == m*m*m*m*m*m*m*m == ((m^2)^2)^2
  True
  ```

  We want to implement binary powering only once, as *generic* code that will apply in all cases.

2.1.3 A bit of help from abstract algebra

The hierarchy of categories

What makes binary powering work in the above examples? In both cases, we have *a set* endowed with a *multiplicative binary operation* which is *associative* and which has a unit element. Such a set is called a *monoid*, and binary powering (to a non-negative power) works generally for any monoid.

Sage knows about monoids:

```
sage: Monoids()
Category of monoids
```

and sure enough, binary powering is defined there:

```
sage: m._pow_int.__module__                                              #␣
↪needs sage.modules
'sage.categories.monoids'
```

That's our bookshelf! And it's used in many places:

```
sage: GL(2, ZZ) in Monoids()                                             #␣
↪needs sage.modules
True
sage: NN in Monoids()
True
```

For a less trivial bookshelf we can consider euclidean rings: once we know how to do euclidean division in some set R, we can compute gcd's in R generically using the Euclidean algorithm.

We are in fact very lucky: abstract algebra provides us right away with a large and robust set of bookshelves which is the result of centuries of work of mathematicians to identify the important concepts. This includes for example:

```
sage: Sets()
Category of sets

sage: Groups()
Category of groups

sage: Rings()
```

(continues on next page)

(continued from previous page)

```
Category of rings

sage: Fields()
Category of fields

sage: HopfAlgebras(QQ)
Category of Hopf algebras over Rational Field
```

Each of the above is called a *category*. It typically specifies what are the operations on the elements, as well as the axioms satisfied by those operations. For example the category of groups specifies that a group is a set endowed with a binary operation (the multiplication) which is associative and admits a unit and inverses.

Each set in Sage knows which bookshelf of generic algorithms it can use, that is to which category it belongs:

```
sage: G = GL(2, ZZ)                                                             #␣
↪needs sage.modules
sage: G.category()                                                              #␣
↪needs sage.modules
Category of infinite groups
```

In fact a group is a semigroup, and Sage knows about this:

```
sage: Groups().is_subcategory(Semigroups())
True
sage: G in Semigroups()                                                         #␣
↪needs sage.modules
True
```

Altogether, our group gets algorithms from a bunch of bookshelves:

```
sage: G.categories()                                                            #␣
↪needs sage.modules
[Category of infinite groups, Category of groups, Category of monoids,
 ...,
 Category of magmas,
 Category of infinite sets, ...]
```

Those can be viewed graphically:

```
sage: g = Groups().category_graph()                                             #␣
↪needs sage.graphs
sage: g.set_latex_options(format="dot2tex")                                     #␣
↪needs sage.graphs sage.modules
sage: view(g)                                       # not tested                #␣
↪needs sage.graphs sage.modules sage.plot
```

In case `dot2tex` is not available, you can use instead:

```
sage: g.show(vertex_shape=None, figsize=20)                                     #␣
↪needs sage.graphs sage.modules sage.plot
```

Here is an overview of all categories in Sage:

```
sage: g = sage.categories.category.category_graph()                             #␣
↪needs sage.graphs sage.modules
sage: g.set_latex_options(format="dot2tex")                                     #␣
↪needs sage.graphs sage.modules
```

(continues on next page)

(continued from previous page)

```
sage: view(g)                           # not tested                         #␣
↪needs sage.graphs sage.modules sage.plot
```

Wrap-up: generic algorithms in Sage are organized in a hierarchy of bookshelves modelled upon the usual hierarchy of categories provided by abstract algebra.

Elements, Parents, Categories

Parent

A *parent* is a Python instance modelling a set of mathematical elements together with its additional (algebraic) structure.

Examples include the ring of integers, the group S_3, the set of prime numbers, the set of linear maps between two given vector spaces, and a given finite semigroup.

These sets are often equipped with additional structure: the set of all integers forms a ring. The main way of encoding this information is specifying which categories a parent belongs to.

It is completely possible to have different Python instances modelling the same set of elements. For example, one might want to consider the ring of integers, or the poset of integers under their standard order, or the poset of integers under divisibility, or the semiring of integers under the operations of maximum and addition. Each of these would be a different instance, belonging to different categories.

For a given model, there should be a unique instance in Sage representing that parent:

```
sage: IntegerRing() is IntegerRing()
True
```

Element

An *element* is a Python instance modelling a mathematical element of a set.

Examples of element include 5 in the integer ring, $x^3 - x$ in the polynomial ring in x over the rationals, $4 + O(3^3)$ in the 3-adics, the transposition (12) in S_3, and the identity morphism in the set of linear maps from $\mathbf{Q}^3$ to $\mathbf{Q}^3$.

Every element in Sage has a parent. The standard idiom in Sage for creating elements is to create their parent, and then provide enough data to define the element:

```
sage: R = PolynomialRing(ZZ, name='x')
sage: R([1,2,3])
3*x^2 + 2*x + 1
```

One can also create elements using various methods on the parent and arithmetic of elements:

```
sage: x = R.gen()
sage: 1 + 2*x + 3*x^2
3*x^2 + 2*x + 1
```

Unlike parents, elements in Sage are not necessarily unique:

```
sage: ZZ(5040) is ZZ(5040)
False
```

Many parents model algebraic structures, and their elements support arithmetic operations. One often further wants to do arithmetic by combining elements from different parents: adding together integers and rationals for example. Sage supports this feature using coercion (see `sage.structure.coerce` for more details).

It is possible for a parent to also have simultaneously the structure of an element. Consider for example the monoid of all finite groups, endowed with the Cartesian product operation. Then, every finite group (which is a parent) is also an element of this monoid. This is not yet implemented, and the design details are not yet fixed but experiments are underway in this direction.

Todo: Give a concrete example, typically using `ElementWrapper`.

Category

A *category* is a Python instance modelling a mathematical category.

Examples of categories include the category of finite semigroups, the category of all (Python) objects, the category of **Z**-algebras, and the category of Cartesian products of **Z**-algebras:

```
sage: FiniteSemigroups()
Category of finite semigroups
sage: Objects()
Category of objects
sage: Algebras(ZZ)
Category of algebras over Integer Ring
sage: Algebras(ZZ).CartesianProducts()
Category of Cartesian products of algebras over Integer Ring
```

Mind the 's' in the names of the categories above; `GroupAlgebra` and `GroupAlgebras` are distinct things.

Every parent belongs to a collection of categories. Moreover, categories are interrelated by the *super categories* relation. For example, the category of rings is a super category of the category of fields, because every field is also a ring.

A category serves two roles:

- to provide a model for the mathematical concept of a category and the associated structures: homsets, morphisms, functorial constructions, axioms.
- to organize and promote generic code, naming conventions, documentation, and tests across similar mathematical structures.

CategoryObject

Objects of a mathematical category are not necessarily parents. Parent has a superclass that provides a means of modeling such.

For example, the category of schemes does not have a faithful forgetful functor to the category of sets, so it does not make sense to talk about schemes as parents.

Morphisms, Homsets

As category theorists will expect, *Morphisms* and *Homsets* will play an ever more important role, as support for them will improve.

Much of the mathematical information in Sage is encoded as relations between elements and their parents, parents and their categories, and categories and their super categories:

```
sage: 1.parent()
Integer Ring

sage: ZZ
Integer Ring

sage: ZZ.category()
Join of Category of Dedekind domains
    and Category of euclidean domains
    and Category of infinite enumerated sets
    and Category of metric spaces

sage: ZZ.categories()
[Join of Category of Dedekind domains
     and Category of euclidean domains
     and Category of infinite enumerated sets
     and Category of metric spaces,
 Category of Dedekind domains,
 Category of euclidean domains, Category of principal ideal domains,
 Category of unique factorization domains, Category of gcd domains,
 Category of integral domains, Category of domains,
 Category of commutative rings, Category of rings, ...
 Category of magmas and additive magmas, ...
 Category of monoids, Category of semigroups,
 Category of commutative magmas, Category of unital magmas, Category of magmas,
 Category of commutative additive groups, ..., Category of additive magmas,
 Category of infinite enumerated sets, Category of enumerated sets,
 Category of infinite sets, Category of metric spaces,
 Category of topological spaces, Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: g = EuclideanDomains().category_graph()                                #␣
↪needs sage.graphs
sage: g.set_latex_options(format="dot2tex")                                  #␣
↪needs sage.graphs sage.plot
sage: view(g)                                   # not tested                 #␣
↪needs sage.graphs sage.plot
```

2.1.4 A bit of help from computer science

Hierarchy of classes

How are the bookshelves implemented in practice?

Sage uses the classical design paradigm of Object Oriented Programming (OOP). Its fundamental principle is that any object that a program is to manipulate should be modelled by an *instance* of a *class*. The class implements:

- a *data structure*: which describes how the object is stored,
- *methods*: which describe the operations on the object.

The instance itself contains the data for the given object, according to the specified data structure.

Hence, all the objects mentioned above should be instances of some classes. For example, an integer in Sage is an instance of the class `Integer` (and it knows about it!):

```
sage: i = 12
sage: type(i)
<class 'sage.rings.integer.Integer'>
```

Applying an operation is generally done by *calling a method*:

```
sage: i.factor()
2^2 * 3

sage: # needs sage.symbolic
sage: x = var('x')
sage: p = 6*x^2 + 12*x + 6
sage: type(p)
<class 'sage.symbolic.expression.Expression'>
sage: p.factor()
6*(x + 1)^2

sage: # needs sage.symbolic
sage: R.<x> = PolynomialRing(QQ, sparse=True)
sage: pQ = R(p)
sage: type(pQ)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_field_with_category.
↪element_class'>
sage: pQ.factor()
(6) * (x + 1)^2

sage: # needs sage.symbolic
sage: pZ = ZZ['x'](p)
sage: type(pZ)
<class 'sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_
↪flint'>
sage: pZ.factor()
2 * 3 * (x + 1)^2
```

Factoring integers, expressions, or polynomials are distinct tasks, with completely different algorithms. Yet, from a user (or caller) point of view, all those objects can be manipulated alike. This illustrates the OOP concepts of *polymorphism*, *data abstraction*, and *encapsulation*.

Let us be curious, and see where some methods are defined. This can be done by introspection:

```
sage: i._mul_??                    # not tested
```

For plain Python methods, one can also just ask in which module they are implemented:

```
sage: i._pow_.__module__   # not tested (Issue #24275)
'sage.categories.semigroups'

sage: pQ._mul_.__module__                                                       #␣
↪needs sage.symbolic
'sage.rings.polynomial.polynomial_element_generic'
sage: pQ._pow_.__module__                   # not tested (Issue #24275)          #␣
↪needs sage.symbolic
'sage.categories.semigroups'
```

We see that integers and polynomials have each their own multiplication method: the multiplication algorithms are indeed unrelated and deeply tied to their respective datastructures. On the other hand, as we have seen above, they share the same powering method because the set $\mathbf{Z}$ of integers, and the set $\mathbf{Q}[x]$ of polynomials are both semigroups. Namely, the class for integers and the class for polynomials both derive from an *abstract class* for semigroup elements, which factors out the *generic* methods like `_pow_`. This illustrates the use of *hierarchy of classes* to share common code between classes having common behaviour.

OOP design is all about isolating the objects that one wants to model together with their operations, and designing an appropriate hierarchy of classes for organizing the code. As we have seen above, the design of the class hierarchy is easy since it can be modelled upon the hierarchy of categories (bookshelves). Here is for example a piece of the hierarchy of classes for an element of a group of permutations:

```
sage: P = Permutations(4)
sage: m = P.an_element()
sage: for cls in m.__class__.mro(): print(cls)
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_class'>
<class 'sage.combinat.permutation.StandardPermutations_n.Element'>
<class 'sage.combinat.permutation.Permutation'>
...
<class 'sage.categories.groups.Groups.element_class'>
<class 'sage.categories.monoids.Monoids.element_class'>
...
<class 'sage.categories.semigroups.Semigroups.element_class'>
...
```

On the top, we see concrete classes that describe the data structure for matrices and provide the operations that are tied to this data structure. Then follow abstract classes that are attached to the hierarchy of categories and provide generic algorithms.

The full hierarchy is best viewed graphically:

```
sage: g = class_graph(m.__class__)                                              #␣
↪needs sage.combinat sage.graphs
sage: g.set_latex_options(format="dot2tex")                                     #␣
↪needs sage.combinat sage.graphs
sage: view(g)                                   # not tested                    #␣
↪needs sage.combinat sage.graphs sage.plot
```

Parallel hierarchy of classes for parents

Let us recall that we do not just want to compute with elements of mathematical sets, but with the sets themselves:

```
sage: ZZ.one()
1

sage: R = QQ['x,y']
sage: R.krull_dimension()
2
sage: A = R.quotient( R.ideal(x^2 - 2) )
sage: A.krull_dimension() # todo: not implemented
```

Here are some typical operations that one may want to carry on various kinds of sets:

- The set of permutations of 5, the set of rational points of an elliptic curve: counting, listing, random generation
- A language (set of words): rationality testing, counting elements, generating series
- A finite semigroup: left/right ideals, center, representation theory
- A vector space, an algebra: Cartesian product, tensor product, quotient

Hence, following the OOP fundamental principle, parents should also be modelled by instances of some (hierarchy of) classes. For example, our group G is an instance of the following class:

```
sage: G = GL(2, ZZ)                                                              #␣
↪needs sage.modules
sage: type(G)                                                                    #␣
↪needs sage.libs.gap sage.modules
<class 'sage.groups.matrix_gps.linear_gap.LinearMatrixGroup_gap_with_category'>
```

Here is a piece of the hierarchy of classes above it:

```
sage: for cls in G.__class__.mro(): print(cls)                                   #␣
↪needs sage.libs.gap sage.modules
<class 'sage.groups.matrix_gps.linear_gap.LinearMatrixGroup_gap_with_category'>
...
<class 'sage.categories.groups.Groups.parent_class'>
<class 'sage.categories.monoids.Monoids.parent_class'>
<class 'sage.categories.semigroups.Semigroups.parent_class'>
...
```

Note that the hierarchy of abstract classes is again attached to categories and parallel to that we had seen for the elements. This is best viewed graphically:

```
sage: # needs sage.graphs sage.modules
sage: g = class_graph(m.__class__)
sage: g.relabel(lambda x: x.replace("_",r"\_"))
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                                      # not tested                  #␣
↪needs sage.plot
```

Note: This is a progress upon systems like Axiom or MuPAD where a parent is modelled by the class of its elements; this oversimplification leads to confusion between methods on parents and elements, and makes parents special; in particular it prevents potentially interesting constructions like “groups of groups”.

2.1.5 Sage categories

Why this business of categories? And to start with, why don’t we just have a good old hierarchy of classes `Group`, `Semigroup`, `Magma`, … ?

Dynamic hierarchy of classes

As we have just seen, when we manipulate groups, we actually manipulate several kinds of objects:

- groups
- group elements
- morphisms between groups
- and even the category of groups itself!

Thus, on the group bookshelf, we want to put generic code for each of the above. We therefore need three, parallel hierarchies of abstract classes:

- Group, Monoid, Semigroup, Magma, …
- GroupElement, MonoidElement, SemigroupElement, MagmaElement, …
- GroupMorphism, MonoidMorphism, SemigroupMorphism, MagmaMorphism, …

(and in fact many more as we will see).

We could implement the above hierarchies as usual:

```
class Group(Monoid):
    # generic methods that apply to all groups

class GroupElement(MonoidElement):
    # generic methods that apply to all group elements

class GroupMorphism(MonoidMorphism):
    # generic methods that apply to all group morphisms
```

And indeed that's how it was done in Sage before 2009, and there are still many traces of this. The drawback of this approach is duplication: the fact that a group is a monoid is repeated three times above!

Instead, Sage now uses the following syntax, where the `Groups` bookshelf is structured into units with *nested classes*:

```
class Groups(Category):

    def super_categories(self):
        return [Monoids(), ...]

    class ParentMethods:
        # generic methods that apply to all groups

    class ElementMethods:
        # generic methods that apply to all group elements

    class MorphismMethods:
        # generic methods that apply to all group morphisms (not yet implemented)

    class SubcategoryMethods:
        # generic methods that apply to all subcategories of Groups()
```

With this syntax, the information that a group is a monoid is specified only once, in the `Category.super_categories()` method. And indeed, when the category of inverse unital magmas was introduced, there was a *single point of truth* to update in order to reflect the fact that a group is an inverse unital magma:

```
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
```

The price to pay (there is no free lunch) is that some magic is required to construct the actual hierarchy of classes for parents, elements, and morphisms. Namely, `Groups.ElementMethods` should be seen as just a bag of methods, and the actual class `Groups().element_class` is constructed from it by adding the appropriate super classes according to `Groups().super_categories()`:

```
sage: Groups().element_class
<class 'sage.categories.groups.Groups.element_class'>

sage: Groups().element_class.__bases__
(<class 'sage.categories.monoids.Monoids.element_class'>,
 <class 'sage.categories.magmas.Magmas.Unital.Inverse.element_class'>)
```

We now see that the hierarchy of classes for parents and elements is parallel to the hierarchy of categories:

```
sage: Groups().all_super_categories()
[Category of groups,
```

(continues on next page)

(continued from previous page)

```
 Category of monoids,
 Category of semigroups,
 ...
 Category of magmas,
 Category of sets,
 ...]

sage: for cls in Groups().element_class.mro(): print(cls)
<class 'sage.categories.groups.Groups.element_class'>
<class 'sage.categories.monoids.Monoids.element_class'>
<class 'sage.categories.semigroups.Semigroups.element_class'>
...
<class 'sage.categories.magmas.Magmas.element_class'>
...
sage: for cls in Groups().parent_class.mro(): print(cls)
<class 'sage.categories.groups.Groups.parent_class'>
<class 'sage.categories.monoids.Monoids.parent_class'>
<class 'sage.categories.semigroups.Semigroups.parent_class'>
...
<class 'sage.categories.magmas.Magmas.parent_class'>
...
```

Another advantage of building the hierarchy of classes dynamically is that, for parametrized categories, the hierarchy may depend on the parameters. For example an algebra over **Q** is a **Q**-vector space, but an algebra over **Z** is not (it is just a **Z**-module)!

Note: At this point this whole infrastructure may feel like overdesigning, right? We felt like this too! But we will see later that, once one gets used to it, this approach scales very naturally.

From a computer science point of view, this infrastructure implements, on top of standard multiple inheritance, a dynamic composition mechanism of mixin classes (Wikipedia article Mixin), governed by mathematical properties.

For implementation details on how the hierarchy of classes for parents and elements is constructed, see *`Category`*.

On the category hierarchy: subcategories and super categories

We have seen above that, for example, the category of sets is a super category of the category of groups. This models the fact that a group can be unambiguously considered as a set by forgetting its group operation. In object-oriented parlance, we want the relation "a group *is a* set", so that groups can directly inherit code implemented on sets.

Formally, a category `Cs()` is a *super category* of a category `Ds()` if Sage considers any object of `Ds()` to be an object of `Cs()`, up to an implicit application of a canonical functor from `Ds()` to `Cs()`. This functor is normally an inclusion of categories or a forgetful functor. Reciprocally, `Ds()` is said to be a *subcategory* of `Cs()`.

Warning: This terminology deviates from the usual mathematical definition of *subcategory* and is subject to change. Indeed, the forgetful functor from the category of groups to the category of sets is not an inclusion of categories, as it is not injective: a given set may admit more than one group structure. See github issue #16183 for more details. The name *supercategory* is also used with a different meaning in certain areas of mathematics.

Categories are instances and have operations

Note that categories themselves are naturally modelled by instances because they can have operations of their own. An important one is:

```
sage: Groups().example()                                                   #␣
↪needs sage.modules
General Linear Group of degree 4 over Rational Field
```

which gives an example of object of the category. Besides illustrating the category, the example provides a minimal template for implementing a new object in the category:

```
sage: S = Semigroups().example(); S
An example of a semigroup: the left zero semigroup
```

Its source code can be obtained by introspection:

```
sage: S??                                            # not tested
```

This example is also typically used for testing generic methods. See `Category.example()` for more.

Other operations on categories include querying the super categories or the axioms satisfied by the operations of a category:

```
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
sage: Groups().axioms()
frozenset({'Associative', 'Inverse', 'Unital'})
```

or constructing the intersection of two categories, or the smallest category containing them:

```
sage: Groups() & FiniteSets()
Category of finite groups
sage: Algebras(QQ) | Groups()
Category of monoids
```

Specifications and generic documentation

Categories do not only contain code but also the specifications of the operations. In particular a list of mandatory and optional methods to be implemented can be found by introspection with:

```
sage: Groups().required_methods()
{'element': {'optional': ['_mul_'], 'required': []},
 'parent': {'optional': [], 'required': ['__contains__']}}
```

Documentation about those methods can be obtained with:

```
sage: G = Groups()
sage: G.element_class._mul_?          # not tested
sage: G.parent_class.one?             # not tested
```

See also the `abstract_method()` decorator.

Warning: Well, more precisely, that's how things should be, but there is still some work to do in this direction. For example, the inverse operation is not specified above. Also, we are still missing a good programmatic syntax to specify the input and output types of the methods. Finally, in many cases the implementer must provide at least one of two methods, each having a default implementation using the other one (e.g. listing or iterating for a finite enumerated set); there is currently no good programmatic way to specify this.

Generic tests

Another feature that parents and elements receive from categories is generic tests; their purpose is to check (at least to some extent) that the parent satisfies the required mathematical properties (is my semigroup indeed associative?) and is implemented according to the specifications (does the method `an_element` indeed return an element of the parent?):

```
sage: S = FiniteSemigroups().example(alphabet=('a', 'b'))
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
    running ._test_elements_eq_reflexive() . . . pass
    running ._test_elements_eq_symmetric() . . . pass
    running ._test_elements_eq_transitive() . . . pass
    running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

Tests can be run individually:

```
sage: S._test_associativity()
```

Here is how to access the code of this test:

```
sage: S._test_associativity?? # not tested
```

Here is how to run the test on all elements:

```
sage: L = S.list()
sage: S._test_associativity(elements=L)
```

See `TestSuite` for more information.

Let us see what happens when a test fails. Here we redefine the product of S to something definitely not associative:

```
sage: S.product = lambda x, y: S("("+x.value +y.value+")")
```

And rerun the test:

```
sage: S._test_associativity(elements=L)
Traceback (most recent call last):
...
  File ".../sage/categories/semigroups.py", line ..., in _test_associativity
    tester.assertTrue((x * y) * z == x * (y * z))
...
AssertionError: '((aa)a)' != '(a(aa))'
```

We can recover instantly the actual values of `x`, `y`, `z`, that is, a counterexample to the associativity of our broken semigroup, using post mortem introspection with the Python debugger `pdb` (this does not work yet in the notebook):

```
sage: import pdb
sage: pdb.pm()                            # not tested
> /opt/sage-5.11.rc1/local/lib/python/unittest/case.py(424)assertTrue()
-> raise self.failureException(msg)
(Pdb) u
> /opt/sage-5.11.rc1/local/lib/python2.7/site-packages/sage/categories/semigroups.
↪py(145)_test_associativity()
-> tester.assertTrue((x * y) * z == x * (y * z))
(Pdb) p x, y, z
('a', 'a', 'a')
(Pdb) p (x * y) * z
'((aa)a)'
(Pdb) p x * (y * z)
'(a(aa))'
```

Wrap-up

- Categories provide a natural hierarchy of bookshelves to organize not only code, but also specifications and testing tools.
- Everything about, say, algebras with a distinguished basis is gathered in *`AlgebrasWithBasis`* or its super categories. This includes properties and algorithms for elements, parents, morphisms, but also, as we will see, for constructions like Cartesian products or quotients.
- The mathematical relations between elements, parents, and categories translate dynamically into a traditional hierarchy of classes.
- This design enforces robustness and consistency, which is particularly welcome given that Python is an interpreted language without static type checking.

2.1.6 Case study

In this section, we study an existing parent in detail; a good followup is to go through the *`sage.categories.tutorial`* or the thematic tutorial on coercion and categories (“How to implement new algebraic structures in Sage”) to learn how to implement a new one!

We consider the example of finite semigroup provided by the category:

```
sage: S = FiniteSemigroups().example(); S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c',
↪'d')
sage: S?                        # not tested
```

Where do all the operations on S and its elements come from?

```
sage: x = S('a')
```

repr is a technical method which comes with the data structure (ElementWrapper); since it's implemented in Cython, we need to use Sage's introspection tools to recover where it's implemented:

```
sage: x._repr_.__module__
sage: sage.misc.sageinspect.sage_getfile(x._repr_)
'.../sage/structure/element_wrapper.pyx'
```

_pow_int is a generic method for all finite semigroups:

```
sage: x._pow_int.__module__
'sage.categories.semigroups'
```

__mul__ is a generic method provided by the *Magmas* category (a *magma* is a set with an inner law $*$, not necessarily associative). If the two arguments are in the same parent, it will call the method _mul_, and otherwise let the coercion model try to discover how to do the multiplication:

```
sage: x.__mul__??                              # not tested
```

Since it is a speed critical method, it is implemented in Cython in a separate file:

```
sage: x._mul_.__module__
'sage.categories.coercion_methods'
```

mul is a default implementation, also provided by the *Magmas* category, that delegates the work to the method product of the parent (following the advice: if you do not know what to do, ask your parent); it's also a speed critical method:

```
sage: x._mul_??                                # not tested
sage: x._mul_.__module__
'sage.categories.coercion_methods'
sage: x._mul_.__func__ is Magmas.ElementMethods._mul_parent
True
```

product is a mathematical method implemented by the parent:

```
sage: S.product.__module__
'sage.categories.examples.finite_semigroups'
```

cayley_graph is a generic method on the parent, provided by the *FiniteSemigroups* category:

```
sage: S.cayley_graph.__module__
'sage.categories.semigroups'
```

multiplication_table is a generic method on the parent, provided by the *Magmas* category (it does not require associativity):

```
sage: S.multiplication_table.__module__
'sage.categories.magmas'
```

Consider now the implementation of the semigroup:

```
sage: S??                                        # not tested
```

This implementation specifies a data structure for the parents and the elements, and makes a promise: the implemented parent is a finite semigroup. Then it fulfills the promise by implementing the basic operation `product`. It also implements the optional method `semigroup_generators`. In exchange, S and its elements receive generic implementations of all the other operations. S may override any of those by more efficient ones. It may typically implement the element method `is_idempotent` to always return `True`.

A (not yet complete) list of mandatory and optional methods to be implemented can be found by introspection with:

```
sage: FiniteSemigroups().required_methods()
{'element': {'optional': ['_mul_'], 'required': []},
 'parent': {'optional': ['semigroup_generators'],
  'required': ['__contains__']}}
```

`product` does not appear in the list because a default implementation is provided in term of the method `_mul_` on elements. Of course, at least one of them should be implemented. On the other hand, a default implementation for `__contains__` is provided by `Parent`.

Documentation about those methods can be obtained with:

```
sage: C = FiniteSemigroups().element_class
sage: C._mul_?                                        # not tested
```

See also the `abstract_method()` decorator.

Here is the code for the finite semigroups category:

```
sage: FiniteSemigroups??                              # not tested
```

2.1.7 Specifying the category of a parent

Some parent constructors (not enough!) allow to specify the desired category for the parent. This can typically be used to specify additional properties of the parent that we know to hold a priori. For example, permutation groups are by default in the category of finite permutation groups (no surprise):

```
sage: P = PermutationGroup([[(1,2,3)]]); P                                      #␣
↪needs sage.combinat sage.groups
Permutation Group with generators [(1,2,3)]
sage: P.category()                                                              #␣
↪needs sage.combinat
Category of finite enumerated permutation groups
```

In this case, the group is commutative, so we can specify this:

```
sage: P = PermutationGroup([[(1,2,3)]],                                         #␣
↪needs sage.combinat sage.groups
....:                      category=PermutationGroups().Finite().Commutative()); P
Permutation Group with generators [(1,2,3)]
sage: P.category()                                                              #␣
↪needs sage.combinat
Category of finite enumerated commutative permutation groups
```

This feature can even be used, typically in experimental code, to add more structure to existing parents, and in particular to add methods for the parents or the elements, without touching the code base:

```
sage: class Foos(Category):
....:     def super_categories(self):
```

(continues on next page)

(continued from previous page)

```
....:             return [PermutationGroups().Finite().Commutative()]
....:         class ParentMethods:
....:             def foo(self): print("foo")
....:         class ElementMethods:
....:             def bar(self): print("bar")

sage: # needs sage.combinat
sage: P = PermutationGroup([[(1,2,3)]], category=Foos())                        #␣
↪needs sage.groups
sage: P.foo()
foo
sage: p = P.an_element()
sage: p.bar()
bar
```

In the long run, it would be thinkable to use this idiom to implement forgetful functors; for example the above group could be constructed as a plain set with:

```
sage: P = PermutationGroup([[(1,2,3)]], category=Sets())     # not implemented, needs␣
↪sage.combinat
```

At this stage though, this is still to be explored for robustness and practicality. For now, most parents that accept a category argument only accept a subcategory of the default one.

2.1.8 Scaling further: functorial constructions, axioms, ...

In this section, we explore more advanced features of categories. Along the way, we illustrate that a large hierarchy of categories is desirable to model complicated mathematics, and that scaling to support such a large hierarchy is the driving motivation for the design of the category infrastructure.

Functorial constructions

Sage has support for a certain number of so-called *covariant functorial constructions* which can be used to construct new parents from existing ones while carrying over as much as possible of their algebraic structure. This includes:

- Cartesian products: See `cartesian_product`.
- Tensor products: See `tensor`.
- Subquotients / quotients / subobjects / isomorphic objects: See:
  - `Sets().Subquotients`,
  - `Sets().Quotients`,
  - `Sets().Subobjects`,
  - `Sets().IsomorphicObjects`
- Dual objects: See `Modules().DualObjects`.
- Algebras, as in group algebras, monoid algebras, ...: See: `Sets.ParentMethods.algebra()`.

Let for example A and B be two parents, and let us construct the Cartesian product $A \times B \times B$:

```
sage: A = AlgebrasWithBasis(QQ).example();     A.rename("A")                    #␣
↪needs sage.combinat sage.modules
sage: B = HopfAlgebrasWithBasis(QQ).example(); B.rename("B")                    #␣
↪needs sage.combinat sage.modules
sage: C = cartesian_product([A, B, B]); C                                       #␣
↪needs sage.combinat sage.modules
A (+) B (+) B
```

In which category should this new parent be? Since A and B are vector spaces, the result is, as a vector space, the direct sum $A \oplus B \oplus B$, hence the notation. Also, since both A and B are monoids, $A \times B \times B$ is naturally endowed with a monoid structure for pointwise multiplication:

```
sage: C in Monoids()                                                            #␣
↪needs sage.combinat sage.modules
True
```

the unit being the Cartesian product of the units of the operands:

```
sage: C.one()                                                                   #␣
↪needs sage.combinat sage.modules
B[(0, word: )] + B[(1, ())] + B[(2, ())]
sage: cartesian_product([A.one(), B.one(), B.one()])                            #␣
↪needs sage.combinat sage.modules
B[(0, word: )] + B[(1, ())] + B[(2, ())]
```

The pointwise product can be implemented generically for all magmas (i.e. sets endowed with a multiplicative operation) that are constructed as Cartesian products. It's thus implemented in the `Magmas` category:

```
sage: C.product.__module__                                                      #␣
↪needs sage.combinat sage.modules
'sage.categories.magmas'
```

More specifically, keeping on using nested classes to structure the code, the product method is put in the nested class `Magmas.CartesianProducts.ParentMethods`:

```
class Magmas(Category):
    class ParentMethods:
        # methods for magmas
    class ElementMethods:
        # methods for elements of magmas
    class CartesianProduct(CartesianProductCategory):
        class ParentMethods:
            # methods for magmas that are constructed as Cartesian products
            def product(self, x, y):
                # ...
        class ElementMethods:
            # ...
```

Note: The support for nested classes in Python is relatively recent. Their intensive use for the category infrastructure did reveal some glitches in their implementation, in particular around class naming and introspection. Sage currently works around the more annoying ones but some remain visible. See e.g. `sage.misc.test_nested_class`.

Let us now look at the categories of `C`:

```
sage: C.categories()                                                          #␣
↪needs sage.combinat sage.modules
[Category of finite dimensional Cartesian products of algebras with basis over␣
↪Rational Field, ...
 Category of Cartesian products of algebras over Rational Field, ...
 Category of Cartesian products of semigroups, Category of semigroups, ...
 Category of Cartesian products of magmas, ..., Category of magmas, ...
 Category of Cartesian products of additive magmas, ..., Category of additive magmas,
 Category of Cartesian products of sets, Category of sets, ...]
```

This reveals the parallel hierarchy of categories for Cartesian products of semigroups magmas, … We are thus glad that Sage uses its knowledge that a monoid is a semigroup to automatically deduce that a Cartesian product of monoids is a Cartesian product of semigroups, and build the hierarchy of classes for parents and elements accordingly.

In general, the Cartesian product of A and B can potentially be an algebra, a coalgebra, a differential module, and be finite dimensional, or graded, or …. This can only be decided at runtime, by introspection into the properties of A and B; furthermore, the number of possible combinations (e.g. finite dimensional differential algebra) grows exponentially with the number of properties.

Axioms

First examples

We have seen that Sage is aware of the axioms satisfied by, for example, groups:

```
sage: Groups().axioms()
frozenset({'Associative', 'Inverse', 'Unital'})
```

In fact, the category of groups can be *defined* by stating that a group is a magma, that is a set endowed with an internal binary multiplication, which satisfies the above axioms. Accordingly, we can construct the category of groups from the category of magmas:

```
sage: Magmas().Associative().Unital().Inverse()
Category of groups
```

In general, we can construct new categories in Sage by specifying the axioms that are satisfied by the operations of the super categories. For example, starting from the category of magmas, we can construct all the following categories just by specifying the axioms satisfied by the multiplication:

```
sage: Magmas()
Category of magmas
sage: Magmas().Unital()
Category of unital magmas
```

```
sage: Magmas().Commutative().Unital()
Category of commutative unital magmas
sage: Magmas().Unital().Commutative()
Category of commutative unital magmas
```

```
sage: Magmas().Associative()
Category of semigroups
```

```
sage: Magmas().Associative().Unital()
Category of monoids
```

```
sage: Magmas().Associative().Unital().Commutative()
Category of commutative monoids
```

```
sage: Magmas().Associative().Unital().Inverse()
Category of groups
```

Axioms and categories with axioms

Here, `Associative`, `Unital`, `Commutative` are axioms. In general, any category `Cs` in Sage can declare a new axiom `A`. Then, the *category with axiom* `Cs.A()` models the subcategory of the objects of `Cs` satisfying the axiom `A`. Similarly, for any subcategory `Ds` of `Cs`, `Ds.A()` models the subcategory of the objects of `Ds` satisfying the axiom `A`. In most cases, it's a *full subcategory* (see Wikipedia article Subcategory).

For example, the category of sets defines the `Finite` axiom, and this axiom is available in the subcategory of groups:

```
sage: Sets().Finite()
Category of finite sets
sage: Groups().Finite()
Category of finite groups
```

The meaning of each axiom is described in the documentation of the corresponding method, which can be obtained as usual by instrospection:

```
sage: C = Groups()
sage: C.Finite?                # not tested
```

The purpose of categories with axioms is no different from other categories: to provide bookshelves of code, documentation, mathematical knowledge, tests, for their objects. The extra feature is that, when intersecting categories, axioms are automatically combined together:

```
sage: C = Magmas().Associative() & Magmas().Unital().Inverse() & Sets().Finite(); C
Category of finite groups
sage: sorted(C.axioms())
['Associative', 'Finite', 'Inverse', 'Unital']
```

For a more advanced example, Sage knows that a ring is a set C endowed with a multiplication which distributes over addition, such that $(C, +)$ is a commutative additive group and $(C, *)$ is a monoid:

```
sage: C = (CommutativeAdditiveGroups() & Monoids()).Distributive(); C
Category of rings

sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Unital']
```

The infrastructure allows for specifying further deduction rules, in order to encode mathematical facts like Wedderburn's theorem:

```
sage: DivisionRings() & Sets().Finite()
Category of finite enumerated fields
```

Note: When an axiom specifies the properties of some operations in Sage, the notations for those operations are tied to this axiom. For example, as we have seen above, we need two distinct axioms for associativity: the axiom "Additive-

Associative" is about the properties of the addition $+$, whereas the axiom "Associative" is about the properties of the multiplication $*$.

We are touching here an inherent limitation of the current infrastructure. There is indeed no support for providing generic code that is independent of the notations. In particular, the category hierarchy about additive structures (additive monoids, additive groups, ...) is completely duplicated by that for multiplicative structures (monoids, groups, ...).

As far as we know, none of the existing computer algebra systems has a good solution for this problem. The difficulty is that this is not only about a single notation but a bunch of operators and methods: `+, -, zero, summation, sum, ...` in one case, `*, /, one, product, prod, factor, ...` in the other. Sharing something between the two hierarchies of categories would only be useful if one could write generic code that applies in both cases; for that one needs to somehow automatically substitute the right operations in the right spots in the code. That's kind of what we are doing manually between e.g. *`AdditiveMagmas.ParentMethods.addition_table()`* and *`Magmas.ParentMethods.multiplication_table()`*, but doing this systematically is a different beast from what we have been doing so far with just usual inheritance.

Single entry point and name space usage

A nice feature of the notation `Cs.A()` is that, from a single entry point (say the category *`Magmas`* as above), one can explore a whole range of related categories, typically with the help of introspection to discover which axioms are available, and without having to import new Python modules. This feature will be used in github issue #15741 to unclutter the global name space from, for example, the many variants of the category of algebras like:

```
sage: FiniteDimensionalAlgebrasWithBasis(QQ)
Category of finite dimensional algebras with basis over Rational Field
```

There will of course be a deprecation step, but it's recommended to prefer right away the more flexible notation:

```
sage: Algebras(QQ).WithBasis().FiniteDimensional()
Category of finite dimensional algebras with basis over Rational Field
```

Design discussion

How far should this be pushed? *`Fields`* should definitely stay, but should *`FiniteGroups`* or *`DivisionRings`* be removed from the global namespace? Do we want to further completely deprecate the notation `FiniteGroups()` \``in favor of ``Groups().Finite()`?

On the potential combinatorial explosion of categories with axioms

Even for a very simple category like `Magmas`, there are about 2^5 potential combinations of the axioms! Think about what this becomes for a category with two operations $+$ and $*$:

```
sage: C = (Magmas() & AdditiveMagmas()).Distributive(); C
Category of distributive magmas and additive magmas

sage: CAA = C.Associative().AdditiveAssociative()
sage: CAA.AdditiveCommutative().AdditiveUnital().AdditiveInverse()
Category of rngs

sage: CAA.AdditiveCommutative().AdditiveUnital().Unital()
Category of semirings
```

(continues on next page)

(continued from previous page)

```
sage: CAA.AdditiveCommutative().AdditiveUnital().AdditiveInverse().Unital()
Category of rings

sage: Rings().Division()
Category of division rings

sage: Rings().Division().Commutative()
Category of fields

sage: Rings().Division().Finite()
Category of finite enumerated fields
```

or for more advanced categories:

```
sage: g = HopfAlgebras(QQ).WithBasis().Graded().Connected().category_graph()         #␣
↪needs sage.graphs
sage: g.set_latex_options(format="dot2tex")                                          #␣
↪needs sage.graphs
sage: view(g)                                # not tested                            #␣
↪needs sage.graphs sage.plot
```

Difference between axioms and regressive covariant functorial constructions

Our running examples here will be the axiom `FiniteDimensional` and the regressive covariant functorial construction `Graded`. Let `Cs` be some subcategory of `Modules`, say the category of modules itself:

```
sage: Cs = Modules(QQ)
```

Then, `Cs.FiniteDimensional()` (respectively `Cs.Graded()`) is the subcategory of the objects `O` of `Cs` which are finite dimensional (respectively graded).

Let also `Ds` be a subcategory of `Cs`, say:

```
sage: Ds = Algebras(QQ)
```

A finite dimensional algebra is also a finite dimensional module:

```
sage: Algebras(QQ).FiniteDimensional().is_subcategory(Modules(QQ).FiniteDimensional())
True
```

Similarly a graded algebra is also a graded module:

```
sage: Algebras(QQ).Graded().is_subcategory( Modules(QQ).Graded() )
True
```

This is the *covariance* property: for `A` an axiom or a covariant functorial construction, if `Ds` is a subcategory of `Cs`, then `Ds.A()` is a subcategory of `Cs.A()`.

What happens if we consider reciprocally an object of `Cs.A()` which is also in `Ds`? A finite dimensional module which is also an algebra is a finite dimensional algebra:

```
sage: Modules(QQ).FiniteDimensional() & Algebras(QQ)
Category of finite dimensional algebras over Rational Field
```

On the other hand, a graded module O which is also an algebra is not necessarily a graded algebra! Indeed, the grading on O may not be compatible with the product on O:

```
sage: Modules(QQ).Graded() & Algebras(QQ)
Join of Category of algebras over Rational Field
 and Category of graded vector spaces over Rational Field
```

The relevant difference between `FiniteDimensional` and `Graded` is that `FiniteDimensional` is a statement about the properties of `O` seen as a module (and thus does not depend on the given category), whereas `Graded` is a statement about the properties of `O` and all its operations in the given category.

In general, if a category satisfies a given axiom, any subcategory also satisfies that axiom. Another formulation is that, for an axiom `A` defined in a super category `Cs` of `Ds`, `Ds.A()` is the intersection of the categories `Ds` and `Cs.A()`:

```
sage: As = Algebras(QQ).FiniteDimensional(); As
Category of finite dimensional algebras over Rational Field
sage: Bs = Algebras(QQ) & Modules(QQ).FiniteDimensional(); As
Category of finite dimensional algebras over Rational Field
sage: As is Bs
True
```

An immediate consequence is that, as we have already noticed, axioms commute:

```
sage: As = Algebras(QQ).FiniteDimensional().WithBasis(); As
Category of finite dimensional algebras with basis over Rational Field
sage: Bs = Algebras(QQ).WithBasis().FiniteDimensional(); Bs
Category of finite dimensional algebras with basis over Rational Field
sage: As is Bs
True
```

On the other hand, axioms do not necessarily commute with functorial constructions, even if the current printout may missuggest so:

```
sage: As = Algebras(QQ).Graded().WithBasis(); As
Category of graded algebras with basis over Rational Field
sage: Bs = Algebras(QQ).WithBasis().Graded(); Bs
Category of graded algebras with basis over Rational Field
sage: As is Bs
False
```

This is because `Bs` is the category of algebras endowed with basis, which are further graded; in particular the basis must respect the grading (i.e. be made of homogeneous elements). On the other hand, `As` is the category of graded algebras, which are further endowed with some basis; that basis need not respect the grading. In fact `As` is really a join category:

```
sage: type(As)
<class 'sage.categories.category.JoinCategory_with_category'>
sage: As._repr_(as_join=True)
'Join of Category of algebras with basis over Rational Field and Category of graded␣
↪algebras over Rational Field'
```

Todo: Improve the printing of functorial constructions and joins to raise this potentially dangerous ambiguity.

Further reading on axioms

We refer to *sage.categories.category_with_axiom* for how to implement axioms.

Wrap-up

As we have seen, there is a combinatorial explosion of possible classes. Constructing by hand the full class hierarchy would not scale unless one would restrict to a very rigid subset. Even if it was possible to construct automatically the full hierarchy, this would not scale with respect to system resources.

When designing software systems with large hierarchies of abstract classes for business objects, the difficulty is usually to identify a proper set of key concepts. Here we are lucky, as the key concepts have been long identified and are relatively few:

- Operations ($+$, $*$, ...)
- Axioms on those operations (associativity, ...)
- Constructions (Cartesian products, ...)

Better, those concepts are sufficiently well known so that a user can reasonably be expected to be familiar with the concepts that are involved for his own needs.

Instead, the difficulty is concentrated in the huge number of possible combinations, an unpredictable large subset of which being potentially of interest; at the same time, only a small – but moving – subset has code naturally attached to it.

This has led to the current design, where one focuses on writing the relatively few classes for which there is actual code or mathematical information, and lets Sage *compose dynamically and lazily* those building blocks to construct the minimal hierarchy of classes needed for the computation at hand. This allows for the infrastructure to scale smoothly as bookshelves are added, extended, or reorganized.

2.1.9 Writing a new category

Each category C **must** be provided with a method `C.super_categories()` and *can* be provided with a method `C._subcategory_hook_(D)`. Also, it may be needed to insert C into the output of the `super_categories()` method of some other category. This determines the position of C in the category graph.

A category *may* provide methods that can be used by all its objects, respectively by all elements of its objects.

Each category *should* come with a good example, in `sage.categories.examples`.

Inserting the new category into the category graph

`C.super_categories()` *must* return a list of categories, namely the *immediate* super categories of C. Of course, if you know that your new category C is an immediate super category of some existing category D, then you should also update the method `D.super_categories` to include C.

The immediate super categories of C *should not* be *join categories*. Furthermore, one always should have:

```
Cs().is_subcategory( Category.join(Cs().super_categories()) )

Cs()._cmp_key  >  other._cmp_key  for other in Cs().super_categories()
```

This is checked by *_test_category()*.

In several cases, the category C is directly provided with a generic implementation of `super_categories`; a typical example is when C implements an axiom or a functorial construction; in such a case, C may implement `C.extra_super_categories()` to complement the super categories discovered by the generic implementation. This method needs not return immediate super categories; instead it's usually best to specify the largest super category providing the desired mathematical information. For example, the category *`Magmas.Commutative.Algebras`* just states that the algebra of a commutative magma is a commutative magma. This is sufficient to let Sage deduce that it's in fact a commutative algebra.

Methods for objects and elements

Different objects of the same category share some algebraic features, and very often these features can be encoded in a method, in a generic way. For example, for every commutative additive monoid, it makes sense to ask for the sum of a list of elements. Sage's category framework allows to provide a generic implementation for all objects of a category.

If you want to provide your new category with generic methods for objects (or elements of objects), then you simply add a nested class called `ParentMethods` (or `ElementMethods`). The methods of that class will automatically become methods of the objects (or the elements). For instance:

```
sage: P.<x,y> = ZZ[]
sage: P.prod([x,y,2])
2*x*y
sage: P.prod.__module__
'sage.categories.monoids'
sage: P.prod.__func__ is raw_getattr(Monoids().ParentMethods, "prod")
True
```

We recommend to study the code of one example:

```
sage: C = CommutativeAdditiveMonoids()
sage: C??                              # not tested
```

On the order of super categories

The generic method `C.all_super_categories()` determines recursively the list of *all* super categories of C.

The order of the categories in this list does influence the inheritance of methods for parents and elements. Namely, if P is an object in the category C and if C_1 and C_2 are both super categories of C defining some method `foo` in `ParentMethods`, then P will use C_1's version of `foo` if and only if C_1 appears in `C.all_super_categories()` before C_2.

However this must be considered as an *implementation detail*: if C_1 and C_2 are incomparable categories, then the order in which they appear must be mathematically irrelevant: in particular, the methods `foo` in C_1 and C_2 must have the same semantic. Code should not rely on any specific order, as it is subject to later change. Whenever one of the implementations is preferred in some common subcategory of C_1 and C_2, for example for efficiency reasons, the ambiguity should be resolved explicitly by defining a method `foo` in this category. See the method `some_elements` in the code of the category *`FiniteCoxeterGroups`* for an example.

Since github issue #11943, `C.all_super_categories()` is computed by the so-called `C3` algorithm used by Python to compute Method Resolution Order of new-style classes. Thus the order in `C.all_super_categories()`, `C.parent_class.mro()` and `C.element_class.mro()` are guaranteed to be consistent.

Since github issue #13589, the `C3` algorithm is put under control of some total order on categories. This order is not necessarily meaningful, but it guarantees that `C3` always finds a consistent Method Resolution Order. For background, see `sage.misc.c3_controlled`. A visible effect is that the order in which categories are specified in `C.super_categories()`, or in a join category, no longer influences the result of `C.all_super_categories()`.

Subcategory hook (advanced optimization feature)

The default implementation of the method `C.is_subcategory(D)` is to look up whether D appears in `C.all_super_categories()`. However, building the list of all the super categories of C is an expensive operation that is sometimes best avoided. For example, if both C and D are categories defined over a base, but the bases differ, then one knows right away that they can not be subcategories of each other.

When such a short-path is known, one can implement a method `_subcategory_hook_`. Then, `C.is_subcategory(D)` first calls `D._subcategory_hook_(C)`. If this returns `Unknown`, then `C.is_subcategory(D)` tries to find `D` in `C.all_super_categories()`. Otherwise, `C.is_subcategory(D)` returns the result of `D._subcategory_hook_(C)`.

By default, `D._subcategory_hook_(C)` tests whether `issubclass(C.parent_class,D.parent_class)`, which is very often giving the right answer:

```
sage: Rings()._subcategory_hook_(Algebras(QQ))
True
sage: HopfAlgebras(QQ)._subcategory_hook_(Algebras(QQ))
False
sage: Algebras(QQ)._subcategory_hook_(HopfAlgebras(QQ))
True
```

2.2 Categories

AUTHORS:

- David Kohel, William Stein and Nicolas M. Thiery

Every Sage object lies in a category. Categories in Sage are modeled on the mathematical idea of category, and are distinct from Python classes, which are a programming construct.

In most cases, typing `x.category()` returns the category to which `x` belongs. If `C` is a category and `x` is any object, `C(x)` tries to make an object in `C` from `x`. Checking if `x` belongs to `C` is done as usually by `x in C`.

See *`Category`* and *`sage.categories.primer`* for more details.

EXAMPLES:

We create a couple of categories:

```
sage: Sets()
Category of sets
sage: GSets(AbelianGroup([2, 4, 9]))                                    #␣
↪needs sage.groups
Category of G-sets for Multiplicative Abelian group isomorphic to C2 x C4 x C9
sage: Semigroups()
Category of semigroups
sage: VectorSpaces(FiniteField(11))
Category of vector spaces over Finite Field of size 11
sage: Ideals(IntegerRing())
Category of ring ideals in Integer Ring
```

Let's request the category of some objects:

```
sage: V = VectorSpace(RationalField(), 3)                               #␣
↪needs sage.modules
sage: V.category()                                                      #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.modules
Category of finite dimensional vector spaces with basis
 over (number fields and quotient fields and metric spaces)

sage: G = SymmetricGroup(9)                                                   #␣
↪needs sage.groups
sage: G.category()                                                            #␣
↪needs sage.groups
Join of
 Category of finite enumerated permutation groups and
 Category of finite Weyl groups and
 Category of well generated finite irreducible complex reflection groups

sage: P = PerfectMatchings(3)                                                 #␣
↪needs sage.combinat
sage: P.category()                                                            #␣
↪needs sage.combinat
Category of finite enumerated sets
```

Let's check some memberships:

```
sage: V in VectorSpaces(QQ)                                                   #␣
↪needs sage.modules
True
sage: V in VectorSpaces(FiniteField(11))                                      #␣
↪needs sage.modules
False
sage: G in Monoids()                                                          #␣
↪needs sage.groups
True
sage: P in Rings()                                                            #␣
↪needs sage.combinat
False
```

For parametrized categories one can use the following shorthand:

```
sage: V in VectorSpaces                                                       #␣
↪needs sage.modules
True
sage: G in VectorSpaces                                                       #␣
↪needs sage.groups
False
```

A parent `P` is in a category `C` if `P.category()` is a subcategory of `C`.

Note: Any object of a category should be an instance of `CategoryObject`.

For backward compatibility this is not yet enforced:

```
sage: class A:
....:     def category(self):
....:         return Fields()
sage: A() in Rings()
True
```

By default, the category of an element x of a parent P is the category of all objects of P (this is dubious and may be deprecated):

```
sage: V = VectorSpace(RationalField(), 3)                                       #␣
↪needs sage.modules
sage: v = V.gen(1)                                                              #␣
↪needs sage.modules
sage: v.category()                                                              #␣
↪needs sage.modules
Category of elements of Vector space of dimension 3 over Rational Field
```

class sage.categories.category.**Category**

Bases: `UniqueRepresentation`, `SageObject`

The base class for modeling mathematical categories, like for example:

- `Groups()`: the category of groups
- `EuclideanDomains()`: the category of euclidean rings
- `VectorSpaces(QQ)`: the category of vector spaces over the field of rationals

See *sage.categories.primer* for an introduction to categories in Sage, their relevance, purpose, and usage. The documentation below will focus on their implementation.

Technically, a category is an instance of the class *Category* or some of its subclasses. Some categories, like *VectorSpaces*, are parametrized: `VectorSpaces(QQ)` is one of many instances of the class *VectorSpaces*. On the other hand, `EuclideanDomains()` is the single instance of the class *EuclideanDomains*.

Recall that an algebraic structure (say, the ring $\mathbf{Q}[x]$) is modelled in Sage by an object which is called a parent. This object belongs to certain categories (here `EuclideanDomains()` and `Algebras()`). The elements of the ring are themselves objects.

The class of a category (say *EuclideanDomains*) can define simultaneously:

- Operations on the category itself (what is its super categories? its category of morphisms? its dual category?).
- Generic operations on parents in this category, like the ring $\mathbf{Q}[x]$.
- Generic operations on elements of such parents (e. g., the Euclidean algorithm for computing gcds).
- Generic operations on morphisms of this category.

This is achieved as follows:

```
sage: from sage.categories.category import Category
sage: class EuclideanDomains(Category):
....:     # operations on the category itself
....:     def super_categories(self):
....:         [Rings()]
....:
....:     def dummy(self): # TODO: find some good examples
....:          pass
....:
....:     class ParentMethods: # holds the generic operations on parents
....:          # TODO: find a good example of an operation
....:          pass
....:
....:     class ElementMethods:# holds the generic operations on elements
....:          def gcd(x,y):
....:              # Euclid algorithms
....:              pass
....:
```

(continues on next page)

(continued from previous page)

```
....:     class MorphismMethods: # holds the generic operations on morphisms
....:         # TODO: find a good example of an operation
....:         pass
....:
```

Note that the nested class `ParentMethods` is merely a container of operations, and does not inherit from anything. Instead, the hierarchy relation is defined once at the level of the categories, and the actual hierarchy of classes is built in parallel from all the `ParentMethods` nested classes, and stored in the attributes `parent_class`. Then, a parent in a category `C` receives the appropriate operations from all the super categories by usual class inheritance from `C.parent_class`.

Similarly, two other hierarchies of classes, for elements and morphisms respectively, are built from all the `ElementMethods` and `MorphismMethods` nested classes.

EXAMPLES:

We define a hierarchy of four categories `As()`, `Bs()`, `Cs()`, `Ds()` with a diamond inheritance. Think for example:

- `As()`: the category of sets
- `Bs()`: the category of additive groups
- `Cs()`: the category of multiplicative monoids
- `Ds()`: the category of rings

```
sage: from sage.categories.category import Category
sage: from sage.misc.lazy_attribute import lazy_attribute
sage: class As (Category):
....:     def super_categories(self):
....:         return []
....:
....:     class ParentMethods:
....:         def fA(self):
....:             return "A"
....:         f = fA

sage: class Bs (Category):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class ParentMethods:
....:         def fB(self):
....:             return "B"

sage: class Cs (Category):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class ParentMethods:
....:         def fC(self):
....:             return "C"
....:         f = fC

sage: class Ds (Category):
....:     def super_categories(self):
....:         return [Bs(),Cs()]
....:
```

(continues on next page)

(continued from previous page)

```
....:     class ParentMethods:
....:         def fD(self):
....:             return "D"
```

Categories should always have unique representation; by github issue #12215, this means that it will be kept in cache, but only if there is still some strong reference to it.

We check this before proceeding:

```
sage: import gc
sage: idAs = id(As())
sage: _ = gc.collect()
sage: n == id(As())
False
sage: a = As()
sage: id(As()) == id(As())
True
sage: As().parent_class == As().parent_class
True
```

We construct a parent in the category `Ds()` (that, is an instance of `Ds().parent_class`), and check that it has access to all the methods provided by all the categories, with the appropriate inheritance order:

```
sage: D = Ds().parent_class()
sage: [ D.fA(), D.fB(), D.fC(), D.fD() ]
['A', 'B', 'C', 'D']
sage: D.f()
'C'
```

```
sage: C = Cs().parent_class()
sage: [ C.fA(), C.fC() ]
['A', 'C']
sage: C.f()
'C'
```

Here is the parallel hierarchy of classes which has been built automatically, together with the method resolution order (`.mro()`):

```
sage: As().parent_class
<class '__main__.As.parent_class'>
sage: As().parent_class.__bases__
(<... 'object'>,)
sage: As().parent_class.mro()
[<class '__main__.As.parent_class'>, <... 'object'>]
```

```
sage: Bs().parent_class
<class '__main__.Bs.parent_class'>
sage: Bs().parent_class.__bases__
(<class '__main__.As.parent_class'>,)
sage: Bs().parent_class.mro()
[<class '__main__.Bs.parent_class'>, <class '__main__.As.parent_class'>, <...
↪'object'>]
```

```
sage: Cs().parent_class
<class '__main__.Cs.parent_class'>
```

(continues on next page)

(continued from previous page)

```
sage: Cs().parent_class.__bases__
(<class '__main__.As.parent_class'>,)
sage: Cs().parent_class.__mro__
(<class '__main__.Cs.parent_class'>, <class '__main__.As.parent_class'>, <...
↪'object'>)
```

```
sage: Ds().parent_class
<class '__main__.Ds.parent_class'>
sage: Ds().parent_class.__bases__
(<class '__main__.Cs.parent_class'>, <class '__main__.Bs.parent_class'>)
sage: Ds().parent_class.mro()
[<class '__main__.Ds.parent_class'>, <class '__main__.Cs.parent_class'>,
 <class '__main__.Bs.parent_class'>, <class '__main__.As.parent_class'>, <...
↪'object'>]
```

Note that two categories in the same class need not have the same `super_categories`. For example, `Algebras(QQ)` has `VectorSpaces(QQ)` as super category, whereas `Algebras(ZZ)` only has `Modules(ZZ)` as super category. In particular, the constructed parent class and element class will differ (inheriting, or not, methods specific for vector spaces):

```
sage: Algebras(QQ).parent_class is Algebras(ZZ).parent_class
False
sage: issubclass(Algebras(QQ).parent_class, VectorSpaces(QQ).parent_class)
True
```

On the other hand, identical hierarchies of classes are, preferably, built only once (e.g. for categories over a base ring):

```
sage: Algebras(GF(5)).parent_class is Algebras(GF(7)).parent_class
True
sage: F = FractionField(ZZ['t'])
sage: Coalgebras(F).parent_class is Coalgebras(FractionField(F['x'])).parent_class
True
```

We now construct a parent in the usual way:

```
sage: class myparent(Parent):
....:     def __init__(self):
....:         Parent.__init__(self, category=Ds())
....:     def g(self):
....:         return "myparent"
....:     class Element():
....:         pass
sage: D = myparent()
sage: D.__class__
<class '__main__.myparent_with_category'>
sage: D.__class__.__bases__
(<class '__main__.myparent'>, <class '__main__.Ds.parent_class'>)
sage: D.__class__.mro()
[<class '__main__.myparent_with_category'>,
<class '__main__.myparent'>,
<class 'sage.structure.parent.Parent'>,
<class 'sage.structure.category_object.CategoryObject'>,
<class 'sage.structure.sage_object.SageObject'>,
<class '__main__.Ds.parent_class'>,
<class '__main__.Cs.parent_class'>,
```

(continues on next page)

(continued from previous page)

```
<class '__main__.Bs.parent_class'>,
<class '__main__.As.parent_class'>,
<... 'object'>]
sage: D.fA()
'A'
sage: D.fB()
'B'
sage: D.fC()
'C'
sage: D.fD()
'D'
sage: D.f()
'C'
sage: D.g()
'myparent'
```

```
sage: D.element_class
<class '__main__.myparent_with_category.element_class'>
sage: D.element_class.mro()
[<class '__main__.myparent_with_category.element_class'>,
<class ...__main__....Element...>,
<class '__main__.Ds.element_class'>,
<class '__main__.Cs.element_class'>,
<class '__main__.Bs.element_class'>,
<class '__main__.As.element_class'>,
<... 'object'>]
```

_super_categories()

The immediate super categories of this category.

This lazy attribute caches the result of the mandatory method *super_categories()* for speed. It also does some mangling (flattening join categories, sorting, …).

Whenever speed matters, developers are advised to use this lazy attribute rather than calling *super_categories()*.

Note: This attribute is likely to eventually become a tuple. When this happens, we might as well use *Category._sort()*, if not *Category._sort_uniq()*.

EXAMPLES:

```
sage: Rings()._super_categories
[Category of rngs, Category of semirings]
```

_super_categories_for_classes()

The super categories of this category used for building classes.

This is a close variant of *_super_categories()* used for constructing the list of the bases for *parent_class()*, *element_class()*, and friends. The purpose is ensure that Python will find a proper Method Resolution Order for those classes. For background, see `sage.misc.c3_controlled`.

See also:

`_cmp_key()`.

Note: This attribute is calculated as a by-product of computing *_all_super_categories()*.

EXAMPLES:

```
sage: Rings()._super_categories_for_classes
[Category of rngs, Category of semirings]
```

_all_super_categories()

All the super categories of this category, including this category.

Since github issue #11943, the order of super categories is determined by Python's method resolution order C3 algorithm.

See also:

all_super_categories()

Note: this attribute is likely to eventually become a tuple.

Note: this sets *_super_categories_for_classes()* as a side effect

EXAMPLES:

```
sage: C = Rings(); C
Category of rings
sage: C._all_super_categories
[Category of rings, Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]
```

_all_super_categories_proper()

All the proper super categories of this category.

Since github issue #11943, the order of super categories is determined by Python's method resolution order C3 algorithm.

See also:

all_super_categories()

Note: this attribute is likely to eventually become a tuple.

EXAMPLES:

```
sage: C = Rings(); C
Category of rings
sage: C._all_super_categories_proper
[Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
```

(continues on next page)

(continued from previous page)

```
 Category of sets, Category of sets with partial maps,
 Category of objects]
```

_set_of_super_categories()

The frozen set of all proper super categories of this category.

Note: this is used for speeding up category containment tests.

See also:

all_super_categories()

EXAMPLES:

```
sage: sorted(Groups()._set_of_super_categories, key=str)
[Category of inverse unital magmas,
 Category of magmas,
 Category of monoids,
 Category of objects,
 Category of semigroups,
 Category of sets,
 Category of sets with partial maps,
 Category of unital magmas]
sage: sorted(Groups()._set_of_super_categories, key=str)
[Category of inverse unital magmas, Category of magmas, Category of monoids,
 Category of objects, Category of semigroups, Category of sets,
 Category of sets with partial maps, Category of unital magmas]
```

_make_named_class(*name*, *method_provider*, *cache=False*, *picklable=True*)

Construction of the parent/element/… class of `self`.

INPUT:

- `name` – a string; the name of the class as an attribute of `self`. E.g. "parent_class"
- `method_provider` – a string; the name of an attribute of `self` that provides methods for the new class (in addition to those coming from the super categories). E.g. "ParentMethods"
- `cache` – a boolean or `ignore_reduction` (default: `False`) (passed down to dynamic_class; for internal use only)
- `picklable` – a boolean (default: `True`)

ASSUMPTION:

It is assumed that this method is only called from a lazy attribute whose name coincides with the given `name`.

OUTPUT:

A dynamic class with bases given by the corresponding named classes of `self`'s super_categories, and methods taken from the class `getattr(self,method_provider)`.

Note:

- In this default implementation, the reduction data of the named class makes it depend on `self`. Since the result is going to be stored in a lazy attribute of `self` anyway, we may as well disable the caching in `dynamic_class` (hence the default value `cache=False`).

- *CategoryWithParameters* overrides this method so that the same parent/element/… classes can be shared between closely related categories.
- The bases of the named class may also contain the named classes of some indirect super categories, according to *_super_categories_for_classes()*. This is to guarantee that Python will build consistent method resolution orders. For background, see sage.misc.c3_controlled.

See also:

`CategoryWithParameters._make_named_class()`

EXAMPLES:

```
sage: PC = Rings()._make_named_class("parent_class", "ParentMethods"); PC
<class 'sage.categories.rings.Rings.parent_class'>
sage: type(PC)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
sage: PC.__bases__
(<class 'sage.categories.rngs.Rngs.parent_class'>,
 <class 'sage.categories.semirings.Semirings.parent_class'>)
```

Note that, by default, the result is not cached:

```
sage: PC is Rings()._make_named_class("parent_class", "ParentMethods")
False
```

Indeed this method is only meant to construct lazy attributes like `parent_class` which already handle this caching:

```
sage: Rings().parent_class
<class 'sage.categories.rings.Rings.parent_class'>
```

Reduction for pickling also assumes the existence of this lazy attribute:

```
sage: PC._reduction
(<built-in function getattr>, (Category of rings, 'parent_class'))
sage: loads(dumps(PC)) is Rings().parent_class
True
```

repr()

Return the print representation of this category.

EXAMPLES:

```
sage: Sets() # indirect doctest
Category of sets
```

_repr_object_names()

Return the name of the objects of this category.

EXAMPLES:

```
sage: FiniteGroups()._repr_object_names()
'finite groups'
sage: AlgebrasWithBasis(QQ)._repr_object_names()
'algebras with basis over Rational Field'
```

_test_category(***options*)

Run generic tests on this category

See also:

`TestSuite`.

EXAMPLES:

```
sage: Sets()._test_category()
```

Let us now write a couple broken categories:

```
sage: class MyObjects(Category):
....:     pass
sage: MyObjects()._test_category()
Traceback (most recent call last):
...
NotImplementedError: <abstract method super_categories at ...>

sage: class MyObjects(Category):
....:     def super_categories(self):
....:         return tuple()
sage: MyObjects()._test_category()
Traceback (most recent call last):
...
AssertionError: Category of my objects.super_categories() should return a list

sage: class MyObjects(Category):
....:     def super_categories(self):
....:         return []
sage: MyObjects()._test_category()
Traceback (most recent call last):
...
AssertionError: Category of my objects is not a subcategory of Objects()
```

_with_axiom(*axiom*)

Return the subcategory of the objects of `self` satisfying the given `axiom`.

INPUT:

- `axiom` – a string, the name of an axiom

EXAMPLES:

```
sage: Sets()._with_axiom("Finite")
Category of finite sets

sage: type(Magmas().Finite().Commutative())
<class 'sage.categories.category.JoinCategory_with_category'>
sage: Magmas().Finite().Commutative().super_categories()
[Category of commutative magmas, Category of finite sets]
sage: C = Algebras(QQ).WithBasis().Commutative()
sage: C is Algebras(QQ).Commutative().WithBasis()
True
```

When `axiom` is not defined for `self`, `self` is returned:

```
sage: Sets()._with_axiom("Associative")
Category of sets
```

Warning: This may be changed in the future to raising an error.

`_with_axiom_as_tuple`(*axiom*)

Return a tuple of categories whose join is `self._with_axiom()`.

INPUT:

- `axiom` – a string, the name of an axiom

This is a lazy version of *`_with_axiom()`* which is used to avoid recursion loops during join calculations.

Note: The order in the result is irrelevant.

EXAMPLES:

```
sage: Sets()._with_axiom_as_tuple('Finite')
(Category of finite sets,)
sage: Magmas()._with_axiom_as_tuple('Finite')
(Category of magmas, Category of finite sets)
sage: Rings().Division()._with_axiom_as_tuple('Finite')
(Category of division rings,
 Category of finite monoids,
 Category of commutative magmas,
 Category of finite additive groups)
sage: HopfAlgebras(QQ)._with_axiom_as_tuple('FiniteDimensional')
(Category of Hopf algebras over Rational Field,
 Category of finite dimensional vector spaces over Rational Field)
```

`_without_axioms`(*named=False*)

Return the category without the axioms that have been added to create it.

INPUT:

- `named` – a boolean (default: `False`)

Todo: Improve this explanation.

If `named` is `True`, then this stops at the first category that has an explicit name of its own. See *`category_with_axiom.CategoryWithAxiom._without_axioms()`*

EXAMPLES:

```
sage: Sets()._without_axioms()
Category of sets
sage: Semigroups()._without_axioms()
Category of magmas
sage: Algebras(QQ).Commutative().WithBasis()._without_axioms()
Category of magmatic algebras over Rational Field
sage: Algebras(QQ).Commutative().WithBasis()._without_axioms(named=True)
Category of algebras over Rational Field
```

`static _sort`(*categories*)

Return the categories after sorting them decreasingly according to their comparison key.

See also:

`_cmp_key()`

INPUT:

- `categories` – a list (or iterable) of non-join categories

OUTPUT:

A sorted tuple of categories, possibly with repeats.

Note: The auxiliary function `_flatten_categories` used in the test below expects a second argument, which is a type such that instances of that type will be replaced by its super categories. Usually, this type is *`JoinCategory`*.

EXAMPLES:

```
sage: Category._sort([Sets(), Objects(), Coalgebras(QQ), Monoids(), Sets().
↪Finite()])
(Category of monoids,
 Category of coalgebras over Rational Field,
 Category of finite sets,
 Category of sets,
 Category of objects)
sage: Category._sort([Sets().Finite(), Semigroups().Finite(), Sets().Facade(),
↪Magmas().Commutative()])
(Category of finite semigroups,
 Category of commutative magmas,
 Category of finite sets,
 Category of facade sets)
sage: Category._sort(Category._flatten_categories([Sets().Finite(),␣
↪Algebras(QQ).WithBasis(), Semigroups().Finite(),
....:                                                    Sets().Facade(),␣
↪Algebras(QQ).Commutative(), Algebras(QQ).Graded().WithBasis()],
....:                                                    sage.categories.category.
↪JoinCategory))
(Category of algebras with basis over Rational Field,
 Category of algebras with basis over Rational Field,
 Category of graded algebras over Rational Field,
 Category of commutative algebras over Rational Field,
 Category of finite semigroups,
 Category of finite sets,
 Category of facade sets)
```

static _sort_uniq(*categories*)

Return the categories after sorting them and removing redundant categories.

Redundant categories include duplicates and categories which are super categories of other categories in the input.

INPUT:

- `categories` – a list (or iterable) of categories

OUTPUT: a sorted tuple of mutually incomparable categories

EXAMPLES:

```
sage: Category._sort_uniq([Rings(), Monoids(), Coalgebras(QQ)])
(Category of rings, Category of coalgebras over Rational Field)
```

Note that, in the above example, `Monoids()` does not appear in the result because it is a super category of `Rings()`.

static __classcall__(**args*, ***options*)

Input mangling for unique representation.

Let `C = Cs(...)` be a category. Since github issue #12895, the class of `C` is a dynamic subclass `Cs_with_category` of `Cs` in order for `C` to inherit code from the `SubcategoryMethods` nested classes of its super categories.

The purpose of this `__classcall__` method is to ensure that reconstructing `C` from its class with `Cs_with_category(...)` actually calls properly `Cs(...)` and gives back `C`.

See also:

`subcategory_class()`

EXAMPLES:

```
sage: A = Algebras(QQ)
sage: A.__class__
<class 'sage.categories.algebras.Algebras_with_category'>
sage: A is Algebras(QQ)
True
sage: A is A.__class__(QQ)
True
```

__init__()

Initialize this category.

EXAMPLES:

```
sage: class SemiprimitiveRings(Category):
....:     def super_categories(self):
....:         return [Rings()]
....:     class ParentMethods:
....:         def jacobson_radical(self):
....:             return self.ideal(0)
sage: C = SemiprimitiveRings()
sage: C
Category of semiprimitive rings
sage: C.__class__
<class '__main__.SemiprimitiveRings_with_category'>
```

Note: If the default name of the category (built from the name of the class) is not adequate, please implement *`_repr_object_names()`* to customize it.

Realizations()

Return the category of realizations of the parent `self` or of objects of the category `self`

INPUT:

- `self` – a parent or a concrete category

Note: this *function* is actually inserted as a *method* in the class *Category* (see *Realizations()*). It is defined here for code locality reasons.

EXAMPLES:

The category of realizations of some algebra:

```
sage: Algebras(QQ).Realizations()
Join of Category of algebras over Rational Field
    and Category of realizations of unital magmas
```

The category of realizations of a given algebra:

```
sage: A = Sets().WithRealizations().example(); A
↪   # needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.Realizations()
↪   # needs sage.modules
Category of realizations of
 The subset algebra of {1, 2, 3} over Rational Field

sage: C = GradedHopfAlgebrasWithBasis(QQ).Realizations(); C
Join of Category of graded Hopf algebras with basis over Rational Field
    and Category of realizations of Hopf algebras over Rational Field
sage: C.super_categories()
[Category of graded Hopf algebras with basis over Rational Field,
 Category of realizations of Hopf algebras over Rational Field]

sage: TestSuite(C).run()
```

See also:

- *Sets().WithRealizations*
- `ClasscallMetaclass`

Todo: Add an optional argument to allow for:

```
sage: Realizations(A, category=Blahs()) # todo: not implemented
```

WithRealizations()

Return the category of parents in `self` endowed with multiple realizations.

INPUT:

- `self` – a category

See also:

- The documentation and code (*sage.categories.examples.with_realizations*) of `Sets().WithRealizations().example()` for more on how to use and implement a parent with several realizations.
- Various use cases:
  - `SymmetricFunctions`

  - QuasiSymmetricFunctions
  - NonCommutativeSymmetricFunctions
  - SymmetricFunctionsNonCommutingVariables
  - DescentAlgebra
  - algebras.Moebius
  - IwahoriHeckeAlgebra
  - ExtendedAffineWeylGroup
- The Implementing Algebraic Structures thematic tutorial.
- *sage.categories.realizations*

Note: this *function* is actually inserted as a *method* in the class *Category* (see *WithRealizations()*). It is defined here for code locality reasons.

EXAMPLES:

```
sage: Sets().WithRealizations()
Category of sets with realizations
```

Parent with realizations

Let us now explain the concept of realizations. A *parent with realizations* is a facade parent (see *Sets.Facade*) admitting multiple concrete realizations where its elements are represented. Consider for example an algebra A which admits several natural bases:

```
sage: A = Sets().WithRealizations().example(); A                           ␣
↪   # needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
```

For each such basis B one implements a parent P_B which realizes A with its elements represented by expanding them on the basis B:

```
sage: # needs sage.modules
sage: A.F()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: A.Out()
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
sage: A.In()
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: A.an_element()
F[{}] + 2*F[{1}] + 3*F[{2}] + F[{1, 2}]
```

If B and B' are two bases, then the change of basis from B to B' is implemented by a canonical coercion between P_B and $P_{B'}$:

```
sage: # needs sage.combinat sage.modules
sage: F = A.F(); In = A.In(); Out = A.Out()
sage: i = In.an_element(); i
In[{}] + 2*In[{1}] + 3*In[{2}] + In[{1, 2}]
sage: F(i)
```

(continues on next page)

(continued from previous page)

```
7*F[{}] + 3*F[{1}] + 4*F[{2}] + F[{1, 2}]
sage: F.coerce_map_from(Out)
Generic morphism:
  From: The subset algebra of {1, 2, 3} over Rational Field in the Out basis
  To:   The subset algebra of {1, 2, 3} over Rational Field in the␣
↪Fundamental basis
```

allowing for mixed arithmetic:

```
sage: (1 + Out.from_set(1)) * In.from_set(2,3)                              ␣
↪   # needs sage.combinat sage.modules
Out[{}] + 2*Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}]
+ 2*Out[{1, 3}] + 4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

In our example, there are three realizations:

```
sage: A.realizations()                                                      ␣
↪   # needs sage.modules
[The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis,
 The subset algebra of {1, 2, 3} over Rational Field in the In basis,
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis]
```

Instead of manually defining the shorthands `F`, `In`, and `Out`, as above one can just do:

```
sage: A.inject_shorthands()                                                 ␣
↪   # needs sage.combinat sage.modules
Defining F as shorthand for
 The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
Defining In as shorthand for
 The subset algebra of {1, 2, 3} over Rational Field in the In basis
Defining Out as shorthand for
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis
```

Rationale

Besides some goodies described below, the role of A is threefold:

- To provide, as illustrated above, a single entry point for the algebra as a whole: documentation, access to its properties and different realizations, etc.
- To provide a natural location for the initialization of the bases and the coercions between, and other methods that are common to all bases.
- To let other objects refer to A while allowing elements to be represented in any of the realizations.

We now illustrate this second point by defining the polynomial ring with coefficients in A:

```
sage: P = A['x']; P                                                         ␣
↪   # needs sage.modules
Univariate Polynomial Ring in x over
 The subset algebra of {1, 2, 3} over Rational Field
sage: x = P.gen()                                                           ␣
↪   # needs sage.modules
```

In the following examples, the coefficients turn out to be all represented in the F basis:

```
sage: P.one()                                                          ␣
↪   # needs sage.modules
F[{}]
sage: (P.an_element() + 1)^2                                           ␣
↪   # needs sage.modules
F[{}]*x^2 + 2*F[{}]*x + F[{}]
```

However we can create a polynomial with mixed coefficients, and compute with it:

```
sage: p = P([1, In[{1}], Out[{2}] ]); p                                ␣
↪   # needs sage.combinat sage.modules
Out[{2}]*x^2 + In[{1}]*x + F[{}]
sage: p^2                                                              ␣
↪   # needs sage.combinat sage.modules
Out[{2}]*x^4
+ (-8*In[{}] + 4*In[{1}] + 8*In[{2}] + 4*In[{3}]
   - 4*In[{1, 2}] - 2*In[{1, 3}] - 4*In[{2, 3}] + 2*In[{1, 2, 3}])*x^3
+ (F[{}] + 3*F[{1}] + 2*F[{2}] - 2*F[{1, 2}] - 2*F[{2, 3}] + 2*F[{1, 2, 3}
↪])*x^2
+ (2*F[{}] + 2*F[{1}])*x
+ F[{}]
```

Note how each coefficient involves a single basis which need not be that of the other coefficients. Which basis is used depends on how coercion happened during mixed arithmetic and needs not be deterministic.

One can easily coerce all coefficient to a given basis with:

```
sage: p.map_coefficients(In)                                           ␣
↪   # needs sage.combinat sage.modules
(-4*In[{}] + 2*In[{1}] + 4*In[{2}] + 2*In[{3}]
 - 2*In[{1, 2}] - In[{1, 3}] - 2*In[{2, 3}] + In[{1, 2, 3}])*x^2
+ In[{1}]*x + In[{}]
```

Alas, the natural notation for constructing such polynomials does not yet work:

```
sage: In[{1}] * x                                                      ␣
↪   # needs sage.combinat sage.modules
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *:
'The subset algebra of {1, 2, 3} over Rational Field in the In basis'
and 'Univariate Polynomial Ring in x over
The subset algebra of {1, 2, 3} over Rational Field'
```

The category of realizations of A

The set of all realizations of A, together with the coercion morphisms is a category (whose class inherits from *Category_realization_of_parent*):

```
sage: A.Realizations()                                                 ␣
↪   # needs sage.modules
Category of realizations of
 The subset algebra of {1, 2, 3} over Rational Field
```

The various parent realizing A belong to this category:

```
sage: A.F() in A.Realizations()                                         ␣
↪   # needs sage.modules
True
```

A itself is in the category of algebras with realizations:

```
sage: A in Algebras(QQ).WithRealizations()                              ␣
↪   # needs sage.modules
True
```

The (mostly technical) `WithRealizations` categories are the analogs of the `*WithSeveralBases` categories in MuPAD-Combinat. They provide support tools for handling the different realizations and the morphisms between them.

Typically, `VectorSpaces(QQ).FiniteDimensional().WithRealizations()` will eventually be in charge, whenever a coercion $\phi : A \mapsto B$ is registered, to register ϕ^{-1} as coercion $B \mapsto A$ if there is none defined yet. To achieve this, `FiniteDimensionalVectorSpaces` would provide a nested class `WithRealizations` implementing the appropriate logic.

`WithRealizations` is a *`regressive covariant functorial construction`*. On our example, this simply means that A is automatically in the category of rings with realizations (covariance):

```
sage: A in Rings().WithRealizations()                                   ␣
↪   # needs sage.modules
True
```

and in the category of algebras (regressiveness):

```
sage: A in Algebras(QQ)                                                 ␣
↪   # needs sage.modules
True
```

Note: For `C` a category, `C.WithRealizations()` in fact calls `sage.categories.with_realizations.WithRealizations(C)`. The later is responsible for building the hierarchy of the categories with realizations in parallel to that of their base categories, optimizing away those categories that do not provide a `WithRealizations` nested class. See *`sage.categories.covariant_functorial_construction`* for the technical details.

Note: Design question: currently `WithRealizations` is a regressive construction. That is `self.WithRealizations()` is a subcategory of `self` by default:

```
sage: Algebras(QQ).WithRealizations().super_categories()
[Category of algebras over Rational Field,
 Category of monoids with realizations,
 Category of additive unital additive magmas with realizations]
```

Is this always desirable? For example, `AlgebrasWithBasis(QQ).WithRealizations()` should certainly be a subcategory of `Algebras(QQ)`, but not of `AlgebrasWithBasis(QQ)`. This is because `AlgebrasWithBasis(QQ)` is specifying something about the concrete realization.

`additional_structure``()`

Return whether `self` defines additional structure.

OUTPUT:

- `self` if `self` defines additional structure and `None` otherwise. This default implementation returns `self`.

A category C *defines additional structure* if C-morphisms shall preserve more structure (e.g. operations) than that specified by the super categories of C. For example, the category of magmas defines additional structure, namely the operation $*$ that shall be preserved by magma morphisms. On the other hand the category of rings does not define additional structure: a function between two rings that is both a unital magma morphism and a unital additive magma morphism is automatically a ring morphism.

Formally speaking C *defines additional structure*, if C is *not* a full subcategory of the join of its super categories: the morphisms need to preserve more structure, and thus the homsets are smaller.

By default, a category is considered as defining additional structure, unless it is a *category with axiom*.

EXAMPLES:

Here are some typical structure categories, with the additional structure they define:

```
sage: Sets().additional_structure()
Category of sets
sage: Magmas().additional_structure()          # `*`
Category of magmas
sage: AdditiveMagmas().additional_structure()  # `+`
Category of additive magmas
sage: LeftModules(ZZ).additional_structure()   # left multiplication by scalar
Category of left modules over Integer Ring
sage: Coalgebras(QQ).additional_structure()    # coproduct
Category of coalgebras over Rational Field
sage: Crystals().additional_structure()        # crystal operators
Category of crystals
```

On the other hand, the category of semigroups is not a structure category, since its operation $+$ is already defined by the category of magmas:

```
sage: Semigroups().additional_structure()
```

Most *categories with axiom* don't define additional structure:

```
sage: Sets().Finite().additional_structure()
sage: Rings().Commutative().additional_structure()
sage: Modules(QQ).FiniteDimensional().additional_structure()
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: MagmaticAlgebras(QQ).Unital().additional_structure()
```

As of Sage 6.4, the only exceptions are the category of unital magmas or the category of unital additive magmas (both define a unit which shall be preserved by morphisms):

```
sage: Magmas().Unital().additional_structure()
Category of unital magmas
sage: AdditiveMagmas().AdditiveUnital().additional_structure()
Category of additive unital additive magmas
```

Similarly, *functorial construction categories* don't define additional structure, unless the construction is actually defined by their base category. For example, the category of graded modules defines a grading which shall be preserved by morphisms:

```
sage: Modules(ZZ).Graded().additional_structure()
Category of graded modules over Integer Ring
```

On the other hand, the category of graded algebras does not define additional structure; indeed an algebra morphism which is also a module morphism is a graded algebra morphism:

```
sage: Algebras(ZZ).Graded().additional_structure()
```

Similarly, morphisms are requested to preserve the structure given by the following constructions:

```
sage: Sets().Quotients().additional_structure()
Category of quotients of sets
sage: Sets().CartesianProducts().additional_structure()
Category of Cartesian products of sets
sage: Modules(QQ).TensorProducts().additional_structure()
```

This might change, as we are lacking enough data points to guarantee that this was the correct design decision.

Note: In some cases a category defines additional structure, where the structure can be useful to manipulate morphisms but where, in most use cases, we don't want the morphisms to necessarily preserve it. For example, in the context of finite dimensional vector spaces, having a distinguished basis allows for representing morphisms by matrices; yet considering only morphisms that preserve that distinguished basis would be boring.

In such cases, we might want to eventually have two categories, one where the additional structure is preserved, and one where it's not necessarily preserved (we would need to find an idiom for this).

At this point, a choice is to be made each time, according to the main use cases. Some of those choices are yet to be settled. For example, should by default:

- an euclidean domain morphism preserve euclidean division?

  ```
  sage: EuclideanDomains().additional_structure()
  Category of euclidean domains
  ```

- an enumerated set morphism preserve the distinguished enumeration?

  ```
  sage: EnumeratedSets().additional_structure()
  ```

- a module with basis morphism preserve the distinguished basis?

  ```
  sage: Modules(QQ).WithBasis().additional_structure()
  ```

See also:

This method together with the methods overloading it provide the basic data to determine, for a given category, the super categories that define some structure (see `structure()`), and to test whether a category is a full subcategory of some other category (see `is_full_subcategory()`). For example, the category of Coxeter groups is not full subcategory of the category of groups since morphisms need to preserve the distinguished generators:

```
sage: CoxeterGroups().is_full_subcategory(Groups())
False
```

The support for modeling full subcategories has been introduced in github issue #16340.

all_super_categories(*proper=False*)

Returns the list of all super categories of this category.

INPUT:

- `proper` – a boolean (default: `False`); whether to exclude this category.

Since github issue #11943, the order of super categories is determined by Python's method resolution order C3 algorithm.

Note: Whenever speed matters, the developers are advised to use instead the lazy attributes *_all_super_categories()*, *_all_super_categories_proper()*, or *_set_of_super_categories()*, as appropriate. Simply because lazy attributes are much faster than any method.

EXAMPLES:

```
sage: C = Rings(); C
Category of rings
sage: C.all_super_categories()
[Category of rings, Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]

sage: C.all_super_categories(proper = True)
[Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]

sage: Sets().all_super_categories()
[Category of sets, Category of sets with partial maps, Category of objects]
sage: Sets().all_super_categories(proper=True)
[Category of sets with partial maps, Category of objects]
sage: Sets().all_super_categories() is Sets()._all_super_categories
True
sage: Sets().all_super_categories(proper=True) is Sets()._all_super_
↪categories_proper
True
```

classmethod an_instance()

Return an instance of this class.

EXAMPLES:

```
sage: Rings.an_instance()
Category of rings
```

Parametrized categories should overload this default implementation to provide appropriate arguments:

```
sage: Algebras.an_instance()
Category of algebras over Rational Field
sage: Bimodules.an_instance()                                                    #␣
↪needs sage.rings.real_mpfr
Category of bimodules over Rational Field on the left
 and Real Field with 53 bits of precision on the right
sage: AlgebraIdeals.an_instance()
Category of algebra ideals
 in Univariate Polynomial Ring in x over Rational Field
```

axioms()

Return the axioms known to be satisfied by all the objects of `self`.

Technically, this is the set of all the axioms `A` such that, if `Cs` is the category defining `A`, then `self` is a subcategory of `Cs().A()`. Any additional axiom `A` would yield a strict subcategory of `self`, at the very least `self & Cs().A()` where `Cs` is the category defining `A`.

EXAMPLES:

```
sage: Monoids().axioms()
frozenset({'Associative', 'Unital'})
sage: (EnumeratedSets().Infinite() & Sets().Facade()).axioms()
frozenset({'Enumerated', 'Facade', 'Infinite'})
```

category()

Return the category of this category. So far, all categories are in the category of objects.

EXAMPLES:

```
sage: Sets().category()
Category of objects
sage: VectorSpaces(QQ).category()
Category of objects
```

category_graph()

Returns the graph of all super categories of this category

EXAMPLES:

```
sage: C = Algebras(QQ)
sage: G = C.category_graph()                                                    #␣
↪needs sage.graphs
sage: G.is_directed_acyclic()                                                   #␣
↪needs sage.graphs
True
```

The girth of a directed acyclic graph is infinite, however, the girth of the underlying undirected graph is 4 in this case:

```
sage: Graph(G).girth()                                                          #␣
↪needs sage.graphs
4
```

element_class()

A common super class for all elements of parents in this category (and its subcategories).

This class contains the methods defined in the nested class `self.ElementMethods` (if it exists), and has as bases the element classes of the super categories of `self`.

See also:

- *parent_class()*, *morphism_class()*
- *Category* for details

EXAMPLES:

```
sage: C = Algebras(QQ).element_class; C
<class 'sage.categories.algebras.Algebras.element_class'>
sage: type(C)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

By github issue #11935, some categories share their element classes. For example, the element class of an algebra only depends on the category of the base. A typical example is the category of algebras over a field versus algebras over a non-field:

```
sage: Algebras(GF(5)).element_class is Algebras(GF(3)).element_class
True
sage: Algebras(QQ).element_class is Algebras(ZZ).element_class
False
sage: Algebras(ZZ['t']).element_class is Algebras(ZZ['t','x']).element_class
True
```

These classes are constructed with `__slots__ = ()`, so instances may not have a `__dict__`:

```
sage: E = FiniteEnumeratedSets().element_class
sage: E.__dictoffset__
0
```

See also:

`parent_class()`

`example`(**args*, ***keywords*)

Returns an object in this category. Most of the time, this is a parent.

This serves three purposes:

- Give a typical example to better explain what the category is all about. (and by the way prove that the category is non empty :-))
- Provide a minimal template for implementing other objects in this category
- Provide an object on which to test generic code implemented by the category

For all those applications, the implementation of the object shall be kept to a strict minimum. The object is therefore not meant to be used for other applications; most of the time a full featured version is available elsewhere in Sage, and should be used instead.

Technical note: by default `FooBar(...).example()` is constructed by looking up `sage.categories.examples.foo_bar.Example` and calling it as `Example()`. Extra positional or named parameters are also passed down. For a category over base ring, the base ring is further passed down as an optional argument.

Categories are welcome to override this default implementation.

EXAMPLES:

```
sage: Semigroups().example()
An example of a semigroup: the left zero semigroup

sage: Monoids().Subquotients().example()
NotImplemented
```

`full_super_categories`()

Return the *immediate* full super categories of `self`.

See also:

- *super_categories()*
- *is_full_subcategory()*

Warning: The current implementation selects the full subcategories among the immediate super categories of `self`. This assumes that, if $C \subset B \subset A$ is a chain of categories and C is a full subcategory of A, then C is a full subcategory of B and B is a full subcategory of A.

This assumption is guaranteed to hold with the current model and implementation of full subcategories in Sage. However, mathematically speaking, this is too restrictive. This indeed prevents the complete modelling of situations where any A morphism between elements of C automatically preserves the B structure. See below for an example.

EXAMPLES:

A semigroup morphism between two finite semigroups is a finite semigroup morphism:

```
sage: Semigroups().Finite().full_super_categories()
[Category of semigroups]
```

On the other hand, a semigroup morphism between two monoids is not necessarily a monoid morphism (which must map the unit to the unit):

```
sage: Monoids().super_categories()
[Category of semigroups, Category of unital magmas]
sage: Monoids().full_super_categories()
[Category of unital magmas]
```

Any semigroup morphism between two groups is automatically a monoid morphism (in a group the unit is the unique idempotent, so it has to be mapped to the unit). Yet, due to the limitation of the model advertised above, Sage currently cannot be taught that the category of groups is a full subcategory of the category of semigroups:

```
sage: Groups().full_super_categories()     # todo: not implemented
[Category of monoids, Category of semigroups, Category of inverse unital␣
↪magmas]
sage: Groups().full_super_categories()
[Category of monoids, Category of inverse unital magmas]
```

is_abelian()

Return whether this category is abelian.

An abelian category is a category satisfying:

- It has a zero object;
- It has all pullbacks and pushouts;
- All monomorphisms and epimorphisms are normal.

Equivalently, one can define an increasing sequence of conditions:

- A category is pre-additive if it is enriched over abelian groups (all homsets are abelian groups and composition is bilinear);
- A pre-additive category is additive if every finite set of objects has a biproduct (we can form direct sums and direct products);
- An additive category is pre-abelian if every morphism has both a kernel and a cokernel;

- A pre-abelian category is abelian if every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism.

EXAMPLES:

```
sage: Modules(ZZ).is_abelian()
True
sage: FreeModules(ZZ).is_abelian()
False
sage: FreeModules(QQ).is_abelian()
True
sage: CommutativeAdditiveGroups().is_abelian()
True
sage: Semigroups().is_abelian()
Traceback (most recent call last):
...
NotImplementedError: is_abelian
```

is_full_subcategory(*other*)

Return whether `self` is a full subcategory of `other`.

A subcategory B of a category A is a *full subcategory* if any A-morphism between two objects of B is also a B-morphism (the reciprocal always holds: any B-morphism between two objects of B is an A-morphism).

This is computed by testing whether `self` is a subcategory of `other` and whether they have the same structure, as determined by *structure()* from the result of *additional_structure()* on the super categories.

Warning: A positive answer is guaranteed to be mathematically correct. A negative answer may mean that Sage has not been taught enough information (or can not yet within the current model) to derive this information. See *full_super_categories()* for a discussion.

See also:

- *is_subcategory()*
- *full_super_categories()*

EXAMPLES:

```
sage: Magmas().Associative().is_full_subcategory(Magmas())
True
sage: Magmas().Unital().is_full_subcategory(Magmas())
False
sage: Rings().is_full_subcategory(Magmas().Unital() & AdditiveMagmas().
↪AdditiveUnital())
True
```

Here are two typical examples of false negatives:

```
sage: Groups().is_full_subcategory(Semigroups())
False
sage: Groups().is_full_subcategory(Semigroups()) # todo: not implemented
True
sage: Fields().is_full_subcategory(Rings())
False
```

(continues on next page)

(continued from previous page)

```
sage: Fields().is_full_subcategory(Rings())       # todo: not implemented
True
```

Todo: The latter is a consequence of *EuclideanDomains* currently being a structure category. Is this what we want?

```
sage: EuclideanDomains().is_full_subcategory(Rings())
False
```

---

is_subcategory(*c*)

Returns True if self is naturally embedded as a subcategory of c.

EXAMPLES:

```
sage: AbGrps = CommutativeAdditiveGroups()
sage: Rings().is_subcategory(AbGrps)
True
sage: AbGrps.is_subcategory(Rings())
False
```

The `is_subcategory` function takes into account the base.

```
sage: M3 = VectorSpaces(FiniteField(3))
sage: M9 = VectorSpaces(FiniteField(9, 'a'))                                  #␣
↪needs sage.rings.finite_rings
sage: M3.is_subcategory(M9)                                                   #␣
↪needs sage.rings.finite_rings
False
```

Join categories are properly handled:

```
sage: CatJ = Category.join((CommutativeAdditiveGroups(), Semigroups()))
sage: Rings().is_subcategory(CatJ)
True
```

```
sage: V3 = VectorSpaces(FiniteField(3))
sage: POSet = PartiallyOrderedSets()
sage: PoV3 = Category.join((V3, POSet))
sage: A3 = AlgebrasWithBasis(FiniteField(3))
sage: PoA3 = Category.join((A3, POSet))
sage: PoA3.is_subcategory(PoV3)
True
sage: PoV3.is_subcategory(PoV3)
True
sage: PoV3.is_subcategory(PoA3)
False
```

static join(*categories*, *as_list=False*, *ignore_axioms=()*, *axioms=()*)

Return the join of the input categories in the lattice of categories.

At the level of objects and morphisms, this operation corresponds to intersection: the objects and morphisms of a join category are those that belong to all its super categories.

INPUT:

- `categories` – a list (or iterable) of categories
- `as_list` – a boolean (default: `False`); whether the result should be returned as a list
- `axioms` – a tuple of strings; the names of some supplementary axioms

See also:

`__and__()` for a shortcut

EXAMPLES:

```
sage: J = Category.join((Groups(), CommutativeAdditiveMonoids())); J
Join of Category of groups and Category of commutative additive monoids
sage: J.super_categories()
[Category of groups, Category of commutative additive monoids]
sage: J.all_super_categories(proper=True)
[Category of groups, ..., Category of magmas,
 Category of commutative additive monoids, ..., Category of additive magmas,
 Category of sets, ...]
```

As a short hand, one can use:

```
sage: Groups() & CommutativeAdditiveMonoids()
Join of Category of groups and Category of commutative additive monoids
```

This is a commutative and associative operation:

```
sage: Groups() & Posets()
Join of Category of groups and Category of posets
sage: Posets() & Groups()
Join of Category of groups and Category of posets

sage: Groups() & (CommutativeAdditiveMonoids() & Posets())
Join of Category of groups
    and Category of commutative additive monoids
    and Category of posets
sage: (Groups() & CommutativeAdditiveMonoids()) & Posets()
Join of Category of groups
    and Category of commutative additive monoids
    and Category of posets
```

The join of a single category is the category itself:

```
sage: Category.join([Monoids()])
Category of monoids
```

Similarly, the join of several mutually comparable categories is the smallest one:

```
sage: Category.join((Sets(), Rings(), Monoids()))
Category of rings
```

In particular, the unit is the top category *`Objects`*:

```
sage: Groups() & Objects()
Category of groups
```

If the optional parameter `as_list` is `True`, this returns the super categories of the join as a list, without constructing the join category itself:

```
sage: Category.join((Groups(), CommutativeAdditiveMonoids()), as_list=True)
[Category of groups, Category of commutative additive monoids]
sage: Category.join((Sets(), Rings(), Monoids()), as_list=True)
[Category of rings]
sage: Category.join((Modules(ZZ), FiniteFields()), as_list=True)
[Category of finite enumerated fields, Category of modules over Integer Ring]
sage: Category.join([], as_list=True)
[]
sage: Category.join([Groups()], as_list=True)
[Category of groups]
sage: Category.join([Groups() & Posets()], as_list=True)
[Category of groups, Category of posets]
```

Support for axiom categories (TODO: put here meaningful examples):

```
sage: Sets().Facade() & Sets().Infinite()
Category of facade infinite sets
sage: Magmas().Infinite() & Sets().Facade()
Category of facade infinite magmas

sage: FiniteSets() & Monoids()
Category of finite monoids
sage: Rings().Commutative() & Sets().Finite()
Category of finite commutative rings
```

Note that several of the above examples are actually join categories; they are just nicely displayed:

```
sage: AlgebrasWithBasis(QQ) & FiniteSets().Algebras(QQ)
Join of Category of finite dimensional algebras with basis over Rational Field
    and Category of finite set algebras over Rational Field

sage: UniqueFactorizationDomains() & Algebras(QQ)
Join of Category of unique factorization domains
    and Category of commutative algebras over Rational Field
```

static meet(*categories*)

Returns the meet of a list of categories

INPUT:

- `categories` - a non empty list (or iterable) of categories

See also:

`__or__()` for a shortcut

EXAMPLES:

```
sage: Category.meet([Algebras(ZZ), Algebras(QQ), Groups()])
Category of monoids
```

That meet of an empty list should be a category which is a subcategory of all categories, which does not make practical sense:

```
sage: Category.meet([])
Traceback (most recent call last):
...
ValueError: The meet of an empty list of categories is not implemented
```

morphism_class()

A common super class for all morphisms between parents in this category (and its subcategories).

This class contains the methods defined in the nested class `self.MorphismMethods` (if it exists), and has as bases the morphism classes of the super categories of `self`.

See also:

- *parent_class()*, *element_class()*
- *Category* for details

EXAMPLES:

```
sage: C = Algebras(QQ).morphism_class; C
<class 'sage.categories.algebras.Algebras.morphism_class'>
sage: type(C)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

or_subcategory(*category=None*, *join=False*)

Return `category` or `self` if `category` is `None`.

INPUT:

- `category` – a sub category of `self`, tuple/list thereof, or `None`
- `join` – a boolean (default: `False`)

OUTPUT:

- a category

EXAMPLES:

```
sage: Monoids().or_subcategory(Groups())
Category of groups
sage: Monoids().or_subcategory(None)
Category of monoids
```

If category is a list/tuple, then a join category is returned:

```
sage: Monoids().or_subcategory((CommutativeAdditiveMonoids(), Groups()))
Join of Category of groups and Category of commutative additive monoids
```

If `join` is `False`, an error if raised if category is not a subcategory of `self`:

```
sage: Monoids().or_subcategory(EnumeratedSets())
Traceback (most recent call last):
...
ValueError: Subcategory of `Category of monoids` required;
got `Category of enumerated sets`
```

Otherwise, the two categories are joined together:

```
sage: Monoids().or_subcategory(EnumeratedSets(), join=True)
Category of enumerated monoids
```

parent_class()

A common super class for all parents in this category (and its subcategories).

This class contains the methods defined in the nested class `self.ParentMethods` (if it exists), and has as bases the parent classes of the super categories of `self`.

See also:

- *element_class()*, *morphism_class()*
- *Category* for details

EXAMPLES:

```
sage: C = Algebras(QQ).parent_class; C
<class 'sage.categories.algebras.Algebras.parent_class'>
sage: type(C)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

By github issue #11935, some categories share their parent classes. For example, the parent class of an algebra only depends on the category of the base ring. A typical example is the category of algebras over a finite field versus algebras over a non-field:

```
sage: Algebras(GF(7)).parent_class is Algebras(GF(5)).parent_class
True
sage: Algebras(QQ).parent_class is Algebras(ZZ).parent_class
False
sage: Algebras(ZZ['t']).parent_class is Algebras(ZZ['t','x']).parent_class
True
```

See *CategoryWithParameters* for an abstract base class for categories that depend on parameters, even though the parent and element classes only depend on the parent or element classes of its super categories. It is used in *Bimodules*, *Category_over_base* and *sage.categories.category.JoinCategory*.

required_methods()

Returns the methods that are required and optional for parents in this category and their elements.

EXAMPLES:

```
sage: Algebras(QQ).required_methods()
{'element': {'optional': ['_add_', '_mul_'], 'required': ['__bool__']},
 'parent': {'optional': ['algebra_generators'], 'required': ['__contains__']}}
```

structure()

Return the structure `self` is endowed with.

This method returns the structure that morphisms in this category shall be preserving. For example, it tells that a ring is a set endowed with a structure of both a unital magma and an additive unital magma which satisfies some further axioms. In other words, a ring morphism is a function that preserves the unital magma and additive unital magma structure.

In practice, this returns the collection of all the super categories of `self` that define some additional structure, as a frozen set.

EXAMPLES:

```
sage: Objects().structure()
frozenset()

sage: def structure(C):
```

(continues on next page)

(continued from previous page)

```
....:     return Category._sort(C.structure())

sage: structure(Sets())
(Category of sets, Category of sets with partial maps)
sage: structure(Magmas())
(Category of magmas, Category of sets, Category of sets with partial maps)
```

In the following example, we only list the smallest structure categories to get a more readable output:

```
sage: def structure(C):
....:     return Category._sort_uniq(C.structure())

sage: structure(Magmas())
(Category of magmas,)
sage: structure(Rings())
(Category of unital magmas, Category of additive unital additive magmas)
sage: structure(Fields())
(Category of euclidean domains,)
sage: structure(Algebras(QQ))
(Category of unital magmas,
 Category of right modules over Rational Field,
 Category of left modules over Rational Field)
sage: structure(HopfAlgebras(QQ).Graded().WithBasis().Connected())
(Category of Hopf algebras over Rational Field,
 Category of graded modules over Rational Field)
```

This method is used in *is_full_subcategory()* for deciding whether a category is a full subcategory of some other category, and for documentation purposes. It is computed recursively from the result of *additional_structure()* on the super categories of `self`.

subcategory_class()

A common superclass for all subcategories of this category (including this one).

This class derives from `D.subcategory_class` for each super category D of `self`, and includes all the methods from the nested class `self.SubcategoryMethods`, if it exists.

See also:

- github issue #12895
- *parent_class()*
- *element_class()*
- *_make_named_class()*

EXAMPLES:

```
sage: cls = Rings().subcategory_class; cls
<class 'sage.categories.rings.Rings.subcategory_class'>
sage: type(cls)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

`Rings()` is an instance of this class, as well as all its subcategories:

```
sage: isinstance(Rings(), cls)
True
```

(continues on next page)

(continued from previous page)

```
sage: isinstance(AlgebrasWithBasis(QQ), cls)
True
```

super_categories()

Return the *immediate* super categories of `self`.

OUTPUT:

- a duplicate-free list of categories.

Every category should implement this method.

EXAMPLES:

```
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
sage: Objects().super_categories()
[]
```

Note: Since github issue #10963, the order of the categories in the result is irrelevant. For details, see *On the order of super categories*.

Note: Whenever speed matters, developers are advised to use the lazy attribute `_super_categories()` instead of calling this method.

class sage.categories.category.**CategoryWithParameters**

Bases: `Category`

A parametrized category whose parent/element classes depend only on its super categories.

Many categories in Sage are parametrized, like `C = Algebras(K)` which takes a base ring as parameter. In many cases, however, the operations provided by `C` in the parent class and element class depend only on the super categories of `C`. For example, the vector space operations are provided if and only if `K` is a field, since `VectorSpaces(K)` is a super category of `C` only in that case. In such cases, and as an optimization (see github issue #11935), we want to use the same parent and element class for all fields. This is the purpose of this abstract class.

Currently, `JoinCategory`, `Category_over_base` and `Bimodules` inherit from this class.

EXAMPLES:

```
sage: C1 = Algebras(GF(5))
sage: C2 = Algebras(GF(3))
sage: C3 = Algebras(ZZ)
sage: from sage.categories.category import CategoryWithParameters
sage: isinstance(C1, CategoryWithParameters)
True
sage: C1.parent_class is C2.parent_class
True
sage: C1.parent_class is C3.parent_class
False
```

Category.**_make_named_class**(*name*, *method_provider*, *cache=False*, *picklable=True*)

Construction of the parent/element/… class of `self`.

INPUT:

- `name` – a string; the name of the class as an attribute of `self`. E.g. "parent_class"
- `method_provider` – a string; the name of an attribute of `self` that provides methods for the new class (in addition to those coming from the super categories). E.g. "ParentMethods"
- `cache` – a boolean or `ignore_reduction` (default: `False`) (passed down to dynamic_class; for internal use only)
- `picklable` – a boolean (default: `True`)

ASSUMPTION:

It is assumed that this method is only called from a lazy attribute whose name coincides with the given `name`.

OUTPUT:

A dynamic class with bases given by the corresponding named classes of `self`'s super_categories, and methods taken from the class `getattr(self,method_provider)`.

Note:

- In this default implementation, the reduction data of the named class makes it depend on `self`. Since the result is going to be stored in a lazy attribute of `self` anyway, we may as well disable the caching in `dynamic_class` (hence the default value `cache=False`).
- *`CategoryWithParameters`* overrides this method so that the same parent/element/… classes can be shared between closely related categories.
- The bases of the named class may also contain the named classes of some indirect super categories, according to *`_super_categories_for_classes()`*. This is to guarantee that Python will build consistent method resolution orders. For background, see `sage.misc.c3_controlled`.

See also:

`CategoryWithParameters._make_named_class()`

EXAMPLES:

```
sage: PC = Rings()._make_named_class("parent_class", "ParentMethods"); PC
<class 'sage.categories.rings.Rings.parent_class'>
sage: type(PC)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
sage: PC.__bases__
(<class 'sage.categories.rngs.Rngs.parent_class'>,
 <class 'sage.categories.semirings.Semirings.parent_class'>)
```

Note that, by default, the result is not cached:

```
sage: PC is Rings()._make_named_class("parent_class", "ParentMethods")
False
```

Indeed this method is only meant to construct lazy attributes like `parent_class` which already handle this caching:

```
sage: Rings().parent_class
<class 'sage.categories.rings.Rings.parent_class'>
```

Reduction for pickling also assumes the existence of this lazy attribute:

```
sage: PC._reduction
(<built-in function getattr>, (Category of rings, 'parent_class'))
sage: loads(dumps(PC)) is Rings().parent_class
True
```

class sage.categories.category.**JoinCategory**(*super_categories*, ***kwds*)

Bases: *CategoryWithParameters*

A class for joins of several categories. Do not use directly; see Category.join instead.

EXAMPLES:

```
sage: from sage.categories.category import JoinCategory
sage: J = JoinCategory((Groups(), CommutativeAdditiveMonoids())); J
Join of Category of groups and Category of commutative additive monoids
sage: J.super_categories()
[Category of groups, Category of commutative additive monoids]
sage: J.all_super_categories(proper=True)
[Category of groups, ..., Category of magmas,
 Category of commutative additive monoids, ..., Category of additive magmas,
 Category of sets, Category of sets with partial maps, Category of objects]
```

By github issue #11935, join categories and categories over base rings inherit from *CategoryWithParameters*. This allows for sharing parent and element classes between similar categories. For example, since group algebras belong to a join category and since the underlying implementation is the same for all finite fields, we have:

```
sage: # needs sage.groups sage.rings.finite_rings
sage: G = SymmetricGroup(10)
sage: A3 = G.algebra(GF(3))
sage: A5 = G.algebra(GF(5))
sage: type(A3.category())
<class 'sage.categories.category.JoinCategory_with_category'>
sage: type(A3) is type(A5)
True
```

Category.**_repr_object_names**()

Return the name of the objects of this category.

EXAMPLES:

```
sage: FiniteGroups()._repr_object_names()
'finite groups'
sage: AlgebrasWithBasis(QQ)._repr_object_names()
'algebras with basis over Rational Field'
```

Category.**_repr_**()

Return the print representation of this category.

EXAMPLES:

```
sage: Sets() # indirect doctest
Category of sets
```

Category.**_without_axioms**(*named=False*)

Return the category without the axioms that have been added to create it.

INPUT:

- named – a boolean (default: False)

Todo: Improve this explanation.

If `named` is `True`, then this stops at the first category that has an explicit name of its own. See *category_with_axiom.CategoryWithAxiom._without_axioms()*

EXAMPLES:

```
sage: Sets()._without_axioms()
Category of sets
sage: Semigroups()._without_axioms()
Category of magmas
sage: Algebras(QQ).Commutative().WithBasis()._without_axioms()
Category of magmatic algebras over Rational Field
sage: Algebras(QQ).Commutative().WithBasis()._without_axioms(named=True)
Category of algebras over Rational Field
```

additional_structure()

Return `None`.

Indeed, a join category defines no additional structure.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: Modules(ZZ).additional_structure()
```

is_subcategory(*C*)

Check whether this join category is subcategory of another category `C`.

EXAMPLES:

```
sage: Category.join([Rings(),Modules(QQ)]).is_subcategory(Category.
↪join([Rngs(),Bimodules(QQ,QQ)]))
True
```

super_categories()

Returns the immediate super categories, as per *Category.super_categories()*.

EXAMPLES:

```
sage: from sage.categories.category import JoinCategory
sage: JoinCategory((Semigroups(), FiniteEnumeratedSets())).super_categories()
[Category of semigroups, Category of finite enumerated sets]
```

`sage.categories.category.`**category_graph**(*categories=None*)

Return the graph of the categories in Sage.

INPUT:

- `categories` – a list (or iterable) of categories

If `categories` is specified, then the graph contains the mentioned categories together with all their super categories. Otherwise the graph contains (an instance of) each category in *sage.categories.all* (e.g. `Algebras(QQ)` for algebras).

For readability, the names of the category are shortened.

Todo: Further remove the base ring (see also github issue #15801).

EXAMPLES:

```
sage: G = sage.categories.category.category_graph(categories=[Groups()])          #␣
↪needs sage.graphs
sage: G.vertices(sort=True)                                                       #␣
↪needs sage.graphs
['groups', 'inverse unital magmas', 'magmas', 'monoids', 'objects',
 'semigroups', 'sets', 'sets with partial maps', 'unital magmas']
sage: G.plot()                                                                    #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 20 graphics primitives

sage: sage.categories.category.category_graph().plot()                            #␣
↪needs sage.graphs sage.plot
Graphics object consisting of ... graphics primitives
```

sage.categories.category.**category_sample**()

Return a sample of categories.

It is constructed by looking for all concrete category classes declared in `sage.categories.all`, calling *Category.an_instance()* on those and taking all their super categories.

EXAMPLES:

```
sage: from sage.categories.category import category_sample
sage: sorted(category_sample(), key=str)                                          #␣
↪needs sage.groups
[Category of Coxeter groups,
 Category of Dedekind domains,
 Category of G-sets for Symmetric group of order 8! as a permutation group,
 Category of Hecke modules over Rational Field,
 Category of Hopf algebras over Rational Field,
 Category of Hopf algebras with basis over Rational Field,
 Category of Lie algebras over Rational Field,
 Category of Weyl groups,
 Category of abelian varieties over Rational Field,
 Category of additive magmas, ...,
 Category of fields, ...,
 Category of graded Hopf algebras with basis over Rational Field, ...,
 Category of modular abelian varieties over Rational Field, ...,
 Category of simplicial complexes, ...,
 Category of vector spaces over Rational Field, ...
```

sage.categories.category.**is_Category**(*x*)

Returns True if x is a category.

EXAMPLES:

```
sage: sage.categories.category.is_Category(CommutativeAdditiveSemigroups())
True
sage: sage.categories.category.is_Category(ZZ)
False
```

2.3 Axioms

This documentation covers how to implement axioms and proceeds with an overview of the implementation of the axiom infrastructure. It assumes that the reader is familiar with the *category primer*, and in particular its *section about axioms*.

2.3.1 Implementing axioms

Simple case involving a single predefined axiom

Suppose that one wants to provide code (and documentation, tests, ...) for the objects of some existing category `Cs()` that satisfy some predefined axiom `A`.

The first step is to open the hood and check whether there already exists a class implementing the category `Cs().A()`. For example, taking `Cs=Semigroups` and the `Finite` axiom, there already exists a class for the category of finite semigroups:

```
sage: Semigroups().Finite()
Category of finite semigroups
sage: type(Semigroups().Finite())
<class 'sage.categories.finite_semigroups.FiniteSemigroups_with_category'>
```

In this case, we say that the category of semigroups *implements* the axiom `Finite`, and code about finite semigroups should go in the class *`FiniteSemigroups`* (or, as usual, in its nested classes `ParentMethods`, `ElementMethods`, and so on).

On the other hand, there is no class for the category of infinite semigroups:

```
sage: Semigroups().Infinite()
Category of infinite semigroups
sage: type(Semigroups().Infinite())
<class 'sage.categories.category.JoinCategory_with_category'>
```

This category is indeed just constructed as the intersection of the categories of semigroups and of infinite sets respectively:

```
sage: Semigroups().Infinite().super_categories()
[Category of semigroups, Category of infinite sets]
```

In this case, one needs to create a new class to implement the axiom `Infinite` for this category. This boils down to adding a nested class `Semigroups.Infinite` inheriting from *`CategoryWithAxiom`*.

In the following example, we implement a category `Cs`, with a subcategory for the objects satisfying the `Finite` axiom defined in the super category `Sets` (we will see later on how to *define* new axioms):

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom
sage: class Cs(Category):
....:     def super_categories(self):
....:         return [Sets()]
....:     class Finite(CategoryWithAxiom):
....:         class ParentMethods:
....:             def foo(self):
....:                 print("I am a method on finite C's")
```

```
sage: Cs().Finite()
Category of finite cs
sage: Cs().Finite().super_categories()
```

(continues on next page)

(continued from previous page)

```
[Category of finite sets, Category of cs]
sage: Cs().Finite().all_super_categories()
[Category of finite cs, Category of finite sets,
 Category of cs, Category of sets, ...]
sage: Cs().Finite().axioms()
frozenset({'Finite'})
```

Now a parent declared in the category `Cs().Finite()` inherits from all the methods of finite sets and of finite *C*'s, as desired:

```
sage: P = Parent(category=Cs().Finite())
sage: P.is_finite()              # Provided by Sets.Finite.ParentMethods
True
sage: P.foo()                    # Provided by Cs.Finite.ParentMethods
I am a method on finite C's
```

Note:

- This follows the same idiom as for *Covariant Functorial Constructions*.
- From an object oriented point of view, any subcategory `Cs()` of *`Sets`* inherits a `Finite` method. Usually `Cs` could complement this method by overriding it with a method `Cs.Finite` which would make a super call to `Sets.Finite` and then do extra stuff.

  In the above example, `Cs` also wants to complement `Sets.Finite`, though not by doing more stuff, but by providing it with an additional mixin class containing the code for finite `Cs`. To keep the analogy, this mixin class is to be put in `Cs.Finite`.
- By defining the axiom `Finite`, *`Sets`* fixes the semantic of `Cs.Finite()` for all its subcategories `Cs`: namely "the category of `Cs` which are finite as sets". Hence, for example, `Modules.Free.Finite` cannot be used to model the category of free modules of finite rank, even though their traditional name "finite free modules" might suggest it.
- It may come as a surprise that we can actually use the same name `Finite` for the mixin class and for the method defining the axiom; indeed, by default a class does not have a binding behavior and would completely override the method. See the section *Defining a new axiom* for details and the rationale behind it.

  An alternative would have been to give another name to the mixin class, like `FiniteCategory`. However this would have resulted in more namespace pollution, whereas using `Finite` is already clear, explicit, and easier to remember.
- Under the hood, the category `Cs().Finite()` is aware that it has been constructed from the category `Cs()` by adding the axiom `Finite`:

  ```
  sage: Cs().Finite().base_category()
  Category of cs
  sage: Cs().Finite()._axiom
  'Finite'
  ```

Over time, the nested class `Cs.Finite` may become large and too cumbersome to keep as a nested subclass of `Cs`. Or the category with axiom may have a name of its own in the literature, like *semigroups* rather than *associative magmas*, or *fields* rather than *commutative division rings*. In this case, the category with axiom can be put elsewhere, typically in a separate file, with just a link from `Cs`:

```
sage: class Cs(Category):
....:     def super_categories(self):
....:         return [Sets()]
sage: class FiniteCs(CategoryWithAxiom):
....:     class ParentMethods:
....:         def foo(self):
....:             print("I am a method on finite C's")
sage: Cs.Finite = FiniteCs
sage: Cs().Finite()
Category of finite cs
```

For a real example, see the code of the class *FiniteGroups* and the link to it in *Groups*. Note that the link is implemented using LazyImport; this is highly recommended: it makes sure that *FiniteGroups* is imported after *Groups* it depends upon, and makes it explicit that the class *Groups* can be imported and is fully functional without importing *FiniteGroups*.

Note: Some categories with axioms are created upon Sage's startup. In such a case, one needs to pass the `at_startup=True` option to LazyImport, in order to quiet the warning about that lazy import being resolved upon startup. See for example `Sets.Finite`.

This is undoubtedly a code smell. Nevertheless, it is preferable to stick to lazy imports, first to resolve the import order properly, and more importantly as a reminder that the category would be best not constructed upon Sage's startup. This is to spur developers to reduce the number of parents (and therefore categories) that are constructed upon startup. Each `at_startup=True` that will be removed will be a measure of progress in this direction.

Note: In principle, due to a limitation of LazyImport with nested classes (see github issue #15648), one should pass the option `as_name` to LazyImport:

```
Finite = LazyImport('sage.categories.finite_groups', 'FiniteGroups',
                    as_name='Finite')
```

in order to prevent `Groups.Finite` to keep on reimporting `FiniteGroups`.

Given that passing this option introduces some redundancy and is error prone, the axiom infrastructure includes a little workaround which makes the `as_name` unnecessary in this case.

Making the category with axiom directly callable

If desired, a category with axiom can be constructed directly through its class rather than through its base category:

```
sage: Semigroups()
Category of semigroups
sage: Semigroups() is Magmas().Associative()
True

sage: FiniteGroups()
Category of finite groups
sage: FiniteGroups() is Groups().Finite()
True
```

For this notation to work, the class *Semigroups* needs to be aware of the base category class (here, *Magmas*) and of the axiom (here, `Associative`):

```
sage: Semigroups._base_category_class_and_axiom
(<class 'sage.categories.magmas.Magmas'>, 'Associative')
sage: Fields._base_category_class_and_axiom
(<class 'sage.categories.division_rings.DivisionRings'>, 'Commutative')
sage: FiniteGroups._base_category_class_and_axiom
(<class 'sage.categories.groups.Groups'>, 'Finite')
sage: FiniteDimensionalAlgebrasWithBasis._base_category_class_and_axiom
(<class 'sage.categories.algebras_with_basis.AlgebrasWithBasis'>, 'FiniteDimensional')
```

In our example, the attribute `_base_category_class_and_axiom` was set upon calling `Cs().Finite()`, which makes the notation seemingly work:

```
sage: FiniteCs()
Category of finite cs
sage: FiniteCs._base_category_class_and_axiom
(<class '__main__.Cs'>, 'Finite')
sage: FiniteCs._base_category_class_and_axiom_origin
'set by __classget__'
```

But calling `FiniteCs()` right after defining the class would have failed (try it!). In general, one needs to set the attribute explicitly:

```
sage: class FiniteCs(CategoryWithAxiom):
....:     _base_category_class_and_axiom = (Cs, 'Finite')
....:     class ParentMethods:
....:         def foo(self):
....:             print("I am a method on finite C's")
```

Having to set explicitly this link back from `FiniteCs` to `Cs` introduces redundancy in the code. It would therefore be desirable to have the infrastructure set the link automatically instead (a difficulty is to achieve this while supporting lazy imported categories with axiom).

As a first step, the link is set automatically upon accessing the class from the base category class:

```
sage: Algebras.WithBasis._base_category_class_and_axiom
(<class 'sage.categories.algebras.Algebras'>, 'WithBasis')
sage: Algebras.WithBasis._base_category_class_and_axiom_origin
'set by __classget__'
```

Hence, for whatever this notation is worth, one can currently do:

```
sage: Algebras.WithBasis(QQ)
Category of algebras with basis over Rational Field
```

We don't recommend using syntax like `Algebras.WithBasis(QQ)`, as it may eventually be deprecated.

As a second step, Sage tries some obvious heuristics to deduce the link from the name of the category with axiom (see *`base_category_class_and_axiom()`* for the details). This typically covers the following examples:

```
sage: FiniteCoxeterGroups()
Category of finite Coxeter groups
sage: FiniteCoxeterGroups() is CoxeterGroups().Finite()
True
sage: FiniteCoxeterGroups._base_category_class_and_axiom_origin
'deduced by base_category_class_and_axiom'

sage: FiniteDimensionalAlgebrasWithBasis(QQ)
```

(continues on next page)

(continued from previous page)

```
Category of finite dimensional algebras with basis over Rational Field
sage: FiniteDimensionalAlgebrasWithBasis(QQ) is Algebras(QQ).FiniteDimensional().
↪WithBasis()
True
```

If the heuristic succeeds, the result is guaranteed to be correct. If it fails, typically because the category has a name of its own like *Fields*, the attribute `_base_category_class_and_axiom` should be set explicitly. For more examples, see the code of the classes *Semigroups* or *Fields*.

Note: When printing out a category with axiom, the heuristic determines whether a category has a name of its own by checking out how `_base_category_class_and_axiom` was set:

```
sage: Fields._base_category_class_and_axiom_origin
'hardcoded'
```

See *CategoryWithAxiom._without_axioms()*, *CategoryWithAxiom._repr_object_names_static()*.

In our running example `FiniteCs`, Sage failed to deduce automatically the base category class and axiom because the class `Cs` is not in the standard location `sage.categories.cs`.

Design discussion

The above deduction, based on names, is undoubtedly inelegant. But it's safe (either the result is guaranteed to be correct, or an error is raised), it saves on some redundant information, and it is only used for the simple shorthands like `FiniteGroups()` for `Groups().Finite()`. Finally, most if not all of these shorthands are likely to eventually disappear (see github issue #15741 and the *related discussion in the primer*).

Defining a new axiom

We describe now how to define a new axiom. The first step is to figure out the largest category where the axiom makes sense. For example `Sets` for `Finite`, `Magmas` for `Associative`, or `Modules` for `FiniteDimensional`. Here we define the axiom `Green` for the category `Cs` and its subcategories:

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom
sage: class Cs(Category):
....:     def super_categories(self):
....:         return [Sets()]
....:     class SubcategoryMethods:
....:         def Green(self):
....:             '<documentation of the axiom Green>'
....:             return self._with_axiom("Green")
....:     class Green(CategoryWithAxiom):
....:         class ParentMethods:
....:             def foo(self):
....:                 print("I am a method on green C's")
```

With the current implementation, the name of the axiom must also be added to a global container:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: all_axioms += ("Green",)
```

We can now use the axiom as usual:

```
sage: Cs().Green()
Category of green cs

sage: P = Parent(category=Cs().Green())
sage: P.foo()
I am a method on green C's
```

Compared with our first example, the only newcomer is the method `.Green()` that can be used by any subcategory `Ds()` of `Cs()` to add the axiom `Green`. Note that the expression `Ds().Green` always evaluates to this method, regardless of whether `Ds` has a nested class `Ds.Green` or not (an implementation detail):

```
sage: Cs().Green
<bound method Cs.SubcategoryMethods.Green of Category of cs>
```

Thanks to this feature (implemented in *`CategoryWithAxiom.__classget__()`*), the user is systematically referred to the documentation of this method when doing introspection on `Ds().Green`:

```
sage: C = Cs()
sage: C.Green?              # not tested
sage: Cs().Green.__doc__
'<documentation of the axiom Green>'
```

It is therefore the natural spot for the documentation of the axiom.

Note: The presence of the nested class `Green` in `Cs` is currently mandatory even if it is empty.

Todo: Specify whether or not one should systematically use @cached_method in the definition of the axiom. And make sure all the definition of axioms in Sage are consistent in this respect!

Todo: We could possibly define an @axiom decorator? This could hide two little implementation details: whether or not to make the method a cached method, and the call to _with_axiom(…) under the hood. It could do possibly do some more magic. The gain is not obvious though.

Note: `all_axioms` is only used marginally, for sanity checks and when trying to derive automatically the base category class. The order of the axioms in this tuple also controls the order in which they appear when printing out categories with axioms (see *`CategoryWithAxiom._repr_object_names_static()`*).

During a Sage session, new axioms should only be added at the *end* of `all_axioms`, as above, so as to not break the cache of `axioms_rank()`. Otherwise, they can be inserted statically anywhere in the tuple. For axioms defined within the Sage library, the name is best inserted by editing directly the definition of `all_axioms` in *`sage.categories.category_with_axiom`*.

Design note

Let us state again that, unlike what the existence of `all_axioms` might suggest, the definition of an axiom is local to a category and its subcategories. In particular, two independent categories `Cs()` and `Ds()` can very well define axioms with the same name and different semantics. As long as the two hierarchies of subcategories don't intersect, this is not a problem. And if they do intersect naturally (that is if one is likely to create a parent belonging to both categories), this probably means that the categories `Cs` and `Ds` are about related enough areas of mathematics that one should clear the ambiguity by having either the same semantic or different names.

This caveat is no different from that of name clashes in hierarchy of classes involving multiple inheritance.

Todo: Explore ways to get rid of this global `all_axioms` tuple, and/or have automatic registration there, and/or having a register_axiom(…) method.

Special case: defining an axiom depending on several categories

In some cases, the largest category where the axiom makes sense is the intersection of two categories. This is typically the case for axioms specifying compatibility conditions between two otherwise unrelated operations, like `Distributive` which specifies a compatibility between $*$ and $+$. Ideally, we would want the `Distributive` axiom to be defined by:

```
sage: Magmas() & AdditiveMagmas()
Join of Category of magmas and Category of additive magmas
```

The current infrastructure does not support this perfectly: indeed, defining an axiom for a category C requires C to have a class of its own; hence a *`JoinCategory`* as above won't do; we need to implement a new class like *`MagmasAndAdditiveMagmas`*; furthermore, we cannot yet model the fact that `MagmasAndAdditiveMagmas()` *is* the intersection of `Magmas()` and `AdditiveMagmas()` rather than a mere subcategory:

```
sage: from sage.categories.magmas_and_additive_magmas import MagmasAndAdditiveMagmas
sage: Magmas() & AdditiveMagmas() is MagmasAndAdditiveMagmas()
False
sage: Magmas() & AdditiveMagmas()              # todo: not implemented
Category of magmas and additive magmas
```

Still, there is a workaround to get the natural notations:

```
sage: (Magmas() & AdditiveMagmas()).Distributive()
Category of distributive magmas and additive magmas
sage: (Monoids() & CommutativeAdditiveGroups()).Distributive()
Category of rings
```

The trick is to define `Distributive` as usual in *`MagmasAndAdditiveMagmas`*, and to add a method *`Magmas.SubcategoryMethods.Distributive()`* which checks that `self` is a subcategory of both `Magmas()` and `AdditiveMagmas()`, complains if not, and otherwise takes the intersection of `self` with `MagmasAndAdditiveMagmas()` before calling `Distributive`.

The downsides of this workaround are:

- Creation of an otherwise empty class *`MagmasAndAdditiveMagmas`*.
- Pollution of the namespace of `Magmas()` (and subcategories like `Groups()`) with a method that is irrelevant (but safely complains if called).
- `C._with_axiom('Distributive')` is not strictly equivalent to `C.Distributive()`, which can be unpleasantly surprising:

  ```
  sage: (Monoids() & CommutativeAdditiveGroups()).Distributive()
  Category of rings
  ```

(continues on next page)

(continued from previous page)

```
sage: (Monoids() & CommutativeAdditiveGroups())._with_axiom('Distributive')
Join of Category of monoids and Category of commutative additive groups
```

Todo: Other categories that would be better implemented via an axiom depending on a join category include:

- *Algebras*: defining an associative unital algebra as a ring and a module satisfying the suitable compatibility axiom between inner multiplication and multiplication by scalars (bilinearity). Of course this should be implemented at the level of *MagmaticAlgebras*, if not higher.
- *Bialgebras*: defining a bialgebra as an algebra and coalgebra where the coproduct is a morphism for the product.
- *Bimodules*: defining a bimodule as a left and right module where the two actions commute.

Todo:

- Design and implement an idiom for the definition of an axiom by a join category.
- Or support more advanced joins, through some hook or registration process to specify that a given category *is* the intersection of two (or more) categories.
- Or at least improve the above workaround to avoid the last issue; this possibly could be achieved using a class `Magmas.Distributive` with a bit of `__classcall__` magic.

Handling multiple axioms, arborescence structure of the code

Prelude

Let us consider the category of magmas, together with two of its axioms, namely `Associative` and `Unital`. An associative magma is a *semigroup* and a unital semigroup is a *monoid*. We have also seen that axioms commute:

```
sage: Magmas().Unital()
Category of unital magmas
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative().Unital()
Category of monoids
sage: Magmas().Unital().Associative()
Category of monoids
```

At the level of the classes implementing these categories, the following comes as a general naturalization of the previous section:

```
sage: Magmas.Unital
<class 'sage.categories.magmas.Magmas.Unital'>
sage: Magmas.Associative
<class 'sage.categories.semigroups.Semigroups'>
sage: Magmas.Associative.Unital
<class 'sage.categories.monoids.Monoids'>
```

However, the following may look suspicious at first:

```
sage: Magmas.Unital.Associative
Traceback (most recent call last):
...
AttributeError: type object 'Magmas.Unital' has no attribute 'Associative'...
```

The purpose of this section is to explain the design of the code layout and the rationale for this mismatch.

Abstract model

As we have seen in the *Primer*, the objects of a category `Cs()` can usually satisfy, or not, many different axioms. Out of all combinations of axioms, only a small number are relevant in practice, in the sense that we actually want to provide features for the objects satisfying these axioms.

Therefore, in the context of the category class `Cs`, we want to provide the system with a collection $(D_S)_{S\in\mathcal{S}}$ where each S is a subset of the axioms and the corresponding D_S is a class for the subcategory of the objects of `Cs()` satisfying the axioms in S. For example, if `Cs()` is the category of magmas, the pairs (S, D_S) would include:

```
{Associative}                  : Semigroups
{Associative, Unital}          : Monoids
{Associative, Unital, Inverse}: Groups
{Associative, Commutative}     : Commutative Semigroups
{Unital,       Inverse}        : Loops
```

Then, given a subset T of axioms, we want the system to be able to select automatically the relevant classes $(D_S)_{S\in\mathcal{S}, S\subset T}$, and build from them a category for the objects of `Cs` satisfying the axioms in T, together with its hierarchy of super categories. If T is in the indexing set $\mathcal{S}$, then the class of the resulting category is directly D_T:

```
sage: C = Magmas().Unital().Inverse().Associative(); C
Category of groups
sage: type(C)
<class 'sage.categories.groups.Groups_with_category'>
```

Otherwise, we get a join category:

```
sage: C = Magmas().Infinite().Unital().Associative(); C
Category of infinite monoids
sage: type(C)
<class 'sage.categories.category.JoinCategory_with_category'>
sage: C.super_categories()
[Category of monoids, Category of infinite sets]
```

Concrete model as an arborescence of nested classes

We further want the construction to be efficient and amenable to laziness. This led us to the following design decision: the collection $(D_S)_{S\in\mathcal{S}}$ of classes should be structured as an arborescence (or equivalently a *rooted forest*). The root is `Cs`, corresponding to $S = \emptyset$. Any other class D_S should be the child of a single class $D_{S'}$ where S' is obtained from S by removing a single axiom A. Of course, $D_{S'}$ and A are respectively the base category class and axiom of the category with axiom D_S that we have met in the first section.

At this point, we urge the reader to explore the code of `Magmas` and `DistributiveMagmasAndAdditiveMagmas` and see how the arborescence structure on the categories with axioms is reflected by the nesting of category classes.

Discussion of the design

Performance

Thanks to the arborescence structure on subsets of axioms, constructing the hierarchy of categories and computing intersections can be made efficient with, roughly speaking, a linear/quadratic complexity in the size of the involved category hierarchy multiplied by the number of axioms (see Section *Algorithms*). This is to be put in perspective with the manipulation of arbitrary collections of subsets (aka boolean functions) which can easily raise NP-hard problems.

Furthermore, thanks to its locality, the algorithms can be made suitably lazy: in particular, only the involved category classes need to be imported.

Flexibility

This design also brings in quite some flexibility, with the possibility to support features such as defining new axioms depending on other axioms and deduction rules. See below.

Asymmetry

As we have seen at the beginning of this section, this design introduces an asymmetry. It's not so bad in practice, since in most practical cases, we want to work incrementally. It's for example more natural to describe *FiniteFields* as *Fields* with the axiom `Finite` rather than *Magmas* and *AdditiveMagmas* with all (or at least sufficiently many) of the following axioms:

```
sage: sorted(Fields().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Commutative', 'Distributive',
 'Division', 'NoZeroDivisors', 'Unital']
```

The main limitation is that the infrastructure currently imposes to be incremental by steps of a single axiom.

In practice, among the roughly 60 categories with axioms that are currently implemented in Sage, most admitted a (rather) natural choice of a base category and single axiom to add. For example, one usually thinks more naturally of a monoid as a semigroup which is unital rather than as a unital magma which is associative. Modeling this asymmetry in the code actually brings a bonus: it is used for printing out categories in a (heuristically) mathematician-friendly way:

```
sage: Magmas().Commutative().Associative()
Category of commutative semigroups
```

Only in a few cases is a choice made that feels mathematically arbitrary. This is essentially in the chain of nested classes *distributive_magmas_and_additive_magmas.DistributiveMagmasAndAdditiveMagmas.AdditiveAssociative.AdditiveCommutative.AdditiveUnital.Associative*.

Placeholder classes

Given that we can only add a single axiom at a time when implementing a `CategoryWithAxiom`, we need to create a few category classes that are just placeholders. For the worst example, see the chain of nested classes `distributive_magmas_and_additive_magmas.DistributiveMagmasAndAdditiveMagmas.AdditiveAssociative.AdditiveCommutative.AdditiveUnital.Associative`.

This is suboptimal, but fits within the scope of the axiom infrastructure which is to reduce a potentially exponential number of placeholder category classes to just a couple.

Note also that, in the above example, it's likely that some of the intermediate classes will grow to non placeholder ones, as people will explore more weaker variants of rings.

Mismatch between the arborescence of nested classes and the hierarchy of categories

The fact that the hierarchy relation between categories is not reflected directly as a relation between the classes may sound suspicious at first! However, as mentioned in the primer, this is actually a big selling point of the axioms infrastructure: by calculating automatically the hierarchy relation between categories with axioms one avoids the nightmare of maintaining it by hand. Instead, only a rather minimal number of links needs to be maintained in the code (one per category with axiom).

Besides, with the flexibility introduced by runtime deduction rules (see below), the hierarchy of categories may depend on the parameters of the categories and not just their class. So it's fine to make it clear from the onset that the two relations do not match.

Evolutivity

At this point, the arborescence structure has to be hardcoded by hand with the annoyances we have seen. This does not preclude, in a future iteration, to design and implement some idiom for categories with axioms that adds several axioms at once to a base category; maybe some variation around:

```
class DistributiveMagmasAndAdditiveMagmas:
    ...

    @category_with_axiom(
        AdditiveAssociative,
        AdditiveCommutative,
        AdditiveUnital,
        AdditiveInverse,
        Associative)
    def _(): return LazyImport('sage.categories.rngs', 'Rngs', at_startup=True)
```

or:

```
register_axiom_category(DistributiveMagmasAndAdditiveMagmas,
                        {AdditiveAssociative,
                         AdditiveCommutative,
                         AdditiveUnital,
                         AdditiveInverse,
                         Associative},
                        'sage.categories.rngs', 'Rngs', at_startup=True)
```

The infrastructure would then be in charge of building the appropriate arborescence under the hood. Or rely on some database (see discussion on github issue #10963, in particular at the end of comment 332).

Axioms defined upon other axioms

Sometimes an axiom can only be defined when some other axiom holds. For example, the axiom `NoZeroDivisors` only makes sense if there is a zero, that is if the axiom `AdditiveUnital` holds. Hence, for the category *`MagmasAndAdditiveMagmas`*, we consider in the abstract model only those subsets of axioms where the presence of `NoZeroDivisors` implies that of `AdditiveUnital`. We also want the axiom to be only available if meaningful:

```
sage: Rings().NoZeroDivisors()
Category of domains
sage: Rings().Commutative().NoZeroDivisors()
Category of integral domains
sage: Semirings().NoZeroDivisors()
Traceback (most recent call last):
...
AttributeError: 'Semirings_with_category' object has no attribute 'NoZeroDivisors'...
```

Concretely, this is to be implemented by defining the new axiom in the (`SubcategoryMethods` nested class of the) appropriate category with axiom. For example the axiom `NoZeroDivisors` would be naturally defined in `magmas_and_additive_magmas.MagmasAndAdditiveMagmas.Distributive.AdditiveUnital`.

Note: The axiom `NoZeroDivisors` is currently defined in *`Rings`*, by simple lack of need for the feature; it should be lifted up as soon as relevant, that is when some code will be available for parents with no zero divisors that are not necessarily rings.

Deduction rules

A similar situation is when an axiom `A` of a category `Cs` implies some other axiom `B`, with the same consequence as above on the subsets of axioms appearing in the abstract model. For example, a division ring necessarily has no zero divisors:

```
sage: 'NoZeroDivisors' in Rings().Division().axioms()
True
sage: 'NoZeroDivisors' in Rings().axioms()
False
```

This deduction rule is implemented by the method `Rings.Division.extra_super_categories()`:

```
sage: Rings().Division().extra_super_categories()
(Category of domains,)
```

In general, this is to be implemented by a method `Cs.A.extra_super_categories` returning a tuple `(Cs().B(),)`, or preferably `(Ds().B(),)` where `Ds` is the category defining the axiom `B`.

This follows the same idiom as for deduction rules about functorial constructions (see `covariant_functorial_construction.CovariantConstructionCategory.extra_super_categories()`). For example, the fact that a Cartesian product of associative magmas (i.e. of semigroups) is an associative magma is implemented in *`Semigroups.CartesianProducts.extra_super_categories()`*:

```
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative().CartesianProducts().extra_super_categories()
[Category of semigroups]
```

Similarly, the fact that the algebra of a commutative magma is commutative is implemented in *Magmas.Commutative.Algebras.extra_super_categories()*:

```
sage: Magmas().Commutative().Algebras(QQ).extra_super_categories()
[Category of commutative magmas]
```

Warning: In some situations this idiom is inapplicable as it would require to implement two classes for the same category. This is the purpose of the next section.

Special case

In the previous examples, the deduction rule only had an influence on the super categories of the category with axiom being constructed. For example, when constructing `Rings().Division()`, the rule `Rings.Division.extra_super_categories()` simply adds `Rings().NoZeroDivisors()` as a super category thereof.

In some situations this idiom is inapplicable because a class for the category with axiom under construction already exists elsewhere. Take for example Wedderburn's theorem: any finite division ring is commutative, i.e. is a finite field. In other words, `DivisionRings().Finite()` *coincides* with `Fields().Finite()`:

```
sage: DivisionRings().Finite()
Category of finite enumerated fields
sage: DivisionRings().Finite() is Fields().Finite()
True
```

Therefore we cannot create a class `DivisionRings.Finite` to hold the desired `extra_super_categories` method, because there is already a class for this category with axiom, namely `Fields.Finite`.

A natural idiom would be to have `DivisionRings.Finite` be a link to `Fields.Finite` (locally introducing an undirected cycle in the arborescence of nested classes). It would be a bit tricky to implement though, since one would need to detect, upon constructing `DivisionRings().Finite()`, that `DivisionRings.Finite` is actually `Fields.Finite`, in order to construct appropriately `Fields().Finite()`; and reciprocally, upon computing the super categories of `Fields().Finite()`, to not try to add `DivisionRings().Finite()` as a super category.

Instead the current idiom is to have a method `DivisionRings.Finite_extra_super_categories` which mimics the behavior of the would-be `DivisionRings.Finite.extra_super_categories`:

```
sage: DivisionRings().Finite_extra_super_categories()
(Category of commutative magmas,)
```

This idiom is admittedly rudimentary, but consistent with how mathematical facts specifying non trivial inclusion relations between categories are implemented elsewhere in the various `extra_super_categories` methods of axiom categories and covariant functorial constructions. Besides, it gives a natural spot (the docstring of the method) to document and test the modeling of the mathematical fact. Finally, Wedderburn's theorem is arguably a theorem about division rings (in the context of division rings, finiteness implies commutativity) and therefore lives naturally in *DivisionRings*.

An alternative would be to implement the category of finite division rings (i.e. finite fields) in a class `DivisionRings.Finite` rather than `Fields.Finite`:

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom

sage: class MyDivisionRings(Category):
....:     def super_categories(self):
....:         return [Rings()]
```

(continues on next page)

(continued from previous page)

```
sage: class MyFields(Category):
....:     def super_categories(self):
....:         return [MyDivisionRings()]

sage: class MyFiniteFields(CategoryWithAxiom):
....:     _base_category_class_and_axiom = (MyDivisionRings, "Finite")
....:     def extra_super_categories(self): # Wedderburn's theorem
....:         return [MyFields()]

sage: MyDivisionRings.Finite = MyFiniteFields

sage: MyDivisionRings().Finite()
Category of my finite fields
sage: MyFields().Finite() is MyDivisionRings().Finite()
True
```

In general, if several categories `C1s()`, `C2s()`, … are mapped to the same category when applying some axiom `A` (that is `C1s().A() == C2s().A() == ...`), then one should be careful to implement this category in a single class `Cs.A`, and set up methods `extra_super_categories` or `A_extra_super_categories` methods as appropriate. Each such method should return something like `[C2s()]` and not `[C2s().A()]` for the latter would likely lead to an infinite recursion.

Design discussion

Supporting similar deduction rules will be an important feature in the future, with quite a few occurrences already implemented in upcoming issues. For the time being though there is a single occurrence of this idiom outside of the tests. So this would be an easy thing to refactor after github issue #10963 if a better idiom is found.

Larger synthetic examples

We now consider some larger synthetic examples to check that the machinery works as expected. Let us start with a category defining a bunch of axioms, using *`axiom()`* for conciseness (don't do it for real axioms; they deserve a full documentation!):

```
sage: from sage.categories.category_singleton import Category_singleton
sage: from sage.categories.category_with_axiom import axiom
sage: import sage.categories.category_with_axiom
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: all_axioms += ("B","C","D","E","F")

sage: class As(Category_singleton):
....:     def super_categories(self):
....:         return [Objects()]
....:
....:     class SubcategoryMethods:
....:         B = axiom("B")
....:         C = axiom("C")
....:         D = axiom("D")
....:         E = axiom("E")
....:         F = axiom("F")
....:
....:     class B(CategoryWithAxiom):
```

(continues on next page)

(continued from previous page)

```
....:         pass
....:     class C(CategoryWithAxiom):
....:         pass
....:     class D(CategoryWithAxiom):
....:         pass
....:     class E(CategoryWithAxiom):
....:         pass
....:     class F(CategoryWithAxiom):
....:         pass
```

Now we construct a subcategory where, by some theorem of William, axioms `B` and `C` together are equivalent to `E` and `F` together:

```
sage: class A1s(Category_singleton):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class B(CategoryWithAxiom):
....:         def C_extra_super_categories(self):
....:             return [As().E(), As().F()]
....:
....:     class E(CategoryWithAxiom):
....:         def F_extra_super_categories(self):
....:             return [As().B(), As().C()]

sage: A1s().B().C()
Category of e f a1s
```

The axioms `B` and `C` do not show up in the name of the obtained category because, for concision, the printing uses some heuristics to not show axioms that are implied by others. But they are satisfied:

```
sage: sorted(A1s().B().C().axioms())
['B', 'C', 'E', 'F']
```

Note also that this is a join category:

```
sage: type(A1s().B().C())
<class 'sage.categories.category.JoinCategory_with_category'>
sage: A1s().B().C().super_categories()
[Category of e a1s,
 Category of f as,
 Category of b a1s,
 Category of c as]
```

As desired, William's theorem holds:

```
sage: A1s().B().C() is A1s().E().F()
True
```

and propagates appropriately to subcategories:

```
sage: C =  A1s().E().F().D().B().C()
sage: C is A1s().B().C().E().F().D()   # commutativity
True
sage: C is A1s().E().F().E().F().D()   # William's theorem
True
```

(continues on next page)

(continued from previous page)

```
sage: C is A1s().E().E().F().F().D()   # commutativity
True
sage: C is A1s().E().F().D()           # idempotency
True
sage: C is A1s().D().E().F()
True
```

In this quick variant, we actually implement the category of `b c a2s`, and choose to do so in `A2s.B.C`:

```
sage: class A2s(Category_singleton):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class B(CategoryWithAxiom):
....:         class C(CategoryWithAxiom):
....:             def extra_super_categories(self):
....:                 return [As().E(), As().F()]
....:
....:     class E(CategoryWithAxiom):
....:         def F_extra_super_categories(self):
....:             return [As().B(), As().C()]


sage: A2s().B().C()
Category of e f a2s
sage: sorted(A2s().B().C().axioms())
['B', 'C', 'E', 'F']
sage: type(A2s().B().C())
<class '__main__.A2s.B.C_with_category'>
```

As desired, William's theorem and its consequences hold:

```
sage: A2s().B().C() is A2s().E().F()
True
sage: C =  A2s().E().F().D().B().C()
sage: C is A2s().B().C().E().F().D()   # commutativity
True
sage: C is A2s().E().F().E().F().D()   # William's theorem
True
sage: C is A2s().E().E().F().F().D()   # commutativity
True
sage: C is A2s().E().F().D()           # idempotency
True
sage: C is A2s().D().E().F()
True
```

Finally, we "accidentally" implement the category of `b c a1s`, both in `A3s.B.C` and `A3s.E.F`:

```
sage: class A3s(Category_singleton):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class B(CategoryWithAxiom):
....:         class C(CategoryWithAxiom):
....:             def extra_super_categories(self):
....:                 return [As().E(), As().F()]
....:
```

(continues on next page)

(continued from previous page)

```
....:     class E(CategoryWithAxiom):
....:         class F(CategoryWithAxiom):
....:             def extra_super_categories(self):
....:                 return [As().B(), As().C()]
```

We can still construct, say:

```
sage: A3s().B()
Category of b a3s
sage: A3s().C()
Category of c a3s
```

However,

```
sage: A3s().B().C()            # not tested
```

runs into an infinite recursion loop, as `A3s().B().C()` wants to have `A3s().E().F()` as super category and reciprocally.

Todo: The above example violates the specifications (a category should be modelled by at most one class), so it's appropriate that it fails. Yet, the error message could be usefully complemented by some hint at what the source of the problem is (a category implemented in two distinct classes). Leaving a large enough piece of the backtrace would be useful though, so that one can explore where the issue comes from (e.g. with post mortem debugging).

2.3.2 Specifications

After fixing some vocabulary, we summarize here some specifications about categories and axioms.

The lattice of constructible categories

A mathematical category C is *implemented* if there is a class in Sage modelling it; it is *constructible* if it is either implemented, or is the intersection of *implemented* categories; in the latter case it is modelled by a `JoinCategory`. The comparison of two constructible categories with the `Category.is_subcategory()` method is supposed to model the comparison of the corresponding mathematical categories for inclusion of the objects (see *On the category hierarchy: subcategories and super categories* for details). For example:

```
sage: Fields().is_subcategory(Rings())
True
```

However this modelling may be incomplete. It can happen that a mathematical fact implying that a category A is a subcategory of a category B is not implemented. Still, the comparison should endow the set of constructible categories with a poset structure and in fact a lattice structure.

In this lattice, the join of two categories (`Category.join()`) is supposed to model their intersection. Given that we compare categories for inclusion, it would be more natural to call this operation the *meet*; blames go to me (Nicolas) for originally comparing categories by *amount of structure* rather than by *inclusion*. In practice, the join of two categories may be a strict super category of their intersection; first because this intersection might not be constructible; second because Sage might miss some mathematical information to recover the smallest constructible super category of the intersection.

Axioms

We say that an axiom `A` is *defined by* a category `Cs()` if `Cs` defines an appropriate method `Cs.SubcategoryMethods.A`, with the semantic of the axiom specified in the documentation; for any subcategory `Ds()`, `Ds().A()` models the subcategory of the objects of `Ds()` satisfying `A`. In this case, we say that the axiom `A` is *defined for* the category `Ds()`. Furthermore, `Ds` *implements the axiom* `A` if `Ds` has a category with axiom as nested class `Ds.A`. The category `Ds()` *satisfies* the axiom if `Ds()` is a subcategory of `Cs().A()` (meaning that all the objects of `Ds()` are known to satisfy the axiom `A`).

A digression on the structure of fibers when adding an axiom

Consider the application ϕ_A which maps a category to its category of objects satisfying A. Equivalently, ϕ_A is computing the intersection with the defining category with axiom of A. It follows immediately from the latter that ϕ_A is a regressive endomorphism of the lattice of categories. It restricts to a regressive endomorphism `Cs() |-> Cs().A()` on the lattice of constructible categories.

This endomorphism may have non trivial fibers, as in our favorite example: `DivisionRings()` and `Fields()` are in the same fiber for the axiom `Finite`:

```
sage: DivisionRings().Finite() is Fields().Finite()
True
```

Consider the intersection S of such a fiber of ϕ_A with the upper set I_A of categories that do not satisfy `A`. The fiber itself is a sublattice. However I_A is not guaranteed to be stable under intersection (though exceptions should be rare). Therefore, there is a priori no guarantee that S would be stable under intersection. Also it's presumably finite, in fact small, but this is not guaranteed either.

Specifications

- Any constructible category `C` should admit a finite number of larger constructible categories.
- The methods `super_categories`, `extra_super_categories`, and friends should always return strict supercategories.

  For example, to specify that a finite division ring is a finite field, `DivisionRings.Finite_extra_super_categories` should not return `Fields().Finite()`! It could possibly return `Fields()`; but it's preferable to return the largest category that contains the relevant information, in this case `Magmas().Commutative()`, and to let the infrastructure apply the derivations.
- The base category of a *`CategoryWithAxiom`* should be an implemented category (i.e. not a *`JoinCategory`*). This is checked by *`CategoryWithAxiom._test_category_with_axiom()`*.
- Arborescent structure: Let `Cs()` be a category, and S be some set of axioms defined in some super categories of `Cs()` but not satisfied by `Cs()`. Suppose we want to provide a category with axiom for the elements of `Cs()` satisfying the axioms in S. Then, there should be a single enumeration `A1, A2, ..., Ak` without repetition of axioms in S such that `Cs.A1.A2....Ak` is an implemented category. Furthermore, every intermediate step `Cs.A1.A2....Ai` with $i \leq k$ should be a category with axiom having `Ai` as axiom and `Cs.A1.A2....Ai-1` as base category class; this base category class should not satisfy `Ai`. In particular, when some axioms of S can be deduced from previous ones by deduction rules, they should not appear in the enumeration `A1, A2, ..., Ak`.
- In particular, if `Cs()` is a category that satisfies some axiom `A` (e.g. from one of its super categories), then it should not implement that axiom. For example, a category class `Cs` can never have a nested class `Cs.A.A`. Similarly, applying the specification recursively, a category satisfying `A` cannot have a nested class `Cs.A1.A2.A3.A` where `A1`, `A2`, `A3` are axioms.

- A category can only implement an axiom if this axiom is defined by some super category. The code has not been systematically checked to support having two super categories defining the same axiom (which should of course have the same semantic). You are welcome to try, at your own risk. :-)
- When a category defines an axiom or functorial construction `A`, this fixes the semantic of `A` for all the subcategories. In particular, if two categories define `A`, then these categories should be independent, and either the semantic of `A` should be the same, or there should be no natural intersection between the two hierarchies of subcategories.
- Any super category of a *`CategoryWithParameters`* should either be a *`CategoryWithParameters`* or a *`Category_singleton`*.
- A *`CategoryWithAxiom`* having a *`Category_singleton`* as base category should be a *`CategoryWithAxiom_singleton`*. This is handled automatically by *`CategoryWithAxiom.__init__()`* and checked in *`CategoryWithAxiom._test_category_with_axiom()`*.
- A *`CategoryWithAxiom`* having a *`Category_over_base_ring`* as base category should be a *`Category_over_base_ring`*. This currently has to be handled by hand, using *`CategoryWithAxiom_over_base_ring`*. This is checked in *`CategoryWithAxiom._test_category_with_axiom()`*.

Todo: The following specifications would be desirable but are not yet implemented:

- A functorial construction category (Graded, CartesianProducts, …) having a *`Category_singleton`* as base category should be a *`CategoryWithAxiom_singleton`*.

  Nothing difficult to implement, but this will need to rework the current "no subclass of a concrete class" assertion test of `Category_singleton.__classcall__()`.
- Similarly, a covariant functorial construction category having a *`Category_over_base_ring`* as base category should be a *`Category_over_base_ring`*.

The following specification might be desirable, or not:

- A join category involving a *`Category_over_base_ring`* should be a *`Category_over_base_ring`*. In the mean time, a `base_ring` method is automatically provided for most of those by *`Modules.SubcategoryMethods.base_ring()`*.

2.3.3 Design goals

As pointed out in the primer, the main design goal of the axioms infrastructure is to subdue the potential combinatorial explosion of the category hierarchy by letting the developer focus on implementing a few bookshelves for which there is actual code or mathematical information, and let Sage *compose dynamically and lazily* these building blocks to construct the minimal hierarchy of classes needed for the computation at hand. This allows for the infrastructure to scale smoothly as bookshelves are added, extended, or reorganized.

Other design goals include:

- Flexibility in the code layout: the category of, say, finite sets can be implemented either within the Sets category (in a nested class `Sets.Finite`), or in a separate file (typically in a class `FiniteSets` in a lazily imported module sage.categories.finite_sets).
- Single point of truth: a theorem, like Wedderburn's, should be implemented in a single spot.
- Single entry point: for example, from the entry *`Rings`*, one can explore a whole range of related categories just by applying axioms and constructions:

```
sage: Rings().Commutative().Finite().NoZeroDivisors()
Category of finite integral domains
sage: Rings().Finite().Division()
Category of finite enumerated fields
```

This will allow for progressively getting rid of all the entries like *GradedHopfAlgebrasWithBasis* which are polluting the global name space.

Note that this is not about precluding the existence of multiple natural ways to construct the same category:

```
sage: Groups().Finite()
Category of finite groups
sage: Monoids().Finite().Inverse()
Category of finite groups
sage: Sets().Finite() & Monoids().Inverse()
Category of finite groups
```

- Concise idioms for the users (adding axioms, . . .)
- Concise idioms and well highlighted hierarchy of bookshelves for the developer (especially with code folding)
- Introspection friendly (listing the axioms, recovering the mixins)

Note: The constructor for instances of this class takes as input the base category. Hence, they should in principle be constructed as:

```
sage: FiniteSets(Sets())
Category of finite sets

sage: Sets.Finite(Sets())
Category of finite sets
```

None of these idioms are really practical for the user. So instead, this object is to be constructed using any of the following idioms:

```
sage: Sets()._with_axiom('Finite')
Category of finite sets
sage: FiniteSets()
Category of finite sets
sage: Sets().Finite()
Category of finite sets
```

The later two are implemented using respectively *CategoryWithAxiom.__classcall__()* and *CategoryWithAxiom.__classget__()*.

2.3.4 Upcoming features

Todo:

- Implement compatibility axiom / functorial constructions. For example, one would want to have:

```
A.CartesianProducts() & B.CartesianProducts() = (A&B).CartesianProducts()
```

- Once full subcategories are implemented (see github issue #10668), make the relevant categories with axioms be such. This can be done systematically for, e.g., the axioms `Associative` or `Commutative`, but not for the axiom `Unital`: a semigroup morphism between two monoids need not preserve the unit.

  Should all full subcategories be implemented in term of axioms?

2.3.5 Algorithms

Computing joins

The workhorse of the axiom infrastructure is the algorithm for computing the join J of a set $C_1, \ldots, C_k$ of categories (see `Category.join()`). Formally, J is defined as the largest constructible category such that $J \subset C_i$ for all i, and $J \subset C.A()$ for every constructible category $C \supset J$ and any axiom A satisfied by J.

The join J is naturally computed as a closure in the lattice of constructible categories: it starts with the C_i's, gathers the set S of all the axioms satisfied by them, and repeatedly adds each axiom A to those categories that do not yet satisfy A using `Category._with_axiom()`. Due to deduction rules or (extra) super categories, new categories or new axioms may appear in the process. The process stops when each remaining category has been combined with each axiom. In practice, only the smallest categories are kept along the way; this is correct because adding an axiom is covariant: `C.A()` is a subcategory of `D.A()` whenever `C` is a subcategory of `D`.

As usual in such closure computations, the result does not depend on the order of execution. Furthermore, given that adding an axiom is an idempotent and regressive operation, the process is guaranteed to stop in a number of steps which is bounded by the number of super categories of J. In particular, it is a finite process.

Todo: Detail this a bit. What could typically go wrong is a situation where, for some category `C1`, `C1.A()` specifies a category `C2` as super category such that `C2.A()` specifies `C3` as super category such that …; this would clearly cause an infinite execution. Note that this situation violates the specifications since `C1.A()` is supposed to be a subcategory of `C2.A()`, … so we would have an infinite increasing chain of constructible categories.

It's reasonable to assume that there is a finite number of axioms defined in the code. There remains to use this assumption to argue that any infinite execution of the algorithm would give rise to such an infinite sequence.

Adding an axiom

Let `Cs` be a category and `A` an axiom defined for this category. To compute `Cs().A()`, there are two cases.

Adding an axiom `A` to a category `Cs()` not implementing it

In this case, `Cs().A()` returns the join of:

- `Cs()`
- `Bs().A()` for every direct super category `Bs()` of `Cs()`
- the categories appearing in `Cs().A_extra_super_categories()`

This is a highly recursive process. In fact, as such, it would run right away into an infinite loop! Indeed, the join of `Cs()` with `Bs().A()` would trigger the construction of `Cs().A()` and reciprocally. To avoid this, the `Category.join()` method itself does not use `Category._with_axiom()` to add axioms, but its sister `Category._with_axiom_as_tuple()`; the latter builds a tuple of categories that should be joined together but leaves the computation of the join to its caller, the master join calculation.

Adding an axiom `A` to a category `Cs()` implementing it

In this case `Cs().A()` simply constructs an instance D of `Cs.A` which models the desired category. The non trivial part is the construction of the super categories of D. Very much like above, this includes:

- `Cs()`
- `Bs().A()` for every super category `Bs()` of `Cs()`
- the categories appearing in `D.extra_super_categories()`

This by itself may not be sufficient, due in particular to deduction rules. On may for example discover a new axiom `A1` satisfied by D, imposing to add `A1` to all of the above categories. Therefore the super categories are computed as the join of the above categories. Up to one twist: as is, the computation of this join would trigger recursively a recalculation of `Cs().A()`! To avoid this, `Category.join()` is given an optional argument to specify that the axiom `A` should *not* be applied to `Cs()`.

Sketch of proof of correctness and evaluation of complexity

As we have seen, this is a highly recursive process! In particular, one needs to argue that, as long as the specifications are satisfied, the algorithm won't run in an infinite recursion, in particular in case of deduction rule.

Theorem

Consider the construction of a category C by adding an axiom to a category (or computing of a join). Let H be the hierarchy of implemented categories above C. Let n and m be respectively the number of categories and the number of inheritance edges in H.

Assuming that the specifications are satisfied, the construction of C involves constructing the categories in H exactly once (and no other category), and at most n join calculations. In particular, the time complexity should be, roughly speaking, bounded by n^2. In particular, it's finite.

Remark

It's actually to be expected that the complexity is more of the order of magnitude of $na + m$, where a is the number of axioms satisfied by C. But this is to be checked in detail, in particular due to the many category inclusion tests involved.

The key argument is that `Category.join` cannot call itself recursively without going through the construction of some implemented category. In turn, the construction of some implemented category C only involves constructing strictly smaller categories, and possibly a direct join calculation whose result is strictly smaller than C. This statement is obvious if C implements the `super_categories` method directly, and easy to check for functorial construction categories. It requires a proof for categories with axioms since there is a recursive join involved.

Lemma

Let C be a category implementing an axiom A. Recall that the construction of `C.A()` involves a single direct join calculation for computing the super categories. No other direct join calculation occur, and the calculation involves only implemented categories that are strictly smaller than `C.A()`.

Proof

Let D be a category involved in the join calculation for the super categories of `C.A()`, and assume by induction that D is strictly smaller than `C.A()`. A category E newly constructed from D can come from:

- `D.(extra_)super_categories()`

  In this case, the specifications impose that E should be strictly smaller than D and therefore strictly smaller than C.

- `D.with_axiom_as_tuple('B')` or `D.B_extra_super_categories()` for some axiom B

  In this case, the axiom B is satisfied by some subcategory of `C.A()`, and therefore must be satisfied by `C.A()` itself. Since adding an axiom is a regressive construction, E must be a subcategory of `C.A()`. If there is equality, then E and `C.A()` must have the same class, and therefore, E must be directly constructed as `C.A()`. However the join construction explicitly prevents this call.

Note that a call to `D.with_axiom_as_tuple('B')` does not trigger a direct join calculation; but of course, if D implements B, the construction of the implemented category `E = D.B()` will involve a strictly smaller join calculation.

2.3.6 Conclusion

This is the end of the axioms documentation. Congratulations on having read that far!

2.3.7 Tests

Note: Quite a few categories with axioms are constructed early on during Sage's startup. Therefore, when playing around with the implementation of the axiom infrastructure, it is easy to break Sage. The following sequence of tests is designed to test the infrastructure from the ground up even in a partially broken Sage. Please don't remove the imports!

class `sage.categories.category_with_axiom.`**`Bars`**

Bases: *`Category_singleton`*

A toy singleton category, for testing purposes.

See also:

`Blahs`

`Unital_extra_super_categories``()`

Return extraneous super categories for the unital objects of `self`.

This method specifies that a unital bar is a test object. Thus, the categories of unital bars and of unital test objects coincide.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import Bars, TestObjects
sage: Bars().Unital_extra_super_categories()
[Category of test objects]
sage: Bars().Unital()
Category of unital test objects
sage: TestObjects().Unital().all_super_categories()
[Category of unital test objects,
```

(continues on next page)

(continued from previous page)

```
Category of unital blahs,
Category of test objects,
Category of bars,
Category of blahs,
Category of sets,
Category of sets with partial maps,
Category of objects]
```

super_categories()

class sage.categories.category_with_axiom.**Blahs**

Bases: *Category_singleton*

A toy singleton category, for testing purposes.

This is the root of a hierarchy of mathematically meaningless categories, used for testing Sage's category framework:

- *Bars*
- *TestObjects*
- *TestObjectsOverBaseRing*

Blue_extra_super_categories()

Illustrates a current limitation in the way to have an axiom imply another one.

Here, we would want `Blue` to imply `Unital`, and to put the class for the category of unital blue blahs in `Blahs.Unital.Blue` rather than `Blahs.Blue`.

This currently fails because `Blahs` is the category where the axiom `Blue` is defined, and the specifications currently impose that a category defining an axiom should also implement it (here in a category with axiom `Blahs.Blue`). In practice, due to this violation of the specifications, the axiom is lost during the join calculation.

Todo: Decide whether we care about this feature. In such a situation, we are not really defining a new axiom, but just defining an axiom as an alias for a couple others, which might not be that useful.

Todo: Improve the infrastructure to detect and report this violation of the specifications, if this is easy. Otherwise, it's not so bad: when defining an axiom A in a category `Cs` the first thing one is supposed to doctest is that `Cs().A()` works. So the problem should not go unnoticed.

class Commutative(*base_category*)

Bases: *CategoryWithAxiom*

class Connected(*base_category*)

Bases: *CategoryWithAxiom*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom*

class Flying(*base_category*)

Bases: *CategoryWithAxiom*

extra_super_categories()

This illustrates a way to have an axiom imply another one.

Here, we want `Flying` to imply `Unital`, and to put the class for the category of unital flying blahs in `Blahs.Flying` rather than `Blahs.Unital.Flying`.

class SubcategoryMethods

Bases: `object`

Blue()

Commutative()

Connected()

FiniteDimensional()

Flying()

Unital()

class Unital(*base_category*)

Bases: *CategoryWithAxiom*

class Blue(*base_category*)

Bases: *CategoryWithAxiom*

super_categories()

class sage.categories.category_with_axiom.**CategoryWithAxiom**(*base_category*)

Bases: *Category*

An abstract class for categories obtained by adding an axiom to a base category.

See the *category primer*, and in particular its *section about axioms* for an introduction to axioms, and *CategoryWithAxiom* for how to implement axioms and the documentation of the axiom infrastructure.

static __classcall__(**args*, ***options*)

Make `FoosBar(**)` an alias for `Foos(**)._with_axiom("Bar")`.

EXAMPLES:

```
sage: FiniteGroups()
Category of finite groups
sage: ModulesWithBasis(ZZ)
Category of modules with basis over Integer Ring
sage: AlgebrasWithBasis(QQ)
Category of algebras with basis over Rational Field
```

This is relevant when e.g. `Foos(**)` does some non trivial transformations:

```
sage: Modules(QQ) is VectorSpaces(QQ)
True
sage: type(Modules(QQ))
<class 'sage.categories.vector_spaces.VectorSpaces_with_category'>

sage: ModulesWithBasis(QQ) is VectorSpaces(QQ).WithBasis()
True
sage: type(ModulesWithBasis(QQ))
<class 'sage.categories.vector_spaces.VectorSpaces.WithBasis_with_category'>
```

static __classget__(*base_category*, *base_category_class*)

Implement the binding behavior for categories with axioms.

This method implements a binding behavior on category with axioms so that, when a category `Cs` implements an axiom `A` with a nested class `Cs.A`, the expression `Cs().A` evaluates to the method defining the axiom `A` and not the nested class. See those design notes for the rationale behind this behavior.

EXAMPLES:

```
sage: Sets().Infinite()
Category of infinite sets
sage: Sets().Infinite
Cached version of <function ...Infinite at ...>
sage: Sets().Infinite.f == Sets.SubcategoryMethods.Infinite.f
True
```

We check that this also works when the class is implemented in a separate file, and lazy imported:

```
sage: Sets().Finite
Cached version of <function ...Finite at ...>
```

There is no binding behavior when accessing `Finite` or `Infinite` from the class of the category instead of the category itself:

```
sage: Sets.Finite
<class 'sage.categories.finite_sets.FiniteSets'>
sage: Sets.Infinite
<class 'sage.categories.sets_cat.Sets.Infinite'>
```

This method also initializes the attribute `_base_category_class_and_axiom` if not already set:

```
sage: Sets.Infinite._base_category_class_and_axiom
(<class 'sage.categories.sets_cat.Sets'>, 'Infinite')
sage: Sets.Infinite._base_category_class_and_axiom_origin
'set by __classget__'
```

__init__(*base_category*)

_repr_object_names()

The names of the objects of this category, as used by `_repr_`.

See also:

`Category._repr_object_names()`

EXAMPLES:

```
sage: FiniteSets()._repr_object_names()
'finite sets'
sage: AlgebrasWithBasis(QQ).FiniteDimensional()._repr_object_names()
'finite dimensional algebras with basis over Rational Field'
sage: Monoids()._repr_object_names()
'monoids'
sage: Semigroups().Unital().Finite()._repr_object_names()
'finite monoids'
sage: Algebras(QQ).Commutative()._repr_object_names()
'commutative algebras over Rational Field'
```

Note: This is implemented by taking _repr_object_names from self._without_axioms(named=True), and adding the names of the relevant axioms in appropriate order.

static _repr_object_names_static(*category*, *axioms*)

INPUT:

- `base_category` – a category
- `axioms` – a list or iterable of strings

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom
sage: CategoryWithAxiom._repr_object_names_static(Semigroups(), ["Flying",
↪"Blue"])
'flying blue semigroups'
sage: CategoryWithAxiom._repr_object_names_static(Algebras(QQ), ["Flying",
↪"WithBasis", "Blue"])
'flying blue algebras with basis over Rational Field'
sage: CategoryWithAxiom._repr_object_names_static(Algebras(QQ), ["WithBasis"])
'algebras with basis over Rational Field'
sage: CategoryWithAxiom._repr_object_names_static(Sets().Finite().
↪Subquotients(), ["Finite"])
'subquotients of finite sets'
sage: CategoryWithAxiom._repr_object_names_static(Monoids(), ["Unital"])
'monoids'
sage: CategoryWithAxiom._repr_object_names_static(Algebras(QQ['x']['y']), [
↪"Flying", "WithBasis", "Blue"])
'flying blue algebras with basis over Univariate Polynomial Ring in y over␣
↪Univariate Polynomial Ring in x over Rational Field'
```

If the axioms is a set or frozen set, then they are first sorted using *`canonicalize_axioms()`*:

```
sage: CategoryWithAxiom._repr_object_names_static(Semigroups(), set(["Finite",
↪ "Commutative", "Facade"]))
'facade finite commutative semigroups'
```

See also:

`_repr_object_names()`

Note: The logic here is shared between *`_repr_object_names()`* and `category.JoinCategory._repr_object_names()`

_test_category_with_axiom(***options*)

Run generic tests on this category with axioms.

See also:

`TestSuite`.

This check that an axiom category of a *`Category_singleton`* is a singleton category, and similarwise for *`Category_over_base_ring`*.

EXAMPLES:

```
sage: Sets().Finite()._test_category_with_axiom()
sage: Modules(ZZ).FiniteDimensional()._test_category_with_axiom()
```

`_without_axioms`(*named=False*)

Return the category without the axioms that have been added to create it.

EXAMPLES:

```
sage: Sets().Finite()._without_axioms()
Category of sets
sage: Monoids().Finite()._without_axioms()
Category of magmas
```

This is because:

```
sage: Semigroups().Unital() is Monoids()
True
```

If `named` is `True`, then `_without_axioms` stops at the first category that has an explicit name of its own:

```
sage: Sets().Finite()._without_axioms(named=True)
Category of sets
sage: Monoids().Finite()._without_axioms(named=True)
Category of monoids
```

Technically we test this by checking if the class specifies explicitly the attribute `_base_category_class_and_axiom` by looking up `_base_category_class_and_axiom_origin`.

Some more examples:

```
sage: Algebras(QQ).Commutative()._without_axioms()
Category of magmatic algebras over Rational Field
sage: Algebras(QQ).Commutative()._without_axioms(named=True)
Category of algebras over Rational Field
```

`additional_structure`()

Return the additional structure defined by `self`.

OUTPUT: `None`

By default, a category with axiom defines no additional structure.

See also:

`Category.additional_structure()`.

EXAMPLES:

```
sage: Sets().Finite().additional_structure()
sage: Monoids().additional_structure()
```

`axioms`()

Return the axioms known to be satisfied by all the objects of `self`.

See also:

`Category.axioms()`

EXAMPLES:

```
sage: C = Sets.Finite(); C
Category of finite sets
sage: C.axioms()
frozenset({'Finite'})

sage: C = Modules(GF(5)).FiniteDimensional(); C
Category of finite dimensional vector spaces over Finite Field of size 5
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Finite', 'FiniteDimensional']

sage: sorted(FiniteMonoids().Algebras(QQ).axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive',
 'FiniteDimensional', 'Unital', 'WithBasis']
sage: sorted(FiniteMonoids().Algebras(GF(3)).axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Finite',
 'FiniteDimensional', 'Unital', 'WithBasis']

sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: MagmasAndAdditiveMagmas().Distributive().Unital().axioms()
frozenset({'Distributive', 'Unital'})

sage: D = MagmasAndAdditiveMagmas().Distributive()
sage: X = D.AdditiveAssociative().AdditiveCommutative().Associative()
sage: X.Unital().super_categories()[1]
Category of monoids
sage: X.Unital().super_categories()[1] is Monoids()
True
```

base_category()

Return the base category of `self`.

EXAMPLES:

```
sage: C = Sets.Finite(); C
Category of finite sets
sage: C.base_category()
Category of sets
sage: C._without_axioms()
Category of sets
```

extra_super_categories()

Return the extra super categories of a category with axiom.

Default implementation which returns `[]`.

EXAMPLES:

```
sage: FiniteSets().extra_super_categories()
[]
```

super_categories()

Return a list of the (immediate) super categories of `self`, as per *Category.super_categories()*.

This implements the property that if `As` is a subcategory of `Bs`, then the intersection of `As` with `FiniteSets()` is a subcategory of `As` and of the intersection of `Bs` with `FiniteSets()`.

EXAMPLES:

A finite magma is both a magma and a finite set:

```
sage: Magmas().Finite().super_categories()
[Category of magmas, Category of finite sets]
```

Variants:

```
sage: Sets().Finite().super_categories()
[Category of sets]

sage: Monoids().Finite().super_categories()
[Category of monoids, Category of finite semigroups]
```

EXAMPLES:

class sage.categories.category_with_axiom.**CategoryWithAxiom_over_base_ring**(*base_category*)

Bases: *CategoryWithAxiom*, *Category_over_base_ring*

class sage.categories.category_with_axiom.**CategoryWithAxiom_singleton**(*base_category*)

Bases: *Category_singleton*, *CategoryWithAxiom*

class sage.categories.category_with_axiom.**TestObjects**

Bases: *Category_singleton*

A toy singleton category, for testing purposes.

See also:

Blahs

class Commutative(*base_category*)

Bases: *CategoryWithAxiom*

class Facade(*base_category*)

Bases: *CategoryWithAxiom*

class Finite(*base_category*)

Bases: *CategoryWithAxiom*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom*

class Finite(*base_category*)

Bases: *CategoryWithAxiom*

class Unital(*base_category*)

Bases: *CategoryWithAxiom*

class Commutative(*base_category*)

Bases: *CategoryWithAxiom*

class Unital(*base_category*)

Bases: *CategoryWithAxiom*

super_categories()

class sage.categories.category_with_axiom.**TestObjectsOverBaseRing**(*base*, *name=None*)

Bases: *Category_over_base_ring*

A toy singleton category, for testing purposes.

See also:

Blahs

class Commutative(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class Facade(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class Finite(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class Finite(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class Unital(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class Commutative(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class Unital(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

super_categories()

sage.categories.category_with_axiom.**axiom**(*axiom*)

Return a function/method `self -> self._with_axiom(axiom)`.

This can used as a shorthand to define axioms, in particular in the tests below. Usually one will want to attach documentation to an axiom, so the need for such a shorthand in real life might not be that clear, unless we start creating lots of axioms.

In the long run maybe this could evolve into an `@axiom` decorator.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import axiom
sage: axiom("Finite")(Semigroups())
Category of finite semigroups
```

Upon assigning the result to a class this becomes a method:

```
sage: class As:
....:     def _with_axiom(self, axiom): return self, axiom
....:     Finite = axiom("Finite")
sage: As().Finite()
(<__main__.As ... at ...>, 'Finite')
```

sage.categories.category_with_axiom.**axiom_of_nested_class**(*nested_cls*)

Given a class and a nested axiom class, return the axiom.

EXAMPLES:

This uses some heuristics like checking if the nested_cls carries the name of the axiom, or is built by appending or prepending the name of the axiom to that of the class:

```
sage: from sage.categories.category_with_axiom import TestObjects, axiom_of_
↪nested_class
sage: axiom_of_nested_class(TestObjects, TestObjects.FiniteDimensional)
'FiniteDimensional'
sage: axiom_of_nested_class(TestObjects.FiniteDimensional,
....:                       TestObjects.FiniteDimensional.Finite)
'Finite'
sage: axiom_of_nested_class(Sets, FiniteSets)
'Finite'
sage: axiom_of_nested_class(Algebras, AlgebrasWithBasis)
'WithBasis'
```

In all other cases, the nested class should provide an attribute `_base_category_class_and_axiom`:

```
sage: Semigroups._base_category_class_and_axiom
(<class 'sage.categories.magmas.Magmas'>, 'Associative')
sage: axiom_of_nested_class(Magmas, Semigroups)
'Associative'
```

sage.categories.category_with_axiom.**base_category_class_and_axiom**(*cls*)

Try to deduce the base category and the axiom from the name of `cls`.

The heuristic is to try to decompose the name as the concatenation of the name of a category and the name of an axiom, and looking up that category in the standard location (i.e. in *sage.categories.hopf_algebras* for *HopfAlgebras*, and in *sage.categories.sets_cat* as a special case for *Sets*).

If the heuristic succeeds, the result is guaranteed to be correct. Otherwise, an error is raised.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import base_category_class_and_
↪axiom, CategoryWithAxiom
sage: base_category_class_and_axiom(FiniteSets)
(<class 'sage.categories.sets_cat.Sets'>, 'Finite')
sage: Sets.Finite
<class 'sage.categories.finite_sets.FiniteSets'>
sage: base_category_class_and_axiom(Sets.Finite)
(<class 'sage.categories.sets_cat.Sets'>, 'Finite')

sage: base_category_class_and_axiom(FiniteDimensionalHopfAlgebrasWithBasis)
(<class 'sage.categories.hopf_algebras_with_basis.HopfAlgebrasWithBasis'>,
 'FiniteDimensional')

sage: base_category_class_and_axiom(HopfAlgebrasWithBasis)
(<class 'sage.categories.hopf_algebras.HopfAlgebras'>, 'WithBasis')
```

Along the way, this does some sanity checks:

```
sage: class FacadeSemigroups(CategoryWithAxiom):
....:     pass
sage: base_category_class_and_axiom(FacadeSemigroups)
Traceback (most recent call last):
...
AssertionError: Missing (lazy import) link
for <class 'sage.categories.semigroups.Semigroups'>
to <class '__main__.FacadeSemigroups'> for axiom Facade?

sage: Semigroups.Facade = FacadeSemigroups
sage: base_category_class_and_axiom(FacadeSemigroups)
(<class 'sage.categories.semigroups.Semigroups'>, 'Facade')
```

Note: In the following example, we could possibly retrieve `Sets` from the class name. However this cannot be implemented robustly until github issue #9107 is fixed. Anyway this feature has not been needed so far:

```
sage: Sets.Infinite
<class 'sage.categories.sets_cat.Sets.Infinite'>
sage: base_category_class_and_axiom(Sets.Infinite)
Traceback (most recent call last):
...
TypeError: Could not retrieve the base category class and axiom
for <class 'sage.categories.sets_cat.Sets.Infinite'>.
...
```

`sage.categories.category_with_axiom.`**`uncamelcase`**`(`*s*, *separator*=`' ')`

EXAMPLES:

```
sage: sage.categories.category_with_axiom.uncamelcase("FiniteDimensionalAlgebras")
'finite dimensional algebras'
sage: sage.categories.category_with_axiom.uncamelcase("JTrivialMonoids")
'j trivial monoids'
sage: sage.categories.category_with_axiom.uncamelcase("FiniteDimensionalAlgebras",
↪ "_")
'finite_dimensional_algebras'
```

2.4 Functors

AUTHORS:

- David Kohel and William Stein
- David Joyner (2005-12-17): examples
- Robert Bradshaw (2007-06-23): Pyrexify
- Simon King (2010-04-30): more examples, several bug fixes, re-implementation of the default call method, making functors applicable to morphisms (not only to objects)
- Simon King (2010-12): Pickling of functors without losing domain and codomain

`sage.categories.functor.`**`ForgetfulFunctor`**`(`*domain*, *codomain*`)`

Construct the forgetful function from one category to another.

INPUT:

`C`, `D` - two categories

OUTPUT:

A functor that returns the corresponding object of `D` for any element of `C`, by forgetting the extra structure.

ASSUMPTION:

The category `C` must be a sub-category of `D`.

EXAMPLES:

```
sage: rings = Rings()
sage: abgrps = CommutativeAdditiveGroups()
sage: F = ForgetfulFunctor(rings, abgrps)
sage: F
The forgetful functor
 from Category of rings
   to Category of commutative additive groups
```

It would be a mistake to call it in opposite order:

```
sage: F = ForgetfulFunctor(abgrps, rings)
Traceback (most recent call last):
...
ValueError: Forgetful functor not supported for domain
Category of commutative additive groups
```

If both categories are equal, the forgetful functor is the same as the identity functor:

```
sage: ForgetfulFunctor(abgrps, abgrps) == IdentityFunctor(abgrps)
True
```

class `sage.categories.functor.`**`ForgetfulFunctor_generic`**

Bases: *`Functor`*

The forgetful functor, i.e., embedding of a subcategory.

NOTE:

Forgetful functors should be created using *`ForgetfulFunctor()`*, since the init method of this class does not check whether the domain is a subcategory of the codomain.

EXAMPLES:

```
sage: F = ForgetfulFunctor(FiniteFields(), Fields())  # indirect doctest
sage: F
The forgetful functor
 from Category of finite enumerated fields
   to Category of fields
sage: F(GF(3))
Finite Field of size 3
```

class `sage.categories.functor.`**`Functor`**

Bases: `SageObject`

A class for functors between two categories

NOTE:

- In the first place, a functor is given by its domain and codomain, which are both categories.

- When defining a sub-class, the user should not implement a call method. Instead, one should implement three methods, which are composed in the default call method:
  - `_coerce_into_domain(self, x)`: Return an object of `self`'s domain, corresponding to `x`, or raise a `TypeError`.
    - Default: Raise `TypeError` if `x` is not in `self`'s domain.
  - `_apply_functor(self, x)`: Apply `self` to an object `x` of `self`'s domain.
    - Default: Conversion into `self`'s codomain.
  - `_apply_functor_to_morphism(self, f)`: Apply `self` to a morphism `f` in `self`'s domain. - Default: Return `self(f.domain()).hom(f,self(f.codomain()))`.

EXAMPLES:

```
sage: rings  = Rings()
sage: abgrps = CommutativeAdditiveGroups()
sage: F = ForgetfulFunctor(rings, abgrps)
sage: F.domain()
Category of rings
sage: F.codomain()
Category of commutative additive groups
sage: from sage.categories.functor import is_Functor
sage: is_Functor(F)
True
sage: I = IdentityFunctor(abgrps)
sage: I
The identity functor on Category of commutative additive groups
sage: I.domain()
Category of commutative additive groups
sage: is_Functor(I)
True
```

Note that by default, an instance of the class Functor is coercion from the domain into the codomain. The above subclasses overloaded this behaviour. Here we illustrate the default:

```
sage: from sage.categories.functor import Functor
sage: F = Functor(Rings(), Fields())
sage: F
Functor from Category of rings to Category of fields
sage: F(ZZ)
Rational Field
sage: F(GF(2))
Finite Field of size 2
```

Functors are not only about the objects of a category, but also about their morphisms. We illustrate it, again, with the coercion functor from rings to fields.

```
sage: R1.<x> = ZZ[]
sage: R2.<a,b> = QQ[]
sage: f = R1.hom([a + b], R2)
sage: f
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Multivariate Polynomial Ring in a, b over Rational Field
  Defn: x |--> a + b
sage: F(f)
Ring morphism:
```

(continues on next page)

(continued from previous page)

```
  From: Fraction Field of Univariate Polynomial Ring in x over Integer Ring
  To:   Fraction Field of Multivariate Polynomial Ring in a, b over Rational Field
  Defn: x |--> a + b
sage: F(f)(1/x)
1/(a + b)
```

We can also apply a polynomial ring construction functor to our homomorphism. The result is a homomorphism that is defined on the base ring:

```
sage: F = QQ['t'].construction()[0]
sage: F
Poly[t]
sage: F(f)
Ring morphism:
  From: Univariate Polynomial Ring in t
         over Univariate Polynomial Ring in x over Integer Ring
  To:   Univariate Polynomial Ring in t
         over Multivariate Polynomial Ring in a, b over Rational Field
  Defn: Induced from base ring by
        Ring morphism:
          From: Univariate Polynomial Ring in x over Integer Ring
          To:   Multivariate Polynomial Ring in a, b over Rational Field
          Defn: x |--> a + b
sage: p = R1['t']('(-x^2 + x)*t^2 + (x^2 - x)*t - 4*x^2 - x + 1')
sage: F(f)(p)
(-a^2 - 2*a*b - b^2 + a + b)*t^2 + (a^2 + 2*a*b + b^2 - a - b)*t
- 4*a^2 - 8*a*b - 4*b^2 - a - b + 1
```

codomain()

The codomain of self

EXAMPLES:

```
sage: F = ForgetfulFunctor(FiniteFields(), Fields())
sage: F.codomain()
Category of fields
```

domain()

The domain of self

EXAMPLES:

```
sage: F = ForgetfulFunctor(FiniteFields(), Fields())
sage: F.domain()
Category of finite enumerated fields
```

sage.categories.functor.**IdentityFunctor**(*C*)

Construct the identity functor of the given category.

INPUT:

A category, C.

OUTPUT:

The identity functor in C.

EXAMPLES:

```
sage: rings = Rings()
sage: F = IdentityFunctor(rings)
sage: F(ZZ['x','y']) is ZZ['x','y']
True
```

class sage.categories.functor.**IdentityFunctor_generic**(*C*)

Bases: *ForgetfulFunctor_generic*

Generic identity functor on any category

NOTE:

This usually is created using *IdentityFunctor()*.

EXAMPLES:

```
sage: F = IdentityFunctor(Fields()) #indirect doctest
sage: F
The identity functor on Category of fields
sage: F(RR) is RR
True
sage: F(ZZ)
Traceback (most recent call last):
...
TypeError: x (=Integer Ring) is not in Category of fields
```

sage.categories.functor.**is_Functor**(*x*)

Test whether the argument is a functor

NOTE:

There is a deprecation warning when using it from top level. Therefore we import it in our doc test.

EXAMPLES:

```
sage: from sage.categories.functor import is_Functor
sage: F1 = QQ.construction()[0]
sage: F1
FractionField
sage: is_Functor(F1)
True
sage: is_Functor(FractionField)
False
sage: F2 = ForgetfulFunctor(Fields(), Rings())
sage: F2
The forgetful functor from Category of fields to Category of rings
sage: is_Functor(F2)
True
```

2.5 Implementing a new parent: a tutorial

The easiest approach for implementing a new parent is to start from a close example in sage.categories.examples. Here, we will get through the process of implementing a new finite semigroup, taking as starting point the provided example:

```
sage: S = FiniteSemigroups().example()
sage: S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c',
↪'d')
```

You may lookup the implementation of this example with:

```
sage: S??                                    # not tested
```

Or by browsing the source code of *sage.categories.examples.finite_semigroups.LeftRegularBand*.

Copy-paste this code into, say, a cell of the notebook, and replace every occurrence of `FiniteSemigroups().example(...)` in the documentation by `LeftRegularBand`. This will be equivalent to:

```
sage: from sage.categories.examples.finite_semigroups import LeftRegularBand
```

Now, try:

```
sage: S = LeftRegularBand(); S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c',
↪'d')
```

and play around with the examples in the documentation of `S` and of *FiniteSemigroups*.

Rename the class to `ShiftSemigroup`, and modify the product to implement the semigroup generated by the given alphabet such that $au = u$ for any u of length 3.

Use `TestSuite` to test the newly implemented semigroup; draw its Cayley graph.

Add another option to the constructor to generalize the construction to any u of length k.

Lookup the Sloane for the sequence of the sizes of those semigroups.

Now implement the commutative monoid of subsets of $\{1, \dots, n\}$ endowed with union as product. What is its category? What are the extra functionalities available there? Implement iteration and cardinality.

Todo: The tutorial should explain there how to reuse the enumerated set of subsets, and endow it with more structure.

CHAPTER

THREE

MAPS AND MORPHISMS

3.1 Base class for maps

AUTHORS:

- Robert Bradshaw: initial implementation
- Sebastien Besnier (2014-05-5): *FormalCompositeMap* contains a list of Map instead of only two Map. See github issue #16291.
- Sebastian Oehms (2019-01-19): `section()` added to *FormalCompositeMap*. See github issue #27081.

class sage.categories.map.**FormalCompositeMap**

Bases: *Map*

Formal composite maps.

A formal composite map is formed by two maps, so that the codomain of the first map is contained in the domain of the second map.

Note: When calling a composite with additional arguments, these arguments are *only* passed to the second underlying map.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<a> = QQ[]
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(R, S, Rings()), lambda p: p[0]*a^p.degree())
sage: g = S.hom([2*x])
sage: f*g
Composite map:
  From: Univariate Polynomial Ring in a over Rational Field
  To:   Univariate Polynomial Ring in a over Rational Field
  Defn:   Ring morphism:
          From: Univariate Polynomial Ring in a over Rational Field
          To:   Univariate Polynomial Ring in x over Rational Field
          Defn: a |--> 2*x
        then
          Generic morphism:
          From: Univariate Polynomial Ring in x over Rational Field
          To:   Univariate Polynomial Ring in a over Rational Field
sage: g*f
Composite map:
```

(continues on next page)

(continued from previous page)

```
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
  Defn:   Generic morphism:
          From: Univariate Polynomial Ring in x over Rational Field
          To:   Univariate Polynomial Ring in a over Rational Field
        then
          Ring morphism:
          From: Univariate Polynomial Ring in a over Rational Field
          To:   Univariate Polynomial Ring in x over Rational Field
          Defn: a |--> 2*x
sage: (f*g)(2*a^2+5)
5*a^2
sage: (g*f)(2*x^2+5)
20*x^2
```

domains()

Iterate over the domains of the factors of this map.

(This is useful in particular to check for loops in coercion maps.)

See also:

Map.domains()

EXAMPLES:

```
sage: f = QQ.coerce_map_from(ZZ)
sage: g = MatrixSpace(QQ, 2, 2).coerce_map_from(QQ)                      #␣
↪needs sage.modules
sage: list((g * f).domains())                                            #␣
↪needs sage.modules
[Integer Ring, Rational Field]
```

first()

Return the first map in the formal composition.

If `self` represents $f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0$, then `self.first()` returns f_0. We have `self == self.then() * self.first()`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<a> = QQ[]
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(R, S, Rings()), lambda p: p[0]*a^p.degree())
sage: g = S.hom([2*x])
sage: fg = f * g
sage: fg.first() == g
True
sage: fg == fg.then() * fg.first()
True
```

is_injective()

Tell whether `self` is injective.

It raises `NotImplementedError` if it cannot be determined.

EXAMPLES:

```
sage: # needs sage.modules
sage: V1 = QQ^2
sage: V2 = QQ^3
sage: phi1 = (QQ^1).hom(Matrix([[1, 1]]), V1)
sage: phi2 = V1.hom(Matrix([[1, 2, 3], [4, 5, 6]]), V2)
```

If both constituents are injective, the composition is injective:

```
sage: from sage.categories.map import FormalCompositeMap
sage: c1 = FormalCompositeMap(Hom(QQ^1, V2, phi1.category_for()),           #␣
↪needs sage.modules
....:                          phi1, phi2)
sage: c1.is_injective()                                                     #␣
↪needs sage.modules
True
```

If it cannot be determined whether the composition is injective, an error is raised:

```
sage: psi1 = V2.hom(Matrix([[1, 2], [3, 4], [5, 6]]), V1)                   #␣
↪needs sage.modules
sage: c2 = FormalCompositeMap(Hom(V1, V1, phi2.category_for()),             #␣
↪needs sage.modules
....:                          phi2, psi1)
sage: c2.is_injective()                                                     #␣
↪needs sage.modules
Traceback (most recent call last):
...
NotImplementedError: not enough information to deduce injectivity
```

If the first map is surjective and the second map is not injective, then the composition is not injective:

```
sage: psi2 = V1.hom([[1], [1]], QQ^1)                                       #␣
↪needs sage.modules
sage: c3 = FormalCompositeMap(Hom(V2, QQ^1, phi2.category_for()),           #␣
↪needs sage.modules
....:                          psi2, psi1)
sage: c3.is_injective()                                                     #␣
↪needs sage.modules
False
```

is_surjective()

Tell whether `self` is surjective.

It raises `NotImplementedError` if it cannot be determined.

EXAMPLES:

```
sage: from sage.categories.map import FormalCompositeMap
sage: V3 = QQ^3                                                             #␣
↪needs sage.modules
sage: V2 = QQ^2                                                             #␣
↪needs sage.modules
sage: V1 = QQ^1                                                             #␣
↪needs sage.modules
```

If both maps are surjective, the composition is surjective:

```
sage: # needs sage.modules
sage: phi32 = V3.hom(Matrix([[1, 2], [3, 4], [5, 6]]), V2)
sage: phi21 = V2.hom(Matrix([[1], [1]]), V1)
sage: c_phi = FormalCompositeMap(Hom(V3, V1, phi32.category_for()),
....:                            phi32, phi21)
sage: c_phi.is_surjective()
True
```

If the second map is not surjective, the composition is not surjective:

```
sage: FormalCompositeMap(Hom(V3, V1, phi32.category_for()),                 #␣
↪needs sage.modules
....:                    phi32,
....:                    V2.hom(Matrix([[0], [0]]), V1)).is_surjective()
False
```

If the second map is an isomorphism and the first map is not surjective, then the composition is not surjective:

```
sage: FormalCompositeMap(Hom(V2, V1, phi32.category_for()),                 #␣
↪needs sage.modules
....:                    V2.hom(Matrix([[0], [0]]), V1),
....:                    V1.hom(Matrix([[1]]), V1)).is_surjective()
False
```

Otherwise, surjectivity of the composition cannot be determined:

```
sage: FormalCompositeMap(Hom(V2, V1, phi32.category_for()),                 #␣
↪needs sage.modules
....:     V2.hom(Matrix([[1, 1], [1, 1]]), V2),
....:     V2.hom(Matrix([[1], [1]]), V1)).is_surjective()
Traceback (most recent call last):
...
NotImplementedError: not enough information to deduce surjectivity
```

section()

Compute a section map from sections of the factors of `self` if they have been implemented.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: incl = P.coerce_map_from(ZZ)
sage: sect = incl.section(); sect
Composite map:
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Integer Ring
  Defn:   Generic map:
          From: Univariate Polynomial Ring in x over Rational Field
          To:   Rational Field
        then
          Generic map:
          From: Rational Field
          To:   Integer Ring
sage: p = x + 5; q = x + 2
sage: sect(p-q)
3
```

the following example has been attached to `_integer_()` of `sage.rings.polynomial.polynomial_element.Polynomial` before (see comment there):

```
sage: k = GF(47)
sage: R.<x> = PolynomialRing(k)
sage: R.coerce_map_from(ZZ).section()
Composite map:
  From: Univariate Polynomial Ring in x over Finite Field of size 47
  To:   Integer Ring
  Defn:   Generic map:
          From: Univariate Polynomial Ring in x over Finite Field of size 47
          To:   Finite Field of size 47
        then
          Lifting map:
          From: Finite Field of size 47
          To:   Integer Ring
sage: ZZ(R(45))                 # indirect doctest
45
sage: ZZ(3*x + 45)              # indirect doctest
Traceback (most recent call last):
...
TypeError: not a constant polynomial
```

then()

Return the tail of the list of maps.

If self represents $f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0$, then self.first() returns $f_n \circ f_{n-1} \circ \cdots \circ f_1$. We have self == self.then() * self.first().

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<a> = QQ[]
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(R, S, Rings()), lambda p: p[0]*a^p.degree())
sage: g = S.hom([2*x])
sage: (f*g).then() == f
True

sage: f = QQ.coerce_map_from(ZZ)
sage: f = f.extend_domain(ZZ).extend_codomain(QQ)
sage: f.then()
Composite map:
From: Integer Ring
To:   Rational Field
Defn:   Natural morphism:
From: Integer Ring
To:   Rational Field
then
Identity endomorphism of Rational Field
```

class sage.categories.map.**Map**

Bases: Element

Basic class for all maps.

Note: The call method is of course not implemented in this base class. This must be done in the sub classes, by overloading _call_ and possibly also _call_with_args.

EXAMPLES:

Usually, instances of this class will not be constructed directly, but for example like this:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: phi = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi(x^2+2*x-1)
-1
sage: R.<x,y> = QQ[]
sage: f = R.hom([x+y, x-y], R)
sage: f(x^2+2*x-1)
x^2 + 2*x*y + y^2 + 2*x + 2*y - 1
```

category_for()

Returns the category self is a morphism for.

Note: This is different from the category of maps to which this map belongs *as an object*.

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: phi = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi.category_for()
Category of rings
sage: phi.category()
Category of homsets of unital magmas and additive unital additive magmas
sage: R.<x,y> = QQ[]
sage: f = R.hom([x+y, x-y], R)
sage: f.category_for()
Join of Category of unique factorization domains
and Category of commutative algebras
over (number fields and quotient fields and metric spaces)
and Category of infinite sets
sage: f.category()
Category of endsets of unital magmas
 and right modules over (number fields and quotient fields and metric spaces)
 and left modules over (number fields and quotient fields and metric spaces)
```

FIXME: find a better name for this method

codomain

domain

domains()

Iterate over the domains of the factors of a (composite) map.

This default implementation simply yields the domain of this map.

See also:

FormalCompositeMap.domains()

EXAMPLES:

```
sage: list(QQ.coerce_map_from(ZZ).domains())
[Integer Ring]
```

extend_codomain(*new_codomain*)

INPUT:

- `self` – a member of Hom(X, Y)
- `new_codomain` – an object Z such that there is a canonical coercion ϕ in Hom(Y, Z)

OUTPUT:

An element of Hom(X, Z) obtained by composing self with ϕ. If no canonical ϕ exists, a `TypeError` is raised.

EXAMPLES:

```
sage: mor = QQ.coerce_map_from(ZZ)
sage: mor.extend_codomain(RDF)
Composite map:
  From: Integer Ring
  To:   Real Double Field
  Defn:   Natural morphism:
          From: Integer Ring
          To:   Rational Field
        then
          Native morphism:
          From: Rational Field
          To:   Real Double Field
sage: mor.extend_codomain(GF(7))
Traceback (most recent call last):
...
TypeError: No coercion from Rational Field to Finite Field of size 7
```

extend_domain(*new_domain*)

INPUT:

- `self` – a member of Hom(Y, Z)
- `new_codomain` – an object X such that there is a canonical coercion ϕ in Hom(X, Y)

OUTPUT:

An element of Hom(X, Z) obtained by composing self with ϕ. If no canonical ϕ exists, a `TypeError` is raised.

EXAMPLES:

```
sage: # needs sage.rings.complex_double
sage: mor = CDF.coerce_map_from(RDF)
sage: mor.extend_domain(QQ)
Composite map:
  From: Rational Field
  To:   Complex Double Field
  Defn:   Native morphism:
          From: Rational Field
          To:   Real Double Field
        then
          Native morphism:
          From: Real Double Field
```

(continues on next page)

(continued from previous page)

```
          To:   Complex Double Field
sage: mor.extend_domain(ZZ['x'])
Traceback (most recent call last):
...
TypeError: No coercion from Univariate Polynomial Ring in x over Integer Ring
to Real Double Field
```

is_surjective()

Tells whether the map is surjective (not implemented in the base class).

parent()

Return the homset containing this map.

Note: The method `_make_weak_references()`, that is used for the maps found by the coercion system, needs to remove the usual strong reference from the coercion map to the homset containing it. As long as the user keeps strong references to domain and codomain of the map, we will be able to reconstruct the homset. However, a strong reference to the coercion map does not prevent the domain from garbage collection!

EXAMPLES:

```
sage: Q = QuadraticField(-5)                                                    #␣
↪needs sage.rings.number_field
sage: phi = CDF._internal_convert_map_from(Q)                                   #␣
↪needs sage.rings.number_field
sage: print(phi.parent())                                                       #␣
↪needs sage.rings.number_field
Set of field embeddings
 from Number Field in a with defining polynomial x^2 + 5
      with a = 2.236067977499790?*I
   to Complex Double Field
```

We now demonstrate that the reference to the coercion map ϕ does not prevent Q from being garbage collected:

```
sage: import gc
sage: del Q                                                                     #␣
↪needs sage.rings.number_field
sage: _ = gc.collect()
sage: phi.parent()                                                              #␣
↪needs sage.rings.number_field
Traceback (most recent call last):
...
ValueError: This map is in an invalid state,
the domain has been garbage collected
```

You can still obtain copies of the maps used by the coercion system with strong references:

```
sage: # needs sage.rings.number_field
sage: Q = QuadraticField(-5)
sage: phi = CDF.convert_map_from(Q)
sage: print(phi.parent())
Set of field embeddings
 from Number Field in a with defining polynomial x^2 + 5
```

(continues on next page)

(continued from previous page)

```
      with a = 2.236067977499790?*I
   to Complex Double Field
sage: import gc
sage: del Q
sage: _ = gc.collect()
sage: phi.parent()
Set of field embeddings
 from Number Field in a with defining polynomial x^2 + 5
      with a = 2.236067977499790?*I
   to Complex Double Field
```

post_compose(*left*)

INPUT:

- `self` – a Map in some `Hom(X, Y, category_right)`
- `left` – a Map in some `Hom(Y, Z, category_left)`

Returns the composition of `self` followed by `left` as a morphism in `Hom(X, Z, category)` where `category` is the meet of `category_left` and `category_right`.

Caveat: see the current restrictions on *`Category.meet()`*

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: Z = QQ
sage: phi_xy = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi_yz = SetMorphism(Hom(Y, Z, Monoids()), lambda y: QQ(y**2))
sage: phi_xz = phi_xy.post_compose(phi_yz); phi_xz
Composite map:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Rational Field
  Defn:   Generic morphism:
          From: Univariate Polynomial Ring in x over Integer Ring
          To:   Integer Ring
        then
          Generic morphism:
          From: Integer Ring
          To:   Rational Field
sage: phi_xz.category_for()
Category of monoids
```

pre_compose(*right*)

INPUT:

- `self` – a Map in some `Hom(Y, Z, category_left)`
- `left` – a Map in some `Hom(X, Y, category_right)`

Returns the composition of `right` followed by `self` as a morphism in `Hom(X, Z, category)` where `category` is the meet of `category_left` and `category_right`.

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
```

(continues on next page)

(continued from previous page)

```
sage: Y = ZZ
sage: Z = QQ
sage: phi_xy = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi_yz = SetMorphism(Hom(Y, Z, Monoids()), lambda y: QQ(y**2))
sage: phi_xz = phi_yz.pre_compose(phi_xy); phi_xz
Composite map:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Rational Field
  Defn:   Generic morphism:
          From: Univariate Polynomial Ring in x over Integer Ring
          To:   Integer Ring
        then
          Generic morphism:
          From: Integer Ring
          To:   Rational Field
sage: phi_xz.category_for()
Category of monoids
```

section()

Return a section of self.

Note: By default, it returns `None`. You may override it in subclasses.

class sage.categories.map.**Section**

Bases: *Map*

A formal section of a map.

Note: Call methods are not implemented for the base class `Section`.

EXAMPLES:

```
sage: from sage.categories.map import Section
sage: R.<x,y> = ZZ[]
sage: S.<a,b> = QQ[]
sage: f = R.hom([a+b, a-b])
sage: sf = Section(f); sf
Section map:
  From: Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in x, y over Integer Ring
sage: sf(a)
Traceback (most recent call last):
...
NotImplementedError: <class 'sage.categories.map.Section'>
```

inverse()

Return inverse of `self`.

sage.categories.map.**is_Map**(*x*)

Auxiliary function: Is the argument a map?

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = R.hom([x+y, x-y], R)
sage: from sage.categories.map import is_Map
sage: is_Map(f)
True
```

`sage.categories.map.`**`unpickle_map`**`(`*_class*, *parent*, *_dict*, *_slots*`)`

Auxiliary function for unpickling a map.

3.2 Homsets

The class *Hom* is the base class used to represent sets of morphisms between objects of a given category. *Hom* objects are usually "weakly" cached upon creation so that they don't have to be generated over and over but can be garbage collected together with the corresponding objects when these are not strongly ref'ed anymore.

EXAMPLES:

In the following, the *Hom* object is indeed cached:

```
sage: K = GF(17)
sage: H = Hom(ZZ, K)
sage: H
Set of Homomorphisms from Integer Ring to Finite Field of size 17
sage: H is Hom(ZZ, K)
True
```

Nonetheless, garbage collection occurs when the original references are overwritten:

```
sage: # needs sage.libs.pari
sage: for p in prime_range(200):
....:     K = GF(p)
....:     H = Hom(ZZ, K)
sage: import gc
sage: _ = gc.collect()
sage: from sage.rings.finite_rings.finite_field_prime_modn import FiniteField_prime_
↪modn as FF
sage: L = [x for x in gc.get_objects() if isinstance(x, FF)]
sage: len(L)
1
sage: L
[Finite Field of size 199]
```

AUTHORS:

- David Kohel and William Stein
- David Joyner (2005-12-17): added examples
- William Stein (2006-01-14): Changed from Homspace to Homset.
- Nicolas M. Thiery (2008-12-): Updated for the new category framework
- Simon King (2011-12): Use a weak cache for homsets
- Simon King (2013-02): added examples

sage.categories.homset.**End**(*X*, *category=None*)

Create the set of endomorphisms of X in the category category.

INPUT:

- X – anything
- category – (optional) category in which to coerce X

OUTPUT:

A set of endomorphisms in category

EXAMPLES:

```
sage: V = VectorSpace(QQ, 3)                                              #␣
↪needs sage.modules
sage: End(V)                                                              #␣
↪needs sage.modules
Set of Morphisms (Linear Transformations)
 from Vector space of dimension 3 over Rational Field
 to Vector space of dimension 3 over Rational Field
```

```
sage: # needs sage.groups
sage: G = AlternatingGroup(3)
sage: S = End(G); S
Set of Morphisms
 from Alternating group of order 3!/2 as a permutation group
 to Alternating group of order 3!/2 as a permutation group
 in Category of finite enumerated permutation groups
sage: from sage.categories.homset import is_Endset
sage: is_Endset(S)
True
sage: S.domain()
Alternating group of order 3!/2 as a permutation group
```

To avoid creating superfluous categories, a homset in a category Cs() is in the homset category of the lowest full super category Bs() of Cs() that implements Bs.Homsets (or the join thereof if there are several). For example, finite groups form a full subcategory of unital magmas: any unital magma morphism between two finite groups is a finite group morphism. Since finite groups currently implement nothing more than unital magmas about their homsets, we have:

```
sage: # needs sage.groups
sage: G = GL(3, 3)
sage: G.category()
Category of finite groups
sage: H = Hom(G, G)
sage: H.homset_category()
Category of finite groups
sage: H.category()
Category of endsets of unital magmas
```

Similarly, a ring morphism just needs to preserve addition, multiplication, zero, and one. Accordingly, and since the category of rings implements nothing specific about its homsets, a ring homset is currently constructed in the category of homsets of unital magmas and unital additive magmas:

```
sage: H = Hom(ZZ,ZZ,Rings())
sage: H.category()
Category of endsets of unital magmas and additive unital additive magmas
```

`sage.categories.homset.Hom(X, Y, category=None, check=True)`

Create the space of homomorphisms from X to Y in the category `category`.

INPUT:

- `X` – an object of a category
- `Y` – an object of a category
- `category` – a category in which the morphisms must be. (default: the meet of the categories of `X` and `Y`) Both `X` and `Y` must belong to that category.
- `check` – a boolean (default: `True`): whether to check the input, and in particular that `X` and `Y` belong to `category`.

OUTPUT: a homset in category

EXAMPLES:

```
sage: V = VectorSpace(QQ, 3)                                                  #␣
↪needs sage.modules
sage: Hom(V, V)                                                               #␣
↪needs sage.modules
Set of Morphisms (Linear Transformations) from
Vector space of dimension 3 over Rational Field to
Vector space of dimension 3 over Rational Field
sage: G = AlternatingGroup(3)                                                 #␣
↪needs sage.groups
sage: Hom(G, G)                                                               #␣
↪needs sage.groups
Set of Morphisms
 from Alternating group of order 3!/2 as a permutation group
   to Alternating group of order 3!/2 as a permutation group
   in Category of finite enumerated permutation groups
sage: Hom(ZZ, QQ, Sets())
Set of Morphisms from Integer Ring to Rational Field in Category of sets

sage: Hom(FreeModule(ZZ, 1), FreeModule(QQ, 1))                               #␣
↪needs sage.modules
Set of Morphisms
 from Ambient free module of rank 1 over the principal ideal domain Integer Ring
   to Vector space of dimension 1 over Rational Field
   in Category of commutative additive groups
sage: Hom(FreeModule(QQ, 1), FreeModule(ZZ, 1))                               #␣
↪needs sage.modules
Set of Morphisms
 from Vector space of dimension 1 over Rational Field
   to Ambient free module of rank 1 over the principal ideal domain Integer Ring
   in Category of commutative additive groups
```

Here, we test against a memory leak that has been fixed at github issue #11521 by using a weak cache:

```
sage: # needs sage.libs.pari
sage: for p in prime_range(10^3):
....:  K = GF(p)
....:  a = K(0)
sage: import gc
sage: gc.collect()       # random
624
sage: from sage.rings.finite_rings.finite_field_prime_modn import FiniteField_
```

(continues on next page)

(continued from previous page)

```
↪prime_modn as FF
sage: L = [x for x in gc.get_objects() if isinstance(x, FF)]
sage: len(L), L[0]
(1, Finite Field of size 997)
```

To illustrate the choice of the category, we consider the following parents as running examples:

```
sage: X = ZZ; X
Integer Ring
sage: Y = SymmetricGroup(3); Y                                          #␣
↪needs sage.groups
Symmetric group of order 3! as a permutation group
```

By default, the smallest category containing both `X` and `Y`, is used:

```
sage: Hom(X, Y)                                                         #␣
↪needs sage.groups
Set of Morphisms from Integer Ring
 to Symmetric group of order 3! as a permutation group
 in Category of enumerated monoids
```

Otherwise, if `category` is specified, then `category` is used, after checking that `X` and `Y` are indeed in `category`:

```
sage: Hom(X, Y, Magmas())                                               #␣
↪needs sage.groups
Set of Morphisms
 from Integer Ring
 to Symmetric group of order 3! as a permutation group
 in Category of magmas

sage: Hom(X, Y, Groups())                                               #␣
↪needs sage.groups
Traceback (most recent call last):
...
ValueError: Integer Ring is not in Category of groups
```

A parent (or a parent class of a category) may specify how to construct certain homsets by implementing a method `_Hom_(self, codomain, category)`. This method should either construct the requested homset or raise a `TypeError`. This hook is currently mostly used to create homsets in some specific subclass of *`Homset`* (e.g. `sage.rings.homset.RingHomset`):

```
sage: Hom(QQ,QQ).__class__
<class 'sage.rings.homset.RingHomset_generic_with_category'>
```

Do not call this hook directly to create homsets, as it does not handle unique representation:

```
sage: Hom(QQ,QQ) == QQ._Hom_(QQ, category=QQ.category())
True
sage: Hom(QQ,QQ) is QQ._Hom_(QQ, category=QQ.category())
False
```

Todo:

- Design decision: how much of the homset comes from the category of `X` and `Y`, and how much from the specific `X` and `Y`. In particular, do we need several parent classes depending on `X` and `Y`, or does the difference only lie in the elements (i.e. the morphism), and of course how the parent calls their constructors.

- Specify the protocol for the `_Hom_` hook in case of ambiguity (e.g. if both a parent and some category thereof provide one).

class sage.categories.homset.**Homset**(*X*, *Y*, *category=None*, *base=None*, *check=True*)

Bases: `Set_generic`

The class for collections of morphisms in a category.

EXAMPLES:

```
sage: H = Hom(QQ^2, QQ^3)                                                    #␣
↪needs sage.modules
sage: loads(H.dumps()) is H                                                  #␣
↪needs sage.modules
True
```

Homsets of unique parents are unique as well:

```
sage: H = End(AffineSpace(2, names='x,y'))
sage: loads(dumps(AffineSpace(2, names='x,y'))) is AffineSpace(2, names='x,y')
True
sage: loads(dumps(H)) is H
True
```

Conversely, homsets of non-unique parents are non-unique:

```
sage: P11 = ProductProjectiveSpaces(QQ, [1, 1])
sage: H = End(P11)
sage: loads(dumps(P11)) is ProductProjectiveSpaces(QQ, [1, 1])
False
sage: loads(dumps(P11)) == ProductProjectiveSpaces(QQ, [1, 1])
True
sage: loads(dumps(H)) is H
False
sage: loads(dumps(H)) == H
True
```

codomain()

Return the codomain of this homset.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: f.parent().codomain()
Univariate Polynomial Ring in t over Rational Field
sage: f.codomain() is f.parent().codomain()
True
```

domain()

Return the domain of this homset.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
```

(continues on next page)

(continued from previous page)

```
sage: f.parent().domain()
Univariate Polynomial Ring in t over Integer Ring
sage: f.domain() is f.parent().domain()
True
```

element_class_set_morphism()

A base class for elements of this homset which are also `SetMorphism`, i.e. implemented by mean of a Python function.

This is currently plain `SetMorphism`, without inheritance from categories.

Todo: Refactor during the upcoming homset cleanup.

EXAMPLES:

```
sage: H = Hom(ZZ, ZZ)
sage: H.element_class_set_morphism
<class 'sage.categories.morphism.SetMorphism'>
```

homset_category()

Return the category that this is a Hom in, i.e., this is typically the category of the domain or codomain object.

EXAMPLES:

```
sage: H = Hom(AlternatingGroup(4), AlternatingGroup(7))                      #␣
↪needs sage.groups
sage: H.homset_category()                                                     #␣
↪needs sage.groups
Category of finite enumerated permutation groups
```

identity()

The identity map of this homset.

Note: Of course, this only exists for sets of endomorphisms.

EXAMPLES:

```
sage: H = Hom(QQ,QQ)
sage: H.identity()
Identity endomorphism of Rational Field
sage: H = Hom(ZZ,QQ)
sage: H.identity()
Traceback (most recent call last):
...
TypeError: identity map only defined for endomorphisms; try natural_map()␣
↪instead
sage: H.natural_map()
Natural morphism:
  From: Integer Ring
  To:   Rational Field
```

natural_map()

Return the "natural map" of this homset.

Note: By default, a formal coercion morphism is returned.

EXAMPLES:

```
sage: H = Hom(ZZ['t'],QQ['t'], CommutativeAdditiveGroups())
sage: H.natural_map()
Coercion morphism:
  From: Univariate Polynomial Ring in t over Integer Ring
  To:   Univariate Polynomial Ring in t over Rational Field
sage: H = Hom(QQ['t'], GF(3)['t'])
sage: H.natural_map()
Traceback (most recent call last):
...
TypeError: natural coercion morphism
from Univariate Polynomial Ring in t over Rational Field
to Univariate Polynomial Ring in t over Finite Field of size 3 not defined
```

one()

The identity map of this homset.

Note: Of course, this only exists for sets of endomorphisms.

EXAMPLES:

```
sage: K = GaussianIntegers()                                                 #␣
↪needs sage.rings.number_field
sage: End(K).one()                                                           #␣
↪needs sage.rings.number_field
Identity endomorphism of Gaussian Integers generated by I
 in Number Field in I with defining polynomial x^2 + 1 with I = 1*I
```

reversed()

Return the corresponding homset, but with the domain and codomain reversed.

EXAMPLES:

```
sage: # needs sage.modules
sage: H = Hom(ZZ^2, ZZ^3); H
Set of Morphisms from Ambient free module of rank 2 over
 the principal ideal domain Integer Ring to Ambient free module
 of rank 3 over the principal ideal domain Integer Ring in
 Category of finite dimensional modules with basis over (Dedekind
 domains and euclidean domains
 and infinite enumerated sets and metric spaces)
sage: type(H)
<class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: H.reversed()
Set of Morphisms from Ambient free module of rank 3 over
 the principal ideal domain Integer Ring to Ambient free module
 of rank 2 over the principal ideal domain Integer Ring in
 Category of finite dimensional modules with basis over (Dedekind
 domains and euclidean domains
 and infinite enumerated sets and metric spaces)
sage: type(H.reversed())
<class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
```

class `sage.categories.homset.`**`HomsetWithBase`**(*X*, *Y*, *category=None*, *check=True*, *base=None*)

Bases: `Homset`

`sage.categories.homset.`**`end`**(*X*, *f*)

Return `End(X)(f)`, where `f` is data that defines an element of `End(X)`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: phi = end(R, [x + 1])
sage: phi
Ring endomorphism of Univariate Polynomial Ring in x over Rational Field
  Defn: x |--> x + 1
sage: phi(x^2 + 5)
x^2 + 2*x + 6
```

`sage.categories.homset.`**`hom`**(*X*, *Y*, *f*)

Return `Hom(X,Y)(f)`, where `f` is data that defines an element of `Hom(X,Y)`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: phi = hom(R, QQ, [2])
sage: phi(x^2 + 3)
7
```

`sage.categories.homset.`**`is_Endset`**(*x*)

Return `True` if `x` is a set of endomorphisms in a category.

EXAMPLES:

```
sage: from sage.categories.homset import is_Endset
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: is_Endset(f.parent())
False
sage: g = P.hom([2*t])
sage: is_Endset(g.parent())
True
```

`sage.categories.homset.`**`is_Homset`**(*x*)

Return `True` if `x` is a set of homomorphisms in a category.

EXAMPLES:

```
sage: from sage.categories.homset import is_Homset
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: is_Homset(f)
False
sage: is_Homset(f.category())
False
sage: is_Homset(f.parent())
True
```

3.3 Morphisms

This module defines the base classes of morphisms between objects of a given category.

EXAMPLES:

Typically, a morphism is defined by the images of the generators of the domain.

```
sage: X.<a, b> = ZZ[]
sage: Y.<c> = ZZ[]
sage: X.hom([c, c^2])
Ring morphism:
  From: Multivariate Polynomial Ring in a, b over Integer Ring
  To:   Univariate Polynomial Ring in c over Integer Ring
  Defn: a |--> c
        b |--> c^2
```

AUTHORS:

- William Stein (2005): initial version
- David Joyner (2005-12-17): added examples
- Robert Bradshaw (2007-06-25): Pyrexification

class sage.categories.morphism.**CallMorphism**

Bases: *Morphism*

class sage.categories.morphism.**FormalCoercionMorphism**

Bases: *Morphism*

class sage.categories.morphism.**IdentityMorphism**

Bases: *Morphism*

is_identity()

Return True if this morphism is the identity morphism.

EXAMPLES:

```
sage: E = End(Partitions(5))                                          #␣
↪needs sage.combinat
sage: E.identity().is_identity()                                      #␣
↪needs sage.combinat
True
```

Check that github issue #15478 is fixed:

```
sage: # needs sage.rings.finite_rings
sage: K.<z> = GF(4)
sage: phi = End(K)([z^2])
sage: R.<t> = K[]
sage: psi = End(R)(phi)
sage: psi.is_identity()
False
```

is_injective()

Return whether this morphism is injective.

EXAMPLES:

```
sage: Hom(ZZ, ZZ).identity().is_injective()
True
```

is_surjective()

Return whether this morphism is surjective.

EXAMPLES:

```
sage: Hom(ZZ, ZZ).identity().is_surjective()
True
```

section()

Return a section of this morphism.

EXAMPLES:

```
sage: T = Hom(ZZ, ZZ).identity()
sage: T.section() is T
True
```

class sage.categories.morphism.**Morphism**

Bases: *Map*

category()

Return the category of the parent of this morphism.

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = R.hom([t**2])
sage: f.category()
Category of endsets of unital magmas and right modules over
 (Dedekind domains and euclidean domains
  and infinite enumerated sets and metric spaces)
 and left modules over (Dedekind domains and euclidean domains
 and infinite enumerated sets and metric spaces)

sage: # needs sage.rings.number_field
sage: K = CyclotomicField(12)
sage: L = CyclotomicField(132)
sage: phi = L._internal_coerce_map_from(K)
sage: phi.category()
Category of homsets of number fields
```

is_endomorphism()

Return `True` if this morphism is an endomorphism.

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = R.hom([t])
sage: f.is_endomorphism()
True

sage: # needs sage.rings.number_field
sage: K = CyclotomicField(12)
sage: L = CyclotomicField(132)
```

(continues on next page)

(continued from previous page)

```
sage: phi = L._internal_coerce_map_from(K)
sage: phi.is_endomorphism()
False
```

is_identity()

Return `True` if this morphism is the identity morphism.

Note: Implemented only when the domain has a method gens()

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = R.hom([t])
sage: f.is_identity()
True
sage: g = R.hom([t + 1])
sage: g.is_identity()
False
```

A morphism between two different spaces cannot be the identity:

```
sage: R2.<t2> = QQ[]
sage: h = R.hom([t2])
sage: h.is_identity()
False
```

pushforward(*I*)

register_as_coercion()

Register this morphism as a coercion to Sage's coercion model (see `sage.structure.coerce`).

EXAMPLES:

By default, adding polynomials over different variables triggers an error:

```
sage: X.<x> = ZZ[]
sage: Y.<y> = ZZ[]
sage: x^2 + y
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +:
'Univariate Polynomial Ring in x over Integer Ring' and
'Univariate Polynomial Ring in y over Integer Ring'
```

Let us declare a coercion from $\mathbf{Z}[x]$ to $\mathbf{Z}[z]$:

```
sage: Z.<z> = ZZ[]
sage: phi = Hom(X, Z)(z)
sage: phi(x^2+1)
z^2 + 1
sage: phi.register_as_coercion()
```

Now we can add elements from $\mathbf{Z}[x]$ and $\mathbf{Z}[z]$, because the elements of the former are allowed to be implicitly coerced into the later:

```
sage: x^2 + z
z^2 + z
```

Caveat: the registration of the coercion must be done before any other coercion is registered or discovered:

```
sage: phi = Hom(X, Z)(z^2)
sage: phi.register_as_coercion()
Traceback (most recent call last):
...
AssertionError: coercion from Univariate Polynomial Ring in x over Integer␣
↪Ring
to Univariate Polynomial Ring in z over Integer Ring
already registered or discovered
```

register_as_conversion()

Register this morphism as a conversion to Sage's coercion model

(see `sage.structure.coerce`).

EXAMPLES:

Let us declare a conversion from the symmetric group to **Z** through the sign map:

```
sage: # needs sage.groups
sage: S = SymmetricGroup(4)
sage: phi = Hom(S, ZZ)(lambda x: ZZ(x.sign()))
sage: x = S.an_element(); x
(2,3,4)
sage: phi(x)
1
sage: phi.register_as_conversion()
sage: ZZ(x)
1
```

class sage.categories.morphism.**SetMorphism**

Bases: *Morphism*

INPUT:

- `parent` – a Homset
- `function` – a Python function that takes elements of the domain as input and returns elements of the domain.

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(QQ, ZZ, Sets()), numerator)
sage: f.parent()
Set of Morphisms from Rational Field to Integer Ring in Category of sets
sage: f.domain()
Rational Field
sage: f.codomain()
Integer Ring
sage: TestSuite(f).run()
```

sage.categories.morphism.**is_Morphism**(*x*)

3.4 Coercion via construction functors

class sage.categories.pushout.**AlgebraicClosureFunctor**

Bases: `ConstructionFunctor`

Algebraic Closure.

EXAMPLES:

```
sage: # needs sage.rings.complex_double sage.rings.number_field
sage: F = CDF.construction()[0]
sage: F(QQ)
Algebraic Field
sage: F(RR)                                                                  #␣
↪needs sage.rings.real_mpfr
Complex Field with 53 bits of precision
sage: F(F(QQ)) is F(QQ)
True
```

merge(*other*)

Mathematically, Algebraic Closure subsumes Algebraic Extension. However, it seems that people do want to work with algebraic extensions of `RR`. Therefore, we do not merge with algebraic extension.

rank = 3

class sage.categories.pushout.**AlgebraicExtensionFunctor**(*polys*, *names*, *embeddings=None*, *structures=None*, *cyclotomic=None*, *precs=None*, *implementations=None*, ***, *residue=None*, *latex_names=None*, ***kwds*)

Bases: `ConstructionFunctor`

Algebraic extension (univariate polynomial ring modulo principal ideal).

EXAMPLES:

```
sage: x = polygen(QQ, 'x')
sage: K.<a> = NumberField(x^3 + x^2 + 1)                                     #␣
↪needs sage.rings.number_field
sage: F = K.construction()[0]                                                #␣
↪needs sage.rings.number_field
sage: F(ZZ['t'])                                                             #␣
↪needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
 over Univariate Polynomial Ring in t over Integer Ring
 with modulus a^3 + a^2 + 1
```

Note that, even if a field is algebraically closed, the algebraic extension will be constructed as the quotient of a univariate polynomial ring:

```
sage: F(CC)                                                                  #␣
↪needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
 over Complex Field with 53 bits of precision
 with modulus a^3 + a^2 + 1.00000000000000
sage: F(RR)                                                                  #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
 over Real Field with 53 bits of precision
 with modulus a^3 + a^2 + 1.00000000000000
```

Note that the construction functor of a number field applied to the integers returns an order (not necessarily maximal) of that field, similar to the behaviour of `ZZ.extension(...)`:

```
sage: F(ZZ)                                                                  #␣
↪needs sage.rings.number_field
Order generated by a in Number Field in a with defining polynomial x^3 + x^2 + 1
```

This also holds for non-absolute number fields:

```
sage: # needs sage.rings.number_field
sage: x = polygen(QQ, 'x')
sage: K.<a,b> = NumberField([x^3 + x^2 + 1, x^2 + x + 1])
sage: F = K.construction()[0]
sage: O = F(ZZ); O
Relative Order
 generated by [(b - 2)*a^2 + (3*b - 1)*a + 3*b + 4, a - b]
 in Number Field in a with defining polynomial x^3 + x^2 + 1
 over its base field
sage: O.ambient() is K
True
```

Special cases are made for cyclotomic fields and residue fields:

```
sage: # needs sage.rings.number_field
sage: C = CyclotomicField(8)
sage: F, R = C.construction()
sage: F
AlgebraicExtensionFunctor
sage: R
Rational Field
sage: F(R)
Cyclotomic Field of order 8 and degree 4
sage: F(ZZ)
Maximal Order generated by zeta8 in Cyclotomic Field of order 8 and degree 4
```

```
sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(7)
sage: P = K.factor(17)[0][0]
sage: k = K.residue_field(P)
sage: F, R = k.construction()
sage: F
AlgebraicExtensionFunctor
sage: R
Cyclotomic Field of order 7 and degree 6
sage: F(R) is k
True
sage: F(ZZ)
Residue field of Integers modulo 17
sage: F(CyclotomicField(49))
Residue field in zbar of Fractional ideal (17)
```

expand()

Decompose the functor F into sub-functors, whose product returns F.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: P.<x> = QQ[]
sage: K.<a> = NumberField(x^3 - 5, embedding=0)
sage: L.<b> = K.extension(x^2 + a)
sage: F, R = L.construction()
sage: prod(F.expand())(R) == L
True
sage: K = NumberField([x^2 - 2, x^2 - 3],'a')
sage: F, R = K.construction()
sage: F
AlgebraicExtensionFunctor
sage: L = F.expand(); L
[AlgebraicExtensionFunctor, AlgebraicExtensionFunctor]
sage: L[-1](QQ)
Number Field in a1 with defining polynomial x^2 - 3
```

merge(*other*)

Merging with another *AlgebraicExtensionFunctor*.

INPUT:

`other` – Construction Functor.

OUTPUT:

- If `self==other`, `self` is returned.
- If `self` and `other` are simple extensions and both provide an embedding, then it is tested whether one of the number fields provided by the functors coerces into the other; the functor associated with the target of the coercion is returned. Otherwise, the construction functor associated with the pushout of the codomains of the two embeddings is returned, provided that it is a number field.
- If these two extensions are defined by Conway polynomials over finite fields, merges them into a single extension of degree the lcm of the two degrees.
- Otherwise, `None` is returned.

REMARK:

Algebraic extension with embeddings currently only works when applied to the rational field. This is why we use the admittedly strange rule above for merging.

EXAMPLES:

The following demonstrate coercions for finite fields using Conway or pseudo-Conway polynomials:

```
sage: k = GF(3^2, prefix='z'); a = k.gen()                                  #␣
↪needs sage.rings.finite_rings
sage: l = GF(3^3, prefix='z'); b = l.gen()                                  #␣
↪needs sage.rings.finite_rings
sage: a + b   # indirect doctest                                            #␣
↪needs sage.rings.finite_rings
z6^5 + 2*z6^4 + 2*z6^3 + z6^2 + 2*z6 + 1
```

Note that embeddings are compatible in lattices of such finite fields:

```
sage: # needs sage.rings.finite_rings
sage: m = GF(3^5, prefix='z'); c = m.gen()
sage: (a + b) + c == a + (b + c) # indirect doctest
True
sage: from sage.categories.pushout import pushout
sage: n = pushout(k, l)
sage: o = pushout(l, m)
sage: q = pushout(n, o)
sage: q(o(b)) == q(n(b)) # indirect doctest
True
```

Coercion is also available for number fields:

```
sage: # needs sage.rings.number_field
sage: P.<x> = QQ[]
sage: L.<b> = NumberField(x^8 - x^4 + 1, embedding=CDF.0)
sage: M1.<c1> = NumberField(x^2 + x + 1, embedding=b^4 - 1)
sage: M2.<c2> = NumberField(x^2 + 1, embedding=-b^6)
sage: M1.coerce_map_from(M2)
sage: M2.coerce_map_from(M1)
sage: c1 + c2; parent(c1 + c2)     #indirect doctest
-b^6 + b^4 - 1
Number Field in b with defining polynomial x^8 - x^4 + 1
 with b = -0.2588190451025208? + 0.9659258262890683?*I
sage: pushout(M1['x'], M2['x'])                                               #␣
↪needs sage.rings.finite_rings
Univariate Polynomial Ring in x
 over Number Field in b with defining polynomial x^8 - x^4 + 1
  with b = -0.2588190451025208? + 0.9659258262890683?*I
```

In the previous example, the number field `L` becomes the pushout of `M1` and `M2` since both are provided with an embedding into `L`, *and* since `L` is a number field. If two number fields are embedded into a field that is not a numberfield, no merging occurs:

```
sage: # needs sage.rings.complex_double sage.rings.number_field
sage: cbrt2 = CDF(2)^(1/3)
sage: zeta3 = CDF.zeta(3)
sage: K.<a> = NumberField(x^3 - 2, embedding=cbrt2 * zeta3)
sage: L.<b> = NumberField(x^6 - 2, embedding=1.1)
sage: L.coerce_map_from(K)
sage: K.coerce_map_from(L)
sage: pushout(K, L)                                                           #␣
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', Number Field in a with
defining polynomial x^3 - 2 with a = -0.6299605249474365? + 1.091123635971722?
↪*I,
Number Field in b with defining polynomial x^6 - 2 with b = 1.122462048309373?
↪)
```

rank = 3

class sage.categories.pushout.**BlackBoxConstructionFunctor**(*box*)

Bases: `ConstructionFunctor`

Construction functor obtained from any callable object.

EXAMPLES:

```
sage: from sage.categories.pushout import BlackBoxConstructionFunctor
sage: FG = BlackBoxConstructionFunctor(gap)
sage: FS = BlackBoxConstructionFunctor(singular)
sage: FG
BlackBoxConstructionFunctor
sage: FG(ZZ)                                                                    #␣
↪needs sage.libs.gap
Integers
sage: FG(ZZ).parent()                                                           #␣
↪needs sage.libs.gap
Gap
sage: FS(QQ['t'])                                                               #␣
↪needs sage.libs.singular
polynomial ring, over a field, global ordering
//   coefficients: QQ
//   number of vars : 1
//        block   1 : ordering lp
//                  : names    t
//        block   2 : ordering C
sage: FG == FS                                                                  #␣
↪needs sage.libs.gap sage.libs.singular
False
sage: FG == loads(dumps(FG))                                                    #␣
↪needs sage.libs.gap
True
```

rank = 100

class sage.categories.pushout.**CompletionFunctor**(*p*, *prec*, *extras=None*)

Bases: *ConstructionFunctor*

Completion of a ring with respect to a given prime (including infinity).

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: R = Zp(5)
sage: R
5-adic Ring with capped relative precision 20
sage: F1 = R.construction()[0]
sage: F1
Completion[5, prec=20]
sage: F1(ZZ) is R
True
sage: F1(QQ)
5-adic Field with capped relative precision 20

sage: F2 = RR.construction()[0]
sage: F2
Completion[+Infinity, prec=53]
sage: F2(QQ) is RR
True

sage: P.<x> = ZZ[]
sage: Px = P.completion(x) # currently the only implemented completion of P
sage: Px
Power Series Ring in x over Integer Ring
sage: F3 = Px.construction()[0]
```

(continues on next page)

(continued from previous page)

```
sage: F3(GF(3)['x'])
Power Series Ring in x over Finite Field of size 3
```

commutes(*other*)

Completion commutes with fraction fields.

EXAMPLES:

```
sage: F1 = Zp(5).construction()[0]                                          #␣
↪needs sage.rings.padics
sage: F2 = QQ.construction()[0]
sage: F1.commutes(F2)                                                       #␣
↪needs sage.rings.padics
True
```

merge(*other*)

Two Completion functors are merged, if they are equal. If the precisions of both functors coincide, then a Completion functor is returned that results from updating the `extras` dictionary of `self` by `other.extras`. Otherwise, if the completion is at infinity then merging does not increase the set precision, and if the completion is at a finite prime, merging does not decrease the capped precision.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: R1.<a> = Zp(5, prec=20)[]
sage: R2 = Qp(5, prec=40)
sage: R2(1) + a         # indirect doctest
(1 + O(5^20))*a + 1 + O(5^40)
sage: R3 = RealField(30)
sage: R4 = RealField(50)
sage: R3(1) + R4(1)     # indirect doctest
2.0000000
sage: (R3(1) + R4(1)).parent()
Real Field with 30 bits of precision
```

rank = 4

class sage.categories.pushout.**CompositeConstructionFunctor**(**args*)

Bases: `ConstructionFunctor`

A Construction Functor composed by other Construction Functors.

INPUT:

`F1, F2,...`: A list of Construction Functors. The result is the composition `F1` followed by `F2` followed by ...

EXAMPLES:

```
sage: from sage.categories.pushout import CompositeConstructionFunctor
sage: F = CompositeConstructionFunctor(QQ.construction()[0], ZZ['x'].
↪construction()[0],
....:                                  QQ.construction()[0], ZZ['y'].
↪construction()[0])
sage: F
Poly[y](FractionField(Poly[x](FractionField(...))))
sage: F == loads(dumps(F))
True
```

(continues on next page)

(continued from previous page)

```
sage: F == CompositeConstructionFunctor(*F.all)
True
sage: F(GF(2)['t'])                                                      #␣
↪needs sage.libs.ntl
Univariate Polynomial Ring in y
 over Fraction Field of Univariate Polynomial Ring in x
  over Fraction Field of Univariate Polynomial Ring in t
   over Finite Field of size 2 (using GF2X)
```

expand()

Return expansion of a CompositeConstructionFunctor.

Note: The product over the list of components, as returned by the `expand()` method, is equal to `self`.

EXAMPLES:

```
sage: from sage.categories.pushout import CompositeConstructionFunctor
sage: F = CompositeConstructionFunctor(QQ.construction()[0],
....:                                  ZZ['x'].construction()[0],
....:                                  QQ.construction()[0],
....:                                  ZZ['y'].construction()[0])
sage: F
Poly[y](FractionField(Poly[x](FractionField(...))))
sage: prod(F.expand()) == F
True
```

class sage.categories.pushout.**ConstructionFunctor**

Bases: *Functor*

Base class for construction functors.

A construction functor is a functorial algebraic construction, such as the construction of a matrix ring over a given ring or the fraction field of a given ring.

In addition to the class *Functor*, construction functors provide rules for combining and merging constructions. This is an important part of Sage's coercion model, namely the pushout of two constructions: When a polynomial `p` in a variable `x` with integer coefficients is added to a rational number `q`, then Sage finds that the parents `ZZ['x']` and `QQ` are obtained from `ZZ` by applying a polynomial ring construction respectively the fraction field construction. Each construction functor has an attribute `rank`, and the rank of the polynomial ring construction is higher than the rank of the fraction field construction. This means that the pushout of `QQ` and `ZZ['x']`, and thus a common parent in which `p` and `q` can be added, is `QQ['x']`, since the construction functor with a lower rank is applied first.

```
sage: F1, R = QQ.construction()
sage: F1
FractionField
sage: R
Integer Ring
sage: F2, R = (ZZ['x']).construction()
sage: F2
Poly[x]
sage: R
Integer Ring
sage: F3 = F2.pushout(F1)
sage: F3
```

(continues on next page)

(continued from previous page)

```
Poly[x](FractionField(...))
sage: F3(R)
Univariate Polynomial Ring in x over Rational Field
sage: from sage.categories.pushout import pushout
sage: P.<x> = ZZ[]
sage: pushout(QQ,P)
Univariate Polynomial Ring in x over Rational Field
sage: ((x+1) + 1/2).parent()
Univariate Polynomial Ring in x over Rational Field
```

When composing two construction functors, they are sometimes merged into one, as is the case in the Quotient construction:

```
sage: Q15, R = (ZZ.quo(15*ZZ)).construction()
sage: Q15
QuotientFunctor
sage: Q35, R = (ZZ.quo(35*ZZ)).construction()
sage: Q35
QuotientFunctor
sage: Q15.merge(Q35)
QuotientFunctor
sage: Q15.merge(Q35)(ZZ)
Ring of integers modulo 5
```

Functors can not only be applied to objects, but also to morphisms in the respective categories. For example:

```
sage: P.<x,y> = ZZ[]
sage: F = P.construction()[0]; F
MPoly[x,y]
sage: A.<a,b> = GF(5)[]
sage: f = A.hom([a + b, a - b], A)
sage: F(A)
Multivariate Polynomial Ring in x, y
 over Multivariate Polynomial Ring in a, b over Finite Field of size 5
sage: F(f)
Ring endomorphism of Multivariate Polynomial Ring in x, y
 over Multivariate Polynomial Ring in a, b over Finite Field of size 5
  Defn: Induced from base ring by
        Ring endomorphism of Multivariate Polynomial Ring in a, b
         over Finite Field of size 5
          Defn: a |--> a + b
                b |--> a - b
sage: F(f)(F(A)(x)*a)
(a + b)*x
```

coercion_reversed = False

common_base(*other_functor*, *self_bases*, *other_bases*)

This function is called by `pushout()` when no common parent is found in the construction tower.

Note: The main use is for multivariate construction functors, which use this function to implement recursion for `pushout()`.

INPUT:

- `other_functor` – a construction functor.

- `self_bases` – the arguments passed to this functor.
- `other_bases` – the arguments passed to the functor `other_functor`.

OUTPUT:

Nothing, since a `CoercionException` is raised.

Note: Overload this function in derived class, see e.e. *`MultivariateConstructionFunctor`*.

commutes(*other*)

Determine whether `self` commutes with another construction functor.

Note: By default, `False` is returned in all cases (even if the two functors are the same, since in this case *`merge()`* will apply anyway). So far there is no construction functor that overloads this method. Anyway, this method only becomes relevant if two construction functors have the same rank.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: P = ZZ['t'].construction()[0]
sage: F.commutes(P)
False
sage: P.commutes(F)
False
sage: F.commutes(F)
False
```

expand()

Decompose `self` into a list of construction functors.

Note: The default is to return the list only containing `self`.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: F.expand()
[FractionField]
sage: Q = ZZ.quo(2).construction()[0]
sage: Q.expand()
[QuotientFunctor]
sage: P = ZZ['t'].construction()[0]
sage: FP = F*P
sage: FP.expand()
[FractionField, Poly[t]]
```

merge(*other*)

Merge `self` with another construction functor, or return `None`.

Note: The default is to merge only if the two functors coincide. But this may be overloaded for subclasses, such as the quotient functor.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: P = ZZ['t'].construction()[0]
sage: F.merge(F)
FractionField
sage: F.merge(P)
sage: P.merge(F)
sage: P.merge(P)
Poly[t]
```

pushout (*other*)

Composition of two construction functors, ordered by their ranks.

Note:

- This method seems not to be used in the coercion model.
- By default, the functor with smaller rank is applied first.

class `sage.categories.pushout.`**EquivariantSubobjectConstructionFunctor** (*S*, *action=<built-in function mul>*, *side='left'*, *other_action=None*, *other_side='left'*)

Bases: `ConstructionFunctor`

Constructor for subobjects invariant or equivariant under given semigroup actions.

Let S be a semigroup that - acts on a parent X as $s \cdot x$ (`action`, `side='left'`) or - acts on X as $x \cdot s$ (`action`, `side='right'`), and (possibly trivially) - acts on X as $s * x$ (`other_action`, `other_side='left'`) or - acts on X as $x * s$ (`other_action`, `other_side='right'`).

The S-equivariant subobject is the subobject

$$X^S := \{x \in X : s \cdot x = s * x, \forall s \in S\}$$

when `side = other_side = 'left'` and mutatis mutandis for the other values of `side` and `other_side`.

When `other_action` is trivial, X^S is called the S-invariant subobject.

EXAMPLES:

Monoterm symmetries of a tensor, here only for matrices: row (index 0), column (index 1); the order of the extra element 2 in a permutation determines whether it is a symmetry or an antisymmetry:

```
sage: # needs sage.groups sage.modules
sage: GSym01 = PermutationGroup([[(0,1),(2,),(3,)]]); GSym01
Permutation Group with generators [(0,1)]
sage: GASym01 = PermutationGroup([[(0,1),(2,3)]]); GASym01
Permutation Group with generators [(0,1)(2,3)]
sage: from sage.categories.action import Action
sage: from sage.structure.element import Matrix
sage: class TensorIndexAction(Action):
....:     def _act_(self, g, x):
```

(continues on next page)

(continued from previous page)

```
....:         if isinstance(x, Matrix):
....:             if g(0) == 1:
....:                 if g(2) == 2:
....:                     return x.transpose()
....:                 else:
....:                     return -x.transpose()
....:             else:
....:                 return x
....:         raise NotImplementedError
sage: M = matrix([[1, 2], [3, 4]]); M
[1 2]
[3 4]
sage: GSym01_action = TensorIndexAction(GSym01, M.parent())
sage: GASym01_action = TensorIndexAction(GASym01, M.parent())
sage: GSym01_action.act(GSym01.0, M)
[1 3]
[2 4]
sage: GASym01_action.act(GASym01.0, M)
[-1 -3]
[-2 -4]
sage: Sym01 = M.parent().invariant_module(GSym01, action=GSym01_action); Sym01
(Permutation Group with generators [(0,1)])-invariant submodule
 of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: list(Sym01.basis())
[B[0], B[1], B[2]]
sage: list(Sym01.basis().map(Sym01.lift))
[
[1 0]  [0 1]  [0 0]
[0 0], [1 0], [0 1]
]
sage: ASym01 = M.parent().invariant_module(GASym01, action=GASym01_action)
sage: ASym01
(Permutation Group with generators [(0,1)(2,3)])-invariant submodule
 of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: list(ASym01.basis())
[B[0]]
sage: list(ASym01.basis().map(ASym01.lift))
[
[ 0  1]
[-1  0]
]
sage: from sage.categories.pushout import pushout
sage: pushout(Sym01, QQ)
(Permutation Group with generators [(0,1)])-invariant submodule
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

class sage.categories.pushout.**FractionField**

Bases: *ConstructionFunctor*

Construction functor for fraction fields.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: F
FractionField
sage: F.domain()
```

(continues on next page)

(continued from previous page)

```
Category of integral domains
sage: F.codomain()
Category of fields
sage: F(GF(5)) is GF(5)
True
sage: F(ZZ['t'])
Fraction Field of Univariate Polynomial Ring in t over Integer Ring
sage: P.<x,y> = QQ[]
sage: f = P.hom([x+2*y,3*x-y],P)
sage: F(f)
Ring endomorphism of
 Fraction Field of Multivariate Polynomial Ring in x, y over Rational Field
  Defn: x |--> x + 2*y
        y |--> 3*x - y
sage: F(f)(1/x)
1/(x + 2*y)
sage: F == loads(dumps(F))
True
```

rank = 5

class sage.categories.pushout.**IdentityConstructionFunctor**

Bases: *ConstructionFunctor*

A construction functor that is the identity functor.

rank = -100

class sage.categories.pushout.**InfinitePolynomialFunctor**(*gens*, *order*, *implementation*)

Bases: *ConstructionFunctor*

A Construction Functor for Infinite Polynomial Rings (see `infinite_polynomial_ring`).

AUTHOR:

– Simon King

This construction functor is used to provide uniqueness of infinite polynomial rings as parent structures. As usual, the construction functor allows for constructing pushouts.

Another purpose is to avoid name conflicts of variables of the to-be-constructed infinite polynomial ring with variables of the base ring, and moreover to keep the internal structure of an Infinite Polynomial Ring as simple as possible: If variables $v_1, ..., v_n$ of the given base ring generate an *ordered* sub-monoid of the monomials of the ambient Infinite Polynomial Ring, then they are removed from the base ring and merged with the generators of the ambient ring. However, if the orders don't match, an error is raised, since there was a name conflict without merging.

EXAMPLES:

```
sage: A.<a,b> = InfinitePolynomialRing(ZZ['t'])
sage: A.construction()
[InfPoly{[a,b], "lex", "dense"},
 Univariate Polynomial Ring in t over Integer Ring]
sage: type(_[0])
<class 'sage.categories.pushout.InfinitePolynomialFunctor'>
sage: B.<x,y,a_3,a_1> = PolynomialRing(QQ, order='lex')
sage: B.construction()
(MPoly[x,y,a_3,a_1], Rational Field)
```

(continues on next page)

(continued from previous page)

```
sage: A.construction()[0] * B.construction()[0]
InfPoly{[a,b], "lex", "dense"}(MPoly[x,y](...))
```

Apparently the variables a_1, a_3 of the polynomial ring are merged with the variables $a_0, a_1, a_2, ...$ of the infinite polynomial ring; indeed, they form an ordered sub-structure. However, if the polynomial ring was given a different ordering, merging would not be allowed, resulting in a name conflict:

```
sage: R = PolynomialRing(QQ, names=['x','y','a_3','a_1'])
sage: A.construction()[0] * R.construction()[0]
Traceback (most recent call last):
...
CoercionException: Incompatible term orders lex, degrevlex
```

In an infinite polynomial ring with generator a_*, the variable a_3 will always be greater than the variable a_1. Hence, the orders are incompatible in the next example as well:

```
sage: R = PolynomialRing(QQ, names=['x','y','a_1','a_3'], order='lex')
sage: A.construction()[0] * R.construction()[0]
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('a', 'b'),['a_1', 'a_3'])
are incompatible
```

Another requirement is that after merging the order of the remaining variables must be unique. This is not the case in the following example, since it is not clear whether the variables x, y should be greater or smaller than the variables b_*:

```
sage: R = PolynomialRing(QQ, names=['a_3','a_1','x','y'], order='lex')
sage: A.construction()[0] * R.construction()[0]
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('a', 'b'),['a_3', 'a_1'])
are incompatible
```

Since the construction functors are actually used to construct infinite polynomial rings, the following result is no surprise:

```
sage: C.<a,b> = InfinitePolynomialRing(B); C
Infinite polynomial ring in a, b
 over Multivariate Polynomial Ring in x, y over Rational Field
```

There is also an overlap in the next example:

```
sage: X.<w,x,y> = InfinitePolynomialRing(ZZ)
sage: Y.<x,y,z> = InfinitePolynomialRing(QQ)
```

X and Y have an overlapping generators x_*, y_*. Since the default lexicographic order is used in both rings, it gives rise to isomorphic sub-monoids in both X and Y. They are merged in the pushout, which also yields a common parent for doing arithmetic:

```
sage: P = sage.categories.pushout.pushout(Y,X); P
Infinite polynomial ring in w, x, y, z over Rational Field
sage: w[2]+z[3]
w_2 + z_3
sage: _.parent() is P
True
```

expand()

Decompose the functor F into sub-functors, whose product returns F.

EXAMPLES:

```
sage: A = InfinitePolynomialRing(QQ, ['x','y'], order='degrevlex')
sage: F = A.construction()[0]; F
InfPoly{[x,y], "degrevlex", "dense"}
sage: F.expand()
[InfPoly{[y], "degrevlex", "dense"}, InfPoly{[x], "degrevlex", "dense"}]
sage: A = InfinitePolynomialRing(QQ, ['x','y','z'], order='degrevlex')
sage: F = A.construction()[0]; F
InfPoly{[x,y,z], "degrevlex", "dense"}
sage: F.expand()
[InfPoly{[z], "degrevlex", "dense"},
 InfPoly{[y], "degrevlex", "dense"},
 InfPoly{[x], "degrevlex", "dense"}]
sage: prod(F.expand())==F
True
```

merge(*other*)

Merge two construction functors of infinite polynomial rings, regardless of monomial order and implementation.

The purpose is to have a pushout (and thus, arithmetic) even in cases when the parents are isomorphic as rings, but not as ordered rings.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: Y.<x,y> = InfinitePolynomialRing(QQ, order='degrevlex')
sage: X.construction()
[InfPoly{[x,y], "lex", "sparse"}, Rational Field]
sage: Y.construction()
[InfPoly{[x,y], "degrevlex", "dense"}, Rational Field]
sage: Y.construction()[0].merge(Y.construction()[0])
InfPoly{[x,y], "degrevlex", "dense"}
sage: y[3] + X(x[2])
x_2 + y_3
sage: _.parent().construction()
[InfPoly{[x,y], "degrevlex", "dense"}, Rational Field]
```

rank = 9.5

class sage.categories.pushout.**LaurentPolynomialFunctor**(*var*, *multi_variate=False*)

Bases: *ConstructionFunctor*

Construction functor for Laurent polynomial rings.

EXAMPLES:

```
sage: L.<t> = LaurentPolynomialRing(ZZ)
sage: F = L.construction()[0]
sage: F
LaurentPolynomialFunctor
sage: F(QQ)
Univariate Laurent Polynomial Ring in t over Rational Field
sage: K.<x> = LaurentPolynomialRing(ZZ)
sage: F(K)
```

(continues on next page)

(continued from previous page)

```
Univariate Laurent Polynomial Ring in t
 over Univariate Laurent Polynomial Ring in x over Integer Ring
sage: P.<x,y> = ZZ[]
sage: f = P.hom([x + 2*y, 3*x - y],P)
sage: F(f)
Ring endomorphism of Univariate Laurent Polynomial Ring in t
 over Multivariate Polynomial Ring in x, y over Integer Ring
  Defn: Induced from base ring by
        Ring endomorphism of Multivariate Polynomial Ring in x, y over Integer␣
↪Ring
          Defn: x |--> x + 2*y
                y |--> 3*x - y
sage: F(f)(x*F(P).gen()^-2 + y*F(P).gen()^3)
(x + 2*y)*t^-2 + (3*x - y)*t^3
```

merge(*other*)

Two Laurent polynomial construction functors merge if the variable names coincide.

The result is multivariate if one of the arguments is multivariate.

EXAMPLES:

```
sage: from sage.categories.pushout import LaurentPolynomialFunctor
sage: F1 = LaurentPolynomialFunctor('t')
sage: F2 = LaurentPolynomialFunctor('t', multi_variate=True)
sage: F1.merge(F2)
LaurentPolynomialFunctor
sage: F1.merge(F2)(LaurentPolynomialRing(GF(2), 'a'))                        #␣
↪needs sage.modules
Multivariate Laurent Polynomial Ring in a, t over Finite Field of size 2
sage: F1.merge(F1)(LaurentPolynomialRing(GF(2), 'a'))                        #␣
↪needs sage.modules
Univariate Laurent Polynomial Ring in t over
 Univariate Laurent Polynomial Ring in a over Finite Field of size 2
```

rank = 9

class sage.categories.pushout.**MatrixFunctor**(*nrows*, *ncols*, *is_sparse=False*)

Bases: *ConstructionFunctor*

A construction functor for matrices over rings.

EXAMPLES:

```
sage: # needs sage.modules
sage: MS = MatrixSpace(ZZ, 2, 3)
sage: F = MS.construction()[0]; F
MatrixFunctor
sage: MS = MatrixSpace(ZZ, 2)
sage: F = MS.construction()[0]; F
MatrixFunctor
sage: P.<x,y> = QQ[]
sage: R = F(P); R
Full MatrixSpace of 2 by 2 dense matrices
 over Multivariate Polynomial Ring in x, y over Rational Field
sage: f = P.hom([x+y, x-y], P); F(f)
Ring endomorphism
```

(continues on next page)

(continued from previous page)

```
 of Full MatrixSpace of 2 by 2 dense matrices
  over Multivariate Polynomial Ring in x, y over Rational Field
  Defn: Induced from base ring by
        Ring endomorphism
         of Multivariate Polynomial Ring in x, y over Rational Field
          Defn: x |--> x + y
                y |--> x - y
sage: M = R([x, y, x*y, x + y])
sage: F(f)(M)
[    x + y      x - y]
[x^2 - y^2        2*x]
```

merge (*other*)

Merging is only happening if both functors are matrix functors of the same dimension.

The result is sparse if and only if both given functors are sparse.

EXAMPLES:

```
sage: # needs sage.modules
sage: F1 = MatrixSpace(ZZ, 2, 2).construction()[0]
sage: F2 = MatrixSpace(ZZ, 2, 3).construction()[0]
sage: F3 = MatrixSpace(ZZ, 2, 2, sparse=True).construction()[0]
sage: F1.merge(F2)
sage: F1.merge(F3)
MatrixFunctor
sage: F13 = F1.merge(F3)
sage: F13.is_sparse
False
sage: F1.is_sparse
False
sage: F3.is_sparse
True
sage: F3.merge(F3).is_sparse
True
```

rank = 10

class sage.categories.pushout.**MultiPolynomialFunctor** (*vars*, *term_order*)

Bases: *ConstructionFunctor*

A constructor for multivariate polynomial rings.

EXAMPLES:

```
sage: P.<x,y> = ZZ[]
sage: F = P.construction()[0]; F
MPoly[x,y]
sage: A.<a,b> = GF(5)[]
sage: F(A)
Multivariate Polynomial Ring in x, y
 over Multivariate Polynomial Ring in a, b over Finite Field of size 5
sage: f = A.hom([a+b, a-b], A)
sage: F(f)
Ring endomorphism of Multivariate Polynomial Ring in x, y
 over Multivariate Polynomial Ring in a, b over Finite Field of size 5
  Defn: Induced from base ring by
```

(continues on next page)

(continued from previous page)

```
        Ring endomorphism of Multivariate Polynomial Ring in a, b over Finite␣
↪Field of size 5
          Defn: a |--> a + b
                b |--> a - b
sage: F(f)(F(A)(x)*a)
(a + b)*x
```

expand()

Decompose `self` into a list of construction functors.

EXAMPLES:

```
sage: F = QQ['x,y,z,t'].construction()[0]; F
MPoly[x,y,z,t]
sage: F.expand()
[MPoly[t], MPoly[z], MPoly[y], MPoly[x]]
```

Now an actual use case:

```
sage: R.<x,y,z> = ZZ[]
sage: S.<z,t> = QQ[]
sage: x+t
x + t
sage: parent(x+t)
Multivariate Polynomial Ring in x, y, z, t over Rational Field
sage: T.<y,s> = QQ[]
sage: x + s
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +:
'Multivariate Polynomial Ring in x, y, z over Integer Ring' and
'Multivariate Polynomial Ring in y, s over Rational Field'
sage: R = PolynomialRing(ZZ, 'x', 50)
sage: S = PolynomialRing(GF(5), 'x', 20)
sage: R.gen(0) + S.gen(0)
2*x0
```

merge(*other*)

Merge `self` with another construction functor, or return `None`.

EXAMPLES:

```
sage: F = sage.categories.pushout.MultiPolynomialFunctor(['x','y'], None)
sage: G = sage.categories.pushout.MultiPolynomialFunctor(['t'], None)
sage: F.merge(G) is None
True
sage: F.merge(F)
MPoly[x,y]
```

rank = 9

class sage.categories.pushout.**MultivariateConstructionFunctor**

Bases: *ConstructionFunctor*

An abstract base class for functors that take multiple inputs (e.g. Cartesian products).

common_base(*other_functor*, *self_bases*, *other_bases*)

This function is called by *pushout()* when no common parent is found in the construction tower.

INPUT:

- `other_functor` – a construction functor.
- `self_bases` – the arguments passed to this functor.
- `other_bases` – the arguments passed to the functor `other_functor`.

OUTPUT:

A parent.

If no common base is found a `sage.structure.coerce_exceptions.CoercionException` is raised.

Note: Overload this function in derived class, see e.g. *MultivariateConstructionFunctor*.

class sage.categories.pushout.**PermutationGroupFunctor**(*gens*, *domain*)

Bases: *ConstructionFunctor*

EXAMPLES:

```
sage: from sage.categories.pushout import PermutationGroupFunctor
sage: PF = PermutationGroupFunctor([PermutationGroupElement([(1,2)])],      #␣
↪needs sage.groups
....:                              [1,2]); PF
PermutationGroupFunctor[(1,2)]
```

gens()

EXAMPLES:

```
sage: P1 = PermutationGroup([[(1,2)]])                                      #␣
↪needs sage.groups
sage: PF, P = P1.construction()                                             #␣
↪needs sage.groups
sage: PF.gens()                                                             #␣
↪needs sage.groups
((1,2),)
```

merge(*other*)

Merge `self` with another construction functor, or return `None`.

EXAMPLES:

```
sage: # needs sage.groups
sage: P1 = PermutationGroup([[(1,2)]])
sage: PF1, P = P1.construction()
sage: P2 = PermutationGroup([[(1,3)]])
sage: PF2, P = P2.construction()
sage: PF1.merge(PF2)
PermutationGroupFunctor[(1,2), (1,3)]
```

rank = 10

class sage.categories.pushout.**PolynomialFunctor**(*var*, *multi_variate=False*, *sparse=False*, *implementation=None*)

Bases: `ConstructionFunctor`

Construction functor for univariate polynomial rings.

EXAMPLES:

```
sage: P = ZZ['t'].construction()[0]
sage: P(GF(3))
Univariate Polynomial Ring in t over Finite Field of size 3
sage: P == loads(dumps(P))
True
sage: R.<x,y> = GF(5)[]
sage: f = R.hom([x + 2*y, 3*x - y], R)
sage: P(f)((x+y) * P(R).0)
(-x + y)*t
```

By github issue #9944, the construction functor distinguishes sparse and dense polynomial rings. Before, the following example failed:

```
sage: R.<x> = PolynomialRing(GF(5), sparse=True)
sage: F, B = R.construction()
sage: F(B) is R
True
sage: S.<x> = PolynomialRing(ZZ)
sage: R.has_coerce_map_from(S)
False
sage: S.has_coerce_map_from(R)
False
sage: S.0 + R.0
2*x
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Finite Field of size 5
sage: (S.0 + R.0).parent().is_sparse()
False
```

merge(*other*)

Merge `self` with another construction functor, or return `None`.

Note: Internally, the merging is delegated to the merging of multipolynomial construction functors. But in effect, this does the same as the default implementation, that returns `None` unless the to-be-merged functors coincide.

EXAMPLES:

```
sage: P = ZZ['x'].construction()[0]
sage: Q = ZZ['y','x'].construction()[0]
sage: P.merge(Q)
sage: P.merge(P) is P
True
```

rank = 9

class sage.categories.pushout.**QuotientFunctor**(*I*, *names=None*, *as_field=False*, *domain=None*, *codomain=None*, ***kwds*)

Bases: `ConstructionFunctor`

Construction functor for quotient rings.

Note: The functor keeps track of variable names. Optionally, it may keep track of additional properties of the quotient, such as its category or its implementation.

EXAMPLES:

```
sage: P.<x,y> = ZZ[]
sage: Q = P.quo([x^2 + y^2] * P)
sage: F = Q.construction()[0]
sage: F(QQ['x','y'])
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
 by the ideal (x^2 + y^2)
sage: F(QQ['x','y']) == QQ['x','y'].quo([x^2 + y^2] * QQ['x','y'])
True
sage: F(QQ['x','y','z'])
Traceback (most recent call last):
...
CoercionException: Cannot apply this quotient functor to
 Multivariate Polynomial Ring in x, y, z over Rational Field
sage: F(QQ['y','z'])                                                    #␣
↪needs sage.rings.finite_rings
Traceback (most recent call last):
...
TypeError: Could not find a mapping of the passed element to this ring.
```

merge(*other*)

Two quotient functors with coinciding names are merged by taking the gcd of their moduli, the meet of their domains, and the join of their codomains.

In particular, if one of the functors being merged knows that the quotient is going to be a field, then the merged functor will return fields as well.

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: P.<x> = QQ[]
sage: Q1 = P.quo([(x^2+1)^2*(x^2-3)])
sage: Q2 = P.quo([(x^2+1)^2*(x^5+3)])
sage: from sage.categories.pushout import pushout
sage: pushout(Q1,Q2)     # indirect doctest
Univariate Quotient Polynomial Ring in xbar over Rational Field
 with modulus x^4 + 2*x^2 + 1
```

The following was fixed in github issue #8800:

```
sage: pushout(GF(5), Integers(5))                                       #␣
↪needs sage.libs.pari
Finite Field of size 5
```

rank = 4.5

class sage.categories.pushout.**SubspaceFunctor**(*basis*)

Bases: *ConstructionFunctor*

Constructing a subspace of an ambient free module, given by a basis.

Note: This construction functor keeps track of the basis. It can only be applied to free modules into which this basis coerces.

EXAMPLES:

```
sage: # needs sage.modules
sage: M = ZZ^3
sage: S = M.submodule([(1,2,3), (4,5,6)]); S
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
sage: F = S.construction()[0]
sage: F(GF(2)^3)
Vector space of degree 3 and dimension 2 over Finite Field of size 2
User basis matrix:
[1 0 1]
[0 1 0]
```

coercion_reversed = True

merge(*other*)

Two Subspace Functors are merged into a construction functor of the sum of two subspaces.

EXAMPLES:

```
sage: # needs sage.modules
sage: M = GF(5)^3
sage: S1 = M.submodule([(1,2,3),(4,5,6)])
sage: S2 = M.submodule([(2,2,3)])
sage: F1 = S1.construction()[0]
sage: F2 = S2.construction()[0]
sage: F1.merge(F2)
SubspaceFunctor
sage: F1.merge(F2)(GF(5)^3) == S1 + S2
True
sage: F1.merge(F2)(GF(5)['t']^3)
Free module of degree 3 and rank 3
 over Univariate Polynomial Ring in t over Finite Field of size 5
User basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]
```

rank = 11

class sage.categories.pushout.**VectorFunctor**(*n=None*, *is_sparse=False*, *inner_product_matrix=None*, ***, *with_basis='standard'*, *basis_keys=None*, *name_mapping=None*, *latex_name_mapping=None*)

Bases: `ConstructionFunctor`

A construction functor for free modules over commutative rings.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = (ZZ^3).construction()[0]
sage: F
VectorFunctor
sage: F(GF(2)['t'])                                                         #␣
↪needs sage.libs.ntl
Ambient free module of rank 3
 over the principal ideal domain Univariate Polynomial Ring in t
  over Finite Field of size 2 (using GF2X)
```

merge(*other*)

Two constructors of free modules merge, if the module ranks and the inner products coincide. If both have explicitly given inner product matrices, they must coincide as well.

EXAMPLES:

Two modules without explicitly given inner product allow coercion:

```
sage: M1 = QQ^3                                                         #␣
↪needs sage.modules
sage: P.<t> = ZZ[]
sage: M2 = FreeModule(P, 3)                                             #␣
↪needs sage.modules
sage: M1([1,1/2,1/3]) + M2([t,t^2+t,3])     # indirect doctest          #␣
↪needs sage.modules
(t + 1, t^2 + t + 1/2, 10/3)
```

If only one summand has an explicit inner product, the result will be provided with it:

```
sage: M3 = FreeModule(P, 3, inner_product_matrix=Matrix(3, 3, range(9)))    #␣
↪needs sage.modules
sage: M1([1,1/2,1/3]) + M3([t,t^2+t,3])                                     #␣
↪needs sage.modules
(t + 1, t^2 + t + 1/2, 10/3)
sage: (M1([1,1/2,1/3]) + M3([t,t^2+t,3])).parent().inner_product_matrix()   #␣
↪needs sage.modules
[0 1 2]
[3 4 5]
[6 7 8]
```

If both summands have an explicit inner product (even if it is the standard inner product), then the products must coincide. The only difference between `M1` and `M4` in the following example is the fact that the default inner product was *explicitly* requested for `M4`. It is therefore not possible to coerce with a different inner product:

```
sage: # needs sage.modules
sage: M4 = FreeModule(QQ, 3, inner_product_matrix=Matrix(3, 3, 1))
sage: M4 == M1
True
sage: M4.inner_product_matrix() == M1.inner_product_matrix()
True
sage: M4([1,1/2,1/3]) + M3([t,t^2+t,3])      # indirect doctest
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +:
'Ambient quadratic space of dimension 3 over Rational Field
Inner product matrix:
```

(continues on next page)

(continued from previous page)

```
[1 0 0]
[0 1 0]
[0 0 1]' and
'Ambient free quadratic module of rank 3 over the integral domain
Univariate Polynomial Ring in t over Integer Ring
Inner product matrix:
[0 1 2]
[3 4 5]
[6 7 8]'
```

Names are removed when they conflict:

```
sage: # needs sage.modules
sage: from sage.categories.pushout import VectorFunctor, pushout
sage: M_ZZx = FreeModule(ZZ['x'], 4, with_basis=None, name='M_ZZx')
sage: N_ZZx = FreeModule(ZZ['x'], 4, with_basis=None, name='N_ZZx')
sage: pushout(M_ZZx, QQ)
Rank-4 free module M_ZZx_base_ext
 over the Univariate Polynomial Ring in x over Rational Field
sage: pushout(M_ZZx, N_ZZx)
Rank-4 free module
 over the Univariate Polynomial Ring in x over Integer Ring
sage: pushout(pushout(M_ZZx, N_ZZx), QQ)
Rank-4 free module
 over the Univariate Polynomial Ring in x over Rational Field
```

rank = 10

sage.categories.pushout.**construction_tower**(*R*)

An auxiliary function that is used in `pushout()` and `pushout_lattice()`.

INPUT:

An object

OUTPUT:

A constructive description of the object from scratch, by a list of pairs of a construction functor and an object to which the construction functor is to be applied. The first pair is formed by `None` and the given object.

EXAMPLES:

```
sage: from sage.categories.pushout import construction_tower
sage: construction_tower(MatrixSpace(FractionField(QQ['t']), 2))            #␣
↪needs sage.modules
[(None, Full MatrixSpace of 2 by 2 dense matrices over Fraction Field
         of Univariate Polynomial Ring in t over Rational Field),
 (MatrixFunctor, Fraction Field
                  of Univariate Polynomial Ring in t over Rational Field),
 (FractionField, Univariate Polynomial Ring in t over Rational Field),
 (Poly[t], Rational Field), (FractionField, Integer Ring)]
```

sage.categories.pushout.**expand_tower**(*tower*)

An auxiliary function that is used in `pushout()`.

INPUT:

A construction tower as returned by `construction_tower()`.

OUTPUT:

A new construction tower with all the construction functors expanded.

EXAMPLES:

```
sage: from sage.categories.pushout import construction_tower, expand_tower
sage: construction_tower(QQ['x,y,z'])
[(None, Multivariate Polynomial Ring in x, y, z over Rational Field),
 (MPoly[x,y,z], Rational Field),
 (FractionField, Integer Ring)]
sage: expand_tower(construction_tower(QQ['x,y,z']))
[(None, Multivariate Polynomial Ring in x, y, z over Rational Field),
 (MPoly[z], Univariate Polynomial Ring in y
             over Univariate Polynomial Ring in x over Rational Field),
 (MPoly[y], Univariate Polynomial Ring in x over Rational Field),
 (MPoly[x], Rational Field),
 (FractionField, Integer Ring)]
```

sage.categories.pushout.**pushout**(R, S)

Given a pair of objects R and S, try to construct a reasonable object Y and return maps such that canonically $R \leftarrow Y \rightarrow S$.

ALGORITHM:

This incorporates the idea of functors discussed at Sage Days 4. Every object R can be viewed as an initial object and a series of functors (e.g. polynomial, quotient, extension, completion, vector/matrix, etc.). Call the series of increasingly simple objects (with the associated functors) the "tower" of R. The construction method is used to create the tower.

Given two objects R and S, try to find a common initial object Z. If the towers of R and S meet, let Z be their join. Otherwise, see if the top of one coerces naturally into the other.

Now we have an initial object and two ordered lists of functors to apply. We wish to merge these in an unambiguous order, popping elements off the top of one or the other tower as we apply them to Z.

- If the functors are of distinct types, there is an absolute ordering given by the rank attribute. Use this.
- Otherwise:
  - If the tops are equal, we (try to) merge them.
  - If exactly one occurs lower in the other tower, we may unambiguously apply the other (hoping for a later merge).
  - If the tops commute, we can apply either first.
  - Otherwise fail due to ambiguity.

The algorithm assumes by default that when a construction F is applied to an object X, the object $F(X)$ admits a coercion map from X. However, the algorithm can also handle the case where $F(X)$ has a coercion map *to* X instead. In this case, the attribute `coercion_reversed` of the class implementing F should be set to `True`.

EXAMPLES:

Here our "towers" are $R = Complete_7(Frac(\mathbf{Z}))$ and $Frac(Poly_x(\mathbf{Z}))$, which give us $Frac(Poly_x(Complete_7(Frac(\mathbf{Z}))))$:

```
sage: from sage.categories.pushout import pushout
sage: pushout(Qp(7), Frac(ZZ['x']))                                     #␣
↪needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
 over 7-adic Field with capped relative precision 20
```

Note we get the same thing with

```
sage: pushout(Zp(7), Frac(QQ['x']))                                    #␣
↪needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
 over 7-adic Field with capped relative precision 20
sage: pushout(Zp(7)['x'], Frac(QQ['x']))                               #␣
↪needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
 over 7-adic Field with capped relative precision 20
```

Note that polynomial variable ordering must be unambiguously determined.

```
sage: pushout(ZZ['x,y,z'], QQ['w,z,t'])
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension',
Multivariate Polynomial Ring in x, y, z over Integer Ring,
Multivariate Polynomial Ring in w, z, t over Rational Field)
sage: pushout(ZZ['x,y,z'], QQ['w,x,z,t'])
Multivariate Polynomial Ring in w, x, y, z, t over Rational Field
```

Some other examples:

```
sage: pushout(Zp(7)['y'], Frac(QQ['t'])['x,y,z'])                      #␣
↪needs sage.rings.padics
Multivariate Polynomial Ring in x, y, z
 over Fraction Field of Univariate Polynomial Ring in t
  over 7-adic Field with capped relative precision 20
sage: pushout(ZZ['x,y,z'], Frac(ZZ['x'])['y'])
Multivariate Polynomial Ring in y, z
 over Fraction Field of Univariate Polynomial Ring in x over Integer Ring
sage: pushout(MatrixSpace(RDF, 2, 2), Frac(ZZ['x']))                   #␣
↪needs sage.modules
Full MatrixSpace of 2 by 2 dense matrices
 over Fraction Field of Univariate Polynomial Ring in x over Real Double Field
sage: pushout(ZZ, MatrixSpace(ZZ[['x']], 3, 3))                        #␣
↪needs sage.modules
Full MatrixSpace of 3 by 3 dense matrices
 over Power Series Ring in x over Integer Ring
sage: pushout(QQ['x,y'], ZZ[['x']])
Univariate Polynomial Ring in y
 over Power Series Ring in x over Rational Field
sage: pushout(Frac(ZZ['x']), QQ[['x']])
Laurent Series Ring in x over Rational Field
```

A construction with `coercion_reversed=True` (currently only the *`SubspaceFunctor`* construction) is only applied if it leads to a valid coercion:

```
sage: # needs sage.modules
sage: A = ZZ^2
sage: V = span([[1, 2]], QQ)
sage: P = sage.categories.pushout.pushout(A, V)
sage: P
Vector space of dimension 2 over Rational Field
sage: P.has_coerce_map_from(A)
True

sage: # needs sage.modules
```

(continues on next page)

(continued from previous page)

```
sage: V = (QQ^3).span([[1, 2, 3/4]])
sage: A = ZZ^3
sage: pushout(A, V)
Vector space of dimension 3 over Rational Field
sage: B = A.span([[0, 0, 2/3]])
sage: pushout(B, V)
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 2 0]
[0 0 1]
```

Some more tests with `coercion_reversed=True`:

```
sage: from sage.categories.pushout import ConstructionFunctor
sage: class EvenPolynomialRing(type(QQ['x'])):
....:     def __init__(self, base, var):
....:         super().__init__(base, var)
....:         self.register_embedding(base[var])
....:     def __repr__(self):
....:         return "Even Power " + super().__repr__()
....:     def construction(self):
....:         return EvenPolynomialFunctor(), self.base()[self.variable_name()]
....:     def _coerce_map_from_(self, R):
....:         return self.base().has_coerce_map_from(R)
sage: class EvenPolynomialFunctor(ConstructionFunctor):
....:     rank = 10
....:     coercion_reversed = True
....:     def __init__(self):
....:         ConstructionFunctor.__init__(self, Rings(), Rings())
....:     def _apply_functor(self, R):
....:         return EvenPolynomialRing(R.base(), R.variable_name())
sage: pushout(EvenPolynomialRing(QQ, 'x'), ZZ)
Even Power Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), QQ)
Even Power Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), RR)                               #␣
↪needs sage.rings.real_mpfr
Even Power Univariate Polynomial Ring in x over Real Field with 53 bits of␣
↪precision

sage: pushout(EvenPolynomialRing(QQ, 'x'), ZZ['x'])
Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), QQ['x'])
Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), RR['x'])                          #␣
↪needs sage.rings.real_mpfr
Univariate Polynomial Ring in x over Real Field with 53 bits of precision

sage: pushout(EvenPolynomialRing(QQ, 'x'), EvenPolynomialRing(QQ, 'x'))
Even Power Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), EvenPolynomialRing(RR, 'x'))      #␣
↪needs sage.rings.real_mpfr
Even Power Univariate Polynomial Ring in x over Real Field with 53 bits of␣
↪precision

sage: pushout(EvenPolynomialRing(QQ, 'x')^2, RR^2)                           #␣
↪needs sage.modules sage.rings.real_mpfr
```

(continues on next page)

(continued from previous page)

```
Ambient free module of rank 2
 over the principal ideal domain Even Power Univariate Polynomial Ring in x
  over Real Field with 53 bits of precision
sage: pushout(EvenPolynomialRing(QQ, 'x')^2, RR['x']^2)                      #␣
↪needs sage.modules sage.rings.real_mpfr
Ambient free module of rank 2
 over the principal ideal domain Univariate Polynomial Ring in x
 over Real Field with 53 bits of precision
```

Some more tests related to univariate/multivariate constructions. We consider a generalization of polynomial rings, where in addition to the coefficient ring C we also specify an additive monoid E for the exponents of the indeterminate. In particular, the elements of such a parent are given by

$$\sum_{i=0}^{I} c_i X^{e_i}$$

with $c_i \in C$ and $e_i \in E$. We define

```
sage: class GPolynomialRing(Parent):
....:     def __init__(self, coefficients, var, exponents):
....:         self.coefficients = coefficients
....:         self.var = var
....:         self.exponents = exponents
....:         super().__init__(category=Rings())
....:     def _repr_(self):
....:         return 'Generalized Polynomial Ring in %s^(%s) over %s' % (
....:                 self.var, self.exponents, self.coefficients)
....:     def construction(self):
....:         return GPolynomialFunctor(self.var, self.exponents), self.
↪coefficients
....:     def _coerce_map_from_(self, R):
....:         return self.coefficients.has_coerce_map_from(R)
```

and

```
sage: class GPolynomialFunctor(ConstructionFunctor):
....:     rank = 10
....:     def __init__(self, var, exponents):
....:         self.var = var
....:         self.exponents = exponents
....:         ConstructionFunctor.__init__(self, Rings(), Rings())
....:     def _repr_(self):
....:         return 'GPoly[%s^(%s)]' % (self.var, self.exponents)
....:     def _apply_functor(self, coefficients):
....:         return GPolynomialRing(coefficients, self.var, self.exponents)
....:     def merge(self, other):
....:         if isinstance(other, GPolynomialFunctor) and self.var == other.var:
....:             exponents = pushout(self.exponents, other.exponents)
....:             return GPolynomialFunctor(self.var, exponents)
```

We can construct a parent now in two different ways:

```
sage: GPolynomialRing(QQ, 'X', ZZ)
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
sage: GP_ZZ = GPolynomialFunctor('X', ZZ); GP_ZZ
GPoly[X^(Integer Ring)]
```

(continues on next page)

(continued from previous page)

```
sage: GP_ZZ(QQ)
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
```

Since the construction

```
sage: GP_ZZ(QQ).construction()
(GPoly[X^(Integer Ring)], Rational Field)
```

uses the coefficient ring, we have the usual coercion with respect to this parameter:

```
sage: pushout(GP_ZZ(ZZ), GP_ZZ(QQ))
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
sage: pushout(GP_ZZ(ZZ['t']), GP_ZZ(QQ))
Generalized Polynomial Ring in X^(Integer Ring)
  over Univariate Polynomial Ring in t over Rational Field
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(ZZ['b,c']))
Generalized Polynomial Ring in X^(Integer Ring)
  over Multivariate Polynomial Ring in a, b, c over Integer Ring
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(QQ['b,c']))
Generalized Polynomial Ring in X^(Integer Ring)
  over Multivariate Polynomial Ring in a, b, c over Rational Field
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(ZZ['c,d']))
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', ...)
```

```
sage: GP_QQ = GPolynomialFunctor('X', QQ)
sage: pushout(GP_ZZ(ZZ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Integer Ring
sage: pushout(GP_QQ(ZZ), GP_ZZ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Integer Ring
```

```
sage: GP_ZZt = GPolynomialFunctor('X', ZZ['t'])
sage: pushout(GP_ZZt(ZZ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t
  over Rational Field) over Integer Ring
```

```
sage: pushout(GP_ZZ(ZZ), GP_QQ(QQ))
Generalized Polynomial Ring in X^(Rational Field) over Rational Field
sage: pushout(GP_ZZ(QQ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Rational Field
sage: pushout(GP_ZZt(QQ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t
  over Rational Field) over Rational Field
sage: pushout(GP_ZZt(ZZ), GP_QQ(QQ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t
  over Rational Field) over Rational Field
sage: pushout(GP_ZZt(ZZ['a,b']), GP_QQ(ZZ['c,d']))
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', ...)
sage: pushout(GP_ZZt(ZZ['a,b']), GP_QQ(ZZ['b,c']))
Generalized Polynomial Ring
 in X^(Univariate Polynomial Ring in t over Rational Field)
  over Multivariate Polynomial Ring in a, b, c over Integer Ring
```

Some tests with Cartesian products:

```
sage: from sage.sets.cartesian_product import CartesianProduct
sage: A = CartesianProduct((ZZ['x'], QQ['y'], QQ['z']),
....:                       Sets().CartesianProducts())
sage: B = CartesianProduct((ZZ['x'], ZZ['y'], ZZ['t']['z']),
....:                       Sets().CartesianProducts())
sage: A.construction()
(The cartesian_product functorial construction,
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Rational Field,
  Univariate Polynomial Ring in z over Rational Field))
sage: pushout(A, B)
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Rational Field,
  Univariate Polynomial Ring in z over
   Univariate Polynomial Ring in t over Rational Field)
sage: pushout(ZZ, cartesian_product([ZZ, QQ]))
Traceback (most recent call last):
...
CoercionException: 'NoneType' object is not iterable
```

```
sage: from sage.categories.pushout import PolynomialFunctor
sage: from sage.sets.cartesian_product import CartesianProduct
sage: class CartesianProductPoly(CartesianProduct):
....:     def __init__(self, polynomial_rings):
....:         sort = sorted(polynomial_rings,
....:                       key=lambda P: P.variable_name())
....:         super().__init__(sort, Sets().CartesianProducts())
....:     def vars(self):
....:         return tuple(P.variable_name()
....:                      for P in self.cartesian_factors())
....:     def _pushout_(self, other):
....:         if isinstance(other, CartesianProductPoly):
....:             s_vars = self.vars()
....:             o_vars = other.vars()
....:             if s_vars == o_vars:
....:                 return
....:             return pushout(CartesianProductPoly(
....:                     self.cartesian_factors() +
....:                     tuple(f for f in other.cartesian_factors()
....:                           if f.variable_name() not in s_vars)),
....:                 CartesianProductPoly(
....:                     other.cartesian_factors() +
....:                     tuple(f for f in self.cartesian_factors()
....:                           if f.variable_name() not in o_vars)))
....:         C = other.construction()
....:         if C is None:
....:             return
....:         elif isinstance(C[0], PolynomialFunctor):
....:             return pushout(self, CartesianProductPoly((other,)))
```

```
sage: pushout(CartesianProductPoly((ZZ['x'],)),
....:         CartesianProductPoly((ZZ['y'],)))
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
```

(continues on next page)

(continued from previous page)

```
  Univariate Polynomial Ring in y over Integer Ring)
sage: pushout(CartesianProductPoly((ZZ['x'], ZZ['y'])),
....:         CartesianProductPoly((ZZ['x'], ZZ['z'])))
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Integer Ring,
  Univariate Polynomial Ring in z over Integer Ring)
sage: pushout(CartesianProductPoly((QQ['a,b']['x'], QQ['y'])),              #␣
↪needs sage.symbolic
....:         CartesianProductPoly((ZZ['b,c']['x'], SR['z'])))
The Cartesian product of
 (Univariate Polynomial Ring in x over
    Multivariate Polynomial Ring in a, b, c over Rational Field,
  Univariate Polynomial Ring in y over Rational Field,
  Univariate Polynomial Ring in z over Symbolic Ring)
```

```
sage: pushout(CartesianProductPoly((ZZ['x'],)), ZZ['y'])
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Integer Ring)
sage: pushout(QQ['b,c']['y'], CartesianProductPoly((ZZ['a,b']['x'],)))
The Cartesian product of
 (Univariate Polynomial Ring in x over
    Multivariate Polynomial Ring in a, b over Integer Ring,
  Univariate Polynomial Ring in y over
    Multivariate Polynomial Ring in b, c over Rational Field)
```

```
sage: pushout(CartesianProductPoly((ZZ['x'],)), ZZ)
Traceback (most recent call last):
...
CoercionException: No common base ("join") found for
The cartesian_product functorial construction(...) and None(Integer Ring):
(Multivariate) functors are incompatible.
```

AUTHORS:

- Robert Bradshaw
- Peter Bruin
- Simon King
- Daniel Krenn
- David Roe

sage.categories.pushout.**pushout_lattice**(R, S)

Given a pair of objects R and S, try to construct a reasonable object Y and return maps such that canonically $R \leftarrow Y \rightarrow S$.

ALGORITHM:

This is based on the model that arose from much discussion at Sage Days 4. Going up the tower of constructions of R and S (e.g. the reals come from the rationals come from the integers), try to find a common parent, and then try to fill in a lattice with these two towers as sides with the top as the common ancestor and the bottom will be the desired ring.

See the code for a specific worked-out example.

EXAMPLES:

```
sage: from sage.categories.pushout import pushout_lattice
sage: A, B = pushout_lattice(Qp(7), Frac(ZZ['x']))                              #␣
↪needs sage.rings.padics
sage: A.codomain()                                                              #␣
↪needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
 over 7-adic Field with capped relative precision 20
sage: A.codomain() is B.codomain()                                              #␣
↪needs sage.rings.padics
True
sage: A, B = pushout_lattice(ZZ, MatrixSpace(ZZ[['x']], 3, 3))                  #␣
↪needs sage.modules
sage: B                                                                         #␣
↪needs sage.modules
Identity endomorphism of Full MatrixSpace of 3 by 3 dense matrices
 over Power Series Ring in x over Integer Ring
```

AUTHOR:

- Robert Bradshaw

`sage.categories.pushout.`**`type_to_parent`**`(P)`

An auxiliary function that is used in `pushout()`.

INPUT:

A type

OUTPUT:

A Sage parent structure corresponding to the given type

CHAPTER

FOUR

INDIVIDUAL CATEGORIES

4.1 Group, ring, etc. actions on objects

The terminology and notation used is suggestive of groups acting on sets, but this framework can be used for modules, algebras, etc.

A group action $G \times S \to S$ is a functor from G to Sets.

Warning: An *Action* object only keeps a weak reference to the underlying set which is acted upon. This decision was made in github issue #715 in order to allow garbage collection within the coercion framework (this is where actions are mainly used) and avoid memory leaks.

```
sage: from sage.categories.action import Action
sage: class P: pass
sage: A = Action(P(),P())
sage: import gc
sage: _ = gc.collect()
sage: A
<repr(<sage.categories.action.Action at 0x...>) failed:
 RuntimeError: This action acted on a set that became garbage collected>
```

To avoid garbage collection of the underlying set, it is sufficient to create a strong reference to it before the action is created.

```
sage: _ = gc.collect()
sage: from sage.categories.action import Action
sage: class P: pass
sage: q = P()
sage: A = Action(P(),q)
sage: gc.collect()
0
sage: A
Left action by <__main__.P ... at ...> on <__main__.P ... at ...>
```

AUTHOR:

- Robert Bradshaw: initial version

class sage.categories.action.**Action**

Bases: *Functor*

The action of `G` on `S`.

INPUT:

- `G` – a parent or Python type
- `S` – a parent or Python type
- `is_left` – (boolean, default: `True`) whether elements of `G` are on the left
- `op` – (default: `None`) operation. This is not used by *`Action`* itself, but other classes may use it

G

act (*g*, *x*)

This is a consistent interface for acting on `x` by `g`, regardless of whether it's a left or right action.

If needed, `g` and `x` are converted to the correct parent.

EXAMPLES:

```
sage: R.<x> = ZZ []
sage: from sage.structure.coerce_actions import IntegerMulAction
sage: A = IntegerMulAction(ZZ, R, True)    # Left action
sage: A.act(5, x)
5*x
sage: A.act(int(5), x)
5*x
sage: A = IntegerMulAction(ZZ, R, False)   # Right action
sage: A.act(5, x)
5*x
sage: A.act(int(5), x)
5*x
```

actor ()

codomain ()

domain ()

is_left ()

left_domain ()

op

operation ()

right_domain ()

class sage.categories.action.**ActionEndomorphism**

Bases: *`Morphism`*

The endomorphism defined by the action of one element.

EXAMPLES:

```
sage: A = ZZ['x'].get_action(QQ, self_on_left=False, op=operator.mul)
sage: A
Left scalar multiplication by Rational Field
 on Univariate Polynomial Ring in x over Integer Ring
sage: A(1/2)
Action of 1/2 on Univariate Polynomial Ring in x over Integer Ring
under Left scalar multiplication by Rational Field on Univariate
Polynomial Ring in x over Integer Ring.
```

class sage.categories.action.**InverseAction**

Bases: *Action*

An action that acts as the inverse of the given action.

EXAMPLES:

```
sage: V = QQ^3                                                              #␣
↪needs sage.modules
sage: v = V((1, 2, 3))                                                      #␣
↪needs sage.modules
sage: cm = get_coercion_model()

sage: # needs sage.modules
sage: a = cm.get_action(V, QQ, operator.mul)
sage: a
Right scalar multiplication by Rational Field
 on Vector space of dimension 3 over Rational Field
sage: ~a
Right inverse action by Rational Field
 on Vector space of dimension 3 over Rational Field
sage: (~a)(v, 1/3)
(3, 6, 9)

sage: # needs sage.modules
sage: b = cm.get_action(QQ, V, operator.mul)
sage: b
Left scalar multiplication by Rational Field
 on Vector space of dimension 3 over Rational Field
sage: ~b
Left inverse action by Rational Field
 on Vector space of dimension 3 over Rational Field
sage: (~b)(1/3, v)
(3, 6, 9)

sage: c = cm.get_action(ZZ, list, operator.mul)
sage: c
Left action by Integer Ring on <... 'list'>
sage: ~c
Traceback (most recent call last):
...
TypeError: no inverse defined for Left action by Integer Ring on <... 'list'>
```

codomain()

class sage.categories.action.**PrecomposedAction**

Bases: *Action*

A precomposed action first applies given maps, and then applying an action to the return values of the maps.

EXAMPLES:

We demonstrate that an example discussed on github issue #14711 did not become a problem:

```
sage: # needs sage.modular
sage: E = ModularSymbols(11).2
sage: s = E.modular_symbol_rep()
sage: del E,s
sage: import gc
```

(continues on next page)

(continued from previous page)

```
sage: _ = gc.collect()
sage: E = ModularSymbols(11).2
sage: v = E.manin_symbol_rep()
sage: c,x = v[0]
sage: y = x.modular_symbol_rep()
sage: coercion_model.get_action(QQ, parent(y), op=operator.mul)
Left scalar multiplication by Rational Field
 on Abelian Group of all Formal Finite Sums over Rational Field
 with precomposition on right by Coercion map:
  From: Abelian Group of all Formal Finite Sums over Integer Ring
  To:   Abelian Group of all Formal Finite Sums over Rational Field
```

codomain()

domain()

left_precomposition

The left map to precompose with, or `None` if there is no left precomposition map.

right_precomposition

The right map to precompose with, or `None` if there is no right precomposition map.

4.2 Additive groups

class sage.categories.additive_groups.**AdditiveGroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of additive groups.

An *additive group* is a set with an internal binary operation $+$ which is associative, admits a zero, and where every element can be negated.

EXAMPLES:

```
sage: from sage.categories.additive_groups import AdditiveGroups
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: AdditiveGroups()
Category of additive groups
sage: AdditiveGroups().super_categories()
[Category of additive inverse additive unital additive magmas,
 Category of additive monoids]
sage: AdditiveGroups().all_super_categories()
[Category of additive groups,
 Category of additive inverse additive unital additive magmas,
 Category of additive monoids,
 Category of additive unital additive magmas,
 Category of additive semigroups,
 Category of additive magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: AdditiveGroups().axioms()
frozenset({'AdditiveAssociative', 'AdditiveInverse', 'AdditiveUnital'})
```

(continues on next page)

(continued from previous page)

```
sage: AdditiveGroups() is AdditiveMonoids().AdditiveInverse()
True
```

AdditiveCommutative

alias of *CommutativeAdditiveGroups*

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: `object`

group()

Return the underlying group of the group algebra.

EXAMPLES:

```
sage: GroupAlgebras(QQ).example(GL(3, GF(11))).group()                    ␣
↪ # needs sage.groups sage.modules
General Linear Group of degree 3 over Finite Field of size 11
sage: SymmetricGroup(10).algebra(QQ).group()                              ␣
↪ # needs sage.groups sage.modules
Symmetric group of order 10! as a permutation group
```

class Finite(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: `object`

extra_super_categories()

Implement Maschke's theorem.

In characteristic 0 all finite group algebras are semisimple.

EXAMPLES:

```
sage: FiniteGroups().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Semisimple())
True
sage: FiniteGroups().Algebras(FiniteField(7)).is_
↪subcategory(Algebras(FiniteField(7)).Semisimple())
False
sage: FiniteGroups().Algebras(ZZ).is_subcategory(Algebras(ZZ).
↪Semisimple())
False
sage: FiniteGroups().Algebras(Fields()).is_
↪subcategory(Algebras(Fields()).Semisimple())
False

sage: Cat = CommutativeAdditiveGroups().Finite()
sage: Cat.Algebras(QQ).is_subcategory(Algebras(QQ).Semisimple())
True
sage: Cat.Algebras(GF(7)).is_subcategory(Algebras(GF(7)).Semisimple())
```

(continues on next page)

(continued from previous page)

```
False
sage: Cat.Algebras(ZZ).is_subcategory(Algebras(ZZ).Semisimple())
False
sage: Cat.Algebras(Fields()).is_subcategory(Algebras(Fields()).
↪Semisimple())
False
```

4.3 Additive magmas

class sage.categories.additive_magmas.**AdditiveMagmas**

Bases: *Category_singleton*

The category of additive magmas.

An additive magma is a set endowed with a binary operation $+$.

EXAMPLES:

```
sage: AdditiveMagmas()
Category of additive magmas
sage: AdditiveMagmas().super_categories()
[Category of sets]
sage: AdditiveMagmas().all_super_categories()
[Category of additive magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

The following axioms are defined by this category:

```
sage: AdditiveMagmas().AdditiveAssociative()
Category of additive semigroups
sage: AdditiveMagmas().AdditiveUnital()
Category of additive unital additive magmas
sage: AdditiveMagmas().AdditiveCommutative()
Category of additive commutative additive magmas
sage: AdditiveMagmas().AdditiveUnital().AdditiveInverse()
Category of additive inverse additive unital additive magmas
sage: C = AdditiveMagmas().AdditiveAssociative().AdditiveCommutative(); C
Category of commutative additive semigroups
sage: C.AdditiveUnital()
Category of commutative additive monoids
sage: C.AdditiveUnital().AdditiveInverse()
Category of commutative additive groups
```

AdditiveAssociative

alias of *AdditiveSemigroups*

class AdditiveCommutative(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

extra_super_categories()

Implement the fact that the algebra of a commutative additive magmas is commutative.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveCommutative().Algebras(QQ)
sage: C.extra_super_categories()
[Category of commutative magmas]

sage: C.super_categories()
[Category of additive magma algebras over Rational Field,
 Category of commutative magmas]
```

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

extra_super_categories()

Implement the fact that a Cartesian product of commutative additive magmas is a commutative additive magma.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveCommutative().CartesianProducts()
sage: C.extra_super_categories()
[Category of additive commutative additive magmas]
sage: C.axioms()
frozenset({'AdditiveCommutative'})
```

class AdditiveUnital(*base_category*)

Bases: `CategoryWithAxiom_singleton`

class AdditiveInverse(*base_category*)

Bases: `CategoryWithAxiom_singleton`

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

class ElementMethods

Bases: `object`

extra_super_categories()

Implement the fact that a Cartesian product of additive magmas with inverses is an additive magma with inverse.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveUnital().AdditiveInverse().
↪CartesianProducts()
sage: C.extra_super_categories()
[Category of additive inverse additive unital additive magmas]
sage: sorted(C.axioms())
['AdditiveInverse', 'AdditiveUnital']
```

class Algebras(*category*, **args*)

Bases: `AlgebrasCategory`

class ParentMethods

Bases: `object`

one_basis()

Return the zero of this additive magma, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: # needs sage.modules
sage: S = CommutativeAdditiveMonoids().example(); S
An example of a commutative monoid:
 the free commutative monoid generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(ZZ)
sage: A.one_basis()
0
sage: A.one()
B[0]
sage: A(3)
3*B[0]
```

extra_super_categories()

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveUnital().Algebras(QQ)
sage: C.extra_super_categories()
[Category of unital magmas]

sage: C.super_categories()
[Category of unital algebras with basis over Rational Field,
 Category of additive magma algebras over Rational Field]
```

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

class ParentMethods

Bases: `object`

zero()

Returns the zero of this group

EXAMPLES:

```
sage: GF(8, 'x').cartesian_product(GF(5)).zero()                  #␣
↪needs sage.rings.finite_rings
(0, 0)
```

extra_super_categories()

Implement the fact that a Cartesian product of unital additive magmas is a unital additive magma.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveUnital().CartesianProducts()
sage: C.extra_super_categories()
[Category of additive unital additive magmas]
sage: C.axioms()
frozenset({'AdditiveUnital'})
```

class ElementMethods

Bases: `object`

class Homsets(*category*, **args*)

Bases: `HomsetsCategory`

class ParentMethods

Bases: `object`

zero()

EXAMPLES:

```
sage: R = QQ['x']
sage: H = Hom(ZZ, R, AdditiveMagmas().AdditiveUnital())
sage: f = H.zero()
sage: f
Generic morphism:
  From: Integer Ring
  To:   Univariate Polynomial Ring in x over Rational Field
sage: f(3)
0
sage: f(3) is R.zero()
True
```

extra_super_categories()

Implement the fact that a homset between two unital additive magmas is a unital additive magma.

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital().Homsets().extra_super_
↪categories()
[Category of additive unital additive magmas]
sage: AdditiveMagmas().AdditiveUnital().Homsets().super_categories()
[Category of additive unital additive magmas, Category of homsets]
```

class ParentMethods

Bases: `object`

is_empty()

Return whether this set is empty.

Since this set is an additive magma it has a zero element and hence is not empty. This method thus always returns `False`.

EXAMPLES:

```
sage: # needs sage.modules
sage: A = AdditiveAbelianGroup([3, 3])
sage: A in AdditiveMagmas()
True
sage: A.is_empty()
False

sage: B = CommutativeAdditiveMonoids().example()
sage: B.is_empty()
False
```

zero()

Return the zero of this additive magma, that is the unique neutral element for $+$.

The default implementation is to coerce `0` into `self`.

It is recommended to override this method because the coercion from the integers:

- is not always meaningful (except for 0), and
- often uses `self.zero()` otherwise.

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example()
sage: S.zero()
0
```

class SubcategoryMethods

Bases: `object`

AdditiveInverse()

Return the full subcategory of the additive inverse objects of `self`.

An inverse *additive magma* is a `unital additive magma` such that every element admits both an additive inverse on the left and on the right. Such an additive magma is also called an *additive loop*.

See also:

Wikipedia article Inverse_element, Wikipedia article Quasigroup

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital().AdditiveInverse()
Category of additive inverse additive unital additive magmas
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: AdditiveMonoids().AdditiveInverse()
Category of additive groups
```

class WithRealizations(*category*, **args*)

Bases: *WithRealizationsCategory*

class ParentMethods

Bases: `object`

zero()

Return the zero of this unital additive magma.

This default implementation returns the zero of the realization of `self` given by *a_realization()*.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                    #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.zero.__module__                                             #␣
↪needs sage.modules
'sage.categories.additive_magmas'
sage: A.zero()                                                      #␣
↪needs sage.modules
0
```

additional_structure()

Return whether `self` is a structure category.

See also:

Category.additional_structure()

The category of unital additive magmas defines the zero as additional structure, and this zero shall be preserved by morphisms.

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital().additional_structure()
Category of additive unital additive magmas
```

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: `object`

algebra_generators()

The generators of this algebra, as per *MagmaticAlgebras.ParentMethods.algebra_generators()*.

They correspond to the generators of the additive semigroup.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative semigroup:
 the free commutative semigroup generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)                                                     #␣
↪needs sage.modules
sage: A.algebra_generators()                                                #␣
↪needs sage.modules
Family (B[a], B[b], B[c], B[d])
```

Todo: This doctest does not actually test this method, but rather the method of the same name for `AdditiveSemigroups`. Find a better doctest!

product_on_basis(*g1*, *g2*)

Product, on basis elements, as per *MagmaticAlgebras.WithBasis.ParentMethods.product_on_basis()*.

The product of two basis elements is induced by the addition of the corresponding elements of the group.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative semigroup:
 the free commutative semigroup generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)                                                     #␣
↪needs sage.modules
sage: a, b, c, d = A.algebra_generators()                                   #␣
↪needs sage.modules
sage: a * d * b                                                             #␣
↪needs sage.modules
B[a + b + d]
```

Todo: This doctest does not actually test this method, but rather the method of the same name for `AdditiveSemigroups`. Find a better doctest!

extra_super_categories()

EXAMPLES:

```
sage: AdditiveMagmas().Algebras(QQ).extra_super_categories()
[Category of magmatic algebras with basis over Rational Field]

sage: AdditiveMagmas().Algebras(QQ).super_categories()
[Category of magmatic algebras with basis over Rational Field,
 Category of set algebras over Rational Field]
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

class ElementMethods

Bases: `object`

extra_super_categories()

Implement the fact that a Cartesian product of additive magmas is an additive magma.

EXAMPLES:

```
sage: C = AdditiveMagmas().CartesianProducts()
sage: C.extra_super_categories()
[Category of additive magmas]
sage: C.super_categories()
[Category of additive magmas, Category of Cartesian products of sets]
sage: C.axioms()
frozenset()
```

class ElementMethods

Bases: `object`

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

extra_super_categories()

Implement the fact that a homset between two magmas is a magma.

EXAMPLES:

```
sage: AdditiveMagmas().Homsets().extra_super_categories()
[Category of additive magmas]
sage: AdditiveMagmas().Homsets().super_categories()
[Category of additive magmas, Category of homsets]
```

class ParentMethods

Bases: `object`

addition_table(*names='letters'*, *elements=None*)

Return a table describing the addition operation.

Note: The order of the elements in the row and column headings is equal to the order given by the table's `column_keys()` method. The association can also be retrieved with the `translation()` method.

INPUT:

- names – the type of names used:
  - 'letters' - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by column_keys(), padded to a common width with leading 'a's.
  - 'digits' - base 10 representation of the elements' positions in the list given by column_keys(), padded to a common width with leading zeros.
  - 'elements' - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- elements – (default: None) A list of elements of the additive magma, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements, perhaps when this is used in the context of a particular structure. The default is to use whatever ordering the S.list method returns. Or the elements can be a subset which is closed under the operation. In particular, this can be used when the base set is infinite.

OUTPUT:

The addition table as an object of the class OperationTable which defines several methods for manipulating and displaying the table. See the documentation there for full details to supplement the documentation here.

EXAMPLES:

All that is required is that an algebraic structure has an addition defined.The default is to represent elements as lowercase ASCII letters.

```
sage: R = IntegerModRing(5)
sage: R.addition_table()                                                        #␣
↪needs sage.modules
+  a b c d e
 +----------
a| a b c d e
b| b c d e a
c| c d e a b
d| d e a b c
e| e a b c d
```

The names argument allows displaying the elements in different ways. Requesting elements will use the representation of the elements of the set. Requesting digits will include leading zeros as padding.

```
sage: R = IntegerModRing(11)
sage: P = R.addition_table(names='elements'); P                                 #␣
↪needs sage.modules
 +   0  1  2  3  4  5  6  7  8  9 10
  +---------------------------------
 0|  0  1  2  3  4  5  6  7  8  9 10
 1|  1  2  3  4  5  6  7  8  9 10  0
 2|  2  3  4  5  6  7  8  9 10  0  1
 3|  3  4  5  6  7  8  9 10  0  1  2
 4|  4  5  6  7  8  9 10  0  1  2  3
 5|  5  6  7  8  9 10  0  1  2  3  4
 6|  6  7  8  9 10  0  1  2  3  4  5
 7|  7  8  9 10  0  1  2  3  4  5  6
 8|  8  9 10  0  1  2  3  4  5  6  7
 9|  9 10  0  1  2  3  4  5  6  7  8
10| 10  0  1  2  3  4  5  6  7  8  9

sage: T = R.addition_table(names='digits'); T                                   #␣
↪needs sage.modules
+  00 01 02 03 04 05 06 07 08 09 10
  +---------------------------------
```

(continues on next page)

(continued from previous page)

```
00| 00 01 02 03 04 05 06 07 08 09 10
01| 01 02 03 04 05 06 07 08 09 10 00
02| 02 03 04 05 06 07 08 09 10 00 01
03| 03 04 05 06 07 08 09 10 00 01 02
04| 04 05 06 07 08 09 10 00 01 02 03
05| 05 06 07 08 09 10 00 01 02 03 04
06| 06 07 08 09 10 00 01 02 03 04 05
07| 07 08 09 10 00 01 02 03 04 05 06
08| 08 09 10 00 01 02 03 04 05 06 07
09| 09 10 00 01 02 03 04 05 06 07 08
10| 10 00 01 02 03 04 05 06 07 08 09
```

Specifying the elements in an alternative order can provide more insight into how the operation behaves.

```
sage: S = IntegerModRing(7)
sage: elts = [0, 3, 6, 2, 5, 1, 4]
sage: S.addition_table(elements=elts)                                    #␣
↪needs sage.modules
+  a b c d e f g
 +--------------
a| a b c d e f g
b| b c d e f g a
c| c d e f g a b
d| d e f g a b c
e| e f g a b c d
f| f g a b c d e
g| g a b c d e f
```

The `elements` argument can be used to provide a subset of the elements of the structure. The subset must be closed under the operation. Elements need only be in a form that can be coerced into the set. The `names` argument can also be used to request that the elements be represented with their usual string representation.

```
sage: T = IntegerModRing(12)
sage: elts = [0, 3, 6, 9]
sage: T.addition_table(names='elements', elements=elts)                  #␣
↪needs sage.modules
+  0 3 6 9
 +--------
0| 0 3 6 9
3| 3 6 9 0
6| 6 9 0 3
9| 9 0 3 6
```

The table returned can be manipulated in various ways. See the documentation for `OperationTable` for more comprehensive documentation.

```
sage: # needs sage.modules
sage: R = IntegerModRing(3)
sage: T = R.addition_table()
sage: T.column_keys()
(0, 1, 2)
sage: sorted(T.translation().items())
[('a', 0), ('b', 1), ('c', 2)]
sage: T.change_names(['x', 'y', 'z'])
sage: sorted(T.translation().items())
```

(continues on next page)

(continued from previous page)

```
[('x', 0), ('y', 1), ('z', 2)]
sage: T
+  x y z
 +------
x| x y z
y| y z x
z| z x y
```

summation(*x*, *y*)

Return the sum of `x` and `y`.

The binary addition operator of this additive magma.

INPUT:

- `x`, `y` – elements of this additive magma

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example()
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: S.summation(a, b)
a + b
```

A parent in `AdditiveMagmas()` must either implement *summation()* in the parent class or `_add_` in the element class. By default, the addition method on elements `x._add_(y)` calls `S.summation(x,y)`, and reciprocally.

As a bonus effect, `S.summation` by itself models the binary function from `S` to `S`:

```
sage: bin = S.summation
sage: bin(a,b)
a + b
```

Here, `S.summation` is just a bound method. Whenever possible, it is recommended to enrich `S.summation` with extra mathematical structure. Lazy attributes can come handy for this.

Todo: Add an example.

summation_from_element_class_add(*x*, *y*)

Return the sum of `x` and `y`.

The binary addition operator of this additive magma.

INPUT:

- `x`, `y` – elements of this additive magma

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example()
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: S.summation(a, b)
a + b
```

A parent in `AdditiveMagmas()` must either implement *summation()* in the parent class or `_add_` in the element class. By default, the addition method on elements `x._add_(y)` calls `S.summation(x,y)`, and reciprocally.

As a bonus effect, `S.summation` by itself models the binary function from `S` to `S`:

```
sage: bin = S.summation
sage: bin(a,b)
a + b
```

Here, `S.summation` is just a bound method. Whenever possible, it is recommended to enrich `S.summation` with extra mathematical structure. Lazy attributes can come handy for this.

Todo: Add an example.

class SubcategoryMethods

Bases: `object`

AdditiveAssociative()

Return the full subcategory of the additive associative objects of `self`.

An *additive magma* M is *associative* if, for all $x, y, z \in M$,

$$x + (y + z) = (x + y) + z$$

See also:

Wikipedia article Associative_property

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveAssociative()
Category of additive semigroups
```

AdditiveCommutative()

Return the full subcategory of the commutative objects of `self`.

An *additive magma* M is *commutative* if, for all $x, y \in M$,

$$x + y = y + x$$

See also:

Wikipedia article Commutative_property

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveCommutative()
Category of additive commutative additive magmas
sage: C = AdditiveMagmas().AdditiveAssociative().AdditiveUnital()
sage: C.AdditiveCommutative()
Category of commutative additive monoids
sage: C.AdditiveCommutative() is CommutativeAdditiveMonoids()
True
```

AdditiveUnital()

Return the subcategory of the unital objects of `self`.

An *additive magma* M is *unital* if it admits an element 0, called *neutral element*, such that for all $x \in M$,

$$0 + x = x + 0 = x$$

This element is necessarily unique, and should be provided as `M.zero()`.

See also:

Wikipedia article Unital_magma#unital

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital()
Category of additive unital additive magmas
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: AdditiveSemigroups().AdditiveUnital()
Category of additive monoids
sage: CommutativeAdditiveMonoids().AdditiveUnital()
Category of commutative additive monoids
```

`super_categories()`

EXAMPLES:

```
sage: AdditiveMagmas().super_categories()
[Category of sets]
```

4.4 Additive monoids

class `sage.categories.additive_monoids.`**`AdditiveMonoids`**`(`*base_category*`)`

Bases: *`CategoryWithAxiom_singleton`*

The category of additive monoids.

An *additive monoid* is a unital *`additive semigroup`*, that is a set endowed with a binary operation + which is associative and admits a zero (see Wikipedia article Monoid).

EXAMPLES:

```
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: C = AdditiveMonoids(); C
Category of additive monoids
sage: C.super_categories()
[Category of additive unital additive magmas, Category of additive semigroups]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveUnital']
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: C is AdditiveSemigroups().AdditiveUnital()
True
```

`AdditiveCommutative`

alias of *`CommutativeAdditiveMonoids`*

`AdditiveInverse`

alias of *`AdditiveGroups`*

class `Homsets``(`*category*, **args*`)`

Bases: *`HomsetsCategory`*

`extra_super_categories()`

Implement the fact that a homset between two monoids is associative.

EXAMPLES:

```
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: AdditiveMonoids().Homsets().extra_super_categories()
[Category of additive semigroups]
sage: AdditiveMonoids().Homsets().super_categories()
[Category of homsets of additive unital additive magmas, Category of␣
↪additive monoids]
```

Todo: This could be deduced from *AdditiveSemigroups.Homsets.extra_super_categories()*. See comment in *Objects.SubcategoryMethods.Homsets()*.

class ParentMethods

Bases: `object`

sum(*args*)

Return the sum of the elements in `args`, as an element of `self`.

INPUT:

- `args` – a list (or iterable) of elements of `self`

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example()
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: S.sum((a,b,a,c,a,b))
3*a + 2*b + c
sage: S.sum(())
0
sage: S.sum(()).parent() == S
True
```

4.5 Additive semigroups

class sage.categories.additive_semigroups.**AdditiveSemigroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of additive semigroups.

An *additive semigroup* is an associative *additive magma*, that is a set endowed with an operation $+$ which is associative.

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: C = AdditiveSemigroups(); C
Category of additive semigroups
sage: C.super_categories()
[Category of additive magmas]
sage: C.all_super_categories()
[Category of additive semigroups,
 Category of additive magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

(continues on next page)

(continued from previous page)

```
sage: C.axioms()
frozenset({'AdditiveAssociative'})
sage: C is AdditiveMagmas().AdditiveAssociative()
True
```

AdditiveCommutative

alias of *CommutativeAdditiveSemigroups*

AdditiveUnital

alias of *AdditiveMonoids*

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: object

algebra_generators()

Return the generators of this algebra, as per *MagmaticAlgebras.ParentMethods.algebra_generators()*.

They correspond to the generators of the additive semigroup.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative semigroup:
 the free commutative semigroup generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)                                                #␣
↪needs sage.modules
sage: A.algebra_generators()                                           #␣
↪needs sage.modules
Family (B[a], B[b], B[c], B[d])
```

product_on_basis(*g1*, *g2*)

Product, on basis elements, as per *MagmaticAlgebras.WithBasis.ParentMethods.product_on_basis()*.

The product of two basis elements is induced by the addition of the corresponding elements of the group.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative semigroup:
 the free commutative semigroup generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)                                                #␣
↪needs sage.modules
sage: a, b, c, d = A.algebra_generators()                              #␣
↪needs sage.modules
sage: b * d * c                                                        #␣
↪needs sage.modules
B[b + c + d]
```

extra_super_categories()

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: AdditiveSemigroups().Algebras(QQ).extra_super_categories()
[Category of semigroups]
sage: CommutativeAdditiveSemigroups().Algebras(QQ).super_categories()
[Category of additive semigroup algebras over Rational Field,
 Category of additive commutative additive magma algebras over Rational␣
↪Field]
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that a Cartesian product of additive semigroups is an additive semigroup.

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: C = AdditiveSemigroups().CartesianProducts()
sage: C.extra_super_categories()
[Category of additive semigroups]
sage: C.axioms()
frozenset({'AdditiveAssociative'})
```

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

extra_super_categories()

Implement the fact that a homset between two semigroups is a semigroup.

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: AdditiveSemigroups().Homsets().extra_super_categories()
[Category of additive semigroups]
sage: AdditiveSemigroups().Homsets().super_categories()
[Category of homsets of additive magmas, Category of additive semigroups]
```

class ParentMethods

Bases: `object`

4.6 Affine Weyl groups

class sage.categories.affine_weyl_groups.**AffineWeylGroups**

Bases: *Category_singleton*

The category of affine Weyl groups

Todo: add a description of this category

See also:

- Wikipedia article Affine_weyl_group
- *WeylGroups*, `WeylGroup`

EXAMPLES:

```
sage: C = AffineWeylGroups(); C
Category of affine Weyl groups
sage: C.super_categories()
[Category of infinite Weyl groups]

sage: C.example()
NotImplemented
sage: W = WeylGroup(["A", 4, 1]); W                                          #␣
↪needs sage.combinat sage.groups
Weyl Group of type ['A', 4, 1] (as a matrix group acting on the root space)
sage: W.category()                                                           #␣
↪needs sage.combinat sage.groups
Category of irreducible affine Weyl groups
```

class ElementMethods

Bases: `object`

affine_grassmannian_to_core()

Bijection between affine Grassmannian elements of type $A_k^{(1)}$ and $(k+1)$-cores.

INPUT:

- `self` – an affine Grassmannian element of some affine Weyl group of type $A_k^{(1)}$

Recall that an element w of an affine Weyl group is affine Grassmannian if all its all reduced words end in 0, see *is_affine_grassmannian()*.

OUTPUT:

- a $(k+1)$-core

See also:

affine_grassmannian_to_partition()

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 2, 1])
sage: w = W.from_reduced_word([0,2,1,0])
sage: la = w.affine_grassmannian_to_core(); la
[4, 2]
sage: type(la)
<class 'sage.combinat.core.Cores_length_with_category.element_class'>
sage: la.to_grassmannian() == w
True

sage: w = W.from_reduced_word([0,2,1])                                   #␣
↪needs sage.combinat sage.groups
sage: w.affine_grassmannian_to_core()                                    #␣
↪needs sage.combinat sage.groups
Traceback (most recent call last):
...
ValueError: this only works on type 'A' affine Grassmannian elements
```

affine_grassmannian_to_partition()

Bijection between affine Grassmannian elements of type $A_k^{(1)}$ and k-bounded partitions.

INPUT:

- `self` is affine Grassmannian element of the affine Weyl group of type $A_k^{(1)}$ (i.e. all reduced words end in 0)

OUTPUT:

- k-bounded partition

See also:

affine_grassmannian_to_core()

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: k = 2
sage: W = WeylGroup(['A', k, 1])
sage: w = W.from_reduced_word([0,2,1,0])
sage: la = w.affine_grassmannian_to_partition(); la
[2, 2]
sage: la.from_kbounded_to_grassmannian(k) == w
True
```

is_affine_grassmannian()

Test whether `self` is affine Grassmannian.

An element of an affine Weyl group is *affine Grassmannian* if any of the following equivalent properties holds:

- all reduced words for `self` end with 0.
- `self` is the identity, or 0 is its single right descent.
- `self` is a minimal coset representative for W / cl W.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 3, 1])
sage: w = W.from_reduced_word([2,1,0])
sage: w.is_affine_grassmannian()
True
sage: w = W.from_reduced_word([2,0])
sage: w.is_affine_grassmannian()
False
sage: W.one().is_affine_grassmannian()
True
```

class ParentMethods

Bases: `object`

affine_grassmannian_elements_of_given_length(k)

Return the affine Grassmannian elements of length k.

This is returned as a finite enumerated set.

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1])                                        #␣
↪needs sage.combinat sage.groups
sage: [x.reduced_word()                                                 #␣
↪needs sage.combinat sage.groups
....:  for x in W.affine_grassmannian_elements_of_given_length(3)]
[[2, 1, 0], [3, 1, 0], [2, 3, 0]]
```

See also:

AffineWeylGroups.ElementMethods.is_affine_grassmannian()

special_node()

Return the distinguished special node of the underlying Dynkin diagram.

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1])                                          #␣
↪needs sage.combinat sage.groups
sage: W.special_node()                                                    #␣
↪needs sage.combinat sage.groups
0
```

additional_structure()

Return `None`.

Indeed, the category of affine Weyl groups defines no additional structure: affine Weyl groups are a special class of Weyl groups.

See also:

Category.additional_structure()

Todo: Should this category be a `CategoryWithAxiom`?

EXAMPLES:

```
sage: AffineWeylGroups().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: AffineWeylGroups().super_categories()
[Category of infinite Weyl groups]
```

4.7 Algebra ideals

class sage.categories.algebra_ideals.**AlgebraIdeals**(*A*)

Bases: *Category_ideal*

The category of two-sided ideals in a fixed algebra A.

EXAMPLES:

```
sage: AlgebraIdeals(QQ['a'])
Category of algebra ideals in Univariate Polynomial Ring in a over Rational Field
```

Todo:

- Add support for non commutative rings (this is currently not supported by the subcategory *AlgebraModules*).
- Make `AlgebraIdeals(R)`, return `CommutativeAlgebraIdeals(R)` when `R` is commutative.
- If useful, implement `AlgebraLeftIdeals` and `AlgebraRightIdeals` of which `AlgebraIdeals` would be a subcategory.

algebra()

EXAMPLES:

```
sage: AlgebraIdeals(QQ['x']).algebra()
Univariate Polynomial Ring in x over Rational Field
```

super_categories()

The category of algebra modules should be a super category of this category.

However, since algebra modules are currently only available over commutative rings, we have to omit it if our ring is non-commutative.

EXAMPLES:

```
sage: AlgebraIdeals(QQ['x']).super_categories()
[Category of algebra modules
  over Univariate Polynomial Ring in x over Rational Field]
sage: C = AlgebraIdeals(FreeAlgebra(QQ, 2, 'a,b'))                          #␣
↪needs sage.combinat sage.modules
sage: C.super_categories()                                                  #␣
↪needs sage.combinat sage.modules
[]
```

4.8 Algebra modules

class sage.categories.algebra_modules.**AlgebraModules**(*A*)

Bases: *Category_module*

The category of modules over a fixed algebra A.

EXAMPLES:

```
sage: AlgebraModules(QQ['a'])
Category of algebra modules over Univariate Polynomial Ring in a over Rational␣
↪Field
sage: AlgebraModules(QQ['a']).super_categories()
[Category of modules over Univariate Polynomial Ring in a over Rational Field]
```

Note: as of now, A is required to be commutative, ensuring that the categories of left and right modules are isomorphic. Feedback and use cases for potential generalizations to the non commutative case are welcome.

algebra()

EXAMPLES:

```
sage: AlgebraModules(QQ['x']).algebra()
Univariate Polynomial Ring in x over Rational Field
```

classmethod an_instance()

Returns an instance of this class

EXAMPLES:

```
sage: AlgebraModules.an_instance()
Category of algebra modules over Univariate Polynomial Ring in x over␣
↪Rational Field
```

super_categories()

EXAMPLES:

```
sage: AlgebraModules(QQ['x']).super_categories()
[Category of modules over Univariate Polynomial Ring in x over Rational Field]
```

4.9 Algebras

AUTHORS:

- David Kohel & William Stein (2005): initial revision
- Nicolas M. Thiery (2008-2011): rewrote for the category framework

class sage.categories.algebras.**Algebras**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of associative and unital algebras over a given base ring.

An associative and unital algebra over a ring R is a module over R which is itself a ring.

Warning: *Algebras* will be eventually be replaced by *magmatic_algebras.MagmaticAlgebras* for consistency with e.g. Wikipedia article Algebras which assumes neither associativity nor the existence of a unit (see github issue #15043).

Todo: Should R be a commutative ring?

EXAMPLES:

```
sage: Algebras(ZZ)
Category of algebras over Integer Ring
sage: sorted(Algebras(ZZ).super_categories(), key=str)
[Category of associative algebras over Integer Ring,
 Category of rings,
 Category of unital algebras over Integer Ring]
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

The category of algebras constructed as Cartesian products of algebras

This construction gives the direct product of algebras. See discussion on:

- http://groups.google.fr/group/sage-devel/browse_thread/thread/35a72b1d0a2fc77a/348f42ae77a66d16#348f42ae77a66d16
- Wikipedia article Direct_product

extra_super_categories()

A Cartesian product of algebras is endowed with a natural algebra structure.

EXAMPLES:

```
sage: C = Algebras(QQ).CartesianProducts()
sage: C.extra_super_categories()
[Category of algebras over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of monoids,
 Category of Cartesian products of unital algebras over Rational Field,
 Category of algebras over Rational Field]
```

Commutative

alias of *CommutativeAlgebras*

class DualObjects(*category*, **args*)

Bases: *DualObjectsCategory*

extra_super_categories()

Return the dual category

EXAMPLES:

The category of algebras over the Rational Field is dual to the category of coalgebras over the same field:

```
sage: C = Algebras(QQ)
sage: C.dual()
Category of duals of algebras over Rational Field
sage: C.dual().extra_super_categories()
[Category of coalgebras over Rational Field]
```

Warning: This is only correct in certain cases (finite dimension, …). See github issue #15647.

class ElementMethods

Bases: object

Filtered

alias of *FilteredAlgebras*

Graded

alias of *GradedAlgebras*

class ParentMethods

Bases: object

characteristic()

Return the characteristic of this algebra, which is the same as the characteristic of its base ring.

EXAMPLES:

```
sage: # needs sage.modules
sage: ZZ.characteristic()
0
sage: A = GF(7^3, 'a')                                                 ␣
↪   # needs sage.rings.finite_rings
sage: A.characteristic()                                               ␣
↪   # needs sage.rings.finite_rings
7
```

has_standard_involution()

Return `True` if the algebra has a standard involution and `False` otherwise.

This algorithm follows Algorithm 2.10 from John Voight's *Identifying the Matrix Ring*. Currently the only type of algebra this will work for is a quaternion algebra. Though this function seems redundant, once algebras have more functionality, in particular have a method to construct a basis, this algorithm will have more general purpose.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: B = QuaternionAlgebra(2)
sage: B.has_standard_involution()
True
sage: R.<x> = PolynomialRing(QQ)
sage: K.<u> = NumberField(x**2 - 2)                                         ␣
↪  # needs sage.rings.number_field
sage: A = QuaternionAlgebra(K, -2, 5)                                       ␣
↪  # needs sage.rings.number_field
sage: A.has_standard_involution()                                           ␣
↪  # needs sage.rings.number_field
True
sage: L.<a,b> = FreeAlgebra(QQ, 2)
sage: L.has_standard_involution()
Traceback (most recent call last):
...
NotImplementedError: has_standard_involution is not implemented for this␣
↪algebra
```

class Quotients(*category*, **args*)

Bases: `QuotientsCategory`

class ParentMethods

Bases: `object`

algebra_generators()

Return algebra generators for `self`.

This implementation retracts the algebra generators from the ambient algebra.

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: S = A.semisimple_quotient()
sage: S.algebra_generators()
Finite family {'x': B['x'], 'y': B['y'], 'a': 0, 'b': 0}
```

Todo: this could possibly remove the elements that retract to zero.

Semisimple

alias of `SemisimpleAlgebras`

class SubcategoryMethods

Bases: `object`

Semisimple()

Return the subcategory of semisimple objects of `self`.

Note: This mimics the syntax of axioms for a smooth transition if `Semisimple` becomes one.

EXAMPLES:

```
sage: Algebras(QQ).Semisimple()
Category of semisimple algebras over Rational Field
sage: Algebras(QQ).WithBasis().FiniteDimensional().Semisimple()
Category of finite dimensional semisimple algebras with basis over␣
↪Rational Field
```

Supercommutative()

Return the full subcategory of the supercommutative objects of `self`.

This is shorthand for creating the corresponding super category.

EXAMPLES:

```
sage: Algebras(ZZ).Supercommutative()
Category of supercommutative algebras over Integer Ring
sage: Algebras(ZZ).WithBasis().Supercommutative()
Category of supercommutative super algebras with basis over Integer Ring

sage: Cat = Algebras(ZZ).Supercommutative()
sage: Cat is Algebras(ZZ).Super().Supercommutative()
True
```

Super

alias of *SuperAlgebras*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

extra_super_categories()

EXAMPLES:

```
sage: Algebras(QQ).TensorProducts().extra_super_categories()
[Category of algebras over Rational Field]
sage: Algebras(QQ).TensorProducts().super_categories()
[Category of algebras over Rational Field,
 Category of tensor products of vector spaces over Rational Field]
```

Meaning: a tensor product of algebras is an algebra

WithBasis

alias of *AlgebrasWithBasis*

4.10 Algebras With Basis

class sage.categories.algebras_with_basis.**AlgebrasWithBasis**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of algebras with a distinguished basis.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ); C
Category of algebras with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of unital algebras with basis over Rational Field]
```

We construct a typical parent in this category, and do some computations with it:

```
sage: A = C.example(); A                                                  #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis:
 the free algebra on the generators ('a', 'b', 'c') over Rational Field

sage: A.category()                                                        #␣
↪needs sage.combinat sage.modules
Category of algebras with basis over Rational Field

sage: A.one_basis()                                                       #␣
↪needs sage.combinat sage.modules
word:
sage: A.one()                                                             #␣
↪needs sage.combinat sage.modules
B[word: ]

sage: A.base_ring()                                                       #␣
↪needs sage.combinat sage.modules
Rational Field
sage: A.basis().keys()                                                    #␣
↪needs sage.combinat sage.modules
Finite words over {'a', 'b', 'c'}

sage: (a,b,c) = A.algebra_generators()                                    #␣
↪needs sage.combinat sage.modules
sage: a^3, b^2                                                            #␣
↪needs sage.combinat sage.modules
(B[word: aaa], B[word: bb])
sage: a * c * b                                                           #␣
↪needs sage.combinat sage.modules
B[word: acb]

sage: A.product                                                           #␣
↪needs sage.combinat sage.modules
<bound method MagmaticAlgebras.WithBasis.ParentMethods._product_from_product_on_
↪basis_multiply of
 An example of an algebra with basis:
  the free algebra on the generators ('a', 'b', 'c') over Rational Field>
sage: A.product(a * b, b)                                                 #␣
↪needs sage.combinat sage.modules
```

(continues on next page)

(continued from previous page)

```
B[word: abb]

sage: TestSuite(A).run(verbose=True)                                          #␣
↪needs sage.combinat sage.modules
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_characteristic() . . . pass
running ._test_construction() . . . pass
running ._test_distributivity() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
running ._test_zero() . . . pass
sage: A.__class__                                                             #␣
↪needs sage.combinat sage.modules
<class 'sage.categories.examples.algebras_with_basis.FreeAlgebra_with_category'>
sage: A.element_class                                                         #␣
↪needs sage.combinat sage.modules
<class 'sage.categories.examples.algebras_with_basis.FreeAlgebra_with_category.
↪element_class'>
```

Please see the source code of A (with `A??`) for how to implement other algebras with basis.

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

The category of algebras with basis, constructed as Cartesian products of algebras with basis.

Note: this construction give the direct products of algebras with basis. See comment in `Algebras.CartesianProducts`

class ParentMethods

Bases: `object`

one()

one_from_cartesian_product_of_one_basis()

Return the one of this Cartesian product of algebras, as per `Monoids.ParentMethods.one`

It is constructed as the Cartesian product of the ones of the summands, using their `one_basis()` methods.

This implementation does not require multiplication by scalars nor calling cartesian_product. This might help keeping things as lazy as possible upon initialization.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A                          #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis: the free algebra
 on the generators ('a', 'b', 'c') over Rational Field
sage: A.one_basis()                                                   #␣
↪needs sage.combinat sage.modules
word:

sage: B = cartesian_product((A, A, A))                                #␣
↪needs sage.combinat sage.modules
sage: B.one_from_cartesian_product_of_one_basis()                     #␣
↪needs sage.combinat sage.modules
B[(0, word: )] + B[(1, word: )] + B[(2, word: )]
sage: B.one()                                                         #␣
↪needs sage.combinat sage.modules
B[(0, word: )] + B[(1, word: )] + B[(2, word: )]

sage: cartesian_product([SymmetricGroupAlgebra(QQ, 3),                #␣
↪needs sage.combinat sage.modules
....:                    SymmetricGroupAlgebra(QQ, 4)]).one()
B[(0, [1, 2, 3])] + B[(1, [1, 2, 3, 4])]
```

extra_super_categories()

A Cartesian product of algebras with basis is endowed with a natural algebra with basis structure.

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).CartesianProducts().extra_super_categories()
[Category of algebras with basis over Rational Field]
sage: AlgebrasWithBasis(QQ).CartesianProducts().super_categories()
[Category of algebras with basis over Rational Field,
 Category of Cartesian products of algebras over Rational Field,
 Category of Cartesian products of vector spaces with basis over Rational␣
↪Field]
```

class ElementMethods

Bases: `object`

Filtered

alias of *FilteredAlgebrasWithBasis*

FiniteDimensional

alias of *FiniteDimensionalAlgebrasWithBasis*

Graded

alias of *GradedAlgebrasWithBasis*

class ParentMethods

Bases: `object`

hochschild_complex(*M*)

Return the Hochschild complex of `self` with coefficients in `M`.

See also:

`HochschildComplex`

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: A = algebras.DifferentialWeyl(R)                                         #␣
↪needs sage.modules
sage: H = A.hochschild_complex(A)                                              #␣
↪needs sage.modules

sage: SGA = SymmetricGroupAlgebra(QQ, 3)                                       #␣
↪needs sage.combinat sage.modules
sage: T = SGA.trivial_representation()                                         #␣
↪needs sage.combinat sage.modules
sage: H = SGA.hochschild_complex(T)                                            #␣
↪needs sage.combinat sage.modules
```

one()

Return the multiplicative unit element.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()                                 #␣
↪needs sage.combinat sage.modules
sage: A.one_basis()                                                       #␣
↪needs sage.combinat sage.modules
word:
sage: A.one()                                                             #␣
↪needs sage.combinat sage.modules
B[word: ]
```

Super

alias of *SuperAlgebrasWithBasis*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of algebras with basis constructed by tensor product of algebras with basis

class ElementMethods

Bases: `object`

Implements operations on elements of tensor products of algebras with basis

class ParentMethods

Bases: `object`

implements operations on tensor products of algebras with basis

one_basis()

Returns the index of the one of this tensor product of algebras, as per `AlgebrasWithBasis.ParentMethods.one_basis`

It is the tuple whose operands are the indices of the ones of the operands, as returned by their *one_basis()* methods.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra
 on the generators ('a', 'b', 'c') over Rational Field
sage: A.one_basis()
word:
sage: B = tensor((A, A, A))
sage: B.one_basis()
(word: , word: , word: )
sage: B.one()
B[word: ] # B[word: ] # B[word: ]
```

product_on_basis(*t1*, *t2*)

The product of the algebra on the basis, as per `AlgebrasWithBasis.ParentMethods.product_on_basis`.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A                               #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis: the free algebra
 on the generators ('a', 'b', 'c') over Rational Field
sage: (a,b,c) = A.algebra_generators()                                     #␣
↪needs sage.combinat sage.modules

sage: x = tensor((a, b, c)); x                                             #␣
↪needs sage.combinat sage.modules
B[word: a] # B[word: b] # B[word: c]
sage: y = tensor((c, b, a)); y                                             #␣
↪needs sage.combinat sage.modules
B[word: c] # B[word: b] # B[word: a]
sage: x * y                                                                #␣
↪needs sage.combinat sage.modules
B[word: ac] # B[word: bb] # B[word: ca]

sage: x = tensor(((a + 2*b), c)); x                                        #␣
↪needs sage.combinat sage.modules
B[word: a] # B[word: c] + 2*B[word: b] # B[word: c]
sage: y = tensor((c, a)) + 1; y                                            #␣
↪needs sage.combinat sage.modules
B[word: ] # B[word: ] + B[word: c] # B[word: a]
sage: x * y                                                                #␣
↪needs sage.combinat sage.modules
B[word: a] # B[word: c] + B[word: ac] # B[word: ca]
 + 2*B[word: b] # B[word: c] + 2*B[word: bc] # B[word: ca]
```

TODO: optimize this implementation!

extra_super_categories()

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).TensorProducts().extra_super_categories()
[Category of algebras with basis over Rational Field]
sage: AlgebrasWithBasis(QQ).TensorProducts().super_categories()
[Category of algebras with basis over Rational Field,
 Category of tensor products of algebras over Rational Field,
```

(continues on next page)

(continued from previous page)

```
Category of tensor products of vector spaces with basis over Rational␣
↪Field]
```

example (*alphabet=('a', 'b', 'c')*)

Return an example of algebra with basis.

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).example()                                       #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis:
 the free algebra on the generators ('a', 'b', 'c') over Rational Field
```

An other set of generators can be specified as optional argument:

```
sage: AlgebrasWithBasis(QQ).example((1,2,3))                                #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis:
 the free algebra on the generators (1, 2, 3) over Rational Field
```

4.11 Aperiodic semigroups

class sage.categories.aperiodic_semigroups.**AperiodicSemigroups** (*base_category*)

Bases: *CategoryWithAxiom*

extra_super_categories ()

Implement the fact that an aperiodic semigroup is H-trivial.

EXAMPLES:

```
sage: Semigroups().Aperiodic().extra_super_categories()
[Category of h trivial semigroups]
```

4.12 Associative algebras

class sage.categories.associative_algebras.**AssociativeAlgebras** (*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of associative algebras over a given base ring.

An associative algebra over a ring R is a module over R which is also a not necessarily unital ring.

Warning: Until github issue #15043 is implemented, *Algebras* is the category of associative unital algebras; thus, unlike the name suggests, *AssociativeAlgebras* is not a subcategory of *Algebras* but of *MagmaticAlgebras*.

EXAMPLES:

```
sage: from sage.categories.associative_algebras import AssociativeAlgebras
sage: C = AssociativeAlgebras(ZZ); C
Category of associative algebras over Integer Ring
```

Unital

alias of *Algebras*

4.13 Bialgebras

class sage.categories.bialgebras.**Bialgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of bialgebras

EXAMPLES:

```
sage: Bialgebras(ZZ)
Category of bialgebras over Integer Ring
sage: Bialgebras(ZZ).super_categories()
[Category of algebras over Integer Ring, Category of coalgebras over Integer Ring]
```

class ElementMethods

Bases: object

is_grouplike()

Return whether self is a grouplike element.

EXAMPLES:

```
sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.modules
sage: s([5]).is_grouplike()                                                 #␣
↪needs sage.modules
False
sage: s([]).is_grouplike()                                                  #␣
↪needs sage.modules
True
```

is_primitive()

Return whether self is a primitive element.

EXAMPLES:

```
sage: # needs sage.modules
sage: s = SymmetricFunctions(QQ).schur()
sage: s([5]).is_primitive()
False
sage: p = SymmetricFunctions(QQ).powersum()
sage: p([5]).is_primitive()
True
```

class Super(*base_category*)

Bases: *SuperModulesCategory*

WithBasis

alias of *BialgebrasWithBasis*

additional_structure()

Return `None`.

Indeed, the category of bialgebras defines no additional structure: a morphism of coalgebras and of algebras between two bialgebras is a bialgebra morphism.

See also:

Category.additional_structure()

Todo: This category should be a `CategoryWithAxiom`.

EXAMPLES:

```
sage: Bialgebras(QQ).additional_structure()
```

super_categories()

EXAMPLES:

```
sage: Bialgebras(QQ).super_categories()
[Category of algebras over Rational Field, Category of coalgebras over␣
↪Rational Field]
```

4.14 Bialgebras with basis

class sage.categories.bialgebras_with_basis.**BialgebrasWithBasis**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of bialgebras with a distinguished basis.

EXAMPLES:

```
sage: C = BialgebrasWithBasis(QQ); C
Category of bialgebras with basis over Rational Field

sage: sorted(C.super_categories(), key=str)
[Category of algebras with basis over Rational Field,
 Category of bialgebras over Rational Field,
 Category of coalgebras with basis over Rational Field]
```

class ElementMethods

Bases: `object`

adams_operator(**args*, ***kwds*)

Deprecated: Use *convolution_power_of_id()* instead. See github issue #36396 for details.

convolution_power_of_id(*n*)

Compute the n-th convolution power of the identity morphism Id on `self`.

INPUT:

- `n` – a nonnegative integer

OUTPUT:

- the image of `self` under the convolution power Id^{*n}

Note: In the literature, this is also called a Hopf power or Sweedler power, cf. [AL2015].

See also:

`sage.categories.bialgebras.ElementMethods.convolution_product()`

Todo: Remove dependency on `modules_with_basis` methods.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: h = SymmetricFunctions(QQ).h()
sage: h[5].convolution_power_of_id(2)
2*h[3, 2] + 2*h[4, 1] + 2*h[5]
sage: h[5].plethysm(2*h[1])
2*h[3, 2] + 2*h[4, 1] + 2*h[5]
sage: h([]).convolution_power_of_id(0)
h[]
sage: h([]).convolution_power_of_id(1)
h[]
sage: h[3,2].convolution_power_of_id(0)
0
sage: h[3,2].convolution_power_of_id(1)
h[3, 2]
```

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()                          #␣
↪needs sage.combinat sage.modules
sage: S[4].convolution_power_of_id(5)                                        ␣
↪       # needs sage.combinat sage.modules
5*S[1, 1, 1, 1] + 10*S[1, 1, 2] + 10*S[1, 2, 1]
 + 10*S[1, 3] + 10*S[2, 1, 1] + 10*S[2, 2] + 10*S[3, 1] + 5*S[4]
```

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()                   #␣
↪needs sage.combinat sage.modules
sage: m[[1,3],[2]].convolution_power_of_id(-2)                               ␣
↪       # needs sage.combinat sage.modules
3*m{{1}, {2, 3}} + 3*m{{1, 2}, {3}} + 6*m{{1, 2, 3}} - 2*m{{1, 3}, {2}}
```

convolution_product(**maps*)

Return the image of `self` under the convolution product (map) of the maps.

Let A and B be bialgebras over a commutative ring R. Given maps $f_i : A \to B$ for $1 \leq i < n$, define the convolution product

$$(f_1 * f_2 * \cdots * f_n) := \mu^{(n-1)} \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta^{(n-1)},$$

where $\Delta^{(k)} := \left(\Delta \otimes \mathrm{Id}^{\otimes(k-1)}\right) \circ \Delta^{(k-1)}$, with $\Delta^{(1)} = \Delta$ (the ordinary coproduct in A) and $\Delta^{(0)} = \mathrm{Id}$; and with $\mu^{(k)} := \mu \circ \left(\mu^{(k-1)} \otimes \mathrm{Id}\right)$ and $\mu^{(1)} = \mu$ (the ordinary product in B). See [Swe1969].

(In the literature, one finds, e.g., $\Delta^{(2)}$ for what we denote above as $\Delta^{(1)}$. See [KMN2012].)

INPUT:

- `maps` – any number $n \geq 0$ of linear maps $f_1, f_2, \ldots, f_n$ on `self.parent()`; or a single `list` or `tuple` of such maps

OUTPUT:

- the convolution product of `maps` applied to `self`

AUTHORS:

- Amy Pang - 12 June 2015 - Sage Days 65

Todo: Remove dependency on `modules_with_basis` methods.

EXAMPLES:

We compute convolution products of the identity and antipode maps on Schur functions:

```
sage: Id = lambda x: x
sage: Antipode = lambda x: x.antipode()
sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: s[3].convolution_product(Id, Id)                                      #␣
↪needs sage.combinat sage.modules
2*s[2, 1] + 4*s[3]
sage: s[3,2].convolution_product(Id) == s[3,2]                              #␣
↪needs sage.combinat sage.modules
True
```

The method accepts multiple arguments, or a single argument consisting of a list of maps:

```
sage: s[3,2].convolution_product(Id, Id)                                    #␣
↪needs sage.combinat sage.modules
2*s[2, 1, 1, 1] + 6*s[2, 2, 1] + 6*s[3, 1, 1] + 12*s[3, 2] + 6*s[4, 1] +␣
↪2*s[5]
sage: s[3,2].convolution_product([Id, Id])                                  #␣
↪needs sage.combinat sage.modules
2*s[2, 1, 1, 1] + 6*s[2, 2, 1] + 6*s[3, 1, 1] + 12*s[3, 2] + 6*s[4, 1] +␣
↪2*s[5]
```

We test the defining property of the antipode morphism; namely, that the antipode is the inverse of the identity map in the convolution algebra whose identity element is the composition of the counit and unit:

```
sage: (s[3,2].convolution_product()                                         #␣
↪needs sage.combinat sage.modules
....:   == s[3,2].convolution_product(Antipode, Id)
....:   == s[3,2].convolution_product(Id, Antipode))
True
```

```
sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()                      #␣
↪needs sage.combinat sage.modules
sage: Psi[2,1].convolution_product(Id, Id, Id)                              #␣
↪needs sage.combinat sage.modules
3*Psi[1, 2] + 6*Psi[2, 1]
sage: (Psi[5,1] - Psi[1,5]).convolution_product(Id, Id, Id)                 #␣
↪needs sage.combinat sage.modules
-3*Psi[1, 5] + 3*Psi[5, 1]
```

```
sage: # needs sage.combinat sage.modules
sage: G = SymmetricGroup(3)
sage: QG = GroupAlgebra(G, QQ)
sage: x = QG.sum_of_terms([(p, p.length())
....:                      for p in Permutations(3)]); x
```

(continues on next page)

(continued from previous page)

```
[1, 3, 2] + [2, 1, 3] + 2*[2, 3, 1] + 2*[3, 1, 2] + 3*[3, 2, 1]
sage: x.convolution_product(Id, Id)
5*[1, 2, 3] + 2*[2, 3, 1] + 2*[3, 1, 2]
sage: x.convolution_product(Id, Id, Id)
4*[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + 3*[3, 2, 1]
sage: x.convolution_product([Id] * 6)
9*[1, 2, 3]
```

class ParentMethods

Bases: `object`

convolution_product(**maps*)

Return the convolution product (a map) of the given maps.

Let A and B be bialgebras over a commutative ring R. Given maps $f_i : A \to B$ for $1 \leq i < n$, define the convolution product

$$(f_1 * f_2 * \cdots * f_n) := \mu^{(n-1)} \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta^{(n-1)},$$

where $\Delta^{(k)} := \left(\Delta \otimes \mathrm{Id}^{\otimes(k-1)}\right) \circ \Delta^{(k-1)}$, with $\Delta^{(1)} = \Delta$ (the ordinary coproduct in A) and $\Delta^{(0)} = \mathrm{Id}$; and with $\mu^{(k)} := \mu \circ \left(\mu^{(k-1)} \otimes \mathrm{Id}\right)$ and $\mu^{(1)} = \mu$ (the ordinary product in B). See [Swe1969].

(In the literature, one finds, e.g., $\Delta^{(2)}$ for what we denote above as $\Delta^{(1)}$. See [KMN2012].)

INPUT:

- `maps` – any number $n \geq 0$ of linear maps $f_1, f_2, \ldots, f_n$ on `self`; or a single `list` or `tuple` of such maps

OUTPUT:

- the new map $f_1 * f_2 * \cdots * f_2$ representing their convolution product

See also:

`sage.categories.bialgebras.ElementMethods.convolution_product()`

AUTHORS:

- Aaron Lauve - 12 June 2015 - Sage Days 65

Todo: Remove dependency on `modules_with_basis` methods.

EXAMPLES:

We construct some maps: the identity, the antipode and projection onto the homogeneous component of degree 2:

```
sage: Id = lambda x: x
sage: Antipode = lambda x: x.antipode()
sage: Proj2 = lambda x: x.parent().sum_of_terms([(m, c) for (m, c) in x␣
↪if m.size() == 2])
```

Compute the convolution product of the identity with itself and with the projection `Proj2` on the Hopf algebra of non-commutative symmetric functions:

```
sage: # needs sage.combinat sage.modules
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: T = R.convolution_product([Id, Id])
sage: [T(R(comp)) for comp in Compositions(3)]
[4*R[1, 1, 1] + R[1, 2] + R[2, 1],
```

(continues on next page)

(continued from previous page)

```
 2*R[1, 1, 1] + 4*R[1, 2] + 2*R[2, 1] + 2*R[3],
 2*R[1, 1, 1] + 2*R[1, 2] + 4*R[2, 1] + 2*R[3],
 R[1, 2] + R[2, 1] + 4*R[3]]
sage: T = R.convolution_product(Proj2, Id)
sage: [T(R([i])) for i in range(1, 5)]
[0, R[2], R[2, 1] + R[3], R[2, 2] + R[4]]
```

Compute the convolution product of no maps on the Hopf algebra of symmetric functions in non-commuting variables. This is the composition of the counit with the unit:

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()                    #␣
↪needs sage.combinat sage.modules
sage: T = m.convolution_product()                                            #␣
↪needs sage.combinat sage.modules
sage: [T(m(lam))                                                             #␣
↪needs sage.combinat sage.modules
....:  for lam in SetPartitions(0).list() + SetPartitions(2).list()]
[m{}, 0, 0]
```

Compute the convolution product of the projection `Proj2` with the identity on the Hopf algebra of symmetric functions in non-commuting variables:

```
sage: T = m.convolution_product(Proj2, Id)                                   #␣
↪needs sage.combinat sage.modules
sage: [T(m(lam)) for lam in SetPartitions(3)]                                #␣
↪needs sage.combinat sage.modules
[0,
 m{{1, 2}, {3}} + m{{1, 2, 3}},
 m{{1, 2}, {3}} + m{{1, 2, 3}},
 m{{1, 2}, {3}} + m{{1, 2, 3}},
 3*m{{1}, {2}, {3}} + 3*m{{1}, {2, 3}} + 3*m{{1, 3}, {2}}]
```

Compute the convolution product of the antipode with itself and the identity map on group algebra of the symmetric group:

```
sage: # needs sage.combinat sage.groups
sage: G = SymmetricGroup(3)
sage: QG = GroupAlgebra(G, QQ)
sage: x = QG.sum_of_terms(
....:         [(p, p.number_of_peaks() + p.number_of_inversions())
....:          for p in Permutations(3)]
....:     ); x
2*[1, 3, 2] + [2, 1, 3] + 3*[2, 3, 1] + 2*[3, 1, 2] + 3*[3, 2, 1]
sage: T = QG.convolution_product(Antipode, Antipode, Id)
sage: T(x)
2*[1, 3, 2] + [2, 1, 3] + 2*[2, 3, 1] + 3*[3, 1, 2] + 3*[3, 2, 1]
```

4.15 Bimodules

class sage.categories.bimodules.**Bimodules**(*left_base*, *right_base*, *name=None*)

Bases: *CategoryWithParameters*

The category of (R, S)-bimodules

For R and S rings, a (R, S)-bimodule X is a left R-module and right S-module such that the left and right actions commute: $r * (x * s) = (r * x) * s$.

EXAMPLES:

```
sage: Bimodules(QQ, ZZ)
Category of bimodules over Rational Field on the left and Integer Ring on the␣
↪right
sage: Bimodules(QQ, ZZ).super_categories()
[Category of left modules over Rational Field, Category of right modules over␣
↪Integer Ring]
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

additional_structure()

Return None.

Indeed, the category of bimodules defines no additional structure: a left and right module morphism between two bimodules is a bimodule morphism.

See also:

Category.additional_structure()

Todo: Should this category be a CategoryWithAxiom?

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).additional_structure()
```

classmethod an_instance()

Return an instance of this class.

EXAMPLES:

```
sage: Bimodules.an_instance()                                              #␣
↪needs sage.rings.real_mpfr
Category of bimodules over Rational Field on the left and Real Field with 53␣
↪bits of precision on the right
```

left_base_ring()

Return the left base ring over which elements of this category are defined.

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).left_base_ring()
Rational Field
```

right_base_ring()

Return the right base ring over which elements of this category are defined.

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).right_base_ring()
Integer Ring
```

super_categories()

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).super_categories()
[Category of left modules over Rational Field, Category of right modules over␣
↪Integer Ring]
```

4.16 Classical Crystals

class sage.categories.classical_crystals.**ClassicalCrystals**

Bases: *Category_singleton*

The category of classical crystals, that is crystals of finite Cartan type.

EXAMPLES:

```
sage: C = ClassicalCrystals()
sage: C
Category of classical crystals
sage: C.super_categories()
[Category of regular crystals,
 Category of finite crystals,
 Category of highest weight crystals]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

class ElementMethods

Bases: object

lusztig_involution()

Return the Lusztig involution on the classical highest weight crystal self.

The Lusztig involution on a finite-dimensional highest weight crystal $B(\lambda)$ of highest weight λ maps the highest weight vector to the lowest weight vector and the Kashiwara operator f_i to e_{i^*}, where i^* is defined as $\alpha_{i^*} = -w_0(\alpha_i)$. Here w_0 is the longest element of the Weyl group acting on the i-th simple root α_i.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',3],shape=[2,1])
sage: b = B(rows=[[1,2],[4]])
sage: b.lusztig_involution()
[[1, 4], [3]]
sage: b.to_tableau().schuetzenberger_involution(n=4)
[[1, 4], [3]]
```

(continues on next page)

(continued from previous page)

```
sage: all(b.lusztig_involution().to_tableau() == b.to_tableau().
↪schuetzenberger_involution(n=4) for b in B)
True

sage: B = crystals.Tableaux(['D',4],shape=[1])
sage: [[b,b.lusztig_involution()] for b in B]
[[[[1]], [[-1]]], [[[2]], [[-2]]], [[[3]], [[-3]]], [[[4]], [[-4]]], [[[-
↪4]],
[[4]]], [[[-3]], [[3]]], [[[-2]], [[2]]], [[[-1]], [[1]]]]

sage: B = crystals.Tableaux(['D',3],shape=[1])
sage: [[b,b.lusztig_involution()] for b in B]
[[[[1]], [[-1]]], [[[2]], [[-2]]], [[[3]], [[3]]], [[[-3]], [[-3]]],
[[[-2]], [[2]]], [[[-1]], [[1]]]]

sage: C = CartanType(['E',6])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[1])
sage: t = T[3]; t
[(-4, 2, 5)]
sage: t.lusztig_involution()
[(-2, -3, 4)]
```

class ParentMethods

Bases: `object`

cardinality()

Returns the number of elements of the crystal, using Weyl's dimension formula on each connected component.

EXAMPLES:

```
sage: C = ClassicalCrystals().example(5)
sage: C.cardinality()
6
```

character(*R=None*)

Returns the character of this crystal.

INPUT:

- `R` – a `WeylCharacterRing` (default: the default `WeylCharacterRing` for this Cartan type)

Returns the character of `self` as an element of `R`.

EXAMPLES:

```
sage: C = crystals.Tableaux("A2", shape=[2,1])
sage: chi = C.character(); chi
A2(2,1,0)

sage: T = crystals.TensorProduct(C,C)
sage: chiT = T.character(); chiT
A2(2,2,2) + 2*A2(3,2,1) + A2(3,3,0) + A2(4,1,1) + A2(4,2,0)
sage: chiT == chi^2
True
```

One may specify an alternate `WeylCharacterRing`:

```
sage: R = WeylCharacterRing("A2", style="coroots")
sage: chiT = T.character(R); chiT
A2(0,0) + 2*A2(1,1) + A2(0,3) + A2(3,0) + A2(2,2)
sage: chiT in R
True
```

It should have the same Cartan type and use the same realization of the weight lattice as `self`:

```
sage: R = WeylCharacterRing("A3", style="coroots")
sage: T.character(R)
Traceback (most recent call last):
...
ValueError: Weyl character ring does not have the right Cartan type
```

demazure_character(*w*, *f=None*)

Return the Demazure character associated to `w`.

INPUT:

- `w` – an element of the ambient weight lattice realization of the crystal, or a reduced word, or an element in the associated Weyl group

OPTIONAL:

- `f` – a function from the crystal to a module

This is currently only supported for crystals whose underlying weight space is the ambient space.

The Demazure character is obtained by applying the Demazure operator D_w (see `sage.categories.regular_crystals.RegularCrystals.ParentMethods.demazure_operator()`) to the highest weight element of the classical crystal. The simple Demazure operators D_i (see `sage.categories.regular_crystals.RegularCrystals.ElementMethods.demazure_operator_simple()`) do not braid on the level of crystals, but on the level of characters they do. That is why it makes sense to input `w` either as a weight, a reduced word, or as an element of the underlying Weyl group.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape = [2,1])
sage: e = T.weight_lattice_realization().basis()
sage: weight = e[0] + 2*e[2]
sage: weight.reduced_word()
[2, 1]
sage: T.demazure_character(weight)
x1^2*x2 + x1*x2^2 + x1^2*x3 + x1*x2*x3 + x1*x3^2

sage: T = crystals.Tableaux(['A',3],shape=[2,1])
sage: T.demazure_character([1,2,3])
x1^2*x2 + x1*x2^2 + x1^2*x3 + x1*x2*x3 + x2^2*x3
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([1,2,3])
sage: T.demazure_character(w)
x1^2*x2 + x1*x2^2 + x1^2*x3 + x1*x2*x3 + x2^2*x3

sage: T = crystals.Tableaux(['B',2], shape = [2])
sage: e = T.weight_lattice_realization().basis()
sage: weight = -2*e[1]
sage: T.demazure_character(weight)
x1^2 + x1*x2 + x2^2 + x1 + x2 + x1/x2 + 1/x2 + 1/x2^2 + 1

sage: T = crystals.Tableaux("B2",shape=[1/2,1/2])
```

(continues on next page)

(continued from previous page)

```
sage: b2=WeylCharacterRing("B2",base_ring=QQ).ambient()
sage: T.demazure_character([1,2],f=lambda x:b2(x.weight()))
b2(-1/2,1/2) + b2(1/2,-1/2) + b2(1/2,1/2)
```

REFERENCES:

- [De1974]
- [Ma2009]

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of classical crystals constructed by tensor product of classical crystals.

extra_super_categories()

EXAMPLES:

```
sage: ClassicalCrystals().TensorProducts().extra_super_categories()
[Category of classical crystals]
```

additional_structure()

Return `None`.

Indeed, the category of classical crystals defines no additional structure: it only states that its objects are $U_q(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is of finite type.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: ClassicalCrystals().additional_structure()
```

example(*n=3*)

Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = ClassicalCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

super_categories()

EXAMPLES:

```
sage: ClassicalCrystals().super_categories()
[Category of regular crystals,
 Category of finite crystals,
 Category of highest weight crystals]
```

4.17 Coalgebras

class sage.categories.coalgebras.**Coalgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of coalgebras

EXAMPLES:

```
sage: Coalgebras(QQ)
Category of coalgebras over Rational Field
sage: Coalgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

class Cocommutative(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Category of cocommutative coalgebras.

class DualObjects(*category*, **args*)

Bases: *DualObjectsCategory*

extra_super_categories()

Return the dual category.

EXAMPLES:

The category of coalgebras over the Rational Field is dual to the category of algebras over the same field:

```
sage: C = Coalgebras(QQ)
sage: C.dual()
Category of duals of coalgebras over Rational Field
sage: C.dual().super_categories() # indirect doctest
[Category of algebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

Warning: This is only correct in certain cases (finite dimension, …). See github issue #15647.

class ElementMethods

Bases: object

coproduct()

Return the coproduct of self.

EXAMPLES:

```
sage: # needs sage.modules
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6 as a permutation␣
↪group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, a.coproduct()
(B[(1,2,3)], B[(1,2,3)] # B[(1,2,3)])
sage: b, b.coproduct()
(B[(1,3)], B[(1,3)] # B[(1,3)])
```

counit()

Return the counit of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6 as a permutation␣
↪group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, a.counit()
(B[(1,2,3)], 1)
sage: b, b.counit()
(B[(1,3)], 1)
```

class Filtered(*base_category*)

Bases: *FilteredModulesCategory*

Category of filtered coalgebras.

Graded

alias of *GradedCoalgebras*

class ParentMethods

Bases: `object`

coproduct(*x*)

Return the coproduct of `x`.

Eventually, there will be a default implementation, delegating to the overloading mechanism and forcing the conversion back

EXAMPLES:

```
sage: # needs sage.modules
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6 as a permutation␣
↪group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, A.coproduct(a)
(B[(1,2,3)], B[(1,2,3)] # B[(1,2,3)])
sage: b, A.coproduct(b)
(B[(1,3)], B[(1,3)] # B[(1,3)])
```

counit(*x*)

Return the counit of `x`.

Eventually, there will be a default implementation, delegating to the overloading mechanism and forcing the conversion back

EXAMPLES:

```
sage: # needs sage.modules
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6 as a permutation␣
↪group over Rational Field
```

(continues on next page)

(continued from previous page)

```
sage: [a,b] = A.algebra_generators()
sage: a, A.counit(a)
(B[(1,2,3)], 1)
sage: b, A.counit(b)
(B[(1,3)], 1)
```

TODO: implement some tests of the axioms of coalgebras, bialgebras and Hopf algebras using the counit.

class Realizations(*category*, **args*)

Bases: *RealizationsCategory*

class ParentMethods

Bases: `object`

coproduct_by_coercion(*x*)

Return the coproduct by coercion if `coproduct_by_basis` is not implemented.

EXAMPLES:

```
sage: # needs sage.modules
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.monomial()
sage: f = m[2,1]
sage: f.coproduct.__module__
'sage.categories.coalgebras'
sage: m.coproduct_on_basis
NotImplemented
sage: m.coproduct == m.coproduct_by_coercion
True
sage: f.coproduct()
m[] # m[2, 1] + m[1] # m[2] + m[2] # m[1] + m[2, 1] # m[]
```

```
sage: # needs sage.modules
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: R.coproduct_by_coercion.__module__
'sage.categories.coalgebras'
sage: R.coproduct_on_basis
NotImplemented
sage: R.coproduct == R.coproduct_by_coercion
True
sage: R[1].coproduct()
R[] # R[1] + R[1] # R[]
```

counit_by_coercion(*x*)

Return the counit of `x` if `counit_by_basis` is not implemented.

EXAMPLES:

```
sage: sp = SymmetricFunctions(QQ).sp()                                    #␣
↪needs sage.modules
sage: sp.an_element()                                                     #␣
↪needs sage.modules
2*sp[] + 2*sp[1] + 3*sp[2]
sage: sp.counit(sp.an_element())                                          #␣
↪needs sage.modules
2
```

(continues on next page)

(continued from previous page)

```
sage: o = SymmetricFunctions(QQ).o()                                    #␣
↪needs sage.modules
sage: o.an_element()                                                    #␣
↪needs sage.modules
2*o[] + 2*o[1] + 3*o[2]
sage: o.counit(o.an_element())                                          #␣
↪needs sage.modules
-1
```

class SubcategoryMethods

Bases: `object`

Cocommutative()

Return the full subcategory of the cocommutative objects of `self`.

A coalgebra C is said to be *cocommutative* if

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}$$

in Sweedler's notation for all $c \in C$.

EXAMPLES:

```
sage: C1 = Coalgebras(ZZ).Cocommutative().WithBasis(); C1
Category of cocommutative coalgebras with basis over Integer Ring
sage: C2 = Coalgebras(ZZ).WithBasis().Cocommutative()
sage: C1 is C2
True
sage: BialgebrasWithBasis(QQ).Cocommutative()
Category of cocommutative bialgebras with basis over Rational Field
```

class Super(*base_category*)

Bases: *SuperModulesCategory*

class SubcategoryMethods

Bases: `object`

Supercocommutative()

Return the full subcategory of the supercocommutative objects of `self`.

EXAMPLES:

```
sage: Coalgebras(ZZ).WithBasis().Super().Supercocommutative()
Category of supercocommutative super coalgebras with basis over␣
↪Integer Ring
sage: BialgebrasWithBasis(QQ).Super().Supercocommutative()
Join of Category of super algebras with basis over Rational Field
 and Category of super bialgebras over Rational Field
 and Category of super coalgebras with basis over Rational Field
 and Category of supercocommutative super coalgebras over Rational␣
↪Field
```

class Supercocommutative(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Category of supercocommutative coalgebras.

extra_super_categories()

EXAMPLES:

```
sage: Coalgebras(ZZ).Super().extra_super_categories()
[Category of graded coalgebras over Integer Ring]
sage: Coalgebras(ZZ).Super().super_categories()
[Category of graded coalgebras over Integer Ring,
 Category of super modules over Integer Ring]
```

Compare this with the situation for bialgebras:

```
sage: Bialgebras(ZZ).Super().extra_super_categories()
[]
sage: Bialgebras(ZZ).Super().super_categories()
[Category of super algebras over Integer Ring,
 Category of super coalgebras over Integer Ring]
```

The category of bialgebras does not occur in these results, since super bialgebras are not bialgebras.

class TensorProducts(*category*, **args*)

Bases: `TensorProductsCategory`

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

extra_super_categories()

EXAMPLES:

```
sage: Coalgebras(QQ).TensorProducts().extra_super_categories()
[Category of coalgebras over Rational Field]
sage: Coalgebras(QQ).TensorProducts().super_categories()
[Category of tensor products of vector spaces over Rational Field,
 Category of coalgebras over Rational Field]
```

Meaning: a tensor product of coalgebras is a coalgebra

WithBasis

alias of `CoalgebrasWithBasis`

class WithRealizations(*category*, **args*)

Bases: `WithRealizationsCategory`

class ParentMethods

Bases: `object`

coproduct(*x*)

Return the coproduct of `x`.

EXAMPLES:

```
sage: # needs sage.modules
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: N.coproduct.__module__
'sage.categories.coalgebras'
```

(continues on next page)

(continued from previous page)

```
sage: N.coproduct(S[2])
S[] # S[2] + S[1] # S[1] + S[2] # S[]
```

counit(*x*)

Return the counit of `x`.

EXAMPLES:

```
sage: # needs sage.modules
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: f = s[2,1]
sage: f.counit.__module__
'sage.categories.coalgebras'
sage: f.counit()
0
```

```
sage: # needs sage.modules
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: N.counit.__module__
'sage.categories.coalgebras'
sage: N.counit(N.one())
1
sage: x = N.an_element(); x
2*S[] + 2*S[1] + 3*S[1, 1]
sage: N.counit(x)
2
```

super_categories()

EXAMPLES:

```
sage: Coalgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

4.18 Coalgebras with basis

class sage.categories.coalgebras_with_basis.**CoalgebrasWithBasis**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of coalgebras with a distinguished basis.

EXAMPLES:

```
sage: CoalgebrasWithBasis(ZZ)
Category of coalgebras with basis over Integer Ring
sage: sorted(CoalgebrasWithBasis(ZZ).super_categories(), key=str)
[Category of coalgebras over Integer Ring,
 Category of modules with basis over Integer Ring]
```

class ElementMethods

Bases: `object`

coproduct_iterated(*n=1*)

Apply `n` coproducts to `self`.

Todo: Remove dependency on `modules_with_basis` methods.

EXAMPLES:

```
sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()                   #␣
↪needs sage.combinat sage.modules
sage: Psi[2,2].coproduct_iterated(0)                                     #␣
↪needs sage.combinat sage.modules
Psi[2, 2]
sage: Psi[2,2].coproduct_iterated(2)                                     #␣
↪needs sage.combinat sage.modules
Psi[] # Psi[] # Psi[2, 2] + 2*Psi[] # Psi[2] # Psi[2]
 + Psi[] # Psi[2, 2] # Psi[] + 2*Psi[2] # Psi[] # Psi[2]
 + 2*Psi[2] # Psi[2] # Psi[] + Psi[2, 2] # Psi[] # Psi[]
```

class Filtered(*base_category*)

Bases: *FilteredModulesCategory*

Category of filtered coalgebras.

Graded

alias of *GradedCoalgebrasWithBasis*

class ParentMethods

Bases: `object`

coproduct()

If *coproduct_on_basis()* is available, construct the coproduct morphism from `self` to `self` ⊗ `self` by extending it by linearity. Otherwise, use *coproduct_by_coercion()*, if available.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6
  as a permutation group over Rational Field
sage: a, b = A.algebra_generators()
sage: a, A.coproduct(a)
(B[(1,2,3)], B[(1,2,3)] # B[(1,2,3)])
sage: b, A.coproduct(b)
(B[(1,3)], B[(1,3)] # B[(1,3)])
```

coproduct_on_basis(*i*)

The coproduct of the algebra on the basis (optional).

INPUT:

- `i` – the indices of an element of the basis of `self`

Returns the coproduct of the corresponding basis elements If implemented, the coproduct of the algebra is defined from it by linearity.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A                          #␣
↪needs sage.groups sage.modules
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6
  as a permutation group over Rational Field
sage: (a, b) = A._group.gens()                                            #␣
↪needs sage.groups sage.modules
sage: A.coproduct_on_basis(a)                                             #␣
↪needs sage.groups sage.modules
B[(1,2,3)] # B[(1,2,3)]
```

counit()

If *counit_on_basis()* is available, construct the counit morphism from `self` to `self` ⊗ `self` by extending it by linearity

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6
  as a permutation group over Rational Field
sage: a, b = A.algebra_generators()
sage: a, A.counit(a)
(B[(1,2,3)], 1)
sage: b, A.counit(b)
(B[(1,3)], 1)
```

counit_on_basis(*i*)

The counit of the algebra on the basis (optional).

INPUT:

- `i` – the indices of an element of the basis of `self`

Returns the counit of the corresponding basis elements If implemented, the counit of the algebra is defined from it by linearity.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A                          #␣
↪needs sage.groups sage.modules
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6
  as a permutation group over Rational Field
sage: (a, b) = A._group.gens()                                            #␣
↪needs sage.groups sage.modules
sage: A.counit_on_basis(a)                                                #␣
↪needs sage.groups sage.modules
1
```

class Super(*base_category*)

Bases: *SuperModulesCategory*

extra_super_categories()

EXAMPLES:

```
sage: C = Coalgebras(ZZ).WithBasis().Super()
sage: sorted(C.super_categories(), key=str)  # indirect doctest
```

(continues on next page)

(continued from previous page)

```
[Category of graded coalgebras with basis over Integer Ring,
 Category of super coalgebras over Integer Ring,
 Category of super modules with basis over Integer Ring]
```

4.19 Commutative additive groups

class sage.categories.commutative_additive_groups.**CommutativeAdditiveGroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*, *AbelianCategory*

The category of abelian groups, i.e. additive abelian monoids where each element has an inverse.

EXAMPLES:

```
sage: C = CommutativeAdditiveGroups(); C
Category of commutative additive groups
sage: C.super_categories()
[Category of additive groups, Category of commutative additive monoids]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse', 'AdditiveUnital
↪']
sage: C is CommutativeAdditiveMonoids().AdditiveInverse()
True
sage: from sage.categories.additive_groups import AdditiveGroups
sage: C is AdditiveGroups().AdditiveCommutative()
True
```

Note: This category is currently empty. It's left there for backward compatibility and because it is likely to grow in the future.

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

class ElementMethods

Bases: object

additive_order()

Return the additive order of this element.

EXAMPLES:

```
sage: G = cartesian_product([Zmod(3), Zmod(6), Zmod(5)])
sage: G((1,1,1)).additive_order()
30
sage: any((i * G((1,1,1))).is_zero() for i in range(1,30))
False
sage: 30 * G((1,1,1))
(0, 0, 0)
```

(continues on next page)

(continued from previous page)

```
sage: G = cartesian_product([ZZ, ZZ])
sage: G((0,0)).additive_order()
1
sage: G((0,1)).additive_order()
+Infinity

sage: # needs sage.rings.finite_rings
sage: K = GF(9)
sage: H = cartesian_product([
....:     cartesian_product([Zmod(2), Zmod(9)]), K])
sage: z = H(((1,2), K.gen()))
sage: z.additive_order()
18
```

4.20 Commutative additive monoids

class sage.categories.commutative_additive_monoids.**CommutativeAdditiveMonoids**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of commutative additive monoids, that is abelian additive semigroups with a unit

EXAMPLES:

```
sage: C = CommutativeAdditiveMonoids(); C
Category of commutative additive monoids
sage: C.super_categories()
[Category of additive monoids, Category of commutative additive semigroups]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveUnital']
sage: C is AdditiveMagmas().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital()
True
```

Note: This category is currently empty and only serves as a place holder to make `C.example()` work.

4.21 Commutative additive semigroups

class sage.categories.commutative_additive_semigroups.**CommutativeAdditiveSemigroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of additive abelian semigroups, i.e. sets with an associative and abelian operation +.

EXAMPLES:

```
sage: C = CommutativeAdditiveSemigroups(); C
Category of commutative additive semigroups
sage: C.example()
An example of a commutative semigroup: the free commutative semigroup generated␣
↪by ('a', 'b', 'c', 'd')

sage: sorted(C.super_categories(), key=str)
[Category of additive commutative additive magmas,
 Category of additive semigroups]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative']
sage: C is AdditiveMagmas().AdditiveAssociative().AdditiveCommutative()
True
```

Note: This category is currently empty and only serves as a place holder to make `C.example()` work.

4.22 Commutative algebra ideals

class `sage.categories.commutative_algebra_ideals.`**`CommutativeAlgebraIdeals`**`(A)`

Bases: *`Category_ideal`*

The category of ideals in a fixed commutative algebra A.

EXAMPLES:

```
sage: C = CommutativeAlgebraIdeals(QQ['x'])
sage: C
Category of commutative algebra ideals in
 Univariate Polynomial Ring in x over Rational Field
```

`algebra``()`

EXAMPLES:

```
sage: CommutativeAlgebraIdeals(QQ['x']).algebra()
Univariate Polynomial Ring in x over Rational Field
```

`super_categories``()`

EXAMPLES:

```
sage: CommutativeAlgebraIdeals(QQ['x']).super_categories()
[Category of algebra ideals in Univariate Polynomial Ring in x over Rational␣
↪Field]
```

4.23 Commutative algebras

class sage.categories.commutative_algebras.**CommutativeAlgebras**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of commutative algebras with unit over a given base ring.

EXAMPLES:

```
sage: M = CommutativeAlgebras(GF(19))
sage: M
Category of commutative algebras over Finite Field of size 19
sage: CommutativeAlgebras(QQ).super_categories()
[Category of algebras over Rational Field, Category of commutative rings]
```

This is just a shortcut for:

```
sage: Algebras(QQ).Commutative()
Category of commutative algebras over Rational Field
```

Todo:

- product (= Cartesian product)
- coproduct (= tensor product over base ring)

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of commutative algebras constructed by tensor product of commutative algebras.

extra_super_categories()

EXAMPLES:

```
sage: Algebras(QQ).Commutative().TensorProducts().extra_super_categories()
[Category of commutative rings]
sage: Algebras(QQ).Commutative().TensorProducts().super_categories()
[Category of tensor products of algebras over Rational Field,
 Category of commutative algebras over Rational Field]
```

4.24 Commutative ring ideals

class sage.categories.commutative_ring_ideals.**CommutativeRingIdeals**(*R*)

Bases: *Category_ideal*

The category of ideals in a fixed commutative ring.

EXAMPLES:

```
sage: C = CommutativeRingIdeals(IntegerRing())
sage: C
Category of commutative ring ideals in Integer Ring
```

super_categories()

EXAMPLES:

```
sage: CommutativeRingIdeals(ZZ).super_categories()
[Category of ring ideals in Integer Ring]
```

4.25 Commutative rings

class sage.categories.commutative_rings.**CommutativeRings**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of commutative rings

commutative rings with unity, i.e. rings with commutative * and a multiplicative identity

EXAMPLES:

```
sage: C = CommutativeRings(); C
Category of commutative rings
sage: C.super_categories()
[Category of rings, Category of commutative monoids]
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Let Sage knows that Cartesian products of commutative rings is a commutative ring.

EXAMPLES:

```
sage: CommutativeRings().Commutative().CartesianProducts().extra_super_
↪categories()
[Category of commutative rings]
sage: cartesian_product([ZZ, Zmod(34),
....:                    QQ, GF(5)]) in CommutativeRings()
True
```

class ElementMethods

Bases: object

class Finite(*base_category*)

Bases: *CategoryWithAxiom_singleton*

Check that Sage knows that Cartesian products of finite commutative rings is a finite commutative ring.

EXAMPLES:

```
sage: cartesian_product([Zmod(34),
....:                    GF(5)]) in Rings().Commutative().Finite()
True
```

class ParentMethods

Bases: object

cyclotomic_cosets(*q*, *cosets=None*)

Return the (multiplicative) orbits of `q` in the ring.

Let R be a finite commutative ring. The group of invertible elements R^* in R gives rise to a group action on R by multiplication. An orbit of the subgroup generated by an invertible element q is called a q-*cyclotomic coset* (since in a finite ring, each invertible element is a root of unity).

These cosets arise in the theory of minimal polynomials of finite fields, duadic codes and combinatorial designs. Fix a primitive element z of $GF(q^k)$. The minimal polynomial of z^s over $GF(q)$ is given by

$$M_s(x) = \prod_{i \in C_s} (x - z^i),$$

where C_s is the q-cyclotomic coset mod n containing s, $n = q^k - 1$.

Note: When $R = \mathbf{Z}/n\mathbf{Z}$ the smallest element of each coset is sometimes called a *coset leader*. This function returns sorted lists so that the coset leader will always be the first element of the coset.

INPUT:

- `q` – an invertible element of the ring
- `cosets` – an optional lists of elements of `self`. If provided, the function only return the list of cosets that contain some element from `cosets`.

OUTPUT:

A list of lists.

EXAMPLES:

```
sage: Zmod(11).cyclotomic_cosets(2)
[[0], [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]]
sage: Zmod(15).cyclotomic_cosets(2)
[[0], [1, 2, 4, 8], [3, 6, 9, 12], [5, 10], [7, 11, 13, 14]]
```

Since the group of invertible elements of a finite field is cyclic, the set of squares is a particular case of cyclotomic coset:

```
sage: # needs sage.rings.finite_rings
sage: K = GF(25, 'z')
sage: a = K.multiplicative_generator()
sage: K.cyclotomic_cosets(a**2, cosets=[1])
[[1, 2, 3, 4, z + 1, z + 3,
  2*z + 1, 2*z + 2, 3*z + 3,
  3*z + 4, 4*z + 2, 4*z + 4]]
sage: sorted(b for b in K if not b.is_zero() and b.is_square())
[1, 2, 3, 4, z + 1, z + 3,
 2*z + 1, 2*z + 2, 3*z + 3,
 3*z + 4, 4*z + 2, 4*z + 4]
```

We compute some examples of minimal polynomials:

```
sage: # needs sage.rings.finite_rings
sage: K = GF(27, 'z')
sage: a = K.multiplicative_generator()
sage: R.<X> = PolynomialRing(K, 'X')
sage: a.minimal_polynomial('X')
X^3 + 2*X + 1
```

(continues on next page)

(continued from previous page)

```
sage: cyc3 = Zmod(26).cyclotomic_cosets(3, cosets=[1]); cyc3
[[1, 3, 9]]
sage: prod(X - a**i for i in cyc3[0])                                   #␣
↪needs sage.rings.finite_rings
X^3 + 2*X + 1
sage: (a**7).minimal_polynomial('X')                                    #␣
↪needs sage.rings.finite_rings
X^3 + X^2 + 2*X + 1
sage: cyc7 = Zmod(26).cyclotomic_cosets(3, cosets=[7]); cyc7
[[7, 11, 21]]
sage: prod(X - a**i for i in cyc7[0])                                   #␣
↪needs sage.rings.finite_rings
X^3 + X^2 + 2*X + 1
```

Cyclotomic cosets of fields are useful in combinatorial design theory to provide so called difference families (see Wikipedia article Difference_set and `difference_family`). This is illustrated on the following examples:

```
sage: K = GF(5)
sage: a = K.multiplicative_generator()                                  #␣
↪needs sage.libs.pari
sage: H = K.cyclotomic_cosets(a**2, cosets=[1, 2]); H                   #␣
↪needs sage.rings.finite_rings
[[1, 4], [2, 3]]
sage: sorted(x - y for D in H for x in D for y in D if x != y)          #␣
↪needs sage.rings.finite_rings
[1, 2, 3, 4]

sage: K = GF(37)
sage: a = K.multiplicative_generator()                                  #␣
↪needs sage.libs.pari
sage: H = K.cyclotomic_cosets(a**4, cosets=[1]); H                      #␣
↪needs sage.rings.finite_rings
[[1, 7, 9, 10, 12, 16, 26, 33, 34]]
sage: sorted(x - y for D in H for x in D for y in D if x != y)          #␣
↪needs sage.rings.finite_rings
[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ..., 33, 34, 34, 35, 35, 36, 36]
```

The method `cyclotomic_cosets` works on any finite commutative ring:

```
sage: R = cartesian_product([GF(7), Zmod(14)])
sage: a = R((3,5))
sage: R.cyclotomic_cosets((3,5), [(1,1)])
[[(1, 1), (2, 11), (3, 5), (4, 9), (5, 3), (6, 13)]]
```

class ParentMethods

Bases: `object`

is_commutative()

Return whether the ring is commutative.

The answer is `True` only if the category is a sub-category of `CommutativeRings`.

It is recommended to use instead `R in Rings().Commutative()`.

EXAMPLES:

```
sage: QQ.is_commutative()
True
sage: QQ['x,y,z'].is_commutative()
True
```

over(*base=None*, *gen=None*, *gens=None*, *name=None*, *names=None*)

Return this ring, considered as an extension of `base`.

INPUT:

- `base` – a commutative ring or a morphism or `None` (default: `None`); the base of this extension or its defining morphism
- `gen` – a generator of this extension (over its base) or `None` (default: `None`);
- `gens` – a list of generators of this extension (over its base) or `None` (default: `None`);
- `name` – a variable name or `None` (default: `None`)
- `names` – a list or a tuple of variable names or `None` (default: `None`)

EXAMPLES:

We construct an extension of finite fields:

```
sage: # needs sage.rings.finite_rings
sage: F = GF(5^2)
sage: k = GF(5^4)
sage: z4 = k.gen()
sage: K = k.over(F); K                                                      #␣
↪needs sage.modules
Field in z4 with defining polynomial
 x^2 + (4*z2 + 3)*x + z2 over its base
```

If not explicitly given, the default generator of the top ring (here k) is used and the same name is kept:

```
sage: K.gen()                                                               #␣
↪needs sage.modules sage.rings.finite_rings
z4
sage: K(z4)                                                                 #␣
↪needs sage.modules sage.rings.finite_rings
z4
```

However, it is possible to specify another generator and/or another name. For example:

```
sage: # needs sage.modules sage.rings.finite_rings
sage: Ka = k.over(F, name='a'); Ka
Field in a with defining polynomial
 x^2 + (4*z2 + 3)*x + z2 over its base
sage: Ka.gen()
a
sage: Ka(z4)
a

sage: # needs sage.modules sage.rings.finite_rings
sage: Kb = k.over(F, gen=-z4+1, name='b')
sage: Kb
Field in b with defining polynomial x^2 + z2*x + 4 over its base
sage: Kb.gen()
b
sage: Kb(-z4+1)
b
```

Note that the shortcut `K.<a>` is also available:

```
sage: KKa.<a> = k.over(F)                                               #␣
↪needs sage.modules sage.rings.finite_rings
sage: KKa is Ka                                                         #␣
↪needs sage.modules sage.rings.finite_rings
True
```

Building an extension on top of another extension is allowed:

```
sage: L = GF(5^12).over(K); L                                           #␣
↪needs sage.modules sage.rings.finite_rings
Field in z12 with defining polynomial
 x^3 + (1 + (4*z2 + 2)*z4)*x^2 + (2 + 2*z4)*x - z4 over its base
sage: L.base_ring()                                                     #␣
↪needs sage.modules sage.rings.finite_rings
Field in z4 with defining polynomial
 x^2 + (4*z2 + 3)*x + z2 over its base
```

The successive bases of an extension are accessible via the method `sage.rings.ring_extension.RingExtension_generic.bases()`:

```
sage: L.bases()                                                         #␣
↪needs sage.modules sage.rings.finite_rings
[Field in z12 with defining polynomial
  x^3 + (1 + (4*z2 + 2)*z4)*x^2 + (2 + 2*z4)*x - z4 over its base,
 Field in z4 with defining polynomial
  x^2 + (4*z2 + 3)*x + z2 over its base,
 Finite Field in z2 of size 5^2]
```

When `base` is omitted, the canonical base of the ring is used:

```
sage: S.<x> = QQ[]
sage: E = S.over(); E                                                   #␣
↪needs sage.modules
Univariate Polynomial Ring in x over Rational Field over its base
sage: E.base_ring()                                                     #␣
↪needs sage.modules
Rational Field
```

Here is an example where `base` is a defining morphism:

```
sage: # needs sage.modules sage.rings.number_field
sage: k.<a> = QQ.extension(x^2 - 2)
sage: l.<b> = QQ.extension(x^4 - 2)
sage: f = k.hom([b^2])
sage: L = l.over(f)
sage: L
Field in b with defining polynomial x^2 - a over its base
sage: L.base_ring()
Number Field in a with defining polynomial x^2 - 2
```

Similarly, one can create a tower of extensions:

```
sage: # needs sage.modules sage.rings.number_field
sage: K = k.over()
sage: L = l.over(Hom(K, l)(f)); L
Field in b with defining polynomial x^2 - a over its base
sage: L.base_ring()
```

(continues on next page)

(continued from previous page)

```
Field in a with defining polynomial x^2 - 2 over its base
sage: L.bases()
[Field in b with defining polynomial x^2 - a over its base,
 Field in a with defining polynomial x^2 - 2 over its base,
 Rational Field]
```

4.26 Complete Discrete Valuation Rings (CDVR) and Fields (CDVF)

class sage.categories.complete_discrete_valuation.**CompleteDiscreteValuationFields**

Bases: *Category_singleton*

The category of complete discrete valuation fields

EXAMPLES:

```
sage: Zp(7) in CompleteDiscreteValuationFields()                          #␣
↪needs sage.rings.padics
False
sage: QQ in CompleteDiscreteValuationFields()
False
sage: LaurentSeriesRing(QQ, 'u') in CompleteDiscreteValuationFields()
True
sage: Qp(7) in CompleteDiscreteValuationFields()                          #␣
↪needs sage.rings.padics
True
sage: TestSuite(CompleteDiscreteValuationFields()).run()
```

class ElementMethods

Bases: object

denominator()

Return the denominator of this element normalized as a power of the uniformizer

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(7)
sage: x = K(1/21)
sage: x.denominator()
7 + O(7^21)
sage: x = K(7)
sage: x.denominator()
1 + O(7^20)
```

Note that the denominator lives in the ring of integers:

```
sage: x.denominator().parent()                                            #␣
↪needs sage.rings.padics
7-adic Ring with capped relative precision 20
```

When the denominator is indistinguishable from 0 and the precision on the input is $O(p^n)$, the return value is 1 if n is nonnegative and $p^(- n)$ otherwise:

```
sage: # needs sage.rings.padics
sage: x = K(0, 5); x
O(7^5)
sage: x.denominator()
1 + O(7^20)
sage: x = K(0, -5); x
O(7^-5)
sage: x.denominator()
7^5 + O(7^25)
```

numerator()

Return the numerator of this element, normalized in such a way that $x = x.numerator()/x.denominator()$ always holds true.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(7, 5)
sage: x = K(1/21)
sage: x.numerator()
5 + 4*7 + 4*7^2 + 4*7^3 + 4*7^4 + O(7^5)
sage: x == x.numerator() / x.denominator()
True
```

Note that the numerator lives in the ring of integers:

```
sage: x.numerator().parent()                                                    #␣
↪needs sage.rings.padics
7-adic Ring with capped relative precision 5
```

valuation()

Return the valuation of this element.

EXAMPLES:

```
sage: K = Qp(7)                                                                 #␣
↪needs sage.rings.padics
sage: x = K(7); x                                                               #␣
↪needs sage.rings.padics
7 + O(7^21)
sage: x.valuation()                                                             #␣
↪needs sage.rings.padics
1
```

super_categories()

EXAMPLES:

```
sage: CompleteDiscreteValuationFields().super_categories()
[Category of discrete valuation fields]
```

class sage.categories.complete_discrete_valuation.**CompleteDiscreteValuationRings**

Bases: *Category_singleton*

The category of complete discrete valuation rings

EXAMPLES:

```
sage: Zp(7) in CompleteDiscreteValuationRings()                         #␣
↪needs sage.rings.padics
True
sage: QQ in CompleteDiscreteValuationRings()
False
sage: QQ[['u']] in CompleteDiscreteValuationRings()
True
sage: Qp(7) in CompleteDiscreteValuationRings()                         #␣
↪needs sage.rings.padics
False
sage: TestSuite(CompleteDiscreteValuationRings()).run()
```

class ElementMethods

Bases: `object`

denominator()

Return the denominator of this element normalized as a power of the uniformizer

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(7)
sage: x = K(1/21)
sage: x.denominator()
7 + O(7^21)
sage: x = K(7)
sage: x.denominator()
1 + O(7^20)
```

Note that the denominator lives in the ring of integers:

```
sage: x.denominator().parent()                                          #␣
↪needs sage.rings.padics
7-adic Ring with capped relative precision 20
```

When the denominator is indistinguishable from 0 and the precision on the input is $O(p^n)$, the return value is 1 if n is nonnegative and $p^(-n)$ otherwise:

```
sage: # needs sage.rings.padics
sage: x = K(0, 5); x
O(7^5)
sage: x.denominator()
1 + O(7^20)
sage: x = K(0, -5); x
O(7^-5)
sage: x.denominator()
7^5 + O(7^25)
```

lift_to_precision(*absprec=None*)

Return another element of the same parent with absolute precision at least `absprec`, congruent to this element modulo the precision of this element.

INPUT:

- `absprec` – an integer or `None` (default: `None`), the absolute precision of the result. If `None`, lifts to the maximum precision allowed.

Note: If setting `absprec` that high would violate the precision cap, raises a precision error. Note that the new digits will not necessarily be zero.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: R = ZpCA(17)
sage: R(-1, 2).lift_to_precision(10)
16 + 16*17 + O(17^10)
sage: R(1, 15).lift_to_precision(10)
1 + O(17^15)
sage: R(1, 15).lift_to_precision(30)
Traceback (most recent call last):
...
PrecisionError: precision higher than allowed by the precision cap
sage: (R(-1, 2).lift_to_precision().precision_absolute()
....:   == R.precision_cap())
True

sage: R = Zp(5); c = R(17, 3); c.lift_to_precision(8)                    #␣
↪needs sage.rings.padics
2 + 3*5 + O(5^8)
sage: c.lift_to_precision().precision_relative() == R.precision_cap()    #␣
↪needs sage.rings.padics
True
```

numerator()

Return the numerator of this element, normalized in such a way that $x = x.numerator()/x.denominator()$ always holds true.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K = Qp(7, 5)
sage: x = K(1/21)
sage: x.numerator()
5 + 4*7 + 4*7^2 + 4*7^3 + 4*7^4 + O(7^5)
sage: x == x.numerator() / x.denominator()
True
```

Note that the numerator lives in the ring of integers:

```
sage: x.numerator().parent()                                              #␣
↪needs sage.rings.padics
7-adic Ring with capped relative precision 5
```

valuation()

Return the valuation of this element.

EXAMPLES:

```
sage: R = Zp(7)                                                           #␣
↪needs sage.rings.padics
sage: x = R(7); x                                                         #␣
↪needs sage.rings.padics
7 + O(7^21)
sage: x.valuation()                                                       #␣
↪needs sage.rings.padics
1
```

super_categories()

EXAMPLES:

```
sage: CompleteDiscreteValuationRings().super_categories()
[Category of discrete valuation rings]
```

4.27 Complex reflection groups

class sage.categories.complex_reflection_groups.**ComplexReflectionGroups**

Bases: *Category_singleton*

The category of complex reflection groups.

Let V be a complex vector space. A *complex reflection* is an element of $\mathrm{GL}(V)$ fixing a hyperplane pointwise and acting by multiplication by a root of unity on a complementary line.

A *complex reflection group* is a group W that is (isomorphic to) a subgroup of some general linear group $\mathrm{GL}(V)$ generated by a distinguished set of complex reflections.

The dimension of V is the *rank* of W.

For a comprehensive treatment of complex reflection groups and many definitions and theorems used here, we refer to [LT2009]. See also Wikipedia article Reflection_group.

See also:

ReflectionGroup() for usage examples of this category.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups()
Category of complex reflection groups
sage: ComplexReflectionGroups().super_categories()
[Category of complex reflection or generalized Coxeter groups]
sage: ComplexReflectionGroups().all_super_categories()
[Category of complex reflection groups,
 Category of complex reflection or generalized Coxeter groups,
 Category of groups,
 Category of monoids,
 Category of finitely generated semigroups,
 Category of semigroups,
 Category of finitely generated magmas,
 Category of inverse unital magmas,
 Category of unital magmas,
 Category of magmas,
 Category of enumerated sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

An example of a reflection group:

```
sage: W = ComplexReflectionGroups().example(); W                           #␣
↪needs sage.combinat
5-colored permutations of size 3
```

W is in the category of complex reflection groups:

```
sage: W in ComplexReflectionGroups()                                           #␣
↪needs sage.combinat
True
```

Finite

alias of *FiniteComplexReflectionGroups*

class ParentMethods

Bases: object

rank()

Return the rank of self.

The rank of self is the dimension of the smallest faithfull reflection representation of self.

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: W.rank()
3
```

additional_structure()

Return None.

Indeed, all the structure complex reflection groups have in addition to groups (simple reflections, …) is already defined in the super category.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().additional_structure()
```

example()

Return an example of a complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().example()                                      #␣
↪needs sage.combinat
5-colored permutations of size 3
```

super_categories()

Return the super categories of self.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().super_categories()
[Category of complex reflection or generalized Coxeter groups]
```

4.28 Common category for Generalized Coxeter Groups or Complex Reflection Groups

class `sage.categories.complex_reflection_or_generalized_coxeter_groups.`**`ComplexReflectionOrGeneralizedCoxeterGroups`**

Bases: *`Category_singleton`*

The category of complex reflection groups or generalized Coxeter groups.

Finite Coxeter groups can be defined equivalently as groups generated by reflections, or by presentations. Over the last decades, the theory has been generalized in both directions, leading to the study of (finite) complex reflection groups on the one hand, and (finite) generalized Coxeter groups on the other hand. Many of the features remain similar, yet, in the current state of the art, there is no general theory covering both directions.

This is reflected by the name of this category which is about factoring out the common code, tests, and declarations.

A group in this category has:

- A distinguished finite set of generators $(s_i)_I$, called *simple reflections*. The set I is called the *index set*. The name "reflection" is somewhat of an abuse as they can have higher order; still, they are all of finite order: $s_i^k = 1$ for some k.
- A collection of *distinguished reflections* which are the conjugates of the simple reflections. For complex reflection groups, they are in one-to-one correspondence with the reflection hyperplanes and share the same index set.
- A collection of *reflections* which are the conjugates of all the non trivial powers of the simple reflections.

The usual notions of reduced words, length, irreducibility, etc can be canonically defined from the above.

The following methods must be implemented:

- *`ComplexReflectionOrGeneralizedCoxeterGroups.ParentMethods.index_set()`*
- *`ComplexReflectionOrGeneralizedCoxeterGroups.ParentMethods.simple_reflection()`*

Optionally one can define analog methods for distinguished reflections and reflections (see below).

At least one of the following methods must be implemented:

- *`ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods.apply_simple_reflection()`*
- *`ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods.apply_simple_reflection_left()`*
- *`ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods.apply_simple_reflection_right()`*
- `ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods._mul_()`

It's recommended to implement either `_mul_` or both `apply_simple_reflection_left` and `apply_simple_reflection_right`.

See also:

- `complex_reflection_groups.ComplexReflectionGroups`
- `generalized_coxeter_groups.GeneralizedCoxeterGroups`

EXAMPLES:

```
sage: from sage.categories.complex_reflection_or_generalized_coxeter_groups␣
↪import ComplexReflectionOrGeneralizedCoxeterGroups
sage: C = ComplexReflectionOrGeneralizedCoxeterGroups(); C
Category of complex reflection or generalized Coxeter groups
sage: C.super_categories()
[Category of finitely generated enumerated groups]

sage: C.required_methods()
{'element': {'optional': ['reflection_length'],
             'required': []},
  'parent': {'optional': ['distinguished_reflection', 'hyperplane_index_set',
                          'irreducible_components',
                          'reflection', 'reflection_index_set'],
            'required':  ['__contains__', 'index_set']}}
```

class ElementMethods

Bases: `object`

apply_conjugation_by_simple_reflection(*i*)

Conjugate `self` by the `i`-th simple reflection.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.apply_conjugation_by_simple_reflection(1).reduced_word()
[3, 2]
```

apply_reflections(*word*, *side='right'*, *word_type='all'*)

Return the result of the (left/right) multiplication of `self` by `word`.

INPUT:

- `word` – a sequence of indices of reflections
- `side` – (default: `'right'`) indicates multiplying from left or right
- `word_type` – (optional, default: `'all'`): either `'simple'`, `'distinguished'`, or `'all'`

EXAMPLES:

```
sage: # optional - gap3
sage: W = ReflectionGroup((1,1,3))
sage: W.one().apply_reflections([1])
(1,4)(2,3)(5,6)
sage: W.one().apply_reflections([2])
(1,3)(2,5)(4,6)
sage: W.one().apply_reflections([2,1])
(1,2,6)(3,4,5)


sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_reflections([0,1], word_type='simple')
(2, 3, 1, 0)
sage: w
(1, 2, 3, 0)
sage: w.apply_reflections([0,1], side='left', word_type='simple')
(0, 1, 3, 2)
```

(continues on next page)

(continued from previous page)

```
sage: W = WeylGroup("A3", prefix='s')
sage: w = W.an_element(); w
s1*s2*s3
sage: AS = W.domain()
sage: r1 = AS.roots()[4]
sage: r1
(0, 1, 0, -1)
sage: r2 = AS.roots()[5]
sage: r2
(0, 0, 1, -1)
sage: w.apply_reflections([r1, r2], word_type='all')
s1


sage: # optional - gap3
sage: W = ReflectionGroup((1,1,3))
sage: W.one().apply_reflections([1], word_type='distinguished')
(1,4)(2,3)(5,6)
sage: W.one().apply_reflections([2],   word_type='distinguished')
(1,3)(2,5)(4,6)
sage: W.one().apply_reflections([3],   word_type='distinguished')
(1,5)(2,4)(3,6)
sage: W.one().apply_reflections([2,1], word_type='distinguished')
(1,2,6)(3,4,5)

sage: W = ReflectionGroup((1,1,3), hyperplane_index_set=['A','B','C']); W␣
↪  # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: W.one().apply_reflections(['A'], word_type='distinguished')   #␣
↪optional - gap3
(1,4)(2,3)(5,6)
```

apply_simple_reflection(*i*, *side='right'*)

Return `self` multiplied by the simple reflection `s[i]`.

INPUT:

- `i` – an element of the index set
- `side` – (default: `"right"`) `"left"` or `"right"`

This default implementation simply calls *apply_simple_reflection_left()* or *apply_simple_reflection_right()*.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflection(0, side = "left")
(0, 2, 3, 1)
sage: w.apply_simple_reflection(1, side = "left")
(2, 1, 3, 0)
sage: w.apply_simple_reflection(2, side = "left")
(1, 3, 2, 0)

sage: w.apply_simple_reflection(0, side = "right")
(2, 1, 3, 0)
sage: w.apply_simple_reflection(1, side = "right")
```

(continues on next page)

(continued from previous page)

```
(1, 3, 2, 0)
sage: w.apply_simple_reflection(2, side = "right")
(1, 2, 0, 3)
```

By default, `side` is `"right"`:

```
sage: w.apply_simple_reflection(0)
(2, 1, 3, 0)
```

Some tests with a complex reflection group:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: W = ComplexReflectionGroups().example(); W
5-colored permutations of size 3
sage: w = W.an_element(); w
[[1, 0, 0], [3, 1, 2]]
sage: w.apply_simple_reflection(1, side="left")
[[0, 1, 0], [1, 3, 2]]
sage: w.apply_simple_reflection(2, side="left")
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection(3, side="left")
[[1, 0, 1], [3, 1, 2]]

sage: w.apply_simple_reflection(1, side="right")
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection(2, side="right")
[[1, 0, 0], [2, 1, 3]]
sage: w.apply_simple_reflection(3, side="right")
[[2, 0, 0], [3, 1, 2]]
```

apply_simple_reflection_left(*i*)

Return `self` multiplied by the simple reflection `s[i]` on the left.

This low level method is used intensively. Coxeter groups are encouraged to override this straightforward implementation whenever a faster approach exists.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflection_left(0)
(0, 2, 3, 1)
sage: w.apply_simple_reflection_left(1)
(2, 1, 3, 0)
sage: w.apply_simple_reflection_left(2)
(1, 3, 2, 0)
```

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: W = ComplexReflectionGroups().example()
sage: w = W.an_element(); w
[[1, 0, 0], [3, 1, 2]]
sage: w.apply_simple_reflection_left(1)
```

(continues on next page)

(continued from previous page)

```
[[0, 1, 0], [1, 3, 2]]
sage: w.apply_simple_reflection_left(2)
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection_left(3)
[[1, 0, 1], [3, 1, 2]]
```

apply_simple_reflection_right(*i*)

Return `self` multiplied by the simple reflection `s[i]` on the right.

This low level method is used intensively. Coxeter groups are encouraged to override this straightforward implementation whenever a faster approach exists.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflection_right(0)
(2, 1, 3, 0)
sage: w.apply_simple_reflection_right(1)
(1, 3, 2, 0)
sage: w.apply_simple_reflection_right(2)
(1, 2, 0, 3)

sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: W = ComplexReflectionGroups().example()
sage: w = W.an_element(); w
[[1, 0, 0], [3, 1, 2]]
sage: w.apply_simple_reflection_right(1)
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection_right(2)
[[1, 0, 0], [2, 1, 3]]
sage: w.apply_simple_reflection_right(3)
[[2, 0, 0], [3, 1, 2]]
```

apply_simple_reflections(*word*, *side='right'*, *type='simple'*)

Return the result of the (left/right) multiplication of `self` by `word`.

INPUT:

- `word` – a sequence of indices of simple reflections
- `side` – (default: `'right'`) indicates multiplying from left or right

This is a specialized implementation of *`apply_reflections()`* for the simple reflections. The rationale for its existence are:

- It can take advantage of `apply_simple_reflection`, which often is less expensive than computing a product.
- It reduced burden on implementations that would want to provide an optimized version of this method.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflections([0,1])
(2, 3, 1, 0)
sage: w
```

(continues on next page)

(continued from previous page)

```
(1, 2, 3, 0)
sage: w.apply_simple_reflections([0,1],side='left')
(0, 1, 3, 2)
```

is_reflection()

Return whether `self` is a reflection.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))                          # optional - gap3
sage: [t.is_reflection() for t in W.reflections()]          # optional - gap3
[True, True, True, True, True, True]
sage: len([t for t in W.reflections() if t.is_reflection()])     #␣
↪optional - gap3
6

sage: W = ReflectionGroup((2,1,3))                          # optional - gap3
sage: [t.is_reflection() for t in W.reflections()]          # optional - gap3
[True, True, True, True, True, True, True, True, True]
sage: len([t for t in W.reflections() if t.is_reflection()])     #␣
↪optional - gap3
9
```

reflection_length()

Return the reflection length of `self`.

This is the minimal length of a factorization of `self` into reflections.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,2))                          # optional - gap3
sage: sorted([t.reflection_length() for t in W])            # optional - gap3
[0, 1]

sage: W = ReflectionGroup((2,1,2))                          # optional - gap3
sage: sorted([t.reflection_length() for t in W])            # optional - gap3
[0, 1, 1, 1, 1, 2, 2, 2]

sage: W = ReflectionGroup((3,1,2))                          # optional - gap3
sage: sorted([t.reflection_length() for t in W])            # optional - gap3
[0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]

sage: W = ReflectionGroup((2,2,2))                          # optional - gap3
sage: sorted([t.reflection_length() for t in W])            # optional - gap3
[0, 1, 1, 2]
```

class Irreducible(*base_category*)

Bases: `CategoryWithAxiom`

class ParentMethods

Bases: `object`

irreducible_components()

Return a list containing all irreducible components of `self` as finite reflection groups.

EXAMPLES:

```
sage: W = ColoredPermutations(4, 3)
sage: W.irreducible_components()
[4-colored permutations of size 3]
```

class ParentMethods

Bases: `object`

distinguished_reflection(*i*)

Return the i-th distinguished reflection of `self`.

INPUT:

- `i` – an element of the index set of the distinguished reflections.

See also:

- *distinguished_reflections()*
- *hyperplane_index_set()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=('a','b','c','d',
↪'e','f'))  # optional - gap3
sage: for i in W.hyperplane_index_set():                    # optional -␣
↪gap3
....:     print('%s %s'%(i, W.distinguished_reflection(i)))
a (1,7)(2,4)(5,6)(8,10)(11,12)
b (1,4)(2,8)(3,5)(7,10)(9,11)
c (2,5)(3,9)(4,6)(8,11)(10,12)
d (1,8)(2,7)(3,6)(4,10)(9,12)
e (1,6)(2,9)(3,8)(5,11)(7,12)
f (1,11)(3,10)(4,9)(5,7)(6,12)
```

distinguished_reflections()

Return a finite family containing the distinguished reflections of `self`, indexed by *hyperplane_index_set()*.

A *distinguished reflection* is a conjugate of a simple reflection. For a Coxeter group, reflections and distinguished reflections coincide. For a Complex reflection groups this is a reflection acting on the complement of the fixed hyperplane H as $\exp(2\pi i/n)$, where n is the order of the reflection subgroup fixing H.

See also:

- *distinguished_reflection()*
- *hyperplane_index_set()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                        # optional - gap3
sage: distinguished_reflections = W.distinguished_reflections()  #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):       #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))
1 (1,4)(2,3)(5,6)
2 (1,3)(2,5)(4,6)
3 (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((1,1,3),hyperplane_index_set=['a','b','c'])   #␣
```

(continues on next page)

(continued from previous page)

```
↪optional - gap3
sage: distinguished_reflections = W.distinguished_reflections()  #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):        #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))
a (1,4)(2,3)(5,6)
b (1,3)(2,5)(4,6)
c (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((3,1,1))                          # optional - gap3
sage: distinguished_reflections = W.distinguished_reflections()  #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):        #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))
1 (1,2,3)

sage: W = ReflectionGroup((1,1,3), (3,1,2))                 # optional - gap3
sage: distinguished_reflections = W.distinguished_reflections()  #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):    #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))
1 (1,6)(2,5)(7,8)
2 (1,5)(2,7)(6,8)
3 (3,9,15)(4,10,16)(12,17,23)(14,18,24)(20,25,29)(21,22,26)(27,28,30)
4 (3,11)(4,12)(9,13)(10,14)(15,19)(16,20)(17,21)(18,22)(23,27)(24,28)(25,
↪26)(29,30)
5 (1,7)(2,6)(5,8)
6 (3,19)(4,25)(9,11)(10,17)(12,28)(13,15)(14,30)(16,18)(20,27)(21,29)(22,
↪23)(24,26)
7 (4,21,27)(10,22,28)(11,13,19)(12,14,20)(16,26,30)(17,18,25)(23,24,29)
8 (3,13)(4,24)(9,19)(10,29)(11,15)(12,26)(14,21)(16,23)(17,30)(18,27)(20,
↪22)(25,28)
```

from_reduced_word(*word*, *word_type='simple'*)

Return an element of `self` from its (reduced) word.

INPUT:

- `word` – a list (or iterable) of elements of the index set of `self` (resp. of the distinguished or of all reflections)
- `word_type` – (optional, default: `'simple'`): either `'simple'`, `'distinguished'`, or `'all'`

If `word` is $[i_1, i_2, \ldots, i_k]$, then this returns the corresponding product of simple reflections $s_{i_1} s_{i_2} \cdots s_{i_k}$.

If `word_type` is `'distinguished'` (resp. `'all'`), then the product of the distinguished reflections (resp. all reflections) is returned.

Note: The main use case is for constructing elements from reduced words, hence the name of this method. However, the input word need *not* be reduced.

See also:

- *index_set()*
- *reflection_index_set()*

- *hyperplane_index_set()*
- *apply_simple_reflections()*
- reduced_word()
- _test_reduced_word()

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: W.from_reduced_word([0,2,0,1])
(0, 3, 1, 2)
sage: W.from_reduced_word((0,2,0,1))
(0, 3, 1, 2)
sage: s[0]*s[2]*s[0]*s[1]
(0, 3, 1, 2)
```

We now experiment with the different values for `word_type` for the colored symmetric group:

```
sage: W = ColoredPermutations(1,4)
sage: W.from_reduced_word([1,2,1,2,1,2])
[[0, 0, 0, 0], [1, 2, 3, 4]]

sage: W.from_reduced_word([1, 2, 3]).reduced_word()
[1, 2, 3]

sage: W = WeylGroup("A3", prefix='s')
sage: AS = W.domain()
sage: r1 = AS.roots()[4]
sage: r1
(0, 1, 0, -1)
sage: r2 = AS.roots()[5]
sage: r2
(0, 0, 1, -1)
sage: W.from_reduced_word([r1, r2], word_type='all')
s3*s2

sage: W = WeylGroup("G2", prefix='s')
sage: W.from_reduced_word(W.domain().positive_roots(), word_type='all')
s1*s2

sage: W = ReflectionGroup((1,1,4))           # optional - gap3
sage: W.from_reduced_word([1,2,3], word_type='all').reduced_word()   #␣
↪optional - gap3
[1, 2, 3]

sage: W.from_reduced_word([1,2,3], word_type='all').reduced_word_in_
↪reflections()    # optional - gap3
[1, 2, 3]

sage: W.from_reduced_word([1,2,3]).reduced_word_in_reflections()     #␣
↪optional - gap3
[1, 2, 3]
```

group_generators()

Return the simple reflections of `self`, as distinguished group generators.

See also:

- *simple_reflections()*
- *Groups.ParentMethods.group_generators()*
- *Semigroups.ParentMethods.semigroup_generators()*

EXAMPLES:

```
sage: D10 = FiniteCoxeterGroups().example(10)
sage: D10.group_generators()
Finite family {1: (1,), 2: (2,)}
sage: SymmetricGroup(5).group_generators()
Finite family {1: (1,2), 2: (2,3), 3: (3,4), 4: (4,5)}

sage: W = ColoredPermutations(3,2)
sage: W.group_generators()
Finite family {1: [[0, 0],
                   [2, 1]],
               2: [[0, 1],
                   [1, 2]]}
```

The simple reflections are also semigroup generators, even for an infinite group:

```
sage: W = WeylGroup(["A",2,1])
sage: W.semigroup_generators()
Finite family {0: [-1  1  1]
                  [ 0  1  0]
                  [ 0  0  1],
               1: [ 1  0  0]
                  [ 1 -1  1]
                  [ 0  0  1],
               2: [ 1  0  0]
                  [ 0  1  0]
                  [ 1  1 -1]}
```

hyperplane_index_set()

Return the index set of the distinguished reflections of `self`.

This is also the index set of the reflection hyperplanes of `self`, hence the name. This name is slightly abusive since the concept of reflection hyperplanes is not defined for all generalized Coxeter groups. However for all practical purposes this is only used for complex reflection groups, and there this is the desirable name.

See also:

- *distinguished_reflection()*
- *distinguished_reflections()*

EXAMPLES:

```
sage: # optional - gap3
sage: W = ReflectionGroup((1,1,4))
sage: W.hyperplane_index_set()
(1, 2, 3, 4, 5, 6)
sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=[1,3,'asdf',7,9,
↪11])
sage: W.hyperplane_index_set()
(1, 3, 'asdf', 7, 9, 11)
sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=('a','b','c','d',
↪'e','f'))
```

(continues on next page)

(continued from previous page)

```
sage: W.hyperplane_index_set()
('a', 'b', 'c', 'd', 'e', 'f')
```

index_set()

Return the index set of (the simple reflections of) `self`, as a list (or iterable).

See also:

- *simple_reflection()*
- *simple_reflections()*

EXAMPLES:

```
sage: W = CoxeterGroups().Finite().example(); W
The 5-th dihedral group of order 10
sage: W.index_set()
(1, 2)

sage: W = ColoredPermutations(1, 4)
sage: W.index_set()
(1, 2, 3)
sage: W = ReflectionGroup((1,1,4), index_set=[1,3,'asdf'])  # optional -␣
↪gap3
sage: W.index_set()                                        # optional - gap3
(1, 3, 'asdf')
sage: W = ReflectionGroup((1,1,4), index_set=('a','b','c')) # optional -␣
↪gap3
sage: W.index_set()                                        # optional - gap3
('a', 'b', 'c')
```

irreducible_component_index_sets()

Return a list containing the index sets of the irreducible components of `self` as finite reflection groups.

EXAMPLES:

```
sage: W = ReflectionGroup([1,1,3], [3,1,3], 4); W         # optional - gap3
Reducible complex reflection group of rank 7 and type A2 x G(3,1,3) x ST4
sage: sorted(W.irreducible_component_index_sets())         # optional - gap3
[[1, 2], [3, 4, 5], [6, 7]]
```

ALGORITHM:

Take the connected components of the graph on the index set with edges `(i,j)`, where `s[i]` and `s[j]` do not commute.

irreducible_components()

Return the irreducible components of `self` as finite reflection groups.

EXAMPLES:

```
sage: W = ReflectionGroup([1,1,3], [3,1,3], 4)             # optional - gap3
sage: W.irreducible_components()                           # optional - gap3
[Irreducible real reflection group of rank 2 and type A2,
 Irreducible complex reflection group of rank 3 and type G(3,1,3),
 Irreducible complex reflection group of rank 2 and type ST4]
```

is_irreducible()

Return `True` if `self` is irreducible.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3); W
1-colored permutations of size 3
sage: W.is_irreducible()
True

sage: W = ReflectionGroup((1,1,3),(2,1,3)); W           # optional - gap3
Reducible real reflection group of rank 5 and type A2 x B3
sage: W.is_irreducible()                                # optional - gap3
False
```

is_reducible()

Return True if self is not irreducible.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3); W
1-colored permutations of size 3
sage: W.is_reducible()
False

sage: W = ReflectionGroup((1,1,3), (2,1,3)); W          # optional - gap3
Reducible real reflection group of rank 5 and type A2 x B3
sage: W.is_reducible()                                  # optional - gap3
True
```

number_of_irreducible_components()

Return the number of irreducible components of self.

EXAMPLES:

```
sage: SymmetricGroup(3).number_of_irreducible_components()
1

sage: ColoredPermutations(1,3).number_of_irreducible_components()
1

sage: ReflectionGroup((1,1,3),(2,1,3)).number_of_irreducible_components()␣
↪   # optional - gap3
2
```

number_of_simple_reflections()

Return the number of simple reflections of self.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.number_of_simple_reflections()
2
sage: W = ColoredPermutations(2,3)
sage: W.number_of_simple_reflections()
3
sage: W = ColoredPermutations(4,3)
sage: W.number_of_simple_reflections()
3
sage: W = ReflectionGroup((4,2,3))                      # optional - gap3
sage: W.number_of_simple_reflections()                  # optional - gap3
4
```

reflection(*i*)

Return the i-th reflection of `self`.

For i in $1, \dots, N$, this gives the i-th reflection of `self`.

See also:

- `reflections_index_set()`
- *`reflections()`*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))                          # optional - gap3
sage: for i in W.reflection_index_set():                    # optional - gap3
....:     print('%s %s'%(i, W.reflection(i)))
1 (1,7)(2,4)(5,6)(8,10)(11,12)
2 (1,4)(2,8)(3,5)(7,10)(9,11)
3 (2,5)(3,9)(4,6)(8,11)(10,12)
4 (1,8)(2,7)(3,6)(4,10)(9,12)
5 (1,6)(2,9)(3,8)(5,11)(7,12)
6 (1,11)(3,10)(4,9)(5,7)(6,12)
```

reflection_index_set()

Return the index set of the reflections of `self`.

See also:

- *`reflection()`*
- *`reflections()`*

EXAMPLES:

```
sage: # optional - gap3
sage: W = ReflectionGroup((1,1,4))
sage: W.reflection_index_set()
(1, 2, 3, 4, 5, 6)
sage: W = ReflectionGroup((1,1,4), reflection_index_set=[1,3,'asdf',7,9,
↪11])
sage: W.reflection_index_set()
(1, 3, 'asdf', 7, 9, 11)
sage: W = ReflectionGroup((1,1,4), reflection_index_set=('a','b','c','d',
↪'e','f'))
sage: W.reflection_index_set()
('a', 'b', 'c', 'd', 'e', 'f')
```

reflections()

Return a finite family containing the reflections of `self`, indexed by *`reflection_index_set()`*.

See also:

- *`reflection()`*
- *`reflection_index_set()`*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                          # optional - gap3
sage: reflections = W.reflections()                         # optional - gap3
sage: for index in sorted(reflections.keys()):              # optional - gap3
....:     print('%s %s'%(index, reflections[index]))
1 (1,4)(2,3)(5,6)
```

(continues on next page)

(continued from previous page)

```
2 (1,3)(2,5)(4,6)
3 (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((1,1,3),reflection_index_set=['a','b','c'])    #␣
↪optional - gap3
sage: reflections = W.reflections()                          # optional - gap3
sage: for index in sorted(reflections.keys()):               # optional - gap3
....:     print('%s %s'%(index, reflections[index]))
a (1,4)(2,3)(5,6)
b (1,3)(2,5)(4,6)
c (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((3,1,1))                           # optional - gap3
sage: reflections = W.reflections()                          # optional - gap3
sage: for index in sorted(reflections.keys()):               # optional - gap3
....:     print('%s %s'%(index, reflections[index]))
1 (1,2,3)
2 (1,3,2)

sage: W = ReflectionGroup((1,1,3), (3,1,2))                  # optional - gap3
sage: reflections = W.reflections()                          # optional - gap3
sage: for index in sorted(reflections.keys()):               # optional - gap3
....:     print('%s %s'%(index, reflections[index]))
1 (1,6)(2,5)(7,8)
2 (1,5)(2,7)(6,8)
3 (3,9,15)(4,10,16)(12,17,23)(14,18,24)(20,25,29)(21,22,26)(27,28,30)
4 (3,11)(4,12)(9,13)(10,14)(15,19)(16,20)(17,21)(18,22)(23,27)(24,28)(25,
↪26)(29,30)
5 (1,7)(2,6)(5,8)
6 (3,19)(4,25)(9,11)(10,17)(12,28)(13,15)(14,30)(16,18)(20,27)(21,29)(22,
↪23)(24,26)
7 (4,21,27)(10,22,28)(11,13,19)(12,14,20)(16,26,30)(17,18,25)(23,24,29)
8 (3,13)(4,24)(9,19)(10,29)(11,15)(12,26)(14,21)(16,23)(17,30)(18,27)(20,
↪22)(25,28)
9 (3,15,9)(4,16,10)(12,23,17)(14,24,18)(20,29,25)(21,26,22)(27,30,28)
10 (4,27,21)(10,28,22)(11,19,13)(12,20,14)(16,30,26)(17,25,18)(23,29,24)
```

semigroup_generators()

Return the simple reflections of `self`, as distinguished group generators.

See also:

- *simple_reflections()*
- *Groups.ParentMethods.group_generators()*
- *Semigroups.ParentMethods.semigroup_generators()*

EXAMPLES:

```
sage: D10 = FiniteCoxeterGroups().example(10)
sage: D10.group_generators()
Finite family {1: (1,), 2: (2,)}
sage: SymmetricGroup(5).group_generators()
Finite family {1: (1,2), 2: (2,3), 3: (3,4), 4: (4,5)}

sage: W = ColoredPermutations(3,2)
sage: W.group_generators()
Finite family {1: [[0, 0],
```

(continues on next page)

(continued from previous page)

```
                [2, 1]],
            2: [[0, 1],
                [1, 2]]}
```

The simple reflections are also semigroup generators, even for an infinite group:

```
sage: W = WeylGroup(["A",2,1])
sage: W.semigroup_generators()
Finite family {0: [-1  1  1]
                  [ 0  1  0]
                  [ 0  0  1],
               1: [ 1  0  0]
                  [ 1 -1  1]
                  [ 0  0  1],
               2: [ 1  0  0]
                  [ 0  1  0]
                  [ 1  1 -1]}
```

simple_reflection(*i*)

Return the i-th simple reflection s_i of `self`.

INPUT:

- `i` – an element from the index set

See also:

- *index_set()*
- *simple_reflections()*

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: W.simple_reflection(1)
(0, 2, 1, 3)
sage: s = W.simple_reflections()
sage: s[1]
(0, 2, 1, 3)

sage: W = ReflectionGroup((1,1,4), index_set=[1,3,'asdf'])  # optional -␣
↪gap3
sage: for i in W.index_set():                             # optional - gap3
....:     print('%s %s'%(i, W.simple_reflection(i)))
1 (1,7)(2,4)(5,6)(8,10)(11,12)
3 (1,4)(2,8)(3,5)(7,10)(9,11)
asdf (2,5)(3,9)(4,6)(8,11)(10,12)
```

simple_reflection_orders()

Return the orders of the simple reflections.

EXAMPLES:

```
sage: W = WeylGroup(['B',3])
sage: W.simple_reflection_orders()
[2, 2, 2]
sage: W = CoxeterGroup(['C',4])
sage: W.simple_reflection_orders()
```

(continues on next page)

(continued from previous page)

```
[2, 2, 2, 2]
sage: SymmetricGroup(5).simple_reflection_orders()
[2, 2, 2, 2]
sage: C = ColoredPermutations(4, 3)
sage: C.simple_reflection_orders()
[2, 2, 4]
```

simple_reflections()

Return the simple reflections $(s_i)_{i\in I}$ of `self` as a family indexed by *index_set()*.

See also:

- *simple_reflection()*
- *index_set()*

EXAMPLES:

For the symmetric group, we recognize the simple transpositions:

```
sage: W = SymmetricGroup(4); W
Symmetric group of order 4! as a permutation group
sage: s = W.simple_reflections()
sage: s
Finite family {1: (1,2), 2: (2,3), 3: (3,4)}
sage: s[1]
(1,2)
sage: s[2]
(2,3)
sage: s[3]
(3,4)
```

Here are the simple reflections for a colored symmetric group and a reflection group:

```
sage: W = ColoredPermutations(1,3)
sage: W.simple_reflections()
Finite family {1: [[0, 0, 0], [2, 1, 3]], 2: [[0, 0, 0], [1, 3, 2]]}

sage: W = ReflectionGroup((1,1,3), index_set=['a','b']) # optional - gap3
sage: W.simple_reflections()                             # optional - gap3
Finite family {'a': (1,4)(2,3)(5,6), 'b': (1,3)(2,5)(4,6)}
```

This default implementation uses *index_set()* and *simple_reflection()*.

some_elements()

Implement *Sets.ParentMethods.some_elements()* by returning some typical elements of `self`.

The result is currently composed of the simple reflections together with the unit and the result of `an_element()`.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: W.some_elements()
[
[0 1 0 0]  [1 0 0 0]  [1 0 0 0]  [1 0 0 0]  [0 0 0 1]
[1 0 0 0]  [0 0 1 0]  [0 1 0 0]  [0 1 0 0]  [1 0 0 0]
[0 0 1 0]  [0 1 0 0]  [0 0 0 1]  [0 0 1 0]  [0 1 0 0]
```

(continues on next page)

(continued from previous page)

```
[0 0 0 1], [0 0 0 1], [0 0 1 0], [0 0 0 1], [0 0 1 0]
]

sage: W = ColoredPermutations(1,4)
sage: W.some_elements()
[[[0, 0, 0, 0], [2, 1, 3, 4]],
 [[0, 0, 0, 0], [1, 3, 2, 4]],
 [[0, 0, 0, 0], [1, 2, 4, 3]],
 [[0, 0, 0, 0], [1, 2, 3, 4]],
 [[0, 0, 0, 0], [4, 1, 2, 3]]]
```

class SubcategoryMethods

Bases: `object`

Irreducible()

Return the full subcategory of irreducible objects of `self`.

A complex reflection group, or generalized Coxeter group is *reducible* if its simple reflections can be split in two sets X and Y such that the elements of X commute with that of Y. In particular, the group is then direct product of $\langle X\rangle$ and $\langle Y\rangle$. It's *irreducible* otherwise.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Irreducible()
Category of irreducible complex reflection groups
sage: CoxeterGroups().Irreducible()
Category of irreducible Coxeter groups
```

super_categories()

Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().super_categories()
[Category of complex reflection or generalized Coxeter groups]
```

4.29 Coxeter Group Algebras

class sage.categories.coxeter_group_algebras.**CoxeterGroupAlgebras**(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: `object`

demazure_lusztig_eigenvectors(*q1*, *q2*)

Return the family of eigenvectors for the Cherednik operators.

INPUT:

- `self` – a finite Coxeter group W
- `q1,q2` – two elements of the ground ring K

The affine Hecke algebra $H_{q_1,q_2}(\tilde{W})$ acts on the group algebra of W through the Demazure-Lusztig operators T_i. Its Cherednik operators Y^λ can be simultaneously diagonalized as long as q_1/q_2 is not a small root of unity [HST2008].

This method returns the family of joint eigenvectors, indexed by W.

See also:

- *demazure_lusztig_operators()*
- sage.combinat.root_system.hecke_algebra_representation.CherednikOperatorsEigenvectors

EXAMPLES:

```
sage: W = WeylGroup(["B",2])
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_
↪word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1,q2)
sage: E.keys()                                                          #␣
↪needs sage.rings.number_field
Weyl Group of type ['B', 2] (as a matrix group acting on the ambient␣
↪space)
sage: w = W.an_element()
sage: E[w]                                                              #␣
↪needs sage.rings.number_field
(q2/(-q1+q2))*2121 + ((-q2)/(-q1+q2))*121 - 212 + 12
```

demazure_lusztig_operator_on_basis(*w*, *i*, *q1*, *q2*, *side='right'*)

Return the result of applying the i-th Demazure Lusztig operator on `w`.

INPUT:

- `w` – an element of the Coxeter group
- `i` – an element of the index set
- `q1,q2` – two elements of the ground ring
- `bar` – a boolean (default `False`)

See *demazure_lusztig_operators()* for details.

EXAMPLES:

```
sage: W = WeylGroup(["B",3])
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_
↪word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: w = W.an_element()
sage: KW.demazure_lusztig_operator_on_basis(w, 0, q1, q2)
(-q2)*323123 + (q1+q2)*123
sage: KW.demazure_lusztig_operator_on_basis(w, 1, q1, q2)
q1*1231
sage: KW.demazure_lusztig_operator_on_basis(w, 2, q1, q2)
q1*1232
sage: KW.demazure_lusztig_operator_on_basis(w, 3, q1, q2)
(q1+q2)*123 + (-q2)*12
```

At $q_1 = 1$ and $q_2 = 0$ we recover the action of the isobaric divided differences π_i:

```
sage: KW.demazure_lusztig_operator_on_basis(w, 0, 1, 0)
123
sage: KW.demazure_lusztig_operator_on_basis(w, 1, 1, 0)
1231
sage: KW.demazure_lusztig_operator_on_basis(w, 2, 1, 0)
1232
sage: KW.demazure_lusztig_operator_on_basis(w, 3, 1, 0)
123
```

At $q_1 = 1$ and $q_2 = -1$ we recover the action of the simple reflection s_i:

```
sage: KW.demazure_lusztig_operator_on_basis(w, 0, 1, -1)
323123
sage: KW.demazure_lusztig_operator_on_basis(w, 1, 1, -1)
1231
sage: KW.demazure_lusztig_operator_on_basis(w, 2, 1, -1)
1232
sage: KW.demazure_lusztig_operator_on_basis(w, 3, 1, -1)
12
```

demazure_lusztig_operators(*q1*, *q2*, *side='right'*, *affine=True*)

Return the Demazure Lusztig operators acting on `self`.

INPUT:

- `q1,q2` – two elements of the ground ring K
- `side` – `"left"` or `"right"` (default: `"right"`); which side to act upon
- `affine` – a boolean (default: `True`)

The Demazure-Lusztig operator T_i is the linear map $R \to R$ obtained by interpolating between the simple projection π_i (see `CoxeterGroups.ElementMethods.simple_projection()`) and the simple reflection s_i so that T_i has eigenvalues q_1 and q_2:

$$(q_1 + q_2)\pi_i - q_2 s_i.$$

The Demazure-Lusztig operators give the usual representation of the operators T_i of the q_1, q_2 Hecke algebra associated to the Coxeter group.

For a finite Coxeter group, and if `affine=True`, the Demazure-Lusztig operators $T_1, \ldots, T_n$ are completed by T_0 to implement the level 0 action of the affine Hecke algebra.

EXAMPLES:

```
sage: W = WeylGroup(["B",3])
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_
↪word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: T = KW.demazure_lusztig_operators(q1, q2, affine=True)
sage: x = KW.monomial(W.an_element()); x
123
sage: T[0](x)
(-q2)*323123 + (q1+q2)*123
sage: T[1](x)
q1*1231
sage: T[2](x)
q1*1232
sage: T[3](x)
```

(continues on next page)

(continued from previous page)

```
(q1+q2)*123 + (-q2)*12

sage: T._test_relations()
```

Note: For a finite Weyl group W, the level 0 action of the affine Weyl group $\tilde{W}$ only depends on the Coxeter diagram of the affinization, not its Dynkin diagram. Hence it is possible to explore all cases using only untwisted affinizations.

4.30 Coxeter Groups

class sage.categories.coxeter_groups.**CoxeterGroups**

Bases: *Category_singleton*

The category of Coxeter groups.

A *Coxeter group* is a group W with a distinguished (finite) family of involutions $(s_i)_{i\in I}$, called the *simple reflections*, subject to relations of the form $(s_i s_j)^{m_{i,j}} = 1$.

I is the *index set* of W and $|I|$ is the *rank* of W.

See Wikipedia article Coxeter_group for details.

EXAMPLES:

```
sage: C = CoxeterGroups(); C
Category of Coxeter groups
sage: C.super_categories()
[Category of generalized Coxeter groups]

sage: W = C.example(); W
The symmetric group on {0, ..., 3}

sage: W.simple_reflections()
Finite family {0: (1, 0, 2, 3), 1: (0, 2, 1, 3), 2: (0, 1, 3, 2)}
```

Here are some further examples:

```
sage: FiniteCoxeterGroups().example()
The 5-th dihedral group of order 10
sage: FiniteWeylGroups().example()
The symmetric group on {0, ..., 3}
sage: WeylGroup(["B", 3])                                                  #␣
↪needs sage.combinat sage.groups
Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space)

sage: S4 = SymmetricGroup(4); S4                                          #␣
↪needs sage.groups
Symmetric group of order 4! as a permutation group
sage: S4 in CoxeterGroups().Finite()                                      #␣
↪needs sage.groups
True
```

Those will eventually be also in this category:

```
sage: DihedralGroup(5)                                                     #␣
↪needs sage.groups
Dihedral group of order 10 as a permutation group
```

Todo: add a demo of usual computations on Coxeter groups.

See also:

- `sage.combinat.root_system`
- *WeylGroups*
- *GeneralizedCoxeterGroups*

Warning: It is assumed that morphisms in this category preserve the distinguished choice of simple reflections. In particular, subobjects in this category are parabolic subgroups. In this sense, this category might be better named `Coxeter Systems`. In the long run we might want to have two distinct categories, one for Coxeter groups (with morphisms being just group morphisms) and one for Coxeter systems:

```
sage: CoxeterGroups().is_full_subcategory(Groups())
False
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: CoxeterGroups().is_full_subcategory(GeneralizedCoxeterGroups())
True
```

`Algebras`

alias of *CoxeterGroupAlgebras*

`class ElementMethods`

Bases: `object`

`absolute_chain()`

Return a (saturated) chain in absolute order from `1` to `self`.

See also:

absolute_chain_reflections()

EXAMPLES:

```
sage: W = CoxeterGroup(['A', 2, 1])
sage: (r, s, t) = W.simple_reflections()
sage: (r * s * r * t).absolute_chain()
[
[1 0 0]   [ 0 -1  2]   [ 2  1 -2]
[0 1 0]   [-1  0  2]   [ 1  2 -2]
[0 0 1],  [ 0  0  1],  [ 1  1 -1]
]
```

`absolute_chain_reflections()`

Return a list of reflection which, when (left) multiplied in order, give `self`.

This method is based on Theorem 1.1 in [Dy2001], combined with the strong exchange condition. As an example, if W is a type A_2 Coxeter group with simple reflections a, b, then the absolute chain reflections for the element $w = ab$ is the list $[a, aba]$ as $w = (aba)a = ab$.

See also:

absolute_length(), *absolute_chain()*

EXAMPLES:

```
sage: W = CoxeterGroup(["A",2,1])
sage: W.one().absolute_chain_reflections()
[]
sage: (r, s, t) = W.simple_reflections()
sage: r.absolute_chain_reflections()
[
[-1  1  1]
[ 0  1  0]
[ 0  0  1]
]
sage: (r * s).absolute_chain_reflections()
[
[-1  1  1]  [ 0 -1  2]
[ 0  1  0]  [-1  0  2]
[ 0  0  1], [ 0  0  1]
]
sage: (r * s * r * t).absolute_chain_reflections()
[
[ 0 -1  2]  [-1 -2  4]
[-1  0  2]  [-2 -1  4]
[ 0  0  1], [-1 -1  3]
]
sage: W = CoxeterGroup(['A', 3, 1])
sage: (r, s, t, u) = W.simple_reflections()
sage: (r * s * t * u).absolute_chain_reflections()
[
[-1  1  0  1]  [ 0 -1  1  1]  [ 0  0 -1  2]  [-3  2  0  2]
[ 0  1  0  0]  [-1  0  1  1]  [-1  1 -1  2]  [-2  2  0  1]
[ 0  0  1  0]  [ 0  0  1  0]  [-1  0  0  2]  [-2  1  1  1]
[ 0  0  0  1], [ 0  0  0  1], [ 0  0  0  1], [-2  1  0  2]
]
sage: (r * s * t * u * s).absolute_chain_reflections()
[
[-1  1  0  1]  [ 0  0 -1  2]  [-3  2  0  2]
[ 0  1  0  0]  [-1  1 -1  2]  [-2  2  0  1]
[ 0  0  1  0]  [-1  0  0  2]  [-2  1  1  1]
[ 0  0  0  1], [ 0  0  0  1], [-2  1  0  2]
]
```

absolute_covers()

Return the list of covers of `self` in absolute order.

See also:

absolute_length()

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(["A", 3])
```

(continues on next page)

(continued from previous page)

```
sage: s = W.simple_reflections()
sage: w0 = s[1]
sage: w1 = s[1]*s[2]*s[3]
sage: list(w0.absolute_covers())
[
[0 0 1 0]  [0 1 0 0]  [0 1 0 0]  [0 0 0 1]  [0 1 0 0]
[1 0 0 0]  [1 0 0 0]  [0 0 1 0]  [1 0 0 0]  [0 0 0 1]
[0 1 0 0]  [0 0 0 1]  [1 0 0 0]  [0 0 1 0]  [0 0 1 0]
[0 0 0 1], [0 0 1 0], [0 0 0 1], [0 1 0 0], [1 0 0 0]
]
```

absolute_le(*other*)

Return whether `self` is smaller than `other` in the absolute order.

A general reflection is an element of the form ws_iw^{-1}, where s_i is a simple reflection. The absolute order is defined analogously to the weak order but using general reflections rather than just simple reflections.

This partial order can be used to define noncrossing partitions associated with this Coxeter group.

See also:

absolute_length()

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(["A", 3])
sage: s = W.simple_reflections()
sage: w0 = s[1]
sage: w1 = s[1]*s[2]*s[3]
sage: w0.absolute_le(w1)
True
sage: w1.absolute_le(w0)
False
sage: w1.absolute_le(w1)
True
```

absolute_length()

Return the absolute length of `self`.

The absolute length is the length of the shortest expression of the element as a product of reflections. In general, we use Theorem 1.1 in [Dy2001].

See also:

absolute_le(), *absolute_chain()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])                                          #␣
↪needs sage.combinat sage.groups
sage: s = W.simple_reflections()                                       #␣
↪needs sage.combinat sage.groups
sage: (s[1]*s[2]*s[3]).absolute_length()                               #␣
↪needs sage.combinat sage.groups
3

sage: W = SymmetricGroup(4)                                            #␣
↪needs sage.groups
```

(continues on next page)

(continued from previous page)

```
sage: s = W.simple_reflections()                                            #␣
↪needs sage.groups
sage: (s[3]*s[2]*s[1]).absolute_length()                                    #␣
↪needs sage.combinat sage.groups
3

sage: W = CoxeterGroup(["A",2,1])
sage: (r, s, t) = W.simple_reflections()
sage: (r * s * r * t).absolute_length()
2
sage: W.one().absolute_length()
0
sage: r.absolute_length()
1
sage: (r * s).absolute_length()
2
sage: (r * s * r).absolute_length()
1
sage: W = CoxeterGroup(['A', 3, 1])
sage: (r, s, t, u) = W.simple_reflections()
sage: (r * s * t * u).absolute_length()
4
sage: (r * s * t * u * s).absolute_length()
3
```

apply_demazure_product (*element*, *side='right'*, *length_increasing=True*)

Return the Demazure or 0-Hecke product of `self` with another Coxeter group element.

See *CoxeterGroups.ParentMethods.simple_projections()*.

INPUT:

- **element – either an element of the same Coxeter**
  group as `self` or a tuple or a list (such as a reduced word) of elements from the index set of the Coxeter group.
- **side – 'left' or 'right' (default: 'right'); the**
  side of `self` on which the element should be applied. If `side` is 'left' then the operation is applied on the left.
- **length_increasing – a boolean (default True)**
  whether to act length increasingly or decreasingly

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['C', 4], prefix="s")
sage: v = W.from_reduced_word([1,2,3,4,3,1])
sage: v.apply_demazure_product([1,3,4,3,3])
s4*s1*s2*s3*s4*s3*s1
sage: v.apply_demazure_product([1,3,4,3], side='left')
s3*s4*s1*s2*s3*s4*s2*s3*s1
sage: v.apply_demazure_product((1,3,4,3), side='left')
s3*s4*s1*s2*s3*s4*s2*s3*s1
sage: v.apply_demazure_product(v)
s2*s3*s4*s1*s2*s3*s4*s2*s3*s2*s1
```

apply_simple_projection (*i*, *side='right'*, *length_increasing=True*)

Return the result of the application of the simple projection π_i (resp. $\overline{\pi}_i$) on `self`.

INPUT:

- `i` - an element of the index set of the Coxeter group
- `side` - 'left' or 'right' (default: 'right')
- `length_increasing` - a boolean (default: True) specifying the direction of the projection

See *CoxeterGroups.ParentMethods.simple_projections()* for the definition of the simple projections.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element()
sage: w
(1, 2, 3, 0)
sage: w.apply_simple_projection(2)
(1, 2, 3, 0)
sage: w.apply_simple_projection(2, length_increasing=False)
(1, 2, 0, 3)

sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['C', 4], prefix="s")
sage: v = W.from_reduced_word([1,2,3,4,3,1])
sage: v
s1*s2*s3*s4*s3*s1
sage: v.apply_simple_projection(2)
s1*s2*s3*s4*s3*s1*s2
sage: v.apply_simple_projection(2, side='left')
s1*s2*s3*s4*s3*s1
sage: v.apply_simple_projection(1, length_increasing=False)
s1*s2*s3*s4*s3
```

binary_factorizations(*predicate=The constant function (...) -> True*)

Return the set of all the factorizations $self = uv$ such that $l(self) = l(u) + l(v)$.

Iterating through this set is Constant Amortized Time (counting arithmetic operations in the Coxeter group as constant time) complexity, and memory linear in the length of $self$.

One can pass as optional argument a predicate p such that $p(u)$ implies $p(u')$ for any u left factor of $self$ and u' left factor of u. Then this returns only the factorizations $self = uv$ such $p(u)$ holds.

EXAMPLES:

We construct the set of all factorizations of the maximal element of the group:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 3])
sage: s = W.simple_reflections()
sage: w0 = W.from_reduced_word([1,2,3,1,2,1])
sage: w0.binary_factorizations().cardinality()
24
```

The same number of factorizations, by bounded length:

```
sage: [w0.binary_factorizations(                                        #␣
↪needs sage.combinat sage.groups
....:      lambda u: u.length() <= l
....:  ).cardinality()
....:  for l in [-1,0,1,2,3,4,5,6]]
[0, 1, 4, 9, 15, 20, 23, 24]
```

The number of factorizations of the elements just below the maximal element:

```
sage: [(s[i]*w0).binary_factorizations().cardinality()                    #␣
↪needs sage.combinat sage.groups
....:  for i in [1,2,3]]
[12, 12, 12]
sage: w0.binary_factorizations(lambda u: False).cardinality()             #␣
↪needs sage.combinat sage.groups
0
```

bruhat_le(*other*)

Return whether `self` <= `other` in the Bruhat order.

INPUT:

- other – an element of the same Coxeter group

OUTPUT: a boolean

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(["A", 3])
sage: u = W.from_reduced_word([1,2,1])
sage: v = W.from_reduced_word([1,2,3,2,1])
sage: u.bruhat_le(u)
True
sage: u.bruhat_le(v)
True
sage: v.bruhat_le(u)
False
sage: v.bruhat_le(v)
True
sage: s = W.simple_reflections()
sage: s[1].bruhat_le(W.one())
False
```

The implementation uses the equivalent condition that any reduced word for `other` contains a reduced word for `self` as subword. See Stembridge, A short derivation of the Möbius function for the Bruhat order. J. Algebraic Combin. 25 (2007), no. 2, 141–148, Proposition 1.1.

Complexity: $O(l * c)$, where l is the minimum of the lengths of u and of v, and c is the cost of the low level methods *first_descent()*, *has_descent()*, *apply_simple_reflection()*), etc. Those are typically $O(n)$, where n is the rank of the Coxeter group.

bruhat_lower_covers()

Return all elements that `self` covers in (strong) Bruhat order.

If `w = self` has a descent at i, then the elements that w covers are exactly $\{ws_i, u_1s_i, u_2s_i, ..., u_js_i\}$, where the u_k are elements that ws_i covers that also do not have a descent at i.

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])                                             #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,2,3])                                    #␣
↪needs sage.combinat sage.groups
sage: print([v.reduced_word() for v in w.bruhat_lower_covers()])          #␣
↪needs sage.combinat sage.groups
[[3, 2], [2, 3]]

sage: # needs sage.combinat sage.groups
```

(continues on next page)

(continued from previous page)

```
sage: W = WeylGroup(["A", 3])
sage: print([v.reduced_word()
....:        for v in W.simple_reflection(1).bruhat_lower_covers()])
[[]]
sage: print([v.reduced_word()
....:        for v in W.one().bruhat_lower_covers()])
[]
sage: W = WeylGroup(["B", 4, 1])
sage: w = W.from_reduced_word([0,2])
sage: print([v.reduced_word() for v in w.bruhat_lower_covers()])
[[2], [0]]
sage: W = WeylGroup("A3", prefix="s", implementation="permutation")
sage: s1, s2, s3 = W.simple_reflections()
sage: (s1*s2*s3*s1).bruhat_lower_covers()
[s2*s1*s3, s1*s2*s1, s1*s2*s3]
```

We now show how to construct the Bruhat poset:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(["A", 3])
sage: covers = tuple([u, v]
....:                for v in W for u in v.bruhat_lower_covers())
sage: P = Poset((W, covers), cover_relations=True)                    #␣
↪needs sage.graphs
sage: P.show()                                                        #␣
↪needs sage.graphs sage.plot
```

Alternatively, one can just use:

```
sage: P = W.bruhat_poset()                                            #␣
↪needs sage.combinat sage.graphs sage.groups
```

The algorithm is taken from Stembridge's 'coxeter/weyl' package for Maple.

bruhat_lower_covers_reflections()

Return all 2-tuples of lower_covers and reflections (v, r) where v is covered by self and r is the reflection such that self = v r.

ALGORITHM:

See *bruhat_lower_covers()*

EXAMPLES:

```
sage: W = WeylGroup(['A', 3], prefix="s")                             #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,1,2,1])                              #␣
↪needs sage.combinat sage.groups
sage: w.bruhat_lower_covers_reflections()                             #␣
↪needs sage.combinat sage.groups
[(s1*s2*s1, s1*s2*s3*s2*s1), (s3*s2*s1, s2), (s3*s1*s2, s1)]
```

bruhat_upper_covers()

Return all elements that cover self in (strong) Bruhat order.

The algorithm works recursively, using the 'inverse' of the method described for lower covers *bruhat_lower_covers()*. Namely, it runs through all i in the index set. Let w equal self.

If w has no right descent i, then ws_i is a cover; if w has a decent at i, then $u_j s_i$ is a cover of w where u_j is a cover of ws_i.

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1], prefix="s")                            #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([1,2,1])                                  #␣
↪needs sage.combinat sage.groups
sage: w.bruhat_upper_covers()                                           #␣
↪needs sage.combinat sage.groups
[s1*s2*s1*s0, s1*s2*s0*s1, s0*s1*s2*s1, s3*s1*s2*s1, s2*s3*s1*s2,␣
↪s1*s2*s3*s1]

sage: W = WeylGroup(['A', 3])                                           #␣
↪needs sage.combinat sage.groups
sage: w = W.long_element()                                              #␣
↪needs sage.combinat sage.groups
sage: w.bruhat_upper_covers()                                           #␣
↪needs sage.combinat sage.groups
[]

sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 3])
sage: w = W.from_reduced_word([1,2,1])
sage: S = [v for v in W if w in v.bruhat_lower_covers()]
sage: C = w.bruhat_upper_covers()
sage: set(S) == set(C)
True
```

bruhat_upper_covers_reflections()

Return all 2-tuples of covers and reflections (`v`, `r`) where `v` covers `self` and `r` is the reflection such that `self` = `v r`.

ALGORITHM:

See *bruhat_upper_covers()*

EXAMPLES:

```
sage: W = WeylGroup(['A', 4], prefix="s")                               #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,1,2,1])                                #␣
↪needs sage.combinat sage.groups
sage: w.bruhat_upper_covers_reflections()                               #␣
↪needs sage.combinat sage.groups
[(s1*s2*s3*s2*s1, s3), (s2*s3*s1*s2*s1, s2*s3*s2),
 (s3*s4*s1*s2*s1, s4), (s4*s3*s1*s2*s1, s1*s2*s3*s4*s3*s2*s1)]
```

canonical_matrix()

Return the matrix of `self` in the canonical faithful representation.

This is an n-dimension real faithful essential representation, where n is the number of generators of the Coxeter group. Note that this is not always the most natural matrix representation, for instance in type A_n.

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])                                          #␣
↪needs sage.combinat sage.groups
sage: s = W.simple_reflections()                                       #␣
↪needs sage.combinat sage.groups
sage: (s[1]*s[2]*s[3]).canonical_matrix()                              #␣
↪needs sage.combinat sage.groups
[ 0  0 -1]
[ 1  0 -1]
[ 0  1 -1]
```

coset_representative(*index_set*, *side='right'*)

Return the unique shortest element of the Coxeter group W which is in the same left (resp. right) coset as `self`, with respect to the parabolic subgroup W_I.

INPUT:

- `index_set` - a subset (or iterable) of the nodes of the Dynkin diagram
- `side` - 'left' or 'right'

EXAMPLES:

```
sage: W = CoxeterGroups().example(5)
sage: s = W.simple_reflections()
sage: w = s[2]*s[1]*s[3]
sage: w.coset_representative([]).reduced_word()
[2, 3, 1]
sage: w.coset_representative([1]).reduced_word()
[2, 3]
sage: w.coset_representative([1,2]).reduced_word()
[2, 3]
sage: w.coset_representative([1,3]                 ).reduced_word()
[2]
sage: w.coset_representative([2,3]                 ).reduced_word()
[2, 1]
sage: w.coset_representative([1,2,3]               ).reduced_word()
[]
sage: w.coset_representative([],      side='left').reduced_word()
[2, 3, 1]
sage: w.coset_representative([1],     side='left').reduced_word()
[2, 3, 1]
sage: w.coset_representative([1,2],   side='left').reduced_word()
[3]
sage: w.coset_representative([1,3],   side='left').reduced_word()
[2, 3, 1]
sage: w.coset_representative([2,3],   side='left').reduced_word()
[1]
sage: w.coset_representative([1,2,3], side='left').reduced_word()
[]
```

cover_reflections(*side='right'*)

Return the set of reflections `t` such that `self t` covers `self`.

If `side` is 'left', `t self` covers `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 4], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
```

(continues on next page)

(continued from previous page)

```
sage: w.cover_reflections()
[s3, s2*s3*s2, s4, s1*s2*s3*s4*s3*s2*s1]
sage: w.cover_reflections(side='left')
[s4, s2, s1*s2*s1, s3*s4*s3]
```

coxeter_sorting_word(*c*)

Return the c-sorting word of self.

For a Coxeter element c and an element w, the c-sorting word of w is the lexicographic minimal reduced expression of w in the infinite word c^∞.

INPUT:

- c– a Coxeter element.

OUTPUT:

the c-sorting word of self as a list of integers.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: c = W.from_reduced_word([0,2,1])
sage: w = W.from_reduced_word([1,2,1,0,1])
sage: w.coxeter_sorting_word(c)
[2, 1, 2, 0, 1]
```

deodhar_factor_element(*w*, *index_set*)

Return Deodhar's Bruhat order factoring element.

INPUT:

- w is an element of the same Coxeter group W as self
- index_set is a subset of Dynkin nodes defining a parabolic subgroup W' of W

It is assumed that v = self and w are minimum length coset representatives for W/W' such that v $\leq$ w in Bruhat order.

OUTPUT:

Deodhar's element f(v,w) is the unique element of W' such that, for all v' and w' in W', vv' $\leq$ ww' in W if and only if v' $\leq$ f(v,w) * w' in W' where * is the Demazure product.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 5], prefix="s")
sage: v = W.from_reduced_word([5])
sage: w = W.from_reduced_word([4,5,2,3,1,2])
sage: v.deodhar_factor_element(w, [1,3,4])
s3*s1
sage: W = WeylGroup(['C', 2])
sage: w = W.from_reduced_word([2,1])
sage: w.deodhar_factor_element(W.from_reduced_word([2]),[1])
Traceback (most recent call last):
...
ValueError: [2, 1] is not of minimum length in its coset
for the parabolic subgroup with index set [1]
```

REFERENCES:

- [Deo1987a]

deodhar_lift_down(*w*, *index_set*)

Letting `v = self`, given a Bruhat relation `v W'` $\geq$ `w W'` among cosets with respect to the subgroup `W'` given by the Dynkin node subset `index_set`, returns the Bruhat-maximum lift `x` of `wW'` such that `v` $\geq$ `x`.

INPUT:

- `w` is an element of the same Coxeter group `W` as `self`.
- `index_set` is a subset of Dynkin nodes defining a parabolic subgroup `W'`.

OUTPUT:

The unique Bruhat-maximum element `x` in `W` such that `x W' = w W'` and `v` $\geq$ `x`.

See also:

sage.categories.coxeter_groups.CoxeterGroups.ElementMethods.deodhar_lift_up()

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 3], prefix="s")
sage: v = W.from_reduced_word([1,2,3,2])
sage: w = W.from_reduced_word([3,2])
sage: v.deodhar_lift_down(w, [3])
s2*s3*s2
```

deodhar_lift_up(*w*, *index_set*)

Letting `v = self`, given a Bruhat relation `v W'` $\leq$ `w W'` among cosets with respect to the subgroup `W'` given by the Dynkin node subset `index_set`, returns the Bruhat-minimum lift `x` of `wW'` such that `v` $\leq$ `x`.

INPUT:

- `w` is an element of the same Coxeter group `W` as `self`.
- `index_set` is a subset of Dynkin nodes defining a parabolic subgroup `W'`.

OUTPUT:

The unique Bruhat-minimum element `x` in `W` such that `x W' = w W'` and `v` $\leq$ `x`.

See also:

sage.categories.coxeter_groups.CoxeterGroups.ElementMethods.deodhar_lift_down()

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 3], prefix="s")
sage: v = W.from_reduced_word([1,2,3])
sage: w = W.from_reduced_word([1,3,2])
sage: v.deodhar_lift_up(w, [3])
s1*s2*s3*s2
```

descents(*side='right'*, *index_set=None*, *positive=False*)

Return the descents of self, as a list of elements of the index_set.

INPUT:

- `index_set` - a subset (as a list or iterable) of the nodes of the Dynkin diagram; (default: all of them)
- `side` - 'left' or 'right' (default: 'right')
- `positive` - a boolean (default: `False`)

The `index_set` option can be used to restrict to the parabolic subgroup indexed by `index_set`.

If positive is `True`, then returns the non-descents instead

Todo: find a better name for `positive`: complement? non_descent?

Caveat: the return type may change to some other iterable (tuple, …) in the future. Please use keyword arguments also, as the order of the arguments may change as well.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0]*s[1]
sage: w.descents()
[1]
sage: w = s[0]*s[2]
sage: w.descents()
[0, 2]
```

Todo: side, index_set, positive

---

first_descent (*side='right'*, *index_set=None*, *positive=False*)

Return the first left (resp. right) descent of self, as an element of `index_set`, or `None` if there is none.

See *descents()* for a description of the options.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[2]*s[0]
sage: w.first_descent()
0
sage: w = s[0]*s[2]
sage: w.first_descent()
0
sage: w = s[0]*s[1]
sage: w.first_descent()
1
```

has_descent (*i*, *side='right'*, *positive=False*)

Return whether i is a (left/right) descent of self.

See *descents()* for a description of the options.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0] * s[1] * s[2]
sage: w.has_descent(2)
True
sage: [ w.has_descent(i)                  for i in [0,1,2] ]
[False, False, True]
sage: [ w.has_descent(i, side='left')    for i in [0,1,2] ]
```

(continues on next page)

(continued from previous page)

```
[True, False, False]
sage: [ w.has_descent(i, positive=True) for i in [0,1,2] ]
[True, True, False]
```

This default implementation delegates the work to `has_left_descent()` and `has_right_descent()`.

`has_full_support()`

Return whether `self` has full support.

An element is said to have full support if its support contains all simple reflections.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.from_reduced_word([1,2,1])
sage: w.has_full_support()
False
sage: w = W.from_reduced_word([1,2,1,0,1])
sage: w.has_full_support()
True
```

`has_left_descent(i)`

Return whether i is a left descent of self.

This default implementation uses that a left descent of w is a right descent of w^{-1}.

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.has_left_descent(0)
True
sage: w.has_left_descent(1)
False
sage: w.has_left_descent(2)
False
```

`has_right_descent(i)`

Return whether `i` is a right descent of self.

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.has_right_descent(0)
False
sage: w.has_right_descent(1)
False
sage: w.has_right_descent(2)
True
```

`inversions_as_reflections()`

Return the set of reflections `r` such that `self r < self`.

EXAMPLES:

```
sage: W = WeylGroup(['A', 3], prefix="s")                                   #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,1,2,1])                                    #␣
↪needs sage.combinat sage.groups
sage: w.inversions_as_reflections()                                         #␣
↪needs sage.combinat sage.groups
[s1, s1*s2*s1, s2, s1*s2*s3*s2*s1]
```

is_coxeter_sortable(*c*, *sorting_word=None*)

Return whether `self` is `c`-sortable.

Given a Coxeter element c, an element w is c-sortable if its c-sorting word decomposes into a sequence of weakly decreasing subwords of c.

INPUT:

- `c` – a Coxeter element.
- `sorting_word` – sorting word (default: None) used to not recompute the `c`-sorting word if already computed.

OUTPUT:

is `self` `c`-sortable

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: c = W.from_reduced_word([0,2,1])
sage: w = W.from_reduced_word([1,2,1,0,1])
sage: w.coxeter_sorting_word(c)
[2, 1, 2, 0, 1]
sage: w.is_coxeter_sortable(c)
False
sage: w = W.from_reduced_word([0,2,1,0,2])
sage: w.coxeter_sorting_word(c)
[2, 0, 1, 2, 0]
sage: w.is_coxeter_sortable(c)
True

sage: W = CoxeterGroup(['A', 3])                                            #␣
↪needs sage.combinat sage.groups
sage: c = W.from_reduced_word([1,2,3])                                      #␣
↪needs sage.combinat sage.groups
```

Number of c-sortable elements in A_3 (Catalan number):

```
sage: len([w for w in W if w.is_coxeter_sortable(c)])
14
```

is_fully_commutative()

Check if `self` is a fully-commutative element.

We use the characterization that an element w in a Coxeter system (W, S) is fully-commutative if and only if for every pair of generators $s, t \in S$ for which $m(s, t) > 2$, no reduced word of w contains the 'braid' word $sts...$ of length $m(s, t)$ as a contiguous subword. See [Ste1996].

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = CoxeterGroup(['A', 3])
sage: len([1 for w in W if w.is_fully_commutative()])
14
sage: W = CoxeterGroup(['B', 3])
sage: len([1 for w in W if w.is_fully_commutative()])
24
```

is_grassmannian(*side='right'*)

Return whether `self` is Grassmannian.

INPUT:

- `side` – "left" or "right" (default: "right")

An element is Grassmannian if it has at most one descent on the right (resp. on the left).

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: W.one().is_grassmannian()
True
sage: s[1].is_grassmannian()
True
sage: (s[1]*s[2]).is_grassmannian()
True
sage: (s[0]*s[1]).is_grassmannian()
True
sage: (s[1]*s[2]*s[1]).is_grassmannian()
False

sage: (s[0]*s[2]*s[1]).is_grassmannian(side="left")
False
sage: (s[0]*s[2]*s[1]).is_grassmannian(side="right")
True
sage: (s[0]*s[2]*s[1]).is_grassmannian()
True
```

kazhdan_lusztig_cell(*side='left'*)

Compute the left, right, or two-sided Kazhdan-Lusztig cell containing the element `self` depending on the specified `side`.

Let C' denote the Kazhdan-Lusztig C'-basis of the Iwahori-Hecke algebra H of a Coxeter system (W, S). Two elements x, y of the Coxeter group W are said to lie in the same left Kazhdan-Lusztig cell if there exist sequences $x = w_1, w_2, \ldots, w_k = y$ and $y = u_1, u_2, \ldots, u_l = x$ such that for all $1 \leq i < k$ and all $1 \leq j < l$, there exist some Coxeter generators $s, t \in S$ for which $C'_{w_{i+1}}$ appears in $C'_s C'_{w_i}$ and $C'_{u_{j+1}}$ appears in $C'_s C'_{u_j}$ in H. Right and two-sided Kazhdan-Lusztig cells of W are defined similarly; see [Lus2013].

In this function, we compute products in the C' basis by using `IwahoriHeckeAlgebra.Cp`. As mentioned in that class, installing the optional package `coxeter3` is recommended (though not required) before using this function because the package speeds up product computations that are sometimes computationally infeasible without it.

INPUT:

- `w` – an element of `self`
- `side` – (default: `'left'`) the kind of cell to compute; must be either `'left'`, `'right'`, or `'two-sided'`

EXAMPLES:

We compute the left cell of the generator s_1 in type A_3 in three different implementations of the Coxeter group. Note that the choice of implementation affects the representation of elements in the output cell but not the method used for the cell computation:

```
sage: W = CoxeterGroup('A3', implementation='permutation')                #␣
↪needs sage.combinat sage.groups
sage: s1, s2, s3 = W.simple_reflections()                                 #␣
↪needs sage.combinat sage.groups
sage: s1.kazhdan_lusztig_cell()                                           #␣
↪needs sage.combinat sage.groups
{(1,2,3,12)(4,5,10,11)(6,7,8,9),
 (1,2,10)(3,6,5)(4,7,8)(9,12,11),
 (1,7)(2,4)(5,6)(8,10)(11,12)}
```

The cell computation uses the optional package `coxeter3` in the background if available to speed up the computation, even in the different implementations:

```
sage: # optional - coxeter3, needs sage.combinat sage.groups sage.modules
sage: W = WeylGroup('A3', prefix='s')
sage: s1,s2,s3 = W.simple_reflections()
sage: s1.kazhdan_lusztig_cell()
{s3*s2*s1, s2*s1, s1}
sage: W = CoxeterGroup('A3', implementation='coxeter3')
sage: s1,s2,s3 = W.simple_reflections()
sage: s1.kazhdan_lusztig_cell()
{[1], [2, 1], [3, 2, 1]}
```

Next, we compute a right cell and a two-sided cell in A_3:

```
sage: # optional - coxeter3, needs sage.combinat sage.groups sage.modules
sage: W = CoxeterGroup('A3', implementation='coxeter3')
sage: s1,s2,s3 = W.simple_reflections()
sage: w = s1 * s3
sage: w.kazhdan_lusztig_cell(side='right')
{[1, 3], [1, 3, 2]}
sage: w.kazhdan_lusztig_cell(side='two-sided')
{[1, 3], [1, 3, 2], [2, 1, 3], [2, 1, 3, 2]}
```

Some slightly longer computations in B_4:

```
sage: # optional - coxeter3, needs sage.combinat sage.groups sage.modules
sage: W = CoxeterGroup('B4', implementation='coxeter3')
sage: s1,s2,s3,s4 = W.simple_reflections()
sage: s1.kazhdan_lusztig_cell(side='right')      # long time (4 seconds)
{[1],
 [1, 2],
 [1, 2, 3],
 [1, 2, 3, 4],
 [1, 2, 3, 4, 3],
 [1, 2, 3, 4, 3, 2],
 [1, 2, 3, 4, 3, 2, 1]}
sage: (s4*s2*s3*s4).kazhdan_lusztig_cell(side='two-sided')      # long␣
↪time (8 seconds)
{[2, 3, 1],
 [2, 3, 1, 2],
 [2, 3, 4, 1],
```

(continues on next page)

(continued from previous page)

```
 [2, 3, 4, 1, 2],
 [2, 3, 4, 1, 2, 3],
 [2, 3, 4, 1, 2, 3, 4],
 [2, 3, 4, 3, 1],
 [2, 3, 4, 3, 1, 2],
 ...
 [4, 3, 4, 2, 3, 4, 1, 2, 3, 4]}
```

left_inversions_as_reflections()

Return the set of reflections r such that r self < self.

EXAMPLES:

```
sage: W = WeylGroup(['A', 3], prefix="s")                                    #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,1,2,1])                                     #␣
↪needs sage.combinat sage.groups
sage: w.left_inversions_as_reflections()                                     #␣
↪needs sage.combinat sage.groups
[s1, s3, s1*s2*s3*s2*s1, s2*s3*s2]
```

length()

Return the length of self.

This is the minimal length of a product of simple reflections giving self.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s1 = W.simple_reflection(1)
sage: s2 = W.simple_reflection(2)
sage: s1.length()
1
sage: (s1*s2).length()
2
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0]*s[1]*s[0]
sage: w.length()
3
sage: W = CoxeterGroups().example()
sage: R.<x> = ZZ[]
sage: s = sum(x^w.length() for w in W)
sage: p = prod(sum(x^i for i in range(j)) for j in range(1, 5))
sage: s - p
0
```

See also:

reduced_word()

Todo: Should use reduced_word_iterator (or reverse_iterator)

lower_cover_reflections(*side='right'*)

Return the reflections t such that self covers self t.

If side is 'left', self covers t self.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 3],prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.lower_cover_reflections()
[s1*s2*s3*s2*s1, s2, s1]
sage: w.lower_cover_reflections(side='left')
[s2*s3*s2, s3, s1]
```

lower_covers(*side='right'*, *index_set=None*)

Return all elements that `self` covers in weak order.

INPUT:

- `side` – `'left'` or `'right'` (default: `'right'`)
- `index_set` – a list of indices or `None`

OUTPUT: a list

EXAMPLES:

```
sage: W = WeylGroup(['A', 3])                                        #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,2,1])                               #␣
↪needs sage.combinat sage.groups
sage: [x.reduced_word() for x in w.lower_covers()]                   #␣
↪needs sage.combinat sage.groups
[[3, 2]]
```

To obtain covers for left weak order, set the option side to 'left':

```
sage: [x.reduced_word() for x in w.lower_covers(side='left')]        #␣
↪needs sage.combinat sage.groups
[[2, 1]]
sage: w = W.from_reduced_word([3,2,3,1])                             #␣
↪needs sage.combinat sage.groups
sage: [x.reduced_word() for x in w.lower_covers()]                   #␣
↪needs sage.combinat sage.groups
[[2, 3, 2], [3, 2, 1]]
```

Covers w.r.t. a parabolic subgroup are obtained with the option `index_set`:

```
sage: [x.reduced_word() for x in w.lower_covers(index_set=[1,2])]    #␣
↪needs sage.combinat sage.groups
[[2, 3, 2]]
sage: [x.reduced_word() for x in w.lower_covers(side='left')]        #␣
↪needs sage.combinat sage.groups
[[3, 2, 1], [2, 3, 1]]
```

min_demazure_product_greater(*element*)

Find the unique Bruhat-minimum element `u` such that `v` $\leq$ `w` * `u` where `v` is `self`, `w` is `element` and `*` is the Demazure product.

INPUT:

- `element` is either an element of the same Coxeter group as `self` or a list (such as a reduced word) of elements from the index set of the Coxeter group.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 4], prefix="s")
sage: v = W.from_reduced_word([2,3,4,1,2])
sage: u = W.from_reduced_word([2,3,2,1])
sage: v.min_demazure_product_greater(u)
s4*s2
sage: v.min_demazure_product_greater([2,3,2,1])
s4*s2
sage: v.min_demazure_product_greater((2,3,2,1))
s4*s2
```

reduced_word()

Return a reduced word for `self`.

This is a word $[i_1, i_2, \ldots, i_k]$ of minimal length such that $s_{i_1} s_{i_2} \cdots s_{i_k} =$ self, where the s_i are the simple reflections.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0]*s[1]*s[2]
sage: w.reduced_word()
[0, 1, 2]
sage: w = s[0]*s[2]
sage: w.reduced_word()
[2, 0]
```

See also:

- *reduced_words()*, *reduced_word_reverse_iterator()*,
- *length()*, *reduced_word_graph()*

reduced_word_graph()

Return the reduced word graph of `self`.

The reduced word graph of an element w in a Coxeter group is the graph whose vertices are the reduced words for w (see *reduced_word()* for a definition of this term), and which has an m-colored edge between two reduced words x and y whenever x and y differ by exactly one length-m braid move (with $m \geq 2$).

This graph is always connected (a theorem due to Tits) and has no multiple edges.

EXAMPLES:

```
sage: # needs sage.combinat sage.graphs sage.groups
sage: W = WeylGroup(['A', 3], prefix='s')
sage: w0 = W.long_element()
sage: G = w0.reduced_word_graph()
sage: G.num_verts()
16
sage: len(w0.reduced_words())
16
sage: G.num_edges()
18
sage: len([e for e in G.edges(sort=False) if e[2] == 2])
10
sage: len([e for e in G.edges(sort=False) if e[2] == 3])
8
```

See also:

reduced_words(), *reduced_word_reverse_iterator()*, *length()*, *reduced_word()*

reduced_word_reverse_iterator()

Return a reverse iterator on a reduced word for `self`.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: sigma = s[0]*s[1]*s[2]
sage: rI=sigma.reduced_word_reverse_iterator()
sage: [i for i in rI]
[2, 1, 0]
sage: s[0]*s[1]*s[2]==sigma
True
sage: sigma.length()
3
```

See also:

reduced_word()

Default implementation: recursively remove the first right descent until the identity is reached (see *first_descent()* and *apply_simple_reflection()*).

reduced_words()

Return all reduced words for `self`.

See *reduced_word()* for the definition of a reduced word.

The algorithm uses the Matsumoto property that any two reduced expressions are related by braid relations, see Theorem 3.3.1(ii) in [BB2005].

See also:

`braid_orbit()`

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0] * s[2]
sage: sorted(w.reduced_words())                                              #␣
↪needs sage.graphs sage.modules
[[0, 2], [2, 0]]

sage: W = WeylGroup(['E', 6])                                                #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([2,3,4,2])                                     #␣
↪needs sage.combinat sage.groups
sage: sorted(w.reduced_words())                                              #␣
↪needs sage.combinat sage.groups
[[2, 3, 4, 2], [3, 2, 4, 2], [3, 4, 2, 4]]

sage: # optional - gap3, needs sage.combinat sage.groups
sage: W = ReflectionGroup(['A',3],
....:                     index_set=["AA","BB","5"])
sage: w = W.long_element()
```

(continues on next page)

(continued from previous page)

```
sage: w.reduced_words()
[['BB', '5', 'AA', 'BB', '5', 'AA'],
 ['5', 'BB', '5', 'AA', 'BB', '5'],
 ['BB', 'AA', 'BB', '5', 'BB', 'AA'],
 ['AA', '5', 'BB', 'AA', '5', 'BB'],
 ['5', 'AA', 'BB', 'AA', '5', 'BB'],
 ['AA', 'BB', '5', 'AA', 'BB', 'AA'],
 ['AA', 'BB', 'AA', '5', 'BB', 'AA'],
 ['AA', 'BB', '5', 'BB', 'AA', 'BB'],
 ['BB', 'AA', '5', 'BB', 'AA', '5'],
 ['BB', '5', 'AA', 'BB', 'AA', '5'],
 ['AA', '5', 'BB', '5', 'AA', 'BB'],
 ['5', 'BB', 'AA', '5', 'BB', '5'],
 ['5', 'BB', 'AA', 'BB', '5', 'BB'],
 ['5', 'AA', 'BB', '5', 'AA', 'BB'],
 ['BB', '5', 'BB', 'AA', 'BB', '5'],
 ['BB', 'AA', '5', 'BB', '5', 'AA']]
```

Todo: The result should be full featured finite enumerated set (e.g., counting can be done much faster than iterating).

See also:

reduced_word(), *reduced_word_reverse_iterator()*, *length()*, *reduced_word_graph()*

reduced_words_iter()

Iterate over all reduced words for `self`.

See *reduced_word()* for the definition of a reduced word.

The algorithm uses the Matsumoto property that any two reduced expressions are related by braid relations, see Theorem 3.3.1(ii) in [BB2005].

See also:

`braid_orbit_iter()`

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0] * s[2]
sage: sorted(w.reduced_words_iter())                                    #␣
↪needs sage.combinat sage.graphs
[[0, 2], [2, 0]]
```

reflection_length()

Return the reflection length of `self`.

The reflection length is the length of the shortest expression of the element as a product of reflections.

See also:

absolute_length()

EXAMPLES:

```
sage: W = WeylGroup(['A', 3])                                                 #␣
↪needs sage.combinat sage.groups
sage: s = W.simple_reflections()                                              #␣
↪needs sage.combinat sage.groups
sage: (s[1]*s[2]*s[3]).reflection_length()                                    #␣
↪needs sage.combinat sage.groups
3

sage: W = SymmetricGroup(4)                                                   #␣
↪needs sage.groups
sage: s = W.simple_reflections()                                              #␣
↪needs sage.groups
sage: (s[3]*s[2]*s[3]).reflection_length()                                    #␣
↪needs sage.combinat sage.groups
1
```

support()

Return the support of `self`, that is the simple reflections that appear in the reduced expressions of `self`.

OUTPUT:

The support of `self` as a set of integers

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.from_reduced_word([1,2,1])
sage: w.support()
{1, 2}
```

upper_covers(*side='right'*, *index_set=None*)

Return all elements that cover `self` in weak order.

INPUT:

- `side` – `'left'` or `'right'` (default: `'right'`)
- `index_set` – a list of indices or `None`

OUTPUT: a list

EXAMPLES:

```
sage: W = WeylGroup(['A', 3])                                                 #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([2,3])                                          #␣
↪needs sage.combinat sage.groups
sage: [x.reduced_word() for x in w.upper_covers()]                            #␣
↪needs sage.combinat sage.groups
[[2, 3, 1], [2, 3, 2]]
```

To obtain covers for left weak order, set the option `side` to 'left':

```
sage: [x.reduced_word() for x in w.upper_covers(side='left')]                #␣
↪needs sage.combinat sage.groups
[[1, 2, 3], [2, 3, 2]]
```

Covers w.r.t. a parabolic subgroup are obtained with the option `index_set`:

```
sage: [x.reduced_word() for x in w.upper_covers(index_set=[1])]             #␣
↪needs sage.combinat sage.groups
[[2, 3, 1]]
sage: [x.reduced_word()                                                     #␣
↪needs sage.combinat sage.groups
....:  for x in w.upper_covers(side='left', index_set=[1])]
[[1, 2, 3]]
```

weak_covers (*side='right'*, *index_set=None*, *positive=False*)

Return all elements that `self` covers in weak order.

INPUT:

- side – 'left' or 'right' (default: 'right')
- positive – a boolean (default: False)
- index_set – a list of indices or None

OUTPUT: a list

EXAMPLES:

```
sage: W = WeylGroup(['A', 3])                                               #␣
↪needs sage.combinat sage.groups
sage: w = W.from_reduced_word([3,2,1])                                      #␣
↪needs sage.combinat sage.groups
sage: [x.reduced_word() for x in w.weak_covers()]                           #␣
↪needs sage.combinat sage.groups
[[3, 2]]
```

To obtain instead elements that cover self, set `positive=True`:

```
sage: [x.reduced_word() for x in w.weak_covers(positive=True)]              #␣
↪needs sage.combinat sage.groups
[[3, 1, 2, 1], [2, 3, 2, 1]]
```

To obtain covers for left weak order, set the option side to 'left':

```
sage: # needs sage.combinat sage.groups
sage: [x.reduced_word() for x in w.weak_covers(side='left')]
[[2, 1]]
sage: w = W.from_reduced_word([3,2,3,1])
sage: [x.reduced_word() for x in w.weak_covers()]
[[2, 3, 2], [3, 2, 1]]
sage: [x.reduced_word() for x in w.weak_covers(side='left')]
[[3, 2, 1], [2, 3, 1]]
```

Covers w.r.t. a parabolic subgroup are obtained with the option `index_set`:

```
sage: [x.reduced_word() for x in w.weak_covers(index_set=[1,2])]            #␣
↪needs sage.combinat sage.groups
[[2, 3, 2]]
```

weak_le (*other*, *side='right'*)

Comparison in weak order.

INPUT:

- other – an element of the same Coxeter group
- side – 'left' or 'right' (default: 'right')

OUTPUT: a boolean

This returns whether `self` <= `other` in left (resp. right) weak order, that is if 'v' can be obtained from 'v' by length increasing multiplication by simple reflections on the left (resp. right).

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(["A", 3])
sage: u = W.from_reduced_word([1,2])
sage: v = W.from_reduced_word([1,2,3,2])
sage: u.weak_le(u)
True
sage: u.weak_le(v)
True
sage: v.weak_le(u)
False
sage: v.weak_le(v)
True
```

Comparison for left weak order is achieved with the option `side`:

```
sage: u.weak_le(v, side='left')                                          #␣
↪needs sage.combinat sage.groups
False
```

The implementation uses the equivalent condition that any reduced word for u is a right (resp. left) prefix of some reduced word for v.

Complexity: $O(l * c)$, where l is the minimum of the lengths of u and of v, and c is the cost of the low level methods `first_descent()`, `has_descent()`, `apply_simple_reflection()`), etc. Those are typically $O(n)$, where n is the rank of the Coxeter group.

We now run consistency tests with permutations:

```
sage: W = WeylGroup(["A", 3])                                            #␣
↪needs sage.combinat sage.groups
sage: P4 = Permutations(4)
sage: def P4toW(w): return W.from_reduced_word(w.reduced_word())
sage: for u in P4:                # long time (5s on sage.math, 2011),␣
↪needs sage.combinat sage.groups
....:     for v in P4:
....:         assert u.permutohedron_lequal(v) == P4toW(u).weak_
↪le(P4toW(v))
....:         assert u.permutohedron_lequal(v, side='left') == P4toW(u).
↪weak_le(P4toW(v), side='left')
```

`Finite`

alias of `FiniteCoxeterGroups`

`class ParentMethods`

Bases: `object`

`braid_group_as_finitely_presented_group()`

Return the associated braid group.

EXAMPLES:

```
sage: W = CoxeterGroup(['A', 2])                                         #␣
↪needs sage.combinat sage.groups
sage: W.braid_group_as_finitely_presented_group()                        #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.groups
Finitely presented group < S1, S2 | S1*S2*S1*S2^-1*S1^-1*S2^-1 >

sage: W = WeylGroup(['B', 2])                                               #␣
↪needs sage.combinat sage.groups
sage: W.braid_group_as_finitely_presented_group()                           #␣
↪needs sage.combinat sage.groups
Finitely presented group < S1, S2 | (S1*S2)^2*(S1^-1*S2^-1)^2 >

sage: W = ReflectionGroup(['B',3], index_set=["AA","BB","5"])  # optional␣
↪- gap3
sage: W.braid_group_as_finitely_presented_group()              # optional␣
↪- gap3
Finitely presented group < SAA, SBB, S5 |
 (SAA*SBB)^2*(SAA^-1*SBB^-1)^2, SAA*S5*SAA^-1*S5^-1,
 SBB*S5*SBB*S5^-1*SBB^-1*S5^-1 >
```

braid_orbit(*word*)

Return the braid orbit of a word word of indices.

The input word does not need to be a reduced expression of an element.

INPUT:

- word: a list (or iterable) of indices in self.index_set()

OUTPUT:

a list of all lists that can be obtained from word by replacements of braid relations

See *braid_relations()* for the definition of braid relations.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0] * s[1] * s[2] * s[1]
sage: word = w.reduced_word(); word
[0, 1, 2, 1]

sage: sorted(W.braid_orbit(word))                                           #␣
↪needs sage.combinat sage.graphs
[[0, 1, 2, 1], [0, 2, 1, 2], [2, 0, 1, 2]]

sage: sorted(W.braid_orbit([2,1,1,2,1]))                                    #␣
↪needs sage.combinat sage.graphs
[[1, 2, 1, 1, 2], [2, 1, 1, 2, 1], [2, 1, 2, 1, 2], [2, 2, 1, 2, 2]]

sage: # optional - gap3
sage: W = ReflectionGroup(['A',3], index_set=["AA","BB","5"])
sage: w = W.long_element()
sage: W.braid_orbit(w.reduced_word())
[['BB', '5', 'AA', 'BB', '5', 'AA'],
 ['5', 'BB', '5', 'AA', 'BB', '5'],
 ['BB', 'AA', 'BB', '5', 'BB', 'AA'],
 ['AA', '5', 'BB', 'AA', '5', 'BB'],
 ['5', 'AA', 'BB', 'AA', '5', 'BB'],
 ['AA', 'BB', '5', 'AA', 'BB', 'AA'],
 ['AA', 'BB', 'AA', '5', 'BB', 'AA'],
 ['AA', 'BB', '5', 'BB', 'AA', 'BB'],
```

(continues on next page)

(continued from previous page)

```
 ['BB', 'AA', '5', 'BB', 'AA', '5'],
 ['BB', '5', 'AA', 'BB', 'AA', '5'],
 ['AA', '5', 'BB', '5', 'AA', 'BB'],
 ['5', 'BB', 'AA', '5', 'BB', '5'],
 ['5', 'BB', 'AA', 'BB', '5', 'BB'],
 ['5', 'AA', 'BB', '5', 'AA', 'BB'],
 ['BB', '5', 'BB', 'AA', 'BB', '5'],
 ['BB', 'AA', '5', 'BB', '5', 'AA']]
```

Todo: The result should be full featured finite enumerated set (e.g., counting can be done much faster than iterating).

See also:

reduced_words()

braid_orbit_iter(*word*)

Iterate over the braid orbit of a word `word` of indices.

The input word does not need to be a reduced expression of an element.

INPUT:

- `word` – a list (or iterable) of indices in `self.index_set()`

OUTPUT:

all lists that can be obtained from `word` by replacements of braid relations

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: sorted(W.braid_orbit_iter([0, 1, 2, 1]))                              #␣
↪needs sage.combinat sage.graphs
[[0, 1, 2, 1], [0, 2, 1, 2], [2, 0, 1, 2]]
```

braid_relations()

Return the braid relations of `self` as a list of reduced words of the braid relations.

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])                                               #␣
↪needs sage.combinat sage.groups
sage: W.braid_relations()                                                   #␣
↪needs sage.combinat sage.groups
[[[1, 2, 1], [2, 1, 2]]]

sage: W = WeylGroup(["B", 3])                                               #␣
↪needs sage.combinat sage.groups
sage: W.braid_relations()                                                   #␣
↪needs sage.combinat sage.groups
[[[1, 2, 1], [2, 1, 2]], [[1, 3], [3, 1]], [[2, 3, 2, 3], [3, 2, 3, 2]]]
```

bruhat_graph(*x=None*, *y=None*, *edge_labels=False*)

Return the Bruhat graph as a directed graph, with an edge $u \to v$ if and only if $u < v$ in the Bruhat order, and $u = r \cdot v$.

The Bruhat graph $\Gamma(x, y)$, defined if $x \leq y$ in the Bruhat order, has as its vertices the Bruhat interval $\{t | x \leq t \leq y\}$, and as its edges are the pairs (u, v) such that $u = r \cdot v$ where r is a reflection, that is, a conjugate of a simple reflection.

REFERENCES:

Carrell, The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties. Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), 53–61, Proc. Sympos. Pure Math., 56, Part 1, Amer. Math. Soc., Providence, RI, 1994.

EXAMPLES:

```
sage: W = CoxeterGroup(['H', 3])                                          #␣
↪needs sage.combinat sage.graphs sage.groups
sage: G = W.bruhat_graph(); G                                             #␣
↪needs sage.combinat sage.graphs sage.groups
Digraph on 120 vertices

sage: # needs sage.combinat sage.graphs sage.groups
sage: W = CoxeterGroup(['A', 2, 1])
sage: s1, s2, s3 = W.simple_reflections()
sage: W.bruhat_graph(s1, s1*s3*s2*s3)
Digraph on 6 vertices
sage: W.bruhat_graph(s1, s3*s2*s3)
Digraph on 0 vertices

sage: W = WeylGroup("A3", prefix="s")                                     #␣
↪needs sage.combinat sage.graphs sage.groups
sage: s1, s2, s3 = W.simple_reflections()                                 #␣
↪needs sage.combinat sage.graphs sage.groups
sage: G = W.bruhat_graph(s1*s3, s1*s2*s3*s2*s1); G                        #␣
↪needs sage.combinat sage.graphs sage.groups
Digraph on 10 vertices
```

Check that the graph has the correct number of edges (see github issue #17744):

```
sage: len(G.edges(sort=False))                                            #␣
↪needs sage.combinat sage.graphs sage.groups
16
```

bruhat_interval(*x*, *y*)

Return the list of `t` such that `x <= t <= y`.

EXAMPLES:

```
sage: W = WeylGroup("A3", prefix="s")                                     #␣
↪needs sage.combinat sage.groups
sage: s1, s2, s3 = W.simple_reflections()                                 #␣
↪needs sage.combinat sage.groups
sage: W.bruhat_interval(s2, s1*s3*s2*s1*s3)                               #␣
↪needs sage.combinat sage.groups
[s1*s2*s3*s2*s1, s2*s3*s2*s1, s3*s1*s2*s1, s1*s2*s3*s1,
 s1*s2*s3*s2, s3*s2*s1, s2*s3*s1, s2*s3*s2, s1*s2*s1,
 s3*s1*s2, s1*s2*s3, s2*s1, s3*s2, s2*s3, s1*s2, s2]

sage: W = WeylGroup(['A', 2, 1], prefix="s")                              #␣
↪needs sage.combinat sage.groups
sage: s0, s1, s2 = W.simple_reflections()                                 #␣
↪needs sage.combinat sage.groups
sage: W.bruhat_interval(1, s0*s1*s2)                                      #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.groups
[s0*s1*s2, s1*s2, s0*s2, s0*s1, s2, s1, s0, 1]
```

bruhat_interval_poset (*x*, *y*, *facade=False*)

Return the poset of the Bruhat interval between x and y in Bruhat order.

EXAMPLES:

```
sage: W = WeylGroup("A3", prefix="s")                                    #␣
↪needs sage.combinat sage.groups
sage: s1, s2, s3 = W.simple_reflections()                                #␣
↪needs sage.combinat sage.groups
sage: W.bruhat_interval_poset(s2, s1*s3*s2*s1*s3)                        #␣
↪needs sage.combinat sage.groups
Finite poset containing 16 elements

sage: W = WeylGroup(['A', 2, 1], prefix="s")                             #␣
↪needs sage.combinat sage.groups
sage: s0, s1, s2 = W.simple_reflections()                                #␣
↪needs sage.combinat sage.groups
sage: W.bruhat_interval_poset(1, s0*s1*s2)                               #␣
↪needs sage.combinat sage.groups
Finite poset containing 8 elements
```

canonical_representation ()

Return the canonical faithful representation of self.

EXAMPLES:

```
sage: W = WeylGroup("A3")                                                #␣
↪needs sage.combinat sage.groups
sage: W.canonical_representation()                                       #␣
↪needs sage.combinat sage.groups
Finite Coxeter group over Integer Ring with Coxeter matrix:
[1 3 2]
[3 1 3]
[2 3 1]
```

coxeter_diagram ()

Return the Coxeter diagram of self.

EXAMPLES:

```
sage: # needs sage.combinat sage.graphs sage.groups
sage: W = CoxeterGroup(['H', 3], implementation="reflection")
sage: G = W.coxeter_diagram(); G
Graph on 3 vertices
sage: G.edges(sort=True)
[(1, 2, 3), (2, 3, 5)]
sage: CoxeterGroup(G) is W
True
sage: G = Graph([(0, 1, 3), (1, 2, oo)])
sage: W = CoxeterGroup(G)
sage: W.coxeter_diagram() == G
True
sage: CoxeterGroup(W.coxeter_diagram()) is W
True
```

coxeter_element()

Return a Coxeter element.

The result is the product of the simple reflections, in some order.

Note: This implementation is shared with well generated complex reflection groups. It would be nicer to put it in some joint super category; however, in the current state of the art, there is none where it is clear that this is the right construction for obtaining a Coxeter element.

In this context, this is an element having a regular eigenvector (a vector not contained in any reflection hyperplane of `self`).

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: CoxeterGroup(['A', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['B', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['D', 4]).coxeter_element().reduced_word()
[1, 2, 4, 3]
sage: CoxeterGroup(['F', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['E', 8]).coxeter_element().reduced_word()
[1, 3, 2, 4, 5, 6, 7, 8]
sage: CoxeterGroup(['H', 3]).coxeter_element().reduced_word()
[1, 2, 3]
```

This method is also used for well generated finite complex reflection groups:

```
sage: W = ReflectionGroup((1,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3]

sage: W = ReflectionGroup((2,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,4,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]
```

coxeter_matrix()

Return the Coxeter matrix associated to `self`.

EXAMPLES:

```
sage: G = WeylGroup(['A', 3])                                              #␣
↪needs sage.combinat sage.groups
sage: G.coxeter_matrix()                                                   #␣
↪needs sage.combinat sage.groups
[1 3 2]
```

(continues on next page)

(continued from previous page)

```
[3 1 3]
[2 3 1]
```

coxeter_type()

Return the Coxeter type of `self`.

EXAMPLES:

```
sage: W = CoxeterGroup(['H', 3])                                            #␣
↪needs sage.combinat sage.groups
sage: W.coxeter_type()                                                      #␣
↪needs sage.combinat sage.groups
Coxeter type of ['H', 3]
```

demazure_product(*Q*)

Return the Demazure product of the list `Q` in `self`.

INPUT:

- `Q` is a list of elements from the index set of `self`.

This returns the Coxeter group element that represents the composition of 0-Hecke or Demazure operators.

See *CoxeterGroups.ParentMethods.simple_projections()*.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = WeylGroup(['A', 2])
sage: w = W.demazure_product([2,2,1])
sage: w.reduced_word()
[2, 1]
sage: w = W.demazure_product([2,1,2,1,2])
sage: w.reduced_word()
[1, 2, 1]

sage: W = WeylGroup(['B', 2])                                               #␣
↪needs sage.combinat sage.groups
sage: w = W.demazure_product([2,1,2,1,2])                                   #␣
↪needs sage.combinat sage.groups
sage: w.reduced_word()                                                      #␣
↪needs sage.combinat sage.groups
[2, 1, 2, 1]
```

elements_of_length(*n*)

Return all elements of length n.

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A', 2, 1])                               #␣
↪needs sage.combinat
sage: [len(list(A.elements_of_length(i))) for i in [0..5]]                  #␣
↪needs sage.combinat
[1, 3, 6, 9, 12, 15]

sage: W = CoxeterGroup(['H', 3])                                            #␣
↪needs sage.combinat sage.groups
sage: [len(list(W.elements_of_length(i))) for i in range(4)]                #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.groups
[1, 3, 5, 7]

sage: W = CoxeterGroup(['A', 2])                                            #␣
↪needs sage.combinat sage.groups
sage: [len(list(W.elements_of_length(i))) for i in range(6)]               #␣
↪needs sage.combinat sage.groups
[1, 2, 2, 1, 0, 0]
```

fully_commutative_elements()

Return the set of fully commutative elements in this Coxeter group.

See also:

`FullyCommutativeElements`

EXAMPLES:

```
sage: CoxeterGroup(['A', 3]).fully_commutative_elements()                   #␣
↪needs sage.combinat sage.groups
Fully commutative elements of
 Finite Coxeter group over Integer Ring with Coxeter matrix:
[1 3 2]
[3 1 3]
[2 3 1]
```

grassmannian_elements(*side='right'*)

Return the left or right Grassmannian elements of `self` as an enumerated set.

INPUT:

- `side` – (default: `"right"`) `"left"` or `"right"`

EXAMPLES:

```
sage: S = CoxeterGroups().example()
sage: G = S.grassmannian_elements()
sage: G.cardinality()
12
sage: G.list()
[(0, 1, 2, 3), (1, 0, 2, 3), (0, 2, 1, 3), (0, 1, 3, 2),
 (2, 0, 1, 3), (1, 2, 0, 3), (0, 3, 1, 2), (0, 2, 3, 1),
 (3, 0, 1, 2), (1, 3, 0, 2), (1, 2, 3, 0), (2, 3, 0, 1)]
sage: sorted(tuple(w.descents()) for w in G)
[(), (0,), (0,), (0,), (1,), (1,), (1,), (1,), (1,), (2,), (2,), (2,)]
sage: G = S.grassmannian_elements(side = "left")
sage: G.cardinality()
12
sage: sorted(tuple(w.descents(side = "left")) for w in G)
[(), (0,), (0,), (0,), (1,), (1,), (1,), (1,), (1,), (2,), (2,), (2,)]
```

index_set()

Return the index set of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = CoxeterGroup([[1,3],[3,1]])
sage: W.index_set()
```

(continues on next page)

(continued from previous page)

```
(1, 2)
sage: W = CoxeterGroup([[1,3],[3,1]], index_set=['x', 'y'])
sage: W.index_set()
('x', 'y')
sage: W = CoxeterGroup(['H', 3])
sage: W.index_set()
(1, 2, 3)
```

kazhdan_lusztig_cells(*side='left'*)

Compute the left, right, or two-sided Kazhdan-Lusztig cells of `self` if `self` is finite.

The cells are computed by using *kazhdan_lusztig_cell()*.

As detailed there, installation of the optional package `coxeter3` is recommended (though not required) before using this function as it speeds up the computation.

INPUT:

- `side` – (default: `'left'`) either `'left'`, `'right'`, or `'two-sided'`

EXAMPLES:

We compute the right cells in the Coxeter group of type A_2 below. Note that each Coxeter group may be created with multiple implementations, namely, 'reflection' (default), 'permutation', 'matrix', or 'coxeter3'. The choice of implementation affects the representation of elements in the output cells but not the method used for the cell computation:

```
sage: # needs sage.combinat sage.groups
sage: W = CoxeterGroup('A2')
sage: KL_cells = W.kazhdan_lusztig_cells(side='right')
sage: set([tuple(sorted(C, key=lambda w: w.reduced_word()))
....:      for C in KL_cells])
{(
[-1  1]  [ 0 -1]
[ 0  1], [ 1 -1]
),
 (
[ 0 -1]
[-1  0]
),
 (
[1 0]
[0 1]
),
 (
[ 1  0]  [-1  1]
[ 1 -1], [-1  0]
)}
sage: len(KL_cells)
4

sage: W = CoxeterGroup('A2', implementation='permutation')            #␣
↪needs sage.combinat sage.groups
sage: len(W.kazhdan_lusztig_cells(side='right'))                      #␣
↪needs sage.combinat sage.groups
4
```

We compute the left cells in the Coxeter group of type A_3 below. If the optional package `coxeter3` is installed, it runs in the background even if the group is not created with the `'coxeter3'` implementation:

```
sage: # optional - coxeter3, needs sage.combinat sage.groups sage.libs.
↪gap sage.modules sage.rings.number_field
sage: W = CoxeterGroup('A3', implementation='coxeter3')
sage: KL_cells = W.kazhdan_lusztig_cells()
sage: set([tuple(sorted(C)) for C in KL_cells])
{([],),
 ([1], [2, 1], [3, 2, 1]),
 ([1, 2], [2], [3, 2]),
 ([1, 2, 1], [1, 3, 2, 1], [2, 1, 3, 2, 1]),
 ([1, 2, 1, 3], [1, 2, 3, 2, 1], [2, 3, 2, 1]),
 ([1, 2, 1, 3, 2], [1, 2, 3, 2], [2, 3, 2]),
 ([1, 2, 1, 3, 2, 1],),
 ([1, 2, 3], [2, 3], [3]),
 ([1, 3], [2, 1, 3]),
 ([1, 3, 2], [2, 1, 3, 2])}
sage: len(KL_cells)
10
sage: W = CoxeterGroup('A3', implementation='permutation')
sage: len(W.kazhdan_lusztig_cells())
10
```

Computing the two sided cells in B_3:

```
sage: # optional - coxeter3, needs sage.combinat sage.groups sage.libs.
↪gap sage.modules sage.rings.number_field
sage: W = CoxeterGroup('B3', implementation='coxeter3')
sage: b3_cells = W.kazhdan_lusztig_cells('two-sided')
sage: len(b3_cells)
6
sage: set([tuple(sorted(C))
....:     for C in W.kazhdan_lusztig_cells()])
{([],),
 ([1], [1, 2, 3, 2, 1], [2, 1], [2, 3, 2, 1], [3, 2, 1]),
 ([1, 2], [1, 2, 3, 2], [2], [2, 3, 2], [3, 2]),
 ([1, 2, 3], [2, 3], [3], [3, 2, 3]),
 ([2, 1, 2], [2, 3, 2, 1, 2], [3, 2, 1, 2]),
 ([2, 1, 2, 3], [2, 3, 2, 1, 2, 3], [3, 2, 1, 2, 3]),
 ([2, 1, 2, 3, 2], [2, 3, 2, 1, 2, 3, 2], [3, 2, 1, 2, 3, 2]),
 ([2, 1, 2, 3, 2, 1],
  [2, 3, 2, 1, 2, 3, 2, 1],
  [3, 2, 1, 2, 3, 2, 1],
  [3, 2, 3, 2, 1, 2]),
 ([2, 3, 1], [3, 1], [3, 2, 3, 1]),
 ([2, 3, 1, 2], [3, 1, 2], [3, 2, 3, 1, 2]),
 ([2, 3, 1, 2, 3], [3, 1, 2, 3], [3, 2, 3, 1, 2, 3]),
 ([2, 3, 1, 2, 3, 2],
  [3, 1, 2, 3, 2],
  [3, 2, 3, 1, 2, 3, 2],
  [3, 2, 3, 2],
  [3, 2, 3, 2, 1, 2, 3, 2]),
 ([2, 3, 1, 2, 3, 2, 1],
  [3, 1, 2, 3, 2, 1],
  [3, 2, 3, 1, 2, 3, 2, 1],
  [3, 2, 3, 2, 1],
  [3, 2, 3, 2, 1, 2, 3]),
 ([3, 2, 3, 2, 1, 2, 3, 2, 1],)}
```

random_element_of_length(*n*)

Return a random element of length `n` in `self`.

Starts at the identity, then chooses an upper cover at random.

Not very uniform: actually constructs a uniformly random reduced word of length n. Thus we most likely get elements with lots of reduced words!

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: A = AffinePermutationGroup(['A', 7, 1])
sage: p = A.random_element_of_length(10)
sage: p in A
True
sage: p.length() == 10
True

sage: # needs sage.combinat sage.groups
sage: W = CoxeterGroup(['A', 4])
sage: p = W.random_element_of_length(5)
sage: p in W
True
sage: p.length() == 5
True
```

sign_representation(*base_ring=None*, *side='twosided'*)

Return the sign representation of `self` over `base_ring`.

INPUT:

- `base_ring` – (optional) the base ring; the default is $\mathbf{Z}$
- `side` – ignored

EXAMPLES:

```
sage: W = WeylGroup(["A", 1, 1])                                          #␣
↪needs sage.combinat sage.groups
sage: W.sign_representation()                                             #␣
↪needs sage.combinat sage.groups
Sign representation of
 Weyl Group of type ['A', 1, 1] (as a matrix group acting on the root␣
↪space)
 over Integer Ring
```

simple_projection(*i*, *side='right'*, *length_increasing=True*)

Return the simple projection π_i (or $\overline{\pi}_i$ if `length_increasing` is `False`).

INPUT:

- `i` - an element of the index set of `self`

See *simple_projections()* for the options and for the definition of the simple projections.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: sigma = W.an_element()
sage: sigma
(1, 2, 3, 0)
sage: u0 = W.simple_projection(0)
```

(continues on next page)

(continued from previous page)

```
sage: d0 = W.simple_projection(0, length_increasing=False)
sage: sigma.length()
3
sage: pi=sigma*s[0]
sage: pi.length()
4
sage: u0(sigma)
(2, 1, 3, 0)
sage: pi
(2, 1, 3, 0)
sage: u0(pi)
(2, 1, 3, 0)
sage: d0(sigma)
(1, 2, 3, 0)
sage: d0(pi)
(1, 2, 3, 0)
```

simple_projections(*side='right'*, *length_increasing=True*)

Return the family of simple projections, also known as 0-Hecke or Demazure operators.

INPUT:

- `self` – a Coxeter group W
- `side` – 'left' or 'right' (default: 'right')
- `length_increasing` – a boolean (default: `True`) specifying whether the operator increases or decreases length

This returns the simple projections of W, as a family.

To each simple reflection s_i of W, corresponds a *simple projection* π_i from W to W defined by:

$\pi_i(w) = ws_i$ if i is not a descent of w $\pi_i(w) = w$ otherwise.

The simple projections $(\pi_i)_{i\in I}$ move elements down the right permutohedron, toward the maximal element. They satisfy the same braid relations as the simple reflections, but are idempotents $\pi_i^2 = \pi$ not involutions $s_i^2 = 1$. As such, the simple projections generate the 0-Hecke monoid.

By symmetry, one can also define the projections $(\overline{\pi}_i)_{i\in I}$ (when the option `length_increasing` is False):

$\overline{\pi}_i(w) = ws_i$ if i is a descent of w $\overline{\pi}_i(w) = w$ otherwise.

as well as the analogues acting on the left (when the option `side` is 'left').

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: sigma = W.an_element(); sigma
(1, 2, 3, 0)
sage: pi = W.simple_projections(); pi
Finite family {0: <function ...<lambda> at ...>,
               1: <function ...<lambda> at ...>,
               2: <function ...<lambda> ...>}
sage: pi[1](sigma)
(1, 3, 2, 0)
sage: W.simple_projection(1)(sigma)
(1, 3, 2, 0)
```

standard_coxeter_elements()

Return all standard Coxeter elements in `self`.

This is the set of all elements in self obtained from any product of the simple reflections in `self`.

Note:

- `self` is assumed to be well-generated.
- This works even beyond real reflection groups, but the conjugacy class is not unique and we only obtain one such class.

EXAMPLES:

```
sage: W = ReflectionGroup(4)                    # optional - gap3
sage: sorted(W.standard_coxeter_elements())  # optional - gap3
[(1,7,6,12,23,20)(2,8,17,24,9,5)(3,16,10,19,15,21)(4,14,11,22,18,13),
 (1,10,4,12,21,22)(2,11,19,24,13,3)(5,15,7,17,16,23)(6,18,8,20,14,9)]
```

weak_order_ideal(*predicate*, *side='right'*, *category=None*)

Return a weak order ideal defined by a predicate

INPUT:

- `predicate`: a predicate on the elements of `self` defining an weak order ideal in `self`
- `side`: "left" or "right" (default: "right")

OUTPUT: an enumerated set

EXAMPLES:

```
sage: D6 = FiniteCoxeterGroups().example(5)
sage: I = D6.weak_order_ideal(predicate=lambda w: w.length() <= 3)
sage: I.cardinality()
7
sage: list(I)
[(), (1,), (2,), (1, 2), (2, 1), (1, 2, 1), (2, 1, 2)]
```

We now consider an infinite Coxeter group:

```
sage: W = WeylGroup(["A",1,1])                                              #␣
↪needs sage.groups sage.rings.number_field
sage: I = W.weak_order_ideal(predicate=lambda w: w.length() <= 2)          #␣
↪needs sage.groups sage.rings.number_field
sage: list(iter(I))                                                         #␣
↪needs sage.groups sage.rings.number_field
[
[1 0]   [-1  2]   [ 1  0]   [ 3 -2]   [-1  2]
[0 1], [ 0  1], [ 2 -1], [ 2 -1], [-2  3]
]
```

Even when the result is finite, some features of *FiniteEnumeratedSets* are not available:

```
sage: I.cardinality()  # todo: not implemented
5
sage: list(I)          # todo: not implemented
```

unless this finiteness is explicitly specified:

```
sage: I = W.weak_order_ideal(predicate=lambda w: w.length() <= 2,          #␣
↪needs sage.groups sage.rings.number_field
....:                            category=FiniteEnumeratedSets())
sage: I.cardinality()                                                       #␣
↪needs sage.groups sage.rings.number_field
5
```

(continues on next page)

(continued from previous page)

```
sage: list(I)                                                         #␣
↪needs sage.groups sage.rings.number_field
[
[1 0]  [-1  2]  [ 1  0]  [ 3 -2]  [-1  2]
[0 1], [ 0  1], [ 2 -1], [ 2 -1], [-2  3]
]
```

Background

The weak order is returned as a `RecursivelyEnumeratedSet_forest`. This is achieved by assigning to each element $u1$ of the ideal a single ancestor $u = u1s_i$, where i is the smallest descent of u.

This allows for iterating through the elements in roughly Constant Amortized Time and constant memory (taking the operations and size of the generated objects as constants).

additional_structure()

Return `None`.

Indeed, all the structure Coxeter groups have in addition to groups (simple reflections, …) is already defined in the super category.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: CoxeterGroups().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: CoxeterGroups().super_categories()
[Category of generalized Coxeter groups]
```

4.31 Crystals

class sage.categories.crystals.**CrystalHomset**(*X*, *Y*, *category=None*)

Bases: *Homset*

The set of crystal morphisms from one crystal to another.

An $U_q(\mathfrak{g})$ I-crystal morphism $\Psi : B \to C$ is a map $\Psi : B \cup \{0\} \to C \cup \{0\}$ such that:

- $\Psi(0) = 0$.
- If $b \in B$ and $\Psi(b) \in C$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$.
- If $b, b' \in B$, $\Psi(b), \Psi(b') \in C$ and $f_i b = b'$, then $f_i \Psi(b) = \Psi(b')$ and $\Psi(b) = e_i \Psi(b')$ for all $i \in I$.

If the Cartan type is unambiguous, it is suppressed from the notation.

We can also generalize the definition of a crystal morphism by considering a map of σ of the (now possibly different) Dynkin diagrams corresponding to B and C along with scaling factors $\gamma_i \in \mathbf{Z}$ for $i \in I$. Let σ_i denote the orbit of

i under σ. We write objects for B as X with corresponding objects of C as $\widehat{X}$. Then a *virtual* crystal morphism Ψ is a map such that the following holds:

- $\Psi(0) = 0$.
- If $b \in B$ and $\Psi(b) \in C$, then for all $j \in \sigma_i$:

$$\varepsilon_i(b) = \frac{1}{\gamma_j}\widehat{\varepsilon}_j(\Psi(b)), \quad \varphi_i(b) = \frac{1}{\gamma_j}\widehat{\varphi}_j(\Psi(b)), \quad \mathrm{wt}(\Psi(b)) = \sum_i c_i \sum_{j \in \sigma_i} \gamma_j \widehat{\Lambda}_j,$$

where $\mathrm{wt}(b) = \sum_i c_i \Lambda_i$.

- If $b, b' \in B$, $\Psi(b), \Psi(b') \in C$ and $f_i b = b'$, then independent of the ordering of σ_i we have:

$$\Psi(b') = e_i \Psi(b) = \prod_{j \in \sigma_i} \widehat{e}_j^{\gamma_i} \Psi(b), \quad \Psi(b') = f_i \Psi(b) = \prod_{j \in \sigma_i} \widehat{f}_j^{\gamma_i} \Psi(b).$$

If $\gamma_i = 1$ for all $i \in I$ and the Dynkin diagrams are the same, then we call Ψ a *twisted* crystal morphism.

INPUT:

- `X` – the domain
- `Y` – the codomain
- `category` – (optional) the category of the crystal morphisms

See also:

For the construction of an element of the homset, see *CrystalMorphismByGenerators* and *crystal_morphism()*.

EXAMPLES:

We begin with the natural embedding of $B(2\Lambda_1)$ into $B(\Lambda_1) \otimes B(\Lambda_1)$ in type A_1:

```
sage: B = crystals.Tableaux(['A',1], shape=[2])
sage: F = crystals.Tableaux(['A',1], shape=[1])
sage: T = crystals.TensorProduct(F, F)
sage: v = T.highest_weight_vectors()[0]; v
[[[1]], [[1]]]
sage: H = Hom(B, T)
sage: psi = H([v])
sage: b = B.highest_weight_vector(); b
[[1, 1]]
sage: psi(b)
[[[1]], [[1]]]
sage: b.f(1)
[[1, 2]]
sage: psi(b.f(1))
[[[1]], [[2]]]
```

We now look at the decomposition of $B(\Lambda_1) \otimes B(\Lambda_1)$ into $B(2\Lambda_1) \oplus B(0)$:

```
sage: B0 = crystals.Tableaux(['A',1], shape=[])
sage: D = crystals.DirectSum([B, B0])
sage: H = Hom(T, D)
sage: psi = H(D.module_generators)
sage: psi
['A', 1] Crystal morphism:
  From: Full tensor product of the crystals
    [The crystal of tableaux of type ['A', 1] and shape(s) [[1]],
```

(continues on next page)

(continued from previous page)

```
    The crystal of tableaux of type ['A', 1] and shape(s) [[1]]
  To:   Direct sum of the crystals Family
   (The crystal of tableaux of type ['A', 1] and shape(s) [[2]],
    The crystal of tableaux of type ['A', 1] and shape(s) [[]])
  Defn: [[[1]], [[1]]] |--> [[1, 1]]
        [[[2]], [[1]]] |--> []
sage: psi.is_isomorphism()
True
```

We can always construct the trivial morphism which sends everything to 0:

```
sage: Binf = crystals.infinity.Tableaux(['B', 2])
sage: B = crystals.Tableaux(['B',2], shape=[1])
sage: H = Hom(Binf, B)
sage: psi = H(lambda x: None)
sage: psi(Binf.highest_weight_vector())
```

For Kirillov-Reshetikhin crystals, we consider the map to the corresponding classical crystal:

```
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: B = K.classical_decomposition()
sage: H = Hom(K, B)
sage: psi = H(lambda x: x.lift(), cartan_type=['D',4])
sage: L = [psi(mg) for mg in K.module_generators]; L
[[], [[1], [2]]]
sage: all(x.parent() == B for x in L)
True
```

Next we consider a type D_4 crystal morphism where we twist by $3 \leftrightarrow 4$:

```
sage: B = crystals.Tableaux(['D',4], shape=[1])
sage: H = Hom(B, B)
sage: d = {1:1, 2:2, 3:4, 4:3}
sage: psi = H(B.module_generators, automorphism=d)
sage: b = B.highest_weight_vector()
sage: b.f_string([1,2,3])
[[4]]
sage: b.f_string([1,2,4])
[[-4]]
sage: psi(b.f_string([1,2,3]))
[[-4]]
sage: psi(b.f_string([1,2,4]))
[[4]]
```

We construct the natural virtual embedding of a type B_3 into a type D_4 crystal:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: H = Hom(B, C)
sage: psi = H(C.module_generators)
sage: psi
['B', 3] -> ['D', 4] Virtual Crystal morphism:
  From: The crystal of tableaux of type ['B', 3] and shape(s) [[1]]
  To:   The crystal of tableaux of type ['D', 4] and shape(s) [[2]]
  Defn: [[1]] |--> [[1, 1]]
sage: for b in B: print("{} |--> {}".format(b, psi(b)))
[[1]] |--> [[1, 1]]
```

(continues on next page)

(continued from previous page)

```
[[2]] |--> [[2, 2]]
[[3]] |--> [[3, 3]]
[[0]] |--> [[3, -3]]
[[-3]] |--> [[-3, -3]]
[[-2]] |--> [[-2, -2]]
[[-1]] |--> [[-1, -1]]
```

Element

alias of *CrystalMorphismByGenerators*

class sage.categories.crystals.**CrystalMorphism**(*parent*, *cartan_type=None*, *virtualization=None*, *scaling_factors=None*)

Bases: *Morphism*

A crystal morphism.

INPUT:

- parent – a homset
- cartan_type – (optional) a Cartan type; the default is the Cartan type of the domain
- virtualization – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain
- scaling_factors – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, ε and φ

cartan_type()

Return the Cartan type of self.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: psi = Hom(B, B).an_element()
sage: psi.cartan_type()
['A', 2]
```

is_injective()

Return if self is an injective crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: psi = Hom(B, B).an_element()
sage: psi.is_injective()
False
```

is_surjective()

Check if self is a surjective crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_surjective()
False
```

(continues on next page)

(continued from previous page)

```
sage: im_gens = [None, B.module_generators[0]]
sage: psi = C.crystal_morphism(im_gens, codomain=B)
sage: psi.is_surjective()
True

sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: B = crystals.infinity.Tableaux(['A',2])
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: W = crystals.elementary.T(['A',2], La[1]+La[2])
sage: T = W.tensor(B)
sage: mg = T(W.module_generators[0], B.module_generators[0])
sage: psi = Hom(C,T)([mg])
sage: psi.is_surjective()
False
```

scaling_factors()

Return the scaling factors γ_i.

EXAMPLES:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: psi = B.crystal_morphism(C.module_generators)
sage: psi.scaling_factors()
Finite family {1: 2, 2: 2, 3: 1}
```

virtualization()

Return the virtualization sets σ_i.

EXAMPLES:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: psi = B.crystal_morphism(C.module_generators)
sage: psi.virtualization()
Finite family {1: (1,), 2: (2,), 3: (3, 4)}
```

class sage.categories.crystals.**CrystalMorphismByGenerators**(*parent*, *on_gens*, *cartan_type=None*, *virtualization=None*, *scaling_factors=None*, *gens=None*, *check=True*)

Bases: *CrystalMorphism*

A crystal morphism defined by a set of generators which create a virtual crystal inside the codomain.

INPUT:

- `parent` – a homset
- `on_gens` – a function or list that determines the image of the generators (if given a list, then this uses the order of the generators of the domain) of the domain under `self`
- `cartan_type` – (optional) a Cartan type; the default is the Cartan type of the domain
- `virtualization` – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain

- `scaling_factors` – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, ε and φ
- `gens` – (optional) a finite list of generators to define the morphism; the default is to use the highest weight vectors of the crystal
- `check` – (default: `True`) check if the crystal morphism is valid

See also:

`sage.categories.crystals.Crystals.ParentMethods.crystal_morphism()`

`im_gens``()`

Return the image of the generators of `self` as a tuple.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: F = crystals.Tableaux(['A',2], shape=[1])
sage: T = crystals.TensorProduct(F, F, F)
sage: H = Hom(T, B)
sage: b = B.highest_weight_vector()
sage: psi = H((None, b, b, None), generators=T.highest_weight_vectors())
sage: psi.im_gens()
(None, [[1, 1], [2]], [[1, 1], [2]], None)
```

`image``()`

Return the image of `self` in the codomain as a `Subcrystal`.

Warning: This assumes that `self` is a strict crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: H = Hom(B, C)
sage: psi = H(C.module_generators)
sage: psi.image()
Virtual crystal of The crystal of tableaux of type ['D', 4] and shape(s)␣
↪[[2]] of type ['B', 3]
```

`to_module_generator``(x)`

Return a generator `mg` and a path of e_i and f_i operations to `mg`.

OUTPUT:

A tuple consisting of:

- a module generator,
- a list of `'e'` and `'f'` to denote which operation, and
- a list of matching indices.

EXAMPLES:

```
sage: B = crystals.elementary.Elementary(['A',2], 2)
sage: psi = B.crystal_morphism(B.module_generators)
sage: psi.to_module_generator(B(4))
```

(continues on next page)

(continued from previous page)

```
(0, ['f', 'f', 'f', 'f'], [2, 2, 2, 2])
sage: psi.to_module_generator(B(-2))
(0, ['e', 'e'], [2, 2])
```

class sage.categories.crystals.**Crystals**

Bases: *Category_singleton*

The category of crystals.

See sage.combinat.crystals.crystals for an introduction to crystals.

EXAMPLES:

```
sage: C = Crystals()
sage: C
Category of crystals
sage: C.super_categories()
[Category of... enumerated sets]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

Parents in this category should implement the following methods:

- either an attribute `_cartan_type` or a method `cartan_type`
- `module_generators`: a list (or container) of distinct elements which generate the crystal using f_i

Furthermore, their elements `x` should implement the following methods:

- `x.e(i)` (returning $e_i(x)$)
- `x.f(i)` (returning $f_i(x)$)
- `x.epsilon(i)` (returning $\varepsilon_i(x)$)
- `x.phi(i)` (returning $\varphi_i(x)$)

EXAMPLES:

```
sage: from sage.misc.abstract_method import abstract_methods_of_class
sage: abstract_methods_of_class(Crystals().element_class)
{'optional': [], 'required': ['e', 'epsilon', 'f', 'phi', 'weight']}
```

class ElementMethods

Bases: `object`

Epsilon()

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(0).Epsilon()
(0, 0, 0, 0, 0, 0)
sage: C(1).Epsilon()
(0, 0, 0, 0, 0, 0)
sage: C(2).Epsilon()
(1, 0, 0, 0, 0, 0)
```

Phi()

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(0).Phi()
(0, 0, 0, 0, 0, 0)
sage: C(1).Phi()
(1, 0, 0, 0, 0, 0)
sage: C(2).Phi()
(1, 1, 0, 0, 0, 0)
```

all_paths_to_highest_weight(*index_set=None*)

Iterate over all paths to the highest weight from `self` with respect to `index_set`.

INPUT:

- `index_set` – (optional) a subset of the index set of `self`

EXAMPLES:

```
sage: B = crystals.infinity.Tableaux("A2")
sage: b0 = B.highest_weight_vector()
sage: b = b0.f_string([1, 2, 1, 2])
sage: L = b.all_paths_to_highest_weight()
sage: list(L)
[[2, 1, 2, 1], [2, 2, 1, 1]]

sage: Y = crystals.infinity.GeneralizedYoungWalls(3)
sage: y0 = Y.highest_weight_vector()
sage: y = y0.f_string([0, 1, 2, 3, 2, 1, 0])
sage: list(y.all_paths_to_highest_weight())
[[0, 1, 2, 3, 2, 1, 0],
 [0, 1, 3, 2, 2, 1, 0],
 [0, 3, 1, 2, 2, 1, 0],
 [0, 3, 2, 1, 1, 0, 2],
 [0, 3, 2, 1, 1, 2, 0]]

sage: B = crystals.Tableaux("A3", shape=[4,2,1])
sage: b0 = B.highest_weight_vector()
sage: b = b0.f_string([1, 1, 2, 3])
sage: list(b.all_paths_to_highest_weight())
[[1, 3, 2, 1], [3, 1, 2, 1], [3, 2, 1, 1]]
```

cartan_type()

Returns the Cartan type associated to `self`

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C(1).cartan_type()
['A', 5]
```

e(i)

Return e_i of `self` if it exists or `None` otherwise.

This method should be implemented by the element class of the crystal.

EXAMPLES:

```
sage: C = Crystals().example(5)
sage: x = C[2]; x
3
```

(continues on next page)

(continued from previous page)

```
sage: x.e(1), x.e(2), x.e(3)
(None, 2, None)
```

e_string(*list*)

Applies $e_{i_r} \cdots e_{i_1}$ to `self` for `list` as $[i_1, ..., i_r]$

EXAMPLES:

```
sage: C = crystals.Letters(['A',3])
sage: b = C(3)
sage: b.e_string([2,1])
1
sage: b.e_string([1,2])
```

epsilon(*i*)

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).epsilon(1)
0
sage: C(2).epsilon(1)
1
```

f(*i*)

Return f_i of `self` if it exists or `None` otherwise.

This method should be implemented by the element class of the crystal.

EXAMPLES:

```
sage: C = Crystals().example(5)
sage: x = C[1]; x
2
sage: x.f(1), x.f(2), x.f(3)
(None, 3, None)
```

f_string(*list*)

Applies $f_{i_r} \cdots f_{i_1}$ to `self` for `list` as $[i_1, ..., i_r]$

EXAMPLES:

```
sage: C = crystals.Letters(['A',3])
sage: b = C(1)
sage: b.f_string([1,2])
3
sage: b.f_string([2,1])
```

index_set()

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).index_set()
(1, 2, 3, 4, 5)
```

is_highest_weight(*index_set=None*)

Return `True` if `self` is a highest weight.

Specifying the option `index_set` to be a subset I of the index set of the underlying crystal, finds all highest weight vectors for arrows in I.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).is_highest_weight()
True
sage: C(2).is_highest_weight()
False
sage: C(2).is_highest_weight(index_set = [2,3,4,5])
True
```

is_lowest_weight (*index_set=None*)

Returns `True` if `self` is a lowest weight. Specifying the option `index_set` to be a subset I of the index set of the underlying crystal, finds all lowest weight vectors for arrows in I.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).is_lowest_weight()
False
sage: C(6).is_lowest_weight()
True
sage: C(4).is_lowest_weight(index_set = [1,3])
True
```

phi (*i*)

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).phi(1)
1
sage: C(2).phi(1)
0
```

phi_minus_epsilon (*i*)

Return $\varphi_i - \varepsilon_i$ of `self`.

There are sometimes better implementations using the weight for this. It is used for reflections along a string.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).phi_minus_epsilon(1)
1
```

s (*i*)

Return the reflection of `self` along its i-string.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: b = C(rows=[[1,1],[3]])
sage: b.s(1)
[[2, 2], [3]]
sage: b = C(rows=[[1,2],[3]])
```

(continues on next page)

(continued from previous page)

```
sage: b.s(2)
[[1, 2], [3]]
sage: T = crystals.Tableaux(['A',2],shape=[4])
sage: t = T(rows=[[1,2,2,2]])
sage: t.s(1)
[[1, 1, 1, 2]]
```

subcrystal(*index_set=None*, *max_depth=inf*, *direction='both'*, *contained=None*, *cartan_type=None*, *category=None*)

Construct the subcrystal generated by `self` using e_i and/or f_i for all i in `index_set`.

INPUT:

- `index_set` – (default: `None`) the index set; if `None` then use the index set of the crystal
- `max_depth` – (default: infinity) the maximum depth to build
- `direction` – (default: `'both'`) the direction to build the subcrystal; it can be one of the following:
  - `'both'` - using both e_i and f_i
  - `'upper'` - using e_i
  - `'lower'` - using f_i
- `contained` – (optional) a set (or function) defining the containment in the subcrystal
- `cartan_type` – (optional) specify the Cartan type of the subcrystal
- `category` – (optional) specify the category of the subcrystal

See also:

- *Crystals.ParentMethods.subcrystal()*

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['A',3,1], 1, 2)
sage: elt = C(1,4)
sage: list(elt.subcrystal(index_set=[1,3]))
[[[1, 4]], [[2, 4]], [[1, 3]], [[2, 3]]]
sage: list(elt.subcrystal(index_set=[1,3], max_depth=1))
[[[1, 4]], [[2, 4]], [[1, 3]]]
sage: list(elt.subcrystal(index_set=[1,3], direction='upper'))
[[[1, 4]], [[1, 3]]]
sage: list(elt.subcrystal(index_set=[1,3], direction='lower'))
[[[1, 4]], [[2, 4]]]
```

tensor(**elts*)

Return the tensor product of `self` with the crystal elements `elts`.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 3])
sage: B = crystals.infinity.Tableaux(['A', 3])
sage: c = C[0]
sage: b = B.highest_weight_vector()
sage: t = c.tensor(c, b)
sage: ascii_art(t)
        1  1  1
1 # 1 #    2  2
        3
sage: tensor([c, c, b]) == t
True
sage: ascii_art(tensor([b, b, c]))
```

(continues on next page)

(continued from previous page)

```
  1  1  1      1  1  1
  2  2    #    2  2    # 1
  3            3
```

to_highest_weight (*index_set=None*)

Return the highest weight element u and a list $[i_1, ..., i_k]$ such that `self` $= f_{i_1} ... f_{i_k} u$, where $i_1, ..., i_k$ are elements in `index_set`.

By default the `index_set` is assumed to be the full index set of `self`.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape = [1])
sage: t = T(rows = [[3]])
sage: t.to_highest_weight()
[[[1]], [2, 1]]
sage: T = crystals.Tableaux(['A',3], shape = [2,1])
sage: t = T(rows = [[1,2],[4]])
sage: t.to_highest_weight()
[[[1, 1], [2]], [1, 3, 2]]
sage: t.to_highest_weight(index_set = [3])
[[[1, 2], [3]], [3]]
sage: K = crystals.KirillovReshetikhin(['A',3,1],2,1)
sage: t = K(rows=[[2],[3]]); t.to_highest_weight(index_set=[1])
[[[1], [3]], [1]]
sage: t.to_highest_weight()
Traceback (most recent call last):
...
ValueError: this is not a highest weight crystal
```

to_lowest_weight (*index_set=None*)

Return the lowest weight element u and a list $[i_1, ..., i_k]$ such that `self` $= e_{i_1} ... e_{i_k} u$, where $i_1, ..., i_k$ are elements in `index_set`.

By default the `index_set` is assumed to be the full index set of `self`.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape = [1])
sage: t = T(rows = [[3]])
sage: t.to_lowest_weight()
[[[4]], [3]]
sage: T = crystals.Tableaux(['A',3], shape = [2,1])
sage: t = T(rows = [[1,2],[4]])
sage: t.to_lowest_weight()
[[[3, 4], [4]], [1, 2, 2, 3]]
sage: t.to_lowest_weight(index_set = [3])
[[[1, 2], [4]], []]
sage: K = crystals.KirillovReshetikhin(['A',3,1],2,1)
sage: t = K.module_generator(); t
[[1], [2]]
sage: t.to_lowest_weight(index_set=[1,2,3])
[[[3], [4]], [2, 1, 3, 2]]
sage: t.to_lowest_weight()
Traceback (most recent call last):
...
ValueError: this is not a highest weight crystal
```

weight()

Return the weight of this crystal element.

This method should be implemented by the element class of the crystal.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).weight()
(1, 0, 0, 0, 0, 0)
```

Finite

alias of *FiniteCrystals*

class MorphismMethods

Bases: `object`

is_embedding()

Check if `self` is an injective crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_embedding()
True

sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: B = crystals.infinity.Tableaux(['A',2])
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: W = crystals.elementary.T(['A',2], La[1]+La[2])
sage: T = W.tensor(B)
sage: mg = T(W.module_generators[0], B.module_generators[0])
sage: psi = Hom(C,T)([mg])
sage: psi.is_embedding()
True
```

is_isomorphism()

Check if `self` is a crystal isomorphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_isomorphism()
False
```

is_strict()

Check if `self` is a strict crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_strict()
True
```

class ParentMethods

Bases: `object`

Lambda()

Returns the fundamental weights in the weight lattice realization for the root system associated with the crystal

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.Lambda()
Finite family {1: (1, 0, 0, 0, 0, 0), 2: (1, 1, 0, 0, 0, 0), 3: (1, 1, 1,␣
↪0, 0, 0), 4: (1, 1, 1, 1, 0, 0), 5: (1, 1, 1, 1, 1, 0)}
```

an_element()

Returns an element of `self`

sage: C = crystals.Letters([‘A’, 5]) sage: C.an_element() 1

cartan_type()

Returns the Cartan type of the crystal

EXAMPLES:

```
sage: C = crystals.Letters(['A',2])
sage: C.cartan_type()
['A', 2]
```

connected_components()

Return the connected components of `self` as subcrystals.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: T.connected_components()
[Subcrystal of Full tensor product of the crystals
 [The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
  The crystal of letters for type ['A', 2]],
 Subcrystal of Full tensor product of the crystals
 [The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
  The crystal of letters for type ['A', 2]],
 Subcrystal of Full tensor product of the crystals
 [The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
  The crystal of letters for type ['A', 2]]]
```

connected_components_generators()

Return a tuple of generators for each of the connected components of `self`.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: T.connected_components_generators()
([[[1, 1], [2]], 1], [[[1, 2], [2]], 1], [[[1, 2], [3]], 1])
```

crystal_morphism(*on_gens*, *codomain=None*, *cartan_type=None*, *index_set=None*, *generators=None*, *automorphism=None*, *virtualization=None*, *scaling_factors=None*, *category=None*, *check=True*)

Construct a crystal morphism from `self` to another crystal `codomain`.

INPUT:

- `on_gens` – a function or list that determines the image of the generators (if given a list, then this uses the order of the generators of the domain) of `self` under the crystal morphism
- `codomain` – (default: `self`) the codomain of the morphism
- `cartan_type` – (optional) the Cartan type of the morphism; the default is the Cartan type of `self`
- `index_set` – (optional) the index set of the morphism; the default is the index set of the Cartan type
- `generators` – (optional) the generators to define the morphism; the default is the generators of `self`
- `automorphism` – (optional) the automorphism to perform the twisting
- `virtualization` – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain; the default is the identity dictionary
- `scaling_factors` – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, ε and φ; the default are all scaling factors to be one
- `category` – (optional) the category for the crystal morphism; the default is the category of `Crystals`.
- `check` – (default: `True`) check if the crystal morphism is valid

See also:

For more examples, see `sage.categories.crystals.CrystalHomset`.

EXAMPLES:

We construct the natural embedding of a crystal using tableaux into the tensor product of single boxes via the reading word:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: F = crystals.Tableaux(['A',2], shape=[1])
sage: T = crystals.TensorProduct(F, F, F)
sage: mg = T.highest_weight_vectors()[2]; mg
[[[1]], [[2]], [[1]]]
sage: psi = B.crystal_morphism([mg], codomain=T); psi
['A', 2] Crystal morphism:
  From: The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]]
  To:   Full tensor product of the crystals
         [The crystal of tableaux of type ['A', 2] and shape(s) [[1]],
          The crystal of tableaux of type ['A', 2] and shape(s) [[1]],
          The crystal of tableaux of type ['A', 2] and shape(s) [[1]]]
  Defn: [[1, 1], [2]] |--> [[[1]], [[2]], [[1]]]
sage: b = B.module_generators[0]
sage: b.pp()
  1  1
  2
sage: psi(b)
[[[1]], [[2]], [[1]]]
sage: psi(b.f(2))
[[[1]], [[3]], [[1]]]
sage: psi(b.f_string([2,1,1]))
[[[2]], [[3]], [[2]]]
sage: lw = b.to_lowest_weight()[0]
sage: lw.pp()
```

(continues on next page)

(continued from previous page)

```
  2  3
  3
sage: psi(lw)
[[[3]], [[3]], [[2]]]
sage: psi(lw) == mg.to_lowest_weight()[0]
True
```

We now take the other isomorphic highest weight component in the tensor product:

```
sage: mg = T.highest_weight_vectors()[1]; mg
[[[2]], [[1]], [[1]]]
sage: psi = B.crystal_morphism([mg], codomain=T)
sage: psi(lw)
[[[3]], [[2]], [[3]]]
```

We construct a crystal morphism of classical crystals using a Kirillov-Reshetikhin crystal:

```
sage: B = crystals.Tableaux(['D', 4], shape=[1,1])
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: K.module_generators
[[], [[1], [2]], [[1, 1], [2, 2]]]
sage: v = K.module_generators[1]
sage: psi = B.crystal_morphism([v], codomain=K, category=FiniteCrystals())
sage: psi
['D', 4] -> ['D', 4, 1] Virtual Crystal morphism:
  From: The crystal of tableaux of type ['D', 4] and shape(s) [[1, 1]]
  To:   Kirillov-Reshetikhin crystal of type ['D', 4, 1] with (r,s)=(2,2)
  Defn: [[1], [2]] |--> [[1], [2]]
sage: b = B.module_generators[0]
sage: psi(b)
[[1], [2]]
sage: psi(b.to_lowest_weight()[0])
[[-2], [-1]]
```

We can define crystal morphisms using a different set of generators. For example, we construct an example using the lowest weight vector:

```
sage: B = crystals.Tableaux(['A',2], shape=[1])
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: T = crystals.elementary.T(['A',2], La[2])
sage: Bp = T.tensor(B)
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: x = C.module_generators[0].f_string([1,2])
sage: psi = Bp.crystal_morphism([x], generators=Bp.lowest_weight_
↪vectors())
sage: psi(Bp.highest_weight_vector())
[[1, 1], [2]]
```

We can also use a dictionary to specify the generators and their images:

```
sage: psi = Bp.crystal_morphism({Bp.lowest_weight_vectors()[0]: x})
sage: psi(Bp.highest_weight_vector())
[[1, 1], [2]]
```

We construct a twisted crystal morphism induced from the diagram automorphism of type $A_3^{(1)}$:

```
sage: La = RootSystem(['A',3,1]).weight_lattice(extended=True).
↪fundamental_weights()
sage: B0 = crystals.GeneralizedYoungWalls(3, La[0])
sage: B1 = crystals.GeneralizedYoungWalls(3, La[1])
sage: phi = B0.crystal_morphism(B1.module_generators, automorphism={0:1,␣
↪1:2, 2:3, 3:0})
sage: phi
['A', 3, 1] Twisted Crystal morphism:
  From: Highest weight crystal of generalized Young walls of Cartan type [
↪'A', 3, 1] and highest weight Lambda[0]
  To:   Highest weight crystal of generalized Young walls of Cartan type [
↪'A', 3, 1] and highest weight Lambda[1]
  Defn: [] |--> []
sage: x = B0.module_generators[0].f_string([0,1,2,3]); x
[[0, 3], [1], [2]]
sage: phi(x)
[[], [1, 0], [2], [3]]
```

We construct a virtual crystal morphism from type G_2 into type D_4:

```
sage: D = crystals.Tableaux(['D',4], shape=[1,1])
sage: G = crystals.Tableaux(['G',2], shape=[1])
sage: psi = G.crystal_morphism(D.module_generators,
....:                          virtualization={1:[2],2:[1,3,4]},
....:                          scaling_factors={1:1, 2:1})
sage: for x in G:
....:     ascii_art(x, psi(x), sep='  |-->  ')
....:     print("")
            1
  1  |-->   2

            1
  2  |-->   3

            2
  3  |-->  -3

            3
  0  |-->  -3

            3
 -3  |-->  -2

           -3
 -2  |-->  -1

           -2
 -1  |-->  -1
```

digraph(*subset=None*, *index_set=None*)

Return the `DiGraph` associated to `self`.

INPUT:

- `subset` – (optional) a subset of vertices for which the digraph should be constructed
- `index_set` – (optional) the index set to draw arrows

EXAMPLES:

```
sage: C = Crystals().example(5)
sage: C.digraph()
Digraph on 6 vertices
```

The edges of the crystal graph are by default colored using blue for edge 1, red for edge 2, and green for edge 3:

```
sage: C = Crystals().example(3)
sage: G = C.digraph()
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

One may also overwrite the colors:

```
sage: C = Crystals().example(3)
sage: G = C.digraph()
sage: G.set_latex_options(color_by_label = {1:"red", 2:"purple", 3:"blue"}
↪)
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

Or one may add colors to yet unspecified edges:

```
sage: C = Crystals().example(4)
sage: G = C.digraph()
sage: C.cartan_type()._index_set_coloring[4]="purple"
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

Here is an example of how to take the top part up to a given depth of an infinite dimensional crystal:

```
sage: C = CartanType(['C',2,1])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[0])
sage: S = T.subcrystal(max_depth=3)
sage: G = T.digraph(subset=S); G
Digraph on 5 vertices
sage: G.vertices(sort=True, key=str)
[(-Lambda[0] + 2*Lambda[1] - delta,),
 (1/2*Lambda[0] + Lambda[1] - Lambda[2] - 1/2*delta, -1/2*Lambda[0] +␣
↪Lambda[1] - 1/2*delta),
 (1/2*Lambda[0] - Lambda[1] + Lambda[2] - 1/2*delta, -1/2*Lambda[0] +␣
↪Lambda[1] - 1/2*delta),
 (Lambda[0] - 2*Lambda[1] + 2*Lambda[2] - delta,),
 (Lambda[0],)]
```

Here is a way to construct a picture of a Demazure crystal using the `subset` option:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = B.highest_weight_vector()
sage: D = B.demazure_subcrystal(t, [2,1])
sage: list(D)
[[[1, 1], [2]], [[1, 2], [2]], [[1, 1], [3]],
 [[1, 3], [2]], [[1, 3], [3]]]
sage: view(D)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

We can also choose to display particular arrows using the `index_set` option:

```
sage: C = crystals.KirillovReshetikhin(['D',4,1], 2, 1)
sage: G = C.digraph(index_set=[1,3])
sage: len(G.edges(sort=False))
20
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

Todo: Add more tests.

direct_sum(*X*)

Return the direct sum of `self` with `X`.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: B.direct_sum(C)
Direct sum of the crystals Family
(The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
 The crystal of letters for type ['A', 2])
```

As a shorthand, we can use `+`:

```
sage: B + C
Direct sum of the crystals Family
(The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
 The crystal of letters for type ['A', 2])
```

dot_tex()

Return a dot_tex string representation of `self`.

EXAMPLES:

```
sage: C = crystals.Letters(['A',2])
sage: C.dot_tex()
'digraph G { \n  node [ shape=plaintext ];\n  N_0  [ label = " ", texlbl =
↪"$1$" ];\n  N_1  [ label = " ", texlbl = "$2$" ];\n  N_2  [ label = " ",␣
↪texlbl = "$3$" ];\n  N_0 -> N_1 [ label = " ", texlbl = "1" ];\n  N_1 ->
↪ N_2  [ label = " ", texlbl = "2" ];\n}'
```

index_set()

Returns the index set of the Dynkin diagram underlying the crystal

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.index_set()
(1, 2, 3, 4, 5)
```

is_connected()

Return `True` if `self` is a connected crystal.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: B.is_connected()
True
sage: T.is_connected()
False
```

latex(***options*)

Returns the crystal graph as a latex string. This can be exported to a file with `self.latex_file('filename')`.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2],shape=[1])
sage: T._latex_()
'...tikzpicture...'
sage: view(T) # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

One can for example also color the edges using the following options:

```
sage: T = crystals.Tableaux(['A',2],shape=[1])
sage: T._latex_(color_by_label={0:"black", 1:"red", 2:"blue"})
'...tikzpicture...'
```

latex_file(*filename*)

Export a file, suitable for pdflatex, to `filename`.

This requires a proper installation of `dot2tex` in sage-python. For more information see the documentation for `self.latex()`.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: fn = tmp_filename(ext='.tex')
sage: C.latex_file(fn)
```

metapost(*filename*, *thicklines=False*, *labels=True*, *scaling_factor=1.0*, *tallness=1.0*)

Export a file, suitable for MetaPost, to `filename`.

Root operators $e(1)$ or $f(1)$ move along red lines, $e(2)$ or $f(2)$ along green. The highest weight is in the lower left. Vertices with the same weight are kept close together. The concise labels on the nodes are strings introduced by Berenstein and Zelevinsky and Littelmann; see Littelmann's paper Cones, Crystals, Patterns, sections 5 and 6.

For Cartan types B2 or C2, the pattern has the form

$a_2a_3a_4a_1$

where $c*a_2 = a_3 = 2*a_4 = 0$ and $a_1 = 0$, with $c = 2$ for B2, $c = 1$ for C2. Applying $e(2)$ a_1 times, $e(1)$ a_2 times, $e(2)$ a_3 times, $e(1)$ a_4 times returns to the highest weight. (Observe that Littelmann writes the roots in opposite of the usual order, so our $e(1)$ is his $e(2)$ for these Cartan types.) For type A2, the pattern has the form

$a_3a_2a_1$

where applying $e(1)$ a_3 times, $e(2)$ a_2 times then $e(1)$ a_1 times returns to the highest weight. These data determine the vertex and may be translated into a Gelfand-Tsetlin pattern or tableau.

INPUT:

- `filename` – name of the output file, e.g., `'filename.mp'`
- `thicklines` – (default: `True`) for thicker edges
- `labels` – (default: False) to suppress labeling of the vertices
- `scaling_factor` – (default: `1.0`) Increasing or decreasing the scaling factor changes the size of the image
- `tallness` – (default: `1.0`) Increasing makes the image taller without increasing the width

EXAMPLES:

```
sage: C = crystals.Letters(['A', 2])
sage: C.metapost(tmp_filename())
```

```
sage: C = crystals.Letters(['A', 5])
sage: C.metapost(tmp_filename())
Traceback (most recent call last):
...
NotImplementedError
```

number_of_connected_components()

Return the number of connected components of `self`.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: T.number_of_connected_components()
3
```

plot(***options*)

Return the plot of `self` as a directed graph.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: print(C.plot())
Graphics object consisting of 17 graphics primitives
```

plot3d(***options*)

Return the 3-dimensional plot of `self` as a directed graph.

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['A',3,1],2,1)
sage: print(C.plot3d())
Graphics3d Object
```

subcrystal(*index_set=None*, *generators=None*, *max_depth=inf*, *direction='both'*, *contained=None*, *virtualization=None*, *scaling_factors=None*, *cartan_type=None*, *category=None*)

Construct the subcrystal from `generators` using e_i and/or f_i for all i in `index_set`.

INPUT:

- `index_set` – (default: `None`) the index set; if `None` then use the index set of the crystal
- `generators` – (default: `None`) the list of generators; if `None` then use the module generators of the crystal
- `max_depth` – (default: infinity) the maximum depth to build

- `direction` – (default: `'both'`) the direction to build the subcrystal; it can be one of the following:
  - `'both'` - using both e_i and f_i
  - `'upper'` - using e_i
  - `'lower'` - using f_i
- `contained` – (optional) a set or function defining the containment in the subcrystal
- `virtualization`, `scaling_factors` – (optional) dictionaries whose key i corresponds to the sets σ_i and γ_i respectively used to define virtual crystals; see `VirtualCrystal`
- `cartan_type` – (optional) specify the Cartan type of the subcrystal
- `category` – (optional) specify the category of the subcrystal

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['A',3,1], 1, 2)
sage: S = list(C.subcrystal(index_set=[1,2])); S
[[[1, 1]], [[1, 2]], [[2, 2]], [[1, 3]], [[2, 3]], [[3, 3]]]
sage: C.cardinality()
10
sage: len(S)
6
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)]))
[[[1, 4]], [[2, 4]], [[1, 3]], [[2, 3]]]
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)], max_
↪depth=1))
[[[1, 4]], [[2, 4]], [[1, 3]]]
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)], direction=
↪'upper'))
[[[1, 4]], [[1, 3]]]
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)], direction=
↪'lower'))
[[[1, 4]], [[2, 4]]]

sage: G = C.subcrystal(index_set=[1,2,3]).digraph()
sage: GA = crystals.Tableaux('A3', shape=[2]).digraph()
sage: G.is_isomorphic(GA, edge_labels=True)
True
```

We construct the subcrystal which contains the necessary data to construct the corresponding dual equivalence graph:

```
sage: C = crystals.Tableaux(['A',5], shape=[3,3])
sage: is_wt0 = lambda x: all(x.epsilon(i) == x.phi(i) for i in x.parent().
↪index_set())
sage: def check(x):
....:     if is_wt0(x):
....:         return True
....:     for i in x.parent().index_set()[:-1]:
....:         L = [x.e(i), x.e_string([i,i+1]), x.f(i), x.f_string([i,
↪i+1])]
....:         if any(y is not None and is_wt0(y) for y in L):
....:             return True
....:     return False
sage: wt0 = [x for x in C if is_wt0(x)]
sage: S = C.subcrystal(contained=check, generators=wt0)
sage: S.module_generators[0]
[[1, 3, 5], [2, 4, 6]]
sage: S.module_generators[0].e(2).e(3).f(2).f(3)
[[1, 2, 5], [3, 4, 6]]
```

An example of a type B_2 virtual crystal inside of a type A_3 ambient crystal:

```
sage: A = crystals.Tableaux(['A',3], shape=[2,1,1])
sage: S = A.subcrystal(virtualization={1:[1,3], 2:[2]},
....:                  scaling_factors={1:1,2:1}, cartan_type=['B',2])
sage: B = crystals.Tableaux(['B',2], shape=[1])
sage: S.digraph().is_isomorphic(B.digraph(), edge_labels=True)
True
```

tensor(**crystals*, ***options*)

Return the tensor product of `self` with the crystals `B`.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 3])
sage: B = crystals.infinity.Tableaux(['A', 3])
sage: T = C.tensor(C, B); T
Full tensor product of the crystals
 [The crystal of letters for type ['A', 3],
  The crystal of letters for type ['A', 3],
  The infinity crystal of tableaux of type ['A', 3]]
sage: tensor([C, C, B]) is T
True

sage: C = crystals.Letters(['A',2])
sage: T = C.tensor(C, C, generators=[[C(2),C(1),C(1)],[C(1),C(2),C(1)]]);␣
↪T
The tensor product of the crystals
 [The crystal of letters for type ['A', 2],
  The crystal of letters for type ['A', 2],
  The crystal of letters for type ['A', 2]]
sage: T.module_generators
([2, 1, 1], [1, 2, 1])
```

weight_lattice_realization()

Return the weight lattice realization used to express weights in `self`.

This default implementation uses the ambient space of the root system for (non relabelled) finite types and the weight lattice otherwise. This is a legacy from when ambient spaces were partially implemented, and may be changed in the future.

For affine types, this returns the extended weight lattice by default.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.weight_lattice_realization()
Ambient space of the Root system of type ['A', 5]
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.weight_lattice_realization()
Weight lattice of the Root system of type ['A', 2, 1]
```

class SubcategoryMethods

Bases: `object`

Methods for all subcategories.

TensorProducts()

Return the full subcategory of objects of `self` constructed as tensor products.

See also:

- *tensor.TensorProductsCategory*
- RegressiveCovariantFunctorialConstruction.

EXAMPLES:

```
sage: HighestWeightCrystals().TensorProducts()
Category of tensor products of highest weight crystals
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of crystals constructed by tensor product of crystals.

extra_super_categories()

EXAMPLES:

```
sage: Crystals().TensorProducts().extra_super_categories()
[Category of crystals]
```

example(*choice='highwt'*, ***kwds*)

Returns an example of a crystal, as per *Category.example()*.

INPUT:

- choice – str [default: 'highwt']. Can be either 'highwt' for the highest weight crystal of type A, or 'naive' for an example of a broken crystal.
- **kwds – keyword arguments passed onto the constructor for the chosen crystal.

EXAMPLES:

```
sage: Crystals().example(choice='highwt', n=5)
Highest weight crystal of type A_5 of highest weight omega_1
sage: Crystals().example(choice='naive')
A broken crystal, defined by digraph, of dimension five.
```

super_categories()

EXAMPLES:

```
sage: Crystals().super_categories()
[Category of enumerated sets]
```

4.32 CW Complexes

class sage.categories.cw_complexes.**CWComplexes**

Bases: *Category_singleton*

The category of CW complexes.

A CW complex is a Closure-finite cell complex in the Weak topology.

REFERENCES:

- Wikipedia article CW_complex

Note: The notion of "finite" is that the number of cells is finite.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: C = CWComplexes(); C
Category of CW complexes
```

Compact_extra_super_categories()

Return extraneous super categories for `CWComplexes().Compact()`.

A compact CW complex is finite, see Proposition A.1 in [Hat2002].

Todo: Fix the name of finite CW complexes.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: CWComplexes().Compact() # indirect doctest
Category of finite finite dimensional CW complexes
sage: CWComplexes().Compact() is CWComplexes().Finite()
True
```

class Connected(*base_category*)

Bases: *CategoryWithAxiom*

The category of connected CW complexes.

class ElementMethods

Bases: `object`

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.an_element().dimension()
2
```

class Finite(*base_category*)

Bases: *CategoryWithAxiom*

Category of finite CW complexes.

A finite CW complex is a CW complex with a finite number of cells.

class ParentMethods

Bases: `object`

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.dimension()
2
```

extra_super_categories()

Return the extra super categories of `self`.

A finite CW complex is a compact finite-dimensional CW complex.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: C = CWComplexes().Finite()
sage: C.extra_super_categories()
[Category of finite dimensional CW complexes,
 Category of compact topological spaces]
```

class **FiniteDimensional**(*base_category*)

Bases: `CategoryWithAxiom`

Category of finite dimensional CW complexes.

class **ParentMethods**

Bases: `object`

cells()

Return the cells of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: C = X.cells()
sage: sorted((d, C[d]) for d in C.keys())
[(0, (0-cell v,)),
 (1, (0-cell e1, 0-cell e2)),
 (2, (2-cell f,))]
```

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.dimension()
2
```

class **SubcategoryMethods**

Bases: `object`

Connected()

Return the full subcategory of the connected objects of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: CWComplexes().Connected()
Category of connected CW complexes
```

FiniteDimensional()

Return the full subcategory of the finite dimensional objects of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: C = CWComplexes().FiniteDimensional(); C
Category of finite dimensional CW complexes
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: CWComplexes().super_categories()
[Category of topological spaces]
```

4.33 Discrete Valuation Rings (DVR) and Fields (DVF)

class sage.categories.discrete_valuation.**DiscreteValuationFields**

Bases: *Category_singleton*

The category of discrete valuation fields

EXAMPLES:

```
sage: Qp(7) in DiscreteValuationFields()                                    #␣
↪needs sage.rings.padics
True
sage: TestSuite(DiscreteValuationFields()).run()
```

class ElementMethods

Bases: `object`

valuation()

Return the valuation of this element.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: x = Qp(5)(50)
sage: x.valuation()
2
```

class ParentMethods

Bases: `object`

residue_field()

Return the residue field of the ring of integers of this discrete valuation field.

EXAMPLES:

```
sage: Qp(5).residue_field()                                              #␣
↪needs sage.rings.padics
Finite Field of size 5

sage: K.<u> = LaurentSeriesRing(QQ)
sage: K.residue_field()
Rational Field
```

uniformizer()

Return a uniformizer of this ring.

EXAMPLES:

```
sage: Qp(5).uniformizer()                                                #␣
↪needs sage.rings.padics
5 + O(5^21)
```

super_categories()

EXAMPLES:

```
sage: DiscreteValuationFields().super_categories()
[Category of fields]
```

class sage.categories.discrete_valuation.**DiscreteValuationRings**

Bases: *Category_singleton*

The category of discrete valuation rings

EXAMPLES:

```
sage: GF(7)[['x']] in DiscreteValuationRings()
True
sage: TestSuite(DiscreteValuationRings()).run()
```

class ElementMethods

Bases: object

euclidean_degree()

Return the Euclidean degree of this element.

gcd(*other*)

Return the greatest common divisor of self and other, normalized so that it is a power of the distinguished uniformizer.

is_unit()

Return `True` if `self` is invertible.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: x = Zp(5)(50)
sage: x.is_unit()
False

sage: # needs sage.rings.padics
sage: x = Zp(7)(50)
sage: x.is_unit()
True
```

lcm(*other*)

Return the least common multiple of self and other, normalized so that it is a power of the distinguished uniformizer.

quo_rem(*other*)

Return the quotient and remainder for Euclidean division of `self` by `other`.

EXAMPLES:

```
sage: R.<q> = GF(5)[[]]
sage: (q^2 + q).quo_rem(q)
(1 + q, 0)
sage: (q + 1).quo_rem(q^2)
(0, 1 + q)
```

valuation()

Return the valuation of this element.

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: x = Zp(5)(50)
sage: x.valuation()
2
```

class ParentMethods

Bases: `object`

residue_field()

Return the residue field of this ring.

EXAMPLES:

```
sage: Zp(5).residue_field()                                                    #␣
↪needs sage.rings.padics
Finite Field of size 5

sage: K.<u> = QQ[[]]
sage: K.residue_field()
Rational Field
```

uniformizer()

Return a uniformizer of this ring.

EXAMPLES:

```
sage: Zp(5).uniformizer()                                                      #␣
↪needs sage.rings.padics
5 + O(5^21)

sage: K.<u> = QQ[[]]
sage: K.uniformizer()
u
```

super_categories()

EXAMPLES:

```
sage: DiscreteValuationRings().super_categories()
[Category of euclidean domains]
```

4.34 Distributive Magmas and Additive Magmas

class sage.categories.distributive_magmas_and_additive_magmas.**DistributiveMagmasAndAdditive**

Bases: *CategoryWithAxiom_singleton*

The category of sets $(S, +, *)$ with $*$ distributing on $+$.

This is similar to a ring, but $+$ and $*$ are only required to be (additive) magmas.

EXAMPLES:

```
sage: from sage.categories.distributive_magmas_and_additive_magmas import␣
↪DistributiveMagmasAndAdditiveMagmas
sage: C = DistributiveMagmasAndAdditiveMagmas(); C
Category of distributive magmas and additive magmas
sage: C.super_categories()
[Category of magmas and additive magmas]
```

class AdditiveAssociative(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class AdditiveCommutative(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class AdditiveUnital(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class Associative(*base_category*)

Bases: *CategoryWithAxiom_singleton*

AdditiveInverse

alias of *Rngs*

Unital

alias of *Semirings*

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that a Cartesian product of magmas distributing over additive magmas is a magma distributing over an additive magma.

EXAMPLES:

```
sage: C = (Magmas() & AdditiveMagmas()).Distributive().CartesianProducts()
sage: C.extra_super_categories()
[Category of distributive magmas and additive magmas]
sage: C.axioms()
frozenset({'Distributive'})
```

class ParentMethods

Bases: `object`

4.35 Division rings

class `sage.categories.division_rings.`**DivisionRings**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of division rings

A division ring (or skew field) is a not necessarily commutative ring where all non-zero elements have multiplicative inverses

EXAMPLES:

```
sage: DivisionRings()
Category of division rings
sage: DivisionRings().super_categories()
[Category of domains]
```

Commutative

alias of *Fields*

class ElementMethods

Bases: `object`

Finite_extra_super_categories()

Return extraneous super categories for `DivisionRings().Finite()`.

EXAMPLES:

Any field is a division ring:

```
sage: Fields().is_subcategory(DivisionRings())
True
```

This methods specifies that, by Weddeburn theorem, the reciprocal holds in the finite case: a finite division ring is commutative and thus a field:

```
sage: DivisionRings().Finite_extra_super_categories()
(Category of commutative magmas,)
sage: DivisionRings().Finite()
Category of finite enumerated fields
```

Warning: This is not implemented in `DivisionRings.Finite.extra_super_categories` because the categories of finite division rings and of finite fields coincide. See the section *Deduction rules* in the documentation of axioms.

class ParentMethods

Bases: `object`

extra_super_categories()

Return the *Domains* category.

This method specifies that a division ring has no zero divisors, i.e. is a domain.

See also:

The *Deduction rules* section in the documentation of axioms

EXAMPLES:

```
sage: DivisionRings().extra_super_categories()
(Category of domains,)
sage: "NoZeroDivisors" in DivisionRings().axioms()
True
```

4.36 Domains

class sage.categories.domains.**Domains**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of domains

A domain (or non-commutative integral domain), is a ring, not necessarily commutative, with no nonzero zero divisors.

EXAMPLES:

```
sage: C = Domains(); C
Category of domains
sage: C.super_categories()
[Category of rings]
sage: C is Rings().NoZeroDivisors()
True
```

Commutative

alias of *IntegralDomains*

class ElementMethods

Bases: object

class ParentMethods

Bases: object

super_categories()

EXAMPLES:

```
sage: Domains().super_categories()
[Category of rings]
```

4.37 Enumerated sets

class sage.categories.enumerated_sets.**EnumeratedSets**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of enumerated sets

An *enumerated set* is a *finite* or *countable* set or multiset S together with a canonical enumeration of its elements; conceptually, this is very similar to an immutable list. The main difference lies in the names and the return type of the methods, and of course the fact that the list of elements is not supposed to be expanded in memory. Whenever possible one should use one of the two sub-categories *FiniteEnumeratedSets* or *InfiniteEnumeratedSets*.

The purpose of this category is threefold:

- to fix a common interface for all these sets;
- to provide a bunch of default implementations;
- to provide consistency tests.

The standard methods for an enumerated set S are:

- S.cardinality(): the number of elements of the set. This is the equivalent for len on a list except that the return value is specified to be a Sage Integer or infinity, instead of a Python int.
- iter(S): an iterator for the elements of the set;
- S.list(): a fresh list of the elements of the set, when possible; raises a NotImplementedError if the list is predictably too large to be expanded in memory.
- S.tuple(): a tuple of the elements of the set, when possible; raises a NotImplementedError if the tuple is predictably too large to be expanded in memory.
- S.unrank(n): the n-th element of the set when n is a sage Integer. This is the equivalent for l[n] on a list.
- S.rank(e): the position of the element e in the set; This is equivalent to l.index(e) for a list except that the return value is specified to be a Sage Integer, instead of a Python int.
- S.first(): the first object of the set; it is equivalent to S.unrank(0).
- S.next(e): the object of the set which follows e; it is equivalent to S.unrank(S.rank(e) + 1).
- S.random_element(): a random generator for an element of the set. Unless otherwise stated, and for finite enumerated sets, the probability is uniform.

For examples and tests see:

- FiniteEnumeratedSets().example()
- InfiniteEnumeratedSets().example()

EXAMPLES:

```
sage: EnumeratedSets()
Category of enumerated sets
sage: EnumeratedSets().super_categories()
[Category of sets]
sage: EnumeratedSets().all_super_categories()
[Category of enumerated sets, Category of sets,
 Category of sets with partial maps, Category of objects]
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

class ParentMethods

Bases: `object`

first()

Return the first element.

EXAMPLES:

```
sage: cartesian_product([ZZ]*10).first()
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
```

class ElementMethods

Bases: `object`

rank()

Return the rank of `self` in its parent.

See also *EnumeratedSets.ElementMethods.rank()*

EXAMPLES:

```
sage: F = FiniteSemigroups().example(('a','b','c'))
sage: L = list(F)
sage: L[7].rank()
7
sage: all(x.rank() == i for i,x in enumerate(L))
True
```

Finite

alias of *FiniteEnumeratedSets*

Infinite

alias of *InfiniteEnumeratedSets*

class ParentMethods

Bases: `object`

first()

The "first" element of `self`.

`self.first()` returns the first element of the set `self`. This is a generic implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.first() # indirect doctest
1
```

is_empty()

Return whether this set is empty.

EXAMPLES:

```
sage: F = FiniteEnumeratedSet([1,2,3])
sage: F.is_empty()
False
sage: F = FiniteEnumeratedSet([])
sage: F.is_empty()
True
```

iterator_range(*start=None*, *stop=None*, *step=None*)

Iterate over the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

See also:

`unrank()`, `unrank_range()`

EXAMPLES:

```
sage: # needs sage.combinat
sage: P = Partitions()
sage: list(P.iterator_range(stop=5))
[[], [1], [2], [1, 1], [3]]
sage: list(P.iterator_range(0, 5))
[[], [1], [2], [1, 1], [3]]
sage: list(P.iterator_range(3, 5))
[[1, 1], [3]]
sage: list(P.iterator_range(3, 10))
[[1, 1], [3], [2, 1], [1, 1, 1], [4], [3, 1], [2, 2]]
sage: list(P.iterator_range(3, 10, 2))
[[1, 1], [2, 1], [4], [2, 2]]
sage: it = P.iterator_range(3)
sage: [next(it) for x in range(10)]
[[1, 1],
 [3], [2, 1], [1, 1, 1],
 [4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1],
 [5]]
sage: it = P.iterator_range(3, step=2)
sage: [next(it) for x in range(5)]
[[1, 1],
 [2, 1],
 [4], [2, 2], [1, 1, 1, 1]]
sage: next(P.iterator_range(stop=-3))
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: next(P.iterator_range(start=-3))
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

list()

Return a list of the elements of `self`.

The elements of set `x` are created and cached on the first call of `x.list()`. Then each call of `x.list()` returns a new list from the cached result. Thus in looping, it may be better to do `for e in x:`, not `for e in x.list():`.

If `x` is not known to be finite, then an exception is raised.

EXAMPLES:

```
sage: (GF(3)^2).list()                                                      #␣
↪needs sage.modules
[(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2)]
sage: R = Integers(11)
sage: R.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: l = R.list(); l
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: l.remove(0); l
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: R.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

sage: C = FiniteEnumeratedSets().example()
sage: C.list()
[1, 2, 3]
```

map (*f*, *name*, *is_injective=None*)

Return the image $\{f(x)|x \in \text{self}\}$ of this enumerated set by f, as an enumerated set.

INPUT:

- `is_injective` – boolean (default: `True`) whether to assume that `f` is injective.

EXAMPLES:

```
sage: R = Compositions(4).map(attrcall('partial_sums')); R
Image of Compositions of 4 by The map *.partial_sums()
 from Compositions of 4
sage: R.cardinality()
8
sage: R.list()
[[1, 2, 3, 4], [1, 2, 4], [1, 3, 4], [1, 4], [2, 3, 4], [2, 4], [3, 4],␣
↪[4]]
sage: [r for r in R]
[[1, 2, 3, 4], [1, 2, 4], [1, 3, 4], [1, 4], [2, 3, 4], [2, 4], [3, 4],␣
↪[4]]
sage: R.category()
Category of finite enumerated subobjects of sets
```

Warning: If the function is not injective, then there may be repeated elements:

```
sage: P = Compositions(4)
sage: P.list()
[[1, 1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 3], [2, 1, 1], [2, 2], [3, 1],␣
↪[4]]
sage: P.map(attrcall('major_index')).list()
[6, 3, 4, 1, 5, 2, 3, 0]
```

Pass `is_injective=False` to get a correct result in this case:

```
sage: P.map(attrcall('major_index'), is_injective=False).list()
[6, 3, 4, 1, 5, 2, 0]
```

next (*obj*)

The "next" element after `obj` in `self`.

`self.next(e)` returns the element of the set `self` which follows `e`. This is a generic implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided.

Remark: this is the default (brute force) implementation of the category `EnumeratedSets()`. Its complexity is $O(r)$, where r is the rank of `obj`.

EXAMPLES:

```
sage: C = InfiniteEnumeratedSets().example()
sage: C._next_from_iterator(10) # indirect doctest
11
```

TODO: specify the behavior when `obj` is not in `self`.

random_element()

Return a random element in `self`.

Unless otherwise stated, and for finite enumerated sets, the probability is uniform.

This is a generic implementation from the category `EnumeratedSets()`. It raises a `NotImplementedError` since one does not know whether the set is finite.

EXAMPLES:

```
sage: class broken(UniqueRepresentation, Parent):
....:  def __init__(self):
....:      Parent.__init__(self, category = EnumeratedSets())
sage: broken().random_element()
Traceback (most recent call last):
...
NotImplementedError: unknown cardinality
```

rank(x)

The rank of an element of `self`

`self.rank(x)` returns the rank of x, that is its position in the enumeration of `self`. This is an integer between `0` and `n-1` where `n` is the cardinality of `self`, or None if x is not in $self$.

This is the default (brute force) implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided. Its complexity is $O(r)$, where r is the rank of `obj`. For infinite enumerated sets, this won't terminate when x is not in `self`

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: list(C)
[1, 2, 3]
sage: C.rank(3) # indirect doctest
2
sage: C.rank(5) # indirect doctest
```

some_elements()

Return some elements in `self`.

See `TestSuite` for a typical use case.

This is a generic implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided. It returns an iterator for up to the first 100 elements of `self`

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: list(C.some_elements()) # indirect doctest
[1, 2, 3]
```

tuple()

Return a tuple of the elements of `self`.

The tuple of elements of `x` is created and cached on the first call of `x.tuple()`. Each following call of `x.tuple()` returns the same tuple.

For looping, it may be better to do `for e in x:`, not `for e in x.tuple():`.

If `x` is not known to be finite, then an exception is raised.

EXAMPLES:

```
sage: (GF(3)^2).tuple()                                                    #␣
↪needs sage.modules
((0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2))
sage: R = Integers(11)
sage: l = R.tuple(); l
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
sage: l is R.tuple()
True
```

unrank(*r*)

The `r`-th element of `self`

`self.unrank(r)` returns the `r`-th element of `self`, where `r` is an integer between `0` and `n-1` where `n` is the cardinality of `self`.

This is the default (brute force) implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided. Its complexity is $O(r)$, where r is the rank of `obj`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.unrank(2) # indirect doctest
3
sage: C._unrank_from_iterator(5)
Traceback (most recent call last):
...
ValueError: the rank must be in the range from 0 to 2
sage: ZZ._unrank_from_iterator(-1)
Traceback (most recent call last):
...
ValueError: the rank must be greater than or equal to 0
```

unrank_range(*start=None*, *stop=None*, *step=None*)

Return the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

See also:

`unrank()`, `iterator_range()`

EXAMPLES:

```
sage: # needs sage.combinat
sage: P = Partitions()
sage: P.unrank_range(stop=5)
```

(continues on next page)

(continued from previous page)

```
[[], [1], [2], [1, 1], [3]]
sage: P.unrank_range(0, 5)
[[], [1], [2], [1, 1], [3]]
sage: P.unrank_range(3, 5)
[[1, 1], [3]]
sage: P.unrank_range(3, 10)
[[1, 1], [3], [2, 1], [1, 1, 1], [4], [3, 1], [2, 2]]
sage: P.unrank_range(3, 10, 2)
[[1, 1], [2, 1], [4], [2, 2]]
sage: P.unrank_range(3)
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: P.unrank_range(stop=-3)
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: P.unrank_range(start=-3)
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

additional_structure()

Return None.

Indeed, morphisms of enumerated sets are not required to preserve the enumeration.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: EnumeratedSets().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: EnumeratedSets().super_categories()
[Category of sets]
```

4.38 Euclidean domains

AUTHORS:

- Teresa Gomez-Diaz (2008): initial version
- Julian Rueth (2013-09-13): added euclidean degree, quotient remainder, and their tests

class sage.categories.euclidean_domains.**EuclideanDomains**

Bases: *Category_singleton*

The category of constructive euclidean domains, i.e., one can divide producing a quotient and a remainder where the remainder is either zero or its *ElementMethods.euclidean_degree()* is smaller than the divisor.

EXAMPLES:

```
sage: EuclideanDomains()
Category of euclidean domains
sage: EuclideanDomains().super_categories()
[Category of principal ideal domains]
```

class ElementMethods

Bases: `object`

euclidean_degree()

Return the degree of this element as an element of an Euclidean domain, i.e., for elements a, b the euclidean degree f satisfies the usual properties:

1. if b is not zero, then there are elements q and r such that $a = bq + r$ with $r = 0$ or $f(r) < f(b)$
2. if a, b are not zero, then $f(a) \leq f(ab)$

Note: The name `euclidean_degree` was chosen because the euclidean function has different names in different contexts, e.g., absolute value for integers, degree for polynomials.

OUTPUT:

For non-zero elements, a natural number. For the zero element, this might raise an exception or produce some other output, depending on the implementation.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.euclidean_degree()
1
sage: ZZ.one().euclidean_degree()
1
```

gcd(*other*)

Return the greatest common divisor of this element and `other`.

INPUT:

- `other` – an element in the same ring as `self`

ALGORITHM:

Algorithm 3.2.1 in [Coh1993].

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ, sparse=True)
sage: EuclideanDomains().element_class.gcd(x,x+1)
-1
```

quo_rem(*other*)

Return the quotient and remainder of the division of this element by the non-zero element `other`.

INPUT:

- `other` – an element in the same euclidean domain

OUTPUT:

a pair of elements

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.quo_rem(x)
(1, 0)
```

class ParentMethods

Bases: `object`

gcd_free_basis(*elts*)

Compute a set of coprime elements that can be used to express the elements of `elts`.

INPUT:

- `elts` - A sequence of elements of `self`.

OUTPUT:

A GCD-free basis (also called a coprime base) of `elts`; that is, a set of pairwise relatively prime elements of `self` such that any element of `elts` can be written as a product of elements of the set.

ALGORITHM:

Naive implementation of the algorithm described in Section 4.8 of Bach & Shallit [BS1996].

EXAMPLES:

```
sage: ZZ.gcd_free_basis([1])
[]
sage: ZZ.gcd_free_basis([4, 30, 14, 49])
[2, 15, 7]

sage: Pol.<x> = QQ[]
sage: sorted(Pol.gcd_free_basis([
....:      (x+1)^3*(x+2)^3*(x+3), (x+1)*(x+2)*(x+3),
....:      (x+1)*(x+2)*(x+4)]))
[x + 3, x + 4, x^2 + 3*x + 2]
```

is_euclidean_domain()

Return True, since this in an object of the category of Euclidean domains.

EXAMPLES:

```
sage: Parent(QQ,category=EuclideanDomains()).is_euclidean_domain()
True
```

super_categories()

EXAMPLES:

```
sage: EuclideanDomains().super_categories()
[Category of principal ideal domains]
```

4.39 Fields

class sage.categories.fields.**Fields**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of (commutative) fields, i.e. commutative rings where all non-zero elements have multiplicative inverses

EXAMPLES:

```
sage: K = Fields()
sage: K
Category of fields
sage: Fields().super_categories()
[Category of euclidean domains, Category of division rings]

sage: K(IntegerRing())
Rational Field
sage: K(PolynomialRing(GF(3), 'x'))
Fraction Field of Univariate Polynomial Ring in x over
Finite Field of size 3
sage: K(RealField())                                                    #␣
↪needs sage.rings.real_mpfr
Real Field with 53 bits of precision
```

class ElementMethods

Bases: object

euclidean_degree()

Return the degree of this element as an element of an Euclidean domain.

In a field, this returns 0 for all but the zero element (for which it is undefined).

EXAMPLES:

```
sage: QQ.one().euclidean_degree()
0
```

factor()

Return a factorization of self.

Since self is either a unit or zero, this function is trivial.

EXAMPLES:

```
sage: x = GF(7)(5)
sage: x.factor()
5
sage: RR(0).factor()
Traceback (most recent call last):
...
ArithmeticError: factorization of 0.000000000000000 is not defined
```

gcd(*other*)

Greatest common divisor.

Note: Since we are in a field and the greatest common divisor is only determined up to a unit, it is correct to either return zero or one. Note that fraction fields of unique factorization domains provide a more sophisticated gcd.

EXAMPLES:

```
sage: K = GF(5)
sage: K(2).gcd(K(1))
1
sage: K(0).gcd(K(0))
0
sage: all(x.gcd(y) == (0 if x == 0 and y == 0 else 1)
....:     for x in K for y in K)
True
```

For field of characteristic zero, the gcd of integers is considered as if they were elements of the integer ring:

```
sage: gcd(15.0,12.0)
3.00000000000000
```

But for other floating point numbers, the gcd is just 0.0 or 1.0:

```
sage: gcd(3.2, 2.18)
1.00000000000000

sage: gcd(0.0, 0.0)
0.000000000000000
```

AUTHOR:

- Simon King (2011-02) – github issue #10771
- Vincent Delecroix (2015) – github issue #17671

inverse_of_unit()

Return the inverse of this element.

EXAMPLES:

```
sage: x = polygen(ZZ, 'x')
sage: NumberField(x^7 + 2, 'a')(2).inverse_of_unit()                    #␣
↪needs sage.rings.number_field
1/2
```

Trying to invert the zero element typically raises a `ZeroDivisionError`:

```
sage: QQ(0).inverse_of_unit()
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
```

To catch that exception in a way that also works for non-units in more general rings, use something like:

```
sage: try:
....:    QQ(0).inverse_of_unit()
....: except ArithmeticError:
....:    pass
```

Also note that some "fields" allow one to invert the zero element:

```
sage: RR(0).inverse_of_unit()
+infinity
```

is_unit()

Returns True if `self` has a multiplicative inverse.

EXAMPLES:

```
sage: QQ(2).is_unit()
True
sage: QQ(0).is_unit()
False
```

lcm(*other*)

Least common multiple.

Note: Since we are in a field and the least common multiple is only determined up to a unit, it is correct to either return zero or one. Note that fraction fields of unique factorization domains provide a more sophisticated lcm.

EXAMPLES:

```
sage: GF(2)(1).lcm(GF(2)(0))
0
sage: GF(2)(1).lcm(GF(2)(1))
1
```

For field of characteristic zero, the lcm of integers is considered as if they were elements of the integer ring:

```
sage: lcm(15.0,12.0)
60.0000000000000
```

But for others floating point numbers, it is just 0.0 or 1.0:

```
sage: lcm(3.2, 2.18)
1.00000000000000

sage: lcm(0.0, 0.0)
0.000000000000000
```

AUTHOR:

- Simon King (2011-02) – github issue #10771
- Vincent Delecroix (2015) – github issue #17671

quo_rem(*other*)

Return the quotient with remainder of the division of this element by `other`.

INPUT:

- `other` – an element of the field

EXAMPLES:

```
sage: f,g = QQ(1), QQ(2)
sage: f.quo_rem(g)
(1/2, 0)
```

xgcd(*other*)

Compute the extended gcd of `self` and `other`.

INPUT:

- `other` – an element with the same parent as `self`

OUTPUT:

A tuple `(r, s, t)` of elements in the parent of `self` such that `r = s * self + t * other`. Since the computations are done over a field, `r` is zero if `self` and `other` are zero, and one otherwise.

AUTHORS:

- Julian Rueth (2012-10-19): moved here from `sage.structure.element.FieldElement`

EXAMPLES:

```
sage: K = GF(5)
sage: K(2).xgcd(K(1))
(1, 3, 0)
sage: K(0).xgcd(K(4))
(1, 0, 4)
sage: K(1).xgcd(K(1))
(1, 1, 0)
sage: GF(5)(0).xgcd(GF(5)(0))
(0, 0, 0)
```

The xgcd of non-zero floating point numbers will be a triple of floating points. But if the input are two integral floating points the result is a floating point version of the standard gcd on **Z**:

```
sage: xgcd(12.0, 8.0)
(4.00000000000000, 1.00000000000000, -1.00000000000000)

sage: xgcd(3.1, 2.98714)
(1.00000000000000, 0.322580645161290, 0.000000000000000)

sage: xgcd(0.0, 1.1)
(1.00000000000000, 0.000000000000000, 0.909090909090909)
```

Finite

alias of *FiniteFields*

class ParentMethods

Bases: `object`

fraction_field()

Returns the *fraction field* of `self`, which is `self`.

EXAMPLES:

```
sage: QQ.fraction_field() is QQ
True
```

is_field(*proof=True*)

Returns True as `self` is a field.

EXAMPLES:

```
sage: QQ.is_field()
True
sage: Parent(QQ,category=Fields()).is_field()
True
```

is_integrally_closed()

Return True, as per IntegralDomain.is_integrally_closed(): for every field F, F is its own field of fractions, hence every element of F is integral over F.

EXAMPLES:

```
sage: QQ.is_integrally_closed()
True
sage: QQbar.is_integrally_closed()                                          #␣
↪needs sage.rings.number_field
True
sage: Z5 = GF(5); Z5
Finite Field of size 5
sage: Z5.is_integrally_closed()
True
```

is_perfect()

Return whether this field is perfect, i.e., its characteristic is $p = 0$ or every element has a p-th root.

EXAMPLES:

```
sage: QQ.is_perfect()
True
sage: GF(2).is_perfect()
True
sage: FunctionField(GF(2), 'x').is_perfect()
False
```

vector_space(**args*, ***kwds*)

Gives an isomorphism of this field with a vector space over a subfield.

This method is an alias for free_module, which may have more documentation.

INPUT:

- base – a subfield or morphism into this field (defaults to the base field)
- basis – a basis of the field as a vector space over the subfield; if not given, one is chosen automatically
- map – whether to return maps from and to the vector space

OUTPUT:

- V – a vector space over base
- from_V – an isomorphism from V to this field
- to_V – the inverse isomorphism from this field to V

EXAMPLES:

```
sage: # needs sage.rings.padics
sage: K.<a> = Qq(125)
sage: V, fr, to = K.vector_space()
sage: v = V([1, 2, 3])
sage: fr(v, 7)
(3*a^2 + 2*a + 1) + O(5^7)
```

extra_super_categories()

EXAMPLES:

```
sage: Fields().extra_super_categories()
[Category of euclidean domains]
```

4.40 Filtered Algebras

class sage.categories.filtered_algebras.**FilteredAlgebras**(*base_category*)

Bases: *FilteredModulesCategory*

The category of filtered algebras.

An algebra A over a commutative ring R is *filtered* if A is endowed with a structure of a filtered R-module (whose underlying R-module structure is identical with that of the R-algebra A) such that the indexing set I (typically $I = \mathbf{N}$) is also an additive abelian monoid, the unity 1 of A belongs to F_0, and we have $F_i \cdot F_j \subseteq F_{i+j}$ for all $i, j \in I$.

EXAMPLES:

```
sage: Algebras(ZZ).Filtered()
Category of filtered algebras over Integer Ring
sage: Algebras(ZZ).Filtered().super_categories()
[Category of algebras over Integer Ring,
 Category of filtered modules over Integer Ring]
```

REFERENCES:

- Wikipedia article Filtered_algebra

class ParentMethods

Bases: object

graded_algebra()

Return the associated graded algebra to self.

Todo: Implement a version of the associated graded algebra which does not require self to have a distinguished basis.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.graded_algebra()
Graded Algebra of An example of a filtered algebra with basis:
 the universal enveloping algebra of
 Lie algebra of RR^3 with cross product over Integer Ring
```

4.41 Filtered Algebras With Basis

A filtered algebra with basis over a commutative ring R is a filtered algebra over R endowed with the structure of a filtered module with basis (with the same underlying filtered-module structure). See *FilteredAlgebras* and *FilteredModulesWithBasis* for these two notions.

class sage.categories.filtered_algebras_with_basis.**FilteredAlgebrasWithBasis**(*base_category*)

Bases: *FilteredModulesCategory*

The category of filtered algebras with a distinguished homogeneous basis.

A filtered algebra with basis over a commutative ring R is a filtered algebra over R endowed with the structure of a filtered module with basis (with the same underlying filtered-module structure). See *FilteredAlgebras* and *FilteredModulesWithBasis* for these two notions.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(ZZ).Filtered(); C
Category of filtered algebras with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of algebras with basis over Integer Ring,
 Category of filtered algebras over Integer Ring,
 Category of filtered modules with basis over Integer Ring]
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

from_graded_conversion()

Return the inverse of the canonical R-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of A (where $A =$ ``$self$`). This inverse is an isomorphism $\operatorname{gr} A \to A$.

This is an isomorphism of R-modules, not of algebras. See the class documentation `AssociatedGradedAlgebra`.

See also:

to_graded_conversion()

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: p = A.an_element() + A.algebra_generators()['x'] + 2; p
U['x']^2*U['y']^2*U['z']^3 + 3*U['x'] + 3*U['y'] + 3
sage: q = A.to_graded_conversion()(p)
sage: A.from_graded_conversion()(q) == p
True
sage: q.parent() is A.graded_algebra()
True
```

graded_algebra()

Return the associated graded algebra to `self`.

See `AssociatedGradedAlgebra` for the definition and the properties of this.

If the filtered algebra `self` with basis is called A, then this method returns $\operatorname{gr} A$. The method *to_graded_conversion()* returns the canonical R-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of A, and the method *from_graded_conversion()* returns the inverse of this isomorphism. The method *projection()* projects elements of A onto $\operatorname{gr} A$ according to their place in the filtration on A.

Warning: When not overridden, this method returns the default implementation of an associated graded algebra – namely, `AssociatedGradedAlgebra(self)`, where `AssociatedGradedAlgebra` is `AssociatedGradedAlgebra`. But many instances of *FilteredAlgebrasWithBasis* override this method, as the associated graded algebra often is (isomorphic) to a simpler object (for instance, the associated graded algebra of a graded algebra can be identified with the graded algebra itself). Generic code that uses associated graded algebras (such as the code of the *induced_graded_map()* method below) should make sure to only communicate with them via the *to_graded_conversion()*, *from_graded_conversion()*, and *projection()* methods (in particular, do not expect there to be a conversion from `self` to `self.graded_algebra()`; this currently does not work for Clifford algebras). Similarly, when overriding *graded_algebra()*, make sure to accordingly redefine these three methods, unless their definitions below still apply to your case (this will happen whenever the basis of your *graded_algebra()* has the same indexing set as `self`, and the partition of this indexing set according to degree is the same as for `self`).

Todo: Maybe the thing about the conversion from `self` to `self.graded_algebra()` on the Clifford at least could be made to work? (I would still warn the user against ASSUMING that it must work – as there is probably no way to guarantee it in all cases, and we shouldn't require users to mess with element constructors.)

EXAMPLES:

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.graded_algebra()
Graded Algebra of An example of a filtered algebra with basis:
 the universal enveloping algebra of
 Lie algebra of RR^3 with cross product over Integer Ring
```

induced_graded_map(*other*, *f*)

Return the graded linear map between the associated graded algebras of `self` and `other` canonically induced by the filtration-preserving map `f : self -> other`.

Let A and B be two filtered algebras with basis, and let $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ be their filtrations. Let $f : A \to B$ be a linear map which preserves the filtration (i.e., satisfies $f(F_i) \subseteq G_i$ for all $i \in I$). Then, there is a canonically defined graded linear map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ which satisfies

$$(\operatorname{gr} f)(p_i(a)) = p_i(f(a)) \qquad \text{for all } i \in I \text{ and } a \in F_i,$$

where the p_i on the left hand side is the canonical projection from F_i onto the i-th graded component of $\operatorname{gr} A$, while the p_i on the right hand side is the canonical projection from G_i onto the i-th graded component of $\operatorname{gr} B$.

INPUT:

- `other` – a filtered algebra with basis
- `f` – a filtration-preserving linear map from `self` to `other` (can be given as a morphism or as a function)

OUTPUT:

The graded linear map $\operatorname{gr} f$.

EXAMPLES:

Example 1.

We start with the universal enveloping algebra of the Lie algebra $\mathbf{R}^3$ (with the cross product serving as Lie bracket):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example(); A
An example of a filtered algebra with basis: the
 universal enveloping algebra of Lie algebra of RR^3
```

(continues on next page)

(continued from previous page)

```
 with cross product over Rational Field
sage: M = A.indices(); M
Free abelian monoid indexed by {'x', 'y', 'z'}
sage: x,y,z = [A.basis()[M.gens()[i]] for i in "xyz"]
```

Let us define a stupid filtered map from A to itself:

```
sage: def map_on_basis(m):
....:     d = m.dict()
....:     i = d.get('x', 0); j = d.get('y', 0); k = d.get('z', 0)
....:     g = (y ** (i+j)) * (z ** k)
....:     if i > 0:
....:         g += i * (x ** (i-1)) * (y ** j) * (z ** k)
....:     return g
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=A)
sage: f(x)
U['y'] + 1
sage: f(x*y*z)
U['y']^2*U['z'] + U['y']*U['z']
sage: f(x*x*y*z)
U['y']^3*U['z'] + 2*U['x']*U['y']*U['z']
sage: f(A.one())
1
sage: f(y*z)
U['y']*U['z']
```

(There is nothing here that is peculiar to this universal enveloping algebra; we are only using its module structure, and we could just as well be using a polynomial algebra in its stead.)

We now compute gr f

```
sage: grA = A.graded_algebra(); grA
Graded Algebra of An example of a filtered algebra with
 basis: the universal enveloping algebra of Lie algebra
 of RR^3 with cross product over Rational Field
sage: xx, yy, zz = [A.to_graded_conversion()(i) for i in [x, y, z]]
sage: xx+yy*zz
bar(U['y']*U['z']) + bar(U['x'])
sage: grf = A.induced_graded_map(A, f); grf
Generic endomorphism of Graded Algebra of An example
 of a filtered algebra with basis: the universal
 enveloping algebra of Lie algebra of RR^3 with cross
 product over Rational Field
sage: grf(xx)
bar(U['y'])
sage: grf(xx*yy*zz)
bar(U['y']^2*U['z'])
sage: grf(xx*xx*yy*zz)
bar(U['y']^3*U['z'])
sage: grf(grA.one())
bar(1)
sage: grf(yy*zz)
bar(U['y']*U['z'])
sage: grf(yy*zz-2*yy)
bar(U['y']*U['z']) - 2*bar(U['y'])
```

Example 2.

We shall now construct gr f for a different map f out of the same A; the new map f will lead into a graded algebra already, namely into the algebra of symmetric functions:

```
sage: # needs sage.combinat sage.modules
sage: h = SymmetricFunctions(QQ).h()
sage: def map_on_basis(m):  # redefining map_on_basis
....:     d = m.dict()
....:     i = d.get('x', 0); j = d.get('y', 0); k = d.get('z', 0)
....:     g = (h[1] ** i) * (h[2] ** (j // 2) * (h[3] ** (k // 3)))
....:     g += i * (h[1] ** (i+j+k))
....:     return g
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=h)  # redefining f
sage: f(x)
2*h[1]
sage: f(y)
h[]
sage: f(z)
h[]
sage: f(y**2)
h[2]
sage: f(x**2)
3*h[1, 1]
sage: f(x*y*z)
h[1] + h[1, 1, 1]
sage: f(x*x*y*y*z)
2*h[1, 1, 1, 1, 1] + h[2, 1, 1]
sage: f(A.one())
h[]
```

The algebra h of symmetric functions in the h-basis is already graded, so its associated graded algebra is implemented as itself:

```
sage: # needs sage.combinat sage.modules
sage: grh = h.graded_algebra(); grh is h
True
sage: grf = A.induced_graded_map(h, f); grf
Generic morphism:
  From: Graded Algebra of An example of a filtered
   algebra with basis: the universal enveloping
   algebra of Lie algebra of RR^3 with cross
   product over Rational Field
  To:   Symmetric Functions over Rational Field
   in the homogeneous basis
sage: grf(xx)
2*h[1]
sage: grf(yy)
0
sage: grf(zz)
0
sage: grf(yy**2)
h[2]
sage: grf(xx**2)
3*h[1, 1]
sage: grf(xx*yy*zz)
h[1, 1, 1]
sage: grf(xx*xx*yy*yy*zz)
2*h[1, 1, 1, 1, 1]
```

(continues on next page)

(continued from previous page)

```
sage: grf(grA.one())
h[]
```

Example 3.

After having had a graded algebra as the codomain, let us try to have one as the domain instead. Our new `f` will go from `h` to `A`:

```
sage: # needs sage.combinat sage.modules
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return x ** (sum(lam)) + y ** (len(lam))
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                       codomain=A)  # redefining f
sage: f(h[1])
U['x'] + U['y']
sage: f(h[2])
U['x']^2 + U['y']
sage: f(h[1, 1])
U['x']^2 + U['y']^2
sage: f(h[2, 2])
U['x']^4 + U['y']^2
sage: f(h[3, 2, 1])
U['x']^6 + U['y']^3
sage: f(h.one())
2
sage: grf = h.induced_graded_map(A, f); grf
Generic morphism:
  From: Symmetric Functions over Rational Field
   in the homogeneous basis
  To:   Graded Algebra of An example of a filtered
   algebra with basis: the universal enveloping
   algebra of Lie algebra of RR^3 with cross
   product over Rational Field
sage: grf(h[1])
bar(U['x']) + bar(U['y'])
sage: grf(h[2])
bar(U['x']^2)
sage: grf(h[1, 1])
bar(U['x']^2) + bar(U['y']^2)
sage: grf(h[2, 2])
bar(U['x']^4)
sage: grf(h[3, 2, 1])
bar(U['x']^6)
sage: grf(h.one())
2*bar(1)
```

Example 4.

The construct gr f also makes sense when f is a filtration-preserving map between graded algebras.

```
sage: # needs sage.combinat sage.modules
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return h[lam] + h[len(lam)]
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                       codomain=h)  # redefining f
sage: f(h[1])
2*h[1]
```

(continues on next page)

(continued from previous page)

```
sage: f(h[2])
h[1] + h[2]
sage: f(h[1, 1])
h[1, 1] + h[2]
sage: f(h[2, 1])
h[2] + h[2, 1]
sage: f(h.one())
2*h[]
sage: grf = h.induced_graded_map(h, f); grf
Generic endomorphism of Symmetric Functions over Rational
 Field in the homogeneous basis
sage: grf(h[1])
2*h[1]
sage: grf(h[2])
h[2]
sage: grf(h[1, 1])
h[1, 1] + h[2]
sage: grf(h[2, 1])
h[2, 1]
sage: grf(h.one())
2*h[]
```

Example 5.

For another example, let us compute gr f for a map f between two Clifford algebras:

```
sage: # needs sage.modules
sage: Q = QuadraticForm(ZZ, 2, [1,2,3])
sage: B = CliffordAlgebra(Q, names=['u','v']); B
The Clifford algebra of the Quadratic form in 2
 variables over Integer Ring with coefficients:
[ 1 2 ]
[ * 3 ]
sage: m = Matrix(ZZ, [[1, 2], [1, -1]])
sage: f = B.lift_module_morphism(m, names=['x','y'])
sage: A = f.domain(); A
The Clifford algebra of the Quadratic form in 2
 variables over Integer Ring with coefficients:
[ 6 0 ]
[ * 3 ]
sage: x, y = A.gens()
sage: f(x)
u + v
sage: f(y)
2*u - v
sage: f(x**2)
6
sage: f(x*y)
-3*u*v + 3
sage: grA = A.graded_algebra(); grA
The exterior algebra of rank 2 over Integer Ring
sage: A.to_graded_conversion()(x)
x
sage: A.to_graded_conversion()(y)
y
sage: A.to_graded_conversion()(x*y)
x*y
```

(continues on next page)

(continued from previous page)

```
sage: u = A.to_graded_conversion()(x*y+1); u
x*y + 1
sage: A.from_graded_conversion()(u)
x*y + 1
sage: A.projection(2)(x*y+1)
x*y
sage: A.projection(1)(x+2*y-2)
x + 2*y
sage: grf = A.induced_graded_map(B, f); grf
Generic morphism:
  From: The exterior algebra of rank 2 over Integer Ring
  To:   The exterior algebra of rank 2 over Integer Ring
sage: grf(A.to_graded_conversion()(x))
u + v
sage: grf(A.to_graded_conversion()(y))
2*u - v
sage: grf(A.to_graded_conversion()(x**2))
6
sage: grf(A.to_graded_conversion()(x*y))
-3*u*v
sage: grf(grA.one())
1
```

projection(*i*)

Return the i-th projection $p_i : F_i \to G_i$ (in the notations of the class documentation AssociatedGradedAlgebra, where $A = ``self`)$.

This method actually does not return the map p_i itself, but an extension of p_i to the whole R-module A. This extension is the composition of the R-module isomorphism $A \to \operatorname{gr} A$ with the canonical projection of the graded R-module $\operatorname{gr} A$ onto its i-th graded component G_i. The codomain of this map is $\operatorname{gr} A$, although its actual image is G_i. The map p_i is obtained from this map by restricting its domain to F_i and its image to G_i.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: p = A.an_element() + A.algebra_generators()['x'] + 2; p
U['x']^2*U['y']^2*U['z']^3 + 3*U['x'] + 3*U['y'] + 3
sage: q = A.projection(7)(p); q
bar(U['x']^2*U['y']^2*U['z']^3)
sage: q.parent() is A.graded_algebra()
True
sage: A.projection(8)(p)
0
```

to_graded_conversion()

Return the canonical R-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of A (where $A = ``self`)$.

This is an isomorphism of R-modules, not of algebras. See the class documentation AssociatedGradedAlgebra.

See also:

from_graded_conversion()

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: p = A.an_element() + A.algebra_generators()['x'] + 2; p
U['x']^2*U['y']^2*U['z']^3 + 3*U['x'] + 3*U['y'] + 3
sage: q = A.to_graded_conversion()(p); q
bar(U['x']^2*U['y']^2*U['z']^3) + 3*bar(U['x'])
 + 3*bar(U['y']) + 3*bar(1)
sage: q.parent() is A.graded_algebra()
True
```

4.42 Filtered Modules

A *filtered module* over a ring R with a totally ordered indexing set I (typically $I = \mathbf{N}$) is an R-module M equipped with a family $(F_i)_{i\in I}$ of R-submodules satisfying $F_i \subseteq F_j$ for all $i, j \in I$ having $i \leq j$, and $M = \bigcup_{i\in I} F_i$. This family is called a *filtration* of the given module M.

Todo: Implement a notion for decreasing filtrations: where $F_j \subseteq F_i$ when $i \leq j$.

Todo: Implement filtrations for all concrete categories.

Todo: Implement gr as a functor.

class sage.categories.filtered_modules.**FilteredModules**(*base_category*)

Bases: *FilteredModulesCategory*

The category of filtered modules over a given ring R.

A *filtered module* over a ring R with a totally ordered indexing set I (typically $I = \mathbf{N}$) is an R-module M equipped with a family $(F_i)_{i\in I}$ of R-submodules satisfying $F_i \subseteq F_j$ for all $i, j \in I$ having $i \leq j$, and $M = \bigcup_{i\in I} F_i$. This family is called a *filtration* of the given module M.

EXAMPLES:

```
sage: Modules(ZZ).Filtered()
Category of filtered modules over Integer Ring
sage: Modules(ZZ).Filtered().super_categories()
[Category of modules over Integer Ring]
```

REFERENCES:

- Wikipedia article Filtration_(mathematics)

class Connected(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class SubcategoryMethods

Bases: `object`

Connected()

Return the full subcategory of the connected objects of `self`.

A filtered R-module M with filtration $(F_0, F_1, F_2, \ldots)$ (indexed by $\mathbf{N}$) is said to be *connected* if F_0 is isomorphic to R.

EXAMPLES:

```
sage: Modules(ZZ).Filtered().Connected()
Category of filtered connected modules over Integer Ring
sage: Coalgebras(QQ).Filtered().Connected()
Category of filtered connected coalgebras over Rational Field
sage: AlgebrasWithBasis(QQ).Filtered().Connected()
Category of filtered connected algebras with basis over Rational Field
```

`extra_super_categories``()`

Add *`VectorSpaces`* to the super categories of `self` if the base ring is a field.

EXAMPLES:

```
sage: Modules(QQ).Filtered().is_subcategory(VectorSpaces(QQ))
True
sage: Modules(ZZ).Filtered().extra_super_categories()
[]
```

This makes sure that `Modules(QQ).Filtered()` returns an instance of *`FilteredModules`* and not a join category of an instance of this class and of `VectorSpaces(QQ)`:

```
sage: type(Modules(QQ).Filtered())
<class 'sage.categories.vector_spaces.VectorSpaces.Filtered_with_category'>
```

Todo: Get rid of this workaround once there is a more systematic approach for the alias `Modules(QQ)` -> `VectorSpaces(QQ)`. Probably the latter should be a category with axiom, and covariant constructions should play well with axioms.

`class` `sage.categories.filtered_modules.`**`FilteredModulesCategory`**(*base_category*)

Bases: *`RegressiveCovariantConstructionCategory`*, *`Category_over_base_ring`*

EXAMPLES:

```
sage: C = Algebras(QQ).Filtered()
sage: C
Category of filtered algebras over Rational Field
sage: C.base_category()
Category of algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of filtered vector spaces over Rational Field]

sage: AlgebrasWithBasis(QQ).Filtered().base_ring()
Rational Field
sage: HopfAlgebrasWithBasis(QQ).Filtered().base_ring()
Rational Field
```

4.43 Filtered Modules With Basis

A *filtered module with basis* over a ring R means (for the purpose of this code) a filtered R-module M with filtration $(F_i)_{i\in I}$ (typically $I = \mathbf{N}$) endowed with a basis $(b_j)_{j\in J}$ of M and a partition $J = \bigsqcup_{i\in I} J_i$ of the set J (it is allowed that some J_i are empty) such that for every $n \in I$, the subfamily $(b_j)_{j\in U_n}$, where $U_n = \bigcup_{i\leq n} J_i$, is a basis of the R-submodule F_n.

For every $i \in I$, the R-submodule of M spanned by $(b_j)_{j\in J_i}$ is called the *i-th graded component* (aka the *i-th homogeneous component*) of the filtered module with basis M; the elements of this submodule are referred to as *homogeneous elements of degree i*.

See the class documentation `FilteredModulesWithBasis` for further details.

class `sage.categories.filtered_modules_with_basis.`**`FilteredModulesWithBasis`**(*base_category*)

Bases: `FilteredModulesCategory`

The category of filtered modules with a distinguished basis.

A *filtered module with basis* over a ring R means (for the purpose of this code) a filtered R-module M with filtration $(F_i)_{i\in I}$ (typically $I = \mathbf{N}$) endowed with a basis $(b_j)_{j\in J}$ of M and a partition $J = \bigsqcup_{i\in I} J_i$ of the set J (it is allowed that some J_i are empty) such that for every $n \in I$, the subfamily $(b_j)_{j\in U_n}$, where $U_n = \bigcup_{i\leq n} J_i$, is a basis of the R-submodule F_n.

For every $i \in I$, the R-submodule of M spanned by $(b_j)_{j\in J_i}$ is called the *i-th graded component* (aka the *i-th homogeneous component*) of the filtered module with basis M; the elements of this submodule are referred to as *homogeneous elements of degree i*. The R-module M is the direct sum of its i-th graded components over all $i \in I$, and thus becomes a graded R-module with basis. Conversely, any graded R-module with basis canonically becomes a filtered R-module with basis (by defining $F_n = \bigoplus_{i\leq n} G_i$ where G_i is the i-th graded component, and defining J_i as the indexing set of the basis of the i-th graded component). Hence, the notion of a filtered R-module with basis is equivalent to the notion of a graded R-module with basis.

However, the *category* of filtered R-modules with basis is not the category of graded R-modules with basis. Indeed, the *morphisms* of filtered R-modules with basis are defined to be morphisms of R-modules which send each F_n of the domain to the corresponding F_n of the target; in contrast, the morphisms of graded R-modules with basis must preserve each homogeneous component. Also, the notion of a filtered algebra with basis differs from that of a graded algebra with basis.

Note: Currently, to make use of the functionality of this class, an instance of `FilteredModulesWithBasis` should fulfill the contract of a `CombinatorialFreeModule` (most likely by inheriting from it). It should also have the indexing set J encoded as its `_indices` attribute, and `_indices.subset(size=i)` should yield the subset J_i (as an iterable). If the latter conditions are not satisfied, then `basis()` must be overridden.

Note: One should implement a `degree_on_basis` method in the parent class in order to fully utilize the methods of this category. This might become a required abstract method in the future.

EXAMPLES:

```
sage: C = ModulesWithBasis(ZZ).Filtered(); C
Category of filtered modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered modules over Integer Ring,
 Category of modules with basis over Integer Ring]
```

(continues on next page)

(continued from previous page)

```
sage: C is ModulesWithBasis(ZZ).Filtered()
True
```

class ElementMethods

Bases: `object`

degree()

The degree of a nonzero homogeneous element `self` in the filtered module.

Note: This raises an error if the element is not homogeneous. To compute the maximum of the degrees of the homogeneous summands of a (not necessarily homogeneous) element, use `maximal_degree()` instead.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A(Partition((3,2,1)))
sage: y = A(Partition((4,4,1)))
sage: z = A(Partition((2,2,2)))
sage: x.degree()
6
sage: (x + 2*z).degree()
6
sage: (y - x).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

An example in a graded algebra:

```
sage: # needs sage.combinat sage.modules
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.homogeneous_degree()
2
sage: (x^3 + 4*y^2).homogeneous_degree()
6
sage: ((1 + x)^3).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

Let us now test a filtered algebra (but remember that the notion of homogeneity now depends on the choice of a basis):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).homogeneous_degree()
2
sage: (y*x).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

(continues on next page)

(continued from previous page)

```
sage: A.one().homogeneous_degree()
0
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m` in `self`.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(QQ).example()                          #␣
↪needs sage.combinat sage.modules
sage: A.degree_on_basis(Partition((2,1)))                               #␣
↪needs sage.combinat sage.modules
3
sage: A.degree_on_basis(Partition((4,2,1,1,1,1)))                       #␣
↪needs sage.combinat sage.modules
10
```

homogeneous_component(*n*)

Return the homogeneous component of degree `n` of the element `self`.

Let m be an element of a filtered R-module M with basis. Then, m can be uniquely written in the form $m = \sum_{i \in I} m_i$, where each m_i is a homogeneous element of degree i. For $n \in I$, we define the homogeneous component of degree n of the element m to be m_n.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.homogeneous_component(-1)
0
sage: x.homogeneous_component(0)
2*P[]
sage: x.homogeneous_component(1)
2*P[1]
sage: x.homogeneous_component(2)
3*P[2]
sage: x.homogeneous_component(3)
0

sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Graded().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.homogeneous_component(-1)
0
sage: x.homogeneous_component(0)
2*P[]
sage: x.homogeneous_component(1)
2*P[1]
sage: x.homogeneous_component(2)
3*P[2]
sage: x.homogeneous_component(3)
0

sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
```

(continues on next page)

(continued from previous page)

```
sage: G = A.algebra_generators()
sage: g = A.an_element() - 2 * G['x'] * G['y']; g
U['x']^2*U['y']^2*U['z']^3 - 2*U['x']*U['y']
 + 2*U['x'] + 3*U['y'] + 1
sage: g.homogeneous_component(-1)
0
sage: g.homogeneous_component(0)
1
sage: g.homogeneous_component(2)
-2*U['x']*U['y']
sage: g.homogeneous_component(5)
0
sage: g.homogeneous_component(7)
U['x']^2*U['y']^2*U['z']^3
sage: g.homogeneous_component(8)
0
```

homogeneous_degree()

The degree of a nonzero homogeneous element `self` in the filtered module.

Note: This raises an error if the element is not homogeneous. To compute the maximum of the degrees of the homogeneous summands of a (not necessarily homogeneous) element, use *maximal_degree()* instead.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A(Partition((3,2,1)))
sage: y = A(Partition((4,4,1)))
sage: z = A(Partition((2,2,2)))
sage: x.degree()
6
sage: (x + 2*z).degree()
6
sage: (y - x).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

An example in a graded algebra:

```
sage: # needs sage.combinat sage.modules
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.homogeneous_degree()
2
sage: (x^3 + 4*y^2).homogeneous_degree()
6
sage: ((1 + x)^3).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

Let us now test a filtered algebra (but remember that the notion of homogeneity now depends on the choice of a basis):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).homogeneous_degree()
2
sage: (y*x).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
sage: A.one().homogeneous_degree()
0
```

is_homogeneous()

Return whether the element `self` is homogeneous.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A(Partition((3,2,1)))
sage: y = A(Partition((4,4,1)))
sage: z = A(Partition((2,2,2)))
sage: (3*x).is_homogeneous()
True
sage: (x - y).is_homogeneous()
False
sage: (x+2*z).is_homogeneous()
True
```

Here is an example with a graded algebra:

```
sage: # needs sage.combinat sage.modules
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: (3*x).is_homogeneous()
True
sage: (x^3 - y^2).is_homogeneous()
True
sage: ((x + y)^2).is_homogeneous()
False
```

Let us now test a filtered algebra (but remember that the notion of homogeneity now depends on the choice of a basis, or at least on a definition of homogeneous components):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).is_homogeneous()
True
sage: (y*x).is_homogeneous()
False
sage: A.one().is_homogeneous()
True
sage: A.zero().is_homogeneous()
True
sage: (A.one()+x).is_homogeneous()
False
```

maximal_degree()

The maximum of the degrees of the homogeneous components of `self`.

This is also the smallest i such that `self` belongs to F_i. Hence, it does not depend on the basis of the parent of `self`.

See also:

homogeneous_degree()

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A(Partition((3,2,1)))
sage: y = A(Partition((4,4,1)))
sage: z = A(Partition((2,2,2)))
sage: x.maximal_degree()
6
sage: (x + 2*z).maximal_degree()
6
sage: (y - x).maximal_degree()
9
sage: (3*z).maximal_degree()
6
```

Now, we test this on a graded algebra:

```
sage: # needs sage.combinat sage.modules
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.maximal_degree()
2
sage: (x^3 + 4*y^2).maximal_degree()
6
sage: ((1 + x)^3).maximal_degree()
6
```

Let us now test a filtered algebra:

```
sage: # needs sage.combinat sage.modules
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).maximal_degree()
2
sage: (y*x).maximal_degree()
2
sage: A.one().maximal_degree()
0
sage: A.zero().maximal_degree()
Traceback (most recent call last):
...
ValueError: the zero element does not have a well-defined degree
sage: (A.one()+x).maximal_degree()
1
```

truncate(*n*)

Return the sum of the homogeneous components of degree strictly less than `n` of `self`.

See *homogeneous_component()* for the notion of a homogeneous component.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.truncate(0)
0
sage: x.truncate(1)
2*P[]
sage: x.truncate(2)
2*P[] + 2*P[1]
sage: x.truncate(3)
2*P[] + 2*P[1] + 3*P[2]

sage: # needs sage.combinat sage.modules
sage: A = ModulesWithBasis(ZZ).Graded().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.truncate(0)
0
sage: x.truncate(1)
2*P[]
sage: x.truncate(2)
2*P[] + 2*P[1]
sage: x.truncate(3)
2*P[] + 2*P[1] + 3*P[2]

sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: G = A.algebra_generators()
sage: g = A.an_element() - 2 * G['x'] * G['y']; g
U['x']^2*U['y']^2*U['z']^3 - 2*U['x']*U['y']
 + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(-1)
0
sage: g.truncate(0)
0
sage: g.truncate(2)
2*U['x'] + 3*U['y'] + 1
sage: g.truncate(3)
-2*U['x']*U['y'] + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(5)
-2*U['x']*U['y'] + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(7)
-2*U['x']*U['y'] + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(8)
U['x']^2*U['y']^2*U['z']^3 - 2*U['x']*U['y']
 + 2*U['x'] + 3*U['y'] + 1
```

class FiniteDimensional(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

class ParentMethods

Bases: `object`

hilbert_series(*prec=None*)

Return the Hilbert series of `self` as a polynomial.

Let R be a commutative ring (with unit). Let $M = \bigcup_{n=0}^{\infty} M_n$ be a filtered R-module. The *Hilbert* *series* of M is the formal power series

$$H(t) = \sum_{n=0}^{\infty} \ell(M_n/M_{n-1})t^n,$$

where $\ell(N)$ is the *length* of N, which is the longest chain of submodules (over R), and by convention $M_{-1} = \{0\}$. By the assumptions of the category, M_n/M_{n-1} is a free R-module, and so $\ell(M_n/M_{n-1})$ is equal to the rank of M_n/M_{n-1}.

EXAMPLES:

```
sage: OS = hyperplane_arrangements.braid(3).orlik_solomon_algebra(QQ)
sage: OS.hilbert_series()
2*t^2 + 3*t + 1

sage: OS = matroids.Uniform(5, 3).orlik_solomon_algebra(ZZ)
sage: OS.hilbert_series()
t^3 + 3*t^2 + 3*t + 1

sage: OS = matroids.PG(2, 3).orlik_solomon_algebra(ZZ['x','y'])
sage: OS.hilbert_series()
27*t^3 + 39*t^2 + 13*t + 1
```

class ParentMethods

Bases: `object`

basis (*d=None*)

Return the basis for (the `d`-th homogeneous component of) `self`.

INPUT:

- `d` – (optional, default `None`) nonnegative integer or `None`

OUTPUT:

If `d` is `None`, returns the basis of the module. Otherwise, returns the basis of the homogeneous component of degree `d` (i.e., the subfamily of the basis of the whole module which consists only of the basis vectors lying in $F_d \setminus \bigcup_{i<d} F_i$).

The basis is always returned as a family.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions of the integer 4}
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
```

Checking this method on a filtered algebra. Note that this will typically raise a `NotImplementedError` when this feature is not implemented.

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Free abelian monoid indexed by
 {'x', 'y', 'z'} to An example of a filtered algebra with
 basis: the universal enveloping algebra of Lie algebra
 of RR^3 with cross product over Integer Ring(i))_{i in
 Free abelian monoid indexed by {'x', 'y', 'z'}}
```

An example with a graded algebra:

```
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: E.basis()
Lazy family (Term map from Subsets of {0,1} to
 The exterior algebra of rank 2 over Rational Field(i))_{i in
 Subsets of {0,1}}
```

from_graded_conversion()

Return the inverse of the canonical R-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of A (where $A = ``self`$). This inverse is an isomorphism $\operatorname{gr} A \to A$.

This is an isomorphism of R-modules. See the class documentation AssociatedGradedAlgebra.

See also:

to_graded_conversion()

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = Modules(QQ).WithBasis().Filtered().example()
sage: p = -2 * A.an_element(); p
-4*P[] - 4*P[1] - 6*P[2]
sage: q = A.to_graded_conversion()(p); q
-4*Bbar[[]] - 4*Bbar[[1]] - 6*Bbar[[2]]
sage: A.from_graded_conversion()(q) == p
True
sage: q.parent() is A.graded_algebra()
True
```

graded_algebra()

Return the associated graded module to self.

See AssociatedGradedAlgebra for the definition and the properties of this.

If the filtered module self with basis is called A, then this method returns $\operatorname{gr} A$. The method *to_graded_conversion()* returns the canonical R-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of A, and the method *from_graded_conversion()* returns the inverse of this isomorphism. The method *projection()* projects elements of A onto $\operatorname{gr} A$ according to their place in the filtration on A.

Warning: When not overridden, this method returns the default implementation of an associated graded module – namely, `AssociatedGradedAlgebra(self)`, where `AssociatedGradedAlgebra` is `AssociatedGradedAlgebra`. But some instances of *`FilteredModulesWithBasis`* override this method, as the associated graded module often is (isomorphic) to a simpler object (for instance, the associated graded module of a graded module can be identified with the graded module itself). Generic code that uses associated graded modules (such as the code of the *`induced_graded_map()`* method below) should make sure to only communicate with them via the *`to_graded_conversion()`*, *`from_graded_conversion()`* and *`projection()`* methods (in particular, do not expect there to be a conversion from `self` to `self.graded_algebra()`; this currently does not work for Clifford algebras). Similarly, when overriding *`graded_algebra()`*, make sure to accordingly redefine these three methods, unless their definitions below still apply to your case (this will happen whenever the basis of your *`graded_algebra()`* has the same indexing set as `self`, and the partition of this indexing set according to degree is the same as for `self`).

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: A.graded_algebra()
Graded Module of An example of a filtered module with basis:
 the free module on partitions over Integer Ring
```

`hilbert_series`(*prec=None*)

Return the Hilbert series of `self`.

Let R be a commutative ring (with unit). Let $M = \bigcup_{n=0}^{\infty} M_n$ be a filtered R-module. The *Hilbert series* of M is the formal power series

$$H(t) = \sum_{n=0}^{\infty} \ell(M_n/M_{n-1})t^n,$$

where $\ell(N)$ is the *length* of N, which is the longest chain of submodules (over R), and by convention $M_{-1} = \{0\}$. By the assumptions of the category, M_n/M_{n-1} is a free R-module, and so $\ell(M_n/M_{n-1})$ is equal to the rank of M_n/M_{n-1}.

INPUT:

- `prec` – (default: ∞) the precision

OUTPUT:

If the precision is finite, then this returns an element in the `PowerSeriesRing` over `ZZ`. Otherwise it returns an element in the `LazyPowerSeriesRing` over `ZZ`.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(ZZ).example()
sage: A.hilbert_series()
1 + t + 2*t^2 + 3*t^3 + 5*t^4 + 7*t^5 + 11*t^6 + O(t^7)
sage: A.hilbert_series(10)
1 + t + 2*t^2 + 3*t^3 + 5*t^4 + 7*t^5 + 11*t^6 + 15*t^7 + 22*t^8 + 30*t^9␣
↪+ O(t^10)
```

`homogeneous_component`(*d*)

Return the `d`-th homogeneous component of `self`.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(ZZ).example()
sage: A.homogeneous_component(4)
Degree 4 homogeneous component of An example of a graded module
 with basis: the free module on partitions over Integer Ring
```

homogeneous_component_basis(*d*)

Return a basis for the `d`-th homogeneous component of `self`.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(ZZ).example()
sage: A.homogeneous_component_basis(4)
Lazy family (Term map
 from Partitions
   to An example of a graded module with basis: the free module
      on partitions over Integer Ring(i))_{i in Partitions of the integer␣
↪4}

sage: # needs sage.modules
sage: cat = GradedModulesWithBasis(ZZ)
sage: C = CombinatorialFreeModule(ZZ, ['a', 'b'], category=cat)
sage: C.degree_on_basis = lambda x: 1 if x == 'a' else 2
sage: C.homogeneous_component_basis(1)
Finite family {'a': B['a']}
sage: C.homogeneous_component_basis(2)
Finite family {'b': B['b']}
```

induced_graded_map(*other*, *f*)

Return the graded linear map between the associated graded modules of `self` and `other` canonically induced by the filtration-preserving map `f : self -> other`.

Let A and B be two filtered modules with basis, and let $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ be their filtrations. Let $f : A \to B$ be a linear map which preserves the filtration (i.e., satisfies $f(F_i) \subseteq G_i$ for all $i \in I$). Then, there is a canonically defined graded linear map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ which satisfies

$$(\operatorname{gr} f)(p_i(a)) = p_i(f(a)) \qquad \text{for all } i \in I \text{ and } a \in F_i,$$

where the p_i on the left hand side is the canonical projection from F_i onto the i-th graded component of $\operatorname{gr} A$, while the p_i on the right hand side is the canonical projection from G_i onto the i-th graded component of $\operatorname{gr} B$.

INPUT:

- `other` – a filtered algebra with basis
- `f` – a filtration-preserving linear map from `self` to `other` (can be given as a morphism or as a function)

OUTPUT:

The graded linear map $\operatorname{gr} f$.

EXAMPLES:

Example 1.

We start with the free **Q**-module with basis the set of all partitions:

```
sage: A = Modules(QQ).WithBasis().Filtered().example(); A                  #␣
↪needs sage.combinat sage.modules
An example of a filtered module with basis: the free module
```

(continues on next page)

(continued from previous page)

```
 on partitions over Rational Field
sage: M = A.indices(); M                                                     #␣
↪needs sage.combinat sage.modules
Partitions
sage: p1, p2, p21, p321 = [A.basis()[Partition(i)]                           #␣
↪needs sage.combinat sage.modules
....:                      for i in [[1], [2], [2,1], [3,2,1]]]
```

Let us define a map from A to itself which acts on the basis by sending every partition λ to the sum of the conjugates of all partitions μ for which λ/μ is a horizontal strip:

```
sage: # needs sage.combinat sage.modules
sage: def map_on_basis(lam):
....:     def mus(k):
....:         return lam.remove_horizontal_border_strip(k)
....:     return A.sum_of_monomials([Partition(mu).conjugate()
....:                                for k in range(sum(lam) + 1)
....:                                for mu in mus(k)])
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=A)
sage: f(p1)
P[] + P[1]
sage: f(p2)
P[] + P[1] + P[1, 1]
sage: f(p21)
P[1] + P[1, 1] + P[2] + P[2, 1]
sage: f(p21 - p1)
-P[] + P[1, 1] + P[2] + P[2, 1]
sage: f(p321)
P[2, 1] + P[2, 1, 1] + P[2, 2] + P[2, 2, 1]
 + P[3, 1] + P[3, 1, 1] + P[3, 2] + P[3, 2, 1]
```

We now compute gr f

```
sage: # needs sage.combinat sage.modules
sage: grA = A.graded_algebra(); grA
Graded Module of An example of a filtered module with basis:
 the free module on partitions over Rational Field
sage: pp1, pp2, pp21, pp321 = [A.to_graded_conversion()(i)
....:                          for i in [p1, p2, p21, p321]]
sage: pp2 + 4 * pp21
Bbar[[2]] + 4*Bbar[[2, 1]]
sage: grf = A.induced_graded_map(A, f); grf
Generic endomorphism of Graded Module of
 An example of a filtered module with basis:
  the free module on partitions over Rational Field
sage: grf(pp1)
Bbar[[1]]
sage: grf(pp2 + 4 * pp21)
Bbar[[1, 1]] + 4*Bbar[[2, 1]]
```

Example 2.

We shall now construct gr f for a different map f out of the same A; the new map f will lead into a graded algebra already, namely into the algebra of symmetric functions:

```
sage: # needs sage.combinat sage.modules
sage: h = SymmetricFunctions(QQ).h()
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     def mus(k):
....:         return lam.remove_horizontal_border_strip(k)
....:     return h.sum_of_monomials([Partition(mu).conjugate()
....:                                for k in range(sum(lam) + 1)
....:                                for mu in mus(k)])
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=h)  # redefining f
sage: f(p1)
h[] + h[1]
sage: f(p2)
h[] + h[1] + h[1, 1]
sage: f(A.zero())
0
sage: f(p2 - 3*p1)
-2*h[] - 2*h[1] + h[1, 1]
```

The algebra `h` of symmetric functions in the h-basis is already graded, so its associated graded algebra is implemented as itself:

```
sage: # needs sage.combinat sage.modules
sage: grh = h.graded_algebra(); grh is h
True
sage: grf = A.induced_graded_map(h, f); grf
Generic morphism:
  From: Graded Module of An example of a filtered
   module with basis: the free module on partitions
   over Rational Field
  To:   Symmetric Functions over Rational Field
   in the homogeneous basis
sage: grf(pp1)
h[1]
sage: grf(pp2)
h[1, 1]
sage: grf(pp321)
h[3, 2, 1]
sage: grf(pp2 - 3*pp1)
-3*h[1] + h[1, 1]
sage: grf(pp21)
h[2, 1]
sage: grf(grA.zero())
0
```

Example 3.

After having had a graded module as the codomain, let us try to have one as the domain instead. Our new `f` will go from `h` to `A`:

```
sage: # needs sage.combinat sage.modules
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     def mus(k):
....:         return lam.remove_horizontal_border_strip(k)
....:     return A.sum_of_monomials([Partition(mu).conjugate()
....:                                for k in range(sum(lam) + 1)
....:                                for mu in mus(k)])
sage: f = h.module_morphism(on_basis=map_on_basis,
```

(continues on next page)

(continued from previous page)

```
....:                            codomain=A)   # redefining f
sage: f(h[1])
P[] + P[1]
sage: f(h[2])
P[] + P[1] + P[1, 1]
sage: f(h[1, 1])
P[1] + P[2]
sage: f(h[2, 2])
P[1, 1] + P[2, 1] + P[2, 2]
sage: f(h[3, 2, 1])
P[2, 1] + P[2, 1, 1] + P[2, 2] + P[2, 2, 1]
 + P[3, 1] + P[3, 1, 1] + P[3, 2] + P[3, 2, 1]
sage: f(h.one())
P[]
sage: grf = h.induced_graded_map(A, f); grf
Generic morphism:
  From: Symmetric Functions over Rational Field
   in the homogeneous basis
  To:   Graded Module of An example of a filtered
   module with basis: the free module on partitions
   over Rational Field
sage: grf(h[1])
Bbar[[1]]
sage: grf(h[2])
Bbar[[1, 1]]
sage: grf(h[1, 1])
Bbar[[2]]
sage: grf(h[2, 2])
Bbar[[2, 2]]
sage: grf(h[3, 2, 1])
Bbar[[3, 2, 1]]
sage: grf(h.one())
Bbar[[]]
```

Example 4.

The construct gr f also makes sense when f is a filtration-preserving map between graded modules.

```
sage: # needs sage.combinat sage.modules
sage: def map_on_basis(lam):   # redefining map_on_basis
....:     def mus(k):
....:         return lam.remove_horizontal_border_strip(k)
....:     return h.sum_of_monomials([Partition(mu).conjugate()
....:                                for k in range(sum(lam) + 1)
....:                                for mu in mus(k)])
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                            codomain=h)   # redefining f
sage: f(h[1])
h[] + h[1]
sage: f(h[2])
h[] + h[1] + h[1, 1]
sage: f(h[1, 1])
h[1] + h[2]
sage: f(h[2, 1])
h[1] + h[1, 1] + h[2] + h[2, 1]
sage: f(h.one())
h[]
```

(continues on next page)

(continued from previous page)

```
sage: grf = h.induced_graded_map(h, f); grf
Generic endomorphism of
 Symmetric Functions over Rational Field in the homogeneous basis
sage: grf(h[1])
h[1]
sage: grf(h[2])
h[1, 1]
sage: grf(h[1, 1])
h[2]
sage: grf(h[2, 1])
h[2, 1]
sage: grf(h.one())
h[]
```

projection(*i*)

Return the i-th projection $p_i : F_i \to G_i$ (in the notations of the class documentation `AssociatedGradedAlgebra`, where $A = ``self`$).

This method actually does not return the map p_i itself, but an extension of p_i to the whole R-module A. This extension is the composition of the R-module isomorphism $A \to \operatorname{gr} A$ with the canonical projection of the graded R-module $\operatorname{gr} A$ onto its i-th graded component G_i. The codomain of this map is $\operatorname{gr} A$, although its actual image is G_i. The map p_i is obtained from this map by restricting its domain to F_i and its image to G_i.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = Modules(ZZ).WithBasis().Filtered().example()
sage: p = -2 * A.an_element(); p
-4*P[] - 4*P[1] - 6*P[2]
sage: q = A.projection(2)(p); q
-6*Bbar[[2]]
sage: q.parent() is A.graded_algebra()
True
sage: A.projection(3)(p)
0
```

to_graded_conversion()

Return the canonical R-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of A (where $A = ``self`$).

This is an isomorphism of R-modules. See the class documentation `AssociatedGradedAlgebra`.

See also:

from_graded_conversion()

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = Modules(QQ).WithBasis().Filtered().example()
sage: p = -2 * A.an_element(); p
-4*P[] - 4*P[1] - 6*P[2]
sage: q = A.to_graded_conversion()(p); q
-4*Bbar[[]] - 4*Bbar[[1]] - 6*Bbar[[2]]
sage: q.parent() is A.graded_algebra()
True
```

class Subobjects(*category*, **args*)

Bases: `SubobjectsCategory`

class ElementMethods

Bases: `object`

degree()

Return the degree of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: S = E.submodule([x + y, x*y - y*z, y])
sage: B = S.basis()
sage: [B[0].lift(), B[1].lift(), B[2].lift()]
[x, y, x*y - y*z]
sage: B[0].degree()
1
sage: B[1].degree()
1
sage: (B[0] + 3*B[1]).degree()
1
```

The degree of inhomogeneous elements is not defined (following the behavior of the exterior algebra):

```
sage: (B[0] + B[2]).degree()                                          #␣
↪needs sage.modules
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

We can still get the maximal degree:

```
sage: (B[0] + B[2]).maximal_degree()                                  #␣
↪needs sage.modules
2
```

maximal_degree()

The maximum of the degrees of the homogeneous components of `self`.

This is also the smallest i such that `self` belongs to F_i. Hence, it does not depend on the basis of the parent of `self`.

See also:

`homogeneous_degree()`

EXAMPLES:

```
sage: # needs sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: F = E.submodule([x + 1, x*y - 1])
sage: B = F.basis()
sage: [B[0].lift(), B[1].lift()]
[-x*y + 1, x*y + x]
sage: B[0].maximal_degree()
2
```

(continues on next page)

(continued from previous page)

```
sage: B[1].maximal_degree()
2
```

class ParentMethods

Bases: `object`

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m` in `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: S = E.submodule([x + y, x*y - y*z, y])
sage: B = S.basis()
sage: [B[0].lift(), B[1].lift(), B[2].lift()]
[x, y, x*y - y*z]
sage: S.degree_on_basis(0)
1
sage: S.degree_on_basis(1)
1
sage: S.degree_on_basis(2)
2
```

4.44 Finite Complex Reflection Groups

class sage.categories.finite_complex_reflection_groups.**FiniteComplexReflectionGroups**(*base_ca e- gory*)

Bases: *CategoryWithAxiom*

The category of finite complex reflection groups.

See `ComplexReflectionGroups` for the definition of complex reflection group. In the finite case, most of the information about the group can be recovered from its *degrees* and *codegrees*, and to a lesser extent to the explicit realization as subgroup of $GL(V)$. Hence the most important optional methods to implement are:

- `ComplexReflectionGroups.Finite.ParentMethods.degrees()`,
- `ComplexReflectionGroups.Finite.ParentMethods.codegrees()`,
- `ComplexReflectionGroups.Finite.ElementMethods.to_matrix()`.

Finite complex reflection groups are completely classified. In particular, if the group is irreducible, then it's uniquely determined by its degrees and codegrees and whether it's reflection representation is *primitive* or not (see [LT2009] Chapter 2.1 for the definition of primitive).

See also:

Wikipedia article Complex_reflection_groups

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite()
```

(continues on next page)

(continued from previous page)

```
Category of finite complex reflection groups
sage: ComplexReflectionGroups().Finite().super_categories()
[Category of complex reflection groups,
 Category of finite groups,
 Category of finite finitely generated semigroups]
```

An example of a finite reflection group:

```
sage: W = ComplexReflectionGroups().Finite().example(); W       # optional - gap3
Reducible real reflection group of rank 4 and type A2 x B2

sage: W.reflections()                                           # optional - gap3
Finite family {1: (1,8)(2,5)(9,12), 2: (1,5)(2,9)(8,12),
               3: (3,10)(4,7)(11,14), 4: (3,6)(4,11)(10,13),
               5: (1,9)(2,8)(5,12), 6: (4,14)(6,13)(7,11),
               7: (3,13)(6,10)(7,14)}
```

W is in the category of complex reflection groups:

```
sage: W in ComplexReflectionGroups().Finite()                   # optional - gap3
True
```

class ElementMethods

Bases: object

character_value()

Return the value at self of the character of the reflection representation given by *to_matrix()*.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3); W                                   #␣
↪needs sage.combinat
1-colored permutations of size 3
sage: [t.character_value() for t in W]                                  #␣
↪needs sage.combinat sage.groups
[3, 1, 1, 0, 0, 1]
```

Note that this could be a different (faithful) representation than that given by the corresponding root system:

```
sage: W = ReflectionGroup((1,1,3)); W      # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: [t.character_value() for t in W]     # optional - gap3
[2, 0, 0, -1, -1, 0]

sage: W = ColoredPermutations(2,2); W                                   #␣
↪needs sage.combinat
2-colored permutations of size 2
sage: [t.character_value() for t in W]                                  #␣
↪needs sage.combinat sage.groups
[2, 0, 0, -2, 0, 0, 0, 0]

sage: W = ColoredPermutations(3,1); W                                   #␣
↪needs sage.combinat
3-colored permutations of size 1
sage: [t.character_value() for t in W]                                  #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.groups
[1, zeta3, -zeta3 - 1]
```

reflection_length(*in_unitary_group=False*)

Return the reflection length of `self`.

This is the minimal numbers of reflections needed to obtain `self`.

INPUT:

- `in_unitary_group` – (default: `False`) if `True`, the reflection length is computed in the unitary group which is the dimension of the move space of `self`

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                  # optional - gap3
sage: sorted([t.reflection_length() for t in W])   # optional - gap3
[0, 1, 1, 1, 2, 2]

sage: W = ReflectionGroup((2,1,2))                  # optional - gap3
sage: sorted([t.reflection_length() for t in W])   # optional - gap3
[0, 1, 1, 1, 1, 2, 2, 2]

sage: W = ReflectionGroup((2,2,2))                  # optional - gap3
sage: sorted([t.reflection_length() for t in W])   # optional - gap3
[0, 1, 1, 2]

sage: W = ReflectionGroup((3,1,2))                  # optional - gap3
sage: sorted([t.reflection_length() for t in W])   # optional - gap3
[0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
```

to_matrix()

Return the matrix presentation of `self` acting on a vector space V.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))          # optional - gap3
sage: [t.to_matrix() for t in W]            # optional - gap3
[
[1 0]   [ 1  1]   [-1  0]   [-1 -1]   [ 0  1]   [ 0 -1]
[0 1],  [ 0 -1],  [ 1  1],  [ 1  0],  [-1 -1],  [-1  0]
]

sage: W = ColoredPermutations(1,3)                                      #␣
↪needs sage.combinat
sage: [t.to_matrix() for t in W]                                        #␣
↪needs sage.combinat sage.groups
[
[1 0 0]   [1 0 0]   [0 1 0]   [0 0 1]   [0 1 0]   [0 0 1]
[0 1 0]   [0 0 1]   [1 0 0]   [1 0 0]   [0 0 1]   [0 1 0]
[0 0 1],  [0 1 0],  [0 0 1],  [0 1 0],  [1 0 0],  [1 0 0]
]
```

A different representation is given by the colored permutations:

```
sage: W = ColoredPermutations(3, 1)                                     #␣
↪needs sage.combinat
sage: [t.to_matrix() for t in W]                                        #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.groups
[[1], [zeta3], [-zeta3 - 1]]
```

class Irreducible(*base_category*)

Bases: `CategoryWithAxiom`

class ParentMethods

Bases: `object`

absolute_order_ideal(*gens=None*, *in_unitary_group=True*, *return_lengths=False*)

Return all elements in `self` below given elements in the absolute order of `self`.

This order is defined by

$$\omega \leq_R \tau \Leftrightarrow \ell_R(\omega) + \ell_R(\omega^{-1}\tau) = \ell_R(\tau),$$

where ℓ_R denotes the reflection length.

This is, if `in_unitary_group` is `False`, then

$$\ell_R(w) = \min\{\ell : w = r_1 \cdots r_\ell, r_i \in R\},$$

and otherwise

$$\ell_R(w) = \dim \operatorname{im}(w - 1).$$

Note: If `gens` are not given, `self` is assumed to be well-generated.

INPUT:

- `gens` – (default: `None`) if one or more elements are given, the order ideal in the absolute order generated by `gens` is returned. Otherwise, the standard Coxeter element is used as unique maximal element.
- `in_unitary_group` (default:`True`) determines the length function used to compute the order. For real groups, both possible orders coincide, and for complex non-real groups, the order in the unitary group is much faster to compute.
- `return_lengths` (default:`False`) whether or not to also return the lengths of the elements.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))      # optional - gap3

sage: sorted(w.reduced_word()           # optional - gap3
....:         for w in W.absolute_order_ideal())
[[], [1], [1, 2], [1, 2, 1], [2]]

sage: sorted(w.reduced_word()           # optional - gap3
....:         for w in W.absolute_order_ideal(W.from_reduced_word([2,
↪1])))
[[], [1], [1, 2, 1], [2], [2, 1]]

sage: sorted(w.reduced_word()           # optional - gap3
....:         for w in W.absolute_order_ideal(W.from_reduced_word([2])))
[[], [2]]

sage: W = CoxeterGroup(['A', 3])                                    #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.groups
sage: len(list(W.absolute_order_ideal()))                          #␣
↪needs sage.combinat sage.groups
14

sage: W = CoxeterGroup(['A', 2])                                   #␣
↪needs sage.combinat sage.groups
sage: for (w, l) in W.absolute_order_ideal(return_lengths=True):   #␣
↪needs sage.combinat sage.groups
....:     print(w.reduced_word(), l)
[1, 2] 2
[1, 2, 1] 1
[2] 1
[1] 1
[] 0
```

absolute_poset (*in_unitary_group=False*)

Return the poset induced by the absolute order of `self` as a finite lattice.

INPUT:

- `in_unitary_group` – (default: `False`) if `False`, the relation is given by `\sigma \leq \tau` if $l_R(\sigma) + l_R(\sigma^{-1}\tau) = l_R(\tau)$ If `True`, the relation is given by $\sigma \leq \tau$ if $\dim(\mathrm{Fix}(\sigma)) + \dim(\mathrm{Fix}(\sigma^{-1}\tau)) = \dim(\mathrm{Fix}(\tau))$

See also:

noncrossing_partition_lattice()

EXAMPLES:

```
sage: P = ReflectionGroup((1,1,3)).absolute_poset(); P      # optional␣
↪- gap3
Finite poset containing 6 elements

sage: sorted(w.reduced_word() for w in P)                   # optional␣
↪- gap3
[[], [1], [1, 2], [1, 2, 1], [2], [2, 1]]

sage: W = ReflectionGroup(4); W                             # optional␣
↪- gap3
Irreducible complex reflection group of rank 2 and type ST4
sage: W.absolute_poset()                                    # optional␣
↪- gap3
Finite poset containing 24 elements
```

coxeter_number ()

Return the Coxeter number of an irreducible reflection group.

This is defined as $\frac{N+N^*}{n}$ where N is the number of reflections, N^* is the number of reflection hyperplanes, and n is the rank of `self`.

EXAMPLES:

```
sage: W = ReflectionGroup(31)          # optional - gap3
sage: W.coxeter_number()               # optional - gap3
30
```

generalized_noncrossing_partitions (*m, c=None, positive=False*)

Return the set of all chains of length m in the noncrossing partition lattice of self, see *noncrossing_partition_lattice()*.

Note: self is assumed to be well-generated.

INPUT:

- c – (default: None) if an element c in self is given, it is used as the maximal element in the interval
- positive – (default: False) if True, only those generalized noncrossing partitions of full support are returned

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                          # optional␣
↪- gap3

sage: chains = W.generalized_noncrossing_partitions(2)      # optional␣
↪- gap3
sage: sorted([w.reduced_word() for w in chain]              # optional␣
↪- gap3
....:        for chain in chains)
[[[], [], [1, 2]],
 [[], [1], [2]],
 [[], [1, 2], []],
 [[], [1, 2, 1], [1]],
 [[], [2], [1, 2, 1]],
 [[1], [], [2]],
 [[1], [2], []],
 [[1, 2], [], []],
 [[1, 2, 1], [], [1]],
 [[1, 2, 1], [1], []],
 [[2], [], [1, 2, 1]],
 [[2], [1, 2, 1], []]]

sage: chains = W.generalized_noncrossing_partitions(2,      # optional␣
↪- gap3
....:                    positive=True)
sage: sorted([w.reduced_word() for w in chain]              # optional␣
↪- gap3
....:        for chain in chains)
[[[], [1, 2], []],
 [[], [1, 2, 1], [1]],
 [[1], [2], []],
 [[1, 2], [], []],
 [[1, 2, 1], [], [1]],
 [[1, 2, 1], [1], []],
 [[2], [1, 2, 1], []]]
```

noncrossing_partition_lattice(*c=None*, *L=None*, *in_unitary_group=True*)

Return the interval $[1, c]$ in the absolute order of self as a finite lattice.

See also:

absolute_order_ideal()

INPUT:

- c – (default: None) if an element c in self is given, it is used as the maximal element in the interval
- L – (default: None) if a subset L (must be hashable!) of self is given, it is used as the underlying set (only cover relations are checked).
- in_unitary_group – (default: False) if False, the relation is given by $\sigma \leq \tau$ if $l_R(\sigma) + l_R(\sigma^{-1}\tau) = l_R(\tau)$; if True, the relation is given by $\sigma \leq \tau$ if $\dim(\mathrm{Fix}(\sigma)) + \dim(\mathrm{Fix}(\sigma^{-1}\tau)) = \dim(\mathrm{Fix}(\tau))$

Note: If L is given, the parameter c is ignored.

EXAMPLES:

```
sage: W = SymmetricGroup(4)                                                       #␣
↪needs sage.combinat sage.groups
sage: W.noncrossing_partition_lattice()                                           #␣
↪needs sage.combinat sage.groups
Finite lattice containing 14 elements

sage: W = WeylGroup(['G', 2])                                                     #␣
↪needs sage.combinat sage.groups
sage: W.noncrossing_partition_lattice()                                           #␣
↪needs sage.combinat sage.groups
Finite lattice containing 8 elements

sage: W = ReflectionGroup((1,1,3))      # optional - gap3

sage: sorted(w.reduced_word()           # optional - gap3
....:        for w in W.noncrossing_partition_lattice())
[[], [1], [1, 2], [1, 2, 1], [2]]

sage: c21 = W.from_reduced_word([2,1]) # optional - gap3
sage: sorted(w.reduced_word()           # optional - gap3
....:        for w in W.noncrossing_partition_lattice(c21))
[[], [1], [1, 2, 1], [2], [2, 1]]

sage: c2 = W.from_reduced_word([2])     # optional - gap3
sage: sorted(w.reduced_word()           # optional - gap3
....:        for w in W.noncrossing_partition_lattice(c2))
[[], [2]]
```

example()

Return an example of an irreducible complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: C = ComplexReflectionGroups().Finite().Irreducible()
sage: C.example()                             # optional - gap3
Irreducible complex reflection group of rank 3 and type G(4,2,3)
```

class ParentMethods

Bases: object

base_change_matrix()

Return the base change from the standard basis of the vector space of self to the basis given by the independent roots of self.

Todo: For non-well-generated groups there is a conflict with construction of the matrix for an element.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))          # optional - gap3
sage: W.base_change_matrix()                # optional - gap3
[1 0]
[0 1]

sage: W = ReflectionGroup(23)               # optional - gap3
sage: W.base_change_matrix()                # optional - gap3
[1 0 0]
[0 1 0]
[0 0 1]

sage: W = ReflectionGroup((3,1,2))          # optional - gap3
sage: W.base_change_matrix()                # optional - gap3
[1 0]
[1 1]

sage: W = ReflectionGroup((4,2,2))          # optional - gap3
sage: W.base_change_matrix()                # optional - gap3
[   1     0]
[E(4)     1]
```

cardinality()

Return the cardinality of `self`.

It is given by the product of the degrees of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = ColoredPermutations(1,3)
sage: W.cardinality()
6
sage: W = ColoredPermutations(2,3)
sage: W.cardinality()
48
sage: W = ColoredPermutations(4,3)
sage: W.cardinality()
384

sage: # optional - gap3, needs sage.combinat sage.groups
sage: W = ReflectionGroup((4,2,3))
sage: W.cardinality()
192
```

codegrees()

Return the codegrees of `self`.

OUTPUT: a tuple of Sage integers

EXAMPLES:

```
sage: W = ColoredPermutations(1,4)                                           #␣
↪needs sage.combinat
```

(continues on next page)

(continued from previous page)

```
sage: W.codegrees()                                                         #␣
↪needs sage.combinat
(2, 1, 0)

sage: W = ColoredPermutations(3,3)                                          #␣
↪needs sage.combinat
sage: W.codegrees()                                                         #␣
↪needs sage.combinat
(6, 3, 0)

sage: W = ReflectionGroup(31)                    # optional - gap3
sage: W.codegrees()                              # optional - gap3
(28, 16, 12, 0)
```

degrees()

Return the degrees of `self`.

OUTPUT: a tuple of Sage integers

EXAMPLES:

```
sage: W = ColoredPermutations(1,4)                                          #␣
↪needs sage.combinat
sage: W.degrees()                                                           #␣
↪needs sage.combinat
(2, 3, 4)

sage: W = ColoredPermutations(3,3)                                          #␣
↪needs sage.combinat
sage: W.degrees()                                                           #␣
↪needs sage.combinat
(3, 6, 9)

sage: W = ReflectionGroup(31)                    # optional - gap3
sage: W.degrees()                                # optional - gap3
(8, 12, 20, 24)
```

is_real()

Return whether `self` is real.

A complex reflection group is *real* if it is isomorphic to a reflection group in $GL(V)$ over a real vector space V. Equivalently its character table has real entries.

This implementation uses the following statement: an irreducible complex reflection group is real if and only if 2 is a degree of `self` with multiplicity one. Hence, in general we just need to compare the number of occurrences of 2 as degree of `self` and the number of irreducible components.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)                                          #␣
↪needs sage.combinat
sage: W.is_real()                                                           #␣
↪needs sage.combinat
True

sage: W = ColoredPermutations(4,3)                                          #␣
↪needs sage.combinat
```

(continues on next page)

(continued from previous page)

```
sage: W.is_real()                                                        #␣
↪needs sage.combinat sage.groups
False
```

Todo: Add an example of non real finite complex reflection group that is generated by order 2 reflections.

is_well_generated()

Return whether `self` is well-generated.

A finite complex reflection group is *well generated* if the number of its simple reflections coincides with its rank.

See also:

`ComplexReflectionGroups.Finite.WellGenerated()`

Note:

- All finite real reflection groups are well generated.
- The complex reflection groups of type $G(r, 1, n)$ and of type $G(r, r, n)$ are well generated.
- The complex reflection groups of type $G(r, p, n)$ with $1 < p < r$ are *not* well generated.
- The direct product of two well generated finite complex reflection group is still well generated.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)                                       #␣
↪needs sage.combinat
sage: W.is_well_generated()                                              #␣
↪needs sage.combinat
True

sage: W = ColoredPermutations(4,3)                                       #␣
↪needs sage.combinat
sage: W.is_well_generated()                                              #␣
↪needs sage.combinat
True

sage: # optional - gap3, needs sage.combinat sage.groups
sage: W = ReflectionGroup((4,2,3))
sage: W.is_well_generated()
False
sage: W = ReflectionGroup((4,4,3))
sage: W.is_well_generated()
True
```

milnor_fiber_poset()

Return the Milnor fiber poset of `self`.

The *Milnor fiber poset* of a finite complex reflection group W is defined as the poset of (right) standard cosets gW_J, where J is a subset of the index set I of W, ordered by reverse inclusion. This is conjecturally a meet semilattice if and only if W is well-generated.

EXAMPLES:

```
sage: W = ColoredPermutations(3, 2)
sage: P = W.milnor_fiber_poset()
sage: P
Finite meet-semilattice containing 34 elements
sage: R.<x> = ZZ[]
sage: sum(x**P.rank(elt) for elt in P)
18*x^2 + 15*x + 1

sage: # optional - gap3
sage: W = ReflectionGroup(4)
sage: P = W.milnor_fiber_poset(); P
Finite meet-semilattice containing 41 elements
sage: sum(x**P.rank(elt) for elt in P)
24*x^2 + 16*x + 1

sage: # optional - gap3
sage: W = ReflectionGroup([4,2,2])
sage: W.is_well_generated()
False
sage: P = W.milnor_fiber_poset(); P
Finite poset containing 47 elements
sage: sum(x**P.rank(elt) for elt in P)
16*x^3 + 24*x^2 + 6*x + 1
sage: P.is_meet_semilattice()
False
```

number_of_reflection_hyperplanes()

Return the number of reflection hyperplanes of `self`.

This is also the number of distinguished reflections. For real groups, this coincides with the number of reflections.

This implementation uses that it is given by the sum of the codegrees of `self` plus its rank.

See also:

number_of_reflections()

EXAMPLES:

```
sage: # needs sage.combinat
sage: W = ColoredPermutations(1,3)
sage: W.number_of_reflection_hyperplanes()
3
sage: W = ColoredPermutations(2,3)
sage: W.number_of_reflection_hyperplanes()
9
sage: W = ColoredPermutations(4,3)
sage: W.number_of_reflection_hyperplanes()
15
sage: W = ReflectionGroup((4,2,3))          # optional - gap3
sage: W.number_of_reflection_hyperplanes()  # optional - gap3
15
```

number_of_reflections()

Return the number of reflections of `self`.

For real groups, this coincides with the number of reflection hyperplanes.

This implementation uses that it is given by the sum of the degrees of `self` minus its rank.

See also:

number_of_reflection_hyperplanes()

EXAMPLES:

```
sage: [SymmetricGroup(i).number_of_reflections()                            #␣
↪needs sage.groups
....:  for i in range(int(8))]
[0, 0, 1, 3, 6, 10, 15, 21]

sage: # needs sage.combinat sage.groups
sage: W = ColoredPermutations(1,3)
sage: W.number_of_reflections()
3
sage: W = ColoredPermutations(2,3)
sage: W.number_of_reflections()
9
sage: W = ColoredPermutations(4,3)
sage: W.number_of_reflections()
21
sage: W = ReflectionGroup((4,2,3))       # optional - gap3
sage: W.number_of_reflections()          # optional - gap3
15
```

rank()

Return the rank of `self`.

The rank of `self` is the dimension of the smallest faithfull reflection representation of `self`.

This default implementation uses that the rank is the number of *degrees()*.

See also:

`ComplexReflectionGroups.rank()`

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: W = ColoredPermutations(1,3)
sage: W.rank()
2
sage: W = ColoredPermutations(2,3)
sage: W.rank()
3
sage: W = ColoredPermutations(4,3)
sage: W.rank()
3

sage: # optional - gap3, needs sage.combinat sage.groups
sage: W = ReflectionGroup((4,2,3))
sage: W.rank()
3
```

class SubcategoryMethods

Bases: `object`

WellGenerated()

Return the full subcategory of well-generated objects of `self`.

A finite complex generated group is *well generated* if it is isomorphic to a subgroup of the general linear group GL_n generated by n reflections.

See also:

`ComplexReflectionGroups.Finite.ParentMethods.is_well_generated()`

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: C = ComplexReflectionGroups().Finite().WellGenerated(); C
Category of well generated finite complex reflection groups
```

Here is an example of a finite well-generated complex reflection group:

```
sage: W = C.example(); W                        # optional - gap3
Reducible complex reflection group of rank 4 and type A2 x G(3,1,2)
```

All finite Coxeter groups are well generated:

```
sage: CoxeterGroups().Finite().is_subcategory(C)
True
sage: SymmetricGroup(3) in C                                           #␣
↪needs sage.groups
True
```

Note: The category of well generated finite complex reflection groups is currently implemented as an axiom. See discussion on github issue #11187. This may be a bit of overkill. Still it's nice to have a full subcategory.

---

class WellGenerated(*base_category*)

Bases: `CategoryWithAxiom`

class Irreducible(*base_category*)

Bases: `CategoryWithAxiom`

The category of finite irreducible well-generated finite complex reflection groups.

class ParentMethods

Bases: `object`

catalan_number(*positive=False*, *polynomial=False*)

Return the Catalan number associated to `self`.

It is defined by

$$\prod_{i=1}^{n} \frac{d_i + h}{d_i},$$

where $d_1, \ldots, d_n$ are the degrees and where h is the Coxeter number. See [Ar2006] for further information.

INPUT:

- `positive` – optional boolean (default `False`) if `True`, return instead the positive Catalan number
- `polynomial` – optional boolean (default `False`) if `True`, return instead the q-analogue as a polynomial in q

Note:

- For the symmetric group S_n, it reduces to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.
- The Catalan numbers for $G(r,1,n)$ all coincide for $r > 1$.

EXAMPLES:

```
sage: [ColoredPermutations(1,n).catalan_number()                    #␣
↪needs sage.combinat
....:  for n in [3,4,5]]
[5, 14, 42]

sage: [ColoredPermutations(2,n).catalan_number()                    #␣
↪needs sage.combinat
....:  for n in [3,4,5]]
[20, 70, 252]

sage: [ReflectionGroup((2,2,n)).catalan_number()   # optional - gap3
....:  for n in [3,4,5]]
[14, 50, 182]
```

coxeter_number()

Return the Coxeter number of a well-generated, irreducible reflection group. This is defined to be the order of a regular element in `self`, and is equal to the highest degree of `self`.

See also:

`ComplexReflectionGroups.Finite.Irreducible()`

Note: This method overwrites the more general method for complex reflection groups since the expression given here is quicker to compute.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)                                  #␣
↪needs sage.combinat
sage: W.coxeter_number()                                            #␣
↪needs sage.combinat
3

sage: W = ColoredPermutations(4,3)                                  #␣
↪needs sage.combinat
sage: W.coxeter_number()                                            #␣
↪needs sage.combinat
12

sage: W = ReflectionGroup((4,4,3))  # optional - gap3
sage: W.coxeter_number()            # optional - gap3
8
```

fuss_catalan_number(*m*, *positive=False*, *polynomial=False*)

Return the `m`-th Fuss-Catalan number associated to `self`.

This is defined by

$$\prod_{i=1}^{n} \frac{d_i + mh}{d_i},$$

where $d_1, \ldots, d_n$ are the degrees and h is the Coxeter number.

INPUT:

- `positive` – optional boolean (default `False`) if `True`, return instead the positive Fuss-Catalan number
- `polynomial` – optional boolean (default `False`) if `True`, return instead the q-analogue as a polynomial in q

See [Ar2006] for further information.

Note:

- For the symmetric group S_n, it reduces to the Fuss-Catalan number $\frac{1}{mn+1}\binom{(m+1)n}{n}$.
- The Fuss-Catalan numbers for $G(r, 1, n)$ all coincide for $r > 1$.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)                                      #␣
↪needs sage.combinat
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]                       #␣
↪needs sage.combinat
[5, 12, 22]

sage: W = ColoredPermutations(1,4)                                      #␣
↪needs sage.combinat
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]                       #␣
↪needs sage.combinat
[14, 55, 140]

sage: W = ColoredPermutations(1,5)                                      #␣
↪needs sage.combinat
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]                       #␣
↪needs sage.combinat
[42, 273, 969]

sage: W = ColoredPermutations(2,2)                                      #␣
↪needs sage.combinat
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]                       #␣
↪needs sage.combinat
[6, 15, 28]

sage: W = ColoredPermutations(2,3)                                      #␣
↪needs sage.combinat
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]                       #␣
↪needs sage.combinat
[20, 84, 220]

sage: W = ColoredPermutations(2,4)                                      #␣
↪needs sage.combinat
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]                       #␣
↪needs sage.combinat
[70, 495, 1820]
```

number_of_reflections_of_full_support()

Return the number of reflections with full support.

EXAMPLES:

```
sage: W = Permutations(4)
sage: W.number_of_reflections_of_full_support()
1

sage: W = ColoredPermutations(1,4)                                      #␣
↪needs sage.combinat
sage: W.number_of_reflections_of_full_support()
1

sage: W = CoxeterGroup("B3")                                            #␣
↪needs sage.combinat sage.groups
sage: W.number_of_reflections_of_full_support()                         #␣
↪needs sage.combinat sage.groups
3

sage: W = ColoredPermutations(3,3)                                      #␣
↪needs sage.combinat
sage: W.number_of_reflections_of_full_support()                         #␣
↪needs sage.combinat
3
```

rational_catalan_number(*p*, *polynomial=False*)

Return the p-th rational Catalan number associated to self.

It is defined by

$$\prod_{i=1}^{n} \frac{p + (p(d_i - 1)) \mod h)}{d_i},$$

where $d_1, \ldots, d_n$ are the degrees and h is the Coxeter number. See [STW2016] for this formula.

INPUT:

- polynomial – optional boolean (default False) if True, return instead the q-analogue as a polynomial in q

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)                                      #␣
↪needs sage.combinat
sage: [W.rational_catalan_number(p) for p in [5,7,8]]                   #␣
↪needs sage.combinat
[7, 12, 15]

sage: W = ColoredPermutations(2,2)                                      #␣
↪needs sage.combinat
sage: [W.rational_catalan_number(p) for p in [7,9,11]]                  #␣
↪needs sage.combinat
[10, 15, 21]
```

example()

Return an example of an irreducible well-generated complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: C = ComplexReflectionGroups().Finite().WellGenerated().
↪Irreducible()
```

(continues on next page)

(continued from previous page)

```
sage: C.example()                                                    #␣
↪needs sage.combinat
4-colored permutations of size 3
```

class ParentMethods

Bases: `object`

coxeter_element()

Return a Coxeter element.

The result is the product of the simple reflections, in some order.

Note: This implementation is shared with well generated complex reflection groups. It would be nicer to put it in some joint super category; however, in the current state of the art, there is none where it is clear that this is the right construction for obtaining a Coxeter element.

In this context, this is an element having a regular eigenvector (a vector not contained in any reflection hyperplane of `self`).

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: CoxeterGroup(['A', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['B', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['D', 4]).coxeter_element().reduced_word()
[1, 2, 4, 3]
sage: CoxeterGroup(['F', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['E', 8]).coxeter_element().reduced_word()
[1, 3, 2, 4, 5, 6, 7, 8]
sage: CoxeterGroup(['H', 3]).coxeter_element().reduced_word()
[1, 2, 3]
```

This method is also used for well generated finite complex reflection groups:

```
sage: W = ReflectionGroup((1,1,4))           # optional - gap3
sage: W.coxeter_element().reduced_word()     # optional - gap3
[1, 2, 3]

sage: W = ReflectionGroup((2,1,4))           # optional - gap3
sage: W.coxeter_element().reduced_word()     # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,1,4))           # optional - gap3
sage: W.coxeter_element().reduced_word()     # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,4,4))           # optional - gap3
sage: W.coxeter_element().reduced_word()     # optional - gap3
[1, 2, 3, 4]
```

coxeter_elements()

Return the (unique) conjugacy class in `self` containing all Coxeter elements.

A Coxeter element is an element that has an eigenvalue $e^{2\pi i/h}$ where h is the Coxeter number.

In case of finite Coxeter groups, these are exactly the elements that are conjugate to one (or, equivalently, all) standard Coxeter element, this is, to an element that is the product of the simple generators in some order.

See also:

`standard_coxeter_elements()`

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))      # optional - gap3
sage: sorted(c.reduced_word()           # optional - gap3
....:        for c in W.coxeter_elements())
[[1, 2], [2, 1]]

sage: W = ReflectionGroup((1,1,4))      # optional - gap3
sage: sorted(c.reduced_word()           # optional - gap3
....:        for c in W.coxeter_elements())
[[1, 2, 1, 3, 2], [1, 2, 3], [1, 3, 2],
 [2, 1, 3], [2, 1, 3, 2, 1], [3, 2, 1]]
```

is_well_generated()

Return `True` as `self` is well-generated.

EXAMPLES:

```
sage: W = ReflectionGroup((3,1,2))      # optional - gap3
sage: W.is_well_generated()             # optional - gap3
True
```

milnor_fiber_complex()

Return the Milnor fiber complex of `self`.

The *Milnor fiber complex* of a finite well-generated complex reflection group W is the simplicial complex whose face poset is given by `milnor_fiber_poset()`. When W is an irreducible Shephard group, it is also an equivariant strong deformation retract of the Milnor fiber $f_1^{-1}(1)$, where $f_1 : V \to \mathbf{C}$ is the polynomial invariant of smallest degree acting on the reflection representation V.

When W is a Coxeter group, this is isomorphic to the Coxeter complex of W.

EXAMPLES:

```
sage: W = ColoredPermutations(3, 2)
sage: C = W.milnor_fiber_complex()
sage: C.homology()
{0: 0, 1: Z x Z x Z x Z}

sage: W = ReflectionGroup(5)                    # optional - gap3
sage: C = W.milnor_fiber_complex()              # optional - gap3
sage: C.homology()                              # optional - gap3
{0: 0, 1: Z^25}
```

standard_coxeter_elements()

Return all standard Coxeter elements in `self`.

This is the set of all elements in self obtained from any product of the simple reflections in `self`.

Note:

- `self` is assumed to be well-generated.
- This works even beyond real reflection groups, but the conjugacy class is not unique and we only obtain one such class.

EXAMPLES:

```
sage: W = ReflectionGroup(4)                    # optional - gap3
sage: sorted(W.standard_coxeter_elements())   # optional - gap3
[(1,7,6,12,23,20)(2,8,17,24,9,5)(3,16,10,19,15,21)(4,14,11,22,18,13),
 (1,10,4,12,21,22)(2,11,19,24,13,3)(5,15,7,17,16,23)(6,18,8,20,14,9)]
```

example()

Return an example of a well-generated complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import
↪ComplexReflectionGroups
sage: C = ComplexReflectionGroups().Finite().WellGenerated()
sage: C.example()                               # optional - gap3
Reducible complex reflection group of rank 4 and type A2 x G(3,1,2)
```

example()

Return an example of a complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite().example()          # optional - gap3
Reducible real reflection group of rank 4 and type A2 x B2
```

4.45 Finite Coxeter Groups

class sage.categories.finite_coxeter_groups.**FiniteCoxeterGroups**(*base_category*)

Bases: *CategoryWithAxiom*

The category of finite Coxeter groups.

EXAMPLES:

```
sage: CoxeterGroups.Finite()
Category of finite Coxeter groups
sage: FiniteCoxeterGroups().super_categories()
[Category of finite generalized Coxeter groups,
 Category of Coxeter groups]

sage: G = CoxeterGroups().Finite().example()
sage: G.cayley_graph(side = "right").plot()
Graphics object consisting of 40 graphics primitives
```

Here are some further examples:

```
sage: WeylGroups().Finite().example()
The symmetric group on {0, ..., 3}

sage: WeylGroup(["B", 3])
Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space)
```

Those other examples will eventually be also in this category:

```
sage: SymmetricGroup(4)
Symmetric group of order 4! as a permutation group
sage: DihedralGroup(5)
Dihedral group of order 10 as a permutation group
```

class ElementMethods

Bases: `object`

absolute_length()

Return the absolute length of `self`.

The absolute length is the length of the shortest expression of the element as a product of reflections. For finite Coxeter groups, the absolute length is the codimension of the 1-eigenspace of the element (Lemmas 1-3 in [Car1972a]).

For permutations in the symmetric groups, the absolute length is the size minus the number of its disjoint cycles.

See also:

`absolute_le()`

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])                                                #␣
↪needs sage.combinat sage.groups
sage: s = W.simple_reflections()                                             #␣
↪needs sage.combinat sage.groups
sage: (s[1]*s[2]*s[3]).absolute_length()                                     #␣
↪needs sage.combinat sage.groups
3

sage: W = SymmetricGroup(4)                                                  #␣
↪needs sage.groups
sage: s = W.simple_reflections()                                             #␣
↪needs sage.groups
sage: (s[3]*s[2]*s[1]).absolute_length()                                     #␣
↪needs sage.combinat sage.groups
3
```

bruhat_upper_covers()

Returns all the elements that cover `self` in Bruhat order.

EXAMPLES:

```
sage: W = WeylGroup(["A",4])
sage: w = W.from_reduced_word([3,2])
sage: print([v.reduced_word() for v in w.bruhat_upper_covers()])
[[4, 3, 2], [3, 4, 2], [2, 3, 2], [3, 1, 2], [3, 2, 1]]
```

(continues on next page)

(continued from previous page)

```
sage: W = WeylGroup(["B",6])
sage: w = W.from_reduced_word([1,2,1,4,5])
sage: C = w.bruhat_upper_covers()
sage: len(C)
9
sage: print([v.reduced_word() for v in C])
[[6, 4, 5, 1, 2, 1], [4, 5, 6, 1, 2, 1], [3, 4, 5, 1, 2, 1], [2, 3, 4, 5,␣
↪1, 2],
[1, 2, 3, 4, 5, 1], [4, 5, 4, 1, 2, 1], [4, 5, 3, 1, 2, 1], [4, 5, 2, 3,␣
↪1, 2],
[4, 5, 1, 2, 3, 1]]
sage: ww = W.from_reduced_word([5,6,5])
sage: CC = ww.bruhat_upper_covers()
sage: print([v.reduced_word() for v in CC])
[[6, 5, 6, 5], [4, 5, 6, 5], [5, 6, 4, 5], [5, 6, 5, 4], [5, 6, 5, 3], [5,
↪ 6, 5, 2],
[5, 6, 5, 1]]
```

Recursive algorithm: write w for `self`. If i is a non-descent of w, then the covers of w are exactly $\{ws_i, u_1 s_i, u_2 s_i, ..., u_j s_i\}$, where the u_k are those covers of ws_i that have a descent at i.

covered_reflections_subgroup()

Return the subgroup of W generated by the conjugates by w of the simple reflections indexed by right descents of w.

This is used to compute the shard intersection order on W.

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], base_ring=ZZ)
sage: len(W.long_element().covered_reflections_subgroup())
24
sage: s = W.simple_reflection(1)
sage: Gs = s.covered_reflections_subgroup()
sage: len(Gs)
2
sage: s in [u.lift() for u in Gs]
True
sage: len(W.one().covered_reflections_subgroup())
1
```

coxeter_knuth_graph()

Return the Coxeter-Knuth graph of type A.

The Coxeter-Knuth graph of type A is generated by the Coxeter-Knuth relations which are given by $aa+1a \sim a+1aa+1$, $abc \sim acb$ if $b < a < c$ and $abc \sim bac$ if $a < c < b$.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix='s')
sage: w = W.from_reduced_word([1,2,1,3,2])
sage: D = w.coxeter_knuth_graph()
sage: D.vertices(sort=True)
[(1, 2, 1, 3, 2),
(1, 2, 3, 1, 2),
(2, 1, 2, 3, 2),
(2, 1, 3, 2, 3),
```

(continues on next page)

(continued from previous page)

```
(2, 3, 1, 2, 3)]
sage: D.edges(sort=True)
[((1, 2, 1, 3, 2), (1, 2, 3, 1, 2), None),
((1, 2, 1, 3, 2), (2, 1, 2, 3, 2), None),
((2, 1, 2, 3, 2), (2, 1, 3, 2, 3), None),
((2, 1, 3, 2, 3), (2, 3, 1, 2, 3), None)]

sage: w = W.from_reduced_word([1,3])
sage: D = w.coxeter_knuth_graph()
sage: D.vertices(sort=True)
[(1, 3), (3, 1)]
sage: D.edges(sort=False)
[]
```

coxeter_knuth_neighbor(*w*)

Return the Coxeter-Knuth (oriented) neighbors of the reduced word w of `self`.

INPUT:

- `w` – reduced word of `self`

The Coxeter-Knuth relations are given by $aa+1a \sim a+1aa+1$, $abc \sim acb$ if $b < a < c$ and $abc \sim bac$ if $a < c < b$. This method returns all neighbors of `w` under the Coxeter-Knuth relations oriented from left to right.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix='s')
sage: word = [1,2,1,3,2]
sage: w = W.from_reduced_word(word)
sage: w.coxeter_knuth_neighbor(word)
{(1, 2, 3, 1, 2), (2, 1, 2, 3, 2)}

sage: word = [1,2,1,3,2,4,3]
sage: w = W.from_reduced_word(word)
sage: w.coxeter_knuth_neighbor(word)
{(1, 2, 1, 3, 4, 2, 3), (1, 2, 3, 1, 2, 4, 3), (2, 1, 2, 3, 2, 4, 3)}
```

is_coxeter_element()

Return whether this is a Coxeter element.

This is, whether `self` has an eigenvalue $e^{2\pi i/h}$ where h is the Coxeter number.

See also:

`coxeter_elements()`

EXAMPLES:

```
sage: W = CoxeterGroup(['A',2])
sage: c = prod(W.gens())
sage: c.is_coxeter_element()
True
sage: W.one().is_coxeter_element()
False

sage: W = WeylGroup(['G', 2])
sage: c = prod(W.gens())
sage: c.is_coxeter_element()
True
```

(continues on next page)

(continued from previous page)

```
sage: W.one().is_coxeter_element()
False
```

class ParentMethods

Bases: `object`

Ambiguity resolution: the implementation of `some_elements` is preferable to that of *FiniteGroups*. The same holds for `__iter__`, although a breadth first search would be more natural; at least this maintains backward compatibility after github issue #13589.

bhz_poset()

Return the Bergeron-Hohlweg-Zabrocki partial order on the Coxeter group.

This is a partial order on the elements of a finite Coxeter group W, which is distinct from the Bruhat order, the weak order and the shard intersection order. It was defined in [BHZ2005].

This partial order is not a lattice, as there is no unique maximal element. It can be succintly defined as follows.

Let u and v be two elements of the Coxeter group W. Let $S(u)$ be the support of u. Then $u \leq v$ if and only if $v_{S(u)} = u$ (here $v = v^I v_I$ denotes the usual parabolic decomposition with respect to the standard parabolic subgroup W_I).

See also:

bruhat_poset(), *shard_poset()*, *weak_poset()*

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], base_ring=ZZ)
sage: P = W.bhz_poset(); P
Finite poset containing 24 elements
sage: P.relations_number()
103
sage: P.chain_polynomial()
34*q^4 + 90*q^3 + 79*q^2 + 24*q + 1
sage: len(P.maximal_elements())
13
```

bruhat_poset(*facade=False*)

Return the Bruhat poset of `self`.

See also:

bhz_poset(), *shard_poset()*, *weak_poset()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])
sage: P = W.bruhat_poset()
sage: P
Finite poset containing 6 elements
sage: P.show()
```

Here are some typical operations on this poset:

```
sage: W = WeylGroup(["A", 3])
sage: P = W.bruhat_poset()
sage: u = W.from_reduced_word([3,1])
```

(continues on next page)

(continued from previous page)

```
sage: v = W.from_reduced_word([3,2,1,2,3])
sage: P(u) <= P(v)
True
sage: len(P.interval(P(u), P(v)))
10
sage: P.is_join_semilattice()
False
```

By default, the elements of P are aware that they belong to P:

```
sage: P.an_element().parent()
Finite poset containing 24 elements
```

If instead one wants the elements to be plain elements of the Coxeter group, one can use the `facade` option:

```
sage: P = W.bruhat_poset(facade = True)
sage: P.an_element().parent()
Weyl Group of type ['A', 3] (as a matrix group acting on the ambient␣
↪space)
```

See also:

`Poset()` for more on posets and facade posets.

Todo:

- Use the symmetric group in the examples (for nicer output), and print the edges for a stronger test.
- The constructed poset should be lazy, in order to handle large / infinite Coxeter groups.

cambrian_lattice(*c*, *on_roots=False*)

Return the c-Cambrian lattice on delta sequences.

See arXiv 1503.00710 and arXiv math/0611106.

Delta sequences are certain 2-colored minimal factorizations of `c` into reflections.

INPUT:

- `c` – a standard Coxeter element in `self` (as a tuple, or as an element of `self`)
- `on_roots` (optional, default `False`) – if `on_roots` is `True`, the lattice is realized on roots rather than on reflections. In order for this to work, the ElementMethod `reflection_to_root` must be available.

EXAMPLES:

```
sage: CoxeterGroup(["A", 2]).cambrian_lattice((1,2))
Finite lattice containing 5 elements

sage: CoxeterGroup(["B", 2]).cambrian_lattice((1,2))
Finite lattice containing 6 elements

sage: CoxeterGroup(["G", 2]).cambrian_lattice((1,2))
Finite lattice containing 8 elements
```

codegrees()

Return the codegrees of the Coxeter group.

These are just the degrees minus 2.

EXAMPLES:

```
sage: CoxeterGroup(['A', 4]).codegrees()
(0, 1, 2, 3)
sage: CoxeterGroup(['B', 4]).codegrees()
(0, 2, 4, 6)
sage: CoxeterGroup(['D', 4]).codegrees()
(0, 2, 2, 4)
sage: CoxeterGroup(['F', 4]).codegrees()
(0, 4, 6, 10)
sage: CoxeterGroup(['E', 8]).codegrees()
(0, 6, 10, 12, 16, 18, 22, 28)
sage: CoxeterGroup(['H', 3]).codegrees()
(0, 4, 8)

sage: WeylGroup([["A",3], ["A",3], ["B",2]]).codegrees()
(0, 1, 2, 0, 1, 2, 0, 2)
```

coxeter_complex()

Return the Coxeter complex of `self`.

Let W be a Coxeter group, and let X be the corresponding Tits cone, which is constructed as the W orbit of the fundamental chamber in the reflection representation. The *Coxeter complex* of W is the simplicial complex $(X \setminus \{0\})/\mathbf{R}_{>0}$. The face poset of this simplicial complex is given by the *coxeter_poset()*. When W is a finite group, then the Coxeter complex is homeomorphic to an $(n-1)$-dimensional sphere, where n is the rank of W.

EXAMPLES:

```
sage: W = CoxeterGroup(['A', 3])
sage: C = W.coxeter_complex()
sage: C
Simplicial complex with 14 vertices and 24 facets
sage: C.homology()
{0: 0, 1: 0, 2: Z}

sage: W = WeylGroup(['B', 3])
sage: C = W.coxeter_complex()
sage: C
Simplicial complex with 26 vertices and 48 facets
sage: C.homology()
{0: 0, 1: 0, 2: Z}

sage: W = CoxeterGroup(['I', 7])
sage: C = W.coxeter_complex()
sage: C
Simplicial complex with 14 vertices and 14 facets
sage: C.homology()
{0: 0, 1: Z}

sage: W = CoxeterGroup(['H', 3])
sage: C = W.coxeter_complex()
sage: C
Simplicial complex with 62 vertices and 120 facets
sage: C.homology()
{0: 0, 1: 0, 2: Z}

sage: # optional - gap3
```

(continues on next page)

(continued from previous page)

```
sage: W = CoxeterGroup(['H', 3], implementation="permutation")
sage: C = W.coxeter_complex()
sage: C
Simplicial complex with 62 vertices and 120 facets
sage: C.homology()
{0: 0, 1: 0, 2: Z}
```

coxeter_poset()

Return the Coxeter poset of `self`.

Let W be a Coxeter group. The *Coxeter poset* is defined as the set of (right) standard cosets gW_J, where J is a subset of the index set I of W, ordered by reverse inclusion.

This is equal to the face poset of the `Coxeter complex`.

EXAMPLES:

```
sage: W = CoxeterGroup(['A', 3])
sage: P = W.coxeter_poset()
sage: P
Finite meet-semilattice containing 75 elements
sage: P.rank()
3

sage: W = WeylGroup(['B', 3])
sage: P = W.coxeter_poset()
sage: P
Finite meet-semilattice containing 147 elements
sage: P.rank()
3

sage: W = CoxeterGroup(['I', 7])
sage: P = W.coxeter_poset()
sage: P
Finite meet-semilattice containing 29 elements
sage: P.rank()
2

sage: W = CoxeterGroup(['H', 3])
sage: P = W.coxeter_poset()
sage: P
Finite meet-semilattice containing 363 elements
sage: P.rank()
3

sage: # optional - gap3
sage: W = CoxeterGroup(['H', 3], implementation="permutation")
sage: P = W.coxeter_poset()
sage: P
Finite meet-semilattice containing 363 elements
sage: P.rank()
3
```

degrees()

Return the degrees of the Coxeter group.

The output is an increasing list of integers.

EXAMPLES:

```
sage: CoxeterGroup(['A', 4]).degrees()
(2, 3, 4, 5)
sage: CoxeterGroup(['B', 4]).degrees()
(2, 4, 6, 8)
sage: CoxeterGroup(['D', 4]).degrees()
(2, 4, 4, 6)
sage: CoxeterGroup(['F', 4]).degrees()
(2, 6, 8, 12)
sage: CoxeterGroup(['E', 8]).degrees()
(2, 8, 12, 14, 18, 20, 24, 30)
sage: CoxeterGroup(['H', 3]).degrees()
(2, 6, 10)

sage: WeylGroup([["A",3], ["A",3], ["B",2]]).degrees()
(2, 3, 4, 2, 3, 4, 2, 4)
```

inversion_sequence(*word*)

Return the inversion sequence corresponding to the `word` in indices of simple generators of `self`.

If `word` corresponds to $[w_0, w_1, ...w_k]$, the output is $[w_0, w_0w_1w_0, \ldots, w_0w_1\cdots w_k\cdots w_1w_0]$.

INPUT:

- `word` – a word in the indices of the simple generators of `self`.

EXAMPLES:

```
sage: CoxeterGroup(["A", 2]).inversion_sequence([1,2,1])
[
[-1  1]  [ 0 -1]  [ 1  0]
[ 0  1], [-1  0], [ 1 -1]
]

sage: [t.reduced_word() for t in CoxeterGroup(["A",3]).inversion_
↪sequence([2,1,3,2,1,3])]
[[2], [1, 2, 1], [2, 3, 2], [1, 2, 3, 2, 1], [3], [1]]
```

is_real()

Return `True` since `self` is a real reflection group.

EXAMPLES:

```
sage: CoxeterGroup(['F',4]).is_real()
True
sage: CoxeterGroup(['H',4]).is_real()
True
```

long_element(*index_set=None*, *as_word=False*)

Return the longest element of `self`, or of the parabolic subgroup corresponding to the given `index_set`.

INPUT:

- `index_set` – a subset (as a list or iterable) of the nodes of the Dynkin diagram; (default: all of them)
- `as_word` – boolean (default `False`). If `True`, then return instead a reduced decomposition of the longest element.

Should this method be called maximal_element? longest_element?

EXAMPLES:

```
sage: D10 = FiniteCoxeterGroups().example(10)
sage: D10.long_element()
(1, 2, 1, 2, 1, 2, 1, 2, 1, 2)
sage: D10.long_element([1])
(1,)
sage: D10.long_element([2])
(2,)
sage: D10.long_element([])
()

sage: D7 = FiniteCoxeterGroups().example(7)
sage: D7.long_element()
(1, 2, 1, 2, 1, 2, 1)
```

One can require instead a reduced word for w0:

```
sage: A3 = CoxeterGroup(['A', 3])
sage: A3.long_element(as_word=True)
[1, 2, 1, 3, 2, 1]
```

m_cambrian_lattice (*c*, *m=1*, *on_roots=False*)

Return the m-Cambrian lattice on m-delta sequences.

See arXiv 1503.00710 and arXiv math/0611106.

The m-delta sequences are certain m-colored minimal factorizations of c into reflections.

INPUT:

- c – a Coxeter element of `self` (as a tuple, or as an element of `self`)
- m – a positive integer (optional, default 1)
- `on_roots` (optional, default `False`) – if `on_roots` is `True`, the lattice is realized on roots rather than on reflections. In order for this to work, the ElementMethod `reflection_to_root` must be available.

EXAMPLES:

```
sage: CoxeterGroup(["A",2]).m_cambrian_lattice((1,2))
Finite lattice containing 5 elements

sage: CoxeterGroup(["A",2]).m_cambrian_lattice((1,2),2)
Finite lattice containing 12 elements
```

permutahedron (*point=None*, *base_ring=None*)

Return the permutahedron of `self`,

This is the convex hull of the point `point` in the weight basis under the action of `self` on the underlying vector space V.

See also:

`permutahedron()`

INPUT:

- `point` – optional, a point given by its coordinates in the weight basis (default is $(1, 1, 1, \ldots)$)
- `base_ring` – optional, the base ring of the polytope

Note: The result is expressed in the root basis coordinates.

Note: If function is too slow, switching the base ring to RDF will almost certainly speed things up.

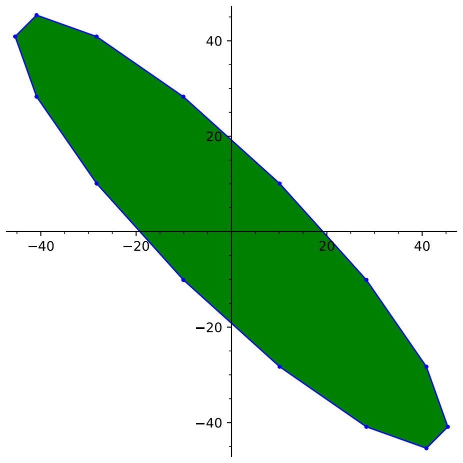
EXAMPLES:

```
sage: W = CoxeterGroup(['H',3], base_ring=RDF)
sage: W.permutahedron()
doctest:warning
...
UserWarning: This polyhedron data is numerically complicated; cdd could␣
↪not convert between the inexact V and H representation without loss of␣
↪data. The resulting object might show inconsistencies.
A 3-dimensional polyhedron in RDF^3 defined as the convex hull of 120␣
↪vertices

sage: W = CoxeterGroup(['I',7])
sage: W.permutahedron()
A 2-dimensional polyhedron in AA^2 defined as the convex hull of 14␣
↪vertices
sage: W.permutahedron(base_ring=RDF)
A 2-dimensional polyhedron in RDF^2 defined as the convex hull of 14␣
↪vertices

sage: W = ReflectionGroup(['A',3])                          # optional -␣
↪gap3
sage: W.permutahedron()                                     # optional -␣
↪gap3
A 3-dimensional polyhedron in QQ^3 defined as the convex hull
of 24 vertices

sage: W = ReflectionGroup(['A',3],['B',2])                  # optional -␣
↪gap3
sage: W.permutahedron()                                     # optional -␣
↪gap3
A 5-dimensional polyhedron in QQ^5 defined as the convex hull of 192␣
↪vertices
```



reflections_from_w0()

Return the reflections of `self` using the inversion set of `w_0`.

EXAMPLES:

```
sage: WeylGroup(['A',2]).reflections_from_w0()
[
[0 1 0]  [0 0 1]  [1 0 0]
[1 0 0]  [0 1 0]  [0 0 1]
[0 0 1], [1 0 0], [0 1 0]
]

sage: WeylGroup(['A',3]).reflections_from_w0()
[
[0 1 0 0]  [0 0 1 0]  [1 0 0 0]  [0 0 0 1]  [1 0 0 0]  [1 0 0 0]
[1 0 0 0]  [0 1 0 0]  [0 0 1 0]  [0 1 0 0]  [0 0 0 1]  [0 1 0 0]
[0 0 1 0]  [1 0 0 0]  [0 1 0 0]  [0 0 1 0]  [0 0 1 0]  [0 0 0 1]
[0 0 0 1], [0 0 0 1], [0 0 0 1], [1 0 0 0], [0 1 0 0], [0 0 1 0]
]
```

shard_poset(*side='right'*)

Return the shard intersection order attached to W.

This is a lattice structure on W, introduced in [Rea2009]. It contains the noncrossing partition lattice, as the induced lattice on the subset of c-sortable elements.

The partial order is given by simultaneous inclusion of inversion sets and subgroups attached to every element.

The precise description used here can be found in [STW2018].

Another implementation for the symmetric groups is available as `shard_poset()`.

See also:

bhz_poset(), *bruhat_poset()*, *weak_poset()*

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], base_ring=ZZ)
sage: SH = W.shard_poset(); SH
Finite lattice containing 24 elements
sage: SH.is_graded()
True
sage: SH.characteristic_polynomial()
q^3 - 11*q^2 + 23*q - 13
sage: SH.f_polynomial()
34*q^3 + 22*q^2 + q
```

w0()

Return the longest element of `self`.

This attribute is deprecated, use *long_element()* instead.

EXAMPLES:

```
sage: D8 = FiniteCoxeterGroups().example(8)
sage: D8.w0
(1, 2, 1, 2, 1, 2, 1, 2)
sage: D3 = FiniteCoxeterGroups().example(3)
sage: D3.w0
(1, 2, 1)
```

weak_lattice(*side='right'*, *facade=False*)

INPUT:

- `side` – “left”, “right”, or “twosided” (default: “right”)
- `facade` – a boolean (default: `False`)

Returns the left (resp. right) poset for weak order. In this poset, u is smaller than v if some reduced word of u is a right (resp. left) factor of some reduced word of v.

See also:

`bhz_poset()`, *`bruhat_poset()`*, *`shard_poset()`*

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])
sage: P = W.weak_poset()
sage: P
Finite lattice containing 6 elements
sage: P.show()
```

This poset is in fact a lattice:

```
sage: W = WeylGroup(["B", 3])
sage: P = W.weak_poset(side = "left")
sage: P.is_lattice()
True
```

so this method has an alias *`weak_lattice()`*:

```
sage: W.weak_lattice(side = "left") is W.weak_poset(side = "left")
True
```

As a bonus feature, one can create the left-right weak poset:

```
sage: W = WeylGroup(["A",2])
sage: P = W.weak_poset(side = "twosided")
sage: P.show()
sage: len(P.hasse_diagram().edges(sort=False))
8
```

This is the transitive closure of the union of left and right order. In this poset, u is smaller than v if some reduced word of u is a factor of some reduced word of v. Note that this is not a lattice:

```
sage: P.is_lattice()
False
```

By default, the elements of P are aware of that they belong to P:

```
sage: P.an_element().parent()
Finite poset containing 6 elements
```

If instead one wants the elements to be plain elements of the Coxeter group, one can use the `facade` option:

```
sage: P = W.weak_poset(facade = True)
sage: P.an_element().parent()
Weyl Group of type ['A', 2] (as a matrix group acting on the ambient␣
↪space)
```

See also:

`Poset()` for more on posets and facade posets.

Todo:

- Use the symmetric group in the examples (for nicer output), and print the edges for a stronger test.
- The constructed poset should be lazy, in order to handle large / infinite Coxeter groups.

weak_poset(*side='right'*, *facade=False*)

INPUT:

- `side` – "left", "right", or "twosided" (default: "right")
- `facade` – a boolean (default: `False`)

Returns the left (resp. right) poset for weak order. In this poset, u is smaller than v if some reduced word of u is a right (resp. left) factor of some reduced word of v.

See also:

bhz_poset(), *bruhat_poset()*, *shard_poset()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])
sage: P = W.weak_poset()
sage: P
Finite lattice containing 6 elements
sage: P.show()
```

This poset is in fact a lattice:

```
sage: W = WeylGroup(["B", 3])
sage: P = W.weak_poset(side = "left")
sage: P.is_lattice()
True
```

so this method has an alias *weak_lattice()*:

```
sage: W.weak_lattice(side = "left") is W.weak_poset(side = "left")
True
```

As a bonus feature, one can create the left-right weak poset:

```
sage: W = WeylGroup(["A",2])
sage: P = W.weak_poset(side = "twosided")
sage: P.show()
sage: len(P.hasse_diagram().edges(sort=False))
8
```

This is the transitive closure of the union of left and right order. In this poset, u is smaller than v if some reduced word of u is a factor of some reduced word of v. Note that this is not a lattice:

```
sage: P.is_lattice()
False
```

By default, the elements of P are aware of that they belong to P:

```
sage: P.an_element().parent()
Finite poset containing 6 elements
```

If instead one wants the elements to be plain elements of the Coxeter group, one can use the `facade` option:

```
sage: P = W.weak_poset(facade = True)
sage: P.an_element().parent()
Weyl Group of type ['A', 2] (as a matrix group acting on the ambient␣
↪space)
```

See also:

`Poset()` for more on posets and facade posets.

Todo:

- Use the symmetric group in the examples (for nicer output), and print the edges for a stronger test.
- The constructed poset should be lazy, in order to handle large / infinite Coxeter groups.

extra_super_categories()

EXAMPLES:

```
sage: CoxeterGroups().Finite().super_categories()
[Category of finite generalized Coxeter groups,
 Category of Coxeter groups]
```

4.46 Finite Crystals

class sage.categories.finite_crystals.**FiniteCrystals**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite crystals.

EXAMPLES:

```
sage: C = FiniteCrystals()
sage: C
Category of finite crystals
sage: C.super_categories()
[Category of crystals, Category of finite enumerated sets]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of finite crystals constructed by tensor product of finite crystals.

extra_super_categories()

EXAMPLES:

```
sage: FiniteCrystals().TensorProducts().extra_super_categories()
[Category of finite crystals]
```

example(*n=3*)

Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = FiniteCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

extra_super_categories()

EXAMPLES:

```
sage: FiniteCrystals().extra_super_categories()
[Category of finite enumerated sets]
```

4.47 Finite dimensional algebras with basis

Todo: Quotients of polynomial rings.

Quotients in general.

Matrix rings.

REFERENCES:

- [CR1962]

class sage.categories.finite_dimensional_algebras_with_basis.**FiniteDimensionalAlgebrasWithI**

Bases: *CategoryWithAxiom_over_base_ring*

The category of finite dimensional algebras with a distinguished basis.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebrasWithBasis(QQ); C
Category of finite dimensional algebras with basis over Rational Field
sage: C.super_categories()
[Category of algebras with basis over Rational Field,
 Category of finite dimensional magmatic algebras with basis over Rational Field]
sage: C.example()                                                         #␣
↪needs sage.modules
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
```

class Cellular(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Cellular algebras.

Let R be a commutative ring. A R-algebra A is a *cellular algebra* if it has a *cell datum*, which is a tuple (Λ, i, M, C), where Λ is finite poset with order $\geq$, if $\mu \in \Lambda$ then $T(\mu)$ is a finite set and

$$C\colon \coprod_{\mu\in\Lambda} T(\mu)\times T(\mu) \longrightarrow A; (\mu, s, t) \mapsto c^{\mu}_{st} \text{ is an injective map}$$

such that the following holds:

- The set $\{c^{\mu}_{st} \mid \mu \in \Lambda, s, t \in T(\mu)\}$ is a basis of A.

- If $a \in A$ and $\mu \in \Lambda, s, t \in T(\mu)$ then:

$$ac^\mu_{st} = \sum_{u \in T(\mu)} r_a(s,u) c^\mu_{ut} \pmod{A^{>\mu}},$$

  where $A^{>\mu}$ is spanned by

$$\{c^\nu_{ab} \mid \nu > \mu \text{ and } a, b \in T(\nu)\}.$$

  Moreover, the scalar $r_a(s,u)$ depends only on a, s and u and, in particular, is independent of t.

- The map $\iota \colon A \longrightarrow A; c^\mu_{st} \mapsto c^\mu_{ts}$ is an algebra anti-isomorphism.

A *cellular basis* for A is any basis of the form $\{c^\mu_{st} \mid \mu \in \Lambda, s, t \in T(\mu)\}$.

Note that in particular, the scalars $r_a(u,s)$ in the second condition do not depend on t.

REFERENCES:

- [GrLe1996]
- [KX1998]
- [Mat1999]
- Wikipedia article Cellular_algebra
- http://webusers.imj-prg.fr/~bernhard.keller/ictp2006/lecturenotes/xi.pdf

class ElementMethods

Bases: `object`

cellular_involution()

Return the cellular involution on `self`.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S = SymmetricGroupAlgebra(QQ, 4)
sage: elt = S([3,1,2,4])
sage: ci = elt.cellular_involution(); ci
7/48*[1, 3, 2, 4] + 49/48*[2, 3, 1, 4]
 - 1/48*[3, 1, 2, 4] - 7/48*[3, 2, 1, 4]
sage: ci.cellular_involution()
[3, 1, 2, 4]
```

class ParentMethods

Bases: `object`

cell_module(*mu*, ***kwds*)

Return the cell module indexed by `mu`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)                                  #␣
↪needs sage.groups sage.modules
sage: S.cell_module(Partition([2,1]))                                   #␣
↪needs sage.combinat sage.groups sage.modules
Cell module indexed by [2, 1] of Cellular basis of
 Symmetric group algebra of order 3 over Rational Field
```

cell_module_indices(*mu*)

Return the indices of the cell module of `self` indexed by `mu` .

This is the finite set $M(\lambda)$.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)                                      #␣
↪needs sage.groups sage.modules
sage: S.cell_module_indices([2,1])                                          #␣
↪needs sage.groups sage.modules
Standard tableaux of shape [2, 1]
```

cell_poset()

Return the cell poset of `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 4)                                      #␣
↪needs sage.groups sage.modules
sage: S.cell_poset()                                                        #␣
↪needs sage.groups sage.modules
Finite poset containing 5 elements
```

cells()

Return the cells of `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)                                      #␣
↪needs sage.groups sage.modules
sage: dict(S.cells())                                                       #␣
↪needs sage.groups sage.modules
{[1, 1, 1]: Standard tableaux of shape [1, 1, 1],
 [2, 1]: Standard tableaux of shape [2, 1],
 [3]: Standard tableaux of shape [3]}
```

cellular_basis()

Return the cellular basis of `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)                                      #␣
↪needs sage.groups sage.modules
sage: S.cellular_basis()                                                    #␣
↪needs sage.groups sage.modules
Cellular basis of Symmetric group algebra of order 3
 over Rational Field
```

cellular_involution(*x*)

Return the cellular involution of `x` in `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)                                      #␣
↪needs sage.groups sage.modules
sage: for b in S.basis(): b, S.cellular_involution(b)                       #␣
↪needs sage.groups sage.modules
```

(continues on next page)

(continued from previous page)

```
([1, 2, 3], [1, 2, 3])
([1, 3, 2], 49/48*[1, 3, 2] + 7/48*[2, 3, 1]
            - 7/48*[3, 1, 2] - 1/48*[3, 2, 1])
([2, 1, 3], [2, 1, 3])
([2, 3, 1], -7/48*[1, 3, 2] - 1/48*[2, 3, 1]
             + 49/48*[3, 1, 2] + 7/48*[3, 2, 1])
([3, 1, 2], 7/48*[1, 3, 2] + 49/48*[2, 3, 1]
             - 1/48*[3, 1, 2] - 7/48*[3, 2, 1])
([3, 2, 1], -1/48*[1, 3, 2] - 7/48*[2, 3, 1]
             + 7/48*[3, 1, 2] + 49/48*[3, 2, 1])
```

simple_module_parameterization()

Return a parameterization of the simple modules of `self`.

The set of simple modules are parameterized by $\lambda \in \Lambda$ such that the cell module bilinear form $\Phi_\lambda \neq 0$.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 4)                                    #␣
↪needs sage.groups sage.modules
sage: S.simple_module_parameterization()                                  #␣
↪needs sage.groups sage.modules
([1, 1, 1, 1], [2, 1, 1], [2, 2], [3, 1], [4])

sage: S = SymmetricGroupAlgebra(GF(3), 4)                                 #␣
↪needs sage.groups sage.modules
sage: S.simple_module_parameterization()                                  #␣
↪needs sage.groups sage.modules
([2, 1, 1], [2, 2], [3, 1], [4])

sage: S = SymmetricGroupAlgebra(GF(4), 4)                                 #␣
↪needs sage.groups sage.modules
sage: S.simple_module_parameterization()                                  #␣
↪needs sage.groups sage.modules
([3, 1], [4])
```

class TensorProducts(*category*, **args*)

Bases: `TensorProductsCategory`

The category of cellular algebras constructed by tensor product of cellular algebras.

class ParentMethods

Bases: `object`

cell_module_indices(*mu*)

Return the indices of the cell module of `self` indexed by `mu` .

This is the finite set $M(\lambda)$.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S2 = SymmetricGroupAlgebra(QQ, 2)
sage: S3 = SymmetricGroupAlgebra(QQ, 3)
sage: T = S2.tensor(S3)
sage: T.cell_module_indices(([1,1], [2,1]))
```

(continues on next page)

(continued from previous page)

```
The Cartesian product of (Standard tableaux of shape [1, 1],
                          Standard tableaux of shape [2, 1])
```

cell_poset()

Return the cell poset of `self`.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S2 = SymmetricGroupAlgebra(QQ, 2)
sage: S3 = SymmetricGroupAlgebra(QQ, 3)
sage: T = S2.tensor(S3)
sage: T.cell_poset()                                                    #␣
↪needs sage.combinat sage.graphs
Finite poset containing 6 elements
```

cellular_involution()

Return the image of the cellular involution of the basis element indexed by `i`.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S2 = SymmetricGroupAlgebra(QQ, 2)
sage: S3 = SymmetricGroupAlgebra(QQ, 3)
sage: T = S2.tensor(S3)
sage: for b in T.basis(): b, T.cellular_involution(b)
([1, 2] # [1, 2, 3], [1, 2] # [1, 2, 3])
([1, 2] # [1, 3, 2],
 49/48*[1, 2] # [1, 3, 2] + 7/48*[1, 2] # [2, 3, 1]
  - 7/48*[1, 2] # [3, 1, 2] - 1/48*[1, 2] # [3, 2, 1])
([1, 2] # [2, 1, 3], [1, 2] # [2, 1, 3])
([1, 2] # [2, 3, 1],
 -7/48*[1, 2] # [1, 3, 2] - 1/48*[1, 2] # [2, 3, 1]
  + 49/48*[1, 2] # [3, 1, 2] + 7/48*[1, 2] # [3, 2, 1])
([1, 2] # [3, 1, 2],
 7/48*[1, 2] # [1, 3, 2] + 49/48*[1, 2] # [2, 3, 1]
  - 1/48*[1, 2] # [3, 1, 2] - 7/48*[1, 2] # [3, 2, 1])
([1, 2] # [3, 2, 1],
 -1/48*[1, 2] # [1, 3, 2] - 7/48*[1, 2] # [2, 3, 1]
  + 7/48*[1, 2] # [3, 1, 2] + 49/48*[1, 2] # [3, 2, 1])
([2, 1] # [1, 2, 3], [2, 1] # [1, 2, 3])
([2, 1] # [1, 3, 2],
 49/48*[2, 1] # [1, 3, 2] + 7/48*[2, 1] # [2, 3, 1]
  - 7/48*[2, 1] # [3, 1, 2] - 1/48*[2, 1] # [3, 2, 1])
([2, 1] # [2, 1, 3], [2, 1] # [2, 1, 3])
([2, 1] # [2, 3, 1],
 -7/48*[2, 1] # [1, 3, 2] - 1/48*[2, 1] # [2, 3, 1]
  + 49/48*[2, 1] # [3, 1, 2] + 7/48*[2, 1] # [3, 2, 1])
([2, 1] # [3, 1, 2],
 7/48*[2, 1] # [1, 3, 2] + 49/48*[2, 1] # [2, 3, 1]
  - 1/48*[2, 1] # [3, 1, 2] - 7/48*[2, 1] # [3, 2, 1])
([2, 1] # [3, 2, 1],
 -1/48*[2, 1] # [1, 3, 2] - 7/48*[2, 1] # [2, 3, 1]
  + 7/48*[2, 1] # [3, 1, 2] + 49/48*[2, 1] # [3, 2, 1])
```

extra_super_categories()

Tensor products of cellular algebras are cellular.

EXAMPLES:

```
sage: cat = Algebras(QQ).FiniteDimensional().WithBasis()
sage: cat.Cellular().TensorProducts().extra_super_categories()
[Category of finite dimensional cellular algebras with basis
 over Rational Field]
```

class ElementMethods

Bases: `object`

on_left_matrix(*base_ring=None*, *action=<built-in function mul>*, *side='left'*)

Return the matrix of the action of `self` on the algebra.

INPUT:

- `base_ring` – the base ring for the matrix to be constructed
- `action` – a bivariate function (default: `operator.mul()`)
- `side` – 'left' or 'right' (default: 'left')

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = QS3([2,1,3])
sage: a.to_matrix(side='left')
[0 0 1 0 0 0]
[0 0 0 0 1 0]
[1 0 0 0 0 0]
[0 0 0 0 0 1]
[0 1 0 0 0 0]
[0 0 0 1 0 0]
sage: a.to_matrix(side='right')
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 0 0 0 1]
[0 0 0 0 1 0]
sage: a.to_matrix(base_ring=RDF, side="left")
[0.0 0.0 1.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 1.0 0.0]
[1.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 1.0]
[0.0 1.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 1.0 0.0 0.0]
```

AUTHORS: Mike Hansen, …

to_matrix(*base_ring=None*, *action=<built-in function mul>*, *side='left'*)

Return the matrix of the action of `self` on the algebra.

INPUT:

- `base_ring` – the base ring for the matrix to be constructed
- `action` – a bivariate function (default: `operator.mul()`)
- `side` – 'left' or 'right' (default: 'left')

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = QS3([2,1,3])
```

(continues on next page)

(continued from previous page)

```
sage: a.to_matrix(side='left')
[0 0 1 0 0 0]
[0 0 0 0 1 0]
[1 0 0 0 0 0]
[0 0 0 0 0 1]
[0 1 0 0 0 0]
[0 0 0 1 0 0]
sage: a.to_matrix(side='right')
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 0 0 0 1]
[0 0 0 0 1 0]
sage: a.to_matrix(base_ring=RDF, side="left")
[0.0 0.0 1.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 1.0 0.0]
[1.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 1.0]
[0.0 1.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 1.0 0.0 0.0]
```

AUTHORS: Mike Hansen, …

class ParentMethods

Bases: `object`

cartan_invariants_matrix()

Return the Cartan invariants matrix of the algebra.

OUTPUT: a matrix of non negative integers

Let A be this finite dimensional algebra and $(S_i)_{i\in I}$ be representatives of the right simple modules of A. Note that their adjoints S_i^* are representatives of the left simple modules.

Let $(P_i^L)_{i\in I}$ and $(P_i^R)_{i\in I}$ be respectively representatives of the corresponding indecomposable projective left and right modules of A. In particular, we assume that the indexing is consistent so that $S_i^* = \text{top } P_i^L$ and $S_i = \text{top } P_i^R$.

The *Cartan invariant matrix* $(C_{i,j})_{i,j\in I}$ is a matrix of non negative integers that encodes much of the representation theory of A; namely:

- $C_{i,j}$ counts how many times $S_i^* \otimes S_j$ appears as composition factor of A seen as a bimodule over itself;
- $C_{i,j} = \dim Hom_A(P_j^R, P_i^R)$;
- $C_{i,j}$ counts how many times S_j appears as composition factor of P_i^R;
- $C_{i,j} = \dim Hom_A(P_i^L, P_j^L)$;
- $C_{i,j}$ counts how many times S_i^* appears as composition factor of P_j^L.

In the commutative case, the Cartan invariant matrix is diagonal. In the context of solving systems of multivariate polynomial equations of dimension zero, A is the quotient of the polynomial ring by the ideal generated by the equations, the simple modules correspond to the roots, and the numbers $C_{i,i}$ give the multiplicities of those roots.

Note: For simplicity, the current implementation assumes that the index set I is of the form $\{0,\dots,n-1\}$. Better indexations will be possible in the future.

ALGORITHM:

The Cartan invariant matrix of A is computed from the dimension of the summands of its Peirce decomposition.

See also:

- *peirce_decomposition()*
- *isotypic_projective_modules()*

EXAMPLES:

For a semisimple algebra, in particular for group algebras in characteristic zero, the Cartan invariants matrix is the identity:

```
sage: A3 = SymmetricGroup(3).algebra(QQ)                                    #␣
↪needs sage.groups sage.modules
sage: A3.cartan_invariants_matrix()                                         #␣
↪needs sage.groups sage.modules
[1 0 0]
[0 1 0]
[0 0 1]
```

For the path algebra of a quiver, the Cartan invariants matrix counts the number of paths between two vertices:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example()
sage: A.cartan_invariants_matrix()                                          #␣
↪needs sage.modules sage.rings.number_field
[1 2]
[0 1]
```

In the commutative case, the Cartan invariant matrix is diagonal:

```
sage: Z12 = Monoids().Finite().example(); Z12
An example of a finite multiplicative monoid: the integers modulo 12
sage: A = Z12.algebra(QQ)                                                   #␣
↪needs sage.modules
sage: A.cartan_invariants_matrix()                                          #␣
↪needs sage.modules sage.rings.number_field
[1 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0]
[0 0 2 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 0 0 2 0 0 0 0]
[0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 2 0]
[0 0 0 0 0 0 0 0 1]
```

With the algebra of the 0-Hecke monoid:

```
sage: # needs sage.combinat sage.groups sage.modules
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: A = HeckeMonoid(SymmetricGroup(4)).algebra(QQ)
sage: A.cartan_invariants_matrix()                                          #␣
↪needs sage.rings.number_field
[1 0 0 0 0 0 0 0]
[0 2 1 0 1 1 0 0]
[0 1 1 0 1 0 0 0]
[0 0 0 1 0 1 1 0]
```

(continues on next page)

(continued from previous page)

```
[0 1 1 0 1 0 0 0]
[0 1 0 1 0 2 1 0]
[0 0 0 1 0 1 1 0]
[0 0 0 0 0 0 0 1]
```

center()

Return the center of `self`.

See also:

center_basis()

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: center = A.center(); center
Center of An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: center in Algebras(QQ).WithBasis().FiniteDimensional().Commutative()
True
sage: center.dimension()
1
sage: center.basis()
Finite family {0: B[0]}
sage: center.ambient() is A
True
sage: [c.lift() for c in center.basis()]
[x + y]
```

The center of a semisimple algebra is semisimple:

```
sage: A = DihedralGroup(6).algebra(QQ)                                          #␣
↪needs sage.groups sage.modules
sage: A.center() in Algebras(QQ).Semisimple()                                   #␣
↪needs sage.groups sage.modules
True
```

Todo:
- Pickling by construction, as `A.center()`?
- Lazy evaluation of `_repr_`

center_basis()

Return a basis of the center of `self`.

OUTPUT:
- a list of elements of `self`.

See also:

center()

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.center_basis()
(x + y,)
```

idempotent_lift(*x*)

Lift an idempotent of the semisimple quotient into an idempotent of `self`.

Let A be this finite dimensional algebra and π be the projection $A \to \overline{A}$ on its semisimple quotient. Let $\overline{x}$ be an idempotent of $\overline{A}$, and x any lift thereof in A. This returns an idempotent e of A such that $\pi(e) = \pi(x)$ and e is a polynomial in x.

INPUT:

- x – an element of A that projects on an idempotent $\overline{x}$ of the semisimple quotient of A. Alternatively one may give as input the idempotent $\overline{x}$, in which case some lift thereof will be taken for x.

OUTPUT: the idempotent e of `self`

ALGORITHM:

Iterate the formula $1 - (1 - x^2)^2$ until having an idempotent.

See [CR1962] for correctness and termination proofs.

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example()
sage: S = A.semisimple_quotient()
sage: A.idempotent_lift(S.basis()['x'])
x
sage: A.idempotent_lift(A.basis()['y'])
y
```

Todo: Add some non trivial example

is_commutative()

Return whether `self` is a commutative algebra.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S4 = SymmetricGroupAlgebra(QQ, 4)
sage: S4.is_commutative()
False
sage: S2 = SymmetricGroupAlgebra(QQ, 2)
sage: S2.is_commutative()
True
```

is_identity_decomposition_into_orthogonal_idempotents(*l*)

Return whether `l` is a decomposition of the identity into orthogonal idempotents.

INPUT:

- `l` – a list or iterable of elements of `self`

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x,y,a,b = A.algebra_generators(); x,y,a,b
(x, y, a, b)
sage: A.is_identity_decomposition_into_orthogonal_idempotents([A.one()])
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents([x, y])
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents([x + a, y -␣
↪a])
True
```

Here the idempotents do not sum up to 1:

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents([x])        ␣
↪          # needs sage.graphs sage.modules
False
```

Here $1 + x$ and $-x$ are neither idempotent nor orthogonal:

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents([1 + x, -
↪x])           # needs sage.graphs sage.modules
False
```

With the algebra of the 0-Hecke monoid:

```
sage: # needs sage.combinat sage.groups sage.modules
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: A = HeckeMonoid(SymmetricGroup(4)).algebra(QQ)
sage: idempotents = A.orthogonal_idempotents_central_mod_radical()       ␣
↪          # needs sage.rings.number_field
sage: A.is_identity_decomposition_into_orthogonal_
↪idempotents(idempotents)           # needs sage.rings.number_field
True
```

Here are some more counterexamples:

1. Some orthogonal elements summing to 1 but not being idempotent:

```
sage: # needs sage.libs.pari sage.modules
sage: class PQAlgebra(CombinatorialFreeModule):
....:     def __init__(self, F, p):
....:         # Construct the quotient algebra F[x] / p,
....:         # where p is a univariate polynomial.
....:         R = parent(p); x = R.gen()
....:         I = R.ideal(p)
....:         self._xbar = R.quotient(I).gen()
....:         basis_keys = [self._xbar**i for i in range(p.degree())]
....:         CombinatorialFreeModule.__init__(self, F, basis_keys,
....:                 category=Algebras(F).FiniteDimensional().
↪WithBasis())
....:     def x(self):
....:         return self(self._xbar)
....:     def one(self):
....:         return self.basis()[self.base_ring().one()]
```

(continues on next page)

(continued from previous page)

```
....:     def product_on_basis(self, w1, w2):
....:         return self.from_vector(vector(w1*w2))
sage: R.<x> = PolynomialRing(QQ)
sage: A = PQAlgebra(QQ, x**3 - x**2 + x + 1); y = A.x()
sage: a, b = y, 1 - y
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, b))
False
```

For comparison:

```
sage: # needs sage.libs.pari sage.modules
sage: A = PQAlgebra(QQ, x**2 - x); y = A.x()
sage: a, b = y, 1-y
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, b))
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, A.
↪zero(), b))
True
sage: A = PQAlgebra(QQ, x**3 - x**2 + x - 1); y = A.x()
sage: a = (y**2 + 1) / 2
sage: b = 1 - a
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, b))
True
```

2. Some idempotents summing to 1 but not orthogonal:

```
sage: # needs sage.libs.pari sage.modules
sage: R.<x> = PolynomialRing(GF(2))
sage: A = PQAlgebra(GF(2), x)
sage: a = A.one()
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a,))
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, a,␣
↪a))
False
```

3. Some orthogonal idempotents not summing to the identity:

```
sage: # needs sage.libs.pari sage.modules
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a,a))
False
sage: A.is_identity_decomposition_into_orthogonal_idempotents(())
False
```

isotypic_projective_modules(*side='left'*)

Return the isotypic projective `side` `self`-modules.

Let P_i be representatives of the indecomposable projective `side`-modules of this finite dimensional algebra A, and S_i be the associated simple modules.

The regular `side` representation of A can be decomposed as a direct sum $A = \bigoplus_i Q_i$ where each Q_i is an isotypic projective module; namely Q_i is the direct sum of dim S_i copies of the indecomposable projective module P_i. This decomposition is not unique.

The isotypic projective modules are constructed as $Q_i = e_i A$, where the $(e_i)_i$ is the decomposition of the identity into orthogonal idempotents obtained by lifting the central orthogonal idempotents of the semisimple quotient of A.

INPUT:

- `side` – 'left' or 'right' (default: 'left')

OUTPUT: a list of subspaces of `self`.

EXAMPLES:

```
sage: # needs sage.graphs sage.modules sage.rings.number_field
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: Q = A.isotypic_projective_modules(side="left"); Q
[Free module generated by {0} over Rational Field,
 Free module generated by {0, 1, 2} over Rational Field]
sage: [[x.lift() for x in Qi.basis()]
....:  for Qi in Q]
[[x],
 [y, a, b]]
```

We check that the sum of the dimensions of the isotypic projective modules is the dimension of `self`:

```
sage: sum([Qi.dimension() for Qi in Q]) == A.dimension()                    #␣
↪needs sage.graphs sage.modules sage.rings.number_field
True
```

See also:

- *orthogonal_idempotents_central_mod_radical()*
- *peirce_decomposition()*

orthogonal_idempotents_central_mod_radical()

Return a family of orthogonal idempotents of `self` that project on the central orthogonal idempotents of the semisimple quotient.

OUTPUT:

- a list of orthogonal idempotents obtained by lifting the central orthogonal idempotents of the semisimple quotient.

ALGORITHM:

The orthogonal idempotents of A are obtained by lifting the central orthogonal idempotents of the semisimple quotient $\overline{A}$.

Namely, let $(\overline{f_i})$ be the central orthogonal idempotents of the semisimple quotient of A. We recursively construct orthogonal idempotents of A by the following procedure: assuming $(f_i)_{i<n}$ is a set of already constructed orthogonal idempotent, we construct f_k by idempotent lifting of $(1-f)g(1-f)$, where g is any lift of $\overline{e_k}$ and $f = \sum_{i<k} f_i$.

See [CR1962] for correctness and termination proofs.

See also:

- `Algebras.SemiSimple.FiniteDimensional.WithBasis.ParentMethods.central_orthogonal_idempotents()`
- *idempotent_lift()*

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
```

(continues on next page)

(continued from previous page)

```
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.orthogonal_idempotents_central_mod_radical()                        #␣
↪needs sage.rings.number_field
(x, y)
```

```
sage: # needs sage.modules sage.rings.number_field
sage: Z12 = Monoids().Finite().example(); Z12
An example of a finite multiplicative monoid: the integers modulo 12
sage: A = Z12.algebra(QQ)
sage: idempotents = A.orthogonal_idempotents_central_mod_radical()
sage: sorted(idempotents, key=str)
[-B[0] + 1/2*B[4] + 1/2*B[8],
 1/2*B[4] - 1/2*B[8],
 1/2*B[9] + 1/2*B[3] - B[0],
 1/2*B[9] - 1/2*B[3],
 1/4*B[1] + 1/4*B[11] - 1/4*B[5] - 1/4*B[7],
 1/4*B[1] - 1/2*B[9] + 1/4*B[5] - 1/4*B[7] + 1/2*B[3] - 1/4*B[11],
 1/4*B[1] - 1/2*B[9] - 1/2*B[3] + 1/4*B[11] + 1/4*B[5] + 1/4*B[7] + B[0] -
↪ 1/2*B[4] - 1/2*B[8],
 1/4*B[1] - 1/4*B[5] + 1/4*B[7] - 1/4*B[11] - 1/2*B[4] + 1/2*B[8],
 B[0]]
sage: sum(idempotents) == 1
True
sage: all(e*e == e for e in idempotents)
True
sage: all(e*f == 0 and f*e == 0
....:     for e in idempotents for f in idempotents if e != f)
True
```

This is best tested with:

```
sage: A.is_identity_decomposition_into_orthogonal_
↪idempotents(idempotents)          # needs sage.graphs sage.modules sage.
↪rings.number_field
True
```

We construct orthogonal idempotents for the algebra of the 0-Hecke monoid:

```
sage: # needs sage.combinat sage.graphs sage.groups sage.modules
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: A = HeckeMonoid(SymmetricGroup(4)).algebra(QQ)
sage: idempotents = A.orthogonal_idempotents_central_mod_radical()
sage: A.is_identity_decomposition_into_orthogonal_idempotents(idempotents)
True
```

peirce_decomposition(*idempotents=None*, *check=True*)

Return a Peirce decomposition of `self`.

Let $(e_i)_i$ be a collection of orthogonal idempotents of A with sum 1. The *Peirce decomposition* of A is the decomposition of A into the direct sum of the subspaces $e_i A e_j$.

With the default collection of orthogonal idempotents, one has

$$\dim e_i A e_j = C_{i,j} \dim S_i \dim S_j$$

where $(S_i)_i$ are the simple modules of A and $(C_{i,j})_{i,j}$ is the Cartan invariants matrix.

INPUT:

- `idempotents` – a list of orthogonal idempotents $(e_i)_{i=0,\ldots,n}$ of the algebra that sum to 1 (default: the idempotents returned by *orthogonal_idempotents_central_mod_radical()*)
- `check` – (default: `True`) whether to check that the idempotents are indeed orthogonal and idempotent and sum to 1

OUTPUT:

A list of lists l such that `l[i][j]` is the subspace $e_i A e_j$.

See also:

- *orthogonal_idempotents_central_mod_radical()*
- *cartan_invariants_matrix()*

EXAMPLES:

```
sage: # needs sage.graphs sage.groups sage.modules sage.rings.number_field
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.orthogonal_idempotents_central_mod_radical()
(x, y)
sage: decomposition = A.peirce_decomposition(); decomposition
[[Free module generated by {0} over Rational Field,
  Free module generated by {0, 1} over Rational Field],
 [Free module generated by {} over Rational Field,
  Free module generated by {0} over Rational Field]]
sage: [ [[x.lift() for x in decomposition[i][j].basis()]
....:     for j in range(2)]
....:   for i in range(2)]
[[[x], [a, b]],
 [[], [y]]]
```

We recover that the group algebra of the symmetric group S_4 is a block matrix algebra:

```
sage: # needs sage.groups sage.modules sage.rings.number_field
sage: A = SymmetricGroup(4).algebra(QQ)
sage: decomposition = A.peirce_decomposition()   # long time
sage: [[decomposition[i][j].dimension()          # long time (4s)
....:   for j in range(len(decomposition))]
....:  for i in range(len(decomposition))]
[[9, 0, 0, 0, 0],
 [0, 9, 0, 0, 0],
 [0, 0, 4, 0, 0],
 [0, 0, 0, 1, 0],
 [0, 0, 0, 0, 1]]
```

The dimension of each block is d^2, where d is the dimension of the corresponding simple module of S_4. The latter are given by:

```
sage: [p.standard_tableaux().cardinality() for p in Partitions(4)]       #␣
↪needs sage.combinat
[1, 3, 2, 3, 1]
```

peirce_summand(*ei*, *ej*)

Return the Peirce decomposition summand $e_i A e_j$.

INPUT:

- `self` – an algebra A

- ei, ej – two idempotents of A

OUTPUT: $e_i A e_j$, as a subspace of A.

See also:

- *peirce_decomposition()*
- *principal_ideal()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example()
sage: idemp = A.orthogonal_idempotents_central_mod_radical()              #␣
↪needs sage.rings.number_field
sage: A.peirce_summand(idemp[0], idemp[1])                                #␣
↪needs sage.rings.number_field
Free module generated by {0, 1} over Rational Field
sage: A.peirce_summand(idemp[1], idemp[0])                                #␣
↪needs sage.rings.number_field
Free module generated by {} over Rational Field
```

We recover the 2×2 block of $\mathbf{Q}[S_4]$ corresponding to the unique simple module of dimension 2 of the symmetric group S_4:

```
sage: A4 = SymmetricGroup(4).algebra(QQ)                                  #␣
↪needs sage.groups
sage: e = A4.central_orthogonal_idempotents()[2]                          #␣
↪needs sage.groups sage.rings.number_field
sage: A4.peirce_summand(e, e)                                             #␣
↪needs sage.groups sage.rings.number_field
Free module generated by {0, 1, 2, 3} over Rational Field
```

principal_ideal(*a*, *side='left'*)

Construct the side principal ideal generated by a.

EXAMPLES:

In order to highlight the difference between left and right principal ideals, our first example deals with a non commutative algebra:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x, y, a, b = A.basis()
```

In this algebra, multiplication on the right by x annihilates all basis elements but x:

```
sage: x*x, y*x, a*x, b*x                                                  #␣
↪needs sage.graphs sage.modules
(x, 0, 0, 0)
```

so the left ideal generated by x is one-dimensional:

```
sage: Ax = A.principal_ideal(x, side='left'); Ax                          #␣
↪needs sage.graphs sage.modules
Free module generated by {0} over Rational Field
sage: [B.lift() for B in Ax.basis()]                                      #␣
↪needs sage.graphs sage.modules
[x]
```

Multiplication on the left by x annihilates only x and fixes the other basis elements:

```
sage: x*x, x*y, x*a, x*b                                                     #␣
↪needs sage.graphs sage.modules
(x, 0, a, b)
```

so the right ideal generated by x is 3-dimensional:

```
sage: xA = A.principal_ideal(x, side='right'); xA                           #␣
↪needs sage.graphs sage.modules
Free module generated by {0, 1, 2} over Rational Field
sage: [B.lift() for B in xA.basis()]                                         #␣
↪needs sage.graphs sage.modules
[x, a, b]
```

See also:

- *peirce_summand()*

radical()

Return the Jacobson radical of `self`.

This uses *radical_basis()*, whose default implementation handles algebras over fields of characteristic zero or fields of characteristic p in which we can compute $x^{1/p}$.

See also:

radical_basis(), *semisimple_quotient()*

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: radical = A.radical(); radical
Radical of An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
```

The radical is an ideal of A, and thus a finite dimensional non unital associative algebra:

```
sage: # needs sage.graphs sage.modules
sage: from sage.categories.associative_algebras import AssociativeAlgebras
sage: radical in AssociativeAlgebras(QQ).WithBasis().FiniteDimensional()
True
sage: radical in Algebras(QQ)
False

sage: # needs sage.graphs sage.modules
sage: radical.dimension()
2
sage: radical.basis()
Finite family {0: B[0], 1: B[1]}
sage: radical.ambient() is A
True
sage: [c.lift() for c in radical.basis()]
[a, b]
```

Todo:

- Tell Sage that the radical is in fact an ideal;
- Pickling by construction, as `A.center()`;
- Lazy evaluation of `_repr_`.

radical_basis()

Return a basis of the Jacobson radical of this algebra.

Note: This implementation handles algebras over fields of characteristic zero (using Dixon's lemma) or fields of characteristic p in which we can compute $x^{1/p}$ [FR1985], [Eb1989].

OUTPUT:

- a list of elements of `self`.

See also:

radical(), *Algebras.Semisimple*

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.radical_basis()
(a, b)
```

We construct the group algebra of the Klein Four-Group over the rationals:

```
sage: A = KleinFourGroup().algebra(QQ)                                  #␣
↪needs sage.groups sage.modules
```

This algebra belongs to the category of finite dimensional algebras over the rationals:

```
sage: A in Algebras(QQ).FiniteDimensional().WithBasis()                 #␣
↪needs sage.groups sage.modules
True
```

Since the field has characteristic 0, Maschke's Theorem tells us that the group algebra is semisimple. So its radical is the zero ideal:

```
sage: A in Algebras(QQ).Semisimple()                                    #␣
↪needs sage.groups sage.modules
True
sage: A.radical_basis()                                                 #␣
↪needs sage.groups sage.modules
()
```

Let's work instead over a field of characteristic 2:

```
sage: A = KleinFourGroup().algebra(GF(2))                               #␣
↪needs sage.groups sage.modules
sage: A in Algebras(GF(2)).Semisimple()                                 #␣
↪needs sage.groups sage.modules
```

(continues on next page)

(continued from previous page)

```
False
sage: A.radical_basis()                                                     #␣
↪needs sage.groups sage.modules
(() + (1,2)(3,4), (3,4) + (1,2)(3,4), (1,2) + (1,2)(3,4))
```

We now implement the algebra $A = K[x]/(x^p - 1)$, where K is a finite field of characteristic p, and check its radical; alas, we currently need to wrap A to make it a proper *ModulesWithBasis*:

```
sage: # needs sage.modules
sage: class AnAlgebra(CombinatorialFreeModule):
....:     def __init__(self, F):
....:         R.<x> = PolynomialRing(F)
....:         I = R.ideal(x**F.characteristic()-F.one())
....:         self._xbar = R.quotient(I).gen()
....:         basis_keys = [self._xbar**i for i in range(F.
↪characteristic())]
....:         CombinatorialFreeModule.__init__(self, F, basis_keys,
....:                 category=Algebras(F).FiniteDimensional().
↪WithBasis())
....:     def one(self):
....:         return self.basis()[self.base_ring().one()]
....:     def product_on_basis(self, w1, w2):
....:         return self.from_vector(vector(w1*w2))
sage: AnAlgebra(GF(3)).radical_basis()
(B[1] + 2*B[xbar^2], B[xbar] + 2*B[xbar^2])
sage: AnAlgebra(GF(16,'a')).radical_basis()                                 #␣
↪needs sage.rings.finite_rings
(B[1] + B[xbar],)
sage: AnAlgebra(GF(49,'a')).radical_basis()                                 #␣
↪needs sage.rings.finite_rings
(B[1] + 6*B[xbar^6], B[xbar] + 6*B[xbar^6], B[xbar^2] + 6*B[xbar^6],
 B[xbar^3] + 6*B[xbar^6], B[xbar^4] + 6*B[xbar^6], B[xbar^5] + 6*B[xbar^
↪6])
```

semisimple_quotient()

Return the semisimple quotient of `self`.

This is the quotient of `self` by its radical.

See also:

radical()

EXAMPLES:

```
sage: # needs sage.graphs sage.modules
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: a,b,x,y = sorted(A.basis())
sage: S = A.semisimple_quotient(); S
Semisimple quotient of An example of a finite dimensional algebra with␣
↪basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: S in Algebras(QQ).Semisimple()
True
```

(continues on next page)

(continued from previous page)

```
sage: S.basis()
Finite family {'x': B['x'], 'y': B['y']}
sage: xs,ys = sorted(S.basis())
sage: (xs + ys) * xs
B['x']
```

Sanity check: the semisimple quotient of the n-th descent algebra of the symmetric group is of dimension the number of partitions of n:

```
sage: [ DescentAlgebra(QQ,n).B().semisimple_quotient().dimension()        #␣
↪needs sage.combinat sage.modules
....:   for n in range(6) ]
[1, 1, 2, 3, 5, 7]
sage: [Partitions(n).cardinality() for n in range(10)]                    #␣
↪needs sage.combinat
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30]
```

Todo:

- Pickling by construction, as `A.semisimple_quotient()`?
- Lazy evaluation of `_repr_`

class SubcategoryMethods

Bases: `object`

Cellular()

Return the full subcategory of the cellular objects of `self`.

See also:

Wikipedia article Cellular_algebra

EXAMPLES:

```
sage: Algebras(QQ).FiniteDimensional().WithBasis().Cellular()
Category of finite dimensional cellular algebras with basis
 over Rational Field
```

4.48 Finite dimensional bialgebras with basis

`sage.categories.finite_dimensional_bialgebras_with_basis.`**FiniteDimensionalBialgebrasWithBas**

The category of finite dimensional bialgebras with a distinguished basis

EXAMPLES:

```
sage: C = FiniteDimensionalBialgebrasWithBasis(QQ); C
Category of finite dimensional bialgebras with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of bialgebras with basis over Rational Field,
 Category of finite dimensional algebras with basis over Rational Field]
sage: C is Bialgebras(QQ).WithBasis().FiniteDimensional()
True
```

4.49 Finite dimensional coalgebras with basis

`sage.categories.finite_dimensional_coalgebras_with_basis.`**`FiniteDimensionalCoalgebrasWithBas`**

The category of finite dimensional coalgebras with a distinguished basis

EXAMPLES:

```
sage: C = FiniteDimensionalCoalgebrasWithBasis(QQ); C
Category of finite dimensional coalgebras with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of coalgebras with basis over Rational Field,
 Category of finite dimensional vector spaces with basis over Rational Field]
sage: C is Coalgebras(QQ).WithBasis().FiniteDimensional()
True
```

4.50 Finite Dimensional Graded Lie Algebras With Basis

AUTHORS:

- Eero Hakavuori (2018-08-16): initial version

class `sage.categories.finite_dimensional_graded_lie_algebras_with_basis.`**`FiniteDimensionalGr`**

Bases: *`CategoryWithAxiom_over_base_ring`*

Category of finite dimensional graded Lie algebras with a basis.

A grading of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition $\mathfrak{g} = \bigoplus_i V_i$ such that $[V_i, V_j] \subset V_{i+j}$.

EXAMPLES:

```
sage: C = LieAlgebras(ZZ).WithBasis().FiniteDimensional().Graded(); C
Category of finite dimensional graded Lie algebras with basis over Integer Ring
sage: C.super_categories()
[Category of graded Lie algebras with basis over Integer Ring,
 Category of finite dimensional filtered modules with basis over Integer Ring,
 Category of finite dimensional Lie algebras with basis over Integer Ring]

sage: C is LieAlgebras(ZZ).WithBasis().FiniteDimensional().Graded()
True
```

class ParentMethods

Bases: `object`

homogeneous_component_as_submodule(*d*)

Return the `d`-th homogeneous component of `self` as a submodule.

EXAMPLES:

```
sage: C = LieAlgebras(QQ).WithBasis().Graded()
sage: C = C.FiniteDimensional().Stratified().Nilpotent()
sage: L = LieAlgebra(QQ, {('x','y'): {'z': 1}},                              #␣
↪needs sage.combinat sage.modules
....:                    nilpotent=True, category=C)
```

(continues on next page)

(continued from previous page)

```
sage: L.homogeneous_component_as_submodule(2)                          #␣
↪needs sage.combinat sage.modules
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 0 1]
```

class Stratified(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

Category of finite dimensional stratified Lie algebras with a basis.

A stratified Lie algebra is a graded Lie algebra that is generated as a Lie algebra by its homogeneous component of degree 1. That is to say, for a graded Lie algebra $L = \bigoplus_{k=1}^{M} L_k$, we have $L_{k+1} = [L_1, L_k]$.

EXAMPLES:

```
sage: C = LieAlgebras(QQ).WithBasis().Graded().Stratified().
↪FiniteDimensional()
sage: C
Category of finite dimensional stratified Lie algebras with basis over␣
↪Rational Field
```

A finite-dimensional stratified Lie algebra is nilpotent:

```
sage: C is C.Nilpotent()
True
```

class ParentMethods

Bases: `object`

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

If the degrees of the basis elements are not defined, they will be computed. By assumption the stratification $L_1 \oplus \cdots \oplus L_s$ of `self` is such that each component L_k is spanned by some subset of the basis.

The degree of a basis element X is therefore the largest index k such that $X \in L_k \oplus \cdots \oplus L_s$. The space $L_k \oplus \cdots \oplus L_s$ is by assumption the k-th term of the lower central series.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: C = LieAlgebras(QQ).WithBasis().Graded()
sage: C = C.FiniteDimensional().Stratified().Nilpotent()
sage: sc = {('X','Y'): {'Z': 1}}
sage: L.<X,Y,Z> = LieAlgebra(QQ, sc, nilpotent=True, category=C)
sage: L.degree_on_basis(X.leading_support())
1
sage: X.degree()
1
sage: Y.degree()
1
sage: L[X, Y]
Z
sage: Z.degree()
2
```

4.51 Finite dimensional Hopf algebras with basis

class sage.categories.finite_dimensional_hopf_algebras_with_basis.**FiniteDimensionalHopfAlge**

Bases: *CategoryWithAxiom_over_base_ring*

The category of finite dimensional Hopf algebras with a distinguished basis.

EXAMPLES:

```
sage: FiniteDimensionalHopfAlgebrasWithBasis(QQ)
Category of finite dimensional Hopf algebras with basis over Rational Field
sage: FiniteDimensionalHopfAlgebrasWithBasis(QQ).super_categories()
[Category of Hopf algebras with basis over Rational Field,
 Category of finite dimensional algebras with basis over Rational Field]
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

4.52 Finite Dimensional Lie Algebras With Basis

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

class sage.categories.finite_dimensional_lie_algebras_with_basis.**FiniteDimensionalLieAlgebr**

Bases: *CategoryWithAxiom_over_base_ring*

Category of finite dimensional Lie algebras with a basis.

Todo: Many of these tests should use non-abelian Lie algebras and need to be added after github issue #16820.

class ElementMethods

Bases: `object`

adjoint_matrix(*sparse=False*)

Return the matrix of the adjoint action of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.an_element().adjoint_matrix()
[0 0 0]
[0 0 0]
[0 0 0]
sage: L.an_element().adjoint_matrix(sparse=True).is_sparse()
True
```

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: x.adjoint_matrix()
[0 1]
[0 0]
sage: y.adjoint_matrix()
[-1  0]
[ 0  0]
```

We verify that this forms a representation:

```
sage: # needs sage.combinat sage.modules
sage: sl3 = lie_algebras.sl(QQ, 3)
sage: e1, e2 = sl3.e(1), sl3.e(2)
sage: e12 = e1.bracket(e2)
sage: E1, E2 = e1.adjoint_matrix(), e2.adjoint_matrix()
sage: E1 * E2 - E2 * E1 == e12.adjoint_matrix()
True
```

to_vector(*order=None*, *sparse=False*)

Return the vector in `g.module()` corresponding to the element `self` of `g` (where `g` is the parent of `self`).

Implement this if you implement `g.module()`. See `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.an_element().to_vector()
(0, 0, 0)
sage: L.an_element().to_vector(sparse=True)
(0, 0, 0)

sage: # needs sage.combinat sage.groupssage.modules
sage: D = DescentAlgebra(QQ, 4).D()
sage: L = LieAlgebra(associative=D)
sage: L.an_element().to_vector()
(1, 1, 1, 1, 1, 1, 1, 1)
```

Nilpotent

alias of *FiniteDimensionalNilpotentLieAlgebrasWithBasis*

class ParentMethods

Bases: `object`

as_finite_dimensional_algebra()

Return `self` as a `FiniteDimensionalAlgebra`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.cross_product(QQ)
sage: x, y, z = L.basis()
sage: F = L.as_finite_dimensional_algebra()
sage: X, Y, Z = F.basis()
```

(continues on next page)

(continued from previous page)

```
sage: x.bracket(y)
Z
sage: X * Y
Z
```

casimir_element(*order=2*, *UEA=None*, *force_generic=False*)

Return the Casimir element in the universal enveloping algebra of `self`.

A *Casimir element* of order k is a distinguished basis element for the center of $U(\mathfrak{g})$ of homogeneous degree k (that is, it is an element of $U_k \setminus U_{k-1}$, where $\{U_i\}_{i=0}^{\infty}$ is the natural filtration of $U(\mathfrak{g})$). When $\mathfrak{g}$ is a simple Lie algebra, then this spans $Z(U(\mathfrak{g}))_k$.

INPUT:

- `order` – (default: `2`) the order of the Casimir element
- `UEA` – (optional) the universal enveloping algebra to return the result in
- `force_generic` – (default: `False`) if `True` for the quadratic order, then this uses the default algorithm; otherwise this is ignored

ALGORITHM:

For the quadratic order (i.e., `order=2`), then this uses K^{ij}, the inverse of the Killing form matrix, to compute $C_{(2)} = \sum_{i,j} K^{ij} X_i \cdots X_j$, where $\{X_1, \ldots, X_n\}$ is a basis for $\mathfrak{g}$. Otherwise this solves the system of equations

$$f_{aj}^{b} \kappa^{jc\cdots d} + f_{aj}^{c} \kappa^{cj\cdots d} \cdots + f_{aj}^{d} \kappa^{bc\cdots j}$$

for the symmetric tensor $\kappa^{i_1\cdots i_k}$, where k is the `order`. This system comes from $[X_i, C_{(k)}] = 0$ with

$$C_{(k)} = \sum_{i_1,\ldots,i_k}^{n} \kappa^{i_1\cdots i_k} X_{i_1} \cdots X_{i_k}.$$

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['A', 1])
sage: C = L.casimir_element(); C
1/8*b1^2 + 1/2*b0*b2 - 1/4*b1
sage: U = L.universal_enveloping_algebra()
sage: all(g * C == C * g for g in U.gens())
True
sage: U = L.pbw_basis()
sage: C = L.casimir_element(UEA=U); C
1/2*PBW[alpha[1]]*PBW[-alpha[1]] + 1/8*PBW[alphacheck[1]]^2
 - 1/4*PBW[alphacheck[1]]
sage: all(g * C == C * g for g in U.algebra_generators())
True

sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['B', 2])
sage: U = L.pbw_basis()
sage: C = L.casimir_element(UEA=U)
sage: all(g * C == C * g for g in U.algebra_generators())
True

sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['C', 3])
sage: U = L.pbw_basis()
```

(continues on next page)

(continued from previous page)

```
sage: C = L.casimir_element(UEA=U)
sage: all(g * C == C * g for g in U.algebra_generators())
True

sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['A', 1])
sage: C4 = L.casimir_element(order=4, UEA=L.pbw_basis()); C4
4*PBW[alpha[1]]^2*PBW[-alpha[1]]^2
 + 2*PBW[alpha[1]]*PBW[alphacheck[1]]^2*PBW[-alpha[1]]
 + 1/4*PBW[alphacheck[1]]^4 - PBW[alphacheck[1]]^3
 - 4*PBW[alpha[1]]*PBW[-alpha[1]] + 2*PBW[alphacheck[1]]
sage: all(g * C4 == C4 * g for g in L.pbw_basis().algebra_generators())
True

sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.Heisenberg(QQ, 2)
sage: L.casimir_element()
0
```

Todo: Use the symmetry of the tensor to reduce the number of equations and/or variables to solve.

center()

Return the center of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: Z = L.center(); Z
An example of a finite dimensional Lie algebra with basis: the
 3-dimensional abelian Lie algebra over Rational Field
sage: Z.basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

centralizer(*S*)

Return the centralizer of `S` in `self`.

INPUT:

- `S` – a subalgebra of `self` or a list of elements that represent generators for a subalgebra

See also:

centralizer_basis()

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.centralizer([a + b, 2*a + c]); S
An example of a finite dimensional Lie algebra with basis:
 the 3-dimensional abelian Lie algebra over Rational Field
sage: S.basis_matrix()
[1 0 0]
```

(continues on next page)

(continued from previous page)

```
[0 1 0]
[0 0 1]
```

centralizer_basis(*S*)

Return a basis of the centralizer of `S` in `self`.

INPUT:

- `S` – a subalgebra of `self` or a list of elements that represent generators for a subalgebra

See also:

centralizer()

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.centralizer_basis([a + b, 2*a + c])
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]

sage: # needs sage.combinat sage.modules
sage: H = lie_algebras.Heisenberg(QQ, 2)
sage: H.centralizer_basis(H)
[z]

sage: # needs sage.combinat sage.groupssage.modules
sage: D = DescentAlgebra(QQ, 4).D()
sage: L = LieAlgebra(associative=D)
sage: L.centralizer_basis(L)
[D{},
 D{1} + D{1, 2} + D{2, 3} + D{3},
 D{1, 2, 3} + D{1, 3} + D{2}]
sage: D.center_basis()
(D{},
 D{1} + D{1, 2} + D{2, 3} + D{3},
 D{1, 2, 3} + D{1, 3} + D{2})
```

chevalley_eilenberg_complex(*M=None*, *dual=False*, *sparse=True*, *ncpus=None*)

Return the Chevalley-Eilenberg complex of `self`.

Let $\mathfrak{g}$ be a Lie algebra and M be a right $\mathfrak{g}$-module. The *Chevalley-Eilenberg complex* is the chain complex on

$$C_\bullet(\mathfrak{g}, M) = M \otimes \bigwedge\nolimits^\bullet \mathfrak{g},$$

where the differential is given by

$$d(m \otimes g_1 \wedge \cdots \wedge g_p) = \sum_{i=1}^{p} (-1)^{i+1}(mg_i) \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_p + \sum_{1 \leq i < j \leq p} (-1)^{i+j} m \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \wedge$$

INPUT:

- `M` – (default: the trivial 1-dimensional module) one of the following:
  - a module M with an action of `self`
  - a dictionary whose keys are basis elements and values are matrices representing a Lie algebra homomorphism defining the representation
- `dual` – (default: `False`) if `True`, causes the dual of the complex to be computed
- `sparse` – (default: `True`) whether to use sparse or dense matrices

- `ncpus` – (optional) how many cpus to use

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.sl(ZZ, 2)
sage: C = L.chevalley_eilenberg_complex(); C
Chain complex with at most 4 nonzero terms over Integer Ring
sage: ascii_art(C)
                          [-2  0  0]       [0]
                          [ 0  1  0]       [0]
            [0 0 0]       [ 0  0 -2]       [0]
 0 <-- C_0 <-------- C_1 <----------- C_2 <---- C_3 <-- 0

sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['C',2])
sage: C = L.chevalley_eilenberg_complex()  # long time
sage: [C.free_module_rank(i) for i in range(11)]  # long time
[1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1]

sage: # needs sage.combinat sage.modules
sage: g = lie_algebras.sl(QQ, 2)
sage: E, F, H = g.basis()
sage: n = g.subalgebra([F, H])
sage: ascii_art(n.chevalley_eilenberg_complex())
                        [ 0]
            [0 0]       [-2]
 0 <-- C_0 <------ C_1 <----- C_2 <-- 0

sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'y':1}})
sage: f = ({x: Matrix([[1,0],[0,0]]), y: Matrix([[0,1],[0,0]])})
sage: C = L.chevalley_eilenberg_complex(f); C
Chain complex with at most 3 nonzero terms over Rational Field
sage: ascii_art(C)
                            [ 0 -1]
                            [ 2  0]
            [1 0 0 1]       [ 0  0]
            [0 0 0 0]       [ 0  1]
 0 <-- C_0 <---------- C_1 <-------- C_2 <-- 0

sage: ascii_art(L.chevalley_eilenberg_complex(f, sparse=False))
                            [ 0 -1]
                            [ 2  0]
            [1 0 0 1]       [ 0  0]
            [0 0 0 0]       [ 0  1]
 0 <-- C_0 <---------- C_1 <-------- C_2 <-- 0
```

REFERENCES:

- Wikipedia article Lie_algebra_cohomology#Chevalley-Eilenberg_complex
- [Wei1994] Chapter 7

cohomology(*deg=None*, *M=None*, *sparse=True*, *ncpus=None*)

Return the Lie algebra cohomology of `self`.

The Lie algebra cohomology is the cohomology of the Chevalley-Eilenberg cochain complex (which is the dual of the Chevalley-Eilenberg chain complex).

Let $\mathfrak{g}$ be a Lie algebra and M a left $\mathfrak{g}$-module. It is known that $H^0(\mathfrak{g}; M)$ is the subspace of $\mathfrak{g}$-invariants of M:

$$H^0(\mathfrak{g}; M) = M^{\mathfrak{g}} = \{m \in M \mid gm = 0 \text{ for all } g \in \mathfrak{g}\}.$$

Additionally, $H^1(\mathfrak{g}; M)$ is the space of derivations $\mathfrak{g} \to M$ modulo the space of inner derivations, and $H^2(\mathfrak{g}; M)$ is the space of equivalence classes of Lie algebra extensions of $\mathfrak{g}$ by M.

INPUT:

- `deg` – the degree of the homology (optional)
- `M` – (default: the trivial module) a right module of `self`
- `sparse` – (default: `True`) whether to use sparse matrices for the Chevalley-Eilenberg chain complex
- `ncpus` – (optional) how many cpus to use when computing the Chevalley-Eilenberg chain complex

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.so(QQ, 4)
sage: L.cohomology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 0 over Rational Field,
 2: Vector space of dimension 0 over Rational Field,
 3: Vector space of dimension 2 over Rational Field,
 4: Vector space of dimension 0 over Rational Field,
 5: Vector space of dimension 0 over Rational Field,
 6: Vector space of dimension 1 over Rational Field}

sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.Heisenberg(QQ, 2)
sage: L.cohomology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 4 over Rational Field,
 2: Vector space of dimension 5 over Rational Field,
 3: Vector space of dimension 5 over Rational Field,
 4: Vector space of dimension 4 over Rational Field,
 5: Vector space of dimension 1 over Rational Field}

sage: # needs sage.combinat sage.modules
sage: d = {('x', 'y'): {'y': 2}}
sage: L.<x,y> = LieAlgebra(ZZ, d)
sage: L.cohomology()
{0: Z, 1: Z, 2: C2}
```

See also:

`chevalley_eilenberg_complex()`

REFERENCES:

- Wikipedia article Lie_algebra_cohomology

`derivations_basis``()`

Return a basis for the Lie algebra of derivations of `self` as matrices.

A derivation D of an algebra is an endomorphism of A such that

$$D([a, b]) = [D(a), b] + [a, D(b)]$$

for all $a, b \in A$. The set of all derivations form a Lie algebra.

EXAMPLES:

We construct the derivations of the Heisenberg Lie algebra:

```
sage: # needs sage.combinat sage.modules
sage: H = lie_algebras.Heisenberg(QQ, 1)
sage: H.derivations_basis()
(
[1 0 0]  [0 1 0]  [0 0 0]  [0 0 0]  [0 0 0]  [0 0 0]
[0 0 0]  [0 0 0]  [1 0 0]  [0 1 0]  [0 0 0]  [0 0 0]
[0 0 1], [0 0 0], [0 0 0], [0 0 1], [1 0 0], [0 1 0]
)
```

We construct the derivations of $\mathfrak{sl}_2$:

```
sage: # needs sage.combinat sage.modules
sage: sl2 = lie_algebras.sl(QQ, 2)
sage: sl2.derivations_basis()
(
[ 1  0  0]  [   0    1    0]  [ 0  0  0]
[ 0  0  0]  [   0    0 -1/2]  [ 1  0  0]
[ 0  0 -1], [   0    0    0], [ 0 -2  0]
)
```

We verify these are derivations:

```
sage: # needs sage.combinat sage.modules
sage: D = [sl2.module_morphism(matrix=M, codomain=sl2)
....:      for M in sl2.derivations_basis()]
sage: all(d(a.bracket(b)) == d(a).bracket(b) + a.bracket(d(b))
....:     for a in sl2.basis() for b in sl2.basis() for d in D)
True
```

REFERENCES:

Wikipedia article Derivation_(differential_algebra)

derived_series()

Return the derived series $(\mathfrak{g}^{(i)})_i$ of `self` where the rightmost $\mathfrak{g}^{(k)} = \mathfrak{g}^{(k+1)} = \cdots$.

We define the derived series of a Lie algebra $\mathfrak{g}$ recursively by $\mathfrak{g}^{(0)} := \mathfrak{g}$ and

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$$

and recall that $\mathfrak{g}^{(k)} \supseteq \mathfrak{g}^{(k+1)}$. Alternatively we can express this as

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq \bigl[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\bigr] \supseteq \Bigl[\bigl[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\bigr], \bigl[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\bigr]\Bigr] \supseteq \cdots .$$

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.derived_series()
(An example of a finite dimensional Lie algebra with basis:
    the 3-dimensional abelian Lie algebra over Rational Field,
 An example of a finite dimensional Lie algebra with basis:
    the 0-dimensional abelian Lie algebra over Rational Field
    with basis matrix: [])
```

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: L.derived_series()         # not implemented
(Lie algebra on 2 generators (x, y) over Rational Field,
 Subalgebra generated of
  Lie algebra on 2 generators (x, y) over Rational Field
  with basis: (x,),
 Subalgebra generated of
  Lie algebra on 2 generators (x, y) over Rational Field
  with basis: ())
```

derived_subalgebra()

Return the derived subalgebra of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.derived_subalgebra()
An example of a finite dimensional Lie algebra with basis:
 the 0-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[]
```

If `self` is semisimple, then the derived subalgebra is `self`:

```
sage: # needs sage.combinat sage.modules
sage: sl3 = LieAlgebra(QQ, cartan_type=['A', 2])
sage: sl3.derived_subalgebra()
Lie algebra of ['A', 2] in the Chevalley basis
sage: sl3 is sl3.derived_subalgebra()
True
```

from_vector(*v*, *order=None*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u                        #␣
↪needs sage.modules
(1, 0, 0)
sage: parent(u) is L                                                     #␣
↪needs sage.modules
True
```

homology(*deg=None*, *M=None*, *sparse=True*, *ncpus=None*)

Return the Lie algebra homology of `self`.

The Lie algebra homology is the homology of the Chevalley-Eilenberg chain complex.

INPUT:

- `deg` – the degree of the homology (optional)
- `M` – (default: the trivial module) a right module of `self`

- `sparse` – (default: `True`) whether to use sparse matrices for the Chevalley-Eilenberg chain complex
- `ncpus` – (optional) how many cpus to use when computing the Chevalley-Eilenberg chain complex

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.cross_product(QQ)
sage: L.homology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 0 over Rational Field,
 2: Vector space of dimension 0 over Rational Field,
 3: Vector space of dimension 1 over Rational Field}

sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.pwitt(GF(5), 5)
sage: L.homology()
{0: Vector space of dimension 1 over Finite Field of size 5,
 1: Vector space of dimension 0 over Finite Field of size 5,
 2: Vector space of dimension 1 over Finite Field of size 5,
 3: Vector space of dimension 1 over Finite Field of size 5,
 4: Vector space of dimension 0 over Finite Field of size 5,
 5: Vector space of dimension 1 over Finite Field of size 5}

sage: # needs sage.combinat sage.modules
sage: d = {('x', 'y'): {'y': 2}}
sage: L.<x,y> = LieAlgebra(ZZ, d)
sage: L.homology()
{0: Z, 1: Z x C2, 2: 0}
```

See also:

`chevalley_eilenberg_complex()`

`ideal`(**gens*, ***kwds*)

Return the ideal of `self` generated by `gens`.

INPUT:

- `gens` – a list of generators of the ideal
- `category` – (optional) a subcategory of subobjects of finite dimensional Lie algebras with basis

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: H = lie_algebras.Heisenberg(QQ, 2)
sage: p1,p2,q1,q2,z = H.basis()
sage: I = H.ideal([p1 - p2, q1 - q2])
sage: I.basis().list()
[-p1 + p2, -q1 + q2, z]
sage: I.reduce(p1 + p2 + q1 + q2 + z)
2*p1 + 2*q1
```

Passing an extra category to an ideal:

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y,z> = LieAlgebra(QQ, abelian=True)
sage: C = LieAlgebras(QQ).FiniteDimensional().WithBasis()
sage: C = C.Subobjects().Graded().Stratified()
sage: I = L.ideal(x, y, category=C)
sage: I.homogeneous_component_basis(1).list()
[x, y]
```

inner_derivations_basis()

Return a basis for the Lie algebra of inner derivations of `self` as matrices.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: H = lie_algebras.Heisenberg(QQ, 1)
sage: H.inner_derivations_basis()
(
[0 0 0]  [0 0 0]
[0 0 0]  [0 0 0]
[1 0 0], [0 1 0]
)
```

is_abelian()

Return if `self` is an abelian Lie algebra.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_abelian()
True
```

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: L.is_abelian()
False
```

is_ideal(*A*)

Return if `self` is an ideal of `A`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: I = L.ideal([2*a - c, b + c])
sage: I.is_ideal(L)
True

sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})                        #␣
↪needs sage.combinat sage.modules
sage: L.is_ideal(L)                                                        #␣
↪needs sage.combinat sage.modules
True

sage: F = LieAlgebra(QQ, 'F', representation='polynomial')                #␣
↪needs sage.combinat sage.modules
sage: L.is_ideal(F)                                                        #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
NotImplementedError: A must be a finite dimensional Lie algebra
 with basis
```

is_nilpotent()

Return if `self` is a nilpotent Lie algebra.

A Lie algebra is nilpotent if the lower central series eventually becomes 0.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_nilpotent()
True
```

is_semisimple()

Return if `self` if a semisimple Lie algebra.

A Lie algebra is semisimple if the solvable radical is zero. In characteristic 0, this is equivalent to saying the Killing form is non-degenerate.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_semisimple()
False
```

is_solvable()

Return if `self` is a solvable Lie algebra.

A Lie algebra is solvable if the derived series eventually becomes 0.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_solvable()
True
```

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: L.is_solvable()            # not implemented
False
```

killing_form(*x*, *y*)

Return the Killing form on `x` and `y`, where `x` and `y` are two elements of `self`.

The Killing form is defined as

$$\langle x \mid y \rangle = \operatorname{tr}\left(\operatorname{ad}_x \circ \operatorname{ad}_y\right).$$

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: a, b, c = L.lie_algebra_generators()                               #␣
↪needs sage.modules
sage: L.killing_form(a, b)                                               #␣
↪needs sage.modules
0
```

killing_form_matrix()

Return the matrix of the Killing form of `self`.

The rows and the columns of this matrix are indexed by the elements of the basis of `self` (in the order provided by `basis()`).

EXAMPLES:

```
sage: # needs sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.killing_form_matrix()
[0 0 0]
[0 0 0]
[0 0 0]
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example(0)
sage: m = L.killing_form_matrix(); m
[]
sage: parent(m)
Full MatrixSpace of 0 by 0 dense matrices over Rational Field
```

killing_matrix(*x*, *y*)

Return the Killing matrix of `x` and `y`, where `x` and `y` are two elements of `self`.

The Killing matrix is defined as the matrix corresponding to the action of $\mathrm{ad}_x \circ \mathrm{ad}_y$ in the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()     #␣
↪needs sage.modules
sage: a, b, c = L.lie_algebra_generators()                              #␣
↪needs sage.modules
sage: L.killing_matrix(a, b)                                            #␣
↪needs sage.modules
[0 0 0]
[0 0 0]
[0 0 0]
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})                    #␣
↪needs sage.combinat sage.modules
sage: L.killing_matrix(y, x)                                            #␣
↪needs sage.combinat sage.modules
[ 0 -1]
[ 0  0]
```

lower_central_series(*submodule=False*)

Return the lower central series $(\mathfrak{g}_i)_i$ of `self` where the rightmost $\mathfrak{g}_k = \mathfrak{g}_{k+1} = \cdots$.

INPUT:

- `submodule` – (default: `False`) if `True`, then the result is given as submodules of `self`

We define the lower central series of a Lie algebra $\mathfrak{g}$ recursively by $\mathfrak{g}_0 := \mathfrak{g}$ and

$$\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k]$$

and recall that $\mathfrak{g}_k \supseteq \mathfrak{g}_{k+1}$. Alternatively we can express this as

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq \bigl[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}\bigr] \supseteq \Bigl[\bigl[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}\bigr], \mathfrak{g}\Bigr] \supseteq \cdots .$$

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.derived_series()
(An example of a finite dimensional Lie algebra with basis:
  the 3-dimensional abelian Lie algebra over Rational Field,
 An example of a finite dimensional Lie algebra with basis:
  the 0-dimensional abelian Lie algebra over Rational Field
   with basis matrix: [])
```

The lower central series as submodules:

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: L.lower_central_series(submodule=True)
(Sparse vector space of dimension 2 over Rational Field,
 Vector space of degree 2 and dimension 1 over Rational Field
  Basis matrix: [1 0])
```

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: L.lower_central_series()  # not implemented
(Lie algebra on 2 generators (x, y) over Rational Field,
 Subalgebra generated of
  Lie algebra on 2 generators (x, y) over Rational Field
  with basis: (x,))
```

module(*R=None*)

Return a dense free module associated to `self` over `R`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()     #␣
↪needs sage.modules
sage: L._dense_free_module()                                            #␣
↪needs sage.modules
Vector space of dimension 3 over Rational Field
```

morphism(*on_generators*, *codomain=None*, *base_map=None*, *check=True*)

Return a Lie algebra morphism defined by images of a Lie generating subset of `self`.

INPUT:

- `on_generators` – dictionary `{X: Y}` of the images Y in `codomain` of elements X of `domain`
- `codomain` – a Lie algebra (optional); this is inferred from the values of `on_generators` if not given
- `base_map` – a homomorphism from the base ring to something coercing into the codomain
- `check` – (default: `True`) boolean; if `False` the values on the Lie brackets implied by `on_generators` will not be checked for contradictory values

Note: The keys of `on_generators` need to generate `domain` as a Lie algebra.

See also:

`sage.algebras.lie_algebras.morphism.LieAlgebraMorphism_from_generators`

EXAMPLES:

A quotient type Lie algebra morphism

```
sage: # needs sage.combinat sage.modules
sage: L.<X,Y,Z,W> = LieAlgebra(QQ, {('X','Y'): {'Z': 1},
....:                                ('X','Z'): {'W': 1}})
sage: K.<A,B> = LieAlgebra(QQ, abelian=True)
sage: L.morphism({X: A, Y: B})
Lie algebra morphism:
  From: Lie algebra on 4 generators (X, Y, Z, W) over Rational Field
  To:   Abelian Lie algebra on 2 generators (A, B) over Rational Field
  Defn: X |--> A
        Y |--> B
        Z |--> 0
        W |--> 0
```

The reverse map $A \mapsto X$, $B \mapsto Y$ does not define a Lie algebra morphism, since $[A, B] = 0$, but $[X, Y] \neq 0$:

```
sage: # needs sage.combinat sage.modules
sage: K.morphism({A:X, B: Y})
Traceback (most recent call last):
...
ValueError: this does not define a Lie algebra morphism;
 contradictory values for brackets of length 2
```

However, it is still possible to create a morphism that acts nontrivially on the coefficients, even though it's not a Lie algebra morphism (since it isn't linear):

```
sage: # needs sage.combinat sage.modules sage.rings.number_fields
sage: R.<x> = ZZ[]
sage: K.<i> = NumberField(x^2 + 1)
sage: cc = K.hom([-i])
sage: L.<X,Y,Z,W> = LieAlgebra(K, {('X','Y'): {'Z': 1},
....:                               ('X','Z'): {'W': 1}})
sage: M.<A,B> = LieAlgebra(K, abelian=True)
sage: phi = L.morphism({X: A, Y: B}, base_map=cc)
sage: phi(X)
A
sage: phi(i*X)
-i*A
```

product_space(*L*, *submodule=False*)

Return the product space `[self, L]`.

INPUT:

- `L` – a Lie subalgebra of `self`
- `submodule` – (default: `False`) if `True`, then the result is forced to be a submodule of `self`

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a,b,c = L.lie_algebra_generators()
sage: X = L.subalgebra([a, b + c])
sage: L.product_space(X)
An example of a finite dimensional Lie algebra with basis:
 the 0-dimensional abelian Lie algebra over Rational Field
  with basis matrix: []
sage: Y = L.subalgebra([a, 2*b - c])
```

(continues on next page)

(continued from previous page)

```
sage: X.product_space(Y)
An example of a finite dimensional Lie algebra with basis:
 the 0-dimensional abelian Lie algebra over Rational Field
  with basis matrix: []
```

```
sage: # needs sage.combinat sage.modules
sage: H = lie_algebras.Heisenberg(ZZ, 4)
sage: Hp = H.product_space(H, submodule=True).basis()
sage: [H.from_vector(v) for v in Hp]
[z]
```

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: Lp = L.product_space(L)    # not implemented
sage: Lp                         # not implemented
Subalgebra generated of
 Lie algebra on 2 generators (x, y) over Rational Field
 with basis: (x,)
sage: Lp.product_space(L)        # not implemented
Subalgebra generated of
 Lie algebra on 2 generators (x, y) over Rational Field
 with basis: (x,)
sage: L.product_space(Lp)        # not implemented
Subalgebra generated of
 Lie algebra on 2 generators (x, y) over Rational Field
 with basis: (x,)
sage: Lp.product_space(Lp)       # not implemented
Subalgebra generated of
 Lie algebra on 2 generators (x, y) over Rational Field
 with basis: ()
```

quotient(*I*, *names=None*, *category=None*)

Return the quotient of `self` by the ideal `I`.

A quotient Lie algebra.

INPUT:

- `I` – an ideal or a list of generators of the ideal
- `names` – (optional) a string or a list of strings; names for the basis elements of the quotient. If `names` is a string, the basis will be named `names_1`,...,``names_n``.

EXAMPLES:

The Engel Lie algebra as a quotient of the free nilpotent Lie algebra of step 3 with 2 generators:

```
sage: # needs sage.combinat sage.modules
sage: L.<X,Y,Z,W,U> = LieAlgebra(QQ, 2, step=3)
sage: E = L.quotient(U); E
Lie algebra quotient L/I of dimension 4 over Rational Field where
 L: Free Nilpotent Lie algebra on 5 generators (X, Y, Z, W, U)
    over Rational Field
 I: Ideal (U)
sage: E.basis().list()
[X, Y, Z, W]
sage: E(X).bracket(E(Y))
Z
sage: Y.bracket(Z)
```

(continues on next page)

(continued from previous page)

```
-U
sage: E(Y).bracket(E(Z))
0
sage: E(U)
0
```

Quotients when the base ring is not a field are not implemented:

```
sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.Heisenberg(ZZ, 1)
sage: L.quotient(L.an_element())
Traceback (most recent call last):
...
NotImplementedError: quotients over non-fields not implemented
```

structure_coefficients(*include_zeros=False*)

Return the structure coefficients of `self`.

INPUT:

- `include_zeros` – (default: `False`) if `True`, then include the $[x, y] = 0$ pairs in the output

OUTPUT:

A dictionary whose keys are pairs of basis indices (i, j) with $i < j$, and whose values are the corresponding *elements* $[b_i, b_j]$ in the Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.structure_coefficients()                                          #␣
↪needs sage.modules
Finite family {}
sage: L.structure_coefficients(True)                                      #␣
↪needs sage.modules
Finite family {(0, 1): (0, 0, 0), (0, 2): (0, 0, 0), (1, 2): (0, 0, 0)}
```

```
sage: # needs sage.combinat sage.groups sage.modules
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L = LieAlgebra(associative=S)
sage: L.structure_coefficients()
Finite family {((2,3), (1,2)): (1,2,3) - (1,3,2),
               ((2,3), (1,3)): -(1,2,3) + (1,3,2),
               ((1,2,3), (2,3)): -(1,2) + (1,3),
               ((1,2,3), (1,2)): (2,3) - (1,3),
               ((1,2,3), (1,3)): -(2,3) + (1,2),
               ((1,3,2), (2,3)): (1,2) - (1,3),
               ((1,3,2), (1,2)): -(2,3) + (1,3),
               ((1,3,2), (1,3)): (2,3) - (1,2),
               ((1,3), (1,2)): -(1,2,3) + (1,3,2)}
```

subalgebra(**gens*, ***kwds*)

Return the subalgebra of `self` generated by `gens`.

INPUT:

- `gens` – a list of generators of the subalgebra
- `category` – (optional) a subcategory of subobjects of finite dimensional Lie algebras with basis

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: H = lie_algebras.Heisenberg(QQ, 2)
sage: p1,p2,q1,q2,z = H.basis()
sage: S = H.subalgebra([p1, q1])
sage: S.basis().list()
[p1, q1, z]
sage: S.basis_matrix()
[1 0 0 0 0]
[0 0 1 0 0]
[0 0 0 0 1]
```

Passing an extra category to a subalgebra:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, 3, step=2)
sage: x,y,z = L.homogeneous_component_basis(1)
sage: C = LieAlgebras(QQ).FiniteDimensional().WithBasis()
sage: C = C.Subobjects().Graded().Stratified()
sage: S = L.subalgebra([x, y], category=C)
sage: S.homogeneous_component_basis(2).list()
[X_12]
```

universal_commutative_algebra()

Return the universal commutative algebra associated to `self`.

Let I be the index set of the basis of `self`. Let $\mathcal{P} = \{P_{a,i,j}\}_{a,i,j\in I}$ denote the universal polynomials of a Lie algebra L. The *universal commutative algebra* associated to L is the quotient ring $R[X_{ij}]_{i,j\in I}/(\mathcal{P})$.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: A = L.universal_commutative_algebra()
sage: a, b, c, d = A.gens()
sage: a, b, c, d
(X00bar, X01bar, 0, X11bar)
sage: a*d - a
0
```

universal_polynomials()

Return the family of universal polynomials of `self`.

The *universal polynomials* of a Lie algebra L with basis $\{e_i\}_{i\in I}$ and structure coefficients $[e_i, e_j] = \tau_{ij}^a e_a$ is given by

$$P_{aij} = \sum_{u\in I} \tau_{ij}^u X_{au} - \sum_{s,t\in I} \tau_{st}^a X_{si} X_{tj},$$

where $a, i, j \in I$.

REFERENCES:

- [AM2020]

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
```

(continues on next page)

(continued from previous page)

```
sage: L.universal_polynomials()
Finite family {('x', 'x', 'y'): X01*X10 - X00*X11 + X00,
               ('y', 'x', 'y'): X10}

sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['A',1])
sage: list(L.universal_polynomials())
[-2*X01*X10 + 2*X00*X11 - 2*X00,
 -2*X02*X10 + 2*X00*X12 + X01,
 -2*X02*X11 + 2*X01*X12 - 2*X02,
 X01*X20 - X00*X21 - 2*X10,
 X02*X20 - X00*X22 + X11,
 X02*X21 - X01*X22 - 2*X12,
 -2*X11*X20 + 2*X10*X21 - 2*X20,
 -2*X12*X20 + 2*X10*X22 + X21,
 -2*X12*X21 + 2*X11*X22 - 2*X22]

sage: # long time, needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type=['B', 2])
sage: al = RootSystem(['B', 2]).root_lattice().simple_roots()
sage: k = list(L.basis().keys())[0]
sage: UP = L.universal_polynomials()
sage: len(UP)
450
sage: UP[al[2], al[1], -al[1]]
X0_7*X4_1 - X0_1*X4_7 - 2*X0_7*X5_1 + 2*X0_1*X5_7 + X2_7*X7_1
 - X2_1*X7_7 - X3_7*X8_1 + X3_1*X8_7 + X0_4
```

class Subobjects(*category*, **args*)

Bases: `SubobjectsCategory`

A category for subalgebras of a finite dimensional Lie algebra with basis.

class ParentMethods

Bases: `object`

ambient()

Return the ambient Lie algebra of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: C = LieAlgebras(QQ).FiniteDimensional().WithBasis()
sage: L = C.example()
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a + b, b + c])
sage: S.ambient() == L
True
```

basis_matrix()

Return the basis matrix of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: C = LieAlgebras(QQ).FiniteDimensional().WithBasis()
sage: L = C.example()
```

(continues on next page)

(continued from previous page)

```
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a + b, b + c])
sage: S.basis_matrix()
[   1    0 -1/2]
[   0    1    1]
```

example (*n=3*)

Return an example of a finite dimensional Lie algebra with basis as per *Category.example*.

EXAMPLES:

```
sage: C = LieAlgebras(QQ).FiniteDimensional().WithBasis()
sage: C.example()                                                              #␣
↪needs sage.modules
An example of a finite dimensional Lie algebra with basis:
 the 3-dimensional abelian Lie algebra over Rational Field
```

Other dimensions can be specified as an optional argument:

```
sage: C.example(5)                                                             #␣
↪needs sage.modules
An example of a finite dimensional Lie algebra with basis:
 the 5-dimensional abelian Lie algebra over Rational Field
```

4.53 Finite dimensional modules with basis

class sage.categories.finite_dimensional_modules_with_basis.**FiniteDimensionalModulesWithBas**

Bases: *CategoryWithAxiom_over_base_ring*

The category of finite dimensional modules with a distinguished basis

EXAMPLES:

```
sage: C = FiniteDimensionalModulesWithBasis(ZZ); C
Category of finite dimensional modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of finite dimensional modules over Integer Ring,
 Category of modules with basis over Integer Ring]
sage: C is Modules(ZZ).WithBasis().FiniteDimensional()
True
```

class ElementMethods

Bases: `object`

dense_coefficient_list (*order=None*)

Return a list of *all* coefficients of `self`.

By default, this list is ordered in the same way as the indexing set of the basis of the parent of `self`.

INPUT:

- `order` – (optional) an ordering of the basis indexing set

EXAMPLES:

```
sage: # needs sage.modules
sage: v = vector([0, -1, -3])
sage: v.dense_coefficient_list()
[0, -1, -3]
sage: v.dense_coefficient_list([2,1,0])
[-3, -1, 0]
sage: sorted(v.coefficients())
[-3, -1]
```

class MorphismMethods

Bases: `object`

image()

Return the image of `self` as a submodule of the codomain.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)                                    #␣
↪needs sage.groups sage.modules
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)            #␣
↪needs sage.groups sage.modules
sage: f.image()                                                             #␣
↪needs sage.groups sage.modules
Free module generated by {0, 1, 2} over Rational Field
```

image_basis()

Return a basis for the image of `self` in echelon form.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)                                    #␣
↪needs sage.groups sage.modules
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)            #␣
↪needs sage.groups sage.modules
sage: f.image_basis()                                                       #␣
↪needs sage.groups sage.modules
([1, 2, 3], [2, 3, 1], [3, 1, 2])
```

kernel()

Return the kernel of `self` as a submodule of the domain.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: SGA = SymmetricGroupAlgebra(QQ, 3)
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)
sage: K = f.kernel()
sage: K
Free module generated by {0, 1, 2} over Rational Field
sage: K.ambient()
Symmetric group algebra of order 3 over Rational Field
```

kernel_basis()

Return a basis of the kernel of `self` in echelon form.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)                                    #␣
↪needs sage.groups sage.modules
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)            #␣
↪needs sage.groups sage.modules
sage: f.kernel_basis()                                                      #␣
↪needs sage.groups sage.modules
([1, 2, 3] - [3, 2, 1], [1, 3, 2] - [3, 2, 1], [2, 1, 3] - [3, 2, 1])
```

matrix(*base_ring=None*, *side='left'*)

Return the matrix of this morphism in the distinguished bases of the domain and codomain.

INPUT:

- `base_ring` – a ring (default: `None`, meaning the base ring of the codomain)
- `side` – "left" or "right" (default: "left")

If `side` is "left", this morphism is considered as acting on the left; i.e. each column of the matrix represents the image of an element of the basis of the domain.

The order of the rows and columns matches with the order in which the bases are enumerated.

See also:

`Modules.WithBasis.ParentMethods.module_morphism()`

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(ZZ, [1,2]); x = X.basis()
sage: Y = CombinatorialFreeModule(ZZ, [3,4]); y = Y.basis()
sage: phi = X.module_morphism(on_basis={1: y[3] + 3*y[4],
....:                                   2: 2*y[3] + 5*y[4]}.__getitem__,
....:                         codomain=Y)
sage: phi.matrix()
[1 2]
[3 5]
sage: phi.matrix(side="right")
[1 3]
[2 5]

sage: phi.matrix().parent()                                                 #␣
↪needs sage.modules
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: phi.matrix(QQ).parent()                                               #␣
↪needs sage.modules
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

The resulting matrix is immutable:

```
sage: phi.matrix().is_mutable()                                             #␣
↪needs sage.modules
False
```

The zero morphism has a zero matrix:

```
sage: Hom(X, Y).zero().matrix()                                             #␣
↪needs sage.modules
[0 0]
[0 0]
```

Todo: Add support for morphisms where the codomain has a different base ring than the domain:

```
sage: Y = CombinatorialFreeModule(QQ, [3,4]); y = Y.basis()          #␣
↪needs sage.modules
sage: phi = X.module_morphism(on_basis={1: y[3] + 3*y[4],            #␣
↪needs sage.modules
....:                                   2: 2*y[3] + 5/2*y[4]}.__getitem__,
....:                         codomain=Y)
sage: phi.matrix().parent()          # not implemented               #␣
↪needs sage.modules
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

This currently does not work because, in this case, the morphism is just in the category of commutative additive groups (i.e. the intersection of the categories of modules over **Z** and over **Q**):

```
sage: phi.parent().homset_category()                                 #␣
↪needs sage.modules
Category of commutative additive semigroups
sage: phi.parent().homset_category()        # not implemented, needs sage.
↪modules
Category of finite dimensional modules with basis over Integer Ring
```

class ParentMethods

Bases: `object`

annihilator(*S*, *action=<built-in function mul>*, *side='right'*, *category=None*)

Return the annihilator of a finite set.

INPUT:

- `S` – a finite set
- `action` – a function (default: `operator.mul`)
- `side` – 'left' or 'right' (default: 'right')
- `category` – a category

Assumptions:

- `action` takes elements of `self` as first argument and elements of `S` as second argument;
- The codomain is any vector space, and `action` is linear on its first argument; typically it is bilinear;
- If `side` is 'left', this is reversed.

OUTPUT:

The subspace of the elements x of `self` such that `action(x,s) = 0` for all $s \in S$. If `side` is 'left' replace the above equation by `action(s,x) = 0`.

If `self` is a ring, `action` an action of `self` on a module M and S is a subset of M, we recover the Wikipedia article Annihilator_%28ring_theory%29. Similarly this can be used to compute torsion or orthogonals.

See also:

`annihilator_basis()` for lots of examples.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = FiniteDimensionalAlgebrasWithBasis(QQ).example(); F
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
```

(continues on next page)

(continued from previous page)

```
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x, y, a, b = F.basis()
sage: A = F.annihilator([a + 3*b + 2*y]); A
Free module generated by {0} over Rational Field
sage: [b.lift() for b in A.basis()]
[-1/2*a - 3/2*b + x]
```

The category can be used to specify other properties of this subspace, like that this is a subalgebra:

```
sage: # needs sage.modules
sage: center = F.annihilator(F.basis(), F.bracket,
....:                        category=Algebras(QQ).Subobjects())
sage: (e,) = center.basis()
sage: e.lift()
x + y
sage: e * e == e
True
```

Taking annihilator is order reversing for inclusion:

```
sage: # needs sage.modules
sage: A   = F.annihilator([]);    A  .rename("A")
sage: Ax  = F.annihilator([x]);   Ax .rename("Ax")
sage: Ay  = F.annihilator([y]);   Ay .rename("Ay")
sage: Axy = F.annihilator([x,y]); Axy.rename("Axy")
sage: P = Poset(([A, Ax, Ay, Axy], attrcall("is_submodule")))              #␣
↪needs sage.combinat sage.graphs
sage: sorted(P.cover_relations(), key=str)                                 #␣
↪needs sage.combinat sage.graphs
[[Ax, A], [Axy, Ax], [Axy, Ay], [Ay, A]]
```

annihilator_basis(*S*, *action=<built-in function mul>*, *side='right'*)

Return a basis of the annihilator of a finite set of elements.

INPUT:

- `S` – a finite set of objects
- `action` – a function (default: `operator.mul`)
- `side` – 'left' or 'right' (default: 'right'): on which side of `self` the elements of S acts.

See *annihilator()* for the assumptions and definition of the annihilator.

EXAMPLES:

By default, the action is the standard $*$ operation. So our first example is about an algebra:

```
sage: # needs sage.graphs sage.modules
sage: F = FiniteDimensionalAlgebrasWithBasis(QQ).example(); F
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x,y,a,b = F.basis()
```

In this algebra, multiplication on the right by x annihilates all basis elements but x:

```
sage: x*x, y*x, a*x, b*x                                                   #␣
↪needs sage.graphs sage.modules
(x, 0, 0, 0)
```

So the annihilator is the subspace spanned by y, a, and b:

```
sage: F.annihilator_basis([x])                                         #␣
↪needs sage.graphs sage.modules
(y, a, b)
```

The same holds for a and b:

```
sage: x*a, y*a, a*a, b*a                                               #␣
↪needs sage.graphs sage.modules
(a, 0, 0, 0)
sage: F.annihilator_basis([a])                                         #␣
↪needs sage.graphs sage.modules
(y, a, b)
```

On the other hand, y annihilates only x:

```
sage: F.annihilator_basis([y])                                         #␣
↪needs sage.graphs sage.modules
(x,)
```

Here is a non trivial annihilator:

```
sage: F.annihilator_basis([a + 3*b + 2*y])                             #␣
↪needs sage.graphs sage.modules
(-1/2*a - 3/2*b + x,)
```

Let's check it:

```
sage: (-1/2*a - 3/2*b + x) * (a + 3*b + 2*y)                           #␣
↪needs sage.graphs sage.modules
0
```

Doing the same calculations on the left exchanges the roles of x and y:

```
sage: # needs sage.graphs sage.modules
sage: F.annihilator_basis([y], side="left")
(x, a, b)
sage: F.annihilator_basis([a], side="left")
(x, a, b)
sage: F.annihilator_basis([b], side="left")
(x, a, b)
sage: F.annihilator_basis([x], side="left")
(y,)
sage: F.annihilator_basis([a + 3*b + 2*x], side="left")
(-1/2*a - 3/2*b + y,)
```

By specifying an inner product, this method can be used to compute the orthogonal of a subspace:

```
sage: # needs sage.graphs sage.modules
sage: x,y,a,b = F.basis()
sage: def scalar(u,v):
....:     return vector([sum(u[i]*v[i] for i in F.basis().keys())])
sage: F.annihilator_basis([x + y, a + b], scalar)
(x - y, a - b)
```

By specifying the standard Lie bracket as action, one can compute the commutator of a subspace of F:

```
sage: F.annihilator_basis([a + b], action=F.bracket)                    #␣
↪needs sage.graphs sage.modules
(x + y, a, b)
```

In particular one can compute a basis of the center of the algebra. In our example, it is reduced to the identity:

```
sage: F.annihilator_basis(F.algebra_generators(), action=F.bracket)     #␣
↪needs sage.graphs sage.modules
(x + y,)
```

But see also *FiniteDimensionalAlgebrasWithBasis.ParentMethods.center_basis()*.

echelon_form(*elements*, *row_reduced=False*, *order=None*)

Return a basis in echelon form of the subspace spanned by a finite set of elements.

INPUT:

- `elements` – a list or finite iterable of elements of `self`
- `row_reduced` – (default: `False`) whether to compute the basis for the row reduced echelon form
- `order` – (optional) either something that can be converted into a tuple or a key function

OUTPUT:

A list of elements of `self` whose expressions as vectors form a matrix in echelon form. If `base_ring` is specified, then the calculation is achieved in this base ring.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: V = X.echelon_form([x[0]-x[1], x[0]-x[2], x[1]-x[2]]); V
[x[0] - x[2], x[1] - x[2]]
sage: matrix(list(map(vector, V)))
[ 1  0 -1]
[ 0  1 -1]
```

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(ZZ, [1,2,3,4])
sage: B = F.basis()
sage: elements = [B[1]-17*B[2]+6*B[3], B[1]-17*B[2]+B[4]]
sage: F.echelon_form(elements)
[B[1] - 17*B[2] + B[4], 6*B[3] - B[4]]
```

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])                    #␣
↪needs sage.modules
sage: a,b,c = F.basis()                                                 #␣
↪needs sage.modules
sage: F.echelon_form([8*a+b+10*c, -3*a+b-c, a-b-c])                     #␣
↪needs sage.modules
[B['a'] + B['c'], B['b'] + 2*B['c']]
```

```
sage: R.<x,y> = QQ[]
sage: C = CombinatorialFreeModule(R, range(3), prefix='x')              #␣
↪needs sage.modules
```

(continues on next page)

(continued from previous page)

```
sage: x = C.basis()                                                      #␣
↪needs sage.modules
sage: C.echelon_form([x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]])        #␣
↪needs sage.modules sage.rings.function_field
[x[0] - x[2], x[1] - x[2]]
```

```
sage: M = MatrixSpace(QQ, 3, 3)                                          #␣
↪needs sage.modules
sage: A = M([[0, 0, 2], [0, 0, 0], [0, 0, 0]])                           #␣
↪needs sage.modules
sage: M.echelon_form([A, A])                                             #␣
↪needs sage.modules
[
[0 0 1]
[0 0 0]
[0 0 0]
]
```

from_vector(*vector*, *order=None*, *coerce=True*)

Build an element of `self` from a vector.

EXAMPLES:

```
sage: # needs sage.modules
sage: p_mult = matrix([[0,0,0], [0,0,-1], [0,0,0]])
sage: q_mult = matrix([[0,0,1], [0,0,0], [0,0,0]])
sage: A = algebras.FiniteDimensional(
....:         QQ, [p_mult, q_mult, matrix(QQ, 3, 3)], 'p,q,z')
sage: A.from_vector(vector([1,0,2]))
p + 2*z
```

gens()

Return the generators of `self`.

OUTPUT:

A tuple containing the basis elements of `self`.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(ZZ, ['a', 'b', 'c'])                   #␣
↪needs sage.modules
sage: F.gens()                                                           #␣
↪needs sage.modules
(B['a'], B['b'], B['c'])
```

invariant_module(*S*, *action=<built-in function mul>*, *action_on_basis=None*, *side='left'*, ***kwargs*)

Return the submodule of `self` invariant under the action of `S`.

For a semigroup S acting on a module M, the invariant submodule is given by

$$M^S = \{m \in M : s \cdot m = m, \forall s \in S\}.$$

INPUT:

- `S` – a finitely-generated semigroup
- `action` – a function (default: `operator.mul`)
- `side` – `'left'` or `'right'` (default: `'right'`); which side of `self` the elements of `S` acts

- `action_on_basis` – (optional) define the action of `S` on the basis of `self`

OUTPUT:

- `FiniteDimensionalInvariantModule`

EXAMPLES:

We build the invariant module of the permutation representation of the symmetric group:

```
sage: # needs sage.groups sage.modules
sage: G = SymmetricGroup(3); G.rename('S3')
sage: M = FreeModule(ZZ, [1,2,3], prefix='M'); M.rename('M')
sage: action = lambda g, x: M.term(g(x))
sage: I = M.invariant_module(G, action_on_basis=action); I
(S3)-invariant submodule of M
sage: I.basis()
Finite family {0: B[0]}
sage: [I.lift(b) for b in I.basis()]
[M[1] + M[2] + M[3]]
sage: G.rename(); M.rename()  # reset the names
```

We can construct the invariant module of any module that has an action of `S`. In this example, we consider the dihedral group $G = D_4$ and the subgroup $H < G$ of all rotations. We construct the H-invariant module of the group algebra $\mathbf{Q}[G]$:

```
sage: # needs sage.groups
sage: G = groups.permutation.Dihedral(4)
sage: H = G.subgroup(G.gen(0))
sage: H
Subgroup generated by [(1,2,3,4)]
 of (Dihedral group of order 8 as a permutation group)
sage: H.cardinality()
4

sage: # needs sage.groups sage.modules
sage: A = G.algebra(QQ)
sage: I = A.invariant_module(H)
sage: [I.lift(b) for b in I.basis()]
[() + (1,2,3,4) + (1,3)(2,4) + (1,4,3,2),
 (2,4) + (1,2)(3,4) + (1,3) + (1,4)(2,3)]
sage: all(h * I.lift(b) == I.lift(b)
....:     for b in I.basis() for h in H)
True
```

`twisted_invariant_module`(*G*, *chi*, *action=<built-in function mul>*, *action_on_basis=None*, *side='left'*, ***kwargs*)

Create the isotypic component of the action of `G` on `self` with irreducible character given by `chi`.

See also:

-`FiniteDimensionalTwistedInvariantModule`

INPUT:

- `G` – a finitely-generated group
- `chi` – a list/tuple of character values or an instance of `ClassFunction_gap`
- `action` – a function (default: `operator.mul`)
- `action_on_basis` – (optional) define the action of `g` on the basis of `self`
- `side` – `'left'` or `'right'` (default: `'right'`); which side of `self` the elements of `S` acts

OUTPUT:

- `FiniteDimensionalTwistedInvariantModule`

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: M = CombinatorialFreeModule(QQ, [1,2,3])
sage: G = SymmetricGroup(3)
sage: def action(g,x): return(M.term(g(x)))   # permute coordinates
sage: T = M.twisted_invariant_module(G, [2,0,-1],
....:                                action_on_basis=action)
sage: import __main__; __main__.action = action
sage: TestSuite(T).run()
```

class TensorProducts(*category*, **args*)

Bases: `TensorProductsCategory`

extra_super_categories()

Implement the fact that a (finite) tensor product of finite dimensional modules is a finite dimensional module.

EXAMPLES:

```
sage: C = ModulesWithBasis(ZZ).FiniteDimensional().TensorProducts()
sage: C.extra_super_categories()
[Category of finite dimensional modules with basis over Integer Ring]
sage: C.FiniteDimensional()
Category of tensor products of
 finite dimensional modules with basis over Integer Ring
```

4.54 Finite Dimensional Nilpotent Lie Algebras With Basis

AUTHORS:

- Eero Hakavuori (2018-08-16): initial version

class sage.categories.finite_dimensional_nilpotent_lie_algebras_with_basis.**FiniteDimensiona**

Bases: `CategoryWithAxiom_over_base_ring`

Category of finite dimensional nilpotent Lie algebras with basis.

class ParentMethods

Bases: `object`

is_nilpotent()

Return `True` since `self` is nilpotent.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, {('x','y'): {'z': 1}}, nilpotent=True)          #␣
↪needs sage.combinat sage.modules
sage: L.is_nilpotent()                                                   #␣
↪needs sage.combinat sage.modules
True
```

lie_group(*name='G'*, ***kwds*)

Return the Lie group associated to `self`.

INPUT:

- `name` – string (default: `'G'`); the name (symbol) given to the Lie group

EXAMPLES:

We define the Heisenberg group:

```
sage: L = lie_algebras.Heisenberg(QQ, 1)                                    #␣
↪needs sage.combinat sage.modules
sage: G = L.lie_group('G'); G                                               #␣
↪needs sage.combinat sage.modules sage.symbolic
Lie group G of Heisenberg algebra of rank 1 over Rational Field
```

We test multiplying elements of the group:

```
sage: # needs sage.combinat sage.modules sage.symbolic
sage: p, q, z = L.basis()
sage: g = G.exp(p); g
exp(p1)
sage: h = G.exp(q); h
exp(q1)
sage: g * h
exp(p1 + q1 + 1/2*z)
```

We extend an element of the Lie algebra to a left-invariant vector field:

```
sage: X = G.left_invariant_extension(2*p + 3*q, name='X'); X                #␣
↪needs sage.combinat sage.modules sage.symbolic
Vector field X on the Lie group G of
 Heisenberg algebra of rank 1 over Rational Field
sage: X.at(G.one()).display()                                               #␣
↪needs sage.combinat sage.modules sage.symbolic
X = 2 ∂/∂x_0 + 3 ∂/∂x_1
sage: X.display()                                                           #␣
↪needs sage.combinat sage.modules sage.symbolic
X = 2 ∂/∂x_0 + 3 ∂/∂x_1 + (3/2*x_0 - x_1) ∂/∂x_2
```

See also:

`NilpotentLieGroup`

step()

Return the nilpotency step of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, {('X','Y'): {'Z': 1}}, nilpotent=True)
sage: L.step()
2
sage: sc = {('X','Y'): {'Z': 1}, ('X','Z'): {'W': 1}}
sage: LieAlgebra(QQ, sc, nilpotent=True).step()
3
```

4.55 Finite dimensional semisimple algebras with basis

class sage.categories.finite_dimensional_semisimple_algebras_with_basis.**FiniteDimensionalSe**

Bases: *CategoryWithAxiom_over_base_ring*

The category of finite dimensional semisimple algebras with a distinguished basis

EXAMPLES:

```
sage: from sage.categories.finite_dimensional_semisimple_algebras_with_basis␣
↪import FiniteDimensionalSemisimpleAlgebrasWithBasis
sage: C = FiniteDimensionalSemisimpleAlgebrasWithBasis(QQ); C
Category of finite dimensional semisimple algebras with basis over Rational Field
```

This category is best constructed as:

```
sage: D = Algebras(QQ).Semisimple().FiniteDimensional().WithBasis(); D
Category of finite dimensional semisimple algebras with basis over Rational Field
sage: D is C
True
```

class Commutative(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class ParentMethods

Bases: object

central_orthogonal_idempotents()

Return the central orthogonal idempotents of this semisimple commutative algebra.

Those idempotents form a maximal decomposition of the identity into primitive orthogonal idempotents.

OUTPUT:

A list of orthogonal idempotents of self.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: A4 = SymmetricGroup(4).algebra(QQ)
sage: Z4 = A4.center()
sage: idempotents = Z4.central_orthogonal_idempotents()
sage: idempotents
(1/24*B[0] + 1/24*B[1] + 1/24*B[2] + 1/24*B[3] + 1/24*B[4],
 3/8*B[0] + 1/8*B[1] - 1/8*B[2] - 1/8*B[4],
 1/6*B[0] + 1/6*B[2] - 1/12*B[3],
 3/8*B[0] - 1/8*B[1] - 1/8*B[2] + 1/8*B[4],
 1/24*B[0] - 1/24*B[1] + 1/24*B[2] + 1/24*B[3] - 1/24*B[4])
```

Lifting those idempotents from the center, we recognize among them the sum and alternating sum of all permutations:

```
sage: [e.lift() for e in idempotents]                                   #␣
↪needs sage.groups sage.modules
[1/24*() + 1/24*(3,4) + 1/24*(2,3) + 1/24*(2,3,4) + 1/24*(2,4,3)
```

(continues on next page)

(continued from previous page)

```
+ 1/24*(2,4) + 1/24*(1,2) + 1/24*(1,2)(3,4) + 1/24*(1,2,3)
+ 1/24*(1,2,3,4) + 1/24*(1,2,4,3) + 1/24*(1,2,4) + 1/24*(1,3,2)
+ 1/24*(1,3,4,2) + 1/24*(1,3) + 1/24*(1,3,4) + 1/24*(1,3)(2,4)
+ 1/24*(1,3,2,4) + 1/24*(1,4,3,2) + 1/24*(1,4,2) + 1/24*(1,4,3)
+ 1/24*(1,4) + 1/24*(1,4,2,3) + 1/24*(1,4)(2,3),
...,
1/24*() - 1/24*(3,4) - 1/24*(2,3) + 1/24*(2,3,4) + 1/24*(2,4,3)
- 1/24*(2,4) - 1/24*(1,2) + 1/24*(1,2)(3,4) + 1/24*(1,2,3)
- 1/24*(1,2,3,4) - 1/24*(1,2,4,3) + 1/24*(1,2,4) + 1/24*(1,3,2)
- 1/24*(1,3,4,2) - 1/24*(1,3) + 1/24*(1,3,4) + 1/24*(1,3)(2,4)
- 1/24*(1,3,2,4) - 1/24*(1,4,3,2) + 1/24*(1,4,2) + 1/24*(1,4,3)
- 1/24*(1,4) - 1/24*(1,4,2,3) + 1/24*(1,4)(2,3)]
```

We check that they indeed form a decomposition of the identity of Z_4 into orthogonal idempotents:

```
sage: Z4.is_identity_decomposition_into_orthogonal_
↪idempotents(idempotents)      # needs sage.groups sage.modules
True
```

class ParentMethods

Bases: `object`

central_orthogonal_idempotents()

Return a maximal list of central orthogonal idempotents of `self`.

Central orthogonal idempotents of an algebra A are idempotents $(e_1, \dots, e_n)$ in the center of A such that $e_i e_j = 0$ whenever $i \neq j$.

With the maximality condition, they sum up to 1 and are uniquely determined (up to order).

EXAMPLES:

For the algebra of the (abelian) alternating group A_3, we recover three idempotents corresponding to the three one-dimensional representations V_i on which $(1, 2, 3)$ acts on V_i as multiplication by the i-th power of a cube root of unity:

```
sage: # needs sage.groups sage.rings.number_field
sage: R = CyclotomicField(3)
sage: A3 = AlternatingGroup(3).algebra(R)
sage: idempotents = A3.central_orthogonal_idempotents()
sage: idempotents
(1/3*() + 1/3*(1,2,3) + 1/3*(1,3,2),
 1/3*() - (1/3*zeta3+1/3)*(1,2,3) - (-1/3*zeta3)*(1,3,2),
 1/3*() - (-1/3*zeta3)*(1,2,3) - (1/3*zeta3+1/3)*(1,3,2))
sage: A3.is_identity_decomposition_into_orthogonal_
↪idempotents(idempotents)
True
```

For the semisimple quotient of a quiver algebra, we recover the vertices of the quiver:

```
sage: # needs sage.graphs sage.modules
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver (containing
the arrows a:x->y and b:x->y) over Rational Field
sage: Aquo = A.semisimple_quotient()
sage: Aquo.central_orthogonal_idempotents()
(B['x'], B['y'])
```

radical_basis(***keywords*)

Return a basis of the Jacobson radical of this algebra.

- `keywords` – for compatibility; ignored.

OUTPUT: the empty list since this algebra is semisimple.

EXAMPLES:

```
sage: A = SymmetricGroup(4).algebra(QQ)                                  #␣
↪needs sage.groups sage.modules
sage: A.radical_basis()                                                  #␣
↪needs sage.groups sage.modules
()
```

4.56 Finite Enumerated Sets

class sage.categories.finite_enumerated_sets.**FiniteEnumeratedSets**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite enumerated sets

EXAMPLES:

```
sage: FiniteEnumeratedSets()
Category of finite enumerated sets
sage: FiniteEnumeratedSets().super_categories()
[Category of enumerated sets, Category of finite sets]
sage: FiniteEnumeratedSets().all_super_categories()
[Category of finite enumerated sets,
 Category of enumerated sets,
 Category of finite sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Todo: `sage.combinat.debruijn_sequence.DeBruijnSequences` should not inherit from this class. If that is solved, then *FiniteEnumeratedSets* shall be turned into a subclass of *Category_singleton*.

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

class ParentMethods

Bases: `object`

cardinality()

Return the cardinality of self.

EXAMPLES:

```
sage: E = FiniteEnumeratedSet([1,2,3])
sage: C = cartesian_product([E, SymmetricGroup(4)])                #␣
↪needs sage.groups
sage: C.cardinality()                                              #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.groups
72

sage: E = FiniteEnumeratedSet([])
sage: C = cartesian_product([E, ZZ, QQ])
sage: C.cardinality()
0

sage: C = cartesian_product([ZZ, QQ])
sage: C.cardinality()
+Infinity

sage: cartesian_product([GF(5), Permutations(10)]).cardinality()
18144000
sage: cartesian_product([GF(71)]*20).cardinality() == 71**20
True
```

last()

Return the last element

EXAMPLES:

```
sage: C = cartesian_product([Zmod(42), Partitions(10),                  #␣
↪needs sage.combinat
....:                         IntegerRange(5)])
sage: C.last()                                                          #␣
↪needs sage.combinat
(41, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1], 4)
```

random_element(**args*)

Return a random element of this Cartesian product.

The extra arguments are passed down to each of the factors of the Cartesian product.

EXAMPLES:

```
sage: C = cartesian_product([Permutations(10)]*5)
sage: C.random_element()           # random
([2, 9, 4, 7, 1, 8, 6, 10, 5, 3],
 [8, 6, 5, 7, 1, 4, 9, 3, 10, 2],
 [5, 10, 3, 8, 2, 9, 1, 4, 7, 6],
 [9, 6, 10, 3, 2, 1, 5, 8, 7, 4],
 [8, 5, 2, 9, 10, 3, 7, 1, 4, 6])

sage: C = cartesian_product([ZZ]*10)
sage: c1 = C.random_element()
sage: c1                      # random
(3, 1, 4, 1, 1, -3, 0, -4, -17, 2)
sage: c2 = C.random_element(4,7)
sage: c2                      # random
(6, 5, 6, 4, 5, 6, 6, 4, 5, 5)
sage: all(4 <= i < 7 for i in c2)
True
```

rank(*x*)

Return the rank of an element of this Cartesian product.

The *rank* of `x` is its position in the enumeration. It is an integer between `0` and `n-1` where `n` is the cardinality of this set.

See also:

- *EnumeratedSets.ParentMethods.rank()*
- *unrank()*

EXAMPLES:

```
sage: C = cartesian_product([GF(2), GF(11), GF(7)])
sage: C.rank(C((1,2,5)))
96
sage: C.rank(C((0,0,0)))
0

sage: for c in C: print(C.rank(c))
0
1
2
3
4
5
...
150
151
152
153

sage: # needs sage.combinat
sage: F1 = FiniteEnumeratedSet('abcdefgh')
sage: F2 = IntegerRange(250)
sage: F3 = Partitions(20)
sage: C = cartesian_product([F1, F2, F3])
sage: c = C(('a', 86, [7,5,4,4]))
sage: C.rank(c)
54213
sage: C.unrank(54213)
('a', 86, [7, 5, 4, 4])
```

unrank(*i*)

Return the `i`-th element of this Cartesian product.

INPUT:

- `i` – integer between `0` and `n-1` where `n` is the cardinality of this set.

See also:

- *EnumeratedSets.ParentMethods.unrank()*
- *rank()*

EXAMPLES:

```
sage: C = cartesian_product([GF(3), GF(11), GF(7), GF(5)])
sage: c = C.unrank(123); c
(0, 3, 3, 3)
sage: C.rank(c)
123

sage: c = C.unrank(857); c
```

(continues on next page)

(continued from previous page)

```
(2, 2, 3, 2)
sage: C.rank(c)
857

sage: C.unrank(2500)
Traceback (most recent call last):
...
IndexError: index i (=2) is greater than the cardinality
```

extra_super_categories()

A Cartesian product of finite enumerated sets is a finite enumerated set.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().CartesianProducts()
sage: C.extra_super_categories()
[Category of finite enumerated sets]
```

class IsomorphicObjects(*category*, **args*)

Bases: *IsomorphicObjectsCategory*

class ParentMethods

Bases: `object`

cardinality()

Returns the cardinality of `self` which is the same as that of the ambient set `self` is isomorphic to.

EXAMPLES:

```
sage: A = FiniteEnumeratedSets().IsomorphicObjects().example(); A
The image by some isomorphism of An example of a finite enumerated␣
↪set: {1,2,3}
sage: A.cardinality()
3
```

example()

Returns an example of isomorphic object of a finite enumerated set, as per *Category.example*.

EXAMPLES:

```
sage: FiniteEnumeratedSets().IsomorphicObjects().example()
The image by some isomorphism of An example of a finite enumerated set:
↪{1,2,3}
```

class ParentMethods

Bases: `object`

cardinality(**ignored_args*, ***ignored_kwds*)

Return the cardinality of `self`.

This brute force implementation of *cardinality()* iterates through the elements of `self` to count them.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example(); C
An example of a finite enumerated set: {1,2,3}
sage: C._cardinality_from_iterator()
3
```

iterator_range(*start=None*, *stop=None*, *step=None*)

Iterate over the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

See also:

`unrank()`, `unrank_range()`

EXAMPLES:

```
sage: F = FiniteEnumeratedSet([1,2,3])
sage: list(F.iterator_range(1))
[2, 3]
sage: list(F.iterator_range(stop=2))
[1, 2]
sage: list(F.iterator_range(stop=2, step=2))
[1]
sage: list(F.iterator_range(start=1, step=2))
[2]
sage: list(F.iterator_range(start=1, stop=2))
[2]
sage: list(F.iterator_range(start=0, stop=1))
[1]
sage: list(F.iterator_range(start=0, stop=3, step=2))
[1, 3]
sage: list(F.iterator_range(stop=-1))
[1, 2]

sage: F = FiniteEnumeratedSet([1,2,3,4])
sage: list(F.iterator_range(start=1, stop=3))
[2, 3]
sage: list(F.iterator_range(stop=10))
[1, 2, 3, 4]
```

last()

The last element of `self`.

`self.last()` returns the last element of `self`.

This is the default (brute force) implementation from the category `FiniteEnumeratedSet()` which can be used when the method `__iter__` is provided. Its complexity is $O(n)$ where n is the size of `self`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.last()
3
sage: C._last_from_iterator()
3
```

random_element()

A random element in `self`.

`self.random_element()` returns a random element in `self` with uniform probability.

This is the default implementation from the category `EnumeratedSet()` which uses the method `unrank`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: n = C.random_element()
sage: n in C
True

sage: n = C._random_element_from_unrank()
sage: n in C
True
```

TODO: implement _test_random which checks uniformness

`tuple`()

Return a `tuple`of the elements of ``self`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.tuple()
(1, 2, 3)
sage: C.tuple() is C.tuple()
True
```

`unrank_range`(*start=None*, *stop=None*, *step=None*)

Return the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

See also `unrank()`.

EXAMPLES:

```
sage: F = FiniteEnumeratedSet([1,2,3])
sage: F.unrank_range(1)
[2, 3]
sage: F.unrank_range(stop=2)
[1, 2]
sage: F.unrank_range(stop=2, step=2)
[1]
sage: F.unrank_range(start=1, step=2)
[2]
sage: F.unrank_range(stop=-1)
[1, 2]

sage: F = FiniteEnumeratedSet([1,2,3,4])
sage: F.unrank_range(stop=10)
[1, 2, 3, 4]
```

4.57 Finite fields

class sage.categories.finite_fields.**FiniteFields**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite fields.

EXAMPLES:

```
sage: K = FiniteFields(); K
Category of finite enumerated fields
```

A finite field is a finite monoid with the structure of a field; it is currently assumed to be enumerated:

```
sage: K.super_categories()
[Category of fields,
 Category of finite commutative rings,
 Category of finite enumerated sets]
```

Some examples of membership testing and coercion:

```
sage: FiniteField(17) in K
True
sage: RationalField() in K
False
sage: K(RationalField())
Traceback (most recent call last):
...
TypeError: unable to canonically associate a finite field to Rational Field
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

extra_super_categories()

Any finite field is assumed to be endowed with an enumeration.

4.58 Finite groups

class sage.categories.finite_groups.**FiniteGroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite (multiplicative) groups.

EXAMPLES:

```
sage: C = FiniteGroups(); C
Category of finite groups
sage: C.super_categories()
[Category of finite monoids, Category of groups]
sage: C.example()
General Linear Group of degree 2 over Finite Field of size 3
```

class Algebras(*category*, **args*)

Bases: `AlgebrasCategory`

class ParentMethods

Bases: `object`

extra_super_categories()

Implement Maschke's theorem.

In characteristic 0 all finite group algebras are semisimple.

EXAMPLES:

```
sage: FiniteGroups().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Semisimple())
True
sage: FiniteGroups().Algebras(FiniteField(7)).is_
↪subcategory(Algebras(FiniteField(7)).Semisimple())
False
sage: FiniteGroups().Algebras(ZZ).is_subcategory(Algebras(ZZ).
↪Semisimple())
False
sage: FiniteGroups().Algebras(Fields()).is_subcategory(Algebras(Fields()).
↪Semisimple())
False

sage: Cat = CommutativeAdditiveGroups().Finite()
sage: Cat.Algebras(QQ).is_subcategory(Algebras(QQ).Semisimple())
True
sage: Cat.Algebras(GF(7)).is_subcategory(Algebras(GF(7)).Semisimple())
False
sage: Cat.Algebras(ZZ).is_subcategory(Algebras(ZZ).Semisimple())
False
sage: Cat.Algebras(Fields()).is_subcategory(Algebras(Fields()).
↪Semisimple())
False
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

cardinality()

Returns the cardinality of `self`, as per `EnumeratedSets.ParentMethods.cardinality()`.

This default implementation calls `order()` if available, and otherwise resorts to `_cardinality_from_iterator()`. This is for backward compatibility only. Finite groups should override this method instead of `order()`.

EXAMPLES:

We need to use a finite group which uses this default implementation of cardinality:

```
sage: G = groups.misc.SemimonomialTransformation(GF(5), 3); G
Semimonomial transformation group over Finite Field of size 5 of degree 3
sage: G.cardinality.__module__
'sage.categories.finite_groups'
```

(continues on next page)

(continued from previous page)

```
sage: G.cardinality()
384
```

cayley_graph_disabled(*connecting_set=None*)

AUTHORS:

- Bobby Moretti (2007-08-10)
- Robert Miller (2008-05-01): editing

conjugacy_classes()

Return a list with all the conjugacy classes of the group.

This will eventually be a fall-back method for groups not defined over GAP. Right now, it just raises a `NotImplementedError`, until we include a non-GAP way of listing the conjugacy classes representatives.

EXAMPLES:

```
sage: from sage.groups.group import FiniteGroup
sage: G = FiniteGroup()
sage: G.conjugacy_classes()
Traceback (most recent call last):
...
NotImplementedError: Listing the conjugacy classes for group <sage.groups.
↪group.FiniteGroup object at ...> is not implemented
```

conjugacy_classes_representatives()

Return a list of the conjugacy classes representatives of the group.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: G.conjugacy_classes_representatives()
[(), (1,2), (1,2,3)]
```

monoid_generators()

Return monoid generators for `self`.

For finite groups, the group generators are also monoid generators. Hence, this default implementation calls *group_generators()*.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.monoid_generators()
Family ((2,3,4), (1,2,3))
```

semigroup_generators()

Returns semigroup generators for self.

For finite groups, the group generators are also semigroup generators. Hence, this default implementation calls *group_generators()*.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.semigroup_generators()
Family ((2,3,4), (1,2,3))
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.some_elements()
Family ((2,3,4), (1,2,3))
```

example()

Return an example of finite group, as per *Category.example()*.

EXAMPLES:

```
sage: G = FiniteGroups().example(); G
General Linear Group of degree 2 over Finite Field of size 3
```

4.59 Finite lattice posets

class sage.categories.finite_lattice_posets.**FiniteLatticePosets**(*base_category*)

Bases: *CategoryWithAxiom*

The category of finite lattices, i.e. finite partially ordered sets which are also lattices.

EXAMPLES:

```
sage: FiniteLatticePosets()
Category of finite lattice posets
sage: FiniteLatticePosets().super_categories()
[Category of lattice posets, Category of finite posets]
sage: FiniteLatticePosets().example()
NotImplemented
```

See also:

FinitePosets, *LatticePosets*, `FiniteLatticePoset`

class ParentMethods

Bases: `object`

irreducibles_poset()

Return the poset of meet- or join-irreducibles of the lattice.

A *join-irreducible* element of a lattice is an element with exactly one lower cover. Dually a *meet-irreducible* element has exactly one upper cover.

This is the smallest poset with completion by cuts being isomorphic to the lattice. As a special case this returns one-element poset from one-element lattice.

See also:

`completion_by_cuts()`.

EXAMPLES:

```
sage: # needs sage.combinat sage.graphs sage.modules
sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [5],
....:                   4: [6], 5: [9, 7], 6: [9, 8], 7: [10],
....:                   8: [10], 9: [10], 10: [11]})
sage: L_ = L.irreducibles_poset()
sage: sorted(L_)
[2, 3, 4, 7, 8, 9, 10, 11]
sage: L_.completion_by_cuts().is_isomorphic(L)
True
```

is_lattice_morphism(*f*, *codomain*)

Return whether `f` is a morphism of posets from `self` to `codomain`.

A map $f : P \to Q$ is a poset morphism if

$$x \leq y \Rightarrow f(x) \leq f(y)$$

for all $x, y \in P$.

INPUT:

- `f` – a function from `self` to `codomain`
- `codomain` – a lattice

EXAMPLES:

We build the boolean lattice of $\{2, 2, 3\}$ and the lattice of divisors of 60, and check that the map $b \mapsto 5 \prod_{x \in b} x$ is a morphism of lattices:

```
sage: D = LatticePoset((divisors(60), attrcall("divides")))              #␣
↪needs sage.graphs sage.modules
sage: B = LatticePoset((Subsets([2,2,3]), attrcall("issubset")))         #␣
↪needs sage.graphs sage.modules
sage: def f(b): return D(5*prod(b))
sage: B.is_lattice_morphism(f, D)                                        #␣
↪needs sage.graphs sage.modules
True
```

We construct the boolean lattice B_2:

```
sage: B = posets.BooleanLattice(2)                                       #␣
↪needs sage.graphs
sage: B.cover_relations()                                                #␣
↪needs sage.graphs
[[0, 1], [0, 2], [1, 3], [2, 3]]
```

And the same lattice with new top and bottom elements numbered respectively -1 and 3:

```
sage: G = DiGraph({-1:[0], 0:[1,2], 1:[3], 2:[3], 3:[4]})                #␣
↪needs sage.graphs
sage: L = LatticePoset(G)                                                #␣
↪needs sage.graphs sage.modules
sage: L.cover_relations()                                                #␣
↪needs sage.graphs sage.modules
[[-1, 0], [0, 1], [0, 2], [1, 3], [2, 3], [3, 4]]

sage: f = {B(0): L(0), B(1): L(1), B(2): L(2), B(3): L(3)}.__getitem__   #␣
↪needs sage.graphs sage.modules
sage: B.is_lattice_morphism(f, L)                                        #␣
↪needs sage.graphs sage.modules
```

(continues on next page)

(continued from previous page)

```
True

sage: f = {B(0): L(-1),B(1): L(1), B(2): L(2), B(3): L(3)}.__getitem__  #␣
↪needs sage.graphs sage.modules
sage: B.is_lattice_morphism(f, L)                                       #␣
↪needs sage.graphs sage.modules
False

sage: f = {B(0): L(0), B(1): L(1), B(2): L(2), B(3): L(4)}.__getitem__  #␣
↪needs sage.graphs sage.modules
sage: B.is_lattice_morphism(f, L)                                       #␣
↪needs sage.graphs sage.modules
False
```

See also:

is_poset_morphism()

join_irreducibles()

Return the join-irreducible elements of this finite lattice.

A *join-irreducible element* of `self` is an element x that is not minimal and that can not be written as the join of two elements different from x.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2],1:[3],2:[3,4],3:[5],4:[5]})             #␣
↪needs sage.graphs sage.modules
sage: L.join_irreducibles()                                             #␣
↪needs sage.graphs sage.modules
[1, 2, 4]
```

See also:

- Dual function: *meet_irreducibles()*
- Other: `double_irreducibles()`, *join_irreducibles_poset()*

join_irreducibles_poset()

Return the poset of join-irreducible elements of this finite lattice.

A *join-irreducible element* of `self` is an element x that is not minimal and can not be written as the join of two elements different from x.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2,3],1:[4],2:[4],3:[4]})                   #␣
↪needs sage.graphs sage.modules
sage: L.join_irreducibles_poset()                                       #␣
↪needs sage.graphs sage.modules
Finite poset containing 3 elements
```

See also:

- Dual function: *meet_irreducibles_poset()*
- Other: *join_irreducibles()*

meet_irreducibles()

Return the meet-irreducible elements of this finite lattice.

A *meet-irreducible element* of `self` is an element x that is not maximal and that can not be written as the meet of two elements different from x.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2],1:[3],2:[3,4],3:[5],4:[5]})                    #␣
↪needs sage.graphs sage.modules
sage: L.meet_irreducibles()                                                    #␣
↪needs sage.graphs sage.modules
[1, 3, 4]
```

See also:

- Dual function: *join_irreducibles()*
- Other: `double_irreducibles()`, *meet_irreducibles_poset()*

meet_irreducibles_poset()

Return the poset of join-irreducible elements of this finite lattice.

A *meet-irreducible element* of `self` is an element x that is not maximal and can not be written as the meet of two elements different from x.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2,3],1:[4],2:[4],3:[4]})                          #␣
↪needs sage.graphs sage.modules
sage: L.join_irreducibles_poset()                                              #␣
↪needs sage.graphs sage.modules
Finite poset containing 3 elements
```

See also:

- Dual function: *join_irreducibles_poset()*
- Other: *meet_irreducibles()*

4.60 Finite monoids

class sage.categories.finite_monoids.**FiniteMonoids**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite (multiplicative) *monoids*.

A finite monoid is a *finite sets* endowed with an associative unital binary operation $*$.

EXAMPLES:

```
sage: FiniteMonoids()
Category of finite monoids
sage: FiniteMonoids().super_categories()
[Category of monoids, Category of finite semigroups]
```

class ElementMethods

Bases: `object`

pseudo_order()

Return the pair $[k, j]$ with k minimal and $0 \leq j < k$ such that `self^k == self^j`.

Note that j is uniquely determined.

EXAMPLES:

```
sage: M = FiniteMonoids().example(); M
An example of a finite multiplicative monoid: the integers modulo 12

sage: x = M(2)
sage: [ x^i for i in range(7) ]
[1, 2, 4, 8, 4, 8, 4]
sage: x.pseudo_order()
[4, 2]

sage: x = M(3)
sage: [ x^i for i in range(7) ]
[1, 3, 9, 3, 9, 3, 9]
sage: x.pseudo_order()
[3, 1]

sage: x = M(4)
sage: [ x^i for i in range(7) ]
[1, 4, 4, 4, 4, 4, 4]
sage: x.pseudo_order()
[2, 1]

sage: x = M(5)
sage: [ x^i for i in range(7) ]
[1, 5, 1, 5, 1, 5, 1]
sage: x.pseudo_order()
[2, 0]
```

Todo: more appropriate name? see, for example, Jean-Eric Pin's lecture notes on semigroups.

class ParentMethods

Bases: `object`

nerve()

The nerve (classifying space) of this monoid.

OUTPUT:

the nerve BG (if G denotes this monoid), as a simplicial set. The k-dimensional simplices of this object are indexed by products of k elements in the monoid:

$$a_1 * a_2 * \cdots * a_k$$

The 0th face of this is obtained by deleting a_1, and the k-th face is obtained by deleting a_k. The other faces are obtained by multiplying elements: the 1st face is

$$(a1 * a_2) * \cdots * a_k$$

and so on. See Wikipedia article Nerve_(category_theory), which describes the construction of the nerve as a simplicial set.

A simplex in this simplicial set will be degenerate if in the corresponding product of k elements, one of those elements is the identity. So we only need to keep track of the products of non-identity elements. Similarly, if a product $a_{i-1}a_i$ is the identity element, then the corresponding face of the simplex will be a degenerate simplex.

EXAMPLES:

The nerve (classifying space) of the cyclic group of order 2 is infinite-dimensional real projective space.

```
sage: Sigma2 = groups.permutation.Cyclic(2)                                     #␣
↪needs sage.groups
sage: BSigma2 = Sigma2.nerve()                                                  #␣
↪needs sage.groups
sage: BSigma2.cohomology(4, base_ring=GF(2))                                    #␣
↪needs sage.groups sage.modules
Vector space of dimension 1 over Finite Field of size 2
```

The k-simplices of the nerve are named after the chains of k non-unit elements to be multiplied. The group Σ_2 has two elements, written `()` (the identity element) and `(1,2)` in Sage. So the 1-cells and 2-cells in $B\Sigma_2$ are:

```
sage: BSigma2.n_cells(1)                                                        #␣
↪needs sage.groups
[(1,2)]
sage: BSigma2.n_cells(2)                                                        #␣
↪needs sage.groups
[(1,2) * (1,2)]
```

Another construction of the group, with different names for its elements:

```
sage: # needs sage.groups
sage: C2 = groups.misc.MultiplicativeAbelian([2])
sage: BC2 = C2.nerve()
sage: BC2.n_cells(0)
[1]
sage: BC2.n_cells(1)
[f]
sage: BC2.n_cells(2)
[f * f]
```

With mod p coefficients, $B\Sigma_p$ should have its first nonvanishing homology group in dimension p:

```
sage: Sigma3 = groups.permutation.Symmetric(3)                                  #␣
↪needs sage.groups
sage: BSigma3 = Sigma3.nerve()                                                  #␣
↪needs sage.groups
sage: BSigma3.homology(range(4), base_ring=GF(3))                               #␣
↪needs sage.groups
{0: Vector space of dimension 0 over Finite Field of size 3,
 1: Vector space of dimension 0 over Finite Field of size 3,
 2: Vector space of dimension 0 over Finite Field of size 3,
 3: Vector space of dimension 1 over Finite Field of size 3}
```

Note that we can construct the n-skeleton for $B\Sigma_2$ for relatively large values of n, while for $B\Sigma_3$, the complexes get large pretty quickly:

```
sage: # needs sage.groups
sage: Sigma2.nerve().n_skeleton(14)
Simplicial set with 15 non-degenerate simplices
sage: BSigma3 = Sigma3.nerve()
sage: BSigma3.n_skeleton(3)
Simplicial set with 156 non-degenerate simplices
sage: BSigma3.n_skeleton(4)
Simplicial set with 781 non-degenerate simplices
```

Finally, note that the classifying space of the order p cyclic group is smaller than that of the symmetric group on p letters, and its first homology group appears earlier:

```
sage: # needs sage.groups
sage: C3 = groups.misc.MultiplicativeAbelian([3])
sage: list(C3)
[1, f, f^2]
sage: BC3 = C3.nerve()
sage: BC3.n_cells(1)
[f, f^2]
sage: BC3.n_cells(2)
[f * f, f * f^2, f^2 * f, f^2 * f^2]
sage: len(BSigma3.n_cells(2))
25
sage: len(BC3.n_cells(3))
8
sage: len(BSigma3.n_cells(3))
125
sage: BC3.homology(range(4), base_ring=GF(3))
{0: Vector space of dimension 0 over Finite Field of size 3,
 1: Vector space of dimension 1 over Finite Field of size 3,
 2: Vector space of dimension 1 over Finite Field of size 3,
 3: Vector space of dimension 1 over Finite Field of size 3}
sage: BC5 = groups.permutation.Cyclic(5).nerve()
sage: BC5.homology(range(4), base_ring=GF(5))
{0: Vector space of dimension 0 over Finite Field of size 5,
 1: Vector space of dimension 1 over Finite Field of size 5,
 2: Vector space of dimension 1 over Finite Field of size 5,
 3: Vector space of dimension 1 over Finite Field of size 5}
```

rhodes_radical_congruence(*base_ring=None*)

Return the Rhodes radical congruence of the semigroup.

The Rhodes radical congruence is the congruence induced on S by the map $S \rightarrow kS \rightarrow kS/radkS$ with k a field.

INPUT:

- `base_ring` (default: **Q**) a field

OUTPUT:

- A list of couples (m, n) with $m \neq n$ in the lexicographic order for the enumeration of the monoid `self`.

EXAMPLES:

```
sage: M = Monoids().Finite().example()
sage: M.rhodes_radical_congruence()                                      #␣
↪needs sage.modules
[(0, 6), (2, 8), (4, 10)]

sage: # needs sage.combinat sage.groups sage.modules
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: H3 = HeckeMonoid(SymmetricGroup(3))
sage: H3.repr_element_method(style="reduced")
sage: H3.rhodes_radical_congruence()
[([1, 2], [2, 1]), ([1, 2], [1, 2, 1]), ([2, 1], [1, 2, 1])]
```

By Maschke's theorem, every group algebra over **Q** is semisimple hence the Rhodes radical of a group must be trivial:

```
sage: SymmetricGroup(3).rhodes_radical_congruence()                        #␣
↪needs sage.groups sage.modules
[]
sage: DihedralGroup(10).rhodes_radical_congruence()                        #␣
↪needs sage.groups sage.modules
[]
```

REFERENCES:

- [Rho69]

4.61 Finite Permutation Groups

class sage.categories.finite_permutation_groups.**FinitePermutationGroups**(*base_category*)

Bases: *CategoryWithAxiom*

The category of finite permutation groups, i.e. groups concretely represented as groups of permutations acting on a finite set.

It is currently assumed that any finite permutation group comes endowed with a distinguished finite set of generators (method group_generators); this is the case for all the existing implementations in Sage.

EXAMPLES:

```
sage: C = PermutationGroups().Finite(); C
Category of finite enumerated permutation groups
sage: C.super_categories()
[Category of permutation groups,
 Category of finite groups,
 Category of finite finitely generated semigroups]

sage: C.example()
Dihedral group of order 6 as a permutation group
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

cycle_index(*parent=None*)

Return the *cycle index* of self.

INPUT:

- self - a permutation group G
- parent – a free module with basis indexed by partitions, or behave as such, with a term and sum method (default: the symmetric functions over the rational field in the p basis)

The *cycle index* of a permutation group G (Wikipedia article Cycle_index) is a gadget counting the elements of G by cycle type, averaged over the group:

$$P = \frac{1}{|G|} \sum_{g \in G} p_{\text{cycle type}(g)}$$

EXAMPLES:

Among the permutations of the symmetric group S_4, there is the identity, 6 cycles of length 2, 3 products of two cycles of length 2, 8 cycles of length 3, and 6 cycles of length 4:

```
sage: S4 = SymmetricGroup(4)
sage: P = S4.cycle_index()
sage: 24 * P
p[1, 1, 1, 1] + 6*p[2, 1, 1] + 3*p[2, 2] + 8*p[3, 1] + 6*p[4]
```

If $l = (l_1, \ldots, l_k)$ is a partition, `|G| P[l]` is the number of elements of G with cycles of length $(p_1, \ldots, p_k)$:

```
sage: 24 * P[ Partition([3,1]) ]
8
```

The cycle index plays an important role in the enumeration of objects modulo the action of a group (Pólya enumeration), via the use of symmetric functions and plethysms. It is therefore encoded as a symmetric function, expressed in the powersum basis:

```
sage: P.parent()
Symmetric Functions over Rational Field in the powersum basis
```

This symmetric function can have some nice properties; for example, for the symmetric group S_n, we get the complete symmetric function h_n:

```
sage: S = SymmetricFunctions(QQ); h = S.h()
sage: h( P )
h[4]
```

Todo: Add some simple examples of Pólya enumeration, once it will be easy to expand symmetric functions on any alphabet.

Here are the cycle indices of some permutation groups:

```
sage: 6 * CyclicPermutationGroup(6).cycle_index()
p[1, 1, 1, 1, 1, 1] + p[2, 2, 2] + 2*p[3, 3] + 2*p[6]

sage: 60 * AlternatingGroup(5).cycle_index()
p[1, 1, 1, 1, 1] + 15*p[2, 2, 1] + 20*p[3, 1, 1] + 24*p[5]

sage: for G in TransitiveGroups(5):               # long time
....:     G.cardinality() * G.cycle_index()
p[1, 1, 1, 1, 1] + 4*p[5]
p[1, 1, 1, 1, 1] + 5*p[2, 2, 1] + 4*p[5]
p[1, 1, 1, 1, 1] + 5*p[2, 2, 1] + 10*p[4, 1] + 4*p[5]
p[1, 1, 1, 1, 1] + 15*p[2, 2, 1] + 20*p[3, 1, 1] + 24*p[5]
p[1, 1, 1, 1, 1] + 10*p[2, 1, 1, 1] + 15*p[2, 2, 1] + 20*p[3, 1, 1] +␣
↪20*p[3, 2] + 30*p[4, 1] + 24*p[5]
```

Permutation groups with arbitrary domains are supported (see github issue #22765):

```
sage: G = PermutationGroup([['b','c','a']], domain=['a','b','c'])
sage: G.cycle_index()
1/3*p[1, 1, 1] + 2/3*p[3]
```

One may specify another parent for the result:

```
sage: F = CombinatorialFreeModule(QQ, Partitions())
sage: P = CyclicPermutationGroup(6).cycle_index(parent = F)
```

(continues on next page)

(continued from previous page)

```
sage: 6 * P
B[[1, 1, 1, 1, 1, 1]] + B[[2, 2, 2]] + 2*B[[3, 3]] + 2*B[[6]]
sage: P.parent() is F
True
```

This parent should be a module with basis indexed by partitions:

```
sage: CyclicPermutationGroup(6).cycle_index(parent = QQ)
Traceback (most recent call last):
  ...
ValueError: `parent` should be a module with basis indexed by partitions
```

REFERENCES:

- [Ke1991]

AUTHORS:

- Nicolas Borie and Nicolas M. Thiéry

profile(*n*, *using_polya=True*)

Return the value in `n` of the profile of the group `self`.

Optional argument `using_polya` allows to change the default method.

INPUT:

- `n` – a nonnegative integer
- `using_polya` (optional) – a boolean: if `True` (default), the computation uses Pólya enumeration (and all values of the profile are cached, so this should be the method used in case several of them are needed); if `False`, uses the GAP interface to compute the orbit.

OUTPUT:

- A nonnegative integer that is the number of orbits of `n`-subsets under the action induced by `self` on the subsets of its domain (i.e. the value of the profile of `self` in `n`)

See also:

- *profile_series()*

EXAMPLES:

```
sage: C6 = CyclicPermutationGroup(6)
sage: C6.profile(2)
3
sage: C6.profile(3)
4
sage: D8 = DihedralGroup(8)
sage: D8.profile(4, using_polya=False)
8
```

profile_polynomial(*variable='z'*)

Return the (finite) generating series of the (finite) profile of the group.

The profile of a permutation group G is the counting function that maps each nonnegative integer n onto the number of orbits of the action induced by G on the n-subsets of its domain. If f is the profile of G, f(n) is thus the number of orbits of n-subsets of G.

INPUT:

- `variable` – a variable, or variable name as a string (default: *'z'*)

OUTPUT:

- A polynomial in `variable` with nonnegative integer coefficients. By default, a polynomial in z over ZZ.

See also:

- *profile()*

EXAMPLES:

```
sage: C8 = CyclicPermutationGroup(8)
sage: C8.profile_series()
z^8 + z^7 + 4*z^6 + 7*z^5 + 10*z^4 + 7*z^3 + 4*z^2 + z + 1
sage: D8 = DihedralGroup(8)
sage: poly_D8 = D8.profile_series()
sage: poly_D8
z^8 + z^7 + 4*z^6 + 5*z^5 + 8*z^4 + 5*z^3 + 4*z^2 + z + 1
sage: poly_D8.parent()
Univariate Polynomial Ring in z over Rational Field
sage: D8.profile_series(variable='y')
y^8 + y^7 + 4*y^6 + 5*y^5 + 8*y^4 + 5*y^3 + 4*y^2 + y + 1
sage: u = var('u')                                                      #␣
↪needs sage.symbolic
sage: D8.profile_series(u).parent()                                     #␣
↪needs sage.symbolic
Symbolic Ring
```

profile_series(*variable='z'*)

Return the (finite) generating series of the (finite) profile of the group.

The profile of a permutation group G is the counting function that maps each nonnegative integer n onto the number of orbits of the action induced by G on the n-subsets of its domain. If f is the profile of G, f(n) is thus the number of orbits of n-subsets of G.

INPUT:

- `variable` – a variable, or variable name as a string (default: $'z'$)

OUTPUT:

- A polynomial in `variable` with nonnegative integer coefficients. By default, a polynomial in z over ZZ.

See also:

- *profile()*

EXAMPLES:

```
sage: C8 = CyclicPermutationGroup(8)
sage: C8.profile_series()
z^8 + z^7 + 4*z^6 + 7*z^5 + 10*z^4 + 7*z^3 + 4*z^2 + z + 1
sage: D8 = DihedralGroup(8)
sage: poly_D8 = D8.profile_series()
sage: poly_D8
z^8 + z^7 + 4*z^6 + 5*z^5 + 8*z^4 + 5*z^3 + 4*z^2 + z + 1
sage: poly_D8.parent()
Univariate Polynomial Ring in z over Rational Field
sage: D8.profile_series(variable='y')
y^8 + y^7 + 4*y^6 + 5*y^5 + 8*y^4 + 5*y^3 + 4*y^2 + y + 1
sage: u = var('u')                                                      #␣
↪needs sage.symbolic
sage: D8.profile_series(u).parent()                                     #␣
↪needs sage.symbolic
Symbolic Ring
```

example()

Returns an example of finite permutation group, as per *Category.example()*.

EXAMPLES:

```
sage: G = FinitePermutationGroups().example(); G
Dihedral group of order 6 as a permutation group
```

extra_super_categories()

Any permutation group is assumed to be endowed with a finite set of generators.

4.62 Finite posets

Here is some terminology used in this file:

- An *order filter* (or *upper set*) of a poset P is a subset S of P such that if $x \leq y$ and $x \in S$ then $y \in S$.
- An *order ideal* (or *lower set*) of a poset P is a subset S of P such that if $x \leq y$ and $y \in S$ then $x \in S$.

class sage.categories.finite_posets.**FinitePosets**(*base_category*)

Bases: *CategoryWithAxiom*

The category of finite posets i.e. finite sets with a partial order structure.

EXAMPLES:

```
sage: FinitePosets()
Category of finite posets
sage: FinitePosets().super_categories()
[Category of posets, Category of finite sets]
sage: FinitePosets().example()
NotImplemented
```

See also:

Posets, Poset()

class ParentMethods

Bases: object

antichains()

Return all antichains of self.

EXAMPLES:

```
sage: A = posets.PentagonPoset().antichains(); A
Set of antichains of Finite lattice containing 5 elements
sage: list(A)
[[], [0], [1], [1, 2], [1, 3], [2], [3], [4]]
```

birational_free_labelling(*linear_extension=None*, *prefix='x'*, *base_field=None*, *reduced=False*, *addvars=None*, *labels=None*, *min_label=None*, *max_label=None*)

Return the birational free labelling of self.

Let us hold back defining this, and introduce birational toggles and birational rowmotion first. These notions have been introduced in [EP2013] as generalizations of the notions of toggles (*order_ideal_toggle()*) and *rowmotion* on order ideals of a finite poset. They have been studied further in [GR2013].

Let $\mathbf{K}$ be a field, and P be a finite poset. Let $\widehat{P}$ denote the poset obtained from P by adding a new element 1 which is greater than all existing elements of P, and a new element 0 which is smaller than all existing elements of P and 1. Now, a $\mathbf{K}$-*labelling of* P will mean any function from $\widehat{P}$ to $\mathbf{K}$. The image of an element v of $\widehat{P}$ under this labelling will be called the *label* of this labelling at v. The set of all $\mathbf{K}$-labellings of P is clearly $\mathbf{K}^{\widehat{P}}$.

For any $v \in P$, we now define a rational map $T_v : \mathbf{K}^{\widehat{P}} \dashrightarrow \mathbf{K}^{\widehat{P}}$ as follows: For every $f \in \mathbf{K}^{\widehat{P}}$, the image $T_v f$ should send every element $u \in \widehat{P}$ distinct from v to $f(u)$ (so the labels at all $u \neq v$ don't change), while v is sent to

$$\frac{1}{f(v)} \cdot \frac{\sum_{u \lessdot v} f(u)}{\sum_{u \gtrdot v} \frac{1}{f(u)}}$$

(both sums are over all $u \in \widehat{P}$ satisfying the respectively given conditions). Here, $\lessdot$ and $\gtrdot$ mean (respectively) "covered by" and "covers", interpreted with respect to the poset $\widehat{P}$. This rational map T_v is an involution and is called the *(birational)* v*-toggle*; see `birational_toggle()` for its implementation.

Now, *birational rowmotion* is defined as the composition $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n}$, where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P (written as a linear ordering of the elements of P). This is a rational map $\mathbf{K}^{\widehat{P}} \dashrightarrow \mathbf{K}^{\widehat{P}}$ which does not depend on the choice of the linear extension; it is denoted by R. See `birational_rowmotion()` for its implementation.

The definitions of birational toggles and birational rowmotion extend to the case of $\mathbf{K}$ being any semifield rather than necessarily a field (although it becomes less clear what constitutes a rational map in this generality). The most useful case is that of the `tropical semiring`, in which case birational rowmotion relates to classical constructions such as promotion of rectangular semistandard Young tableaux (page 5 of [EP2013b] and future work, via the related notion of birational *promotion*) and rowmotion on order ideals of the poset ([EP2013]).

The *birational free labelling* is a special labelling defined for every finite poset P and every linear extension $(v_1, v_2, \ldots, v_n)$ of P. It is given by sending every element v_i in P to x_i, sending the element 0 of $\widehat{P}$ to a, and sending the element 1 of $\widehat{P}$ to b, where the ground field $\mathbf{K}$ is the field of rational functions in $n+2$ indeterminates $a, x_1, x_2, \ldots, x_n, b$ over $\mathbb{Q}$.

In Sage, a labelling f of a poset P is encoded as a 4-tuple $(\mathbf{K}, d, u, v)$, where $\mathbf{K}$ is the ground field of the labelling (i. e., its target), d is the dictionary containing the values of f at the elements of P (the keys being the respective elements of P), u is the label of f at 0, and v is the label of f at 1.

Warning: The dictionary d is labelled by the elements of P. If P is a poset with `facade` option set to `False`, these might not be what they seem to be! (For instance, if `P == Poset({1: [2, 3]}, facade=False)`, then the value of d at 1 has to be accessed by `d[P(1)]`, not by `d[1]`.)

Warning: Dictionaries are mutable. They do compare correctly, but are not hashable and need to be cloned to avoid spooky action at a distance. Be careful!

INPUT:

- `linear_extension` – (default: the default linear extension of `self`) a linear extension of `self` (as a linear extension or as a list), or more generally a list of all elements of all elements of `self` each occurring exactly once
- `prefix` – (default: `'x'`) the prefix to name the indeterminates corresponding to the elements of `self` in the labelling (so, setting it to `'frog'` will result in these indeterminates being called `frog1, frog2, ..., frogn` rather than `x1, x2, ..., xn`).

- `base_field` – (default: `QQ`) the base field to be used instead of **Q** to define the rational function field over; this is not going to be the base field of the labelling, because the latter will have indeterminates adjoined!
- `reduced` – (default: `False`) if set to `True`, the result will be the *reduced* birational free labelling, which differs from the regular one by having 0 and 1 both sent to 1 instead of a and b (the indeterminates a and b then also won't appear in the ground field)
- `addvars` – (default: `''`) a string containing names of extra variables to be adjoined to the ground field (these don't have an effect on the labels)
- `labels` – (default: `'x'`) Either a function that takes an element of the poset and returns a name for the indeterminate corresponding to that element, or a string containing a comma-separated list of indeterminates that will be assigned to elements in the order of `linear_extension`. If the list contains more indeterminates than needed, the excess will be ignored. If it contains too few, then the needed indeterminates will be constructed from `prefix`.
- `min_label` – (default: `'a'`) a string to be used as the label for the element 0 of $\widehat{P}$
- `max_label` – (default: `'b'`) a string to be used as the label for the element 1 of $\widehat{P}$

OUTPUT:

The birational free labelling of the poset `self` and the linear extension `linear_extension`. Or, if `reduced` is set to `True`, the reduced birational free labelling.

EXAMPLES:

We construct the birational free labelling on a simple poset:

```
sage: P = Poset({1: [2, 3]})
sage: l = P.birational_free_labelling(); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(1, x1), (2, x2), (3, x3)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2]); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(1, x1), (2, x3), (3, x2)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2],␣
↪reduced=True, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in x1, x2, x3, spam, eggs␣
↪over Rational Field,
 {...},
 1,
 1)
sage: sorted(l[1].items())
[(1, x1), (2, x3), (3, x2)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2], prefix=
↪"wut", reduced=True, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in wut1, wut2, wut3, spam,
↪ eggs over Rational Field,
 {...},
```

(continues on next page)

(continued from previous page)

```
 1,
 1)
sage: sorted(l[1].items())
[(1, wut1), (2, wut3), (3, wut2)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2],␣
↪reduced=False, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b, spam,
↪ eggs over Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(1, x1), (2, x3), (3, x2)]
sage: l[1][2]
x3
```

Illustrating labelling with a function:

```
sage: P = posets.ChainPoset(2).product(posets.ChainPoset(2))
sage: l = P.birational_free_labelling(labels=lambda e : 'x_' + str(e[0])␣
↪+ str(e[1]))
sage: sorted(l[1].items())
[((0, 0), x_00), ((0, 1), x_01), ((1, 0), x_10), ((1, 1), x_11)]
sage: l[2]
a
```

The same, but with `min_label` and `max_label` provided:

```
sage: P = posets.ChainPoset(2).product(posets.ChainPoset(2))
sage: l = P.birational_free_labelling(labels=lambda e : 'x_' + str(e[0])␣
↪+ str(e[1]), min_label="lambda", max_label="mu")
sage: sorted(l[1].items())
[((0, 0), x_00), ((0, 1), x_01), ((1, 0), x_10), ((1, 1), x_11)]
sage: l[2]
lambda
sage: l[3]
mu
```

Illustrating labelling with a comma separated list of labels:

```
sage: l = P.birational_free_labelling(labels='w,x,y,z')
sage: sorted(l[1].items())
[((0, 0), w), ((0, 1), x), ((1, 0), y), ((1, 1), z)]
sage: l = P.birational_free_labelling(labels='w,x,y,z,m')
sage: sorted(l[1].items())
[((0, 0), w), ((0, 1), x), ((1, 0), y), ((1, 1), z)]
sage: l = P.birational_free_labelling(labels='w')
sage: sorted(l[1].items())
[((0, 0), w), ((0, 1), x1), ((1, 0), x2), ((1, 1), x3)]
```

Illustrating the warning about facade:

```
sage: P = Poset({1: [2, 3]}, facade=False)
sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2],␣
↪reduced=False, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b, spam,
```

(continues on next page)

(continued from previous page)

```
↪ eggs over Rational Field,
 {...},
 a,
 b)
sage: l[1][2]
Traceback (most recent call last):
...
KeyError: 2
sage: l[1][P(2)]
x3
```

Another poset:

```
sage: P = posets.SSTPoset([2,1])
sage: lext = sorted(P)
sage: l = P.birational_free_labelling(linear_extension=lext, addvars="ohai
↪")
sage: l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, x4, x5,␣
↪x6, x7, x8, b, ohai over Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[([[1, 1], [2]], x1), ([[1, 1], [3]], x2), ([[1, 2], [2]], x3), ([[1, 2],␣
↪[3]], x4),
 ([[1, 3], [2]], x5), ([[1, 3], [3]], x6), ([[2, 2], [3]], x7), ([[2, 3],␣
↪[3]], x8)]
```

See `birational_rowmotion()`, `birational_toggle()` and `birational_toggles()` for more substantial examples of what one can do with the birational free labelling.

birational_rowmotion(*labelling*)

Return the result of applying birational rowmotion to the **K**-labelling `labelling` of the poset `self`.

See the documentation of `birational_free_labelling()` for a definition of birational rowmotion and **K**-labellings and for an explanation of how **K**-labellings are to be encoded to be understood by Sage. This implementation allows **K** to be a semifield, not just a field. Birational rowmotion is only a rational map, so an exception (most likely, `ZeroDivisionError`) will be thrown if the denominator is zero.

INPUT:

- `labelling` – a **K**-labelling of `self` in the sense as defined in the documentation of `birational_free_labelling()`

OUTPUT:

The image of the **K**-labelling f under birational rowmotion.

EXAMPLES:

```
sage: P = Poset({1: [2, 3], 2: [4], 3: [4]})
sage: lex = [1, 2, 3, 4]
sage: t = P.birational_free_labelling(linear_extension=lex); t
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, x4, b␣
↪over Rational Field,
 {...},
 a,
```

(continues on next page)

(continued from previous page)

```
 b)
sage: sorted(t[1].items())
[(1, x1), (2, x2), (3, x3), (4, x4)]
sage: t = P.birational_rowmotion(t); t
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, x4, b␣
↪over Rational Field,
 {...},
 a,
 b)
sage: sorted(t[1].items())
[(1, a*b/x4), (2, (x1*x2*b + x1*x3*b)/(x2*x4)),
 (3, (x1*x2*b + x1*x3*b)/(x3*x4)), (4, (x2*b + x3*b)/x4)]
```

A result of [GR2013] states that applying birational rowmotion $n + m$ times to a **K**-labelling f of the poset $[n] \times [m]$ gives back f. Let us check this:

```
sage: def test_rectangle_periodicity(n, m, k):
....:     P = posets.ChainPoset(n).product(posets.ChainPoset(m))
....:     t0 = P.birational_free_labelling(P)
....:     t = t0
....:     for i in range(k):
....:         t = P.birational_rowmotion(t)
....:     return t == t0
sage: test_rectangle_periodicity(2, 2, 4)
True
sage: test_rectangle_periodicity(2, 2, 2)
False
sage: test_rectangle_periodicity(2, 3, 5)  # long time
True
```

While computations with the birational free labelling quickly run out of memory due to the complexity of the rational functions involved, it is computationally cheap to check properties of birational rowmotion on examples in the tropical semiring:

```
sage: def test_rectangle_periodicity_tropical(n, m, k):
....:     P = posets.ChainPoset(n).product(posets.ChainPoset(m))
....:     TT = TropicalSemiring(ZZ)
....:     t0 = (TT, {v: TT(randint(0, 99)) for v in P}, TT(0), TT(124))
....:     t = t0
....:     for i in range(k):
....:         t = P.birational_rowmotion(t)
....:     return t == t0
sage: test_rectangle_periodicity_tropical(7, 6, 13)
True
```

Tropicalization is also what relates birational rowmotion to classical rowmotion on order ideals. In fact, if T denotes the `tropical semiring` of **Z** and P is a finite poset, then we can define an embedding ϕ from the set $J(P)$ of all order ideals of P into the set $T^{\widehat{P}}$ of all T-labellings of P by sending every $I \in J(P)$ to the indicator function of I extended by the value 1 at the element 0 and the value 0 at the element 1. This map ϕ has the property that $R \circ \phi = \phi \circ r$, where R denotes birational rowmotion, and r denotes `classical rowmotion` on $J(P)$. An example:

```
sage: P = posets.IntegerPartitions(5)
sage: TT = TropicalSemiring(ZZ)
sage: def indicator_labelling(I):
....:     # send order ideal `I` to a `T`-labelling of `P`.
```

(continues on next page)

(continued from previous page)

```
....:     dct = {v: TT(v in I) for v in P}
....:     return (TT, dct, TT(1), TT(0))
sage: all(indicator_labelling(P.rowmotion(I))
....:     == P.birational_rowmotion(indicator_labelling(I))
....:     for I in P.order_ideals_lattice(facade=True))
True
```

birational_toggle(*v*, *labelling*)

Return the result of applying the birational v-toggle T_v to the **K**-labelling `labelling` of the poset `self`.

See the documentation of *birational_free_labelling()* for a definition of this toggle and of **K**-labellings as well as an explanation of how **K**-labellings are to be encoded to be understood by Sage. This implementation allows **K** to be a semifield, not just a field. The birational v-toggle is only a rational map, so an exception (most likely, `ZeroDivisionError`) will be thrown if the denominator is zero.

INPUT:

- `v` – an element of `self` (must have `self` as parent if `self` is a `facade=False` poset)
- `labelling` – a **K**-labelling of `self` in the sense as defined in the documentation of *birational_free_labelling()*

OUTPUT:

The **K**-labelling $T_v f$ of `self`, where f is `labelling`.

EXAMPLES:

Let us start with the birational free labelling of the “V”-poset (the three-element poset with Hasse diagram looking like a “V”):

```
sage: V = Poset({1: [2, 3]})
sage: s = V.birational_free_labelling(); s
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s[1].items())
[(1, x1), (2, x2), (3, x3)]
```

The image of s under the 1-toggle T_1 is:

```
sage: s1 = V.birational_toggle(1, s); s1
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s1[1].items())
[(1, a*x2*x3/(x1*x2 + x1*x3)), (2, x2), (3, x3)]
```

Now let us apply the 2-toggle T_2 (to the old `s`):

```
sage: s2 = V.birational_toggle(2, s); s2
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
```

(continues on next page)

(continued from previous page)

```
 b)
sage: sorted(s2[1].items())
[(1, x1), (2, x1*b/x2), (3, x3)]
```

On the other hand, we can also apply T_2 to the image of s under T_1:

```
sage: s12 = V.birational_toggle(2, s1); s12
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s12[1].items())
[(1, a*x2*x3/(x1*x2 + x1*x3)), (2, a*x3*b/(x1*x2 + x1*x3)), (3, x3)]
```

Each toggle is an involution:

```
sage: all( V.birational_toggle(i, V.birational_toggle(i, s)) == s
....:      for i in V )
True
```

We can also start with a less generic labelling:

```
sage: t = (QQ, {1: 3, 2: 6, 3: 7}, 2, 10)
sage: t1 = V.birational_toggle(1, t); t1
(Rational Field, {...}, 2, 10)
sage: sorted(t1[1].items())
[(1, 28/13), (2, 6), (3, 7)]
sage: t13 = V.birational_toggle(3, t1); t13
(Rational Field, {...}, 2, 10)
sage: sorted(t13[1].items())
[(1, 28/13), (2, 6), (3, 40/13)]
```

However, labellings have to be sufficiently generic, lest denominators vanish:

```
sage: t = (QQ, {1: 3, 2: 5, 3: -5}, 1, 15)
sage: t1 = V.birational_toggle(1, t)
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
```

We don't get into zero-division issues in the tropical semiring (unless the zero of the tropical semiring appears in the labelling):

```
sage: TT = TropicalSemiring(QQ)
sage: t = (TT, {1: TT(2), 2: TT(4), 3: TT(1)}, TT(6), TT(0))
sage: t1 = V.birational_toggle(1, t); t1
(Tropical semiring over Rational Field, {...}, 6, 0)
sage: sorted(t1[1].items())
[(1, 8), (2, 4), (3, 1)]
sage: t12 = V.birational_toggle(2, t1); t12
(Tropical semiring over Rational Field, {...}, 6, 0)
sage: sorted(t12[1].items())
[(1, 8), (2, 4), (3, 1)]
sage: t123 = V.birational_toggle(3, t12); t123
(Tropical semiring over Rational Field, {...}, 6, 0)
```

(continues on next page)

(continued from previous page)

```
sage: sorted(t123[1].items())
[(1, 8), (2, 4), (3, 7)]
```

We turn to more interesting posets. Here is the 6-element poset arising from the weak order on S_3:

```
sage: P = posets.SymmetricGroupWeakOrderPoset(3)
sage: sorted(list(P))
['123', '132', '213', '231', '312', '321']
sage: t = (TT, {'123': TT(4), '132': TT(2), '213': TT(3), '231': TT(1),
↪'321': TT(1), '312': TT(2)}, TT(7), TT(1))
sage: t1 = P.birational_toggle('123', t); t1
(Tropical semiring over Rational Field, {...}, 7, 1)
sage: sorted(t1[1].items())
[('123', 6), ('132', 2), ('213', 3), ('231', 1), ('312', 2), ('321', 1)]
sage: t13 = P.birational_toggle('213', t1); t13
(Tropical semiring over Rational Field, {...}, 7, 1)
sage: sorted(t13[1].items())
[('123', 6), ('132', 2), ('213', 4), ('231', 1), ('312', 2), ('321', 1)]
```

Let us verify on this example some basic properties of toggles. First of all, again let us check that T_v is an involution for every v:

```
sage: all( P.birational_toggle(v, P.birational_toggle(v, t)) == t
....:      for v in P )
True
```

Furthermore, two toggles T_v and T_w commute unless one of v or w covers the other:

```
sage: all( P.covers(v, w) or P.covers(w, v)
....:      or P.birational_toggle(v, P.birational_toggle(w, t))
....:         == P.birational_toggle(w, P.birational_toggle(v, t))
....:      for v in P for w in P )
True
```

birational_toggles(*vs*, *labelling*)

Return the result of applying a sequence of birational toggles (specified by `vs`) to the **K**-labelling `labelling` of the poset `self`.

See the documentation of *birational_free_labelling()* for a definition of birational toggles and **K**-labellings and for an explanation of how **K**-labellings are to be encoded to be understood by Sage. This implementation allows **K** to be a semifield, not just a field. The birational v-toggle is only a rational map, so an exception (most likely, `ZeroDivisionError`) will be thrown if the denominator is zero.

INPUT:

- `vs` – an iterable comprising elements of `self` (which must have `self` as parent if `self` is a `facade=False` poset)
- `labelling` – a **K**-labelling of `self` in the sense as defined in the documentation of *birational_free_labelling()*

OUTPUT:

The **K**-labelling $T_{v_n} T_{v_{n-1}} \cdots T_{v_1} f$ of `self`, where f is `labelling` and $(v_1, v_2, \ldots, v_n)$ is `vs` (written as list).

EXAMPLES:

```
sage: P = posets.SymmetricGroupBruhatOrderPoset(3)
sage: sorted(list(P))
```

(continues on next page)

(continued from previous page)

```
['123', '132', '213', '231', '312', '321']
sage: TT = TropicalSemiring(ZZ)
sage: t = (TT, {'123': TT(4), '132': TT(2), '213': TT(3), '231': TT(1),
↪'321': TT(1), '312': TT(2)}, TT(7), TT(1))
sage: tA = P.birational_toggles(['123', '231', '312'], t); tA
(Tropical semiring over Integer Ring, {...}, 7, 1)
sage: sorted(tA[1].items())
[('123', 6), ('132', 2), ('213', 3), ('231', 2), ('312', 1), ('321', 1)]
sage: tAB = P.birational_toggles(['132', '213', '321'], tA); tAB
(Tropical semiring over Integer Ring, {...}, 7, 1)
sage: sorted(tAB[1].items())
[('123', 6), ('132', 6), ('213', 5), ('231', 2), ('312', 1), ('321', 1)]

sage: P = Poset({1: [2, 3], 2: [4], 3: [4]})
sage: Qx = PolynomialRing(QQ, 'x').fraction_field()
sage: x = Qx.gen()
sage: t = (Qx, {1: 1, 2: x, 3: (x+1)/x, 4: x^2}, 1, 1)
sage: t1 = P.birational_toggles((i for i in range(1, 5)), t); t1
(Fraction Field of Univariate Polynomial Ring in x over Rational Field,
 {...},
 1,
 1)
sage: sorted(t1[1].items())
[(1, (x^2 + x)/(x^2 + x + 1)), (2, (x^3 + x^2)/(x^2 + x + 1)), (3, x^4/(x^
↪2 + x + 1)), (4, 1)]
sage: t2 = P.birational_toggles(reversed(range(1, 5)), t)
sage: sorted(t2[1].items())
[(1, 1/x^2), (2, (x^2 + x + 1)/x^4), (3, (x^2 + x + 1)/(x^3 + x^2)), (4,␣
↪(x^2 + x + 1)/x^3)]
```

Facade set to `False` works:

```
sage: P = Poset({'x': ['y', 'w'], 'y': ['z'], 'w': ['z']}, facade=False)
sage: lex = ['x', 'y', 'w', 'z']
sage: t = P.birational_free_labelling(linear_extension=lex)
sage: sorted(P.birational_toggles([P('x'), P('y')], t)[1].items())
[(x, a*x2*x3/(x1*x2 + x1*x3)), (y, a*x3*x4/(x1*x2 + x1*x3)), (w, x3), (z,␣
↪x4)]
```

`directed_subsets`(*direction*)

Return the order filters (resp. order ideals) of `self`, as lists.

If `direction` is 'up', returns the order filters (upper sets).

If `direction` is 'down', returns the order ideals (lower sets).

INPUT:

- `direction` – 'up' or 'down'

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True)
sage: A = P.directed_subsets('up')
sage: sorted(list(A))
[[], [1, 2, 4, 3, 6, 12], [2, 4, 3, 6, 12], [2, 4, 6, 12], [3, 6, 12], [4,
↪ 3, 6, 12], [4, 6, 12], [4, 12], [6, 12], [12]]
```

`is_lattice`()

Return whether the poset is a lattice.

A poset is a lattice if all pairs of elements have both a least upper bound ("join") and a greatest lower bound ("meet") in the poset.

EXAMPLES:

```
sage: P = Poset([[1, 3, 2], [4], [4, 5, 6], [6], [7], [7], [7], []])
sage: P.is_lattice()
True

sage: P = Poset([[1, 2], [3], [3], []])
sage: P.is_lattice()
True

sage: P = Poset({0: [2, 3], 1: [2, 3]})
sage: P.is_lattice()
False

sage: P = Poset({1: [2, 3, 4], 2: [5, 6], 3: [5, 7], 4: [6, 7], 5: [8, 9],
....:            6: [8, 10], 7: [9, 10], 8: [11], 9: [11], 10: [11]})
sage: P.is_lattice()
False
```

See also:

- Weaker properties: `is_join_semilattice()`, `is_meet_semilattice()`

`is_poset_isomorphism`(*f*, *codomain*)

Return whether f is an isomorphism of posets from `self` to `codomain`.

INPUT:
- `f` – a function from `self` to `codomain`
- `codomain` – a poset

EXAMPLES:

We build the poset D of divisors of 30, and check that it is isomorphic to the boolean lattice B of the subsets of $\{2, 3, 5\}$ ordered by inclusion, via the reverse function $f : B \to D, b \mapsto \prod_{x \in b} x$:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: B = Poset(([frozenset(s) for s in Subsets([2,3,5])], attrcall(
↪"issubset")))
sage: def f(b): return D(prod(b))
sage: B.is_poset_isomorphism(f, D)
True
```

On the other hand, f is not an isomorphism to the chain of divisors of 30, ordered by usual comparison:

```
sage: P = Poset((divisors(30), operator.le))
sage: def f(b): return P(prod(b))
sage: B.is_poset_isomorphism(f, P)
False
```

A non surjective case:

```
sage: B = Poset(([frozenset(s) for s in Subsets([2,3])], attrcall(
↪"issubset")))
sage: def f(b): return D(prod(b))
sage: B.is_poset_isomorphism(f, D)
False
```

A non injective case:

```
sage: B = Poset(([frozenset(s) for s in Subsets([2,3,5,6])], attrcall(
↪"issubset")))
sage: def f(b): return D(gcd(prod(b), 30))
sage: B.is_poset_isomorphism(f, D)
False
```

Note: since `D` and `B` are not facade posets, `f` is responsible for the conversions between integers and subsets to elements of `D` and `B` and back.

See also:

FiniteLatticePosets.ParentMethods.is_lattice_morphism()

is_poset_morphism(*f*, *codomain*)

Return whether f is a morphism of posets from `self` to `codomain`, that is

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all x and y in `self`.

INPUT:

- `f` – a function from `self` to `codomain`
- `codomain` – a poset

EXAMPLES:

We build the boolean lattice of the subsets of $\{2, 3, 5, 6\}$ and the lattice of divisors of 30, and check that the map $b \mapsto \gcd(\prod_{x \in b} x, 30)$ is a morphism of posets:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: B = Poset(([frozenset(s) for s in Subsets([2,3,5,6])], attrcall(
↪"issubset")))
sage: def f(b): return D(gcd(prod(b), 30))
sage: B.is_poset_morphism(f, D)
True
```

Note: since `D` and `B` are not facade posets, `f` is responsible for the conversions between integers and subsets to elements of `D` and `B` and back.

f is also a morphism of posets to the chain of divisors of 30, ordered by usual comparison:

```
sage: P = Poset((divisors(30), operator.le))
sage: def f(b): return P(gcd(prod(b), 30))
sage: B.is_poset_morphism(f, P)
True
```

FIXME: should this be `is_order_preserving_morphism`?

See also:

is_poset_isomorphism()

is_self_dual()

Return whether the poset is *self-dual*.

A poset is self-dual if it is isomorphic to its dual poset.

EXAMPLES:

```
sage: P = Poset({1: [3, 4], 2: [3, 4]})
sage: P.is_self_dual()
True

sage: P = Poset({1: [2, 3]})
sage: P.is_self_dual()
False
```

See also:

- Stronger properties: `is_orthocomplemented()` (for lattices)
- Other: `dual()`

`order_filter_generators`(*filter*)

Generators for an order filter

INPUT:

- `filter` – an order filter of `self`, as a list (or iterable)

EXAMPLES:

```
sage: P = Poset((Subsets([1,2,3]), attrcall("issubset")))
sage: I = P.order_filter([Set([1,2]), Set([2,3]), Set([1])])
sage: sorted(sorted(p) for p in I)
[[1], [1, 2], [1, 2, 3], [1, 3], [2, 3]]
sage: gen = P.order_filter_generators(I)
sage: sorted(sorted(p) for p in gen)
[[1], [2, 3]]
```

See also:

`order_ideal_generators()`

`order_ideal_complement_generators`(*antichain*, *direction='up'*)

Return the Panyushev complement of the antichain `antichain`.

Given an antichain A of a poset P, the Panyushev complement of A is defined to be the antichain consisting of the minimal elements of the order filter B, where B is the (set-theoretic) complement of the order ideal of P generated by A.

Setting the optional keyword variable `direction` to `'down'` leads to the inverse Panyushev complement being computed instead of the Panyushev complement. The inverse Panyushev complement of an antichain A is the antichain whose Panyushev complement is A. It can be found as the antichain consisting of the maximal elements of the order ideal C, where C is the (set-theoretic) complement of the order filter of P generated by A.

`panyushev_complement()` is an alias for this method.

Panyushev complementation is related (actually, isomorphic) to rowmotion (*`rowmotion()`*).

INPUT:

- `antichain` – an antichain of `self`, as a list (or iterable), or, more generally, generators of an order ideal (resp. order filter)
- `direction` – 'up' or 'down' (default: 'up')

OUTPUT:

- the generating antichain of the complement order filter (resp. order ideal) of the order ideal (resp. order filter) generated by the antichain `antichain`

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: P.order_ideal_complement_generators([1])
{2}
sage: P.order_ideal_complement_generators([3])
set()
sage: P.order_ideal_complement_generators([1,2])
{3}
sage: P.order_ideal_complement_generators([1,2,3])
set()

sage: P.order_ideal_complement_generators([1], direction="down")
{2}
sage: P.order_ideal_complement_generators([3], direction="down")
{1, 2}
sage: P.order_ideal_complement_generators([1,2], direction="down")
set()
sage: P.order_ideal_complement_generators([1,2,3], direction="down")
set()
```

Warning: This is a brute force implementation, building the order ideal generated by the antichain, and searching for order filter generators of its complement

order_ideal_generators(*ideal*, *direction='down'*)

Return the antichain of (minimal) generators of the order ideal (resp. order filter) `ideal`.

INPUT:

- `ideal` – an order ideal I (resp. order filter) of `self`, as a list (or iterable); this should be an order ideal if `direction` is set to `'down'`, and an order filter if `direction` is set to `'up'`.
- `direction` – `'up'` or `'down'` (default: `'down'`).

The antichain of (minimal) generators of an order ideal I in a poset P is the set of all minimal elements of P. In the case of an order filter, the definition is similar, but with "maximal" used instead of "minimal".

EXAMPLES:

We build the boolean lattice of all subsets of $\{1, 2, 3\}$ ordered by inclusion, and compute an order ideal there:

```
sage: P = Poset((Subsets([1,2,3]), attrcall("issubset")))
sage: I = P.order_ideal([Set([1,2]), Set([2,3]), Set([1])])
sage: sorted(sorted(p) for p in I)
[[], [1], [1, 2], [2], [2, 3], [3]]
```

Then, we retrieve the generators of this ideal:

```
sage: gen = P.order_ideal_generators(I)
sage: sorted(sorted(p) for p in gen)
[[1, 2], [2, 3]]
```

If `direction` is 'up', then this instead computes the minimal generators for an order filter:

```
sage: I = P.order_filter([Set([1,2]), Set([2,3]), Set([1])])
sage: sorted(sorted(p) for p in I)
[[1], [1, 2], [1, 2, 3], [1, 3], [2, 3]]
sage: gen = P.order_ideal_generators(I, direction='up')
sage: sorted(sorted(p) for p in gen)
[[1], [2, 3]]
```

Complexity: $O(n + m)$ where n is the cardinality of I, and m the number of upper covers of elements of I.

order_ideals_lattice(*as_ideals=True*, *facade=None*)

Return the lattice of order ideals of a poset `self`, ordered by inclusion.

The lattice of order ideals of a poset P is usually denoted by $J(P)$. Its underlying set is the set of order ideals of P, and its partial order is given by inclusion.

The order ideals of P are in a canonical bijection with the antichains of P. The bijection maps every order ideal to the antichain formed by its maximal elements. By setting the `as_ideals` keyword variable to `False`, one can make this method apply this bijection before returning the lattice.

INPUT:

- `as_ideals` – Boolean, if `True` (default) returns a poset on the set of order ideals, otherwise on the set of antichains
- `facade` – Boolean or `None` (default). Whether to return a facade lattice or not. By default return facade lattice if the poset is a facade poset.

EXAMPLES:

```
sage: P = posets.PentagonPoset()
sage: P.cover_relations()
[[0, 1], [0, 2], [1, 4], [2, 3], [3, 4]]
sage: J = P.order_ideals_lattice(); J
Finite lattice containing 8 elements
sage: sorted(sorted(e) for e in J)
 [[], [0], [0, 1], [0, 1, 2], [0, 1, 2, 3], [0, 1, 2, 3, 4], [0, 2], [0,␣
↪2, 3]]
```

As a lattice on antichains:

```
sage: J2 = P.order_ideals_lattice(False); J2
Finite lattice containing 8 elements
sage: sorted(J2)
[(), (0,), (1,), (1, 2), (1, 3), (2,), (3,), (4,)]
```

panyushev_complement(*antichain*, *direction='up'*)

Return the Panyushev complement of the antichain `antichain`.

Given an antichain A of a poset P, the Panyushev complement of A is defined to be the antichain consisting of the minimal elements of the order filter B, where B is the (set-theoretic) complement of the order ideal of P generated by A.

Setting the optional keyword variable `direction` to `'down'` leads to the inverse Panyushev complement being computed instead of the Panyushev complement. The inverse Panyushev complement of an antichain A is the antichain whose Panyushev complement is A. It can be found as the antichain consisting of the maximal elements of the order ideal C, where C is the (set-theoretic) complement of the order filter of P generated by A.

`panyushev_complement()` is an alias for this method.

Panyushev complementation is related (actually, isomorphic) to rowmotion (*`rowmotion()`*).

INPUT:

- `antichain` – an antichain of `self`, as a list (or iterable), or, more generally, generators of an order ideal (resp. order filter)
- `direction` – 'up' or 'down' (default: 'up')

OUTPUT:

- the generating antichain of the complement order filter (resp. order ideal) of the order ideal (resp. order filter) generated by the antichain `antichain`

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: P.order_ideal_complement_generators([1])
{2}
sage: P.order_ideal_complement_generators([3])
set()
sage: P.order_ideal_complement_generators([1,2])
{3}
sage: P.order_ideal_complement_generators([1,2,3])
set()

sage: P.order_ideal_complement_generators([1], direction="down")
{2}
sage: P.order_ideal_complement_generators([3], direction="down")
{1, 2}
sage: P.order_ideal_complement_generators([1,2], direction="down")
set()
sage: P.order_ideal_complement_generators([1,2,3], direction="down")
set()
```

Warning: This is a brute force implementation, building the order ideal generated by the antichain, and searching for order filter generators of its complement

panyushev_orbit_iter(*antichain*, *element_constructor=<class 'set'>*, *stop=True*, *check=True*)

Iterate over the Panyushev orbit of an antichain `antichain` of `self`.

The Panyushev orbit of an antichain is its orbit under Panyushev complementation (see *panyushev_complement()*).

INPUT:

- `antichain` – an antichain of `self`, given as an iterable.
- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the antichains before they are yielded.
- `stop` – a Boolean (default: `True`) determining whether the iterator should stop once it completes its cycle (this happens when it is set to `True`) or go on forever (this happens when it is set to `False`).
- `check` – a Boolean (default: `True`) determining whether `antichain` should be checked for being an antichain.

OUTPUT:

- an iterator over the orbit of the antichain `antichain` under Panyushev complementation. This iterator I has the property that `I[0] == antichain` and each i satisfies `self.order_ideal_complement_generators(I[i]) == I[i+1]`, where `I[i+1]` has to be understood as `I[0]` if it is undefined. The entries `I[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: list(P.panyushev_orbit_iter(set([1, 2])))
[{1, 2}, {3}, set()]
sage: list(P.panyushev_orbit_iter([1, 2]))
[{1, 2}, {3}, set()]
sage: list(P.panyushev_orbit_iter([2, 1]))
[{1, 2}, {3}, set()]
sage: list(P.panyushev_orbit_iter(set([1, 2]), element_constructor=list))
[[1, 2], [3], []]
```

(continues on next page)

(continued from previous page)

```
sage: list(P.panyushev_orbit_iter(set([1, 2]), element_
↪constructor=frozenset))
[frozenset({1, 2}), frozenset({3}), frozenset()]
sage: list(P.panyushev_orbit_iter(set([1, 2]), element_constructor=tuple))
[(1, 2), (3,), ()]

sage: P = Poset( {} )
sage: list(P.panyushev_orbit_iter([]))
[set()]

sage: P = Poset({ 1: [2, 3], 2: [4], 3: [4], 4: [] })
sage: Piter = P.panyushev_orbit_iter([2], stop=False)
sage: next(Piter)
{2}
sage: next(Piter)
{3}
sage: next(Piter)
{2}
sage: next(Piter)
{3}
```

panyushev_orbits(*element_constructor=<class 'set'>*)

Return the Panyushev orbits of antichains in `self`.

The Panyushev orbit of an antichain is its orbit under Panyushev complementation (see *panyushev_complement()*).

INPUT:

- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the antichains before they are returned.

OUTPUT:

- the partition of the set of all antichains of `self` into orbits under Panyushev complementation. This is returned as a list of lists `L` such that for each `L` and `i`, cyclically: `self.order_ideal_complement_generators(L[i]) == L[i+1]`. The entries `L[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: orb = P.panyushev_orbits()
sage: sorted(sorted(o) for o in orb)
[[set(), {1, 2}, {3}], [{2}, {1}]]
sage: orb = P.panyushev_orbits(element_constructor=list)
sage: sorted(sorted(o) for o in orb)
[[[], [1, 2], [3]], [[1], [2]]]
sage: orb = P.panyushev_orbits(element_constructor=frozenset)
sage: sorted(sorted(o) for o in orb)
[[frozenset(), frozenset({1, 2}), frozenset({3})],
 [frozenset({2}), frozenset({1})]]
sage: orb = P.panyushev_orbits(element_constructor=tuple)
sage: sorted(sorted(o) for o in orb)
[[(), (1, 2), (3,)], [(1,), (2,)]]
sage: P = Poset( {} )
sage: P.panyushev_orbits()
[[set()]]
```

rowmotion(*order_ideal*)

The image of the order ideal `order_ideal` under rowmotion in `self`.

Rowmotion on a finite poset P is an automorphism of the set $J(P)$ of all order ideals of P. One way to define it is as follows: Given an order ideal $I \in J(P)$, we let F be the set-theoretic complement of I in P. Furthermore we let A be the antichain consisting of all minimal elements of F. Then, the rowmotion of I is defined to be the order ideal of P generated by the antichain A (that is, the order ideal consisting of each element of P which has some element of A above it).

Rowmotion is related (actually, isomorphic) to Panyushev complementation (*`panyushev_complement()`*).

INPUT:

- `order_ideal` – an order ideal of `self`, as a set

OUTPUT:

- the image of `order_ideal` under rowmotion, as a set again

EXAMPLES:

```
sage: P = Poset( {1: [2, 3], 2: [], 3: [], 4: [8], 5: [], 6: [5], 7: [1,␣
↪4], 8: []} )
sage: I = Set({2, 6, 1, 7})
sage: P.rowmotion(I)
{1, 3, 4, 5, 6, 7}

sage: P = Poset( {} )
sage: I = Set({})
sage: P.rowmotion(I)
{}
```

rowmotion_orbit_iter(*oideal*, *element_constructor=<class 'set'>*, *stop=True*, *check=True*)

Iterate over the rowmotion orbit of an order ideal `oideal` of `self`.

The rowmotion orbit of an order ideal is its orbit under rowmotion (see *`rowmotion()`*).

INPUT:

- `oideal` – an order ideal of `self`, given as an iterable.
- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the order ideals before they are yielded.
- `stop` – a Boolean (default: `True`) determining whether the iterator should stop once it completes its cycle (this happens when it is set to `True`) or go on forever (this happens when it is set to `False`).
- `check` – a Boolean (default: `True`) determining whether `oideal` should be checked for being an order ideal.

OUTPUT:

- an iterator over the orbit of the order ideal `oideal` under rowmotion. This iterator I has the property that `I[0] == oideal` and that every i satisfies `self.rowmotion(I[i]) == I[i+1]`, where `I[i+1]` has to be understood as `I[0]` if it is undefined. The entries `I[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: list(P.rowmotion_orbit_iter(set([1, 2])))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.rowmotion_orbit_iter([1, 2]))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.rowmotion_orbit_iter([2, 1]))
[{1, 2}, {1, 2, 3}, set()]
```

(continues on next page)

(continued from previous page)

```
sage: list(P.rowmotion_orbit_iter(set([1, 2]), element_constructor=list))
[[1, 2], [1, 2, 3], []]
sage: list(P.rowmotion_orbit_iter(set([1, 2]), element_
↪constructor=frozenset))
[frozenset({1, 2}), frozenset({1, 2, 3}), frozenset()]
sage: list(P.rowmotion_orbit_iter(set([1, 2]), element_constructor=tuple))
[(1, 2), (1, 2, 3), ()]

sage: P = Poset( {} )
sage: list(P.rowmotion_orbit_iter([]))
[set()]

sage: P = Poset({ 1: [2, 3], 2: [4], 3: [4], 4: [] })
sage: Piter = P.rowmotion_orbit_iter([1, 2, 3], stop=False)
sage: next(Piter)
{1, 2, 3}
sage: next(Piter)
{1, 2, 3, 4}
sage: next(Piter)
set()
sage: next(Piter)
{1}
sage: next(Piter)
{1, 2, 3}

sage: P = Poset({ 1: [4], 2: [4, 5], 3: [5] })
sage: list(P.rowmotion_orbit_iter([1, 2], element_constructor=list))
[[1, 2], [1, 2, 3, 4], [2, 3, 5], [1], [2, 3], [1, 2, 3, 5], [1, 2, 4],␣
↪[3]]
```

rowmotion_orbits(*element_constructor=<class 'set'>*)

Return the rowmotion orbits of order ideals in `self`.

The rowmotion orbit of an order ideal is its orbit under rowmotion (see `rowmotion()`).

INPUT:

- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the antichains before they are returned.

OUTPUT:

- the partition of the set of all order ideals of `self` into orbits under rowmotion. This is returned as a list of lists `L` such that for each `L` and `i`, cyclically: `self.rowmotion(L[i]) == L[i+1]`. The entries `L[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( {1: [2, 3], 2: [], 3: [], 4: [2]} )
sage: sorted(len(o) for o in P.rowmotion_orbits())
[3, 5]
sage: orb = P.rowmotion_orbits(element_constructor=list)
sage: sorted(sorted(e) for e in orb)
[[[], [4, 1], [4, 1, 2, 3]], [[1], [1, 3], [4], [4, 1, 2], [4, 1, 3]]]
sage: orb = P.rowmotion_orbits(element_constructor=tuple)
sage: sorted(sorted(e) for e in orb)
[[(), (4, 1), (4, 1, 2, 3)], [(1,), (1, 3), (4,), (4, 1, 2), (4, 1, 3)]]
sage: P = Poset({})
sage: P.rowmotion_orbits(element_constructor=tuple)
[[()]]
```

rowmotion_orbits_plots()

Return plots of the rowmotion orbits of order ideals in `self`.

The rowmotion orbit of an order ideal is its orbit under rowmotion (see *rowmotion()*).

EXAMPLES:

```
sage: P = Poset( {1: [2, 3], 2: [], 3: [], 4: [2]} )
sage: P.rowmotion_orbits_plots()
Graphics Array of size 2 x 5
sage: P = Poset({})
sage: P.rowmotion_orbits_plots()
Graphics Array of size 1 x 1
```

toggling_orbit_iter(*vs*, *oideal*, *element_constructor=<class 'set'>*, *stop=True*, *check=True*)

Iterate over the orbit of an order ideal `oideal` of `self` under the operation of toggling the vertices `vs[0], vs[1], ...` in this order.

See *order_ideal_toggle()* for a definition of toggling.

Warning: The orbit is that under the composition of toggles, *not* under the single toggles themselves. Thus, for example, if `vs == [1,2]`, then the orbit has the form $(I, T_2T_1I, T_2T_1T_2T_1I, \ldots)$ (where I denotes `oideal` and T_i means toggling at i) rather than $(I, T_1I, T_2T_1I, T_1T_2T_1I, \ldots)$.

INPUT:

- `vs`: a list (or other iterable) of elements of `self` (but since the output depends on the order, sets should not be used as `vs`).
- `oideal` – an order ideal of `self`, given as an iterable.
- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the order ideals before they are yielded.
- `stop` – a Boolean (default: `True`) determining whether the iterator should stop once it completes its cycle (this happens when it is set to `True`) or go on forever (this happens when it is set to `False`).
- `check` – a Boolean (default: `True`) determining whether `oideal` should be checked for being an order ideal.

OUTPUT:

- an iterator over the orbit of the order ideal `oideal` under toggling the vertices in the list `vs` in this order. This iterator I has the property that `I[0] == oideal` and that every i satisfies `self.order_ideal_toggles(I[i], vs) == I[i+1]`, where `I[i+1]` has to be understood as `I[0]` if it is undefined. The entries `I[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: list(P.toggling_orbit_iter([1, 3, 1], set([1, 2])))
[{1, 2}]
sage: list(P.toggling_orbit_iter([1, 2, 3], set([1, 2])))
[{1, 2}, set(), {1, 2, 3}]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2])))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2]), element_
↪constructor=list))
[[1, 2], [1, 2, 3], []]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2]), element_
↪constructor=frozenset))
[frozenset({1, 2}), frozenset({1, 2, 3}), frozenset()]
```

(continues on next page)

(continued from previous page)

```
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2]), element_
↪constructor=tuple))
[(1, 2), (1, 2, 3), ()]
sage: list(P.toggling_orbit_iter([3, 2, 1], [2, 1], element_
↪constructor=tuple))
[(1, 2), (1, 2, 3), ()]

sage: P = Poset( {} )
sage: list(P.toggling_orbit_iter([], []))
[set()]

sage: P = Poset({ 1: [2, 3], 2: [4], 3: [4], 4: [] })
sage: Piter = P.toggling_orbit_iter([1, 2, 4, 3], [1, 2, 3], stop=False)
sage: next(Piter)
{1, 2, 3}
sage: next(Piter)
{1}
sage: next(Piter)
set()
sage: next(Piter)
{1, 2, 3}
sage: next(Piter)
{1}
```

toggling_orbits(*vs*, *element_constructor=<class 'set'>*)

Return the orbits of order ideals in `self` under the operation of toggling the vertices `vs[0]`, `vs[1]`, `...` in this order.

See *order_ideal_toggle()* for a definition of toggling.

Warning: The orbits are those under the composition of toggles, *not* under the single toggles themselves. Thus, for example, if `vs == [1,2]`, then the orbits have the form $(I, T_2 T_1 I, T_2 T_1 T_2 T_1 I, \ldots)$ (where I denotes an order ideal and T_i means toggling at i) rather than $(I, T_1 I, T_2 T_1 I, T_1 T_2 T_1 I, \ldots)$.

INPUT:

- `vs`: a list (or other iterable) of elements of `self` (but since the output depends on the order, sets should not be used as `vs`).

OUTPUT:

- a partition of the order ideals of `self`, as a list of sets `L` such that for each `L` and `i`, cyclically: `self.order_ideal_toggles(L[i], vs) == L[i+1]`.

EXAMPLES:

```
sage: P = Poset( {1: [2, 4], 2: [], 3: [4], 4: []} )
sage: sorted(len(o) for o in P.toggling_orbits([1, 2]))
[2, 3, 3]
sage: P = Poset( {1: [3], 2: [1, 4], 3: [], 4: [3]} )
sage: sorted(len(o) for o in P.toggling_orbits((1, 2, 4, 3)))
[3, 3]
```

toggling_orbits_plots(*vs*)

Return plots of the orbits of order ideals in `self` under the operation of toggling the vertices `vs[0]`, `vs[1]`, `...` in this order.

See *toggling_orbits()* for more information.

EXAMPLES:

```
sage: P = Poset( {1: [2, 3], 2: [], 3: [], 4: [2]} )
sage: P.toggling_orbits_plots([1,2,3,4])
Graphics Array of size 2 x 5
sage: P = Poset({})
sage: P.toggling_orbits_plots([])
Graphics Array of size 1 x 1
```

4.63 Finite semigroups

class sage.categories.finite_semigroups.**FiniteSemigroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite (multiplicative) semigroups.

A finite semigroup is a *finite set* endowed with an associative binary operation $*$.

Warning: Finite semigroups in Sage used to be automatically endowed with an *enumerated set* structure; the default enumeration is then obtained by iteratively multiplying the semigroup generators. This forced any finite semigroup to either implement an enumeration, or provide semigroup generators; this was often inconvenient.

Instead, finite semigroups that provide a distinguished finite set of generators with `semigroup_generators()` should now explicitly declare themselves in the category of `finitely generated semigroups`:

```
sage: Semigroups().FinitelyGenerated()
Category of finitely generated semigroups
```

This is a backward incompatible change.

EXAMPLES:

```
sage: C = FiniteSemigroups(); C
Category of finite semigroups
sage: C.super_categories()
[Category of semigroups, Category of finite sets]
sage: sorted(C.axioms())
['Associative', 'Finite']
sage: C.example()
An example of a finite semigroup:
 the left regular band generated by ('a', 'b', 'c', 'd')
```

class ParentMethods

Bases: `object`

idempotents()

Returns the idempotents of the semigroup

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('x','y'))
sage: sorted(S.idempotents())
['x', 'xy', 'y', 'yx']
```

j_classes()

Returns the J-classes of the semigroup.

Two elements u and v of a monoid are in the same J-class if u divides v and v divides u.

OUTPUT:
All the J-classes of self, as a list of lists.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b', 'c'))
sage: sorted(map(sorted, S.j_classes()))                                    #␣
↪needs sage.graphs
[['a'], ['ab', 'ba'], ['abc', 'acb', 'bac', 'bca', 'cab', 'cba'],
 ['ac', 'ca'], ['b'], ['bc', 'cb'], ['c']]
```

j_classes_of_idempotents()

Returns all the idempotents of self, grouped by J-class.

OUTPUT:
a list of lists.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b', 'c'))
sage: sorted(map(sorted, S.j_classes_of_idempotents()))                     #␣
↪needs sage.graphs
[['a'], ['ab', 'ba'], ['abc', 'acb', 'bac', 'bca', 'cab', 'cba'],
 ['ac', 'ca'], ['b'], ['bc', 'cb'], ['c']]
```

j_transversal_of_idempotents()

Returns a list of one idempotent per regular J-class

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b', 'c'))
```

The chosen elements depend on the order of each J-class, and that order is random when using Python 3.

```
sage: sorted(S.j_transversal_of_idempotents())   # random                   #␣
↪needs sage.graphs
['a', 'ab', 'abc', 'ac', 'b', 'c', 'cb']
```

4.64 Finite sets

class sage.categories.finite_sets.**FiniteSets**(*base_category*)

Bases: `CategoryWithAxiom_singleton`

The category of finite sets.

EXAMPLES:

```
sage: C = FiniteSets(); C
Category of finite sets
sage: C.super_categories()
[Category of sets]
```

(continues on next page)

(continued from previous page)

```
sage: C.all_super_categories()
[Category of finite sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
sage: C.example()
NotImplemented
```

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

extra_super_categories()

EXAMPLES:

```
sage: FiniteSets().Algebras(QQ).extra_super_categories()
[Category of finite dimensional vector spaces with basis over Rational␣
↪Field]
```

This implements the fact that the algebra of a finite set is finite dimensional:

```
sage: FiniteMonoids().Algebras(QQ).is_subcategory(AlgebrasWithBasis(QQ).
↪FiniteDimensional())
True
```

class ParentMethods

Bases: `object`

is_finite()

Return `True` since `self` is finite.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.is_finite()
True
```

class Subquotients(*category*, **args*)

Bases: *SubquotientsCategory*

extra_super_categories()

EXAMPLES:

```
sage: FiniteSets().Subquotients().extra_super_categories()
[Category of finite sets]
```

This implements the fact that a subquotient (and therefore a quotient or subobject) of a finite set is finite:

```
sage: FiniteSets().Subquotients().is_subcategory(FiniteSets())
True
sage: FiniteSets().Quotients   ().is_subcategory(FiniteSets())
True
sage: FiniteSets().Subobjects  ().is_subcategory(FiniteSets())
True
```

4.65 Finite Weyl Groups

class sage.categories.finite_weyl_groups.**FiniteWeylGroups**(*base_category*)

Bases: *CategoryWithAxiom*

The category of finite Weyl groups.

EXAMPLES:

```
sage: C = FiniteWeylGroups()
sage: C
Category of finite Weyl groups
sage: C.super_categories()
[Category of finite Coxeter groups, Category of Weyl groups]
sage: C.example()
The symmetric group on {0, ..., 3}
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

4.66 Finitely Generated Lambda bracket Algebras

AUTHORS:

- Reimundo Heluani (2020-08-21): Initial implementation.

class sage.categories.finitely_generated_lambda_bracket_algebras.**FinitelyGeneratedLambdaBra**

Bases: *CategoryWithAxiom_over_base_ring*

The category of finitely generated lambda bracket algebras.

EXAMPLES:

```
sage: from sage.categories.lambda_bracket_algebras import LambdaBracketAlgebras
sage: LambdaBracketAlgebras(QQbar).FinitelyGenerated()                        #␣
↪needs sage.rings.number_field
Category of finitely generated lambda bracket algebras over Algebraic Field
```

class Graded(*base_category*)

Bases: *GradedModulesCategory*

The category of H-graded finitely generated Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).FinitelyGenerated().Graded()             #␣
↪needs sage.rings.number_field
Category of H-graded finitely generated Lie conformal algebras
 over Algebraic Field
```

class ParentMethods

Bases: `object`

gen(*i*)

The `i`-th generator of this Lie conformal algebra.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: V = lie_conformal_algebras.Affine(QQ, 'A1')
sage: V.gens()
(B[alpha[1]], B[alphacheck[1]], B[-alpha[1]], B['K'])
sage: V.gen(0)
B[alpha[1]]
sage: V.1
B[alphacheck[1]]
```

ngens()

The number of generators of this Lie conformal algebra.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)                           #␣
↪needs sage.combinat sage.modules
sage: Vir.ngens()                                                          #␣
↪needs sage.combinat sage.modules
2

sage: V = lie_conformal_algebras.Affine(QQ, 'A2')                         #␣
↪needs sage.combinat sage.modules
sage: V.ngens()                                                            #␣
↪needs sage.combinat sage.modules
9
```

some_elements()

Some elements of this Lie conformal algebra.

This method returns a list with elements containing at least the generators.

EXAMPLES:

```
sage: V = lie_conformal_algebras.Affine(QQ, 'A1',                         #␣
↪needs sage.combinat sage.modules
....:                                     names=('e', 'h', 'f'))
sage: V.some_elements()                                                    #␣
↪needs sage.combinat sage.modules
[e, h, f, K, ...]
sage: all(v.parent() is V for v in V.some_elements())                     #␣
↪needs sage.combinat sage.modules
True
```

4.67 Finitely Generated Lie Conformal Algebras

AUTHORS:

- Reimundo Heluani (2019-10-05): Initial implementation.

class sage.categories.finitely_generated_lie_conformal_algebras.**FinitelyGeneratedLieConfor**

Bases: *CategoryWithAxiom_over_base_ring*

The category of finitely generated Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).FinitelyGenerated()                          #␣
↪needs sage.rings.number_field
Category of finitely generated Lie conformal algebras over Algebraic Field
```

class Graded(*base_category*)

Bases: *GradedModulesCategory*

The category of H-graded finitely generated Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).FinitelyGenerated().Graded()                 #␣
↪needs sage.rings.number_field
Category of H-graded finitely generated Lie conformal algebras
 over Algebraic Field
```

class ParentMethods

Bases: object

some_elements()

Some elements of this Lie conformal algebra.

Returns a list with elements containing at least the generators.

EXAMPLES:

```
sage: V = lie_conformal_algebras.Affine(QQ, 'A1',                             #␣
↪needs sage.combinat sage.modules
....:                                      names=('e', 'h', 'f'))
sage: V.some_elements()                                                        #␣
↪needs sage.combinat sage.modules
[e, h, f, K, ...]
sage: all(v.parent() is V for v in V.some_elements())                          #␣
↪needs sage.combinat sage.modules
True
```

class Super(*base_category*)

Bases: *SuperModulesCategory*

The category of super finitely generated Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(AA).FinitelyGenerated().Super()                    #␣
↪needs sage.rings.number_field
Category of super finitely generated Lie conformal algebras
 over Algebraic Real Field
```

class Graded(*base_category*)

Bases: `GradedModulesCategory`

The category of H-graded super finitely generated Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).FinitelyGenerated().Super().Graded()   #␣
↪needs sage.rings.number_field
Category of H-graded super finitely generated Lie conformal algebras
 over Algebraic Field
```

4.68 Finitely generated magmas

class sage.categories.finitely_generated_magmas.**FinitelyGeneratedMagmas**(*base_category*)

Bases: `CategoryWithAxiom_singleton`

The category of finitely generated (multiplicative) magmas.

See `Magmas.SubcategoryMethods.FinitelyGeneratedAsMagma()` for details.

EXAMPLES:

```
sage: C = Magmas().FinitelyGeneratedAsMagma(); C
Category of finitely generated magmas
sage: C.super_categories()
[Category of magmas]
sage: sorted(C.axioms())
['FinitelyGeneratedAsMagma']
```

class ParentMethods

Bases: `object`

magma_generators()

Return distinguished magma generators for `self`.

OUTPUT: a finite family

This method should be implemented by all `finitely generated magmas`.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.magma_generators()
Family ('a', 'b', 'c', 'd')
```

4.69 Finitely generated semigroups

class sage.categories.finitely_generated_semigroups.**FinitelyGeneratedSemigroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finitely generated (multiplicative) semigroups.

A *finitely generated semigroup* is a *semigroup* endowed with a distinguished finite set of generators (see *FinitelyGeneratedSemigroups.ParentMethods.semigroup_generators()*). This makes it into an *enumerated set*.

EXAMPLES:

```
sage: C = Semigroups().FinitelyGenerated(); C
Category of finitely generated semigroups
sage: C.super_categories()
[Category of semigroups,
 Category of finitely generated magmas,
 Category of enumerated sets]
sage: sorted(C.axioms())
['Associative', 'Enumerated', 'FinitelyGeneratedAsMagma']
sage: C.example()
An example of a semigroup: the free semigroup generated
by ('a', 'b', 'c', 'd')
```

class Finite(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class ParentMethods

Bases: object

some_elements()

Return an iterable containing some elements of the semigroup.

OUTPUT: the ten first elements of the semigroup, if they exist.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('x','y'))
sage: sorted(S.some_elements())
['x', 'xy', 'y', 'yx']
sage: S = FiniteSemigroups().example(alphabet=('x','y','z'))
sage: X = S.some_elements()
sage: len(X)
10
sage: all(x in S for x in X)
True
```

class ParentMethods

Bases: object

ideal(*gens*, *side='twosided'*)

Return the side-sided ideal generated by gens.

This brute force implementation recursively multiplies the elements of gens by the distinguished generators of this semigroup.

See also:

semigroup_generators()

INPUT:

- `gens` – a list (or iterable) of elements of `self`
- `side` – [default: "twosided"] "left", "right" or "twosided"

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: sorted(S.ideal([S('cab')], side="left"))
['abc', 'abcd', 'abdc', 'acb', 'acbd', 'acdb', 'adbc',
 'adcb', 'bac', 'bacd', 'badc', 'bca', 'bcad', 'bcda',
 'bdac', 'bdca', 'cab', 'cabd', 'cadb', 'cba', 'cbad',
 'cbda', 'cdab', 'cdba', 'dabc', 'dacb', 'dbac', 'dbca',
 'dcab', 'dcba']
sage: list(S.ideal([S('cab')], side="right"))
['cab', 'cabd']
sage: sorted(S.ideal([S('cab')], side="twosided"))
['abc', 'abcd', 'abdc', 'acb', 'acbd', 'acdb', 'adbc',
 'adcb', 'bac', 'bacd', 'badc', 'bca', 'bcad', 'bcda',
 'bdac', 'bdca', 'cab', 'cabd', 'cadb', 'cba', 'cbad',
 'cbda', 'cdab', 'cdba', 'dabc', 'dacb', 'dbac', 'dbca',
 'dcab', 'dcba']
sage: sorted(S.ideal([S('cab')]))
['abc', 'abcd', 'abdc', 'acb', 'acbd', 'acdb', 'adbc',
 'adcb', 'bac', 'bacd', 'badc', 'bca', 'bcad', 'bcda',
 'bdac', 'bdca', 'cab', 'cabd', 'cadb', 'cba', 'cbad',
 'cbda', 'cdab', 'cdba', 'dabc', 'dacb', 'dbac', 'dbca',
 'dcab', 'dcba']
```

semigroup_generators()

Return distinguished semigroup generators for `self`.

OUTPUT: a finite family

This method should be implemented by all semigroups in *FinitelyGeneratedSemigroups*.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

succ_generators(*side='twosided'*)

Return the successor function of the `side`-sided Cayley graph of `self`.

This is a function that maps an element of `self` to all the products of `x` by a generator of this semigroup, where the product is taken on the left, right, or both sides.

INPUT:

- `side`: "left", "right", or "twosided"

Todo: Design choice:

- find a better name for this method
- should we return a set? a family?

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.succ_generators("left" )(S('ca'))
('ac', 'bca', 'ca', 'dca')
sage: S.succ_generators("right")(S('ca'))
('ca', 'cab', 'ca', 'cad')
sage: S.succ_generators("twosided" )(S('ca'))
('ac', 'bca', 'ca', 'dca', 'ca', 'cab', 'ca', 'cad')
```

example()

EXAMPLES:

```
sage: Semigroups().FinitelyGenerated().example()
An example of a semigroup: the free semigroup generated
by ('a', 'b', 'c', 'd')
```

extra_super_categories()

State that a finitely generated semigroup is endowed with a default enumeration.

EXAMPLES:

```
sage: Semigroups().FinitelyGenerated().extra_super_categories()
[Category of enumerated sets]
```

4.70 Function fields

class sage.categories.function_fields.**FunctionFields**

Bases: *Category*

The category of function fields.

EXAMPLES:

We create the category of function fields:

```
sage: C = FunctionFields()
sage: C
Category of function fields
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

super_categories()

Returns the Category of which this is a direct sub-Category For a list off all super categories see all_super_categories

EXAMPLES:

```
sage: FunctionFields().super_categories()
[Category of fields]
```

4.71 G-Sets

class sage.categories.g_sets.**GSets**(G)

Bases: *Category*

The category of G-sets, for a group G.

EXAMPLES:

```
sage: S = SymmetricGroup(3)                                                     #␣
↪needs sage.groups
sage: GSets(S)                                                                  #␣
↪needs sage.groups
Category of G-sets for Symmetric group of order 3! as a permutation group
```

TODO: should this derive from Category_over_base?

classmethod an_instance()

Returns an instance of this class.

EXAMPLES:

```
sage: GSets.an_instance()  # indirect doctest                                   #␣
↪needs sage.groups
Category of G-sets for Symmetric group of order 8! as a permutation group
```

super_categories()

EXAMPLES:

```
sage: GSets(SymmetricGroup(8)).super_categories()                               #␣
↪needs sage.groups
[Category of sets]
```

4.72 Gcd domains

class sage.categories.gcd_domains.**GcdDomains**

Bases: *Category_singleton*

The category of gcd domains domains where gcd can be computed but where there is no guarantee of factorisation into irreducibles

EXAMPLES:

```
sage: GcdDomains()
Category of gcd domains
sage: GcdDomains().super_categories()
[Category of integral domains]
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

additional_structure()

Return None.

Indeed, the category of gcd domains defines no additional structure: a ring morphism between two gcd domains is a gcd domain morphism.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: GcdDomains().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: GcdDomains().super_categories()
[Category of integral domains]
```

4.73 Generalized Coxeter Groups

class sage.categories.generalized_coxeter_groups.**GeneralizedCoxeterGroups**

Bases: *Category_singleton*

The category of generalized Coxeter groups.

A generalized Coxeter group is a group with a presentation of the following form:

$$\langle s_i \mid s_i^{p_i}, s_i s_j \cdots = s_j s_i \cdots \rangle,$$

where $p_i > 1$, $i \in I$, and the factors in the braid relation occur $m_{ij} = m_{ji}$ times for all $i \neq j \in I$.

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: C = GeneralizedCoxeterGroups(); C
Category of generalized Coxeter groups
```

class Finite(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of finite generalized Coxeter groups.

extra_super_categories()

Implement that a finite generalized Coxeter group is a well-generated complex reflection group.

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups

sage: Cat = GeneralizedCoxeterGroups().Finite()
sage: Cat.extra_super_categories()
```

(continues on next page)

(continued from previous page)

```
[Category of well generated finite complex reflection groups]
sage: Cat.is_subcategory(ComplexReflectionGroups().Finite().
↪WellGenerated())
True
```

additional_structure()

Return None.

Indeed, all the structure generalized Coxeter groups have in addition to groups (simple reflections, …) is already defined in the super category.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: GeneralizedCoxeterGroups().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: GeneralizedCoxeterGroups().super_categories()
[Category of complex reflection or generalized Coxeter groups]
```

4.74 Graded Algebras

class sage.categories.graded_algebras.**GradedAlgebras**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded algebras

EXAMPLES:

```
sage: GradedAlgebras(ZZ)
Category of graded algebras over Integer Ring
sage: GradedAlgebras(ZZ).super_categories()
[Category of filtered algebras over Integer Ring,
 Category of graded modules over Integer Ring]
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

graded_algebra()

Return the associated graded algebra to self.

Since self is already graded, this just returns self.

EXAMPLES:

```
sage: m = SymmetricFunctions(QQ).m()                                          #␣
↪needs sage.combinat sage.modules
sage: m.graded_algebra() is m                                                 #␣
↪needs sage.combinat sage.modules
True
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

extra_super_categories()

EXAMPLES:

```
sage: Algebras(QQ).Graded().SignedTensorProducts().extra_super_
↪categories()
[Category of graded algebras over Rational Field]
sage: Algebras(QQ).Graded().SignedTensorProducts().super_categories()
[Category of graded algebras over Rational Field]
```

Meaning: a signed tensor product of algebras is an algebra

class SubcategoryMethods

Bases: `object`

SignedTensorProducts()

Return the full subcategory of objects of `self` constructed as signed tensor products.

See also:

- *SignedTensorProductsCategory*
- *CovariantFunctorialConstruction*

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).Graded().SignedTensorProducts()
Category of signed tensor products of graded algebras with basis
 over Rational Field
```

4.75 Graded algebras with basis

class sage.categories.graded_algebras_with_basis.**GradedAlgebrasWithBasis**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded algebras with a distinguished basis

EXAMPLES:

```
sage: C = GradedAlgebrasWithBasis(ZZ); C
Category of graded algebras with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered algebras with basis over Integer Ring,
 Category of graded algebras over Integer Ring,
 Category of graded modules with basis over Integer Ring]
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

completion()

Return the completion of all formal linear combinations of self with finite linear combinations in each homogeneous degree (computed lazily).

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: S = NCSF.Complete()
sage: L = S.formal_series_ring()
sage: L
Lazy completion of Non-Commutative Symmetric Functions over
 the Rational Field in the Complete basis
```

formal_series_ring()

Return the completion of all formal linear combinations of self with finite linear combinations in each homogeneous degree (computed lazily).

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: S = NCSF.Complete()
sage: L = S.formal_series_ring()
sage: L
Lazy completion of Non-Commutative Symmetric Functions over
 the Rational Field in the Complete basis
```

free_graded_module(*generator_degrees*, *names=None*)

Create a finitely generated free graded module over self

INPUT:

- generator_degrees – tuple of integers defining the number of generators of the module and their degrees
- names – (optional) the names of the generators. If names is a comma-separated string like 'a, b, c', then those will be the names. Otherwise, for example if names is abc, then the names will be abc[d,i].

By default, if all generators are in distinct degrees, then the names of the generators will have the form g[d] where d is the degree of the generator. If the degrees are not distinct, then the generators will be called g[d,i] where d is the degree and i is its index in the list of generators in that degree.

See sage.modules.fp_graded.free_module for more examples and details.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: Q = QuadraticForm(QQ, 3, [1,2,3,4,5,6])
sage: Cl = CliffordAlgebra(Q)
sage: M = Cl.free_graded_module((0, 2, 3))
sage: M.gens()
(g[0], g[2], g[3])
sage: N.<xy, z> = Cl.free_graded_module((1, 2))
```

(continues on next page)

(continued from previous page)

```
sage: N.generators()
(xy, z)
```

graded_algebra()

Return the associated graded algebra to self.

This is self, because self is already graded. See graded_algebra() for the general behavior of this method, and see AssociatedGradedAlgebra for the definition and properties of associated graded algebras.

EXAMPLES:

```
sage: m = SymmetricFunctions(QQ).m()                                        #␣
↪needs sage.combinat sage.modules
sage: m.graded_algebra() is m                                               #␣
↪needs sage.combinat sage.modules
True
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

The category of algebras with basis constructed by signed tensor product of algebras with basis.

class ParentMethods

Bases: object

Implements operations on tensor products of super algebras with basis.

one_basis()

Return the index of the one of this signed tensor product of algebras, as per AlgebrasWithBasis.ParentMethods.one_basis.

It is the tuple whose operands are the indices of the ones of the operands, as returned by their *one_basis()* methods.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y> = ExteriorAlgebra(QQ)
sage: A.one_basis()
0
sage: B = tensor((A, A, A))
sage: B.one_basis()
(0, 0, 0)
sage: B.one()
1 # 1 # 1
```

product_on_basis(*t0*, *t1*)

The product of the algebra on the basis, as per AlgebrasWithBasis.ParentMethods.product_on_basis.

EXAMPLES:

Test the sign in the super tensor product:

```
sage: # needs sage.combinat sage.modules
sage: A = SteenrodAlgebra(3)
sage: x = A.Q(0)
```

(continues on next page)

(continued from previous page)

```
sage: y = x.coproduct()
sage: y^2
0
```

TODO: optimize this implementation!

extra_super_categories()

EXAMPLES:

```
sage: Cat = AlgebrasWithBasis(QQ).Graded()
sage: Cat.SignedTensorProducts().extra_super_categories()
[Category of graded algebras with basis over Rational Field]
sage: Cat.SignedTensorProducts().super_categories()
[Category of graded algebras with basis over Rational Field,
 Category of signed tensor products of graded algebras over Rational␣
↪Field]
```

4.76 Graded bialgebras

sage.categories.graded_bialgebras.**GradedBialgebras**(*base_ring*)

The category of graded bialgebras

EXAMPLES:

```
sage: C = GradedBialgebras(QQ); C
Join of Category of graded algebras over Rational Field
    and Category of bialgebras over Rational Field
    and Category of graded coalgebras over Rational Field
sage: C is Bialgebras(QQ).Graded()
True
```

4.77 Graded bialgebras with basis

sage.categories.graded_bialgebras_with_basis.**GradedBialgebrasWithBasis**(*base_ring*)

The category of graded bialgebras with a distinguished basis

EXAMPLES:

```
sage: C = GradedBialgebrasWithBasis(QQ); C
Join of Category of ...
sage: sorted(C.super_categories(), key=str)
[Category of bialgebras with basis over Rational Field,
 Category of graded algebras with basis over Rational Field,
 Category of graded coalgebras with basis over Rational Field]
```

4.78 Graded Coalgebras

class sage.categories.graded_coalgebras.**GradedCoalgebras**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded coalgebras

EXAMPLES:

```
sage: C = GradedCoalgebras(QQ); C
Category of graded coalgebras over Rational Field
sage: C is Coalgebras(QQ).Graded()
True
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

extra_super_categories()

EXAMPLES:

```
sage: Coalgebras(QQ).Graded().SignedTensorProducts().extra_super_
↪categories()
[Category of graded coalgebras over Rational Field]
sage: Coalgebras(QQ).Graded().SignedTensorProducts().super_categories()
[Category of graded coalgebras over Rational Field]
```

Meaning: a signed tensor product of coalgebras is a coalgebra

class SubcategoryMethods

Bases: object

SignedTensorProducts()

Return the full subcategory of objects of self constructed as signed tensor products.

See also:

- *SignedTensorProductsCategory*
- *CovariantFunctorialConstruction*

EXAMPLES:

```
sage: CoalgebrasWithBasis(QQ).Graded().SignedTensorProducts()
Category of signed tensor products of graded coalgebras with basis
 over Rational Field
```

4.79 Graded coalgebras with basis

class sage.categories.graded_coalgebras_with_basis.**GradedCoalgebrasWithBasis**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded coalgebras with a distinguished basis.

EXAMPLES:

```
sage: C = GradedCoalgebrasWithBasis(QQ); C
Category of graded coalgebras with basis over Rational Field
sage: C is Coalgebras(QQ).WithBasis().Graded()
True
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

The category of coalgebras with basis constructed by signed tensor product of coalgebras with basis.

extra_super_categories()

EXAMPLES:

```
sage: Cat = CoalgebrasWithBasis(QQ).Graded()
sage: Cat.SignedTensorProducts().extra_super_categories()
[Category of graded coalgebras with basis over Rational Field]
sage: Cat.SignedTensorProducts().super_categories()
[Category of graded coalgebras with basis over Rational Field,
 Category of signed tensor products of graded coalgebras over Rational␣
↪Field]
```

4.80 Graded Hopf algebras

sage.categories.graded_hopf_algebras.**GradedHopfAlgebras**(*base_ring*)

The category of graded Hopf algebras.

EXAMPLES:

```
sage: C = GradedHopfAlgebras(QQ); C
Join of Category of Hopf algebras over Rational Field
    and Category of graded algebras over Rational Field
    and Category of graded coalgebras over Rational Field
sage: C is HopfAlgebras(QQ).Graded()
True
```

Note: This is not a graded Hopf algebra as is typically defined in algebraic topology as the product in the tensor square $(x \otimes y)(a \otimes b) = (xa) \otimes (yb)$ does not carry an additional sign. For this, instead use *super Hopf algebras*.

4.81 Graded Hopf algebras with basis

class sage.categories.graded_hopf_algebras_with_basis.**GradedHopfAlgebrasWithBasis**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded Hopf algebras with a distinguished basis.

EXAMPLES:

```
sage: C = GradedHopfAlgebrasWithBasis(ZZ); C
Category of graded Hopf algebras with basis over Integer Ring
sage: C.super_categories()
[Category of filtered Hopf algebras with basis over Integer Ring,
 Category of graded algebras with basis over Integer Ring,
 Category of graded coalgebras with basis over Integer Ring]

sage: C is HopfAlgebras(ZZ).WithBasis().Graded()
True
sage: C is HopfAlgebras(ZZ).Graded().WithBasis()
False
```

class Connected(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

antipode_on_basis(*index*)

The antipode on the basis element indexed by `index`.

INPUT:

- `index` – an element of the index set

For a graded connected Hopf algebra, we can define an antipode recursively by

$$S(x) := -\sum_{x^L \neq x} S(x^L) \times x^R$$

when $|x| > 0$, and by $S(x) = x$ when $|x| = 0$.

counit_on_basis(*i*)

The default counit of a graded connected Hopf algebra.

INPUT:

- `i` – an element of the index set

OUTPUT:

- an element of the base ring

$$c(i) := \begin{cases} 1 & \text{if } i \text{ indexes the } 1 \text{ of the algebra} \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLES:

```
sage: H = GradedHopfAlgebrasWithBasis(QQ).Connected().example()      #␣
↪needs sage.modules
sage: H.monomial(4).counit()   # indirect doctest                    #␣
↪needs sage.modules
0
sage: H.monomial(0).counit()   # indirect doctest                    #␣
↪needs sage.modules
1
```

example()

Return an example of a graded connected Hopf algebra with a distinguished basis.

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

class WithRealizations(*category*, **args*)

Bases: *WithRealizationsCategory*

super_categories()

EXAMPLES:

```
sage: GradedHopfAlgebrasWithBasis(QQ).WithRealizations().super_
↪categories()
[Join of Category of Hopf algebras over Rational Field
 and Category of graded algebras over Rational Field
 and Category of graded coalgebras over Rational Field]
```

example()

Return an example of a graded Hopf algebra with a distinguished basis.

4.82 Graded Lie Algebras

AUTHORS:

- Eero Hakavuori (2018-08-16): initial version

class sage.categories.graded_lie_algebras.**GradedLieAlgebras**(*base_category*)

Bases: *GradedModulesCategory*

Category of graded Lie algebras.

class Stratified(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Category of stratified Lie algebras.

A graded Lie algebra $L = \bigoplus_{k=1}^{M} L_k$ (where possibly $M = \infty$) is called *stratified* if it is generated by L_1; in other words, we have $L_{k+1} = [L_1, L_k]$.

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Category of finite dimensional stratified Lie algebras.

EXAMPLES:

```
sage: LieAlgebras(QQ).Graded().Stratified().FiniteDimensional()
Category of finite dimensional stratified Lie algebras over Rational Field
```

extra_super_categories()

Implements the fact that a finite dimensional stratified Lie algebra is nilpotent.

EXAMPLES:

```
sage: C = LieAlgebras(QQ).Graded().Stratified().FiniteDimensional()
sage: C.extra_super_categories()
[Category of nilpotent Lie algebras over Rational Field]
sage: C is C.Nilpotent()
True
sage: C.is_subcategory(LieAlgebras(QQ).Nilpotent())
True
```

class SubcategoryMethods

Bases: `object`

Stratified()

Return the full subcategory of stratified objects of `self`.

A Lie algebra is stratified if it is graded and generated as a Lie algebra by its component of degree one.

EXAMPLES:

```
sage: LieAlgebras(QQ).Graded().Stratified()
Category of stratified Lie algebras over Rational Field
```

4.83 Graded Lie Algebras With Basis

class sage.categories.graded_lie_algebras_with_basis.**GradedLieAlgebrasWithBasis**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded Lie algebras with a distinguished basis.

EXAMPLES:

```
sage: C = LieAlgebras(ZZ).WithBasis().Graded(); C
Category of graded Lie algebras with basis over Integer Ring
sage: C.super_categories()
[Category of graded modules with basis over Integer Ring,
 Category of Lie algebras with basis over Integer Ring,
 Category of graded Lie algebras over Integer Ring]

sage: C is LieAlgebras(ZZ).WithBasis().Graded()
True
sage: C is LieAlgebras(ZZ).Graded().WithBasis()
False
```

FiniteDimensional

alias of *FiniteDimensionalGradedLieAlgebrasWithBasis*

4.84 Graded Lie Conformal Algebras

AUTHORS:

- Reimundo Heluani (2019-10-05): Initial implementation.

class sage.categories.graded_lie_conformal_algebras.**GradedLieConformalAlgebras**(*base_category*)

Bases: *GradedLieConformalAlgebrasCategory*

The category of graded Lie conformal algebras.

EXAMPLES:

```
sage: C = LieConformalAlgebras(QQbar).Graded(); C                          #␣
↪needs sage.rings.number_field
Category of H-graded Lie conformal algebras over Algebraic Field

sage: CS = LieConformalAlgebras(QQ).Graded().Super(); CS
Category of H-graded super Lie conformal algebras over Rational Field
sage: CS is LieConformalAlgebras(QQ).Super().Graded()
True
```

class sage.categories.graded_lie_conformal_algebras.**GradedLieConformalAlgebrasCategory**(*base_e-gory*)

Bases: *GradedModulesCategory*

Super(*base_ring=None*)

Return the super-analogue category of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: C = LieConformalAlgebras(QQbar)
sage: C.Graded().Super() is C.Super().Graded()
True
sage: Cp = C.WithBasis()
sage: Cp.Graded().Super() is Cp.Super().Graded()
True
```

4.85 Graded modules

class sage.categories.graded_modules.**GradedModules**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded modules.

We consider every graded module $M = \bigoplus_i M_i$ as a filtered module under the (natural) filtration given by

$$F_i = \bigoplus_{j<i} M_j.$$

EXAMPLES:

```
sage: GradedModules(ZZ)
Category of graded modules over Integer Ring
sage: GradedModules(ZZ).super_categories()
[Category of filtered modules over Integer Ring]
```

The category of graded modules defines the graded structure which shall be preserved by morphisms:

```
sage: Modules(ZZ).Graded().additional_structure()
Category of graded modules over Integer Ring
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

class sage.categories.graded_modules.**GradedModulesCategory**(*base_category*)

Bases: *RegressiveCovariantConstructionCategory*, *Category_over_base_ring*

EXAMPLES:

```
sage: C = GradedAlgebras(QQ)
sage: C
Category of graded algebras over Rational Field
sage: C.base_category()
Category of algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of filtered algebras over Rational Field,
 Category of graded vector spaces over Rational Field]

sage: AlgebrasWithBasis(QQ).Graded().base_ring()
Rational Field
sage: GradedHopfAlgebrasWithBasis(QQ).base_ring()
Rational Field
```

classmethod default_super_categories(*category*, **args*)

Return the default super categories of `category.Graded()`.

Mathematical meaning: every graded object (module, algebra, etc.) is a filtered object with the (implicit) filtration defined by $F_i = \bigoplus_{j\leq i} G_j$.

INPUT:

- `cls` – the class `GradedModulesCategory`
- `category` – a category

OUTPUT: a (join) category

In practice, this returns `category.Filtered()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category.Filtered()` and `cat` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Algebras()`, which has `cat=Modules()` as super category. Then, a grading of an algebra G is also a filtration of G:

```
sage: Algebras(QQ).Graded().super_categories()
[Category of filtered algebras over Rational Field,
 Category of graded vector spaces over Rational Field]
```

This resulted from the following call:

```
sage: sage.categories.graded_modules.GradedModulesCategory.default_super_
↪categories(Algebras(QQ))
Join of Category of filtered algebras over Rational Field
 and Category of graded vector spaces over Rational Field
```

4.86 Graded modules with basis

class sage.categories.graded_modules_with_basis.**GradedModulesWithBasis**(*base_category*)

Bases: *GradedModulesCategory*

The category of graded modules with a distinguished basis.

EXAMPLES:

```
sage: C = GradedModulesWithBasis(ZZ); C
Category of graded modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered modules with basis over Integer Ring,
 Category of graded modules over Integer Ring]
sage: C is ModulesWithBasis(ZZ).Graded()
True
```

class ElementMethods

Bases: object

degree_negation()

Return the image of self under the degree negation automorphism of the graded module to which self belongs.

The degree negation is the module automorphism which scales every homogeneous element of degree k by $(-1)^k$ (for all k). This assumes that the module to which self belongs (that is, the module self.parent()) is $\mathbf{Z}$-graded.

EXAMPLES:

```
sage: E.<a,b> = ExteriorAlgebra(QQ)                                    #␣
↪needs sage.modules
sage: ((1 + a) * (1 + b)).degree_negation()                            #␣
↪needs sage.modules
a*b - a - b + 1
sage: E.zero().degree_negation()                                       #␣
↪needs sage.modules
0

sage: P = GradedModulesWithBasis(ZZ).example(); P                      #␣
↪needs sage.combinat sage.modules
An example of a graded module with basis:
 the free module on partitions over Integer Ring
```

(continues on next page)

(continued from previous page)

```
sage: pbp = lambda x: P.basis()[Partition(list(x))]
sage: p = pbp([3,1]) - 2 * pbp([2]) + 4 * pbp([1])                    #␣
↪needs sage.combinat sage.modules
sage: p.degree_negation()                                              #␣
↪needs sage.combinat sage.modules
-4*P[1] - 2*P[2] + P[3, 1]
```

class ParentMethods

Bases: `object`

degree_negation(*element*)

Return the image of `element` under the degree negation automorphism of the graded module `self`.

The degree negation is the module automorphism which scales every homogeneous element of degree k by $(-1)^k$ (for all k). This assumes that the module `self` is $\mathbf{Z}$-graded.

INPUT:

- `element` – element of the module `self`

EXAMPLES:

```
sage: E.<a,b> = ExteriorAlgebra(QQ)                                    #␣
↪needs sage.modules
sage: E.degree_negation((1 + a) * (1 + b))                             #␣
↪needs sage.modules
a*b - a - b + 1
sage: E.degree_negation(E.zero())                                      #␣
↪needs sage.modules
0

sage: P = GradedModulesWithBasis(ZZ).example(); P                      #␣
↪needs sage.combinat sage.modules
An example of a graded module with basis:
 the free module on partitions over Integer Ring
sage: pbp = lambda x: P.basis()[Partition(list(x))]
sage: p = pbp([3,1]) - 2 * pbp([2]) + 4 * pbp([1])                    #␣
↪needs sage.combinat sage.modules
sage: P.degree_negation(p)                                             #␣
↪needs sage.combinat sage.modules
-4*P[1] - 2*P[2] + P[3, 1]
```

quotient_module(*submodule*, *check=True*, *already_echelonized=False*, *category=None*)

Construct the quotient module `self` / `submodule`.

INPUT:

- `submodule` – a submodule with basis of `self`, or something that can be turned into one via `self.submodule(submodule)`
- `check`, `already_echelonized` – passed down to *`ModulesWithBasis.ParentMethods.submodule()`*
- `category` – (optional) the category of the quotient module

Warning: At this point, this only supports quotients by free submodules admitting a basis in unitriangular echelon form. In this case, the quotient is also a free module, with a basis consisting of the retract of a subset of the basis of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: S = E.submodule([x + y, x*y - y*z, y])
sage: Q = E.quotient_module(S)
sage: Q.category()
Join of
 Category of quotients of graded modules with basis over Rational Field␣
↪and
 Category of graded vector spaces with basis over Rational Field and
 Category of finite dimensional filtered modules with basis over Rational␣
↪Field
```

See also:

- `Modules.WithBasis.ParentMethods.submodule()`
- *`Rings.ParentMethods.quotient()`*
- `sage.modules.with_basis.subquotient.QuotientModuleWithBasis`

submodule(*gens*, *check=True*, *already_echelonized=False*, *unitriangular=False*, *support_order=None*, *category=None*, **args*, ***opts*)

Return the submodule spanned by a finite set of elements.

INPUT:

- `gens` – a list or family of elements of `self`
- **`check` – (default: `True`) whether to verify that the**
  elements of `gens` are in `self`
- **`already_echelonized` – (default: `False`) whether**
  the elements of `gens` are already in (not necessarily reduced) echelon form
- `unitriangular` – (default: `False`) whether the lift morphism is unitriangular
- `support_order` – (optional) either something that can be converted into a tuple or a key function
- `category` – (optional) the category of the submodule

If `already_echelonized` is `False`, then the generators are put in reduced echelon form using `echelonize()`, and reindexed by $0, 1, \ldots$.

Warning: At this point, this method only works for finite dimensional submodules and if matrices can be echelonized over the base ring.

If in addition `unitriangular` is `True`, then the generators are made such that the coefficients of the pivots are 1, so that lifting map is unitriangular.

The basis of the submodule uses the same index set as the generators, and the lifting map sends y_i to $gens[i]$.

See also:

- `ModulesWithBasis.FiniteDimensional.ParentMethods.quotient_module()`
- `sage.modules.with_basis.subquotient.SubmoduleWithBasis`

EXAMPLES:

A graded submodule of a graded module generated by homogeneous elements is naturally graded:

```
sage: # needs sage.combinat sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: S = E.submodule([x + y, x*y - y*z])
```

(continues on next page)

(continued from previous page)

```
sage: S.category()
Join of
 Category of graded vector spaces with basis over Rational Field and
 Category of subobjects of filtered modules with basis over Rational␣
↪Field and
 Category of finite dimensional filtered modules with basis over Rational␣
↪Field
sage: S.basis()[0].degree()
1
sage: S.basis()[1].degree()
2
```

We check on the echelonized basis:

```
sage: Sp = E.submodule([1, x + y + 5, x*y - y*z + x + y - 2])               #␣
↪needs sage.combinat sage.modules
sage: Sp.category()                                                        #␣
↪needs sage.combinat sage.modules
Join of
 Category of graded vector spaces with basis over Rational Field and
 Category of subobjects of filtered modules with basis over Rational␣
↪Field and
 Category of finite dimensional filtered modules with basis over Rational␣
↪Field
```

If it is generated by inhomogeneous elements, then it is filtered by default:

```
sage: F = E.submodule([x + y*z, x*z + y*x])                                #␣
↪needs sage.combinat sage.modules
sage: F.category()                                                         #␣
↪needs sage.combinat sage.modules
Join of
 Category of subobjects of filtered modules with basis over Rational␣
↪Field and
 Category of finite dimensional filtered modules with basis over Rational␣
↪Field and
 Category of filtered vector spaces with basis over Rational Field
```

If `category` is specified, then it does not give any extra structure to the submodule (we can think of this as applying the forgetful functor):

```
sage: # needs sage.combinat sage.modules
sage: SM = E.submodule([x + y, x*y - y*z],
....:                  category=ModulesWithBasis(QQ))
sage: SM.category()
Join of
 Category of finite dimensional vector spaces with basis over Rational␣
↪Field and
 Category of subobjects of sets
sage: FM = E.submodule([x + 1, x*y - x*y*z],
....:                  category=ModulesWithBasis(QQ))
sage: FM.category()
Join of
 Category of finite dimensional vector spaces with basis over Rational␣
↪Field and
 Category of subobjects of sets
```

If we have specified that this is a graded submodule of a graded module, then the echelonized elements must be homogeneous:

```
sage: Cat = ModulesWithBasis(QQ).Graded().Subobjects()
sage: E.submodule([x + y, x*y - 1], category=Cat)                          #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
ValueError: all of the generators must be homogeneous
sage: E.submodule([x + y, x*y - x - y], category=Cat)                      #␣
↪needs sage.combinat sage.modules
Free module generated by {0, 1} over Rational Field
```

class Quotients(*category*, **args*)

Bases: QuotientsCategory

class ElementMethods

Bases: object

degree()

Return the degree of self.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: S = E.submodule([x + y, x*y - y*z, y])
sage: Q = E.quotient_module(S)
sage: B = Q.basis()
sage: [B[i].lift() for i in Q.indices()]
[1, z, x*z, y*z, x*y*z]
sage: [B[i].degree() for i in Q.indices()]
[0, 1, 2, 2, 3]
```

class ParentMethods

Bases: object

degree_on_basis(*m*)

Return the degree of the basis element indexed by m in self.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: S = E.submodule([x + y, x*y - y*z, y])
sage: Q = E.quotient_module(S)
sage: B = Q.basis()
sage: [B[i].lift() for i in Q.indices()]
[1, z, x*z, y*z, x*y*z]
sage: [Q.degree_on_basis(i) for i in Q.indices()]
[0, 1, 2, 2, 3]
```

4.87 Graphs

class sage.categories.graphs.**Graphs**

Bases: *Category_singleton*

The category of graphs.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs(); C
Category of graphs
```

class Connected(*base_category*)

Bases: *CategoryWithAxiom*

The category of connected graphs.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().Connected()
sage: TestSuite(C).run()
```

extra_super_categories()

Return the extra super categories of `self`.

A connected graph is also a metric space.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: Graphs().Connected().super_categories() # indirect doctest
[Category of connected topological spaces,
 Category of connected simplicial complexes,
 Category of graphs,
 Category of metric spaces]
```

class ParentMethods

Bases: `object`

dimension()

Return the dimension of `self` as a CW complex.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.dimension()
1
```

edges()

Return the edges of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.edges()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

faces()

Return the faces of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: sorted(C.faces(), key=lambda x: (x.dimension(), x.value))
[0, 1, 2, 3, 4, (0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

facets()

Return the facets of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.facets()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

vertices()

Return the vertices of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.vertices()
[0, 1, 2, 3, 4]
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: Graphs().super_categories()
[Category of simplicial complexes]
```

4.88 Group Algebras

This module implements the category of group algebras for arbitrary groups over arbitrary commutative rings. For details, see *sage.categories.algebra_functor*.

AUTHOR:

- David Loeffler (2008-08-24): initial version
- Martin Raum (2009-08): update to use new coercion model – see github issue #6670.
- John Palmieri (2011-07): more updates to coercion, categories, etc., group algebras constructed using CombinatorialFreeModule – see github issue #6670.

- Nicolas M. Thiéry (2010-2017), Travis Scrimshaw (2017): generalization to a covariant functorial construction for monoid algebras, and beyond – see e.g. github issue #18700.

class sage.categories.group_algebras.**GroupAlgebras**(*category*, **args*)

Bases: `AlgebrasCategory`

The category of group algebras over a given base ring.

EXAMPLES:

```
sage: C = Groups().Algebras(ZZ); C
Category of group algebras over Integer Ring
sage: C.super_categories()
[Category of Hopf algebras with basis over Integer Ring,
 Category of monoid algebras over Integer Ring]
```

We can also construct this category with:

```
sage: C is GroupAlgebras(ZZ)
True
```

Here is how to create the group algebra of a group G:

```
sage: G = DihedralGroup(5)                                              #␣
↪needs sage.groups
sage: QG = G.algebra(QQ); QG                                            #␣
↪needs sage.groups sage.modules
Algebra of
 Dihedral group of order 10 as a permutation group over Rational Field
```

and an example of computation:

```
sage: g = G.an_element(); g                                             #␣
↪needs sage.groups sage.modules
(1,4)(2,3)
sage: (QG.term(g) + 1)**3                                               #␣
↪needs sage.groups sage.modules
4*() + 4*(1,4)(2,3)
```

Todo:

- Check which methods would be better located in `Monoid.Algebras` or `Groups.Finite.Algebras`.

class ElementMethods

Bases: `object`

central_form()

Return `self` expressed in the canonical basis of the center of the group algebra.

INPUT:
- `self` – an element of the center of the group algebra

OUTPUT:
- A formal linear combination of the conjugacy class representatives representing its coordinates in the canonical basis of the center. See `Groups.Algebras.ParentMethods.center_basis()` for details.

Warning:

- This method requires the underlying group to have a method `conjugacy_classes_representatives` (every permutation group has one, thanks GAP!).
- This method does not check that the element is indeed central. Use the method *Monoids.Algebras.ElementMethods.is_central()* for this purpose.
- This function has a complexity linear in the number of conjugacy classes of the group. One could easily implement a function whose complexity is linear in the size of the support of `self`.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: QS3 = SymmetricGroup(3).algebra(QQ)
sage: A = QS3([2,3,1]) + QS3([3,1,2])
sage: A.central_form()
B[(1,2,3)]
sage: QS4 = SymmetricGroup(4).algebra(QQ)
sage: B = sum(len(s.cycle_type()) * QS4(s) for s in Permutations(4))
sage: B.central_form()
4*B[()] + 3*B[(1,2)] + 2*B[(1,2)(3,4)] + 2*B[(1,2,3)] + B[(1,2,3,4)]
```

The following test fails due to a bug involving combinatorial free modules and the coercion system (see github issue #28544):

```
sage: # needs sage.groups sage.modules
sage: G = PermutationGroup([[(1,2,3),(4,5)], [(3,4)]])
sage: QG = GroupAlgebras(QQ).example(G)
sage: s = sum(QG.basis())
sage: s.central_form()          # not tested
B[()] + B[(4,5)] + B[(3,4,5)] + B[(2,3)(4,5)]
+ B[(2,3,4,5)] + B[(1,2)(3,4,5)] + B[(1,2,3,4,5)]
```

See also:

- `Groups.Algebras.ParentMethods.center_basis()`
- *Monoids.Algebras.ElementMethods.is_central()*

class ParentMethods

Bases: `object`

antipode_on_basis(*g*)

Return the antipode of the element `g` of the basis.

Each basis element `g` is group-like, and so has antipode g^{-1}. This method is used to compute the antipode of any element.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A
Algebra of
 Cyclic group of order 6 as a permutation group over Integer Ring
sage: g = CyclicPermutationGroup(6).an_element(); g
(1,2,3,4,5,6)
sage: A.antipode_on_basis(g)
```

(continues on next page)

(continued from previous page)

```
(1,6,5,4,3,2)
sage: a = A.an_element(); a
() + 3*(1,2,3,4,5,6) + 3*(1,3,5)(2,4,6)
sage: a.antipode()
() + 3*(1,5,3)(2,6,4) + 3*(1,6,5,4,3,2)
```

center_basis()

Return a basis of the center of the group algebra.

The canonical basis of the center of the group algebra is the family $(f_\sigma)_{\sigma \in C}$, where C is any collection of representatives of the conjugacy classes of the group, and f_σ is the sum of the elements in the conjugacy class of σ.

OUTPUT:

- `tuple` of elements of `self`

Warning:

- This method requires the underlying group to have a method `conjugacy_classes` (every permutation group has one, thanks GAP!).

EXAMPLES:

```
sage: SymmetricGroup(3).algebra(QQ).center_basis()                    #␣
↪needs sage.groups sage.modules
((), (2,3) + (1,2) + (1,3), (1,2,3) + (1,3,2))
```

See also:

- `Groups.Algebras.ElementMethods.central_form()`
- *Monoids.Algebras.ElementMethods.is_central()*

coproduct_on_basis(*g*)

Return the coproduct of the element `g` of the basis.

Each basis element `g` is group-like. This method is used to compute the coproduct of any element.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A
Algebra of
 Cyclic group of order 6 as a permutation group over Integer Ring
sage: g = CyclicPermutationGroup(6).an_element(); g
(1,2,3,4,5,6)
sage: A.coproduct_on_basis(g)
(1,2,3,4,5,6) # (1,2,3,4,5,6)
sage: a = A.an_element(); a
() + 3*(1,2,3,4,5,6) + 3*(1,3,5)(2,4,6)
sage: a.coproduct()
() # () + 3*(1,2,3,4,5,6) # (1,2,3,4,5,6) + 3*(1,3,5)(2,4,6) # (1,3,5)(2,
↪4,6)
```

counit(*x*)

Return the counit of the element `x` of the group algebra.

This is the sum of all coefficients of `x` with respect to the standard basis of the group algebra.

EXAMPLES:

```
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A                          #␣
↪needs sage.groups sage.modules
Algebra of
 Cyclic group of order 6 as a permutation group over Integer Ring
sage: a = A.an_element(); a                                                  #␣
↪needs sage.groups sage.modules
() + 3*(1,2,3,4,5,6) + 3*(1,3,5)(2,4,6)
sage: a.counit()                                                             #␣
↪needs sage.groups sage.modules
7
```

counit_on_basis(*g*)

Return the counit of the element g of the basis.

Each basis element g is group-like, and so has counit 1. This method is used to compute the counit of any element.

EXAMPLES:

```
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A                          #␣
↪needs sage.groups sage.modules
Algebra of
 Cyclic group of order 6 as a permutation group over Integer Ring
sage: g = CyclicPermutationGroup(6).an_element(); g                          #␣
↪needs sage.groups sage.modules
(1,2,3,4,5,6)
sage: A.counit_on_basis(g)                                                   #␣
↪needs sage.groups sage.modules
1
```

group()

Return the underlying group of the group algebra.

EXAMPLES:

```
sage: GroupAlgebras(QQ).example(GL(3, GF(11))).group()                       #␣
↪needs sage.groups sage.modules
General Linear Group of degree 3 over Finite Field of size 11
sage: SymmetricGroup(10).algebra(QQ).group()                                 #␣
↪needs sage.groups sage.modules
Symmetric group of order 10! as a permutation group
```

is_integral_domain(*proof=True*)

Return `True` if `self` is an integral domain.

This is false unless `self.base_ring()` is an integral domain, and even then it is false unless `self.group()` has no nontrivial elements of finite order. I don't know if this condition suffices, but it obviously does if the group is abelian and finitely generated.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S2 = SymmetricGroup(2)
sage: GroupAlgebra(S2).is_integral_domain()
False
sage: S1 = SymmetricGroup(1)
```

(continues on next page)

(continued from previous page)

```
sage: GroupAlgebra(S1).is_integral_domain()
True
sage: GroupAlgebra(S1, IntegerModRing(4)).is_integral_domain()
False
sage: GroupAlgebra(AbelianGroup(1)).is_integral_domain()
True
sage: GroupAlgebra(AbelianGroup(2, [0,2])).is_integral_domain()
False
sage: GroupAlgebra(GL(2, ZZ)).is_integral_domain()      # not implemented
False
```

example(*G=None*)

Return an example of group algebra.

EXAMPLES:

```
sage: GroupAlgebras(QQ['x']).example()                                    #␣
↪needs sage.groups sage.modules
Algebra of Dihedral group of order 8 as a permutation group
 over Univariate Polynomial Ring in x over Rational Field
```

An other group can be specified as optional argument:

```
sage: GroupAlgebras(QQ).example(AlternatingGroup(4))                      #␣
↪needs sage.groups sage.modules
Algebra of
 Alternating group of order 4!/2 as a permutation group over Rational Field
```

extra_super_categories()

Implement the fact that the algebra of a group is a Hopf algebra.

EXAMPLES:

```
sage: C = Groups().Algebras(QQ)
sage: C.extra_super_categories()
[Category of Hopf algebras over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of Hopf algebras with basis over Rational Field,
 Category of monoid algebras over Rational Field]
```

4.89 Groupoid

class sage.categories.groupoid.**Groupoid**(*G=None*)

Bases: *CategoryWithParameters*

The category of groupoids, for a set (usually a group) G.

FIXME:

- Groupoid or Groupoids ?
- definition and link with Wikipedia article Groupoid
- Should Groupoid inherit from Category_over_base?

EXAMPLES:

```
sage: Groupoid(DihedralGroup(3))
Groupoid with underlying set Dihedral group of order 6 as a permutation group
```

classmethod an_instance()

Returns an instance of this class.

EXAMPLES:

```
sage: Groupoid.an_instance() # indirect doctest
Groupoid with underlying set Symmetric group of order 8! as a permutation␣
↪group
```

super_categories()

EXAMPLES:

```
sage: Groupoid(DihedralGroup(3)).super_categories()
[Category of sets]
```

4.90 Groups

class sage.categories.groups.**Groups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of (multiplicative) groups, i.e. monoids with inverses.

EXAMPLES:

```
sage: Groups()
Category of groups
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
```

Algebras

alias of *GroupAlgebras*

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

The category of groups constructed as Cartesian products of groups.

This construction gives the direct product of groups. See Wikipedia article Direct_product and Wikipedia article Direct_product_of_groups for more information.

class ParentMethods

Bases: object

group_generators()

Return the group generators of self.

EXAMPLES:

```
sage: # needs sage.groups
sage: C5 = CyclicPermutationGroup(5)
sage: C4 = CyclicPermutationGroup(4)
sage: S4 = SymmetricGroup(3)
```

(continues on next page)

(continued from previous page)

```
sage: C = cartesian_product([C5, C4, S4])
sage: C.group_generators()
Family (((1,2,3,4,5), (), ()),
        ((), (1,2,3,4), ()),
        ((), (), (1,2)),
        ((), (), (2,3)))
```

We check the other portion of github issue #16718 is fixed:

```
sage: len(C.j_classes())                                                    #␣
↪needs sage.groups
1
```

An example with an infinitely generated group (a better output is needed):

```
sage: # needs sage.groups
sage: G = Groups.free([1,2])
sage: H = Groups.free(ZZ)
sage: C = cartesian_product([G, H])
sage: C.monoid_generators()
Lazy family (gen(i))_{i in The Cartesian product of (...)}
```

order()

Return the cardinality of self.

EXAMPLES:

```
sage: C = cartesian_product([SymmetricGroup(10), SL(2, GF(3))])       #␣
↪needs sage.groups sage.rings.finite_rings
sage: C.order()                                                             #␣
↪needs sage.groups sage.rings.finite_rings
87091200
```

Todo: this method is just here to prevent `FiniteGroups.ParentMethods` to call `_cardinality_from_iterator`.

extra_super_categories()

A Cartesian product of groups is endowed with a natural group structure.

EXAMPLES:

```
sage: C = Groups().CartesianProducts()
sage: C.extra_super_categories()
[Category of groups]
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of inverse unital magmas,
 Category of Cartesian products of monoids,
 Category of groups]
```

class Commutative(*base_category*)

Bases: `CategoryWithAxiom`

Category of commutative (abelian) groups.

A group G is *commutative* if $xy = yx$ for all $x, y \in G$.

static free(*index_set=None*, *names=None*, ***kwds*)

Return the free commutative group.

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

EXAMPLES:

```
sage: # needs sage.groups
sage: Groups.Commutative.free(index_set=ZZ)
Free abelian group indexed by Integer Ring
sage: Groups().Commutative().free(ZZ)
Free abelian group indexed by Integer Ring
sage: Groups().Commutative().free(5)
Multiplicative Abelian group isomorphic to Z x Z x Z x Z x Z
sage: F.<x,y,z> = Groups().Commutative().free(); F
Multiplicative Abelian group isomorphic to Z x Z x Z
```

class ElementMethods

Bases: `object`

conjugacy_class()

Return the conjugacy class of `self`.

EXAMPLES:

```
sage: D = DihedralGroup(5)                                                  #␣
↪needs sage.groups
sage: g = D((1,3,5,2,4))                                                    #␣
↪needs sage.groups
sage: g.conjugacy_class()                                                   #␣
↪needs sage.groups
Conjugacy class of (1,3,5,2,4)
 in Dihedral group of order 10 as a permutation group

sage: H = MatrixGroup([matrix(GF(5), 2, [1,2, -1,1]),                       #␣
↪needs sage.modules
....:                  matrix(GF(5), 2, [1,1, 0,1])])
sage: h = H(matrix(GF(5), 2, [1,2, -1,1]))                                  #␣
↪needs sage.modules
sage: h.conjugacy_class()                                                   #␣
↪needs sage.groups sage.modules
Conjugacy class of [1 2]
                   [4 1]
in Matrix group over Finite Field of size 5 with 2 generators (
[1 2]  [1 1]
[4 1], [0 1]
)

sage: G = SL(2, GF(2))                                                      #␣
↪needs sage.modules
sage: g = G.gens()[0]                                                       #␣
↪needs sage.modules
sage: g.conjugacy_class()                                                   #␣
↪needs sage.modules
Conjugacy class of [1 1]
[0 1] in Special Linear Group of degree 2 over Finite Field of size 2
```

(continues on next page)

(continued from previous page)

```
sage: G = SL(2, QQ)                                                         #␣
↪needs sage.modules
sage: g = G([[1,1], [0,1]])                                                 #␣
↪needs sage.modules
sage: g.conjugacy_class()                                                   #␣
↪needs sage.groups sage.modules
Conjugacy class of [1 1]
[0 1] in Special Linear Group of degree 2 over Rational Field
```

Finite

alias of *FiniteGroups*

Lie

alias of *LieGroups*

class ParentMethods

Bases: `object`

cayley_table(*names='letters'*, *elements=None*)

Return the "multiplication" table of this multiplicative group, which is also known as the "Cayley table".

Note: The order of the elements in the row and column headings is equal to the order given by the table's `column_keys()` method. The association between the actual elements and the names/symbols used in the table can also be retrieved as a dictionary with the `translation()` method.

For groups, this routine should behave identically to the *multiplication_table()* method for magmas, which applies in greater generality.

INPUT:

- `names` - the type of names used, values are:
  - `'letters'` - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by `list()`, padded to a common width with leading 'a's.
  - `'digits'` - base 10 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading zeros.
  - `'elements'` - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- `elements` - default = `None`. A list of elements of the group, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements, perhaps when this is used in the context of a particular structure. The default is to use whatever ordering is provided by the the group, which is reported by the `column_keys()` method. Or the `elements` can be a subset which is closed under the operation. In particular, this can be used when the base set is infinite.

OUTPUT:

An object representing the multiplication table. This is an `OperationTable` object and even more documentation can be found there.

EXAMPLES:

Permutation groups, matrix groups and abelian groups can all compute their multiplication tables.

```
sage: # needs sage.groups
sage: G = DiCyclicGroup(3)
sage: T = G.cayley_table()
```

(continues on next page)

(continued from previous page)

```
sage: T.column_keys()
((), (5,6,7), ..., (1,4,2,3)(5,7))
sage: T
*  a b c d e f g h i j k l
 +------------------------
a| a b c d e f g h i j k l
b| b c a e f d i g h l j k
c| c a b f d e h i g k l j
d| d e f a b c j k l g h i
e| e f d b c a l j k i g h
f| f d e c a b k l j h i g
g| g h i j k l d e f a b c
h| h i g k l j f d e c a b
i| i g h l j k e f d b c a
j| j k l g h i a b c d e f
k| k l j h i g c a b f d e
l| l j k i g h b c a e f d
```

```
sage: M = SL(2, 2)                                                      #␣
↪needs sage.modules
sage: M.cayley_table()                                                  #␣
↪needs sage.modules
*  a b c d e f
 +------------
a| a b c d e f
b| b a d c f e
c| c e a f b d
d| d f b e a c
e| e c f a d b
f| f d e b c a
```

```
sage: A = AbelianGroup([2, 3])                                          #␣
↪needs sage.groups
sage: A.cayley_table()                                                  #␣
↪needs sage.groups
*  a b c d e f
 +------------
a| a b c d e f
b| b c a e f d
c| c a b f d e
d| d e f a b c
e| e f d b c a
f| f d e c a b
```

Lowercase ASCII letters are the default symbols used for the table, but you can also specify the use of decimal digit strings, or provide your own strings (in the proper order if they have meaning). Also, if the elements themselves are not too complex, you can choose to just use the string representations of the elements themselves.

```
sage: C = CyclicPermutationGroup(11)                                    #␣
↪needs sage.groups
sage: C.cayley_table(names='digits')                                    #␣
↪needs sage.groups
 *  00 01 02 03 04 05 06 07 08 09 10
  +---------------------------------
00| 00 01 02 03 04 05 06 07 08 09 10
```

(continues on next page)

(continued from previous page)

```
01| 01 02 03 04 05 06 07 08 09 10 00
02| 02 03 04 05 06 07 08 09 10 00 01
03| 03 04 05 06 07 08 09 10 00 01 02
04| 04 05 06 07 08 09 10 00 01 02 03
05| 05 06 07 08 09 10 00 01 02 03 04
06| 06 07 08 09 10 00 01 02 03 04 05
07| 07 08 09 10 00 01 02 03 04 05 06
08| 08 09 10 00 01 02 03 04 05 06 07
09| 09 10 00 01 02 03 04 05 06 07 08
10| 10 00 01 02 03 04 05 06 07 08 09
```

```
sage: G = QuaternionGroup()                                                      #␣
↪needs sage.groups
sage: names = ['1', 'I', '-1', '-I', 'J', '-K', '-J', 'K']
sage: G.cayley_table(names=names)                                                #␣
↪needs sage.groups
 *   1  I -1 -I  J -K -J  K
  +------------------------
 1|  1  I -1 -I  J -K -J  K
 I|  I -1 -I  1  K  J -K -J
-1| -1 -I  1  I -J  K  J -K
-I| -I  1  I -1 -K -J  K  J
 J|  J -K -J  K -1 -I  1  I
-K| -K -J  K  J  I -1 -I  1
-J| -J  K  J -K  1  I -1 -I
 K|  K  J -K -J -I  1  I -1
```

```
sage: A = AbelianGroup([2, 2])                                                   #␣
↪needs sage.groups
sage: A.cayley_table(names='elements')                                           #␣
↪needs sage.groups
    *      1    f1    f0 f0*f1
     +------------------------
    1|     1    f1    f0 f0*f1
   f1|    f1     1 f0*f1    f0
   f0|    f0 f0*f1     1    f1
f0*f1| f0*f1    f0    f1     1
```

The `change_names()` routine behaves similarly, but changes an existing table "in-place."

```
sage: # needs sage.groups
sage: G = AlternatingGroup(3)
sage: T = G.cayley_table()
sage: T.change_names('digits')
sage: T
*  0 1 2
 +------
0| 0 1 2
1| 1 2 0
2| 2 0 1
```

For an infinite group, you can still work with finite sets of elements, provided the set is closed under multiplication. Elements will be coerced into the group as part of setting up the table.

```
sage: # needs sage.modules
sage: G = SL(2,ZZ); G
```

(continues on next page)

(continued from previous page)

```
Special Linear Group of degree 2 over Integer Ring
sage: identity = matrix(ZZ, [[1,0], [0,1]])
sage: G.cayley_table(elements=[identity, -identity])
*  a b
 +----
a| a b
b| b a
```

The `OperationTable` class provides even greater flexibility, including changing the operation. Here is one such example, illustrating the computation of commutators. `commutator` is defined as a function of two variables, before being used to build the table. From this, the commutator subgroup seems obvious, and creating a Cayley table with just these three elements confirms that they form a closed subset in the group.

```
sage: # needs sage.groups sage.modules
sage: from sage.matrix.operation_table import OperationTable
sage: G = DiCyclicGroup(3)
sage: commutator = lambda x, y: x*y*x^-1*y^-1
sage: T = OperationTable(G, commutator); T
.  a b c d e f g h i j k l
 +------------------------
a| a a a a a a a a a a a a
b| a a a a a a c c c c c c
c| a a a a a a b b b b b b
d| a a a a a a a a a a a a
e| a a a a a a c c c c c c
f| a a a a a a b b b b b b
g| a b c a b c a c b a c b
h| a b c a b c b a c b a c
i| a b c a b c c b a c b a
j| a b c a b c a c b a c b
k| a b c a b c b a c b a c
l| a b c a b c c b a c b a
sage: trans = T.translation()
sage: comm = [trans['a'], trans['b'], trans['c']]
sage: comm
[(), (5,6,7), (5,7,6)]
sage: P = G.cayley_table(elements=comm)
sage: P
*  a b c
 +------
a| a b c
b| b c a
c| c a b
```

Todo: Arrange an ordering of elements into cosets of a normal subgroup close to size $\sqrt{n}$. Then the quotient group structure is often apparent in the table. See comments on github issue #7555.

AUTHOR:

- Rob Beezer (2010-03-15)

conjugacy_class(*g*)

Return the conjugacy class of the element `g`.

This is a fall-back method for groups not defined over GAP.

EXAMPLES:

```
sage: A = AbelianGroup([2, 2])                                                #␣
↪needs sage.groups
sage: c = A.conjugacy_class(A.an_element())                                   #␣
↪needs sage.groups
sage: type(c)                                                                 #␣
↪needs sage.groups
<class 'sage.groups.conjugacy_classes.ConjugacyClass_with_category'>
```

group_generators()

Return group generators for `self`.

This default implementation calls `gens()`, for backward compatibility.

EXAMPLES:

```
sage: A = AlternatingGroup(4)                                                 #␣
↪needs sage.groups
sage: A.group_generators()                                                    #␣
↪needs sage.groups
Family ((2,3,4), (1,2,3))
```

holomorph()

The holomorph of a group

The holomorph of a group G is the semidirect product $G \rtimes_{id} Aut(G)$, where id is the identity function on $Aut(G)$, the automorphism group of G.

See Wikipedia article Holomorph (mathematics)

EXAMPLES:

```
sage: G = Groups().example()                                                  #␣
↪needs sage.modules
sage: G.holomorph()                                                           #␣
↪needs sage.modules
Traceback (most recent call last):
...
NotImplementedError: holomorph of General Linear Group of degree 4
over Rational Field not yet implemented
```

monoid_generators()

Return the generators of `self` as a monoid.

Let G be a group with generating set X. In general, the generating set of G as a monoid is given by $X \cup X^{-1}$, where X^{-1} is the set of inverses of X. If G is a finite group, then the generating set as a monoid is X.

EXAMPLES:

```
sage: # needs sage.groups
sage: A = AlternatingGroup(4)
sage: A.monoid_generators()
Family ((2,3,4), (1,2,3))
sage: F.<x,y> = FreeGroup()
sage: F.monoid_generators()
Family (x, y, x^-1, y^-1)
```

semidirect_product(*N*, *mapping*, *check=True*)

The semi-direct product of two groups

EXAMPLES:

```
sage: G = Groups().example()                                                 #␣
↪needs sage.modules
sage: G.semidirect_product(G, Morphism(G, G))                                #␣
↪needs sage.modules
Traceback (most recent call last):
...
NotImplementedError: semidirect product of General Linear Group of degree␣
↪4
over Rational Field and General Linear Group of degree 4 over Rational␣
↪Field
not yet implemented
```

class Topological(*category*, **args*)

Bases: `TopologicalSpacesCategory`

Category of topological groups.

A topological group G is a group which has a topology such that multiplication and taking inverses are continuous functions.

REFERENCES:

- Wikipedia article Topological_group

example()

EXAMPLES:

```
sage: Groups().example()                                                     #␣
↪needs sage.modules
General Linear Group of degree 4 over Rational Field
```

static free(*index_set=None*, *names=None*, ***kwds*)

Return the free group.

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

When the index set is an integer or only variable names are given, this returns `FreeGroup_class`, which currently has more features due to the interface with GAP than `IndexedFreeGroup`.

EXAMPLES:

```
sage: # needs sage.groups
sage: Groups.free(index_set=ZZ)
Free group indexed by Integer Ring
sage: Groups().free(ZZ)
Free group indexed by Integer Ring
sage: Groups().free(5)
Free Group on generators {x0, x1, x2, x3, x4}
sage: F.<x,y,z> = Groups().free(); F
Free Group on generators {x, y, z}
```

4.91 Hecke modules

class sage.categories.hecke_modules.**HeckeModules**(*R*)

Bases: *Category_module*

The category of Hecke modules.

A Hecke module is a module M over the emph{anemic} Hecke algebra, i.e., the Hecke algebra generated by Hecke operators T_n with n coprime to the level of M. (Every Hecke module defines a level function, which is a positive integer.) The reason we require that M only be a module over the anemic Hecke algebra is that many natural maps, e.g., degeneracy maps, Atkin-Lehner operators, etc., are **T**-module homomorphisms; but they are homomorphisms over the anemic Hecke algebra.

EXAMPLES:

We create the category of Hecke modules over **Q**:

```
sage: C = HeckeModules(RationalField()); C
Category of Hecke modules over Rational Field
```

TODO: check that this is what we want:

```
sage: C.super_categories()
[Category of vector spaces with basis over Rational Field]
```

# [Category of vector spaces over Rational Field]

Note that the base ring can be an arbitrary commutative ring:

```
sage: HeckeModules(IntegerRing())
Category of Hecke modules over Integer Ring
sage: HeckeModules(FiniteField(5))
Category of Hecke modules over Finite Field of size 5
```

The base ring doesn't have to be a principal ideal domain:

```
sage: HeckeModules(PolynomialRing(IntegerRing(), 'x'))
Category of Hecke modules over Univariate Polynomial Ring in x over Integer Ring
```

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

class ParentMethods

Bases: object

extra_super_categories()

class ParentMethods

Bases: object

super_categories()

EXAMPLES:

```
sage: HeckeModules(QQ).super_categories()
[Category of vector spaces with basis over Rational Field]
```

4.92 Highest Weight Crystals

class sage.categories.highest_weight_crystals.**HighestWeightCrystalHomset**(*X*, *Y*, *category=None*)

Bases: *CrystalHomset*

The set of crystal morphisms from a highest weight crystal to another crystal.

See also:

See *sage.categories.crystals.CrystalHomset* for more information.

Element

alias of *HighestWeightCrystalMorphism*

class sage.categories.highest_weight_crystals.**HighestWeightCrystalMorphism**(*parent*, *on_gens*, *cartan_type=None*, *virtualization=None*, *scaling_factors=None*, *gens=None*, *check=True*)

Bases: *CrystalMorphismByGenerators*

A virtual crystal morphism whose domain is a highest weight crystal.

INPUT:

- `parent` – a homset
- `on_gens` – a function or list that determines the image of the generators (if given a list, then this uses the order of the generators of the domain) of the domain under `self`
- `cartan_type` – (optional) a Cartan type; the default is the Cartan type of the domain
- `virtualization` – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain
- `scaling_factors` – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, ε and φ
- `gens` – (optional) a list of generators to define the morphism; the default is to use the highest weight vectors of the crystal
- `check` – (default: `True`) check if the crystal morphism is valid

class sage.categories.highest_weight_crystals.**HighestWeightCrystals**

Bases: *Category_singleton*

The category of highest weight crystals.

A crystal is highest weight if it is acyclic; in particular, every connected component has a unique highest weight element, and that element generate the component.

EXAMPLES:

```
sage: C = HighestWeightCrystals()
sage: C
Category of highest weight crystals
sage: C.super_categories()
[Category of crystals]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

class ElementMethods

Bases: `object`

string_parameters(*word=None*)

Return the string parameters of `self` corresponding to the reduced word `word`.

Given a reduced expression $w = s_{i_1} \cdots s_{i_k}$, the string parameters of $b \in B$ corresponding to w are $(a_1, \ldots, a_k)$ such that

$$e_{i_m}^{a_m} \cdots e_{i_1}^{a_1} b \neq 0$$
$$e_{i_m}^{a_m+1} \cdots e_{i_1}^{a_1} b = 0$$

for all $1 \leq m \leq k$.

For connected components isomorphic to $B(\lambda)$ or $B(\infty)$, if $w = w_0$ is the longest element of the Weyl group, then the path determined by the string parametrization terminates at the highest weight vector.

INPUT:

- `word` – a word in the alphabet of the index set; if not specified and we are in finite type, then this will be some reduced expression for the long element determined by the Weyl group

EXAMPLES:

```
sage: B = crystals.infinity.NakajimaMonomials(['A',3])
sage: mg = B.highest_weight_vector()
sage: w0 = [1,2,1,3,2,1]
sage: mg.string_parameters(w0)
[0, 0, 0, 0, 0, 0]
sage: mg.f_string([1]).string_parameters(w0)
[1, 0, 0, 0, 0, 0]
sage: mg.f_string([1,1,1]).string_parameters(w0)
[3, 0, 0, 0, 0, 0]
sage: mg.f_string([1,1,1,2,2]).string_parameters(w0)
[1, 2, 2, 0, 0, 0]
sage: mg.f_string([1,1,1,2,2]) == mg.f_string([1,1,2,2,1])
True
sage: x = mg.f_string([1,1,1,2,2,1,3,3,2,1,1,1])
sage: x.string_parameters(w0)
[4, 1, 1, 2, 2, 2]
sage: x.string_parameters([3,2,1,3,2,3])
[2, 3, 7, 0, 0, 0]
sage: x == mg.f_string([1]*7 + [2]*3 + [3]*2)
True
```

```
sage: B = crystals.infinity.Tableaux("A5")
sage: b = B(rows=[[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,3,6,6,6,6,6,6],
....:              [2,2,2,2,2,2,2,2,2,4,5,5,5,6],
....:              [3,3,3,3,3,3,3,5],
....:              [4,4,4,6,6,6],
```

(continues on next page)

(continued from previous page)

```
....:               [5,6]])
sage: b.string_parameters([1,2,1,3,2,1,4,3,2,1,5,4,3,2,1])
[0, 1, 1, 1, 1, 0, 4, 4, 3, 0, 11, 10, 7, 7, 6]

sage: B = crystals.infinity.Tableaux("G2")
sage: b = B(rows=[[1,1,1,1,1,3,3,0,-3,-3,-2,-2,-1,-1,-1,-1],[2,3,3,3]])
sage: b.string_parameters([2,1,2,1,2,1])
[5, 13, 11, 15, 4, 4]
sage: b.string_parameters([1,2,1,2,1,2])
[7, 12, 15, 8, 10, 0]
```

```
sage: C = crystals.Tableaux(['C',2], shape=[2,1])
sage: mg = C.highest_weight_vector()
sage: lw = C.lowest_weight_vectors()[0]
sage: lw.string_parameters([1,2,1,2])
[1, 2, 3, 1]
sage: lw.string_parameters([2,1,2,1])
[1, 3, 2, 1]
sage: lw.e_string([2,1,1,1,2,2,1]) == mg
True
sage: lw.e_string([1,2,2,1,1,1,2]) == mg
True
```

class ParentMethods

Bases: `object`

connected_components_generators()

Returns the highest weight vectors of `self`

This default implementation selects among the module generators those that are highest weight, and caches the result. A crystal element b is highest weight if $e_i(b) = 0$ for all i in the index set.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.highest_weight_vectors()
(1,)
```

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C, C, C, generators=[[C(2),C(1),C(1)],
....:                                                   [C(1),C(2),C(1)]])
sage: T.highest_weight_vectors()
([2, 1, 1], [1, 2, 1])
```

digraph(*subset=None*, *index_set=None*, *depth=None*)

Return the DiGraph associated to `self`.

INPUT:

- `subset` – (optional) a subset of vertices for which the digraph should be constructed
- `index_set` – (optional) the index set to draw arrows
- `depth` – the depth to draw; optional only for finite crystals

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: T.digraph()
Digraph on 8 vertices
```

(continues on next page)

(continued from previous page)

```
sage: S = T.subcrystal(max_depth=2)
sage: len(S)
5
sage: G = T.digraph(subset=list(S))
sage: G.is_isomorphic(T.digraph(depth=2), edge_labels=True)
True
```

highest_weight_vector()

Returns the highest weight vector if there is a single one; otherwise, raises an error.

Caveat: this assumes that *highest_weight_vectors()* returns a list or tuple.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.highest_weight_vector()
1
```

highest_weight_vectors()

Returns the highest weight vectors of `self`

This default implementation selects among the module generators those that are highest weight, and caches the result. A crystal element b is highest weight if $e_i(b) = 0$ for all i in the index set.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.highest_weight_vectors()
(1,)
```

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C, C, C, generators=[[C(2),C(1),C(1)],
....:                                                 [C(1),C(2),C(1)]])
sage: T.highest_weight_vectors()
([2, 1, 1], [1, 2, 1])
```

lowest_weight_vectors()

Return the lowest weight vectors of `self`.

This default implementation selects among all elements of the crystal those that are lowest weight, and cache the result. A crystal element b is lowest weight if $f_i(b) = 0$ for all i in the index set.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.lowest_weight_vectors()
(6,)
```

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C, C, C,generators=[[C(2),C(1),C(1)],
....:                                                [C(1),C(2),C(1)]])
sage: T.lowest_weight_vectors()
([3, 2, 3], [3, 3, 2])
```

q_dimension(*q=None*, *prec=None*, *use_product=False*)

Return the q-dimension of `self`.

Let $B(\lambda)$ denote a highest weight crystal. Recall that the degree of the μ-weight space of $B(\lambda)$ (under the principal gradation) is equal to $\langle \rho^\vee, \lambda - \mu \rangle$ where $\langle \rho^\vee, \alpha_i \rangle = 1$ for all $i \in I$ (in particular, take $\rho^\vee = \sum_{i \in I} h_i$).

The q-dimension of a highest weight crystal $B(\lambda)$ is defined as

$$\dim_q B(\lambda) := \sum_{j \geq 0} \dim(B_j) q^j,$$

where B_j denotes the degree j portion of $B(\lambda)$. This can be expressed as the product

$$\dim_q B(\lambda) = \prod_{\alpha^\vee \in \Delta_+^\vee} \left(\frac{1 - q^{\langle \lambda + \rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}} \right)^{\mathrm{mult}\, \alpha},$$

where $\Delta_+^\vee$ denotes the set of positive coroots. Taking the limit as $q \to 1$ gives the dimension of $B(\lambda)$. For more information, see [Ka1990] Section 10.10.

INPUT:

- `q` – the (generic) parameter q
- `prec` – (default: `None`) The precision of the power series ring to use if the crystal is not known to be finite (i.e. the number of terms returned). If `None`, then the result is returned as a lazy power series.
- `use_product` – (default: `False`) if we have a finite crystal and `True`, use the product formula

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: qdim = C.q_dimension(); qdim
q^4 + 2*q^3 + 2*q^2 + 2*q + 1
sage: qdim(1)
8
sage: len(C) == qdim(1)
True
sage: C.q_dimension(use_product=True) == qdim
True
sage: C.q_dimension(prec=20)
q^4 + 2*q^3 + 2*q^2 + 2*q + 1
sage: C.q_dimension(prec=2)
2*q + 1

sage: R.<t> = QQ[]
sage: C.q_dimension(q=t^2)
t^8 + 2*t^6 + 2*t^4 + 2*t^2 + 1

sage: C = crystals.Tableaux(['A',2], shape=[5,2])
sage: C.q_dimension()
q^10 + 2*q^9 + 4*q^8 + 5*q^7 + 6*q^6 + 6*q^5
 + 6*q^4 + 5*q^3 + 4*q^2 + 2*q + 1

sage: C = crystals.Tableaux(['B',2], shape=[2,1])
sage: qdim = C.q_dimension(); qdim
q^10 + 2*q^9 + 3*q^8 + 4*q^7 + 5*q^6 + 5*q^5
 + 5*q^4 + 4*q^3 + 3*q^2 + 2*q + 1
sage: qdim == C.q_dimension(use_product=True)
True

sage: C = crystals.Tableaux(['D',4], shape=[2,1])
sage: C.q_dimension()
```

(continues on next page)

(continued from previous page)

```
q^16 + 2*q^15 + 4*q^14 + 7*q^13 + 10*q^12 + 13*q^11
 + 16*q^10 + 18*q^9 + 18*q^8 + 18*q^7 + 16*q^6 + 13*q^5
 + 10*q^4 + 7*q^3 + 4*q^2 + 2*q + 1
```

We check with a finite tensor product:

```
sage: TP = crystals.TensorProduct(C, C)
sage: TP.cardinality()
25600
sage: qdim = TP.q_dimension(use_product=True); qdim # long time
q^32 + 2*q^31 + 8*q^30 + 15*q^29 + 34*q^28 + 63*q^27 + 110*q^26
 + 175*q^25 + 276*q^24 + 389*q^23 + 550*q^22 + 725*q^21
 + 930*q^20 + 1131*q^19 + 1362*q^18 + 1548*q^17 + 1736*q^16
 + 1858*q^15 + 1947*q^14 + 1944*q^13 + 1918*q^12 + 1777*q^11
 + 1628*q^10 + 1407*q^9 + 1186*q^8 + 928*q^7 + 720*q^6
 + 498*q^5 + 342*q^4 + 201*q^3 + 117*q^2 + 48*q + 26
sage: qdim(1) # long time
25600
sage: TP.q_dimension() == qdim # long time
True
```

The q-dimensions of infinite crystals are returned as formal power series:

```
sage: C = crystals.LSPaths(['A',2,1], [1,0,0])
sage: C.q_dimension(prec=5)
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + O(q^5)
sage: C.q_dimension(prec=10)
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + 5*q^5 + 7*q^6
 + 9*q^7 + 13*q^8 + 16*q^9 + O(q^10)
sage: qdim = C.q_dimension(); qdim
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + 5*q^5 + 7*q^6 + O(q^7)
sage: qdim[:16]
[1, 1, 2, 2, 4, 5, 7, 9, 13, 16, 22, 27, 36, 44, 57, 70]
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of highest weight crystals constructed by tensor product of highest weight crystals.

class ParentMethods

Bases: `object`

Implements operations on tensor products of crystals.

highest_weight_vectors()

Return the highest weight vectors of `self`.

This works by using a backtracing algorithm since if $b_2 \otimes b_1$ is highest weight then b_1 is highest weight.

EXAMPLES:

```
sage: C = crystals.Tableaux(['D',4], shape=[2,2])
sage: D = crystals.Tableaux(['D',4], shape=[1])
sage: T = crystals.TensorProduct(D, C)
sage: T.highest_weight_vectors()
([[[1]], [[1, 1], [2, 2]]],
 [[[3]], [[1, 1], [2, 2]]],
```

(continues on next page)

(continued from previous page)

```
 [[[-2]], [[1, 1], [2, 2]]])
sage: L = filter(lambda x: x.is_highest_weight(), T)
sage: tuple(L) == T.highest_weight_vectors()
True
```

highest_weight_vectors_iterator()

Iterate over the highest weight vectors of `self`.

This works by using a backtracing algorithm since if $b_2 \otimes b_1$ is highest weight then b_1 is highest weight.

EXAMPLES:

```
sage: C = crystals.Tableaux(['D',4], shape=[2,2])
sage: D = crystals.Tableaux(['D',4], shape=[1])
sage: T = crystals.TensorProduct(D, C)
sage: tuple(T.highest_weight_vectors_iterator())
([[[1]], [[1, 1], [2, 2]]],
 [[[3]], [[1, 1], [2, 2]]],
 [[[-2]], [[1, 1], [2, 2]]])
sage: L = filter(lambda x: x.is_highest_weight(), T)
sage: tuple(L) == tuple(T.highest_weight_vectors_iterator())
True
```

extra_super_categories()

EXAMPLES:

```
sage: HighestWeightCrystals().TensorProducts().extra_super_categories()
[Category of highest weight crystals]
```

additional_structure()

Return `None`.

Indeed, the category of highest weight crystals defines no additional structure: it only guarantees the existence of a unique highest weight element in each component.

See also:

Category.additional_structure()

Todo: Should this category be a `CategoryWithAxiom`?

EXAMPLES:

```
sage: HighestWeightCrystals().additional_structure()
```

example()

Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = HighestWeightCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

super_categories()

EXAMPLES:

```
sage: HighestWeightCrystals().super_categories()
[Category of crystals]
```

4.93 Hopf algebras

class sage.categories.hopf_algebras.**HopfAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of Hopf algebras.

EXAMPLES:

```
sage: HopfAlgebras(QQ)
Category of Hopf algebras over Rational Field
sage: HopfAlgebras(QQ).super_categories()
[Category of bialgebras over Rational Field]
```

class DualCategory(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of Hopf algebras constructed as dual of a Hopf algebra

class ParentMethods

Bases: object

class ElementMethods

Bases: object

antipode()

Return the antipode of self

EXAMPLES:

```
sage: # needs sage.groups
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the
 Dihedral group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, a.antipode()
(B[(1,2,3)], B[(1,3,2)])
sage: b, b.antipode()
(B[(1,3)], B[(1,3)])
```

class Morphism

Bases: *Category*

The category of Hopf algebra morphisms.

class ParentMethods

Bases: object

class Realizations(*category*, **args*)

Bases: *RealizationsCategory*

class ParentMethods

Bases: object

antipode_by_coercion(*x*)

Returns the image of `x` by the antipode

This default implementation coerces to the default realization, computes the antipode there, and coerces the result back.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: R.antipode_by_coercion.__module__
'sage.categories.hopf_algebras'
sage: R.antipode_by_coercion(R[1,3,1])
-R[2, 1, 2]
```

class Super(*base_category*)

Bases: *SuperModulesCategory*

The category of super Hopf algebras.

Note: A super Hopf algebra is *not* simply a Hopf algebra with a $\mathbf{Z}/2\mathbf{Z}$ grading due to the signed bialgebra compatibility conditions.

class ElementMethods

Bases: `object`

antipode()

Return the antipode of `self`.

EXAMPLES:

```
sage: A = SteenrodAlgebra(3)                                              #␣
↪needs sage.combinat sage.modules
sage: a = A.an_element()                                                  #␣
↪needs sage.combinat sage.modules
sage: a, a.antipode()                                                     #␣
↪needs sage.combinat sage.modules
(2 Q_1 Q_3 P(2,1), Q_1 Q_3 P(2,1))
```

dual()

Return the dual category.

EXAMPLES:

The category of super Hopf algebras over any field is self dual:

```
sage: C = HopfAlgebras(QQ).Super()
sage: C.dual()
Category of super Hopf algebras over Rational Field
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of Hopf algebras constructed by tensor product of Hopf algebras

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

extra_super_categories()

EXAMPLES:

```
sage: C = HopfAlgebras(QQ).TensorProducts()
sage: C.extra_super_categories()
[Category of Hopf algebras over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of Hopf algebras over Rational Field,
 Category of tensor products of algebras over Rational Field,
 Category of tensor products of coalgebras over Rational Field]
```

WithBasis

alias of *HopfAlgebrasWithBasis*

dual()

Return the dual category

EXAMPLES:

The category of Hopf algebras over any field is self dual:

```
sage: C = HopfAlgebras(QQ)
sage: C.dual()
Category of Hopf algebras over Rational Field
```

super_categories()

EXAMPLES:

```
sage: HopfAlgebras(QQ).super_categories()
[Category of bialgebras over Rational Field]
```

4.94 Hopf algebras with basis

class sage.categories.hopf_algebras_with_basis.**HopfAlgebrasWithBasis**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of Hopf algebras with a distinguished basis

EXAMPLES:

```
sage: C = HopfAlgebrasWithBasis(QQ)
sage: C
Category of Hopf algebras with basis over Rational Field
sage: C.super_categories()
[Category of Hopf algebras over Rational Field,
 Category of bialgebras with basis over Rational Field]
```

We now show how to use a simple Hopf algebra, namely the group algebra of the dihedral group (see also AlgebrasWithBasis):

```
sage: A = C.example(); A                                                     #␣
↪needs sage.groups
An example of Hopf algebra with basis: the group algebra of the
 Dihedral group of order 6 as a permutation group over Rational Field
sage: A.rename("A")                                                          #␣
↪needs sage.groups
sage: A.category()                                                           #␣
↪needs sage.groups
Category of finite dimensional Hopf algebras with basis over Rational Field

sage: A.one_basis()                                                          #␣
↪needs sage.groups
()
sage: A.one()                                                                #␣
↪needs sage.groups
B[()]

sage: A.base_ring()                                                          #␣
↪needs sage.groups
Rational Field
sage: A.basis().keys()                                                       #␣
↪needs sage.groups
Dihedral group of order 6 as a permutation group

sage: # needs sage.groups
sage: [a,b] = A.algebra_generators()
sage: a, b
(B[(1,2,3)], B[(1,3)])
sage: a^3, b^2
(B[()], B[()])
sage: a*b
B[(1,2)]

sage: A.product           # todo: not quite ...                              #␣
↪needs sage.groups
<bound method MagmaticAlgebras.WithBasis.ParentMethods._product_from_product_on_
↪basis_multiply of A>
sage: A.product(b, b)                                                        #␣
↪needs sage.groups
B[()]

sage: A.zero().coproduct()                                                   #␣
↪needs sage.groups
0
sage: A.zero().coproduct().parent()                                          #␣
↪needs sage.groups
A # A
sage: a.coproduct()                                                          #␣
↪needs sage.groups
B[(1,2,3)] # B[(1,2,3)]

sage: TestSuite(A).run(verbose=True)                                         #␣
↪needs sage.groups
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_antipode() . . . pass
running ._test_associativity() . . . pass
```

(continues on next page)

(continued from previous page)

```
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_characteristic() . . . pass
running ._test_construction() . . . pass
running ._test_distributivity() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
running ._test_zero() . . . pass
sage: A.__class__                                                                  #␣
↪needs sage.groups
<class 'sage.categories.examples.hopf_algebras_with_basis.MyGroupAlgebra_with_
↪category'>
sage: A.element_class                                                              #␣
↪needs sage.groups
<class 'sage.categories.examples.hopf_algebras_with_basis.MyGroupAlgebra_with_
↪category.element_class'>
```

Let us look at the code for implementing A:

```
sage: A??                                # not implemented                         #␣
↪needs sage.groups
```

class ElementMethods

Bases: `object`

Filtered

alias of `FilteredHopfAlgebrasWithBasis`

FiniteDimensional

alias of *`FiniteDimensionalHopfAlgebrasWithBasis`*

Graded

alias of *`GradedHopfAlgebrasWithBasis`*

class ParentMethods

Bases: `object`

antipode()

The antipode of this Hopf algebra.

If `antipode_basis()` is available, this constructs the antipode morphism from `self` to `self` by extending it by linearity. Otherwise, `self.antipode_by_coercion()` is used, if available.

EXAMPLES:

```
sage: # needs sage.groups
sage: A = HopfAlgebrasWithBasis(ZZ).example(); A
An example of Hopf algebra with basis: the group algebra of the
 Dihedral group of order 6 as a permutation group over Integer Ring
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: [a,b] = A.algebra_generators()
sage: a, A.antipode(a)
(B[(1,2,3)], B[(1,3,2)])
sage: b, A.antipode(b)
(B[(1,3)], B[(1,3)])
```

antipode_on_basis(*x*)

The antipode of the Hopf algebra on the basis (optional)

INPUT:

- `x` – an index of an element of the basis of `self`

Returns the antipode of the basis element indexed by `x`.

If this method is implemented, then *antipode()* is defined from this by linearity.

EXAMPLES:

```
sage: # needs sage.groups
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: W = A.basis().keys(); W
Dihedral group of order 6 as a permutation group
sage: w = W.gen(0); w
(1,2,3)
sage: A.antipode_on_basis(w)
B[(1,3,2)]
```

Super

alias of *SuperHopfAlgebrasWithBasis*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of Hopf algebras with basis constructed by tensor product of Hopf algebras with basis

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

extra_super_categories()

EXAMPLES:

```
sage: C = HopfAlgebrasWithBasis(QQ).TensorProducts()
sage: C.extra_super_categories()
[Category of Hopf algebras with basis over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of Hopf algebras with basis over Rational Field,
 Category of tensor products of Hopf algebras over Rational Field,
 Category of tensor products of algebras with basis over Rational Field]
```

example(*G=None*)

Returns an example of algebra with basis:

```
sage: HopfAlgebrasWithBasis(QQ['x']).example()                              #␣
↪needs sage.groups
An example of Hopf algebra with basis: the group algebra of the
Dihedral group of order 6 as a permutation group
over Univariate Polynomial Ring in x over Rational Field
```

An other group can be specified as optional argument:

```
sage: HopfAlgebrasWithBasis(QQ).example(SymmetricGroup(4))                  #␣
↪needs sage.groups
An example of Hopf algebra with basis: the group algebra of the
Symmetric group of order 4! as a permutation group over Rational Field
```

4.95 H-trivial semigroups

class sage.categories.h_trivial_semigroups.**HTrivialSemigroups**(*base_category*)

Bases: *CategoryWithAxiom*

Finite_extra_super_categories()

Implement the fact that a finite H-trivial is aperiodic

EXAMPLES:

```
sage: Semigroups().HTrivial().Finite_extra_super_categories()
[Category of aperiodic semigroups]
sage: Semigroups().HTrivial().Finite() is Semigroups().Aperiodic().Finite()
True
```

Inverse_extra_super_categories()

Implement the fact that an H-trivial inverse semigroup is J-trivial.

Todo: Generalization for inverse semigroups.

Recall that there are two invertibility axioms for a semigroup S:

- One stating the existence, for all x, of a local inverse y satisfying $x = xyx$ and $y = yxy$;
- One stating the existence, for all x, of a global inverse y satisfying $xy = yx = 1$, where 1 is the unit of S (which must of course exist).

It is sufficient to have local inverses for H-triviality to imply J-triviality. However, at this stage, only the second axiom is implemented in Sage (see *Magmas.Unital.SubcategoryMethods.Inverse()*). Therefore this fact is only implemented for semigroups with global inverses, that is groups. However the trivial group is the unique H-trivial group, so this is rather boring.

EXAMPLES:

```
sage: Semigroups().HTrivial().Inverse_extra_super_categories()
[Category of j trivial semigroups]
sage: Monoids().HTrivial().Inverse()
Category of h trivial groups
```

4.96 Infinite Enumerated Sets

AUTHORS:

- Florent Hivert (2009-11): initial revision.

class sage.categories.infinite_enumerated_sets.**InfiniteEnumeratedSets**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of infinite enumerated sets

An infinite enumerated sets is a countable set together with a canonical enumeration of its elements.

EXAMPLES:

```
sage: InfiniteEnumeratedSets()
Category of infinite enumerated sets
sage: InfiniteEnumeratedSets().super_categories()
[Category of enumerated sets, Category of infinite sets]
sage: InfiniteEnumeratedSets().all_super_categories()
[Category of infinite enumerated sets,
 Category of enumerated sets,
 Category of infinite sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

class ParentMethods

Bases: object

list()

Raise an error because self is an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

random_element()

Raise an error because self is an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.random_element()
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

TODO: should this be an optional abstract_method instead?

tuple()

Raise an error because self is an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.tuple()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

4.97 Integral domains

class sage.categories.integral_domains.**IntegralDomains**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of integral domains

An integral domain is commutative ring with no zero divisors, or equivalently a commutative domain.

EXAMPLES:

```
sage: C = IntegralDomains(); C
Category of integral domains
sage: sorted(C.super_categories(), key=str)
[Category of commutative rings, Category of domains]
sage: C is Domains().Commutative()
True
sage: C is Rings().Commutative().NoZeroDivisors()
True
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

is_integral_domain(*proof=True*)

Return True, since this in an object of the category of integral domains.

EXAMPLES:

```
sage: QQ.is_integral_domain()
True
sage: Parent(QQ, category=IntegralDomains()).is_integral_domain()
True

sage: L.<z> = LazyLaurentSeriesRing(QQ)                                 #␣
↪needs sage.combinat
sage: L.is_integral_domain()                                            #␣
↪needs sage.combinat
True
sage: L.is_integral_domain(proof=True)                                  #␣
↪needs sage.combinat
True
```

4.98 J-trivial semigroups

class sage.categories.j_trivial_semigroups.**JTrivialSemigroups**(*base_category*)

Bases: *CategoryWithAxiom*

extra_super_categories()

Implement the fact that a J-trivial semigroup is L and R-trivial.

EXAMPLES:

```
sage: Semigroups().JTrivial().extra_super_categories()
[Category of l trivial semigroups, Category of r trivial semigroups]
```

4.99 Kac-Moody Algebras

AUTHORS:

- Travis Scrimshaw (07-15-2017): Initial implementation

class sage.categories.kac_moody_algebras.**KacMoodyAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

Category of Kac-Moody algebras.

class ParentMethods

Bases: object

cartan_type()

Return the Cartan type of self.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2])                              #␣
↪needs sage.combinat sage.modules
sage: L.cartan_type()                                                       #␣
↪needs sage.combinat sage.modules
['A', 2]
```

weyl_group()

Return the Weyl group of self.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2])                              #␣
↪needs sage.combinat sage.modules
sage: L.weyl_group()                                                        #␣
↪needs sage.combinat sage.modules
Weyl Group of type ['A', 2] (as a matrix group acting on the ambient␣
↪space)
```

example(*n=2*)

Return an example of a Kac-Moody algebra as per *Category.example*.

EXAMPLES:

```
sage: from sage.categories.kac_moody_algebras import KacMoodyAlgebras
sage: KacMoodyAlgebras(QQ).example()                                    #␣
↪needs sage.combinat sage.modules
Lie algebra of ['A', 2] in the Chevalley basis
```

We can specify the rank of the example:

```
sage: KacMoodyAlgebras(QQ).example(4)                                   #␣
↪needs sage.combinat sage.modules
Lie algebra of ['A', 4] in the Chevalley basis
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.kac_moody_algebras import KacMoodyAlgebras
sage: KacMoodyAlgebras(QQ).super_categories()
[Category of Lie algebras over Rational Field]
```

4.100 Lambda Bracket Algebras

AUTHORS:

- Reimundo Heluani (2019-10-05): Initial implementation.

class sage.categories.lambda_bracket_algebras.**LambdaBracketAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of Lambda bracket algebras.

This is an abstract base category for Lie conformal algebras and super Lie conformal algebras.

class ElementMethods

Bases: object

T(*n=1*)

The n-th derivative of self.

INPUT:

- n – integer (default:1); how many times to apply T to this element

OUTPUT:

$T^n a$ where a is this element. Notice that we use the *divided powers* notation $T^{(j)} = \frac{T^j}{j!}$.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: Vir = lie_conformal_algebras.Virasoro(QQ)
sage: Vir.inject_variables()
Defining L, C
sage: L.T()
TL
sage: L.T(3)
6*T^(3)L
sage: C.T()
0
```

bracket(*rhs*)

The λ-bracket of these two elements.

EXAMPLES:

The brackets of the Virasoro Lie conformal algebra:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ); L = Vir.0                      #␣
↪needs sage.combinat sage.modules
sage: L.bracket(L)                                                              #␣
↪needs sage.combinat sage.modules
{0: TL, 1: 2*L, 3: 1/2*C}
sage: L.bracket(L.T())                                                          #␣
↪needs sage.combinat sage.modules
{0: 2*T^(2)L, 1: 3*TL, 2: 4*L, 4: 2*C}
```

Now with a current algebra:

```
sage: # needs sage.combinat sage.modules
sage: V = lie_conformal_algebras.Affine(QQ, 'A1')
sage: V.gens()
(B[alpha[1]], B[alphacheck[1]], B[-alpha[1]], B['K'])
sage: E = V.0; H = V.1; F = V.2
sage: H.bracket(H)
{1: 2*B['K']}
sage: E.bracket(F)
{0: B[alphacheck[1]], 1: B['K']}
```

nproduct(*rhs*, *n*)

The `n`-th product of these two elements.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: Vir = lie_conformal_algebras.Virasoro(QQ); L = Vir.0
sage: L.nproduct(L, 3)
1/2*C
sage: L.nproduct(L.T(), 0)
2*T^(2)L
sage: V = lie_conformal_algebras.Affine(QQ, 'A1')
sage: E = V.0; H = V.1; F = V.2
sage: E.nproduct(H, 0) == - 2*E
True
sage: E.nproduct(F, 1)
B['K']
```

FinitelyGeneratedAsLambdaBracketAlgebra

alias of *FinitelyGeneratedLambdaBracketAlgebras*

class ParentMethods

Bases: `object`

ideal(**gens*, ***kwds*)

The ideal of this Lambda bracket algebra generated by `gens`.

Todo: Ideals of Lie Conformal Algebras are not implemented yet.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)                          #␣
↪needs sage.combinat sage.modules
sage: Vir.ideal()                                                        #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
NotImplementedError: ideals of Lie Conformal algebras are not implemented␣
↪yet
```

class SubcategoryMethods

Bases: `object`

FinitelyGenerated()

The category of finitely generated Lambda bracket algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(QQ).FinitelyGenerated()
Category of finitely generated Lie conformal algebras over Rational Field
```

FinitelyGeneratedAsLambdaBracketAlgebra()

The category of finitely generated Lambda bracket algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(QQ).FinitelyGenerated()
Category of finitely generated Lie conformal algebras over Rational Field
```

WithBasis

alias of *LambdaBracketAlgebrasWithBasis*

super_categories()

The list of super categories of this category.

EXAMPLES:

```
sage: from sage.categories.lambda_bracket_algebras import␣
↪LambdaBracketAlgebras
sage: LambdaBracketAlgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

4.101 Lambda Bracket Algebras With Basis

AUTHORS:

- Reimundo Heluani (2020-08-21): Initial implementation.

class sage.categories.lambda_bracket_algebras_with_basis.**LambdaBracketAlgebrasWithBasis**(*bas e- gor*

Bases: *CategoryWithAxiom_over_base_ring*

The category of Lambda bracket algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).WithBasis()                                  #␣
↪needs sage.rings.number_field
Category of Lie conformal algebras with basis over Algebraic Field
```

class ElementMethods

Bases: `object`

index()

The index of this basis element.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: V = lie_conformal_algebras.NeveuSchwarz(QQ)
sage: V.inject_variables()
Defining L, G, C
sage: G.T(3).index()
('G', 3)
sage: v = V.an_element(); v
L + G + C
sage: v.index()
Traceback (most recent call last):
...
ValueError: index can only be computed for monomials, got L + G + C
```

class FinitelyGeneratedAsLambdaBracketAlgebra(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

The category of finitely generated lambda bracket algebras with basis.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: C = LieConformalAlgebras(QQbar)
sage: C1 = C.WithBasis().FinitelyGenerated(); C1
Category of finitely generated Lie conformal algebras with basis
 over Algebraic Field
sage: C2 = C.FinitelyGenerated().WithBasis(); C2
Category of finitely generated Lie conformal algebras with basis
 over Algebraic Field
sage: C1 is C2
True
```

class Graded(*base_category*)

Bases: `GradedModulesCategory`

The category of H-graded finitely generated lambda bracket algebras with basis.

EXAMPLES:

```
sage: C = LieConformalAlgebras(QQbar)                                   #␣
↪needs sage.rings.number_field
sage: C.WithBasis().FinitelyGenerated().Graded()                        #␣
↪needs sage.rings.number_field
Category of H-graded finitely generated Lie conformal algebras
 with basis over Algebraic Field
```

class ParentMethods

Bases: `object`

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m` in `self`.

EXAMPLES:

```
sage: V = lie_conformal_algebras.Virasoro(QQ)                          #␣
↪needs sage.combinat sage.modules
sage: V.degree_on_basis(('L', 2))                                       #␣
↪needs sage.combinat sage.modules
4
```

4.102 Lattice posets

class `sage.categories.lattice_posets.`**LatticePosets**

Bases: *Category*

The category of lattices, i.e. partially ordered sets in which any two elements have a unique supremum (the elements' least upper bound; called their *join*) and a unique infimum (greatest lower bound; called their *meet*).

EXAMPLES:

```
sage: LatticePosets()
Category of lattice posets
sage: LatticePosets().super_categories()
[Category of posets]
sage: LatticePosets().example()
NotImplemented
```

See also:

Posets, *FiniteLatticePosets*, `LatticePoset()`

Finite

alias of *FiniteLatticePosets*

class ParentMethods

Bases: `object`

join(*x*, *y*)

Returns the join of x and y in this lattice

INPUT:

- `x`, `y` – elements of `self`

EXAMPLES:

```
sage: D = LatticePoset((divisors(60), attrcall("divides")))        #␣
↪needs sage.graphs sage.modules
sage: D.join( D(6), D(10) )                                         #␣
↪needs sage.graphs sage.modules
30
```

meet(*x*, *y*)

Returns the meet of x and y in this lattice

INPUT:

- `x`, `y` – elements of `self`

EXAMPLES:

```
sage: D = LatticePoset((divisors(30), attrcall("divides")))            #␣
↪needs sage.graphs sage.modules
sage: D.meet( D(6), D(15) )                                            #␣
↪needs sage.graphs sage.modules
3
```

super_categories()

Returns a list of the (immediate) super categories of `self`, as per *Category.super_categories()*.

EXAMPLES:

```
sage: LatticePosets().super_categories()
[Category of posets]
```

4.103 Left modules

class sage.categories.left_modules.**LeftModules**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of left modules left modules over an rng (ring not necessarily with unit), i.e. an abelian group with left multiplication by elements of the rng

EXAMPLES:

```
sage: LeftModules(ZZ)
Category of left modules over Integer Ring
sage: LeftModules(ZZ).super_categories()
[Category of commutative additive groups]
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

super_categories()

EXAMPLES:

```
sage: LeftModules(QQ).super_categories()
[Category of commutative additive groups]
```

4.104 Lie Algebras

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

class sage.categories.lie_algebras.**LieAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of Lie algebras.

EXAMPLES:

```
sage: C = LieAlgebras(QQ); C
Category of Lie algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of vector spaces over Rational Field]
```

We construct a typical parent in this category, and do some computations with it:

```
sage: A = C.example(); A                                                    #␣
↪needs sage.groups sage.modules
An example of a Lie algebra: the Lie algebra from the associative
 algebra Symmetric group algebra of order 3 over Rational Field
 generated by ([2, 1, 3], [2, 3, 1])

sage: A.category()                                                          #␣
↪needs sage.groups sage.modules
Category of Lie algebras over Rational Field

sage: A.base_ring()                                                         #␣
↪needs sage.groups sage.modules
Rational Field

sage: a, b = A.lie_algebra_generators()                                     #␣
↪needs sage.groups sage.modules
sage: a.bracket(b)                                                          #␣
↪needs sage.groups sage.modules
-[1, 3, 2] + [3, 2, 1]
sage: b.bracket(2*a + b)                                                    #␣
↪needs sage.groups sage.modules
2*[1, 3, 2] - 2*[3, 2, 1]

sage: A.bracket(a, b)                                                       #␣
↪needs sage.groups sage.modules
-[1, 3, 2] + [3, 2, 1]
```

Please see the source code of A (with `A??`) for how to implement other Lie algebras.

Todo: Many of these tests should use Lie algebras that are not the minimal example and need to be added after github issue #16820 (and github issue #16823).

class ElementMethods

Bases: `object`

bracket(*rhs*)

Return the Lie bracket `[self, rhs]`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).example()
sage: x,y = L.lie_algebra_generators()
sage: x.bracket(y)
-[1, 3, 2] + [3, 2, 1]
sage: x.bracket(0)
0
```

exp(*lie_group=None*)

Return the exponential of `self` in `lie_group`.

INPUT:

- `lie_group` – (optional) the Lie group to map into; If `lie_group` is not given, the Lie group associated to the parent Lie algebra of `self` is used.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L.<X,Y,Z> = LieAlgebra(QQ, 2, step=2)
sage: g = (X + Y + Z).exp(); g                                          #␣
↪needs sage.symbolic
exp(X + Y + Z)
sage: h = X.exp(); h                                                    #␣
↪needs sage.symbolic
exp(X)
sage: g.parent()                                                        #␣
↪needs sage.symbolic
Lie group G of Free Nilpotent Lie algebra on 3 generators (X, Y, Z)
 over Rational Field
sage: g.parent() is h.parent()                                          #␣
↪needs sage.symbolic
True
```

The Lie group can be specified explicitly:

```
sage: # needs sage.combinat sage.modules sage.symbolic
sage: H = L.lie_group('H')
sage: k = Z.exp(lie_group=H); k
exp(Z)
sage: k.parent()
Lie group H of Free Nilpotent Lie algebra on 3 generators (X, Y, Z)
 over Rational Field
sage: g.parent() == k.parent()
False
```

killing_form(*x*)

Return the Killing form of `self` and `x`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()     #␣
↪needs sage.modules
sage: a, b, c = L.lie_algebra_generators()                              #␣
↪needs sage.modules
sage: a.killing_form(b)                                                 #␣
↪needs sage.modules
0
```

lift()

Return the image of `self` under the canonical lift from the Lie algebra to its universal enveloping algebra.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
```

(continues on next page)

(continued from previous page)

```
sage: elt = 3*a + b - c
sage: elt.lift()
3*b0 + b1 - b2
```

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)                                  #␣
↪needs sage.combinat sage.modules
sage: x.lift()                                                                 #␣
↪needs sage.combinat sage.modules
x
```

to_vector(*order=None*)

Return the vector in `g.module()` corresponding to the element `self` of `g` (where `g` is the parent of `self`).

Implement this if you implement `g.module()`. See `LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()          #␣
↪needs sage.modules
sage: u = L((1, 0, 0)).to_vector(); u                                         #␣
↪needs sage.modules
(1, 0, 0)
sage: parent(u)                                                                #␣
↪needs sage.modules
Vector space of dimension 3 over Rational Field
```

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

WithBasis

alias of *FiniteDimensionalLieAlgebrasWithBasis*

extra_super_categories()

Implements the fact that a finite dimensional Lie algebra over a finite ring is finite.

EXAMPLES:

```
sage: LieAlgebras(IntegerModRing(4)).FiniteDimensional().extra_super_
↪categories()
[Category of finite sets]
sage: LieAlgebras(ZZ).FiniteDimensional().extra_super_categories()
[]
sage: C = LieAlgebras(GF(5)).FiniteDimensional()
sage: C.is_subcategory(Sets().Finite())
True
sage: C = LieAlgebras(ZZ).FiniteDimensional()
sage: C.is_subcategory(Sets().Finite())
False
sage: C = LieAlgebras(GF(5)).WithBasis().FiniteDimensional()
sage: C.is_subcategory(Sets().Finite())
True
```

Graded

alias of *GradedLieAlgebras*

class Nilpotent(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Category of nilpotent Lie algebras.

class ParentMethods

Bases: `object`

is_nilpotent()

Return `True` since `self` is nilpotent.

EXAMPLES:

```
sage: h = lie_algebras.Heisenberg(ZZ, oo)                                   #␣
↪needs sage.combinat sage.modules
sage: h.is_nilpotent()                                                      #␣
↪needs sage.combinat sage.modules
True
```

step()

Return the nilpotency step of `self`.

EXAMPLES:

```
sage: h = lie_algebras.Heisenberg(ZZ, oo)                                   #␣
↪needs sage.combinat sage.modules
sage: h.step()                                                              #␣
↪needs sage.combinat sage.modules
2
```

class ParentMethods

Bases: `object`

baker_campbell_hausdorff(*X*, *Y*, *prec=None*)

Return the element $\log(\exp(X)\exp(Y))$.

The BCH formula is an expression for $\log(\exp(X)\exp(Y))$ as a sum of Lie brackets of `X ` and ``Y` with rational coefficients. It is only defined if the base ring of `self` has a coercion from the rationals.

INPUT:

- `X` – an element of `self`
- `Y` – an element of `self`
- `prec` – an integer; the maximum length of Lie brackets to be considered in the formula

EXAMPLES:

The BCH formula for the generators of a free nilpotent Lie algebra of step 4:

```
sage: L = LieAlgebra(QQ, 2, step=4)                                           #␣
↪needs sage.combinat sage.modules
sage: L.inject_variables()                                                    #␣
↪needs sage.combinat sage.modules
Defining X_1, X_2, X_12, X_112, X_122, X_1112, X_1122, X_1222
sage: L.bch(X_1, X_2)                                                         #␣
↪needs sage.combinat sage.modules
X_1 + X_2 + 1/2*X_12 + 1/12*X_112 + 1/12*X_122 + 1/24*X_1122
```

An example of the BCH formula in a quotient:

```
sage: Q = L.quotient(X_112 + X_122)                                     #␣
↪needs sage.combinat sage.modules
sage: x, y = Q.basis().list()[:2]                                       #␣
↪needs sage.combinat sage.modules
sage: Q.bch(x, y)                                                       #␣
↪needs sage.combinat sage.modules
X_1 + X_2 + 1/2*X_12 - 1/24*X_1112
```

The BCH formula for a non-nilpotent Lie algebra requires the precision to be explicitly stated:

```
sage: L.<X,Y> = LieAlgebra(QQ)                                          #␣
↪needs sage.combinat sage.modules
sage: L.bch(X, Y)                                                       #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
ValueError: the Lie algebra is not known to be nilpotent,
so you must specify the precision
sage: L.bch(X, Y, 4)                                                    #␣
↪needs sage.combinat sage.modules
X + 1/12*[X, [X, Y]] + 1/24*[X, [[X, Y], Y]]
+ 1/2*[X, Y] + 1/12*[[X, Y], Y] + Y
```

The BCH formula requires a coercion from the rationals:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, 2, step=2)                             #␣
↪needs sage.combinat sage.modules
sage: L.bch(X, Y)                                                       #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
TypeError: the BCH formula is not well defined
since Integer Ring has no coercion from Rational Field
```

bch(*X*, *Y*, *prec=None*)

Return the element $\log(\exp(X)\exp(Y))$.

The BCH formula is an expression for $\log(\exp(X)\exp(Y))$ as a sum of Lie brackets of `X ` and ``Y` with rational coefficients. It is only defined if the base ring of `self` has a coercion from the rationals.

INPUT:

- `X` – an element of `self`
- `Y` – an element of `self`
- `prec` – an integer; the maximum length of Lie brackets to be considered in the formula

EXAMPLES:

The BCH formula for the generators of a free nilpotent Lie algebra of step 4:

```
sage: L = LieAlgebra(QQ, 2, step=4)                                     #␣
↪needs sage.combinat sage.modules
sage: L.inject_variables()                                              #␣
↪needs sage.combinat sage.modules
Defining X_1, X_2, X_12, X_112, X_122, X_1112, X_1122, X_1222
sage: L.bch(X_1, X_2)                                                   #␣
↪needs sage.combinat sage.modules
X_1 + X_2 + 1/2*X_12 + 1/12*X_112 + 1/12*X_122 + 1/24*X_1122
```

An example of the BCH formula in a quotient:

```
sage: Q = L.quotient(X_112 + X_122)                                     #␣
↪needs sage.combinat sage.modules
sage: x, y = Q.basis().list()[:2]                                        #␣
↪needs sage.combinat sage.modules
sage: Q.bch(x, y)                                                        #␣
↪needs sage.combinat sage.modules
X_1 + X_2 + 1/2*X_12 - 1/24*X_1112
```

The BCH formula for a non-nilpotent Lie algebra requires the precision to be explicitly stated:

```
sage: L.<X,Y> = LieAlgebra(QQ)                                           #␣
↪needs sage.combinat sage.modules
sage: L.bch(X, Y)                                                        #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
ValueError: the Lie algebra is not known to be nilpotent,
so you must specify the precision
sage: L.bch(X, Y, 4)                                                     #␣
↪needs sage.combinat sage.modules
X + 1/12*[X, [X, Y]] + 1/24*[X, [[X, Y], Y]]
+ 1/2*[X, Y] + 1/12*[[X, Y], Y] + Y
```

The BCH formula requires a coercion from the rationals:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, 2, step=2)                              #␣
↪needs sage.combinat sage.modules
sage: L.bch(X, Y)                                                        #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
TypeError: the BCH formula is not well defined
since Integer Ring has no coercion from Rational Field
```

bracket (*lhs*, *rhs*)

Return the Lie bracket `[lhs, rhs]` after coercing `lhs` and `rhs` into elements of `self`.

If `lhs` and `rhs` are Lie algebras, then this constructs the product space, and if only one of them is a Lie algebra, then it constructs the corresponding ideal.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).example()
sage: x, y = L.lie_algebra_generators()
sage: L.bracket(x, x + y)
-[1, 3, 2] + [3, 2, 1]
sage: L.bracket(x, 0)
0
sage: L.bracket(0, x)
0
```

Constructing the product space:

```
sage: L = lie_algebras.Heisenberg(QQ, 1)                                 #␣
↪needs sage.combinat sage.modules
sage: Z = L.bracket(L, L); Z                                             #␣
↪needs sage.combinat sage.modules
```

(continues on next page)

(continued from previous page)

```
Ideal (z) of Heisenberg algebra of rank 1 over Rational Field
sage: L.bracket(L, Z)                                                   #␣
↪needs sage.combinat sage.modules
Ideal () of Heisenberg algebra of rank 1 over Rational Field
```

Constructing ideals:

```
sage: p, q, z = L.basis(); p, q, z                                      #␣
↪needs sage.combinat sage.modules
(p1, q1, z)
sage: L.bracket(3*p, L)                                                 #␣
↪needs sage.combinat sage.modules
Ideal (3*p1) of Heisenberg algebra of rank 1 over Rational Field
sage: L.bracket(L, q + p)                                               #␣
↪needs sage.combinat sage.modules
Ideal (p1 + q1) of Heisenberg algebra of rank 1 over Rational Field
```

from_vector(*v*, *order=None*, *coerce=False*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement `module()`; see the documentation of the latter for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()     #␣
↪needs sage.modules
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u                       #␣
↪needs sage.modules
(1, 0, 0)
sage: parent(u) is L                                                    #␣
↪needs sage.modules
True
```

ideal(**gens*, ***kwds*)

Return the ideal of `self` generated by `gens`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()     #␣
↪needs sage.modules
sage: a, b, c = L.lie_algebra_generators()                              #␣
↪needs sage.modules
sage: L.ideal([2*a - c, b + c])                                         #␣
↪needs sage.modules
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

```
sage: L = LieAlgebras(QQ).example()                                     #␣
↪needs sage.combinat sage.modules
sage: x, y = L.lie_algebra_generators()                                 #␣
↪needs sage.combinat sage.modules
sage: L.ideal([x + y])                                                  #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
NotImplementedError: ideals not yet implemented: see #16824
```

is_abelian()

Return True if this Lie algebra is abelian.

A Lie algebra $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).example()
sage: L.is_abelian()
False
sage: R = QQ['x,y']
sage: L = LieAlgebras(QQ).example(R.gens())
sage: L.is_abelian()
True
```

```
sage: # not implemented, needs sage.combinat sage.modules
sage: L.<x> = LieAlgebra(QQ, 1)
sage: L.is_abelian()
True
sage: L.<x,y> = LieAlgebra(QQ, 2)
sage: L.is_abelian()
False
```

is_commutative()

Return if self is commutative. This is equivalent to self being abelian.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()                                          #␣
↪needs sage.combinat sage.modules
sage: L.is_commutative()                                                     #␣
↪needs sage.combinat sage.modules
False
```

```
sage: L.<x> = LieAlgebra(QQ, 1)         # not implemented                    #␣
↪needs sage.combinat sage.modules
sage: L.is_commutative()         # not implemented                           #␣
↪needs sage.combinat sage.modules
True
```

is_ideal(*A*)

Return if self is an ideal of A.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()                                          #␣
↪needs sage.combinat sage.modules
sage: L.is_ideal(L)                                                          #␣
↪needs sage.combinat sage.modules
True
```

is_nilpotent()

Return if `self` is a nilpotent Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.is_nilpotent()                                                   #␣
↪needs sage.modules
True
```

is_solvable()

Return if `self` is a solvable Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.is_solvable()                                                    #␣
↪needs sage.modules
True
```

killing_form(*x*, *y*)

Return the Killing form of `x` and `y`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: a, b, c = L.lie_algebra_generators()                               #␣
↪needs sage.modules
sage: L.killing_form(a, b + c)                                           #␣
↪needs sage.modules
0
```

lie_group(*name='G'*, ***kwds*)

Return the simply connected Lie group related to `self`.

INPUT:

- `name` – string (default: `'G'`); the name (symbol) given to the Lie group

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 1)                                 #␣
↪needs sage.combinat sage.modules
sage: G = L.lie_group('G'); G                                            #␣
↪needs sage.combinat sage.modules sage.symbolic
Lie group G of Heisenberg algebra of rank 1 over Rational Field
```

lift()

Construct the lift morphism from `self` to the universal enveloping algebra of `self` (the latter is implemented as *universal_enveloping_algebra()*).

This is a Lie algebra homomorphism. It is injective if `self` is a free module over its base ring, or if the base ring is a **Q**-algebra.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: lifted = L.lift(2*a + b - c); lifted
2*b0 + b1 - b2
sage: lifted.parent() is L.universal_enveloping_algebra()
True
```

module()

Return an R-module which is isomorphic to the underlying R-module of `self`.

The rationale behind this method is to enable linear algebraic functionality on `self` (such as computing the span of a list of vectors in `self`) via an isomorphism from `self` to an R-module (typically, although not always, an R-module of the form R^n for an $n \in \mathbf{N}$) on which such functionality already exists. For this method to be of any use, it should return an R-module which has linear algebraic functionality that `self` does not have.

For instance, if `self` has ordered basis (e, f, h), then `self.module()` will be the R-module R^3, and the elements e, f and h of `self` will correspond to the basis vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of `self.module()`.

This method *`module()`* needs to be set whenever a finite-dimensional Lie algebra with basis is intended to support linear algebra (which is, e.g., used in the computation of centralizers and lower central series). One then needs to also implement the R-module isomorphism from `self` to `self.module()` in both directions; that is, implement:

- a `to_vector` ElementMethod which sends every element of `self` to the corresponding element of `self.module()`;
- a `from_vector` ParentMethod which sends every element of `self.module()` to an element of `self`.

The `from_vector` method will automatically serve as an element constructor of `self` (that is, `self(v)` for any `v` in `self.module()` will return `self.from_vector(v)`).

Todo: Ensure that this is actually so.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.module()                                                         #␣
↪needs sage.modules
Vector space of dimension 3 over Rational Field
```

representation(*f=None*, *index_set=None*, *on_basis=False*, ***kwargs*)

Return a representation of `self`.

If no arguments are given, then this returns the trivial representation.

Currently the only implemented method of constructing a representation is by explicitly specifying the action of

- the elements of `self` by matrices;
- the basis elements of `self` using a `dict` or a `Family()`;
- a function on basis elements (either passed as `on_basis` or setting `on_basis=True`).

INPUT:

- `f` – the function that defines the action
- `index_set` – the index set of the representation
- `on_basis` – (optional) see above

See also:

RepresentationByMorphism

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'y':1}})
sage: f = {x: Matrix([[1,0],[0,0]]), y: Matrix([[0,1],[0,0]])}
sage: L.representation(f)
Representation of Lie algebra on 2 generators (x, y) over Rational Field␣
↪defined by:
       [1 0]
x |--> [0 0]
       [0 1]
y |--> [0 0]
sage: L.representation()
Trivial representation of Lie algebra on 2 generators (x, y) over␣
↪Rational Field
```

subalgebra(*gens*, *names=None*, *index_set=None*, *category=None*)

Return the subalgebra of `self` generated by `gens`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: a, b, c = L.lie_algebra_generators()                                #␣
↪needs sage.modules
sage: L.subalgebra([2*a - c, b + c])                                      #␣
↪needs sage.modules
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

```
sage: L = LieAlgebras(QQ).example()                                       #␣
↪needs sage.combinat sage.modules
sage: x,y = L.lie_algebra_generators()                                    #␣
↪needs sage.combinat sage.modules
sage: L.subalgebra([x + y])                                               #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
NotImplementedError: subalgebras not yet implemented: see #17416
```

trivial_representation()

Return the trivial representation of `self`.

EXAMPLES:

```
sage: L = lie_algebras.strictly_upper_triangular_matrices(QQ, 4)
sage: L.trivial_representation()
Trivial representation of Lie algebra of 4-dimensional
 strictly upper triangular matrices over Rational Field
```

universal_enveloping_algebra()

Return the universal enveloping algebra of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.universal_enveloping_algebra()                                   #␣
↪needs sage.combinat sage.modules
Noncommutative Multivariate Polynomial Ring in b0, b1, b2
 over Rational Field, nc-relations: {}
```

```
sage: L = LieAlgebra(QQ, 3, 'x', abelian=True)                           #␣
↪needs sage.combinat sage.modules
sage: L.universal_enveloping_algebra()                                   #␣
↪needs sage.combinat sage.modules
Multivariate Polynomial Ring in x0, x1, x2 over Rational Field
```

See also:

lift()

class SubcategoryMethods

Bases: object

Nilpotent()

Return the full subcategory of nilpotent objects of self.

A Lie algebra L is nilpotent if there exist an integer s such that all iterated brackets of L of length more than s vanish. The integer s is called the nilpotency step. For instance any abelian Lie algebra is nilpotent of step 1.

EXAMPLES:

```
sage: LieAlgebras(QQ).Nilpotent()
Category of nilpotent Lie algebras over Rational Field
sage: LieAlgebras(QQ).WithBasis().Nilpotent()
Category of nilpotent Lie algebras with basis over Rational Field
```

WithBasis

alias of *LieAlgebrasWithBasis*

example(*gens=None*)

Return an example of a Lie algebra as per *Category.example*.

EXAMPLES:

```
sage: LieAlgebras(QQ).example()                                            #␣
↪needs sage.groups sage.modules
An example of a Lie algebra: the Lie algebra from the associative algebra
 Symmetric group algebra of order 3 over Rational Field
 generated by ([2, 1, 3], [2, 3, 1])
```

Another set of generators can be specified as an optional argument:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)                                          #␣
↪needs sage.combinat sage.modules
sage: LieAlgebras(QQ).example(F.gens())                                    #␣
↪needs sage.combinat sage.modules
An example of a Lie algebra: the Lie algebra from the associative algebra
 Free Algebra on 3 generators (x, y, z) over Rational Field
 generated by (x, y, z)
```

super_categories()

EXAMPLES:

```
sage: LieAlgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

class sage.categories.lie_algebras.**LiftMorphism**(*domain*, *codomain*)

Bases: *Morphism*

The natural lifting morphism from a Lie algebra to its enveloping algebra.

4.105 Lie Algebras With Basis

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

class sage.categories.lie_algebras_with_basis.**LieAlgebrasWithBasis**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

Category of Lie algebras with a basis.

class ElementMethods

Bases: `object`

lift()

Lift `self` to the universal enveloping algebra.

EXAMPLES:

```
sage: # needs sage.groups
sage: S = SymmetricGroup(3).algebra(QQ)
sage: L = LieAlgebra(associative=S)
sage: x = L.gen(3)
sage: y = L.gen(1)
sage: x.lift()
b3
sage: y.lift()
b1
sage: x * y
b1*b3 + b4 - b5
```

to_vector(*order=None*)

Return the vector in `g.module()` corresponding to the element `self` of `g` (where `g` is the parent of `self`).

Implement this if you implement `g.module()`. See `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.an_element().to_vector()                                         #␣
↪needs sage.modules
(0, 0, 0)
```

Todo: Doctest this implementation on an example not overshadowed.

Graded

alias of *GradedLieAlgebrasWithBasis*

class ParentMethods

Bases: `object`

bracket_on_basis(*x*, *y*)

Return the bracket of basis elements indexed by `x` and `y` where `x < y`. If this is not implemented, then the method `_bracket_()` for the elements must be overwritten.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()                          #␣
↪needs sage.combinat sage.modules
sage: L.bracket_on_basis(Partition([3,1]), Partition([2,2,1,1]))         #␣
↪needs sage.combinat sage.modules
0
```

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: L.dimension()                                                      #␣
↪needs sage.modules
3
```

```
sage: L = LieAlgebra(QQ, 'x,y', {('x','y'): {'x':1}})                    #␣
↪needs sage.combinat sage.modules
sage: L.dimension()                                                      #␣
↪needs sage.combinat sage.modules
2
```

from_vector(*v*, *order=None*, *coerce=False*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()      #␣
↪needs sage.modules
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u                        #␣
↪needs sage.modules
(1, 0, 0)
sage: parent(u) is L                                                     #␣
↪needs sage.modules
True
```

module()

Return an R-module which is isomorphic to the underlying R-module of `self`.

See `sage.categories.lie_algebras.LieAlgebras.module()` for an explanation.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()                              #␣
↪needs sage.combinat sage.modules
sage: L.module()                                                             #␣
↪needs sage.combinat sage.modules
Free module generated by Partitions over Rational Field
```

pbw_basis(*basis_key=None*, ***kwds*)

Return the Poincare-Birkhoff-Witt basis of the universal enveloping algebra corresponding to `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)                                             #␣
↪needs sage.combinat sage.modules
sage: PBW = L.pbw_basis()                                                    #␣
↪needs sage.combinat sage.modules
```

poincare_birkhoff_witt_basis(*basis_key=None*, ***kwds*)

Return the Poincare-Birkhoff-Witt basis of the universal enveloping algebra corresponding to `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)                                             #␣
↪needs sage.combinat sage.modules
sage: PBW = L.pbw_basis()                                                    #␣
↪needs sage.combinat sage.modules
```

example(*gens=None*)

Return an example of a Lie algebra as per `Category.example`.

EXAMPLES:

```
sage: LieAlgebras(QQ).WithBasis().example()                                      #␣
↪needs sage.combinat sage.modules
An example of a Lie algebra: the abelian Lie algebra on the
 generators indexed by Partitions over Rational Field
```

Another set of generators can be specified as an optional argument:

```
sage: LieAlgebras(QQ).WithBasis().example(Compositions())                        #␣
↪needs sage.combinat sage.modules
An example of a Lie algebra: the abelian Lie algebra on the
 generators indexed by Compositions of non-negative integers
 over Rational Field
```

4.106 Lie Conformal Algebras

Let R be a commutative ring, a *super Lie conformal algebra* [Kac1997] over R (also known as a *vertex Lie algebra*) is an $R[T]$ super module L together with a $\mathbb{Z}/2\mathbb{Z}$-graded R-bilinear operation (called the λ-bracket) $L \otimes L \to L[\lambda]$ (polynomials in λ with coefficients in L), $a \otimes b \mapsto [a_\lambda b]$ satisfying

1. Sesquilinearity:

$$[Ta_\lambda b] = -\lambda[a_\lambda b], \qquad [a_\lambda Tb] = (\lambda + T)[a_\lambda b].$$

2. Skew-Symmetry:

$$[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-T} a],$$

   where $p(a)$ is 0 if a is *even* and 1 if a is *odd*. The bracket in the RHS is computed as follows. First we evaluate $[b_\mu a]$ with the formal parameter μ to the *left*, then replace each appearance of the formal variable μ by $-\lambda - T$. Finally apply T to the coefficients in L.

3. Jacobi identity:

$$[a_\lambda [b_\mu c]] = [[a_{\lambda+\mu} b]_\mu c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]],$$

   which is understood as an equality in $L[\lambda, \mu]$.

   T is usually called the *translation operation* or the *derivative*. For an element $a \in L$ we will say that Ta is the *derivative of* a. We define the *n-th products* $a_{(n)}b$ for $a, b \in L$ by

$$[a_\lambda b] = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)} b.$$

   A Lie conformal algebra is called *H-Graded* [DSK2006] if there exists a decomposition $L = \oplus L_n$ such that the λ-bracket becomes graded of degree -1, that is:

$$a_{(n)}b \in L_{p+q-n-1} \qquad a \in L_p, \ b \in L_q, \ n \geq 0.$$

   In particular this implies that the action of T increases degree by 1.

Note: In the literature arbitrary gradings are allowed. In this implementation we only support non-negative rational gradings.

EXAMPLES:

1. The **Virasoro** Lie conformal algebra Vir over a ring R where 12 is invertible has two generators L, C as an $R[T]$-module. It is the direct sum of a free module of rank 1 generated by L, and a free rank one R module generated by C satisfying $TC = 0$. C is central (the λ-bracket of C with any other vector vanishes). The remaining λ-bracket is given by

$$[L_\lambda L] = TL + 2\lambda L + \frac{\lambda^3}{12} C.$$

2. The **affine** or current Lie conformal algebra $L(\mathfrak{g})$ associated to a finite dimensional Lie algebra $\mathfrak{g}$ with non-degenerate, invariant R-bilinear form $(,)$ is given as a central extension of the free $R[T]$ module generated by $\mathfrak{g}$ by a central element K. The λ-bracket of generators is given by

$$[a_\lambda b] = [a, b] + \lambda(a, b)K, \qquad a, b \in \mathfrak{g}$$

3. The **Weyl** Lie conformal algebra, or $\beta - \gamma$ system is given as the central extension of a free $R[T]$ module with two generators β and γ, by a central element K. The only non-trivial brackets among generators are

$$[\beta_\lambda \gamma] = -[\gamma_\lambda \beta] = K$$

4. The **Neveu-Schwarz** super Lie conformal algebra is a super Lie conformal algebra which is an extension of the Virasoro Lie conformal algebra. It consists of a Virasoro generator L as in example 1 above and an *odd* generator G. The remaining brackets are given by:

$$[L_\lambda G] = \left(T + \frac{3}{2}\lambda\right) G \qquad [G_\lambda G] = 2L + \frac{\lambda^2}{3} C$$

See also:

- `sage.algebras.lie_conformal_algebras.lie_conformal_algebra`
- `sage.algebras.lie_conformal_algebras.examples`

AUTHORS:

- Reimundo Heluani (2019-10-05): Initial implementation.

class `sage.categories.lie_conformal_algebras.`**`LieConformalAlgebras`**(*base*, *name=None*)

Bases: *`Category_over_base_ring`*

The category of Lie conformal algebras.

This is the base category for all Lie conformal algebras. Subcategories with axioms are `FinitelyGenerated` and `WithBasis`. A *finitely generated* Lie conformal algebra is a Lie conformal algebra over R which is finitely generated as an $R[T]$-module. A Lie conformal algebra *with basis* is one with a preferred basis as an R-module.

EXAMPLES:

The base category:

```
sage: C = LieConformalAlgebras(QQ); C
Category of Lie conformal algebras over Rational Field
sage: C.is_subcategory(VectorSpaces(QQ))
True
```

Some subcategories:

```
sage: LieConformalAlgebras(QQbar).FinitelyGenerated().WithBasis()            #␣
↪needs sage.rings.number_field
Category of finitely generated Lie conformal algebras with basis
 over Algebraic Field
```

In addition we support functorial constructions `Graded` and `Super`. These functors commute:

```
sage: CGS = LieConformalAlgebras(AA).Graded().Super(); CGS                   #␣
↪needs sage.rings.number_field
Category of H-graded super Lie conformal algebras over Algebraic Real Field
sage: CGS is LieConformalAlgebras(AA).Super().Graded()                       #␣
↪needs sage.rings.number_field
True
```

That is, we only consider gradings on super Lie conformal algebras that are compatible with the $\mathbf{Z}/2\mathbf{Z}$ grading.

The base ring needs to be a commutative ring:

```
sage: LieConformalAlgebras(QuaternionAlgebra(2))                              #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
ValueError: base must be a commutative ring
got Quaternion Algebra (-1, -1) with base ring Rational Field
```

class ElementMethods

Bases: `object`

is_even_odd()

Return `0` if this element is *even* and `1` if it is *odd*.

Note: This method returns `0` by default since every Lie conformal algebra can be thought as a purely even Lie conformal algebra. In order to implement a super Lie conformal algebra, the user needs to implement this method.

EXAMPLES:

```
sage: R = lie_conformal_algebras.NeveuSchwarz(QQ)                        #␣
↪needs sage.combinat sage.modules
sage: R.inject_variables()                                               #␣
↪needs sage.combinat sage.modules
Defining L, G, C
sage: G.is_even_odd()                                                    #␣
↪needs sage.combinat sage.modules
1
```

FinitelyGeneratedAsLambdaBracketAlgebra

alias of *FinitelyGeneratedLieConformalAlgebras*

Graded

alias of *GradedLieConformalAlgebras*

class ParentMethods

Bases: `object`

Super

alias of *SuperLieConformalAlgebras*

WithBasis

alias of *LieConformalAlgebrasWithBasis*

example()

An example of parent in this category.

EXAMPLES:

```
sage: LieConformalAlgebras(QQ).example()                                     #␣
↪needs sage.combinat sage.modules
The Virasoro Lie conformal algebra over Rational Field
```

super_categories()

The list of super categories of this category.

EXAMPLES:

```
sage: C = LieConformalAlgebras(QQ)
sage: C.super_categories()
[Category of Lambda bracket algebras over Rational Field]
sage: C = LieConformalAlgebras(QQ).FinitelyGenerated(); C
Category of finitely generated Lie conformal algebras over Rational Field
sage: C.super_categories()
[Category of finitely generated lambda bracket algebras over Rational Field,
 Category of Lie conformal algebras over Rational Field]
sage: C.all_super_categories()
[Category of finitely generated Lie conformal algebras over Rational Field,
 Category of finitely generated lambda bracket algebras over Rational Field,
 Category of Lie conformal algebras over Rational Field,
 Category of Lambda bracket algebras over Rational Field,
 Category of vector spaces over Rational Field,
 Category of modules over Rational Field,
 Category of bimodules over Rational Field on the left and Rational Field on␣
↪the right,
 Category of right modules over Rational Field,
 Category of left modules over Rational Field,
 Category of commutative additive groups,
 Category of additive groups,
 Category of additive inverse additive unital additive magmas,
 Category of commutative additive monoids,
 Category of additive monoids,
 Category of additive unital additive magmas,
 Category of commutative additive semigroups,
 Category of additive commutative additive magmas,
 Category of additive semigroups,
 Category of additive magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

4.107 Lie Conformal Algebras With Basis

AUTHORS:

- Reimundo Heluani (2019-10-05): Initial implementation.

class sage.categories.lie_conformal_algebras_with_basis.**LieConformalAlgebrasWithBasis**(*base_cat e- gory*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of Lie conformal algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).WithBasis()                              #␣
↪needs sage.rings.number_field
Category of Lie conformal algebras with basis over Algebraic Field
```

class FinitelyGeneratedAsLambdaBracketAlgebra(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of finitely generated Lie conformal algebras with basis.

EXAMPLES:

```
sage: C = LieConformalAlgebras(QQbar)                                    #␣
↪needs sage.rings.number_field
sage: CWF = C.WithBasis().FinitelyGenerated(); CWF                       #␣
↪needs sage.rings.number_field
Category of finitely generated Lie conformal algebras with basis
 over Algebraic Field
sage: CWF is C.FinitelyGenerated().WithBasis()                           #␣
↪needs sage.rings.number_field
True
```

class Graded(*base_category*)

Bases: `GradedLieConformalAlgebrasCategory`

The category of H-graded finitely generated Lie conformal algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).WithBasis().FinitelyGenerated().
↪Graded()          # needs sage.rings.number_field
Category of H-graded finitely generated Lie conformal algebras with basis
 over Algebraic Field
```

class Super(*base_category*)

Bases: `SuperModulesCategory`

The category of super finitely generated Lie conformal algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(AA).WithBasis().FinitelyGenerated().Super()   #␣
↪needs sage.rings.number_field
Category of super finitely generated Lie conformal algebras with basis
 over Algebraic Real Field
```

class Graded(*base_category*)

Bases: `GradedModulesCategory`

The category of H-graded super finitely generated Lie conformal algebras with basis.

EXAMPLES:

```
sage: C = LieConformalAlgebras(QQbar).WithBasis().FinitelyGenerated() ␣
↪          # needs sage.rings.number_field
sage: C.Graded().Super()                                              ␣
↪          # needs sage.rings.number_field
Category of H-graded super finitely generated Lie conformal algebras
 with basis over Algebraic Field
sage: C.Graded().Super() is C.Super().Graded()                        ␣
↪          # needs sage.rings.number_field
True
```

class Graded(*base_category*)

Bases: `GradedLieConformalAlgebrasCategory`

The category of H-graded Lie conformal algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).WithBasis().Graded()                    #␣
↪needs sage.rings.number_field
Category of H-graded Lie conformal algebras with basis over Algebraic Field
```

class Super(*base_category*)

Bases: *SuperModulesCategory*

The category of super Lie conformal algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(AA).WithBasis().Super()                        #␣
↪needs sage.rings.number_field
Category of super Lie conformal algebras with basis
 over Algebraic Real Field
```

class Graded(*base_category*)

Bases: *GradedLieConformalAlgebrasCategory*

The category of H-graded super Lie conformal algebras with basis.

EXAMPLES:

```
sage: LieConformalAlgebras(QQbar).WithBasis().Super().Graded()          #␣
↪needs sage.rings.number_field
Category of H-graded super Lie conformal algebras with basis
 over Algebraic Field
```

class ParentMethods

Bases: `object`

4.108 Lie Groups

class sage.categories.lie_groups.**LieGroups**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of Lie groups.

A Lie group is a topological group with a smooth manifold structure.

EXAMPLES:

```
sage: from sage.categories.lie_groups import LieGroups
sage: C = LieGroups(QQ); C
Category of Lie groups over Rational Field
```

additional_structure()

Return `None`.

Indeed, the category of Lie groups defines no new structure: a morphism of topological spaces and of smooth manifolds is a morphism as Lie groups.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: from sage.categories.lie_groups import LieGroups
sage: LieGroups(QQ).additional_structure()
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.lie_groups import LieGroups
sage: LieGroups(QQ).super_categories()
[Category of topological groups,
 Category of smooth manifolds over Rational Field]
```

4.109 Loop Crystals

class sage.categories.loop_crystals.**KirillovReshetikhinCrystals**

Bases: *Category_singleton*

Category of Kirillov-Reshetikhin crystals.

class ElementMethods

Bases: object

energy_function()

Return the energy function of self.

Let B be a KR crystal. Let $b^\sharp$ denote the unique element such that $\varphi(b^\sharp) = \ell\Lambda_0$ with $\ell = \min\{\langle c, \varphi(b) \mid b \in B\}$. Let u_B denote the maximal element of B. The *energy* of $b \in B$ is given by

$$D(b) = H(b \otimes b^\sharp) - H(u_B \otimes b^\sharp),$$

where H is the *local energy function*.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: for x in K.classically_highest_weight_vectors():
....:     x, x.energy_function()
([], 1)
([[1], [2]], 0)

sage: K = crystals.KirillovReshetikhin(['D',4,3], 1,2)
sage: for x in K.classically_highest_weight_vectors():
....:     x, x.energy_function()
([], 2)
([[1]], 1)
([[1, 1]], 0)
```

lusztig_involution()

Return the result of the classical Lusztig involution on self.

EXAMPLES:

```
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 2, 3, model='KR')
sage: mg = KRT.module_generators[1]
sage: mg.lusztig_involution()
[[-2, -2, 1], [-1, -1, 2]]
```

(continues on next page)

(continued from previous page)

```
sage: elt = mg.f_string([2,1,3,2]); elt
[[3, -2, 1], [4, -1, 2]]
sage: elt.lusztig_involution()
[[-4, -2, 1], [-3, -1, 2]]
```

class ParentMethods

Bases: `object`

R_matrix(*K*)

Return the combinatorial R-matrix of `self` to `K`.

The *combinatorial* R*-matrix* is the affine crystal isomorphism $R : L \otimes K \to K \otimes L$ which maps $u_L \otimes u_K$ to $u_K \otimes u_L$, where u_K is the unique element in $K = B^{r,s}$ of weight $s\Lambda_r - sc\Lambda_0$ (see *maximal_vector()*).

INPUT:

- `self` – a crystal L
- `K` – a Kirillov-Reshetikhin crystal of the same type as L

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: L = crystals.KirillovReshetikhin(['A',2,1],1,2)
sage: f = K.R_matrix(L)
sage: [[b,f(b)] for b in crystals.TensorProduct(K,L)]
[[[[[1]], [[1, 1]]], [[[1, 1]], [[1]]]],
 [[[[1]], [[1, 2]]], [[[1, 1]], [[2]]]],
 [[[[1]], [[2, 2]]], [[[1, 2]], [[2]]]],
 [[[[1]], [[1, 3]]], [[[1, 1]], [[3]]]],
 [[[[1]], [[2, 3]]], [[[1, 2]], [[3]]]],
 [[[[1]], [[3, 3]]], [[[1, 3]], [[3]]]],
 [[[[2]], [[1, 1]]], [[[1, 2]], [[1]]]],
 [[[[2]], [[1, 2]]], [[[2, 2]], [[1]]]],
 [[[[2]], [[2, 2]]], [[[2, 2]], [[2]]]],
 [[[[2]], [[1, 3]]], [[[2, 3]], [[1]]]],
 [[[[2]], [[2, 3]]], [[[2, 2]], [[3]]]],
 [[[[2]], [[3, 3]]], [[[2, 3]], [[3]]]],
 [[[[3]], [[1, 1]]], [[[1, 3]], [[1]]]],
 [[[[3]], [[1, 2]]], [[[1, 3]], [[2]]]],
 [[[[3]], [[2, 2]]], [[[2, 3]], [[2]]]],
 [[[[3]], [[1, 3]]], [[[3, 3]], [[1]]]],
 [[[[3]], [[2, 3]]], [[[3, 3]], [[2]]]],
 [[[[3]], [[3, 3]]], [[[3, 3]], [[3]]]]]

sage: K = crystals.KirillovReshetikhin(['D',4,1],1,1)
sage: L = crystals.KirillovReshetikhin(['D',4,1],2,1)
sage: f = K.R_matrix(L)
sage: T = crystals.TensorProduct(K,L)
sage: b = T( K(rows=[[1]]), L(rows=[]) )
sage: f(b)
[[[2], [-2]], [[1]]]
```

Alternatively, one can compute the combinatorial R-matrix using the isomorphism method of digraphs:

```
sage: K1 = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: K2 = crystals.KirillovReshetikhin(['A',2,1],2,1)
sage: T1 = crystals.TensorProduct(K1,K2)
sage: T2 = crystals.TensorProduct(K2,K1)
```

(continues on next page)

(continued from previous page)

```
sage: T1.digraph().is_isomorphic(T2.digraph(), edge_labels=True,  # todo:␣
↪not implemented (see #10904 and #10549)
....:                                 certificate=True)
(True, {[[[1]], [[2], [3]]]: [[[1], [3]], [[2]]],
        [[[3]], [[2], [3]]]: [[[2], [3]], [[3]]],
        [[[3]], [[1], [3]]]: [[[1], [3]], [[3]]],
        [[[1]], [[1], [3]]]: [[[1], [3]], [[1]]], [[[1]],
        [[1], [2]]]: [[[1], [2]], [[1]]],
        [[[2]], [[1], [2]]]: [[[1], [2]], [[2]]], [[[3]],
        [[1], [2]]]: [[[2], [3]], [[1]]],
        [[[2]], [[1], [3]]]: [[[1], [2]], [[3]]],
        [[[2]], [[2], [3]]]: [[[2], [3]], [[2]]]})
```

affinization()

Return the corresponding affinization crystal of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.affinization()
Affinization of Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,
↪s)=(1,1)

sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1, model='KR')
sage: K.affinization()
Affinization of Kirillov-Reshetikhin tableaux of type ['A', 2, 1] and␣
↪shape (1, 1)
```

b_sharp()

Return the element $b^\sharp$ of `self`.

Let B be a KR crystal. The element $b^\sharp$ is the unique element such that $\varphi(b^\sharp) = \ell\Lambda_0$ with $\ell = \min\{\langle c, \varphi(b) \rangle \mid b \in B\}$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,1)
sage: K.b_sharp()
[]
sage: K.b_sharp().Phi()
Lambda[0]

sage: K = crystals.KirillovReshetikhin(['C',3,1], 1,3)
sage: K.b_sharp()
[[-1]]
sage: K.b_sharp().Phi()
2*Lambda[0]

sage: K = crystals.KirillovReshetikhin(['D',6,2], 2,2)
sage: K.b_sharp() # long time
[]
sage: K.b_sharp().Phi() # long time
2*Lambda[0]
```

cardinality()

Return the cardinality of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',6,1], 1,1)
sage: K.cardinality()
27
sage: K = crystals.KirillovReshetikhin(['C',6,1], 4,3)
sage: K.cardinality()
4736732
```

classical_decomposition()

Return the classical decomposition of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['A', 3] and shape(s) [[2, 2]]
```

classically_highest_weight_vectors()

Return the classically highest weight elements of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',6,1],1,1)
sage: K.classically_highest_weight_vectors()
([(1,)],)
```

is_perfect(*ell=None*)

Check if `self` is a perfect crystal of level `ell`.

A crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B}$ is isomorphic to the crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. There exists a $\lambda \in X$, such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I} \mathbf{Z}_{\leq 0} \alpha_i$ and there is a unique element in $\mathcal{B}$ of classical weight λ.
4. For all $b \in \mathcal{B}$, $\mathrm{level}(\varepsilon(b)) \geq \ell$.
5. For all Λ dominant weights of level ℓ, there exist unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that $\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$.

Points (1)-(3) are known to hold. This method checks points (4) and (5).

If `self` is the Kirillov-Reshetikhin crystal $B^{r,s}$, then it was proven for non-exceptional types in [FOS2010] that it is perfect if and only if s/c_r is an integer (where c_r is a constant related to the type of the crystal).

It is conjectured this is true for all affine types.

INPUT:

- `ell` – (default: s/c_r) integer; the level

REFERENCES:

[FOS2010]

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.is_perfect()
True

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 1)
sage: K.is_perfect()
```

(continues on next page)

(continued from previous page)

```
False

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 2)
sage: K.is_perfect()
True

sage: K = crystals.KirillovReshetikhin(['E',6,1], 1, 3)
sage: K.is_perfect()
True
```

Todo: Implement a version for tensor products of KR crystals.

level()

Return the level of `self` when `self` is a perfect crystal.

See also:

is_perfect()

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.level()
1
sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 2)
sage: K.level()
1
sage: K = crystals.KirillovReshetikhin(['D',4,1], 1, 3)
sage: K.level()
3

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 1)
sage: K.level()
Traceback (most recent call last):
...
ValueError: this crystal is not perfect
```

local_energy_function(*B*)

Return the local energy function of `self` and `B`.

See *LocalEnergyFunction* for a definition.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,1)
sage: Kp = crystals.KirillovReshetikhin(['A',6,2], 1,1)
sage: H = K.local_energy_function(Kp); H
Local energy function of
 Kirillov-Reshetikhin crystal of type ['BC', 3, 2] with (r,s)=(2,1)
tensor
 Kirillov-Reshetikhin crystal of type ['BC', 3, 2] with (r,s)=(1,1)
```

maximal_vector()

Return the unique element of classical weight $s\Lambda_r$ in `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['C',2,1],1,2)
sage: K.maximal_vector()
[[1, 1]]
sage: K = crystals.KirillovReshetikhin(['E',6,1],1,1)
sage: K.maximal_vector()
[(1,)]

sage: K = crystals.KirillovReshetikhin(['D',4,1],2,1)
sage: K.maximal_vector()
[[1], [2]]
```

module_generator()

Return the unique module generator of classical weight $s\Lambda_r$ of the Kirillov-Reshetikhin crystal $B^{r,s}$.

EXAMPLES:

```
sage: La = RootSystem(['G',2,1]).weight_space().fundamental_weights()
sage: K = crystals.ProjectedLevelZeroLSPaths(La[1])
sage: K.module_generator()
(-Lambda[0] + Lambda[1],)
```

q_dimension(*q=None*, *prec=None*, *use_product=False*)

Return the q-dimension of `self`.

The q-dimension of a KR crystal is defined as the q-dimension of the underlying classical crystal.

EXAMPLES:

```
sage: KRC = crystals.KirillovReshetikhin(['A',2,1], 2,2)
sage: KRC.q_dimension()
q^4 + q^3 + 2*q^2 + q + 1
sage: KRC = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: KRC.q_dimension()
q^10 + q^9 + 3*q^8 + 3*q^7 + 4*q^6 + 4*q^5 + 4*q^4 + 3*q^3 + 3*q^2 + q + 2
```

r()

Return the value r in `self` written as $B^{r,s}$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,4)
sage: K.r()
2
```

s()

Return the value s in `self` written as $B^{r,s}$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,4)
sage: K.s()
4
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of tensor products of Kirillov-Reshetikhin crystals.

class ElementMethods

Bases: `object`

affine_grading()

Return the affine grading of `self`.

The affine grading is calculated by finding a path from `self` to a ground state path (using the helper method *e_string_to_ground_state()*) and counting the number of affine Kashiwara operators e_0 applied on the way.

OUTPUT: an integer

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: t = T.module_generators[0]
sage: t.affine_grading()
1

sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.affine_grading()))
[[[1]], [[1]], [[1]]] 3
[[[2]], [[1]], [[1]]] 2
[[[1]], [[2]], [[1]]] 1
[[[3]], [[2]], [[1]]] 0

sage: K = crystals.KirillovReshetikhin(['C',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.affine_grading()))
[[[1]], [[1]], [[1]]] 2
[[[2]], [[1]], [[1]]] 1
[[[-1]], [[1]], [[1]]] 1
[[[1]], [[2]], [[1]]] 1
[[[-2]], [[2]], [[1]]] 0
[[[1]], [[-1]], [[1]]] 0
```

e_string_to_ground_state()

Return a string of integers in the index set $(i_1, \ldots, i_k)$ such that $e_{i_k} \cdots e_{i_1}$ of `self` is the ground state.

This method calculates a path from `self` to a ground state path using Demazure arrows as defined in Lemma 7.3 in [ST2011].

OUTPUT: a tuple of integers $(i_1, \ldots, i_k)$

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: t = T.module_generators[0]
sage: t.e_string_to_ground_state()
(0, 2)
```

(continues on next page)

(continued from previous page)

```
sage: K = crystals.KirillovReshetikhin(['C',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: t = T.module_generators[0]; t
[[[1]], [[1]]]
sage: t.e_string_to_ground_state()
(0,)
sage: x = t.e(0)
sage: x.e_string_to_ground_state()
()
sage: y = t.f_string([1,2,1,1,0]); y
[[[2]], [[1]]]
sage: y.e_string_to_ground_state()
()
```

energy_function(*algorithm=None*)

Return the energy function of `self`.

ALGORITHM:

definition

Let T be a tensor product of Kirillov-Reshetikhin crystals. Let R_i and H_i be the combinatorial R-matrix and local energy functions, respectively, acting on the i and $i+1$ factors. Let D_B be the energy function of a single Kirillov-Reshetikhin crystal. The *energy function* is given by

$$D = \sum_{j>i} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j} D_B R_1 R_2 \cdots R_{j-1},$$

where D_B acts on the rightmost factor.

grading

If `self` is an element of T, a tensor product of perfect crystals of the same level, then use the affine grading to determine the energy. Specifically, let g denote the affine grading of `self` and d the affine grading of the maximal vector in T. Then the energy of `self` is given by $d - g$.

For more details, see Theorem 7.5 in [ST2011].

INPUT:

- `algorithm` – (default: `None`) use one of the following algorithms to determine the energy function:
  - `'definition'` - use the definition of the energy function;
  - `'grading'` - use the affine grading;

  if not specified, then this uses `'grading'` if all factors are perfect of the same level and otherwise this uses `'definition'`

OUTPUT: an integer

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: T = crystals.TensorProduct(K,K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.energy_function()))
```

(continues on next page)

(continued from previous page)

```
[[[1]], [[1]], [[1]]] 0
[[[2]], [[1]], [[1]]] 1
[[[1]], [[2]], [[1]]] 2
[[[3]], [[2]], [[1]]] 3

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 2)
sage: T = crystals.TensorProduct(K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.energy_function()))
[[], []] 4
[[[1, 1]], []] 3
[[], [[1, 1]]] 1
[[[1, 1]], [[1, 1]]] 0
[[[1, 2]], [[1, 1]]] 1
[[[2, 2]], [[1, 1]]] 2
[[[-1, -1]], [[1, 1]]] 2
[[[1, -1]], [[1, 1]]] 2
[[[2, -1]], [[1, 1]]] 2

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 1)
sage: T = crystals.TensorProduct(K)
sage: t = T.module_generators[0]
sage: t.energy_function('grading')
Traceback (most recent call last):
...
NotImplementedError: all crystals in the tensor product
 need to be perfect of the same level
```

class ParentMethods

Bases: `object`

cardinality()

Return the cardinality of `self`.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [1, 2]])
sage: RC.cardinality()
100
sage: len(RC.list())
100

sage: RC = RiggedConfigurations(['E', 7, 1], [[1,1]])
sage: RC.cardinality()
134
sage: len(RC.list())
134

sage: RC = RiggedConfigurations(['B', 3, 1], [[2,2],[1,2]])
sage: RC.cardinality()
5130
```

classically_highest_weight_vectors()

Return the classically highest weight elements of `self`.

This works by using a backtracking algorithm since if $b_2 \otimes b_1$ is classically highest weight then b_1 is classically highest weight.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: T.classically_highest_weight_vectors()
([[[1]], [[1]], [[1]]],
 [[[2]], [[1]], [[1]]],
 [[[1]], [[2]], [[1]]],
 [[[3]], [[2]], [[1]]])
```

maximal_vector()

Return the maximal vector of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: T.maximal_vector()
[[[1]], [[1]], [[1]]]
```

one_dimensional_configuration_sum(*q=None*, *group_components=True*)

Compute the one-dimensional configuration sum of `self`.

INPUT:

- `q` – (default: `None`) a variable or `None`; if `None`, a variable q is set in the code
- `group_components` – (default: `True`) boolean; if `True`, then the terms are grouped by classical component

The one-dimensional configuration sum is the sum of the weights of all elements in the crystal weighted by the energy function.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: T.one_dimensional_configuration_sum()
B[-2*Lambda[1] + 2*Lambda[2]] + (q+1)*B[-Lambda[1]]
 + (q+1)*B[Lambda[1] - Lambda[2]] + B[2*Lambda[1]]
 + B[-2*Lambda[2]] + (q+1)*B[Lambda[2]]
sage: R.<t> = ZZ[]
sage: T.one_dimensional_configuration_sum(t, False)
B[-2*Lambda[1] + 2*Lambda[2]] + (t+1)*B[-Lambda[1]]
 + (t+1)*B[Lambda[1] - Lambda[2]] + B[2*Lambda[1]]
 + B[-2*Lambda[2]] + (t+1)*B[Lambda[2]]

sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: (LS.one_dimensional_configuration_sum()   # long time
....:  == T.one_dimensional_configuration_sum())
True
```

extra_super_categories()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import␣
↪KirillovReshetikhinCrystals
sage: KirillovReshetikhinCrystals().TensorProducts().extra_super_
```

(continues on next page)

(continued from previous page)

```
↪categories()
[Category of finite regular loop crystals]
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import KirillovReshetikhinCrystals
sage: KirillovReshetikhinCrystals().super_categories()
[Category of finite regular loop crystals]
```

class sage.categories.loop_crystals.**LocalEnergyFunction**(*B*, *Bp*, *normalization=0*)

Bases: *Map*

The local energy function.

Let B and B' be Kirillov-Reshetikhin crystals with maximal vectors u_B and $u_{B'}$ respectively. The *local energy function* $H : B \otimes B' \to \mathbf{Z}$ is the function which satisfies

$$H(e_0(b \otimes b')) = H(b \otimes b') + \begin{cases} 1 & \text{if } i = 0 \text{ and LL,} \\ -1 & \text{if } i = 0 \text{ and RR,} \\ 0 & \text{otherwise,} \end{cases}$$

where LL (resp. RR) denote e_0 acts on the left (resp. right) on both $b \otimes b'$ and $R(b \otimes b')$, and normalized by $H(u_B \otimes u_{B'}) = 0$.

INPUT:

- `B` – a Kirillov-Reshetikhin crystal
- `Bp` – a Kirillov-Reshetikhin crystal
- `normalization` – (default: 0) the normalization value

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['C',2,1], 1,2)
sage: K2 = crystals.KirillovReshetikhin(['C',2,1], 2,1)
sage: H = K.local_energy_function(K2)
sage: T = tensor([K, K2])
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     b, H(b)
([[], [[1], [2]]], 1)
([[[1, 1]], [[1], [2]]], 0)
([[[2, -2]], [[1], [2]]], 1)
([[[1, -2]], [[1], [2]]], 1)
```

REFERENCES:

[KKMMNN1992]

class sage.categories.loop_crystals.**LoopCrystals**

Bases: *Category_singleton*

The category of $U_q'(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is of affine type.

The category is called loop crystals as we can also consider them as crystals corresponding to the loop algebra $\mathfrak{g}_0[t]$, where $\mathfrak{g}_0$ is the corresponding classical type.

EXAMPLES:

```
sage: from sage.categories.loop_crystals import LoopCrystals
sage: C = LoopCrystals()
sage: C
Category of loop crystals
sage: C.super_categories()
[Category of crystals]
sage: C.example()
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(1,1)
```

class ParentMethods

Bases: `object`

digraph(*subset=None*, *index_set=None*)

Return the `DiGraph` associated to `self`.

INPUT:

- `subset` – (optional) a subset of vertices for which the digraph should be constructed
- `index_set` – (optional) the index set to draw arrows

See also:

sage.categories.crystals.Crystals.ParentMethods.digraph()

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['D',4,1], 2, 1)
sage: G = C.digraph()
sage: G.latex_options()  # optional - dot2tex
LaTeX options for Digraph on 29 vertices:
{...'edge_options': <function ... at ...>...}
sage: view(G, tightpage=True)  # optional - dot2tex graphviz, not tested␣
↪(opens external window)
```

weight_lattice_realization()

Return the weight lattice realization used to express weights of elements in `self`.

The default is to use the non-extended affine weight lattice.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.weight_lattice_realization()
Ambient space of the Root system of type ['A', 5]
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.weight_lattice_realization()
Weight lattice of the Root system of type ['A', 2, 1]
```

example(*n=3*)

Return an example of Kirillov-Reshetikhin crystals, as per *Category.example()*.

EXAMPLES:

```
sage: from sage.categories.loop_crystals import LoopCrystals
sage: B = LoopCrystals().example(); B
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(1,1)
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import LoopCrystals
sage: LoopCrystals().super_categories()
[Category of crystals]
```

class sage.categories.loop_crystals.**RegularLoopCrystals**

Bases: *Category_singleton*

The category of regular $U_q'(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is of affine type.

class ElementMethods

Bases: object

classical_weight()

Return the classical weight of self.

EXAMPLES:

```
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: hw = LS.classically_highest_weight_vectors()
sage: [(v.weight(), v.classical_weight()) for v in hw]
[(-2*Lambda[0] + 2*Lambda[1], (2, 0, 0)),
 (-Lambda[0] + Lambda[2], (1, 1, 0))]
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import RegularLoopCrystals
sage: RegularLoopCrystals().super_categories()
[Category of regular crystals,
 Category of loop crystals]
```

4.110 L-trivial semigroups

class sage.categories.l_trivial_semigroups.**LTrivialSemigroups**(*base_category*)

Bases: *CategoryWithAxiom*

Commutative_extra_super_categories()

Implement the fact that a commutative R-trivial semigroup is J-trivial.

EXAMPLES:

```
sage: Semigroups().LTrivial().Commutative_extra_super_categories()
[Category of j trivial semigroups]
```

RTrivial_extra_super_categories()

Implement the fact that an L-trivial and R-trivial semigroup is J-trivial.

EXAMPLES:

```
sage: Semigroups().LTrivial().RTrivial_extra_super_categories()
[Category of j trivial magmas]
```

extra_super_categories()

Implement the fact that a L-trivial semigroup is H-trivial.

EXAMPLES:

```
sage: Semigroups().LTrivial().extra_super_categories()
[Category of h trivial semigroups]
```

4.111 Magmas

class sage.categories.magmas.**Magmas**

Bases: *Category_singleton*

The category of (multiplicative) magmas.

A magma is a set with a binary operation $*$.

EXAMPLES:

```
sage: Magmas()
Category of magmas
sage: Magmas().super_categories()
[Category of sets]
sage: Magmas().all_super_categories()
[Category of magmas, Category of sets,
 Category of sets with partial maps, Category of objects]
```

The following axioms are defined by this category:

```
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Unital()
Category of unital magmas
sage: Magmas().Commutative()
Category of commutative magmas
sage: Magmas().Unital().Inverse()
Category of inverse unital magmas
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative().Unital()
Category of monoids
sage: Magmas().Associative().Unital().Inverse()
Category of groups
```

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: object

is_field(*proof=True*)

Return True if self is a field.

For a magma algebra RS this is always false unless S is trivial and the base ring $R`$ is a field.

EXAMPLES:

```
sage: SymmetricGroup(1).algebra(QQ).is_field()                             #␣
↪needs sage.groups
True
sage: SymmetricGroup(1).algebra(ZZ).is_field()                             #␣
↪needs sage.groups
False
sage: SymmetricGroup(2).algebra(QQ).is_field()                             #␣
↪needs sage.groups
False
```

extra_super_categories()

EXAMPLES:

```
sage: MCA = Magmas().Commutative().Algebras(QQ)
sage: MCA.extra_super_categories()
[Category of commutative magmas]
```

This implements the fact that the algebra of a commutative magma is commutative:

```
sage: MCA.super_categories()
[Category of magma algebras over Rational Field,
 Category of commutative magmas]
```

In particular, commutative monoid algebras are commutative algebras:

```
sage: MoCA = Monoids().Commutative().Algebras(QQ)
sage: MoCA.is_subcategory(Algebras(QQ).Commutative())
True
```

Associative

alias of *Semigroups*

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

class ParentMethods

Bases: object

product(*left*, *right*)

EXAMPLES:

```
sage: C = Magmas().CartesianProducts().example(); C
The Cartesian product of (Rational Field, Integer Ring, Integer Ring)
sage: x = C.an_element(); x
(1/2, 1, 1)
sage: x * x
(1/4, 1, 1)

sage: # needs sage.groups sage.modules
sage: A = SymmetricGroupAlgebra(QQ, 3)
sage: x = cartesian_product([A([1,3,2]), A([2,3,1])])
sage: y = cartesian_product([A([1,3,2]), A([2,3,1])])
sage: cartesian_product([A,A]).product(x,y)
B[(0, [1, 2, 3])] + B[(1, [3, 1, 2])]
sage: x*y
B[(0, [1, 2, 3])] + B[(1, [3, 1, 2])]
```

example()

Return an example of Cartesian product of magmas.

EXAMPLES:

```
sage: C = Magmas().CartesianProducts().example(); C
The Cartesian product of (Rational Field, Integer Ring, Integer Ring)
sage: C.category()
Join of Category of Cartesian products of commutative rings and
Category of Cartesian products of metric spaces
sage: sorted(C.category().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Commutative',
 'Distributive', 'Unital']

sage: TestSuite(C).run()
```

extra_super_categories()

This implements the fact that a subquotient (and therefore a quotient or subobject) of a finite set is finite.

EXAMPLES:

```
sage: Semigroups().CartesianProducts().extra_super_categories()
[Category of semigroups]
sage: Semigroups().CartesianProducts().super_categories()
[Category of semigroups, Category of Cartesian products of magmas]
```

class Commutative(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

extra_super_categories()

EXAMPLES:

```
sage: MCA = Magmas().Commutative().Algebras(QQ)
sage: MCA.extra_super_categories()
[Category of commutative magmas]
```

This implements the fact that the algebra of a commutative magma is commutative:

```
sage: MCA.super_categories()
[Category of magma algebras over Rational Field,
 Category of commutative magmas]
```

In particular, commutative monoid algebras are commutative algebras:

```
sage: MoCA = Monoids().Commutative().Algebras(QQ)
sage: MoCA.is_subcategory(Algebras(QQ).Commutative())
True
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that a Cartesian product of commutative additive magmas is still a commutative additive magmas.

EXAMPLES:

```
sage: C = Magmas().Commutative().CartesianProducts()
sage: C.extra_super_categories()
[Category of commutative magmas]
sage: C.axioms()
frozenset({'Commutative'})
```

class ParentMethods

Bases: `object`

is_commutative()

Return `True`, since commutative magmas are commutative.

EXAMPLES:

```
sage: Parent(QQ, category=CommutativeRings()).is_commutative()
True
```

class ElementMethods

Bases: `object`

is_idempotent()

Test whether `self` is idempotent.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup:
 the free semigroup generated by ('a', 'b', 'c', 'd')
sage: a = S('a')
sage: a^2
'aa'
sage: a.is_idempotent()
False
```

```
sage: L = Semigroups().example("leftzero"); L
An example of a semigroup: the left zero semigroup
sage: x = L('x')
sage: x^2
'x'
sage: x.is_idempotent()
True
```

FinitelyGeneratedAsMagma

alias of *FinitelyGeneratedMagmas*

class JTrivial(*base_category*)

Bases: *CategoryWithAxiom*

class ParentMethods

Bases: `object`

multiplication_table(*names='letters'*, *elements=None*)

Returns a table describing the multiplication operation.

Note: The order of the elements in the row and column headings is equal to the order given by the table's `list()` method. The association can also be retrieved with the `dict()` method.

INPUT:

- `names` - the type of names used
  - `'letters'` - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading 'a's.
  - `'digits'` - base 10 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading zeros.
  - `'elements'` - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- `elements` - default = `None`. A list of elements of the magma, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements, perhaps when this is used in the context of a particular structure. The default is to use whatever ordering the `S.list` method returns. Or the `elements` can be a subset which is closed under the operation. In particular, this can be used when the base set is infinite.

OUTPUT:

The multiplication table as an object of the class `OperationTable` which defines several methods for manipulating and displaying the table. See the documentation there for full details to supplement the documentation here.

EXAMPLES:

The default is to represent elements as lowercase ASCII letters.

```
sage: G = CyclicPermutationGroup(5)                                     #␣
↪needs sage.groups
sage: G.multiplication_table()                                          #␣
↪needs sage.groups
*  a b c d e
 +----------
a| a b c d e
b| b c d e a
c| c d e a b
d| d e a b c
e| e a b c d
```

All that is required is that an algebraic structure has a multiplication defined. A *LeftRegularBand* is an example of a finite semigroup. The `names` argument allows displaying the elements in different ways.

```
sage: from sage.categories.examples.finite_semigroups import␣
↪LeftRegularBand
sage: L = LeftRegularBand(('a', 'b'))
sage: T = L.multiplication_table(names='digits')                        #␣
↪needs sage.modules
sage: T.column_keys()                                                   #␣
↪needs sage.modules
('a', 'ab', 'b', 'ba')
sage: T                                                                 #␣
↪needs sage.modules
*  0 1 2 3
 +--------
0| 0 1 1 1
1| 1 1 1 1
```

(continues on next page)

(continued from previous page)

```
2|  3  3  2  3
3|  3  3  3  3
```

Specifying the elements in an alternative order can provide more insight into how the operation behaves.

```
sage: L = LeftRegularBand(('a', 'b', 'c'))
sage: elts = sorted(L.list())
sage: L.multiplication_table(elements=elts)                               #␣
↪needs sage.modules
*  a b c d e f g h i j k l m n o
 +------------------------------
a| a b c d e b b c c c d d e e e
b| b b c c c b b c c c c c c c c
c| c c c c c c c c c c c c c c c
d| d e e d e e e e e e d d e e e
e| e e e e e e e e e e e e e e e
f| g g h h h f g h i j i j j i j
g| g g h h h g g h h h h h h h h
h| h h h h h h h h h h h h h h h
i| j j j j j i j j i j i j j i j
j| j j j j j j j j j j j j j j j
k| l m m l m n o o n o k l m n o
l| l m m l m m m m m m l l m m m
m| m m m m m m m m m m m m m m m
n| o o o o o n o o n o n o o n o
o| o o o o o o o o o o o o o o o
```

The `elements` argument can be used to provide a subset of the elements of the structure. The subset must be closed under the operation. Elements need only be in a form that can be coerced into the set. The `names` argument can also be used to request that the elements be represented with their usual string representation.

```
sage: L = LeftRegularBand(('a','b','c'))
sage: elts=['a', 'c', 'ac', 'ca']
sage: L.multiplication_table(names='elements', elements=elts)            #␣
↪needs sage.modules
   *   'a'  'c' 'ac' 'ca'
    +--------------------
 'a'|  'a' 'ac' 'ac' 'ac'
 'c'| 'ca'  'c' 'ca' 'ca'
'ac'| 'ac' 'ac' 'ac' 'ac'
'ca'| 'ca' 'ca' 'ca' 'ca'
```

The table returned can be manipulated in various ways. See the documentation for `OperationTable` for more comprehensive documentation.

```
sage: # needs sage.groups sage.modules
sage: G = AlternatingGroup(3)
sage: T = G.multiplication_table()
sage: T.column_keys()
((), (1,2,3), (1,3,2))
sage: T.translation()
{'a': (), 'b': (1,2,3), 'c': (1,3,2)}
sage: T.change_names(['x', 'y', 'z'])
sage: T.translation()
{'x': (), 'y': (1,2,3), 'z': (1,3,2)}
```

(continues on next page)

(continued from previous page)

```
sage: T
*  x y z
 +------
x| x y z
y| y z x
z| z x y
```

product (*x*, *y*)

The binary multiplication of the magma.

INPUT:

- x, y – elements of this magma

OUTPUT:

- an element of the magma (the product of x and y)

EXAMPLES:

```
sage: S = Semigroups().example("free")
sage: x = S('a'); y = S('b')
sage: S.product(x, y)
'ab'
```

A parent in Magmas() must either implement *product()* in the parent class or _mul_ in the element class. By default, the addition method on elements x._mul_(y) calls S.product(x,y), and reciprocally.

As a bonus, S.product models the binary function from S to S:

```
sage: bin = S.product
sage: bin(x,y)
'ab'
```

Currently, S.product is just a bound method:

```
sage: bin
<bound method FreeSemigroup.product of An example of a semigroup:
 the free semigroup generated by ('a', 'b', 'c', 'd')>
```

When Sage will support multivariate morphisms, it will be possible, and in fact recommended, to enrich S.product with extra mathematical structure. This will typically be implemented using lazy attributes.:

```
sage: bin                  # todo: not implemented
Generic binary morphism:
From: (S x S)
To:   S
```

product_from_element_class_mul (*x*, *y*)

The binary multiplication of the magma.

INPUT:

- x, y – elements of this magma

OUTPUT:

- an element of the magma (the product of x and y)

EXAMPLES:

```
sage: S = Semigroups().example("free")
sage: x = S('a'); y = S('b')
sage: S.product(x, y)
'ab'
```

A parent in `Magmas()` must either implement *`product()`* in the parent class or `_mul_` in the element class. By default, the addition method on elements `x._mul_(y)` calls `S.product(x,y)`, and reciprocally.

As a bonus, `S.product` models the binary function from `S` to `S`:

```
sage: bin = S.product
sage: bin(x,y)
'ab'
```

Currently, `S.product` is just a bound method:

```
sage: bin
<bound method FreeSemigroup.product of An example of a semigroup:
 the free semigroup generated by ('a', 'b', 'c', 'd')>
```

When Sage will support multivariate morphisms, it will be possible, and in fact recommended, to enrich `S.product` with extra mathematical structure. This will typically be implemented using lazy attributes.:

```
sage: bin                  # todo: not implemented
Generic binary morphism:
From: (S x S)
To:   S
```

class Realizations(*category*, **args*)

Bases: *`RealizationsCategory`*

class ParentMethods

Bases: `object`

product_by_coercion(*left*, *right*)

Default implementation of product for realizations.

This method coerces to the realization specified by `self.realization_of().a_realization()`, computes the product in that realization, and then coerces back.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: Out = Sets().WithRealizations().example().Out(); Out
The subset algebra of {1, 2, 3} over Rational Field
 in the Out basis
sage: Out.product
<bound method Magmas.Realizations.ParentMethods.product_by_coercion
 of The subset algebra of {1, 2, 3} over Rational Field
  in the Out basis>
sage: Out.product.__module__
'sage.categories.magmas'
sage: x = Out.an_element()
sage: y = Out.an_element()
sage: Out.product(x, y)
Out[{}] + 4*Out[{1}] + 9*Out[{2}] + Out[{1, 2}]
```

class SubcategoryMethods

Bases: `object`

Associative()

Return the full subcategory of the associative objects of `self`.

A (multiplicative) `magma Magmas` M is *associative* if, for all $x, y, z \in M$,

$$x * (y * z) = (x * y) * z$$

See also:

Wikipedia article Associative_property

EXAMPLES:

```
sage: Magmas().Associative()
Category of semigroups
```

Commutative()

Return the full subcategory of the commutative objects of `self`.

A (multiplicative) `magma Magmas` M is *commutative* if, for all $x, y \in M$,

$$x * y = y * x$$

See also:

Wikipedia article Commutative_property

EXAMPLES:

```
sage: Magmas().Commutative()
Category of commutative magmas
sage: Monoids().Commutative()
Category of commutative monoids
```

Distributive()

Return the full subcategory of the objects of `self` where $*$ is distributive on $+$.

INPUT:

- `self` – a subcategory of *`Magmas`* and *`AdditiveMagmas`*

Given that Sage does not yet know that the category `MagmasAndAdditiveMagmas` is the intersection of the categories *`Magmas`* and *`AdditiveMagmas`*, the method `MagmasAndAdditiveMagmas.SubcategoryMethods.Distributive()` is not available, as would be desirable, for this intersection.

This method is a workaround. It checks that `self` is a subcategory of both *`Magmas`* and *`AdditiveMagmas`* and upgrades it to a subcategory of `MagmasAndAdditiveMagmas` before applying the axiom. It complains otherwise, since the `Distributive` axiom does not make sense for a plain magma.

EXAMPLES:

```
sage: (Magmas() & AdditiveMagmas()).Distributive()
Category of distributive magmas and additive magmas
sage: (Monoids() & CommutativeAdditiveGroups()).Distributive()
Category of rings
```

(continues on next page)

(continued from previous page)

```
sage: Magmas().Distributive()
Traceback (most recent call last):
...
ValueError: The distributive axiom only makes sense on a magma
which is simultaneously an additive magma
sage: Semigroups().Distributive()
Traceback (most recent call last):
...
ValueError: The distributive axiom only makes sense on a magma
which is simultaneously an additive magma
```

FinitelyGenerated()

Return the subcategory of the objects of `self` that are endowed with a distinguished finite set of (multiplicative) magma generators.

EXAMPLES:

This is a shorthand for *FinitelyGeneratedAsMagma()*, which see:

```
sage: Magmas().FinitelyGenerated()
Category of finitely generated magmas
sage: Semigroups().FinitelyGenerated()
Category of finitely generated semigroups
sage: Groups().FinitelyGenerated()
Category of finitely generated enumerated groups
```

An error is raised if this is ambiguous:

```
sage: (Magmas() & AdditiveMagmas()).FinitelyGenerated()
Traceback (most recent call last):
...
ValueError: FinitelyGenerated is ambiguous for
Join of Category of magmas and Category of additive magmas.
Please use explicitly one of the FinitelyGeneratedAsXXX methods
```

Note: Checking that there is no ambiguity currently assumes that all the other "finitely generated" axioms involve an additive structure. As of Sage 6.4, this is correct.

The use of this shorthand should be reserved for casual interactive use or when there is no risk of ambiguity.

FinitelyGeneratedAsMagma()

Return the subcategory of the objects of `self` that are endowed with a distinguished finite set of (multiplicative) magma generators.

A set S of elements of a multiplicative magma form a *set of generators* if any element of the magma can be expressed recursively from elements of S and products thereof.

It is not imposed that morphisms shall preserve the distinguished set of generators; hence this is a full subcategory.

See also:

Wikipedia article Unital_magma#unital

EXAMPLES:

```
sage: Magmas().FinitelyGeneratedAsMagma()
Category of finitely generated magmas
```

Being finitely generated does depend on the structure: for a ring, being finitely generated as a magma, as an additive magma, or as a ring are different concepts. Hence the name of this axiom is explicit:

```
sage: Rings().FinitelyGeneratedAsMagma()
Category of finitely generated as magma enumerated rings
```

On the other hand, it does not depend on the multiplicative structure: for example a group is finitely generated if and only if it is finitely generated as a magma. A short hand is provided when there is no ambiguity, and the output tries to reflect that:

```
sage: Semigroups().FinitelyGenerated()
Category of finitely generated semigroups
sage: Groups().FinitelyGenerated()
Category of finitely generated enumerated groups

sage: Semigroups().FinitelyGenerated().axioms()
frozenset({'Associative', 'Enumerated', 'FinitelyGeneratedAsMagma'})
```

Note that the set of generators may depend on the actual category; for example, in a group, one can often use less generators since it is allowed to take inverses.

JTrivial()

Return the full subcategory of the J-trivial objects of `self`.

This axiom is in fact only meaningful for *semigroups*. This stub definition is here as a workaround for github issue #20515, in order to define the J-trivial axiom as the intersection of the L and R-trivial axioms.

See also:

Semigroups.SubcategoryMethods.JTrivial()

Unital()

Return the subcategory of the unital objects of `self`.

A (multiplicative) `magma Magmas` M is *unital* if it admits an element 1, called *unit*, such that for all $x \in M$,

$$1 * x = x * 1 = x$$

This element is necessarily unique, and should be provided as `M.one()`.

See also:

Wikipedia article Unital_magma#unital

EXAMPLES:

```
sage: Magmas().Unital()
Category of unital magmas
sage: Semigroups().Unital()
Category of monoids
sage: Monoids().Unital()
Category of monoids
sage: from sage.categories.associative_algebras import AssociativeAlgebras
sage: AssociativeAlgebras(QQ).Unital()
Category of algebras over Rational Field
```

class Subquotients(*category*, **args*)

Bases: `SubquotientsCategory`

The category of subquotient magmas.

See `Sets.SubcategoryMethods.Subquotients()` for the general setup for subquotients. In the case of a subquotient magma S of a magma G, the condition that r be a morphism in `As` can be rewritten as follows:

- for any two $a, b \in S$ the identity $a \times_S b = r(l(a) \times_G l(b))$ holds.

This is used by this category to implement the product $\times_S$ of S from l and r and the product of G.

EXAMPLES:

```
sage: Semigroups().Subquotients().all_super_categories()
[Category of subquotients of semigroups, Category of semigroups,
 Category of subquotients of magmas, Category of magmas,
 Category of subquotients of sets, Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

class ParentMethods

Bases: `object`

product(*x*, *y*)

Return the product of two elements of `self`.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S
An example of a (sub)quotient semigroup:
a quotient of the left zero semigroup
sage: S.product(S(19), S(3))
19
```

Here is a more elaborate example involving a sub algebra:

```
sage: Z = SymmetricGroup(5).algebra(QQ).center()                          #␣
↪needs sage.groups
sage: B = Z.basis()                                                      #␣
↪needs sage.groups
sage: B[3] * B[2]                                                        #␣
↪needs sage.groups
4*B[2] + 6*B[3] + 5*B[6]
```

class Unital(*base_category*)

Bases: `CategoryWithAxiom_singleton`

class Algebras(*category*, **args*)

Bases: `AlgebrasCategory`

extra_super_categories()

EXAMPLES:

```
sage: MCA = Magmas().Commutative().Algebras(QQ)
sage: MCA.extra_super_categories()
[Category of commutative magmas]
```

This implements the fact that the algebra of a commutative magma is commutative:

```
sage: MCA.super_categories()
[Category of magma algebras over Rational Field,
 Category of commutative magmas]
```

In particular, commutative monoid algebras are commutative algebras:

```
sage: MoCA = Monoids().Commutative().Algebras(QQ)
sage: MoCA.is_subcategory(Algebras(QQ).Commutative())
True
```

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

one()

Return the unit of this Cartesian product.

It is built from the units for the Cartesian factors of `self`.

EXAMPLES:

```
sage: cartesian_product([QQ, ZZ, RR]).one()                    #␣
↪needs sage.rings.real_mpfr
(1, 1, 1.00000000000000)
```

extra_super_categories()

Implement the fact that a Cartesian product of unital magmas is a unital magma

EXAMPLES:

```
sage: C = Magmas().Unital().CartesianProducts()
sage: C.extra_super_categories()
[Category of unital magmas]
sage: C.axioms()
frozenset({'Unital'})

sage: Monoids().CartesianProducts().is_subcategory(Monoids())
True
```

class ElementMethods

Bases: `object`

class Inverse(*base_category*)

Bases: `CategoryWithAxiom_singleton`

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

extra_super_categories()

Implement the fact that a Cartesian product of magmas with inverses is a magma with inverse.

EXAMPLES:

```
sage: C = Magmas().Unital().Inverse().CartesianProducts()
sage: C.extra_super_categories()
[Category of inverse unital magmas]
sage: sorted(C.axioms())
['Inverse', 'Unital']
```

class ParentMethods

Bases: `object`

is_empty()

Return whether `self` is empty.

Since this set is a unital magma it is not empty and this method always return `False`.

EXAMPLES:

```
sage: S = SymmetricGroup(2)                                              #␣
↪needs sage.groups
sage: S.is_empty()                                                       #␣
↪needs sage.groups
False

sage: M = Monoids().example()
sage: M.is_empty()
False
```

one()

Return the unit of the monoid, that is the unique neutral element for $*$.

Note: The default implementation is to coerce 1 into `self`. It is recommended to override this method because the coercion from the integers:

- is not always meaningful (except for 1);
- often uses `self.one()`.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid:
 the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.one()
''
```

class Realizations(*category*, **args*)

Bases: `RealizationsCategory`

class ParentMethods

Bases: `object`

one()

Return the unit element of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.groups
sage: from sage.combinat.root_system.extended_affine_weyl_group␣
↪import ExtendedAffineWeylGroup
```

(continues on next page)

(continued from previous page)

```
sage: PvW0 = ExtendedAffineWeylGroup(['A',2,1]).PvW0()
sage: PvW0 in Magmas().Unital().Realizations()
True
sage: PvW0.one()
1
```

class SubcategoryMethods

Bases: `object`

Inverse()

Return the full subcategory of the inverse objects of `self`.

An inverse :class:`(multiplicative) magma <Magmas>` is a *unital magma* such that every element admits both an inverse on the left and on the right. Such a magma is also called a *loop*.

See also:

Wikipedia article Inverse_element, Wikipedia article Quasigroup

EXAMPLES:

```
sage: Magmas().Unital().Inverse()
Category of inverse unital magmas
sage: Monoids().Inverse()
Category of groups
```

additional_structure()

Return `self`.

Indeed, the category of unital magmas defines an additional structure, namely the unit of the magma which shall be preserved by morphisms.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: Magmas().Unital().additional_structure()
Category of unital magmas
```

super_categories()

EXAMPLES:

```
sage: Magmas().super_categories()
[Category of sets]
```

4.112 Magmas and Additive Magmas

class sage.categories.magmas_and_additive_magmas.**MagmasAndAdditiveMagmas**

Bases: *Category_singleton*

The category of sets $(S, +, *)$ with an additive operation '+' and a multiplicative operation $*$

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: C = MagmasAndAdditiveMagmas(); C
Category of magmas and additive magmas
```

This is the base category for the categories of rings and their variants:

```
sage: C.Distributive()
Category of distributive magmas and additive magmas
sage: C.Distributive().Associative().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().AdditiveInverse()
Category of rngs
sage: C.Distributive().Associative().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().Unital()
Category of semirings
sage: C.Distributive().Associative().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().AdditiveInverse().Unital()
Category of rings
```

This category is really meant to represent the intersection of the categories of *Magmas* and *AdditiveMagmas*; however Sage's infrastructure does not allow yet to model this:

```
sage: Magmas() & AdditiveMagmas()
Join of Category of magmas and Category of additive magmas

sage: Magmas() & AdditiveMagmas()        # todo: not implemented
Category of magmas and additive magmas
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that this structure is stable under Cartesian products.

Distributive

alias of *DistributiveMagmasAndAdditiveMagmas*

class SubcategoryMethods

Bases: `object`

Distributive()

Return the full subcategory of the objects of `self` where $*$ is distributive on $+$.

A *magma* and *additive magma* M is *distributive* if, for all $x, y, z \in M$,

$$x * (y + z) = x * y + x * z \text{ and } (x + y) * z = x * z + y * z$$

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: C = MagmasAndAdditiveMagmas().Distributive(); C
Category of distributive magmas and additive magmas
```

Note: Given that Sage does not know that *MagmasAndAdditiveMagmas* is the intersection of *Magmas* and *AdditiveMagmas*, this method is not available for:

```
sage: Magmas() & AdditiveMagmas()
Join of Category of magmas and Category of additive magmas
```

Still, the natural syntax works:

```
sage: (Magmas() & AdditiveMagmas()).Distributive()
Category of distributive magmas and additive magmas
```

thanks to a workaround implemented in *Magmas.SubcategoryMethods.Distributive()*:

```
sage: (Magmas() & AdditiveMagmas()).Distributive.__module__
'sage.categories.magmas'
```

additional_structure()

Return None.

Indeed, this category is meant to represent the join of *AdditiveMagmas* and *Magmas*. As such, it defines no additional structure.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: MagmasAndAdditiveMagmas().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: MagmasAndAdditiveMagmas().super_categories()
[Category of magmas, Category of additive magmas]
```

4.113 Non-unital non-associative algebras

class sage.categories.magmatic_algebras.**MagmaticAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of algebras over a given base ring.

An algebra over a ring R is a module over R endowed with a bilinear multiplication.

Warning: *MagmaticAlgebras* will eventually replace the current *Algebras* for consistency with e.g. Wikipedia article Algebras which assumes neither associativity nor the existence of a unit (see github issue #15043).

EXAMPLES:

```
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: C = MagmaticAlgebras(ZZ); C
Category of magmatic algebras over Integer Ring
sage: C.super_categories()
[Category of additive commutative additive associative additive
  unital distributive magmas and additive magmas,
 Category of modules over Integer Ring]
```

Associative

alias of *AssociativeAlgebras*

class ParentMethods

Bases: `object`

algebra_generators()

Return a family of generators of this algebra.

EXAMPLES:

```
sage: F = AlgebrasWithBasis(QQ).example(); F                        #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis:
 the free algebra on the generators ('a', 'b', 'c') over Rational Field
sage: F.algebra_generators()                                        #␣
↪needs sage.combinat sage.modules
Family (B[word: a], B[word: b], B[word: c])
```

Unital

alias of *UnitalAlgebras*

class WithBasis(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class ParentMethods

Bases: `object`

derivations_basis()

Return a basis for the Lie algebra of derivations of `self` as matrices.

A derivation D of an algebra is an endomorphism of A such that

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. The set of all derivations form a Lie algebra.

EXAMPLES:

We construct the Heisenberg Lie algebra as a multiplicative algebra:

```
sage: # needs sage.combinat sage.modules
sage: p_mult = matrix([[0,0,0], [0,0,-1], [0,0,0]])
sage: q_mult = matrix([[0,0,1], [0,0,0], [0,0,0]])
sage: A = algebras.FiniteDimensional(QQ,
....:          [p_mult, q_mult, matrix(QQ, 3, 3)], 'p,q,z')
sage: A.inject_variables()
```

(continues on next page)

(continued from previous page)

```
Defining p, q, z
sage: p * q
z
sage: q * p
-z
sage: A.derivations_basis()
(
[1 0 0]  [0 1 0]  [0 0 0]  [0 0 0]  [0 0 0]  [0 0 0]
[0 0 0]  [0 0 0]  [1 0 0]  [0 1 0]  [0 0 0]  [0 0 0]
[0 0 1], [0 0 0], [0 0 0], [0 0 1], [1 0 0], [0 1 0]
)
```

We construct another example using the exterior algebra and verify we obtain a derivation:

```
sage: # needs sage.combinat sage.modules
sage: A = algebras.Exterior(QQ, 1)
sage: A.derivations_basis()
(
[0 0]
[0 1]
)
sage: D = A.module_morphism(matrix=A.derivations_basis()[0],
....:                       codomain=A)
sage: one, e = A.basis()
sage: all(D(a*b) == D(a) * b + a * D(b)
....:     for a in A.basis() for b in A.basis())
True
```

REFERENCES:

Wikipedia article Derivation_(differential_algebra)

class ParentMethods

Bases: `object`

algebra_generators()

Return generators for this algebra.

This default implementation returns the basis of this algebra.

OUTPUT: a family

See also:

- *basis()*
- *MagmaticAlgebras.ParentMethods.algebra_generators()*

EXAMPLES:

```
sage: D4 = DescentAlgebra(QQ, 4).B()                                  #␣
↪needs sage.combinat sage.modules
sage: D4.algebra_generators()                                         #␣
↪needs sage.combinat sage.modules
Lazy family (...)_{i in Compositions of 4}

sage: R.<x> = ZZ[]
sage: P = PartitionAlgebra(1, x, R)                                   #␣
↪needs sage.combinat sage.modules
sage: P.algebra_generators()                                          #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
Lazy family (Term map
 from Partition diagrams of order 1
   to Partition Algebra of rank 1 with parameter x
       over Univariate Polynomial Ring in x
        over Integer Ring(i))_{i in Partition diagrams of order 1}
```

product()

The product of the algebra, as per *Magmas.ParentMethods.product()*

By default, this is implemented using one of the following methods, in the specified order:

- *product_on_basis()*
- *product_by_coercion()*

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()                                  #␣
↪needs sage.combinat sage.modules
sage: a, b, c = A.algebra_generators()                                     #␣
↪needs sage.combinat sage.modules
sage: A.product(a + 2*b, 3*c)                                              #␣
↪needs sage.combinat sage.modules
3*B[word: ac] + 6*B[word: bc]
```

product_on_basis(*i*, *j*)

The product of the algebra on the basis (optional).

INPUT:

- i, j – the indices of two elements of the basis of self

Return the product of the two corresponding basis elements indexed by i and j.

If implemented, *product()* is defined from it by bilinearity.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()                                  #␣
↪needs sage.combinat sage.modules
sage: Word = A.basis().keys()                                              #␣
↪needs sage.combinat sage.modules
sage: A.product_on_basis(Word("abc"), Word("cba"))                         #␣
↪needs sage.combinat sage.modules
B[word: abccba]
```

additional_structure()

Return None.

Indeed, the category of (magmatic) algebras defines no new structure: a morphism of modules and of magmas between two (magmatic) algebras is a (magmatic) algebra morphism.

See also:

Category.additional_structure()

Todo: This category should be a *CategoryWithAxiom*, the axiom specifying the compatibility between the magma and module structure.

EXAMPLES:

```
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: MagmaticAlgebras(ZZ).additional_structure()
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: MA = MagmaticAlgebras(ZZ)
sage: MA.super_categories()
[Category of additive commutative additive associative additive
  unital distributive magmas and additive magmas,
 Category of modules over Integer Ring]

sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: MA.is_subcategory((AdditiveSemigroups() & Magmas()).Distributive())
True
```

4.114 Manifolds

class sage.categories.manifolds.**ComplexManifolds**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of complex manifolds.

A d-dimensional complex manifold is a manifold whose underlying vector space is $\mathbf{C}^d$ and has a holomorphic atlas.

super_categories()

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).super_categories()
[Category of topological spaces]
```

class sage.categories.manifolds.**Manifolds**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of manifolds over any topological field.

Let k be a topological field. A d-dimensional *k-manifold* M is a second countable Hausdorff space such that the neighborhood of any point $x \in M$ is homeomorphic to k^d.

EXAMPLES:

```
sage: # needs sage.rings.real_mpfr
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR); C
Category of manifolds over Real Field with 53 bits of precision
sage: C.super_categories()
[Category of topological spaces]
```

class AlmostComplex(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of almost complex manifolds.

An *almost complex manifold* M is a manifold with a smooth tensor field J of rank $(1, 1)$ such that $J^2 = -1$ when regarded as a vector bundle isomorphism $J : TM \to TM$ on the tangent bundle. The tensor field J is called the *almost complex structure* of M.

extra_super_categories()

Return the extra super categories of `self`.

An almost complex manifold is smooth.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).AlmostComplex().super_categories()   # indirect␣
↪doctest           # needs sage.rings.real_mpfr
[Category of smooth manifolds
 over Real Field with 53 bits of precision]
```

class Analytic(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

The category of complex manifolds.

An analytic manifold is a manifold with an analytic atlas.

extra_super_categories()

Return the extra super categories of `self`.

An analytic manifold is smooth.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Analytic().super_categories()  # indirect doctest    #␣
↪needs sage.rings.real_mpfr
[Category of smooth manifolds
 over Real Field with 53 bits of precision]
```

class Connected(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

The category of connected manifolds.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR).Connected()
sage: TestSuite(C).run(skip="_test_category_over_bases")
```

class Differentiable(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

The category of differentiable manifolds.

A differentiable manifold is a manifold with a differentiable atlas.

class FiniteDimensional(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

Category of finite dimensional manifolds.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR).FiniteDimensional()
sage: TestSuite(C).run(skip="_test_category_over_bases")
```

class ParentMethods

Bases: `object`

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(RR).example()
sage: M.dimension()
3
```

class Smooth(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of smooth manifolds.

A smooth manifold is a manifold with a smooth atlas.

extra_super_categories()

Return the extra super categories of `self`.

A smooth manifold is differentiable.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Smooth().super_categories()  # indirect doctest     #␣
↪needs sage.rings.real_mpfr
[Category of differentiable manifolds
 over Real Field with 53 bits of precision]
```

class SubcategoryMethods

Bases: `object`

AlmostComplex()

Return the subcategory of the almost complex objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).AlmostComplex()                                     #␣
↪needs sage.rings.real_mpfr
Category of almost complex manifolds
 over Real Field with 53 bits of precision
```

Analytic()

Return the subcategory of the analytic objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Analytic()                                            #␣
↪needs sage.rings.real_mpfr
Category of analytic manifolds
 over Real Field with 53 bits of precision
```

Complex()

Return the subcategory of manifolds over **C** of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(CC).Complex()                                             #␣
↪needs sage.rings.real_mpfr
Category of complex manifolds over
 Complex Field with 53 bits of precision
```

Connected()

Return the full subcategory of the connected objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Connected()                                           #␣
↪needs sage.rings.real_mpfr
Category of connected manifolds
 over Real Field with 53 bits of precision
```

Differentiable()

Return the subcategory of the differentiable objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Differentiable()                                      #␣
↪needs sage.rings.real_mpfr
Category of differentiable manifolds
 over Real Field with 53 bits of precision
```

FiniteDimensional()

Return the full subcategory of the finite dimensional objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR).Connected().FiniteDimensional(); C                #␣
↪needs sage.rings.real_mpfr
Category of finite dimensional connected manifolds
 over Real Field with 53 bits of precision
```

Smooth()

Return the subcategory of the smooth objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Smooth()                                              #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.rings.real_mpfr
Category of smooth manifolds
 over Real Field with 53 bits of precision
```

additional_structure()

Return None.

Indeed, the category of manifolds defines no new structure: a morphism of topological spaces between manifolds is a manifold morphism.

See also:

Category.additional_structure()

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).additional_structure()
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).super_categories()
[Category of topological spaces]
```

4.115 Matrix algebras

class sage.categories.matrix_algebras.**MatrixAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of matrix algebras over a field.

EXAMPLES:

```
sage: MatrixAlgebras(RationalField())
Category of matrix algebras over Rational Field
```

super_categories()

EXAMPLES:

```
sage: MatrixAlgebras(QQ).super_categories()
[Category of algebras over Rational Field]
```

4.116 Metric Spaces

class sage.categories.metric_spaces.**MetricSpaces**(*category*, **args*)

Bases: *MetricSpacesCategory*

The category of metric spaces.

A *metric* on a set S is a function $d : S \times S \to \mathbf{R}$ such that:

- $d(a, b) \geq 0$,
- $d(a, b) = 0$ if and only if $a = b$.

A metric space is a set S with a distinguished metric.

Implementation

Objects in this category must implement either a `dist` on the parent or the elements or `metric` on the parent; otherwise this will cause an infinite recursion.

Todo:

- Implement a general geodesics class.
- Implement a category for metric additive groups and move the generic distance $d(a, b) = |a - b|$ there.
- Incorporate the length of a geodesic as part of the default distance cycle.

EXAMPLES:

```
sage: from sage.categories.metric_spaces import MetricSpaces
sage: C = MetricSpaces()
sage: C
Category of metric spaces
sage: TestSuite(C).run()
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

class ParentMethods

Bases: `object`

dist(*a*, *b*)

Return the distance between `a` and `b` in `self`.

It is defined as the maximum of the distances within the Cartesian factors.

EXAMPLES:

```
sage: from sage.categories.metric_spaces import MetricSpaces
sage: Q2 = QQ.cartesian_product(QQ)
sage: Q2.category()
Join of
Category of Cartesian products of commutative rings and
Category of Cartesian products of metric spaces
sage: Q2 in MetricSpaces()
True
```

(continues on next page)

(continued from previous page)

```
sage: Q2.dist((0, 0), (2, 3))
3
```

extra_super_categories()

Implement the fact that a (finite) Cartesian product of metric spaces is a metric space.

EXAMPLES:

```
sage: from sage.categories.metric_spaces import MetricSpaces
sage: C = MetricSpaces().CartesianProducts()
sage: C.extra_super_categories()
[Category of metric spaces]
sage: C.super_categories()
[Category of Cartesian products of topological spaces,
 Category of metric spaces]
sage: C.axioms()
frozenset()
```

class Complete(*base_category*)

Bases: `CategoryWithAxiom`

The category of complete metric spaces.

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

extra_super_categories()

Implement the fact that a (finite) Cartesian product of complete metric spaces is a complete metric space.

EXAMPLES:

```
sage: from sage.categories.metric_spaces import MetricSpaces
sage: C = MetricSpaces().Complete().CartesianProducts()
sage: C.extra_super_categories()
[Category of complete metric spaces]
sage: C.super_categories()
[Category of Cartesian products of metric spaces,
 Category of complete metric spaces]
sage: C.axioms()
frozenset({'Complete'})

sage: R2 = RR.cartesian_product(RR)
sage: R2 in MetricSpaces()
True
sage: R2 in MetricSpaces().Complete()
True

sage: QR = QQ.cartesian_product(RR)
sage: QR in MetricSpaces()
True
sage: QR in MetricSpaces().Complete()
False
```

class ElementMethods

Bases: `object`

abs()

Return the absolute value of `self`.

EXAMPLES:

```
sage: CC(I).abs()                                                    #␣
↪needs sage.rings.real_mpfr sage.symbolic
1.00000000000000
```

dist(*b*)

Return the distance between `self` and `other`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: UHP = HyperbolicPlane().UHP()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1 + I)
sage: p1.dist(p2)
arccosh(33/7)
```

class Homsets(*category*, **args*)

Bases: `HomsetsCategory`

The category of homsets of metric spaces

It consists of the metric maps, that is, the Lipschitz functions with Lipschitz constant 1.

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

dist(*a*, *b*)

Return the distance between `a` and `b` in `self`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: UHP = HyperbolicPlane().UHP()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1.0 + I)
sage: UHP.dist(p1, p2)
2.23230104635820

sage: PD = HyperbolicPlane().PD()                                    #␣
↪needs sage.symbolic
sage: PD.dist(PD.get_point(0), PD.get_point(I/2))                    #␣
↪needs sage.symbolic
arccosh(5/3)
```

metric(**args*, ***kwds*)

Deprecated: Use `metric_function()` instead. See github issue #30062 for details.

metric_function()

Return the metric function of `self`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: UHP = HyperbolicPlane().UHP()
sage: m = UHP.metric_function()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1.0 + I)
sage: m(p1, p2)
2.23230104635820
```

class SubcategoryMethods

Bases: `object`

Complete()

Return the full subcategory of the complete objects of `self`.

EXAMPLES:

```
sage: Sets().Metric().Complete()
Category of complete metric spaces
```

class WithRealizations(*category*, **args*)

Bases: *WithRealizationsCategory*

class ParentMethods

Bases: `object`

dist(*a*, *b*)

Return the distance between `a` and `b` by converting them to a realization of `self` and doing the computation.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: H = HyperbolicPlane()
sage: PD = H.PD()
sage: p1 = PD.get_point(0)
sage: p2 = PD.get_point(I/2)
sage: H.dist(p1, p2)
arccosh(5/3)
```

class sage.categories.metric_spaces.**MetricSpacesCategory**(*category*, **args*)

Bases: *RegressiveCovariantConstructionCategory*

classmethod default_super_categories(*category*)

Return the default super categories of `category.Metric()`.

Mathematical meaning: if A is a metric space in the category C, then A is also a topological space.

INPUT:

- `cls` – the class `MetricSpaces`
- `category` – a category Cat

OUTPUT:

A (join) category

In practice, this returns `category.Metric()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category` and `cat.Metric()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`. Then, a group G with a metric is simultaneously a topological group by itself, and a metric space:

```
sage: Groups().Metric().super_categories()
[Category of topological groups, Category of metric spaces]
```

This resulted from the following call:

```
sage: sage.categories.metric_spaces.MetricSpacesCategory.default_super_
↪categories(Groups())
Join of Category of topological groups and Category of metric spaces
```

4.117 Modular abelian varieties

class sage.categories.modular_abelian_varieties.**ModularAbelianVarieties**(*Y*)

Bases: *Category_over_base*

The category of modular abelian varieties over a given field.

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ)
Category of modular abelian varieties over Rational Field
```

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

class Endset(*base_category*)

Bases: *CategoryWithAxiom*

extra_super_categories()

Implement the fact that an endset of modular abelian variety is a ring.

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ).Endsets().extra_super_categories()
[Category of rings]
```

base_field()

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ).base_field()
Rational Field
```

super_categories()

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ).super_categories()
[Category of sets]
```

4.118 Modules

class `sage.categories.modules.`**`Modules`**(*base*, *name=None*)

Bases: *Category_module*

The category of all modules over a base ring R.

An R-module M is a left and right R-module over a commutative ring R such that:

$$r * (x * s) = (r * x) * s \qquad \forall r, s \in R \text{ and } x \in M$$

INPUT:

- `base_ring` – a ring R or subcategory of `Rings()`
- `dispatch` – a boolean (for internal use; default: `True`)

When the base ring is a field, the category of vector spaces is returned instead (unless `dispatch == False`).

Warning: Outside of the context of symmetric modules over a commutative ring, the specifications of this category are fuzzy and not yet set in stone (see below). The code in this category and its subcategories is therefore prone to bugs or arbitrary limitations in this case.

EXAMPLES:

```
sage: Modules(ZZ)
Category of modules over Integer Ring
sage: Modules(QQ)
Category of vector spaces over Rational Field

sage: Modules(Rings())
Category of modules over rings
sage: Modules(FiniteFields())
Category of vector spaces over finite enumerated fields

sage: Modules(Integers(9))
Category of modules over Ring of integers modulo 9

sage: Modules(Integers(9)).super_categories()
[Category of bimodules over Ring of integers modulo 9 on the left
                        and Ring of integers modulo 9 on the right]

sage: Modules(ZZ).super_categories()
[Category of bimodules over Integer Ring on the left
                        and Integer Ring on the right]

sage: Modules == RingModules
True

sage: Modules(ZZ['x']).is_abelian()    # see #6081
True
```

Todo:

- Clarify the distinction, if any, with `BiModules(R, R)`. In particular, if R is a commutative ring (e.g. a field), some pieces of the code possibly assume that M is a *symmetric `R`-`R`-bimodule*:

$$r * x = x * r \qquad \forall r \in R \text{ and } x \in M$$

- Make sure that non symmetric modules are properly supported by all the code, and advertise it.
- Make sure that non commutative rings are properly supported by all the code, and advertise it.
- Add support for base semirings.
- Implement a `FreeModules(R)` category, when so prompted by a concrete use case: e.g. modeling a free module with several bases (using `Sets.SubcategoryMethods.Realizations()`) or with an atlas of local maps (see e.g. github issue #15916).

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

The category of modules constructed as Cartesian products of modules

This construction gives the direct product of modules. The implementation is based on the following resources:

- http://groups.google.fr/group/sage-devel/browse_thread/thread/35a72b1d0a2fc77a/348f42ae77a66d16#348f42ae77a66d16
- Wikipedia article Direct_product

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

extra_super_categories()

A Cartesian product of modules is endowed with a natural module structure.

EXAMPLES:

```
sage: Modules(ZZ).CartesianProducts().extra_super_categories()
[Category of modules over Integer Ring]
sage: Modules(ZZ).CartesianProducts().super_categories()
[Category of Cartesian products of commutative additive groups,
 Category of modules over Integer Ring]
```

class ElementMethods

Bases: `object`

Filtered

alias of *FilteredModules*

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

extra_super_categories()

Implement the fact that a (finite) tensor product of finite dimensional modules is a finite dimensional module.

EXAMPLES:

```
sage: Modules(ZZ).FiniteDimensional().TensorProducts().extra_super_
↪categories()
[Category of finite dimensional modules over Integer Ring]
sage: Modules(QQ).FiniteDimensional().TensorProducts().
↪FiniteDimensional()
Category of tensor products of finite dimensional vector spaces
 over Rational Field
```

extra_super_categories()

Implement the fact that a finite dimensional module over a finite ring is finite.

EXAMPLES:

```
sage: Modules(IntegerModRing(4)).FiniteDimensional().extra_super_
↪categories()
[Category of finite sets]
sage: Modules(ZZ).FiniteDimensional().extra_super_categories()
[]
sage: Modules(GF(5)).FiniteDimensional().is_subcategory(Sets().Finite())
True
sage: Modules(ZZ).FiniteDimensional().is_subcategory(Sets().Finite())
False

sage: Modules(Rings().Finite()).FiniteDimensional().is_subcategory(Sets().
↪Finite())
True
sage: Modules(Rings()).FiniteDimensional().is_subcategory(Sets().Finite())
False
```

class FinitelyPresented(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

extra_super_categories()

Implement the fact that a finitely presented module over a finite ring is finite.

EXAMPLES:

```
sage: Modules(IntegerModRing(4)).FiniteDimensional().extra_super_
↪categories()
[Category of finite sets]
sage: Modules(ZZ).FiniteDimensional().extra_super_categories()
[]
sage: Modules(GF(5)).FiniteDimensional().is_subcategory(Sets().Finite())
True
sage: Modules(ZZ).FiniteDimensional().is_subcategory(Sets().Finite())
False

sage: Modules(Rings().Finite()).FiniteDimensional().is_subcategory(Sets().
↪Finite())
True
sage: Modules(Rings()).FiniteDimensional().is_subcategory(Sets().Finite())
False
```

Graded

alias of *GradedModules*

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

The category of homomorphism sets $\hom(X, Y)$ for X, Y modules.

class Endset(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of endomorphism sets $End(X)$ for X a module (this is not used yet)

extra_super_categories()

Implement the fact that the endomorphism set of a module is an algebra.

See also:

`CategoryWithAxiom.extra_super_categories()`

EXAMPLES:

```
sage: Modules(ZZ).Endsets().extra_super_categories()
[Category of magmatic algebras over Integer Ring]

sage: End(ZZ^3) in Algebras(ZZ)                                          #␣
↪needs sage.modules
True
```

class ParentMethods

Bases: `object`

base_ring()

Return the base ring of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: E = CombinatorialFreeModule(ZZ, [1,2,3])
sage: F = CombinatorialFreeModule(ZZ, [2,3,4])
sage: H = Hom(E, F)
sage: H.base_ring()
Integer Ring
```

This `base_ring` method is actually overridden by `sage.structure.category_object.CategoryObject.base_ring()`:

```
sage: H.base_ring.__module__                                             #␣
↪needs sage.modules
```

Here we call it directly:

```
sage: method = H.category().parent_class.base_ring                      #␣
↪needs sage.modules
sage: method.__get__(H)()                                                #␣
↪needs sage.modules
Integer Ring
```

zero()

EXAMPLES:

```
sage: # needs sage.modules
sage: E = CombinatorialFreeModule(ZZ, [1,2,3])
sage: F = CombinatorialFreeModule(ZZ, [2,3,4])
sage: H = Hom(E, F)
sage: f = H.zero()
sage: f
Generic morphism:
  From: Free module generated by {1, 2, 3} over Integer Ring
  To:   Free module generated by {2, 3, 4} over Integer Ring
sage: f(E.monomial(2))
0
sage: f(E.monomial(3)) == F.zero()
True
```

base_ring()

EXAMPLES:

```
sage: Modules(ZZ).Homsets().base_ring()
Integer Ring
```

Todo: Generalize this so that any homset category of a full subcategory of modules over a base ring is a category over this base ring.

extra_super_categories()

EXAMPLES:

```
sage: Modules(ZZ).Homsets().extra_super_categories()
[Category of modules over Integer Ring]
```

class ParentMethods

Bases: `object`

linear_combination(*iter_of_elements_coeff*, *factor_on_left=True*)

Return the linear combination $\lambda_1 v_1 + \cdots + \lambda_k v_k$ (resp. the linear combination $v_1 \lambda_1 + \cdots + v_k \lambda_k$) where `iter_of_elements_coeff` iterates through the sequence $((\lambda_1, v_1), ..., (\lambda_k, v_k))$.

INPUT:

- `iter_of_elements_coeff` – iterator of pairs `(element, coeff)` with `element` in `self` and `coeff` in `self.base_ring()`
- `factor_on_left` – (optional) if `True`, the coefficients are multiplied from the left; if `False`, the coefficients are multiplied from the right

EXAMPLES:

```
sage: m = matrix([[0,1], [1,1]])                                          #␣
↪needs sage.modules
sage: J.<a,b,c> = JordanAlgebra(m)                                        #␣
↪needs sage.combinat sage.modules
sage: J.linear_combination(((a+b, 1), (-2*b + c, -1)))                    #␣
↪needs sage.combinat sage.modules
1 + (3, -1)
```

module_morphism(*function*, *category*, *codomain*, ***keywords*)

Construct a module morphism from `self` to `codomain`.

Let `self` be a module X over a ring R. This constructs a morphism $f : X \to Y$.

INPUT:

- `self` – a parent X in `Modules(R)`.
- `function` – a function f from X to Y
- `codomain` – the codomain Y of the morphism (default: `f.codomain()` if it's defined; otherwise it must be specified)
- `category` – a category or `None` (default: `None`)

EXAMPLES:

```
sage: # needs sage.modules
sage: V = FiniteRankFreeModule(QQ, 2)
sage: e = V.basis('e'); e
Basis (e_0,e_1) on the
 2-dimensional vector space over the Rational Field
sage: neg = V.module_morphism(function=operator.neg, codomain=V); neg
Generic endomorphism of
 2-dimensional vector space over the Rational Field
sage: neg(e[0])
Element -e_0 of the 2-dimensional vector space over the Rational Field
```

quotient(*submodule*, *check=True*, ***kwds*)

Construct the quotient module `self` / `submodule`.

INPUT:

- `submodule` – a submodule with basis of `self`, or something that can be turned into one via `self.submodule(submodule)`
- `check`, other keyword arguments: passed on to `quotient_module()`.

This method just delegates to `quotient_module()`. Classes implementing modules should override that method.

Parents in categories with additional structure may override *`quotient()`*. For example, in algebras, *`quotient()`* will be the same as `quotient_ring()`.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])                    #␣
↪needs sage.modules
sage: TA = TensorAlgebra(C)                                             #␣
↪needs sage.combinat sage.modules
sage: TA.quotient                                                       #␣
↪needs sage.combinat sage.modules
<bound method Rings.ParentMethods.quotient of
 Tensor Algebra of Free module generated by {'a', 'b', 'c'}
 over Rational Field>
```

tensor_square()

Returns the tensor square of `self`

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()                           #␣
↪needs sage.groups sage.modules
sage: A.tensor_square()                                                 #␣
↪needs sage.groups sage.modules
```

(continues on next page)

(continued from previous page)

```
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6
 as a permutation group over Rational Field # An example
 of Hopf algebra with basis: the group algebra of the Dihedral
 group of order 6 as a permutation group over Rational Field
```

class SubcategoryMethods

Bases: `object`

DualObjects()

Return the category of spaces constructed as duals of spaces of `self`.

The *dual* of a vector space V is the space consisting of all linear functionals on V (see Wikipedia article Dual_space). Additional structure on V can endow its dual with additional structure; for example, if V is a finite dimensional algebra, then its dual is a coalgebra.

This returns the category of spaces constructed as dual of spaces in `self`, endowed with the appropriate additional structure.

Warning:

- This semantic of `dual` and `DualObject` is imposed on all subcategories, in particular to make `dual` a covariant functorial construction.

  A subcategory that defines a different notion of dual needs to use a different name.

- Typically, the category of graded modules should define a separate `graded_dual` construction (see github issue #15647). For now the two constructions are not distinguished which is an oversimplified model.

See also:

- *dual.DualObjectsCategory*
- *CovariantFunctorialConstruction*.

EXAMPLES:

```
sage: VectorSpaces(QQ).DualObjects()
Category of duals of vector spaces over Rational Field
```

The dual of a vector space is a vector space:

```
sage: VectorSpaces(QQ).DualObjects().super_categories()
[Category of vector spaces over Rational Field]
```

The dual of an algebra is a coalgebra:

```
sage: sorted(Algebras(QQ).DualObjects().super_categories(), key=str)
[Category of coalgebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

The dual of a coalgebra is an algebra:

```
sage: sorted(Coalgebras(QQ).DualObjects().super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

As a shorthand, this category can be accessed with the `dual()` method:

```
sage: VectorSpaces(QQ).dual()
Category of duals of vector spaces over Rational Field
```

Filtered(*base_ring=None*)

Return the subcategory of the filtered objects of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: Modules(ZZ).Filtered()
Category of filtered modules over Integer Ring

sage: Coalgebras(QQ).Filtered()
Category of filtered coalgebras over Rational Field

sage: AlgebrasWithBasis(QQ).Filtered()
Category of filtered algebras with basis over Rational Field
```

Todo:

- Explain why this does not commute with `WithBasis()`
- Improve the support for covariant functorial constructions categories over a base ring so as to get rid of the `base_ring` argument.

FiniteDimensional()

Return the full subcategory of the finite dimensional objects of `self`.

EXAMPLES:

```
sage: Modules(ZZ).FiniteDimensional()
Category of finite dimensional modules over Integer Ring
sage: Coalgebras(QQ).FiniteDimensional()
Category of finite dimensional coalgebras over Rational Field
sage: AlgebrasWithBasis(QQ).FiniteDimensional()
Category of finite dimensional algebras with basis over Rational Field
```

FinitelyPresented()

Return the full subcategory of the finitely presented objects of `self`.

EXAMPLES:

```
sage: Modules(ZZ).FinitelyPresented()
Category of finitely presented modules over Integer Ring
sage: A = SteenrodAlgebra(2)                                                    #␣
↪needs sage.combinat sage.modules
sage: from sage.modules.fp_graded.module import FPModule                        #␣
↪needs sage.combinat sage.modules
sage: FPModule(A, [0, 1], [[Sq(2), Sq(1)]]).category()                          #␣
↪needs sage.combinat sage.modules
Category of finitely presented graded modules
 over mod 2 Steenrod algebra, milnor basis
```

Graded(*base_ring=None*)

Return the subcategory of the graded objects of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: Modules(ZZ).Graded()
Category of graded modules over Integer Ring

sage: Coalgebras(QQ).Graded()
Category of graded coalgebras over Rational Field

sage: AlgebrasWithBasis(QQ).Graded()
Category of graded algebras with basis over Rational Field
```

Todo:

- Explain why this does not commute with *`WithBasis()`*
- Improve the support for covariant functorial constructions categories over a base ring so as to get rid of the `base_ring` argument.

`Super`(*base_ring=None*)

Return the super-analogue category of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: Modules(ZZ).Super()
Category of super modules over Integer Ring

sage: Coalgebras(QQ).Super()
Category of super coalgebras over Rational Field

sage: AlgebrasWithBasis(QQ).Super()
Category of super algebras with basis over Rational Field
```

Todo:

- Explain why this does not commute with *`WithBasis()`*
- Improve the support for covariant functorial constructions categories over a base ring so as to get rid of the `base_ring` argument.

`TensorProducts`()

Return the full subcategory of objects of `self` constructed as tensor products.

See also:

- *`tensor.TensorProductsCategory`*
- `RegressiveCovariantFunctorialConstruction`.

EXAMPLES:

```
sage: ModulesWithBasis(QQ).TensorProducts()
Category of tensor products of vector spaces with basis over Rational␣
↪Field
```

`WithBasis`()

Return the full subcategory of the objects of `self` with a distinguished basis.

EXAMPLES:

```
sage: Modules(ZZ).WithBasis()
Category of modules with basis over Integer Ring
sage: Coalgebras(QQ).WithBasis()
Category of coalgebras with basis over Rational Field
sage: AlgebrasWithBasis(QQ).WithBasis()
Category of algebras with basis over Rational Field
```

base_ring()

Return the base ring (category) for `self`.

This implements a `base_ring` method for all subcategories of `Modules(K)`.

EXAMPLES:

```
sage: C = Modules(QQ) & Semigroups(); C
Join of Category of semigroups
    and Category of vector spaces over Rational Field
sage: C.base_ring()
Rational Field
sage: C.base_ring.__module__
'sage.categories.modules'

sage: C2 = Modules(Rings()) & Semigroups(); C2
Join of Category of semigroups and Category of modules over rings
sage: C2.base_ring()
Category of rings
sage: C2.base_ring.__module__
'sage.categories.modules'

sage: # needs sage.combinat sage.modules
sage: C3 = DescentAlgebra(QQ,3).B().category()
sage: C3.base_ring.__module__
'sage.categories.modules'
sage: C3.base_ring()
Rational Field

sage: # needs sage.combinat sage.modules
sage: C4 = QuasiSymmetricFunctions(QQ).F().category()
sage: C4.base_ring.__module__
'sage.categories.modules'
sage: C4.base_ring()
Rational Field
```

dual()

Return the category of spaces constructed as duals of spaces of `self`.

The *dual* of a vector space V is the space consisting of all linear functionals on V (see Wikipedia article Dual_space). Additional structure on V can endow its dual with additional structure; for example, if V is a finite dimensional algebra, then its dual is a coalgebra.

This returns the category of spaces constructed as dual of spaces in `self`, endowed with the appropriate additional structure.

Warning:

- This semantic of `dual` and `DualObject` is imposed on all subcategories, in particular to make `dual` a covariant functorial construction.

  A subcategory that defines a different notion of dual needs to use a different name.

- Typically, the category of graded modules should define a separate `graded_dual` construction (see github issue #15647). For now the two constructions are not distinguished which is an oversimplified model.

See also:

- *dual.DualObjectsCategory*
- *CovariantFunctorialConstruction*.

EXAMPLES:

```
sage: VectorSpaces(QQ).DualObjects()
Category of duals of vector spaces over Rational Field
```

The dual of a vector space is a vector space:

```
sage: VectorSpaces(QQ).DualObjects().super_categories()
[Category of vector spaces over Rational Field]
```

The dual of an algebra is a coalgebra:

```
sage: sorted(Algebras(QQ).DualObjects().super_categories(), key=str)
[Category of coalgebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

The dual of a coalgebra is an algebra:

```
sage: sorted(Coalgebras(QQ).DualObjects().super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

As a shorthand, this category can be accessed with the *dual()* method:

```
sage: VectorSpaces(QQ).dual()
Category of duals of vector spaces over Rational Field
```

Super

alias of *SuperModules*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of modules constructed by tensor product of modules.

class ParentMethods

Bases: `object`

Implement operations on tensor products of modules.

construction()

Return the construction of `self`.

EXAMPLES:

```
sage: A = algebras.Free(QQ, 2)                                          #␣
↪needs sage.combinat sage.modules
sage: T = A.tensor(A)                                                   #␣
↪needs sage.combinat sage.modules
sage: T.construction()                                                  #␣
↪needs sage.combinat sage.modules
(The tensor functorial construction,
 (Free Algebra on 2 generators (None0, None1) over Rational Field,
  Free Algebra on 2 generators (None0, None1) over Rational Field))
```

tensor_factors()

Return the tensor factors of this tensor product.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(ZZ, [1,2])
sage: F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [3,4])
sage: G.rename("G")
sage: T = tensor([F, G]); T
F # G
sage: T.tensor_factors()
(F, G)
```

extra_super_categories()

EXAMPLES:

```
sage: Modules(ZZ).TensorProducts().extra_super_categories()
[Category of modules over Integer Ring]
sage: Modules(ZZ).TensorProducts().super_categories()
[Category of modules over Integer Ring]
```

WithBasis

alias of *ModulesWithBasis*

additional_structure()

Return None.

Indeed, the category of modules defines no additional structure: a bimodule morphism between two modules is a module morphism.

See also:

Category.additional_structure()

Todo: Should this category be a *CategoryWithAxiom*?

EXAMPLES:

```
sage: Modules(ZZ).additional_structure()
```

super_categories()

EXAMPLES:

```
sage: Modules(ZZ).super_categories()
[Category of bimodules over Integer Ring on the left
                         and Integer Ring on the right]
```

Nota bene:

```
sage: Modules(QQ)
Category of vector spaces over Rational Field
sage: Modules(QQ).super_categories()
[Category of modules over Rational Field]
```

4.119 Modules With Basis

AUTHORS:

- Nicolas M. Thiery (2008-2014): initial revision, axiomatization
- Jason Bandlow and Florent Hivert (2010): Triangular Morphisms
- Christian Stump (2010): github issue #9648 module_morphism's to a wider class of codomains

class sage.categories.modules_with_basis.**ModulesWithBasis**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of modules with a distinguished basis.

The elements are represented by expanding them in the distinguished basis. The morphisms are not required to respect the distinguished basis.

EXAMPLES:

```
sage: ModulesWithBasis(ZZ)
Category of modules with basis over Integer Ring
sage: ModulesWithBasis(ZZ).super_categories()
[Category of modules over Integer Ring]
```

If the base ring is actually a field, this constructs instead the category of vector spaces with basis:

```
sage: ModulesWithBasis(QQ)
Category of vector spaces with basis over Rational Field

sage: ModulesWithBasis(QQ).super_categories()
[Category of modules with basis over Rational Field,
 Category of vector spaces over Rational Field]
```

Let X and Y be two modules with basis. We can build $Hom(X, Y)$:

```
sage: X = CombinatorialFreeModule(QQ, [1,2]); X.rename("X")                    #␣
↪needs sage.modules
sage: Y = CombinatorialFreeModule(QQ, [3,4]); Y.rename("Y")                    #␣
↪needs sage.modules
sage: H = Hom(X, Y); H                                                         #␣
↪needs sage.modules
Set of Morphisms from X to Y
 in Category of finite dimensional vector spaces with basis over Rational Field
```

The simplest morphism is the zero map:

```
sage: H.zero()      # todo: move this test into module once we have an example    #␣
↪needs sage.modules
Generic morphism:
  From: X
  To:   Y
```

which we can apply to elements of X:

```
sage: x = X.monomial(1) + 3 * X.monomial(2)                                       #␣
↪needs sage.modules
sage: H.zero()(x)                                                                 #␣
↪needs sage.modules
0
```

EXAMPLES:

We now construct a more interesting morphism by extending a function by linearity:

```
sage: phi = H(on_basis=lambda i: Y.monomial(i + 2)); phi                          #␣
↪needs sage.modules
Generic morphism:
  From: X
  To:   Y
sage: phi(x)                                                                      #␣
↪needs sage.modules
B[3] + 3*B[4]
```

We can retrieve the function acting on indices of the basis:

```
sage: f = phi.on_basis()                                                          #␣
↪needs sage.modules
sage: f(1), f(2)                                                                  #␣
↪needs sage.modules
(B[3], B[4])
```

$Hom(X, Y)$ has a natural module structure (except for the zero, the operations are not yet implemented though). However since the dimension is not necessarily finite, it is not a module with basis; but see `FiniteDimensionalModulesWithBasis` and `GradedModulesWithBasis`:

```
sage: H in ModulesWithBasis(QQ), H in Modules(QQ)                                 #␣
↪needs sage.modules
(False, True)
```

Some more playing around with categories and higher order homsets:

```
sage: H.category()                                                                #␣
↪needs sage.modules
Category of homsets of modules with basis over Rational Field
sage: Hom(H, H).category()                                                        #␣
↪needs sage.modules
Category of endsets of homsets of modules with basis over Rational Field
```

Todo: `End(X)` is an algebra.

Note: This category currently requires an implementation of an element method `support`. Once github issue

#18066 is merged, an implementation of an `items` method will be required.

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

The category of modules with basis constructed by Cartesian products of modules with basis.

class ParentMethods

Bases: `object`

extra_super_categories()

EXAMPLES:

```
sage: ModulesWithBasis(QQ).CartesianProducts().extra_super_categories()
[Category of vector spaces with basis over Rational Field]
sage: ModulesWithBasis(QQ).CartesianProducts().super_categories()
[Category of Cartesian products of modules with basis over Rational Field,
 Category of vector spaces with basis over Rational Field,
 Category of Cartesian products of vector spaces over Rational Field]
```

class DualObjects(*category*, **args*)

Bases: *DualObjectsCategory*

extra_super_categories()

EXAMPLES:

```
sage: ModulesWithBasis(ZZ).DualObjects().extra_super_categories()
[Category of modules over Integer Ring]
sage: ModulesWithBasis(QQ).DualObjects().super_categories()
[Category of duals of vector spaces over Rational Field,
 Category of duals of modules with basis over Rational Field]
```

class ElementMethods

Bases: `object`

coefficient(*m*)

Return the coefficient of `m` in `self` and raise an error if `m` is not in the basis indexing set.

INPUT:

- `m` – a basis index of the parent of `self`

OUTPUT:

The `B[m]`-coordinate of `self` with respect to the basis `B`. Here, `B` denotes the given basis of the parent of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: s = CombinatorialFreeModule(QQ, Partitions())
sage: z = s([4]) - 2*s([2,1]) + s([1,1,1]) + s([1])
sage: z.coefficient([4])
1
sage: z.coefficient([2,1])
-2
sage: z.coefficient(Partition([2,1]))
-2
sage: z.coefficient([1,2])
```

(continues on next page)

(continued from previous page)

```
Traceback (most recent call last):
...
AssertionError: [1, 2] should be an element of Partitions
sage: z.coefficient(Composition([2,1]))
Traceback (most recent call last):
...
AssertionError: [2, 1] should be an element of Partitions
```

Test that `coefficient` also works for those parents that do not have an `element_class`:

```
sage: # needs sage.modules sage.rings.padics
sage: H = pAdicWeightSpace(3)
sage: F = CombinatorialFreeModule(QQ, H)
sage: hasattr(H, "element_class")
False
sage: h = H.an_element()
sage: (2*F.monomial(h)).coefficient(h)
2
```

coefficients(*sort=True*)

Return a list of the (non-zero) coefficients appearing on the basis elements in `self` (in an arbitrary order).

INPUT:

- `sort` – (default: `True`) to sort the coefficients based upon the default ordering of the indexing set

See also:

dense_coefficient_list()

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.coefficients()
[1, -3]
sage: f = B['c'] - 3*B['a']
sage: f.coefficients()
[-3, 1]
```

```
sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])                          #␣
↪needs sage.combinat sage.modules
sage: z.coefficients()                                                      #␣
↪needs sage.combinat sage.modules
[1, 1, 1, 1]
```

is_zero()

Return `True` if and only if `self == 0`.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
```

(continues on next page)

(continued from previous page)

```
sage: f = B['a'] - 3*B['c']
sage: f.is_zero()
False
sage: F.zero().is_zero()
True
```

```
sage: # needs sage.combinat sage.modules
sage: s = SymmetricFunctions(QQ).schur()
sage: s([2,1]).is_zero()
False
sage: s(0).is_zero()
True
sage: (s([2,1]) - s([2,1])).is_zero()
True
```

leading_coefficient(**args*, ***kwds*)

Return the leading coefficient of `self`.

This is the coefficient of the term whose corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison key, `key(x,y)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.leading_coefficient()
1
sage: def key(x): return -x
sage: x.leading_coefficient(key=key)
3

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                        #␣
↪needs sage.combinat sage.modules
sage: f.leading_coefficient()                                               #␣
↪needs sage.combinat sage.modules
-5
```

leading_item(**args*, ***kwds*)

Return the pair `(k, c)` where

$$c \cdot (\text{the basis element indexed by } k)$$

is the leading term of `self`.

Here 'leading term' means that the corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison function, `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + 4*X.monomial(3)
sage: x.leading_item()
(3, 4)
sage: def key(x): return -x
sage: x.leading_item(key=key)
(1, 3)

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                        #␣
↪needs sage.combinat sage.modules
sage: f.leading_item()                                                      #␣
↪needs sage.combinat sage.modules
([3], -5)
```

leading_monomial(**args*, ***kwds*)

Return the leading monomial of `self`.

This is the monomial whose corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.leading_monomial()
B[3]
sage: def key(x): return -x
sage: x.leading_monomial(key=key)
B[1]

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                        #␣
↪needs sage.combinat sage.modules
sage: f.leading_monomial()                                                  #␣
↪needs sage.combinat sage.modules
s[3]
```

leading_support(**args*, ***kwds*)

Return the maximal element of the support of `self`.

Note that this may not be the term which actually appears first when `self` is printed.

If the default ordering of the basis elements is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3])
sage: X.rename("X"); x = X.basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + 4*X.monomial(3)
sage: x.leading_support()
```

(continues on next page)

(continued from previous page)

```
3
sage: def key(x): return -x
sage: x.leading_support(key=key)
1

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                        #␣
↪needs sage.combinat sage.modules
sage: f.leading_support()                                                   #␣
↪needs sage.combinat sage.modules
[3]
```

leading_term(**args*, ***kwds*)

Return the leading term of `self`.

This is the term whose corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.leading_term()
B[3]
sage: def key(x): return -x
sage: x.leading_term(key=key)
3*B[1]

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                        #␣
↪needs sage.combinat sage.modules
sage: f.leading_term()                                                      #␣
↪needs sage.combinat sage.modules
-5*s[3]
```

length()

Return the number of basis elements whose coefficients in `self` are nonzero.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.length()
2
```

```
sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])                           #␣
↪needs sage.combinat sage.modules
sage: z.length()                                                            #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
4
```

map_coefficients(*f*)

Mapping a function on coefficients.

INPUT:

- `f` – an endofunction on the coefficient ring of the free module

Return a new element of `self.parent()` obtained by applying the function `f` to all of the coefficients of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.map_coefficients(lambda x: x + 5)
6*B['a'] + 2*B['c']
```

Killed coefficients are handled properly:

```
sage: f.map_coefficients(lambda x: 0)                                       #␣
↪needs sage.modules
0
sage: list(f.map_coefficients(lambda x: 0))                                 #␣
↪needs sage.modules
[]
```

```
sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: a = s([2,1]) + 2*s([3,2])                                             #␣
↪needs sage.combinat sage.modules
sage: a.map_coefficients(lambda x: x * 2)                                   #␣
↪needs sage.combinat sage.modules
2*s[2, 1] + 4*s[3, 2]
```

map_item(*f*)

Mapping a function on items.

INPUT:

- `f` – a function mapping pairs `(index, coeff)` to other such pairs

Return a new element of `self.parent()` obtained by applying the function *f* to all items `(index, coeff)` of `self`.

EXAMPLES:

```
sage: B = CombinatorialFreeModule(ZZ, [-1, 0, 1])                           #␣
↪needs sage.modules
sage: x = B.an_element(); x                                                 #␣
↪needs sage.modules
2*B[-1] + 2*B[0] + 3*B[1]
sage: x.map_item(lambda i, c: (-i, 2*c))                                    #␣
↪needs sage.modules
6*B[-1] + 4*B[0] + 4*B[1]
```

`f` needs not be injective:

```
sage: x.map_item(lambda i, c: (1, 2*c))                                    #␣
↪needs sage.modules
14*B[1]

sage: s = SymmetricFunctions(QQ).schur()                                   #␣
↪needs sage.combinat sage.modules
sage: f = lambda m, c: (m.conjugate(), 2 * c)
sage: a = s([2,1]) + s([1,1,1])                                            #␣
↪needs sage.combinat sage.modules
sage: a.map_item(f)                                                        #␣
↪needs sage.combinat sage.modules
2*s[2, 1] + 2*s[3]
```

map_support(*f*)

Mapping a function on the support.

INPUT:

- f – an endofunction on the indices of the free module

Return a new element of self.parent() obtained by applying the function f to all of the objects indexing the basis elements.

EXAMPLES:

```
sage: B = CombinatorialFreeModule(ZZ, [-1, 0, 1])                          #␣
↪needs sage.modules
sage: x = B.an_element(); x                                                #␣
↪needs sage.modules
2*B[-1] + 2*B[0] + 3*B[1]
sage: x.map_support(lambda i: -i)                                          #␣
↪needs sage.modules
3*B[-1] + 2*B[0] + 2*B[1]
```

f needs not be injective:

```
sage: x.map_support(lambda i: 1)                                           #␣
↪needs sage.modules
7*B[1]

sage: s = SymmetricFunctions(QQ).schur()                                   #␣
↪needs sage.combinat sage.modules
sage: a = s([2,1]) + 2*s([3,2])                                            #␣
↪needs sage.combinat sage.modules
sage: a.map_support(lambda x: x.conjugate())                               #␣
↪needs sage.combinat sage.modules
s[2, 1] + 2*s[2, 2, 1]
```

map_support_skip_none(*f*)

Mapping a function on the support.

INPUT:

- f – an endofunction on the indices of the free module

Returns a new element of self.parent() obtained by applying the function f to all of the objects indexing the basis elements.

EXAMPLES:

```
sage: B = CombinatorialFreeModule(ZZ, [-1, 0, 1])                          #␣
↪needs sage.modules
```

(continues on next page)

(continued from previous page)

```
sage: x = B.an_element(); x                                                #␣
↪needs sage.modules
2*B[-1] + 2*B[0] + 3*B[1]
sage: x.map_support_skip_none(lambda i: -i if i else None)                 #␣
↪needs sage.modules
3*B[-1] + 2*B[1]
```

f needs not be injective:

```
sage: x.map_support_skip_none(lambda i: 1 if i else None)                  #␣
↪needs sage.modules
5*B[1]
```

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- `copy` – (default: `True`) if `self` is internally represented by a dictionary `d`, then make a copy of `d`; if `False`, then this can cause undesired behavior by mutating `d`

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 3*B['c']
sage: d = f.monomial_coefficients()
sage: d['a']
1
sage: d['c']
3
```

monomials()

Return a list of the monomials of `self` (in an arbitrary order).

The monomials of an element a are defined to be the basis elements whose corresponding coefficients of a are non-zero.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 2*B['c']
sage: f.monomials()
[B['a'], B['c']]

sage: (F.zero()).monomials()                                               #␣
↪needs sage.modules
[]
```

support()

Return an iterable of the objects indexing the basis of `self.parent()` whose corresponding coefficients of `self` are non-zero.

This method returns these objects in an arbitrary order.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: sorted(f.support())
['a', 'c']
```

```
sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])                           #␣
↪needs sage.combinat sage.modules
sage: sorted(z.support())                                                   #␣
↪needs sage.combinat sage.modules
[[1], [1, 1, 1], [2, 1], [4]]
```

support_of_term()

Return the support of `self`, where `self` is a monomial (possibly with coefficient).

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1,2,3,4]); X.rename("X")            #␣
↪needs sage.modules
sage: X.monomial(2).support_of_term()                                       #␣
↪needs sage.modules
2
sage: X.term(3, 2).support_of_term()                                        #␣
↪needs sage.modules
3
```

An exception is raised if `self` has more than one term:

```
sage: (X.monomial(2) + X.monomial(3)).support_of_term()                     #␣
↪needs sage.modules
Traceback (most recent call last):
...
ValueError: B[2] + B[3] is not a single term
```

tensor(**elements*)

Return the tensor product of its arguments, as an element of the tensor product of the parents of those elements.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example()                                                       #␣
↪needs sage.combinat sage.modules
sage: a, b, c = A.algebra_generators()                                      #␣
↪needs sage.combinat sage.modules
sage: a.tensor(b, c)                                                        #␣
↪needs sage.combinat sage.modules
B[word: a] # B[word: b] # B[word: c]
```

FIXME: is this a policy that we want to enforce on all parents?

terms()

Return a list of the (non-zero) terms of `self` (in an arbitrary order).

See also:

monomials()

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 2*B['c']
sage: f.terms()
[B['a'], 2*B['c']]
```

trailing_coefficient(**args*, ***kwds*)

Return the trailing coefficient of `self`.

This is the coefficient of the monomial whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_coefficient()
3
sage: def key(x): return -x
sage: x.trailing_coefficient(key=key)
1

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                        #␣
↪needs sage.combinat sage.modules
sage: f.trailing_coefficient()                                              #␣
↪needs sage.combinat sage.modules
2
```

trailing_item(**args*, ***kwds*)

Return the pair `(c, k)` where `c*self.parent().monomial(k)` is the trailing term of `self`.

This is the monomial whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_item()
(1, 3)
sage: def key(x): return -x
sage: x.trailing_item(key=key)
(3, 1)

sage: s = SymmetricFunctions(QQ).schur()                                    #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                          #␣
↪needs sage.combinat sage.modules
sage: f.trailing_item()                                                       #␣
↪needs sage.combinat sage.modules
([1], 2)
```

trailing_monomial(**args*, ***kwds*)

Return the trailing monomial of `self`.

This is the monomial whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_monomial()
B[1]
sage: def key(x): return -x
sage: x.trailing_monomial(key=key)
B[3]

sage: s = SymmetricFunctions(QQ).schur()                                      #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                          #␣
↪needs sage.combinat sage.modules
sage: f.trailing_monomial()                                                   #␣
↪needs sage.combinat sage.modules
s[1]
```

trailing_support(**args*, ***kwds*)

Return the minimal element of the support of `self`. Note that this may not be the term which actually appears last when `self` is printed.

If the default ordering of the basis elements is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")              #␣
↪needs sage.modules
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + 4*X.monomial(3)                 #␣
↪needs sage.modules
sage: x.trailing_support()                                                    #␣
↪needs sage.modules
1

sage: def key(x): return -x
sage: x.trailing_support(key=key)                                             #␣
↪needs sage.modules
3

sage: s = SymmetricFunctions(QQ).schur()                                      #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                              #␣
↪needs sage.combinat sage.modules
sage: f.trailing_support()                                                        #␣
↪needs sage.combinat sage.modules
[1]
```

trailing_term(**args*, ***kwds*)

Return the trailing term of `self`.

This is the term whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_term()
3*B[1]
sage: def key(x): return -x
sage: x.trailing_term(key=key)
B[3]

sage: s = SymmetricFunctions(QQ).schur()                                          #␣
↪needs sage.combinat sage.modules
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]                                              #␣
↪needs sage.combinat sage.modules
sage: f.trailing_term()                                                           #␣
↪needs sage.combinat sage.modules
2*s[1]
```

Filtered

alias of *FilteredModulesWithBasis*

FiniteDimensional

alias of *FiniteDimensionalModulesWithBasis*

Graded

alias of *GradedModulesWithBasis*

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

class ParentMethods

Bases: `object`

class MorphismMethods

Bases: `object`

on_basis()

Return the action of this morphism on basis elements.

OUTPUT:

- a function from the indices of the basis of the domain to the codomain

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1,2,3]);   X.rename("X")
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4]); Y.rename("Y")
sage: H = Hom(X, Y)
sage: x = X.basis()
sage: f = H(lambda x: Y.zero()).on_basis()
sage: f(2)
0
sage: f = lambda i: Y.monomial(i) + 2*Y.monomial(i+1)
sage: g = H(on_basis=f).on_basis()
sage: g(2)
B[2] + 2*B[3]
sage: g == f
True
```

class ParentMethods

Bases: `object`

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])                    #␣
↪needs sage.modules
sage: F.basis()                                                           #␣
↪needs sage.modules
Finite family {'a': B['a'], 'b': B['b'], 'c': B['c']}
```

```
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)                                  #␣
↪needs sage.combinat sage.groups sage.modules
sage: list(QS3.basis())                                                   #␣
↪needs sage.combinat sage.groups sage.modules
[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]
```

cardinality()

Return the cardinality of `self`.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: S = SymmetricGroupAlgebra(QQ, 4)
sage: S.cardinality()
+Infinity
sage: S = SymmetricGroupAlgebra(GF(2), 4)
sage: S.cardinality()
16777216
sage: S.cardinality().factor()
2^24

sage: # needs sage.modules
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: E.cardinality()
+Infinity
sage: E.<x,y> = ExteriorAlgebra(GF(3))
sage: E.cardinality()
81
```

(continues on next page)

(continued from previous page)

```
sage: s = SymmetricFunctions(GF(2)).s()                                   #␣
↪needs sage.combinat sage.modules
sage: s.cardinality()                                                     #␣
↪needs sage.combinat sage.modules
+Infinity

sage: M = CombinatorialFreeModule(QQ, [])
sage: M.dimension()
0
sage: M.cardinality()
1
```

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: A.<x,y> = algebras.DifferentialWeyl(QQ)                             #␣
↪needs sage.modules
sage: A.dimension()                                                       #␣
↪needs sage.modules
+Infinity
```

echelon_form(*elements*, *row_reduced=False*, *order=None*)

Return a basis in echelon form of the subspace spanned by a finite set of elements.

INPUT:

- `elements` – a list or finite iterable of elements of `self`
- `row_reduced` – (default: `False`) whether to compute the basis for the row reduced echelon form
- `order` – (optional) either something that can be converted into a tuple or a key function

OUTPUT:

A list of elements of `self` whose expressions as vectors form a matrix in echelon form. If `base_ring` is specified, then the calculation is achieved in this base ring.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: C = CombinatorialFreeModule(R, ZZ, prefix='z')                      #␣
↪needs sage.modules
sage: z = C.basis()                                                       #␣
↪needs sage.modules
sage: C.echelon_form([z[0] - z[1], 2*z[1] - 2*z[2], z[0] - z[2]])        #␣
↪needs sage.libs.singular sage.modules
[z[0] - z[2], z[1] - z[2]]
```

intersection(*other*)

Return the intersection of `self` with `other`.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, range(4)); x = X.basis()
sage: U = X.submodule([x[0]-x[1], x[1]-x[2], x[2]-x[3]])
sage: F = CombinatorialFreeModule(QQ, ['a','b','c','d'])
sage: G = F.submodule([F.basis()['a']])
```

(continues on next page)

(continued from previous page)

```
sage: X.intersection(X) is X
True
sage: X.intersection(U) is U
True
sage: X.intersection(F)
Traceback (most recent call last):
...
TypeError: other must be a submodule
sage: X.intersection(G)
Traceback (most recent call last):
...
ArithmeticError: this module must be the ambient
```

is_finite()

Return whether `self` is finite.

This is true if and only if `self.basis().keys()` and `self.base_ring()` are both finite.

EXAMPLES:

```
sage: GroupAlgebra(SymmetricGroup(2), IntegerModRing(10)).is_finite()    #␣
↪needs sage.combinat sage.groups sage.modules
True
sage: GroupAlgebra(SymmetricGroup(2)).is_finite()                        #␣
↪needs sage.combinat sage.groups sage.modules
False
sage: GroupAlgebra(AbelianGroup(1), IntegerModRing(10)).is_finite()      #␣
↪needs sage.groups sage.modules
False
```

module_morphism(*on_basis=None*, *matrix=None*, *function=None*, *diagonal=None*, *triangular=None*, *unitriangular=False*, ***keywords*)

Construct a module morphism from `self` to `codomain`.

Let `self` be a module X with a basis indexed by I. This constructs a morphism $f : X \to Y$ by linearity from a map $I \to Y$ which is to be its restriction to the basis $(x_i)_{i\in I}$ of X. Some variants are possible too.

INPUT:

- `self` – a parent X in `ModulesWithBasis(R)` with basis $x = (x_i)_{i\in I}$.

Exactly one of the four following options must be specified in order to define the morphism:

- `on_basis` – a function f from I to Y
- `diagonal` – a function d from I to R
- `function` – a function f from X to Y
- `matrix` – a matrix of size $\dim Y \times \dim X$ (if the keyword `side` is set to `'left'`) or $\dim Y \times \dim X$ (if this keyword is `'right'`)

Further options include:

- `codomain` – the codomain Y of the morphism (default: `f.codomain()` if it's defined; otherwise it must be specified)
- `category` – a category or `None` (default: `None`)
- `zero` – the zero of the codomain (default: `codomain.zero()`); can be used (with care) to define affine maps. Only meaningful with `on_basis`.
- `position` – a non-negative integer specifying which positional argument is used as the input of the function f (default: 0); this is currently only used with `on_basis`.
- `triangular` – (default: `None`) `"upper"` or `"lower"` or `None`:
  - `"upper"` - if the *`leading_support()`* of the image of the basis vector x_i is i, or

  - "lower" - if the *trailing_support()* of the image of the basis vector x_i is i.
- unitriangular – (default: False) a boolean. Only meaningful for a triangular morphism. As a shorthand, one may use unitriangular="lower" for triangular="lower", unitriangular=True.
- side – "left" or "right" (default: "left") Only meaningful for a morphism built from a matrix.

EXAMPLES:

With the on_basis option, this returns a function g obtained by extending f by linearity on the position-th positional argument. For example, for position == 1 and a ternary function f, one has:

$$g\left(a, \sum_i \lambda_i x_i, c\right) = \sum_i \lambda_i f(a, i, c).$$

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [1,2,3]);   X.rename("X")
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4]); Y.rename("Y")
sage: def f(i):
....:     return Y.monomial(i) + 2*Y.monomial(i+1)
sage: phi = X.module_morphism(f, codomain=Y)
sage: x = X.basis(); y = Y.basis()
sage: phi(x[1] + x[3])
B[1] + 2*B[2] + B[3] + 2*B[4]
sage: phi
Generic morphism:
From: X
To:   Y
```

By default, the category is the first of Modules(R).WithBasis().FiniteDimensional(), Modules(R).WithBasis(), Modules(R), and CommutativeAdditiveMonoids() that contains both the domain and the codomain:

```
sage: phi.category_for()                                                #␣
↪needs sage.modules
Category of finite dimensional vector spaces with basis
 over Rational Field
```

With the zero argument, one can define affine morphisms:

```
sage: def f(i):
....:     return Y.monomial(i) + 2*Y.monomial(i+1)
sage: phi = X.module_morphism(f, codomain=Y, zero=10*y[1])              #␣
↪needs sage.modules
sage: phi(x[1] + x[3])                                                  #␣
↪needs sage.modules
11*B[1] + 2*B[2] + B[3] + 2*B[4]
```

In this special case, the default category is Sets():

```
sage: phi.category_for()                                                #␣
↪needs sage.modules
Category of sets
```

One can construct morphisms with the base ring as codomain:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(ZZ, [1, -1])
sage: phi = X.module_morphism(on_basis=lambda i: i, codomain=ZZ)
sage: phi(2 * X.monomial(1) + 3 * X.monomial(-1))
-1
sage: phi.category_for()
Category of commutative additive semigroups
sage: phi.category_for()        # not implemented
Category of modules over Integer Ring
```

Or more generally any ring admitting a coercion map from the base ring:

```
sage: # needs sage.modules
sage: phi = X.module_morphism(on_basis=lambda i: i, codomain=RR)
sage: phi(2 * X.monomial(1) + 3 * X.monomial(-1))
-1.00000000000000
sage: phi.category_for()
Category of commutative additive semigroups
sage: phi.category_for()        # not implemented
Category of modules over Integer Ring

sage: phi = X.module_morphism(on_basis=lambda i: i, codomain=Zmod(4))    #␣
↪needs sage.modules
sage: phi(2 * X.monomial(1) + 3 * X.monomial(-1))                        #␣
↪needs sage.modules
3

sage: phi = Y.module_morphism(on_basis=lambda i: i, codomain=Zmod(4))    #␣
↪needs sage.modules
Traceback (most recent call last):
...
ValueError: codomain(=Ring of integers modulo 4) should be a module
over the base ring of the domain(=Y)
```

On can also define module morphisms between free modules over different base rings; here we implement the natural map from $X = \mathbf{R}^2$ to $Y = \mathbf{C}$:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(RR, ['x', 'y'])
sage: Y = CombinatorialFreeModule(CC, ['z'])
sage: x = X.monomial('x')
sage: y = X.monomial('y')
sage: z = Y.monomial('z')
sage: def on_basis(a):
....:     if a == 'x':
....:         return CC(1) * z
....:     elif a == 'y':
....:         return CC(I) * z
sage: phi = X.module_morphism(on_basis=on_basis, codomain=Y)
sage: v = 3 * x + 2 * y; v
3.00000000000000*B['x'] + 2.00000000000000*B['y']
sage: phi(v)                                                             #␣
↪needs sage.symbolic
(3.00000000000000+2.00000000000000*I)*B['z']
sage: phi.category_for()
Category of commutative additive semigroups
sage: phi.category_for()        # not implemented
```

(continues on next page)

(continued from previous page)

```
Category of vector spaces over Real Field with 53 bits of precision

sage: # needs sage.modules
sage: Y = CombinatorialFreeModule(CC['q'], ['z'])
sage: z = Y.monomial('z')
sage: phi = X.module_morphism(on_basis=on_basis, codomain=Y)
sage: phi(v)                                                                #␣
↪needs sage.symbolic
(3.00000000000000+2.00000000000000*I)*B['z']
```

Of course, there should be a coercion between the respective base rings of the domain and the codomain for this to be meaningful:

```
sage: Y = CombinatorialFreeModule(QQ, ['z'])                                #␣
↪needs sage.modules
sage: phi = X.module_morphism(on_basis=on_basis, codomain=Y)                #␣
↪needs sage.modules
Traceback (most recent call last):
...
ValueError: codomain(=Free module generated by {'z'} over Rational Field)
should be a module over the base ring of the domain(=Free module
generated by {'x', 'y'} over Real Field with 53 bits of precision)

sage: Y = CombinatorialFreeModule(RR['q'], ['z'])                           #␣
↪needs sage.modules
sage: phi = Y.module_morphism(on_basis=on_basis, codomain=X)                #␣
↪needs sage.modules
Traceback (most recent call last):
...
ValueError: codomain(=Free module generated by {'x', 'y'}
over Real Field with 53 bits of precision) should be a module over
the base ring of the domain(=Free module generated by {'z'} over
Univariate Polynomial Ring in q over Real Field with 53 bits of precision)
```

With the `diagonal=d` argument, this constructs the module morphism g such that

$$`g(x_i) = d(i)y_i`.$$

This assumes that the respective bases x and y of X and Y have the same index set I:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(ZZ, [1, 2, 3]); X.rename("X")
sage: from sage.arith.misc import factorial
sage: phi = X.module_morphism(diagonal=factorial, codomain=X)
sage: x = X.basis()
sage: phi(x[1]), phi(x[2]), phi(x[3])
(B[1], 2*B[2], 6*B[3])
```

See also: `sage.modules.with_basis.morphism.DiagonalModuleMorphism`.

With the `matrix=m` argument, this constructs the module morphism whose matrix in the distinguished basis of X and Y is m:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); X.rename("X")
sage: x = X.basis()
sage: Y = CombinatorialFreeModule(ZZ, [3,4]); Y.rename("Y")
```

(continues on next page)

(continued from previous page)

```
sage: y = Y.basis()
sage: m = matrix([[0,1,2], [3,5,0]])
sage: phi = X.module_morphism(matrix=m, codomain=Y)
sage: phi(x[1])
3*B[4]
sage: phi(x[2])
B[3] + 5*B[4]
```

See also: `sage.modules.with_basis.morphism.ModuleMorphismFromMatrix`.

With `triangular="upper"`, the constructed module morphism is assumed to be upper triangular; that is its matrix in the distinguished basis of X and Y would be upper triangular with invertible elements on its diagonal. This is used to compute preimages and to invert the morphism:

```
sage: # needs sage.modules
sage: I = list(range(1, 200))
sage: X = CombinatorialFreeModule(QQ, I); X.rename("X"); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, I); Y.rename("Y"); y = Y.basis()
sage: f = Y.sum_of_monomials * divisors
sage: phi = X.module_morphism(f, triangular="upper", codomain=Y)
sage: phi(x[2])
B[1] + B[2]
sage: phi(x[6])
B[1] + B[2] + B[3] + B[6]
sage: phi(x[30])
B[1] + B[2] + B[3] + B[5] + B[6] + B[10] + B[15] + B[30]
sage: phi.preimage(y[2])
-B[1] + B[2]
sage: phi.preimage(y[6])
B[1] - B[2] - B[3] + B[6]
sage: phi.preimage(y[30])
-B[1] + B[2] + B[3] + B[5] - B[6] - B[10] - B[15] + B[30]
sage: (phi^-1)(y[30])
-B[1] + B[2] + B[3] + B[5] - B[6] - B[10] - B[15] + B[30]
```

Since github issue #8678, one can also define a triangular morphism from a function:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, [0,1,2,3,4]); x = X.basis()
sage: from sage.modules.with_basis.morphism import␣
↪TriangularModuleMorphismFromFunction
sage: def f(x): return x + X.term(0, sum(x.coefficients()))
sage: phi = X.module_morphism(function=f, codomain=X,
....:                          triangular="upper")
sage: phi(x[2] + 3*x[4])
4*B[0] + B[2] + 3*B[4]
sage: phi.preimage(_)
B[2] + 3*B[4]
```

For details and further optional arguments, see `sage.modules.with_basis.morphism.TriangularModuleMorphism`.

Warning: As a temporary measure, until multivariate morphisms are implemented, the constructed morphism is in `Hom(codomain, domain, category)`. This is only correct for unary functions.

Todo:

- Should codomain be `self` by default in the diagonal, triangular, and matrix cases?
- Support for diagonal morphisms between modules not sharing the same index set

monomial(*i*)

Return the basis element indexed by `i`.

INPUT:

- `i` – an element of the index set

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])                    #␣
↪needs sage.modules
sage: F.monomial('a')                                                     #␣
↪needs sage.modules
B['a']
```

`F.monomial` is in fact (almost) a map:

```
sage: F.monomial                                                          #␣
↪needs sage.modules
Term map from {'a', 'b', 'c'}
 to Free module generated by {'a', 'b', 'c'} over Rational Field
```

monomial_or_zero_if_none(*i*)

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])                    #␣
↪needs sage.modules
sage: F.monomial_or_zero_if_none('a')                                     #␣
↪needs sage.modules
B['a']
sage: F.monomial_or_zero_if_none(None)                                    #␣
↪needs sage.modules
0
```

quotient_module(*submodule*, *check=True*, *already_echelonized=False*, *category=None*)

Construct the quotient module `self` / `submodule`.

INPUT:

- `submodule` – a submodule with basis of `self`, or something that can be turned into one via `self.submodule(submodule)`
- `check`, `already_echelonized` – passed down to *`ModulesWithBasis.ParentMethods.submodule()`*

Warning: At this point, this only supports quotients by free submodules admitting a basis in unitriangular echelon form. In this case, the quotient is also a free module, with a basis consisting of the retract of a subset of the basis of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
```

(continues on next page)

(continued from previous page)

```
sage: Y = X.quotient_module([x[0] - x[1], x[1] - x[2]],
....:                        already_echelonized=True)
sage: Y.print_options(prefix='y'); Y
Free module generated by {2} over Rational Field
sage: y = Y.basis()
sage: y[2]
y[2]
sage: y[2].lift()
x[2]
sage: Y.retract(x[0] + 2*x[1])
3*y[2]

sage: # needs sage.modules
sage: R.<a,b> = QQ[]
sage: C = CombinatorialFreeModule(R, range(3), prefix='x')
sage: x = C.basis()
sage: gens = [x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]
sage: Y = C.quotient_module(gens)
```

See also:

- `Modules.WithBasis.ParentMethods.submodule()`
- *`Rings.ParentMethods.quotient()`*
- `sage.modules.with_basis.subquotient.QuotientModuleWithBasis`

`random_element`(*n=2*)

Return a 'random' element of `self`.

INPUT:

- `n` – integer (default: 2); number of summands

ALGORITHM:

Return a sum of `n` terms, each of which is formed by multiplying a random element of the base ring by a random element of the group.

EXAMPLES:

```
sage: x = DihedralGroup(6).algebra(QQ).random_element()                 #␣
↪needs sage.groups sage.modules
sage: x.parent() is DihedralGroup(6).algebra(QQ)                        #␣
↪needs sage.groups sage.modules
True
```

Note, this result can depend on the PRNG state in libgap in a way that depends on which packages are loaded, so we must re-seed GAP to ensure a consistent result for this example:

```
sage: libgap.set_seed(0)                                                #␣
↪needs sage.libs.gap
0
sage: m = SU(2, 13).algebra(QQ).random_element(1)                       #␣
↪needs sage.groups sage.libs.pari sage.modules
sage: m.parent() is SU(2, 13).algebra(QQ)                               #␣
↪needs sage.groups sage.libs.pari sage.modules
True
sage: p = CombinatorialFreeModule(ZZ, Partitions(4)).random_element()   #␣
↪needs sage.combinat sage.modules
sage: p.parent() is CombinatorialFreeModule(ZZ, Partitions(4))          #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
True
```

submodule (*gens*, *check=True*, *already_echelonized=False*, *unitriangular=False*, *support_order=None*, *category=None*, **args*, ***opts*)

The submodule spanned by a finite set of elements.

INPUT:

- `gens` – a list or family of elements of `self`
- **`check` – (default: `True`) whether to verify that the**
  elements of `gens` are in `self`
- **`already_echelonized` – (default: `False`) whether**
  the elements of `gens` are already in (not necessarily reduced) echelon form
- `unitriangular` – (default: `False`) whether the lift morphism is unitriangular
- `support_order` – (optional) either something that can be converted into a tuple or a key function
- `category` – (optional) the category of the submodule

If `already_echelonized` is `False`, then the generators are put in reduced echelon form using `echelonize()`, and reindexed by $0, 1, \ldots$.

Warning: At this point, this method only works for finite dimensional submodules and if matrices can be echelonized over the base ring.

If in addition `unitriangular` is `True`, then the generators are made such that the coefficients of the pivots are 1, so that lifting map is unitriangular.

The basis of the submodule uses the same index set as the generators, and the lifting map sends y_i to $gens[i]$.

See also:

- `ModulesWithBasis.FiniteDimensional.ParentMethods.quotient_module()`
- `sage.modules.with_basis.subquotient.SubmoduleWithBasis`

EXAMPLES:

We construct a submodule of the free **Q**-module generated by x_0, x_1, x_2. The submodule is spanned by $y_0 = x_0 - x_1$ and $y_1 - x_1 - x_2$, and its basis elements are indexed by 0 and 1:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: gens = [x[0] - x[1], x[1] - x[2]]; gens
[x[0] - x[1], x[1] - x[2]]
sage: Y = X.submodule(gens, already_echelonized=True)
sage: Y.print_options(prefix='y'); Y
Free module generated by {0, 1} over Rational Field
sage: y = Y.basis()
sage: y[1]
y[1]
sage: y[1].lift()
x[1] - x[2]
sage: Y.retract(x[0] - x[2])
y[0] + y[1]
sage: Y.retract(x[0])
```

(continues on next page)

(continued from previous page)

```
Traceback (most recent call last):
...
ValueError: x[0] is not in the image
```

By using a family to specify a basis of the submodule, we obtain a submodule whose index set coincides with the index set of the family:

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: gens = Family({1: x[0] - x[1], 3: x[1] - x[2]}); gens
Finite family {1: x[0] - x[1], 3: x[1] - x[2]}
sage: Y = X.submodule(gens, already_echelonized=True)
sage: Y.print_options(prefix='y'); Y
Free module generated by {1, 3} over Rational Field
sage: y = Y.basis()
sage: y[1]
y[1]
sage: y[1].lift()
x[0] - x[1]
sage: y[3].lift()
x[1] - x[2]
sage: Y.retract(x[0] - x[2])
y[1] + y[3]
sage: Y.retract(x[0])
Traceback (most recent call last):
...
ValueError: x[0] is not in the image
```

It is not necessary that the generators of the submodule form a basis (an explicit basis will be computed):

```
sage: # needs sage.modules
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: gens = [x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]; gens
[x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]
sage: Y = X.submodule(gens, already_echelonized=False)
sage: Y.print_options(prefix='y')
sage: Y
Free module generated by {0, 1} over Rational Field
sage: [b.lift() for b in Y.basis()]
[x[0] - x[2], x[1] - x[2]]
```

We now implement by hand the center of the algebra of the symmetric group S_3:

```
sage: # needs sage.combinat sage.groups sage.modules
sage: S3 = SymmetricGroup(3)
sage: S3A = S3.algebra(QQ)
sage: basis = S3A.annihilator_basis(S3A.algebra_generators(),
....:                                S3A.bracket)
sage: basis
((), (1,2,3) + (1,3,2), (2,3) + (1,2) + (1,3))
sage: center = S3A.submodule(basis,
....:     category=AlgebrasWithBasis(QQ).Subobjects(),
....:     already_echelonized=True)
sage: center
Free module generated by {0, 1, 2} over Rational Field
```

(continues on next page)

(continued from previous page)

```
sage: center in Algebras
True
sage: center.print_options(prefix='c')
sage: c = center.basis()
sage: c[1].lift()
(1,2,3) + (1,3,2)
sage: c[0]^2
c[0]
sage: e = 1/6 * (c[0]+c[1]+c[2])
sage: e.is_idempotent()
True
```

Of course, this center is best constructed using:

```
sage: center = S3A.center()                                              #␣
↪needs sage.combinat sage.groups sage.modules
```

We can also automatically construct a basis such that the lift morphism is (lower) unitriangular:

```
sage: # needs sage.modules
sage: R.<a,b> = QQ[]
sage: C = CombinatorialFreeModule(R, range(3), prefix='x')
sage: x = C.basis()
sage: gens = [x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]
sage: Y = C.submodule(gens, unitriangular=True)
sage: Y.lift.matrix()
[ 1  0]
[ 0  1]
[-1 -1]
```

We now construct a (finite-dimensional) submodule of an infinite dimensional free module:

```
sage: # needs sage.modules
sage: C = CombinatorialFreeModule(QQ, ZZ, prefix='z')
sage: z = C.basis()
sage: gens = [z[0] - z[1], 2*z[1] - 2*z[2], z[0] - z[2]]
sage: Y = C.submodule(gens)
sage: [Y.lift(b) for b in Y.basis()]
[z[0] - z[2], z[1] - z[2]]
```

sum_of_monomials()

Return the sum of the basis elements with indices in `indices`.

INPUT:

- `indices` – a list (or iterable) of indices of basis elements

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])                   #␣
↪needs sage.modules
sage: F.sum_of_monomials(['a', 'b'])                                     #␣
↪needs sage.modules
B['a'] + B['b']

sage: F.sum_of_monomials(['a', 'b', 'a'])                                #␣
↪needs sage.modules
2*B['a'] + B['b']
```

`F.sum_of_monomials` is in fact (almost) a map:

```
sage: F.sum_of_monomials                                                    #␣
↪needs sage.modules
A map to Free module generated by {'a', 'b', 'c'} over Rational Field
```

sum_of_terms (*terms*)

Construct a sum of terms of `self`.

INPUT:

- `terms` – a list (or iterable) of pairs `(index, coeff)`

OUTPUT:

Sum of `coeff * B[index]` over all `(index, coeff)` in `terms`, where `B` is the basis of `self`.

EXAMPLES:

```
sage: m = matrix([[0,1], [1,1]])                                            #␣
↪needs sage.modules
sage: J.<a,b,c> = JordanAlgebra(m)                                          #␣
↪needs sage.combinat sage.modules
sage: J.sum_of_terms([(0, 2), (2, -3)])                                     #␣
↪needs sage.combinat sage.modules
2 + (0, -3)
```

tensor (**parents*, ***kwargs*)

Return the tensor product of the parents.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example(); A.rename("A")                                        #␣
↪needs sage.combinat sage.modules
sage: A.tensor(A, A)                                                        #␣
↪needs sage.combinat sage.modules
A # A # A
sage: A.rename(None)                                                        #␣
↪needs sage.combinat sage.modules
```

term (*index*, *coeff=None*)

Construct a term in `self`.

INPUT:

- `index` – the index of a basis element
- `coeff` – an element of the coefficient ring (default: one)

OUTPUT:

`coeff * B[index]`, where `B` is the basis of `self`.

EXAMPLES:

```
sage: m = matrix([[0,1], [1,1]])                                            #␣
↪needs sage.modules
sage: J.<a,b,c> = JordanAlgebra(m)                                          #␣
↪needs sage.combinat sage.modules
sage: J.term(1, -2)                                                         #␣
↪needs sage.combinat sage.modules
0 + (-2, 0)
```

Design: should this do coercion on the coefficient ring?

Super

alias of *SuperModulesWithBasis*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of modules with basis constructed by tensor product of modules with basis.

class ElementMethods

Bases: `object`

Implements operations on elements of tensor products of modules with basis.

apply_multilinear_morphism(*f*, *codomain=None*)

Return the result of applying the morphism induced by `f` to `self`.

INPUT:

- `f` – a multilinear morphism from the component modules of the parent tensor product to any module
- `codomain` – the codomain of `f` (optional)

By the universal property of the tensor product, `f` induces a linear morphism from $self.parent()$ to the target module. Returns the result of applying that morphism to `self`.

The codomain is used for optimizations purposes only. If it's not provided, it's recovered by calling `f` on the zero input.

EXAMPLES:

We start with simple (admittedly not so interesting) examples, with two modules A and B:

```
sage: # needs sage.modules
sage: A = CombinatorialFreeModule(ZZ, [1,2], prefix="A")
sage: A.rename("A")
sage: B = CombinatorialFreeModule(ZZ, [3,4], prefix="B")
sage: B.rename("B")
```

and f the bilinear morphism $(a, b) \mapsto b \otimes a$ from $A \times B$ to $B \otimes A$:

```
sage: def f(a,b):
....:     return tensor([b,a])
```

Now, calling applying f on $a \otimes b$ returns the same as $f(a, b)$:

```
sage: # needs sage.modules
sage: a = A.monomial(1) + 2 * A.monomial(2); a
A[1] + 2*A[2]
sage: b = B.monomial(3) - 2 * B.monomial(4); b
B[3] - 2*B[4]
sage: f(a, b)
B[3] # A[1] + 2*B[3] # A[2] - 2*B[4] # A[1] - 4*B[4] # A[2]
sage: tensor([a, b]).apply_multilinear_morphism(f)
B[3] # A[1] + 2*B[3] # A[2] - 2*B[4] # A[1] - 4*B[4] # A[2]
```

f may be a bilinear morphism to any module over the base ring of A and B. Here the codomain is **Z**:

```
sage: def f(a, b):
....:     return sum(a.coefficients(), 0) * sum(b.coefficients(), 0)
sage: f(a, b)                                                           #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.modules
-3
sage: tensor([a, b]).apply_multilinear_morphism(f)                            #␣
↪needs sage.modules
-3
```

Mind the 0 in the sums above; otherwise f would not return 0 in $\mathbf{Z}$:

```
sage: def f(a,b):
....:     return sum(a.coefficients()) * sum(b.coefficients())
sage: type(f(A.zero(), B.zero()))                                             #␣
↪needs sage.modules
<... 'int'>
```

Which would be wrong and break this method:

```
sage: tensor([a, b]).apply_multilinear_morphism(f)                            #␣
↪needs sage.modules
Traceback (most recent call last):
...
AttributeError: 'int' object has no attribute 'parent'...
```

Here we consider an example where the codomain is a module with basis with a different base ring:

```
   sage: # needs sage.modules
   sage: C = CombinatorialFreeModule(QQ, [(1,3),(2,4)], prefix="C")
   sage: C.rename("C")
   sage: def f(a, b):
   ....:     return C.sum_of_terms([((1,3), QQ(a[1]*b[3])),
   ....:                            ((2,4), QQ(a[2]*b[4]))])
   sage: f(a,b)
   C[(1, 3)] - 4*C[(2, 4)]
   sage: tensor([a, b]).apply_multilinear_morphism(f)
   C[(1, 3)] - 4*C[(2, 4)]

We conclude with a real life application, where we
check that the antipode of the Hopf algebra of
Symmetric functions on the Schur basis satisfies its
defining formula::

   sage: # needs lrcalc_python sage.combinat sage.modules
   sage: Sym = SymmetricFunctions(QQ)
   sage: s = Sym.schur()
   sage: def f(a, b): return a * b.antipode()
   sage: x = 4 * s.an_element(); x
   8*s[] + 8*s[1] + 12*s[2]
   sage: x.coproduct().apply_multilinear_morphism(f)
   8*s[]
   sage: x.coproduct().apply_multilinear_morphism(f) == x.counit()
   True
```

We recover the constant term of x, as desired.

Todo: Extract a method to linearize a multilinear morphism, and delegate the work there.

class ParentMethods

Bases: `object`

Implements operations on tensor products of modules with basis.

extra_super_categories()

EXAMPLES:

```
sage: ModulesWithBasis(QQ).TensorProducts().extra_super_categories()
[Category of vector spaces with basis over Rational Field]
sage: ModulesWithBasis(QQ).TensorProducts().super_categories()
[Category of tensor products of modules with basis over Rational Field,
 Category of vector spaces with basis over Rational Field,
 Category of tensor products of vector spaces over Rational Field]
```

is_abelian()

Return whether this category is abelian.

This is the case if and only if the base ring is a field.

EXAMPLES:

```
sage: ModulesWithBasis(QQ).is_abelian()
True
sage: ModulesWithBasis(ZZ).is_abelian()
False
```

4.120 Monoid algebras

`sage.categories.monoid_algebras.`**MonoidAlgebras**(*base_ring*)

The category of monoid algebras over `base_ring`.

EXAMPLES:

```
sage: C = MonoidAlgebras(QQ); C
Category of monoid algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of bialgebras with basis over Rational Field,
 Category of semigroup algebras over Rational Field,
 Category of unital magma algebras over Rational Field]
```

This is just an alias for:

```
sage: C is Monoids().Algebras(QQ)
True
```

4.121 Monoids

class sage.categories.monoids.**Monoids**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of (multiplicative) monoids.

A *monoid* is a unital *semigroup*, that is a set endowed with a multiplicative binary operation $*$ which is associative and admits a unit (see Wikipedia article Monoid).

EXAMPLES:

```
sage: Monoids()
Category of monoids
sage: Monoids().super_categories()
[Category of semigroups, Category of unital magmas]
sage: Monoids().all_super_categories()
[Category of monoids,
 Category of semigroups,
 Category of unital magmas, Category of magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: Monoids().axioms()
frozenset({'Associative', 'Unital'})
sage: Semigroups().Unital()
Category of monoids

sage: Monoids().example()
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
```

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ElementMethods

Bases: object

is_central()

Return whether the element self is central.

EXAMPLES:

```
sage: SG4 = SymmetricGroupAlgebra(ZZ,4)                                  #␣
↪needs sage.groups sage.modules
sage: SG4(1).is_central()                                                #␣
↪needs sage.groups sage.modules
True
sage: SG4(Permutation([1,3,2,4])).is_central()                           #␣
↪needs sage.groups sage.modules
False

sage: A = GroupAlgebras(QQ).example(); A                                 #␣
↪needs sage.groups sage.modules
Algebra of Dihedral group of order 8
 as a permutation group over Rational Field
sage: sum(A.basis()).is_central()                                        #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.groups sage.modules
True
```

class ParentMethods

Bases: `object`

algebra_generators()

Return generators for this algebra.

For a monoid algebra, the algebra generators are built from the monoid generators if available and from the semigroup generators otherwise.

See also:

- *Semigroups.Algebras.ParentMethods.algebra_generators()*
- *MagmaticAlgebras.ParentMethods.algebra_generators()*.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid:
the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.monoid_generators()
Finite family {'a': 'a', 'b': 'b', 'c': 'c', 'd': 'd'}
sage: M.algebra(ZZ).algebra_generators()                                   #␣
↪needs sage.modules
Finite family {'a': B['a'], 'b': B['b'], 'c': B['c'], 'd': B['d']}

sage: Z12 = Monoids().Finite().example(); Z12
An example of a finite multiplicative monoid:
the integers modulo 12
sage: Z12.monoid_generators()
Traceback (most recent call last):
...
AttributeError: 'IntegerModMonoid_with_category' object
has no attribute 'monoid_generators'...
sage: Z12.semigroup_generators()
Family (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
sage: Z12.algebra(QQ).algebra_generators()                                 #␣
↪needs sage.modules
Family (B[0], B[1], B[2], B[3], B[4], B[5], B[6], B[7], B[8], B[9],␣
↪B[10], B[11])


sage: A10 = AlternatingGroup(10)                                           #␣
↪needs sage.groups
sage: GroupAlgebras(QQ).example(A10).algebra_generators()                  #␣
↪needs sage.groups sage.modules
Family ((8,9,10), (1,2,3,4,5,6,7,8,9))

sage: A = DihedralGroup(3).algebra(QQ); A                                  #␣
↪needs sage.groups sage.modules
Algebra of Dihedral group of order 6 as a permutation group
 over Rational Field
sage: A.algebra_generators()                                               #␣
↪needs sage.groups sage.modules
Family ((1,2,3), (1,3))
```

one_basis()

Return the unit of the monoid, which indexes the unit of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: # needs sage.modules
sage: A = Monoids().example().algebra(ZZ)
sage: A.one_basis()
''
sage: A.one()
B['']
sage: A(3)
3*B['']
```

extra_super_categories()

The algebra of a monoid is a bialgebra and a monoid.

EXAMPLES:

```
sage: C = Monoids().Algebras(QQ)
sage: C.extra_super_categories()
[Category of bialgebras over Rational Field,
 Category of monoids]
sage: Monoids().Algebras(QQ).super_categories()
[Category of bialgebras with basis over Rational Field,
 Category of semigroup algebras over Rational Field,
 Category of unital magma algebras over Rational Field]
```

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

The category of monoids constructed as Cartesian products of monoids.

This construction gives the direct product of monoids. See Wikipedia article Direct_product for more information.

class ElementMethods

Bases: `object`

multiplicative_order()

Return the multiplicative order of this element.

EXAMPLES:

```
sage: # needs sage.groups sage.modules
sage: G1 = SymmetricGroup(3)
sage: G2 = SL(2, 3)
sage: G = cartesian_product([G1, G2])
sage: G((G1.gen(0), G2.gen(1))).multiplicative_order()
12
```

class ParentMethods

Bases: `object`

monoid_generators()

Return the generators of `self`.

EXAMPLES:

```
sage: # needs sage.groups
sage: M = Monoids.free([1, 2, 3])
sage: N = Monoids.free(['a', 'b'])
sage: C = cartesian_product([M, N])
sage: C.monoid_generators()
Family ((F[1], 1), (F[2], 1), (F[3], 1),
        (1, F['a']), (1, F['b']))
```

An example with an infinitely generated group (a better output is needed):

```
sage: N = Monoids.free(ZZ)                                                    #␣
↪needs sage.combinat
sage: C = cartesian_product([M, N])                                           #␣
↪needs sage.combinat sage.groups
sage: C.monoid_generators()                                                   #␣
↪needs sage.combinat sage.groups
Lazy family (gen(i))_{i in The Cartesian product of (...)}
```

extra_super_categories()

A Cartesian product of monoids is endowed with a natural group structure.

EXAMPLES:

```
sage: C = Monoids().CartesianProducts()
sage: C.extra_super_categories()
[Category of monoids]
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of semigroups,
 Category of Cartesian products of unital magmas,
 Category of monoids]
```

class Commutative(*base_category*)

Bases: `CategoryWithAxiom_singleton`

Category of commutative (abelian) monoids.

A monoid M is *commutative* if $xy = yx$ for all $x, y \in M$.

static free(*index_set=None*, *names=None*, ***kwds*)

Return a free abelian monoid on n generators or with the generators indexed by a set I.

A free monoid is constructed by specifying either:

- the number of generators and/or the names of the generators, or
- the indexing set for the generators.

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \dots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

EXAMPLES:

```
sage: Monoids.Commutative.free(index_set=ZZ)                                  #␣
↪needs sage.combinat
Free abelian monoid indexed by Integer Ring
sage: Monoids().Commutative().free(ZZ)                                        #␣
↪needs sage.combinat
Free abelian monoid indexed by Integer Ring
sage: F.<x,y,z> = Monoids().Commutative().free(); F                           #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat
Free abelian monoid indexed by {'x', 'y', 'z'}
```

class ElementMethods

Bases: `object`

inverse()

Return the multiplicative inverse of `self`.

This is an alias for inversion, which can also be invoked by `~x` for an element `x`.

Nota Bene: Element classes should implement `__invert__` only.

EXAMPLES:

```
sage: AA(sqrt(~2)).inverse()                                          #␣
↪needs sage.rings.number_field sage.symbolic
1.414213562373095?
```

is_one()

Return whether `self` is the one of the monoid.

The default implementation is to compare with `self.one()`.

powers(n)

Return the list $[x^0, x^1, \ldots, x^{n-1}]$.

EXAMPLES:

```
sage: A = Matrix([[1, 1], [-1, 0]])                                   #␣
↪needs sage.modules
sage: A.powers(6)                                                     #␣
↪needs sage.modules
[
[1 0]   [ 1  1]   [ 0  1]   [-1  0]   [-1 -1]   [ 0 -1]
[0 1], [-1  0], [-1 -1], [ 0 -1], [ 1  0], [ 1  1]
]
```

Finite

alias of *FiniteMonoids*

Inverse

alias of *Groups*

class ParentMethods

Bases: `object`

prod(*args*)

n-ary product of elements of `self`.

INPUT:

- `args` – a list (or iterable) of elements of `self`

Returns the product of the elements in `args`, as an element of `self`.

EXAMPLES:

```
sage: S = Monoids().example()
sage: S.prod([S('a'), S('b')])
'ab'
```

semigroup_generators()

Return the generators of `self` as a semigroup.

The generators of a monoid M as a semigroup are the generators of M as a monoid and the unit.

EXAMPLES:

```
sage: M = Monoids().free([1,2,3])                                       #␣
↪needs sage.combinat
sage: M.semigroup_generators()                                          #␣
↪needs sage.combinat
Family (1, F[1], F[2], F[3])
```

submonoid(*generators*, *category=None*)

Return the multiplicative submonoid generated by `generators`.

INPUT:

- `generators` – a finite family of elements of `self`, or a list, iterable, … that can be converted into one (see `Family`).
- `category` – a category

This is a shorthand for *Semigroups.ParentMethods.subsemigroup()* that specifies that this is a submonoid, and in particular that the unit is `self.one()`.

EXAMPLES:

```
sage: R = IntegerModRing(15)
sage: M = R.submonoid([R(3), R(5)]); M                                  #␣
↪needs sage.combinat
A submonoid of (Ring of integers modulo 15) with 2 generators
sage: M.list()                                                          #␣
↪needs sage.combinat
[1, 3, 5, 9, 0, 10, 12, 6]
```

Not the presence of the unit, unlike in:

```
sage: S = R.subsemigroup([R(3), R(5)]); S                               #␣
↪needs sage.combinat
A subsemigroup of (Ring of integers modulo 15) with 2 generators
sage: S.list()                                                          #␣
↪needs sage.combinat
[3, 5, 9, 0, 10, 12, 6]
```

This method is really a shorthand for subsemigroup:

```
sage: M2 = R.subsemigroup([R(3), R(5)], one=R.one())                    #␣
↪needs sage.combinat
sage: M2 is M                                                           #␣
↪needs sage.combinat
True
```

class Subquotients(*category*, **args*)

Bases: *SubquotientsCategory*

class ParentMethods

Bases: `object`

one()

Returns the multiplicative unit of this monoid, obtained by retracting that of the ambient monoid.

EXAMPLES:

```
sage: S = Monoids().Subquotients().example() # todo: not implemented
sage: S.one()                                # todo: not implemented
```

class WithRealizations(*category*, **args*)

Bases: *WithRealizationsCategory*

class ParentMethods

Bases: `object`

one()

Return the unit of this monoid.

This default implementation returns the unit of the realization of `self` given by *a_realization()*.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                          #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.one.__module__                                                    #␣
↪needs sage.modules
'sage.categories.monoids'
sage: A.one()                                                             #␣
↪needs sage.modules
F[{}]
```

static free(*index_set=None*, *names=None*, ***kwds*)

Return a free monoid on n generators or with the generators indexed by a set I.

A free monoid is constructed by specifying either:

- the number of generators and/or the names of the generators
- the indexing set for the generators

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

EXAMPLES:

```
sage: Monoids.free(index_set=ZZ)                                          #␣
↪needs sage.combinat
Free monoid indexed by Integer Ring
sage: Monoids().free(ZZ)                                                  #␣
↪needs sage.combinat
Free monoid indexed by Integer Ring
sage: F.<x,y,z> = Monoids().free(); F                                     #␣
↪needs sage.combinat
Free monoid indexed by {'x', 'y', 'z'}
```

4.122 Number fields

class sage.categories.number_fields.**NumberFields**

Bases: *Category_singleton*

The category of number fields.

EXAMPLES:

We create the category of number fields:

```
sage: C = NumberFields()
sage: C
Category of number fields
```

By definition, it is infinite:

```
sage: NumberFields().Infinite() is NumberFields()
True
```

Notice that the rational numbers **Q** *are* considered as an object in this category:

```
sage: RationalField() in C
True
```

However, we can define a degree 1 extension of **Q**, which is of course also in this category:

```
sage: x = PolynomialRing(RationalField(), 'x').gen()
sage: K = NumberField(x - 1, 'a'); K                                    #␣
↪needs sage.rings.number_field
Number Field in a with defining polynomial x - 1
sage: K in C                                                            #␣
↪needs sage.rings.number_field
True
```

Number fields all lie in this category, regardless of the name of the variable:

```
sage: K = NumberField(x^2 + 1, 'a')                                     #␣
↪needs sage.rings.number_field
sage: K in C                                                            #␣
↪needs sage.rings.number_field
True
```

class ElementMethods

Bases: object

class ParentMethods

Bases: object

zeta_function(*prec=53*, *max_imaginary_part=0*, *max_asymp_coeffs=40*, *algorithm='pari'*)

Return the Dedekind zeta function of this number field.

Actually, this returns an interface for computing with the Dedekind zeta function $\zeta_F(s)$ of the number field F.

INPUT:

- prec – optional integer (default 53) bits precision
- max_imaginary_part – optional real number (default 0)

- `max_asymp_coeffs` – optional integer (default 40)
- `algorithm` – optional (default "pari") either "gp" or "pari"

OUTPUT: The zeta function of this number field.

If algorithm is "gp", this returns an interface to Tim Dokchitser's gp script for computing with L-functions.

If algorithm is "pari", this returns instead an interface to Pari's own general implementation of L-functions.

EXAMPLES:

```
sage: K.<a> = NumberField(ZZ['x'].0^2 + ZZ['x'].0 - 1)                    #␣
↪needs sage.rings.number_field
sage: Z = K.zeta_function(); Z                                            #␣
↪needs sage.rings.number_field sage.symbolic
PARI zeta function associated to Number Field in a
 with defining polynomial x^2 + x - 1
sage: Z(-1)                                                               #␣
↪needs sage.rings.number_field sage.symbolic
0.0333333333333333

sage: x = polygen(QQ, 'x')
sage: L.<a, b, c> = NumberField([x^2 - 5, x^2 + 3, x^2 + 1])             #␣
↪needs sage.rings.number_field
sage: Z = L.zeta_function()                                               #␣
↪needs sage.rings.number_field sage.symbolic
sage: Z(5)                                                                #␣
↪needs sage.rings.number_field sage.symbolic
1.00199015670185
```

Using the algorithm "pari":

```
sage: K.<a> = NumberField(ZZ['x'].0^2 + ZZ['x'].0 - 1)                    #␣
↪needs sage.rings.number_field
sage: Z = K.zeta_function(algorithm="pari")                               #␣
↪needs sage.rings.number_field sage.symbolic
sage: Z(-1)                                                               #␣
↪needs sage.rings.number_field sage.symbolic
0.0333333333333333

sage: x = polygen(QQ, 'x')
sage: L.<a, b, c> = NumberField([x^2 - 5, x^2 + 3, x^2 + 1])             #␣
↪needs sage.rings.number_field
sage: Z = L.zeta_function(algorithm="pari")                               #␣
↪needs sage.rings.number_field sage.symbolic
sage: Z(5)                                                                #␣
↪needs sage.rings.number_field sage.symbolic
1.00199015670185
```

super_categories()

EXAMPLES:

```
sage: NumberFields().super_categories()
[Category of infinite fields]
```

4.123 Objects

class sage.categories.objects.**Objects**

Bases: *Category_singleton*

The category of all objects the basic category

EXAMPLES:

```
sage: Objects()
Category of objects
sage: Objects().super_categories()
[]
```

class ParentMethods

Bases: object

Methods for all category objects

class SubcategoryMethods

Bases: object

Endsets()

Return the category of endsets between objects of this category.

EXAMPLES:

```
sage: Sets().Endsets()
Category of endsets of sets

sage: Rings().Endsets()
Category of endsets of unital magmas and additive unital additive magmas
```

See also:

- *Homsets()*

Homsets()

Return the category of homsets between objects of this category.

EXAMPLES:

```
sage: Sets().Homsets()
Category of homsets of sets

sage: Rings().Homsets()
Category of homsets of unital magmas and additive unital additive magmas
```

Note: Background

Information, code, documentation, and tests about the category of homsets of a category Cs should go in the nested class Cs.Homsets. They will then be made available to homsets of any subcategory of Cs.

Assume, for example, that homsets of Cs are Cs themselves. This information can be implemented in the method Cs.Homsets.extra_super_categories to make Cs.Homsets() a subcategory of Cs().

Methods about the homsets themselves should go in the nested class `Cs.Homsets.ParentMethods`.

Methods about the morphisms can go in the nested class `Cs.Homsets.ElementMethods`. However it's generally preferable to put them in the nested class `Cs.MorphimMethods`; indeed they will then apply to morphisms of all subcategories of `Cs`, and not only full subcategories.

See also:

`FunctorialConstruction`

Todo:

- Design a mechanism to specify that an axiom is compatible with taking subsets. Examples: `Finite`, `Associative`, `Commutative` (when meaningful), but not `Infinite` nor `Unital`.
- Design a mechanism to specify that, when B is a subcategory of A, a B-homset is a subset of the corresponding A homset. And use it to recover all the relevant axioms from homsets in super categories.
- For instances of redundant code due to this missing feature, see:
  - `AdditiveMonoids.Homsets.extra_super_categories()`
  - `HomsetsCategory.extra_super_categories()` (slightly different nature)
  - plus plenty of spots where this is not implemented.

additional_structure()

Return `None`

Indeed, by convention, the category of objects defines no additional structure.

See also:

`Category.additional_structure()`

EXAMPLES:

```
sage: Objects().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: Objects().super_categories()
[]
```

4.124 Partially ordered monoids

class sage.categories.partially_ordered_monoids.**PartiallyOrderedMonoids**

Bases: *`Category_singleton`*

The category of partially ordered monoids, that is partially ordered sets which are also monoids, and such that multiplication preserves the ordering: $x \leq y$ implies $x * z < y * z$ and $z * x < z * y$.

See Wikipedia article Ordered_monoid

EXAMPLES:

```
sage: PartiallyOrderedMonoids()
Category of partially ordered monoids
sage: PartiallyOrderedMonoids().super_categories()
[Category of posets, Category of monoids]
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

super_categories()

EXAMPLES:

```
sage: PartiallyOrderedMonoids().super_categories()
[Category of posets, Category of monoids]
```

4.125 Permutation groups

class sage.categories.permutation_groups.**PermutationGroups**

Bases: *Category*

The category of permutation groups.

A *permutation group* is a group whose elements are concretely represented by permutations of some set. In other words, the group comes endowed with a distinguished action on some set.

This distinguished action should be preserved by permutation group morphisms. For details, see Wikipedia article Permutation_group#Permutation_isomorphic_groups.

Todo: shall we accept only permutations with finite support or not?

EXAMPLES:

```
sage: PermutationGroups()
Category of permutation groups
sage: PermutationGroups().super_categories()
[Category of groups]
```

The category of permutation groups defines additional structure that should be preserved by morphisms, namely the distinguished action:

```
sage: PermutationGroups().additional_structure()
Category of permutation groups
```

Finite

alias of *FinitePermutationGroups*

super_categories()

Return a list of the immediate super categories of `self`.

EXAMPLES:

```
sage: PermutationGroups().super_categories()
[Category of groups]
```

4.126 Pointed sets

class sage.categories.pointed_sets.**PointedSets**

Bases: *Category_singleton*

The category of pointed sets.

EXAMPLES:

```
sage: PointedSets()
Category of pointed sets
```

super_categories()

EXAMPLES:

```
sage: PointedSets().super_categories()
[Category of sets]
```

4.127 Polyhedral subsets of free ZZ, QQ or RR-modules.

class sage.categories.polyhedra.**PolyhedralSets**(*R*)

Bases: *Category_over_base_ring*

The category of polyhedra over a ring.

EXAMPLES:

We create the category of polyhedra over **Q**:

```
sage: PolyhedralSets(QQ)
Category of polyhedral sets over Rational Field
```

super_categories()

EXAMPLES:

```
sage: PolyhedralSets(QQ).super_categories()
[Category of commutative magmas, Category of additive monoids]
```

4.128 Posets

class sage.categories.posets.**Posets**

Bases: *Category*

The category of posets i.e. sets with a partial order structure.

EXAMPLES:

```
sage: Posets()
Category of posets
sage: Posets().super_categories()
[Category of sets]
sage: P = Posets().example(); P
An example of a poset: sets ordered by inclusion
```

The partial order is implemented by the mandatory method `le()`:

```
sage: x = P(Set([1,3])); y = P(Set([1,2,3]))
sage: x, y
({1, 3}, {1, 2, 3})
sage: P.le(x, y)
True
sage: P.le(x, x)
True
sage: P.le(y, x)
False
```

The other comparison methods are called `lt()`, `ge()`, `gt()`, following Python's naming convention in `operator`. Default implementations are provided:

```
sage: P.lt(x, x)
False
sage: P.ge(y, x)
True
```

Unless the poset is a facade (see `Sets.Facade`), one can compare directly its elements using the usual Python operators:

```
sage: D = Poset((divisors(30), attrcall("divides")), facade = False)
sage: D(3) <= D(6)
True
sage: D(3) <= D(3)
True
sage: D(3) <= D(5)
False
sage: D(3) < D(3)
False
sage: D(10) >= D(5)
True
```

At this point, this has to be implemented by hand. Once github issue #10130 will be resolved, this will be automatically provided by this category:

```
sage: # not implemented
sage: x < y
True
sage: x < x
False
sage: x <= x
True
sage: y >= x
True
```

See also:

`Poset()`, *`FinitePosets`*, *`LatticePosets`*

class ElementMethods

Bases: `object`

Finite

alias of *FinitePosets*

class ParentMethods

Bases: `object`

CartesianProduct

alias of `CartesianProductPoset`

directed_subset (*elements*, *direction*)

Return the order filter or the order ideal generated by a list of elements.

If `direction` is 'up', the order filter (upper set) is being returned.

If `direction` is 'down', the order ideal (lower set) is being returned.

INPUT:

- elements – a list of elements.
- direction – 'up' or 'down'.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.directed_subset([3, 8], 'up')
[3, 7, 8, 9, 10, 11, 12, 13, 14, 15]
sage: B.directed_subset([7, 10], 'down')
[0, 1, 2, 3, 4, 5, 6, 7, 8, 10]
```

ge (*x*, *y*)

Return whether $x \geq y$ in the poset `self`.

INPUT:

- `x`, `y` – elements of `self`.

This default implementation delegates the work to *le()*.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.ge( 6, 3 )
True
sage: D.ge( 3, 3 )
True
sage: D.ge( 3, 5 )
False
```

gt (*x*, *y*)

Return whether $x > y$ in the poset `self`.

INPUT:

- `x`, `y` – elements of `self`.

This default implementation delegates the work to *lt()*.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.gt( 3, 6 )
False
```

(continues on next page)

(continued from previous page)

```
sage: D.gt( 3, 3 )
False
sage: D.gt( 3, 5 )
False
```

is_antichain_of_poset(*o*)

Return whether an iterable `o` is an antichain of `self`.

INPUT:

- `o` – an iterable (e. g., list, set, or tuple) containing some elements of `self`

OUTPUT:

`True` if the subset of `self` consisting of the entries of `o` is an antichain of `self`, and `False` otherwise.

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")),
....:           facade=True, linear_extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_antichain_of_poset([1, 3])
False
sage: P.is_antichain_of_poset([3, 1])
False
sage: P.is_antichain_of_poset([1, 1, 3])
False
sage: P.is_antichain_of_poset([])
True
sage: P.is_antichain_of_poset([1])
True
sage: P.is_antichain_of_poset([1, 1])
True
sage: P.is_antichain_of_poset([3, 4])
True
sage: P.is_antichain_of_poset([3, 4, 12])
False
sage: P.is_antichain_of_poset([6, 4])
True
sage: P.is_antichain_of_poset(i for i in divisors(12)
....:                         if (2 < i and i < 6))
True
sage: P.is_antichain_of_poset(i for i in divisors(12)
....:                         if (2 <= i and i < 6))
False

sage: Q = Poset({2: [3, 1], 3: [4], 1: [4]})
sage: Q.is_antichain_of_poset((1, 2))
False
sage: Q.is_antichain_of_poset((2, 4))
False
sage: Q.is_antichain_of_poset((4, 2))
False
sage: Q.is_antichain_of_poset((2, 2))
True
sage: Q.is_antichain_of_poset((3, 4))
False
sage: Q.is_antichain_of_poset((3, 1))
```

(continues on next page)

(continued from previous page)

```
True
sage: Q.is_antichain_of_poset((1, ))
True
sage: Q.is_antichain_of_poset(())
True
```

An infinite poset:

```
sage: from sage.categories.examples.posets import␣
↪FiniteSetsOrderedByInclusion
sage: R = FiniteSetsOrderedByInclusion()
sage: R.is_antichain_of_poset([R(set([3, 1, 2])),
....:                          R(set([1, 4])), R(set([4, 5]))])
True
sage: R.is_antichain_of_poset([R(set([3, 1, 2, 4])),
....:                          R(set([1, 4])), R(set([4, 5]))])
False
```

is_chain_of_poset(*o*, *ordered=False*)

Return whether an iterable `o` is a chain of `self`, including a check for `o` being ordered from smallest to largest element if the keyword `ordered` is set to `True`.

INPUT:

- `o` – an iterable (e. g., list, set, or tuple) containing some elements of `self`
- `ordered` – a Boolean (default: `False`) which decides whether the notion of a chain includes being ordered

OUTPUT:

If `ordered` is set to `False`, the truth value of the following assertion is returned: The subset of `self` formed by the elements of `o` is a chain in `self`.

If `ordered` is set to `True`, the truth value of the following assertion is returned: Every element of the list `o` is (strictly!) smaller than its successor in `self`. (This makes no sense if `ordered` is a set.)

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")),
....:           facade=True, linear_extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_chain_of_poset([1, 3])
True
sage: P.is_chain_of_poset([3, 1])
True
sage: P.is_chain_of_poset([1, 3], ordered=True)
True
sage: P.is_chain_of_poset([3, 1], ordered=True)
False
sage: P.is_chain_of_poset([])
True
sage: P.is_chain_of_poset([], ordered=True)
True
sage: P.is_chain_of_poset((2, 12, 6))
True
sage: P.is_chain_of_poset((2, 6, 12), ordered=True)
True
sage: P.is_chain_of_poset((2, 12, 6), ordered=True)
```

(continues on next page)

(continued from previous page)

```
False
sage: P.is_chain_of_poset((2, 12, 6, 3))
False
sage: P.is_chain_of_poset((2, 3))
False

sage: Q = Poset({2: [3, 1], 3: [4], 1: [4]})
sage: Q.is_chain_of_poset([1, 2], ordered=True)
False
sage: Q.is_chain_of_poset([1, 2])
True
sage: Q.is_chain_of_poset([2, 1], ordered=True)
True
sage: Q.is_chain_of_poset([2, 1, 1], ordered=True)
False
sage: Q.is_chain_of_poset([3])
True
sage: Q.is_chain_of_poset([4, 2, 3])
True
sage: Q.is_chain_of_poset([4, 2, 3], ordered=True)
False
sage: Q.is_chain_of_poset([2, 3, 4], ordered=True)
True
```

Examples with infinite posets:

```
sage: from sage.categories.examples.posets import␣
↪FiniteSetsOrderedByInclusion
sage: R = FiniteSetsOrderedByInclusion()
sage: R.is_chain_of_poset([R(set([3, 1, 2])),
....:                      R(set([1, 4])),
....:                      R(set([4, 5]))])
False
sage: R.is_chain_of_poset([R(set([3, 1, 2])),
....:                      R(set([1, 2])),
....:                      R(set([1]))], ordered=True)
False
sage: R.is_chain_of_poset([R(set([3, 1, 2])),
....:                      R(set([1, 2])), R(set([1]))])
True

sage: from sage.categories.examples.posets import␣
↪PositiveIntegersOrderedByDivisibilityFacade
sage: T = PositiveIntegersOrderedByDivisibilityFacade()
sage: T.is_chain_of_poset((T(3), T(4), T(7)))
False
sage: T.is_chain_of_poset((T(3), T(6), T(3)))
True
sage: T.is_chain_of_poset((T(3), T(6), T(3)), ordered=True)
False
sage: T.is_chain_of_poset((T(3), T(3), T(6)))
True
sage: T.is_chain_of_poset((T(3), T(3), T(6)), ordered=True)
False
sage: T.is_chain_of_poset((T(3), T(6)), ordered=True)
True
sage: T.is_chain_of_poset((), ordered=True)
```

(continues on next page)

(continued from previous page)

```
True
sage: T.is_chain_of_poset((T(3),), ordered=True)
True
sage: T.is_chain_of_poset((T(q) for q in divisors(27)))
True
sage: T.is_chain_of_poset((T(q) for q in divisors(18)))
False
```

is_order_filter(*o*)

Return whether `o` is an order filter of `self`, assuming `self` has no infinite ascending path.

INPUT:

- `o` – a list (or set, or tuple) containing some elements of `self`

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")),
....:           facade=True, linear_extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_order_filter([4, 12])
True
sage: P.is_order_filter([])
True
sage: P.is_order_filter({3, 4, 12})
False
sage: P.is_order_filter({3, 6, 12})
True
```

is_order_ideal(*o*)

Return whether `o` is an order ideal of `self`, assuming `self` has no infinite descending path.

INPUT:

- `o` – a list (or set, or tuple) containing some elements of `self`

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")),
....:           facade=True, linear_extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_order_ideal([1, 3])
True
sage: P.is_order_ideal([])
True
sage: P.is_order_ideal({1, 3})
True
sage: P.is_order_ideal([1, 3, 4])
False
```

le(*x*, *y*)

Return whether $x \leq y$ in the poset `self`.

INPUT:

- `x`, `y` – elements of `self`.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.le( 3, 6 )
True
sage: D.le( 3, 3 )
True
sage: D.le( 3, 5 )
False
```

lower_covers(*x*)

Return the lower covers of x, that is, the elements y such that $y < x$ and there exists no z such that $y < z < x$.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.lower_covers(15)
[3, 5]
```

lt(*x*, *y*)

Return whether $x < y$ in the poset `self`.

INPUT:

- `x`, `y` – elements of `self`.

This default implementation delegates the work to `le()`.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.lt( 3, 6 )
True
sage: D.lt( 3, 3 )
False
sage: D.lt( 3, 5 )
False
```

order_filter(*elements*)

Return the order filter generated by a list of elements.

A subset I of a poset is said to be an order filter if, for any x in I and y such that $y \geq x$, then y is in I.

This is also called the upper set generated by these elements.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.order_filter([3,8])
[3, 7, 8, 9, 10, 11, 12, 13, 14, 15]
```

order_ideal(*elements*)

Return the order ideal in `self` generated by the elements of an iterable `elements`.

A subset I of a poset is said to be an order ideal if, for any x in I and y such that $y \leq x$, then y is in I.

This is also called the lower set generated by these elements.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.order_ideal([7,10])
[0, 1, 2, 3, 4, 5, 6, 7, 8, 10]
```

order_ideal_toggle(*I*, *v*)

Return the result of toggling the element `v` in the order ideal `I`.

If v is an element of a poset P, then toggling the element v is an automorphism of the set $J(P)$ of all order ideals of P. It is defined as follows: If I is an order ideal of P, then the image of I under toggling the element v is

- the set $I \cup \{v\}$, if $v \notin I$ but every element of P smaller than v is in I;
- the set $I \setminus \{v\}$, if $v \in I$ but no element of P greater than v is in I;
- I otherwise.

This image always is an order ideal of P.

EXAMPLES:

```
sage: P = Poset({1: [2,3], 2: [4], 3: []})
sage: I = Set({1, 2})
sage: I in P.order_ideals_lattice()                                          #␣
↪needs sage.modules
True
sage: P.order_ideal_toggle(I, 1)
{1, 2}
sage: P.order_ideal_toggle(I, 2)
{1}
sage: P.order_ideal_toggle(I, 3)
{1, 2, 3}
sage: P.order_ideal_toggle(I, 4)
{1, 2, 4}
sage: P4 = Posets(4)
sage: all(all(all(P.order_ideal_toggle(P.order_ideal_toggle(I, i), i) ==␣
↪I          # needs sage.modules
....:                 for i in range(4))
....:             for I in P.order_ideals_lattice(facade=True))
....:         for P in P4)
True
```

order_ideal_toggles(*I*, *vs*)

Return the result of toggling the elements of the list (or iterable) `vs` (one by one, from left to right) in the order ideal `I`.

See *order_ideal_toggle()* for a definition of toggling.

EXAMPLES:

```
sage: P = Poset({1: [2,3], 2: [4], 3: []})
sage: I = Set({1, 2})
sage: P.order_ideal_toggles(I, [1,2,3,4])
{1, 3}
sage: P.order_ideal_toggles(I, (1,2,3,4))
{1, 3}
```

principal_lower_set(*x*)

Return the order ideal generated by an element `x`.

This is also called the lower set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_ideal(6)
[0, 2, 4, 6]
```

principal_order_filter(*x*)

Return the order filter generated by an element x.

This is also called the upper set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_filter(2)
[2, 3, 6, 7, 10, 11, 14, 15]
```

principal_order_ideal(*x*)

Return the order ideal generated by an element x.

This is also called the lower set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_ideal(6)
[0, 2, 4, 6]
```

principal_upper_set(*x*)

Return the order filter generated by an element x.

This is also called the upper set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_filter(2)
[2, 3, 6, 7, 10, 11, 14, 15]
```

upper_covers(*x*)

Return the upper covers of x, that is, the elements y such that $x < y$ and there exists no z such that $x < z < y$.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.upper_covers(3)
[6, 15]
```

example(*choice=None*)

Return examples of objects of `Posets()`, as per *`Category.example()`*.

EXAMPLES:

```
sage: Posets().example()
An example of a poset: sets ordered by inclusion

sage: Posets().example("facade")
An example of a facade poset:
 the positive integers ordered by divisibility
```

super_categories()

Return a list of the (immediate) super categories of `self`, as per *Category.super_categories()*.

EXAMPLES:

```
sage: Posets().super_categories()
[Category of sets]
```

4.129 Principal ideal domains

class sage.categories.principal_ideal_domains.**PrincipalIdealDomains**

Bases: *Category_singleton*

The category of (constructive) principal ideal domains

By constructive, we mean that a single generator can be constructively found for any ideal given by a finite set of generators. Note that this constructive definition only implies that finitely generated ideals are principal. It is not clear what we would mean by an infinitely generated ideal.

EXAMPLES:

```
sage: PrincipalIdealDomains()
Category of principal ideal domains
sage: PrincipalIdealDomains().super_categories()
[Category of unique factorization domains]
```

See also Wikipedia article Principal_ideal_domain

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

additional_structure()

Return `None`.

Indeed, the category of principal ideal domains defines no additional structure: a ring morphism between two principal ideal domains is a principal ideal domain morphism.

EXAMPLES:

```
sage: PrincipalIdealDomains().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: PrincipalIdealDomains().super_categories()
[Category of unique factorization domains]
```

4.130 Quotient fields

class sage.categories.quotient_fields.**QuotientFields**

Bases: *Category_singleton*

The category of quotient fields over an integral domain

EXAMPLES:

```
sage: QuotientFields()
Category of quotient fields
sage: QuotientFields().super_categories()
[Category of fields]
```

class ElementMethods

Bases: object

denominator()

Constructor for abstract methods

EXAMPLES:

```
sage: def f(x):
....:     "doc of f"
....:     return 1
sage: x = abstract_method(f); x
<abstract method f at ...>
sage: x.__doc__
'doc of f'
sage: x.__name__
'f'
sage: x.__module__
'__main__'
```

derivative(**args*)

The derivative of this rational function, with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

See also:

_derivative()

EXAMPLES:

```
sage: F.<x> = Frac(QQ['x'])
sage: (1/x).derivative()
-1/x^2
```

```
sage: (x+1/x).derivative(x, 2)
2/x^3
```

```
sage: F.<x,y> = Frac(QQ['x,y'])
sage: (1/(x+y)).derivative(x,y)
2/(x^3 + 3*x^2*y + 3*x*y^2 + y^3)
```

factor(**args*, ***kwds*)

Return the factorization of `self` over the base ring.

INPUT:

- `*args` - Arbitrary arguments suitable over the base ring
- `**kwds` - Arbitrary keyword arguments suitable over the base ring

OUTPUT:

- Factorization of `self` over the base ring

EXAMPLES:

```
sage: K.<x> = QQ[]
sage: f = (x^3+x)/(x-3)
sage: f.factor()                                                            #␣
↪needs sage.libs.pari
(x - 3)^-1 * x * (x^2 + 1)
```

Here is an example to show that github issue #7868 has been resolved:

```
sage: R.<x,y> = GF(2)[]
sage: f = x*y/(x+y)
sage: f.factor()                                                            #␣
↪needs sage.rings.finite_rings
(x + y)^-1 * y * x
```

gcd(*other*)

Greatest common divisor

Note: In a field, the greatest common divisor is not very informative, as it is only determined up to a unit. But in the fraction field of an integral domain that provides both gcd and lcm, it is possible to be a bit more specific and define the gcd uniquely up to a unit of the base ring (rather than in the fraction field).

AUTHOR:

- Simon King (2011-02): See github issue #10771

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: R.<x> = QQ['x']
sage: p = (1+x)^3*(1+2*x^2)/(1-x^5)
sage: q = (1+x)^2*(1+3*x^2)/(1-x^4)
sage: factor(p)
(-2) * (x - 1)^-1 * (x + 1)^3 * (x^2 + 1/2) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(q)
(-3) * (x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^2 + 1/3)
sage: gcd(p, q)
(x + 1)/(x^7 + x^5 - x^2 - 1)
sage: factor(gcd(p, q))
(x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(gcd(p, 1 + x))
(x - 1)^-1 * (x + 1) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(gcd(1 + x, q))
(x - 1)^-1 * (x + 1) * (x^2 + 1)^-1
```

lcm(*other*)

Least common multiple

In a field, the least common multiple is not very informative, as it is only determined up to a unit. But in the fraction field of an integral domain that provides both gcd and lcm, it is reasonable to be a bit more specific and to define the least common multiple so that it restricts to the usual least common multiple in the base ring and is unique up to a unit of the base ring (rather than up to a unit of the fraction field).

The least common multiple is easily described in terms of the prime decomposition. A rational number can be written as a product of primes with integer (positive or negative) powers in a unique way. The least common multiple of two rational numbers x and y can then be defined by specifying that the exponent of every prime p in $lcm(x, y)$ is the supremum of the exponents of p in x, and the exponent of p in y (where the primes that does not appear in the decomposition of x or y are considered to have exponent zero).

AUTHOR:

- Simon King (2011-02): See github issue #10771

EXAMPLES:

```
sage: lcm(2/3, 1/5)
2
```

Indeed $2/3 = 2^1 3^{-1} 5^0$ and $1/5 = 2^0 3^0 5^{-1}$, so $lcm(2/3, 1/5) = 2^1 3^0 5^0 = 2$.

sage: lcm(1/3, 1/5) 1 sage: lcm(1/3, 1/6) 1/3

Some more involved examples:

```
sage: # needs sage.libs.pari
sage: R.<x> = QQ[]
sage: p = (1+x)^3*(1+2*x^2)/(1-x^5)
sage: q = (1+x)^2*(1+3*x^2)/(1-x^4)
sage: factor(p)
(-2) * (x - 1)^-1 * (x + 1)^3 * (x^2 + 1/2) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(q)
(-3) * (x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^2 + 1/3)
sage: factor(lcm(p, q))
(x - 1)^-1 * (x + 1)^3 * (x^2 + 1/3) * (x^2 + 1/2)
sage: factor(lcm(p, 1 + x))
(x + 1)^3 * (x^2 + 1/2)
sage: factor(lcm(1 + x, q))
(x + 1) * (x^2 + 1/3)
```

numerator()

Constructor for abstract methods

EXAMPLES:

```
sage: def f(x):
....:     "doc of f"
....:     return 1
sage: x = abstract_method(f); x
<abstract method f at ...>
sage: x.__doc__
'doc of f'
sage: x.__name__
'f'
sage: x.__module__
'__main__'
```

partial_fraction_decomposition(*decompose_powers=True*)

Decompose fraction field element into a whole part and a list of fraction field elements over prime power denominators.

The sum will be equal to the original fraction.

INPUT:

- `decompose_powers` – boolean (default: `True`); whether to decompose prime power denominators as opposed to having a single term for each irreducible factor of the denominator

OUTPUT:

Partial fraction decomposition of `self` over the base ring.

AUTHORS:

- Robert Bradshaw (2007-05-31)

EXAMPLES:

```
sage: # needs sage.libs.pari
sage: S.<t> = QQ[]
sage: q = 1/(t+1) + 2/(t+2) + 3/(t-3); q
(6*t^2 + 4*t - 6)/(t^3 - 7*t - 6)
sage: whole, parts = q.partial_fraction_decomposition(); parts
[3/(t - 3), 1/(t + 1), 2/(t + 2)]
sage: sum(parts) == q
True
sage: q = 1/(t^3+1) + 2/(t^2+2) + 3/(t-3)^5
sage: whole, parts = q.partial_fraction_decomposition(); parts
[1/3/(t + 1), 3/(t^5 - 15*t^4 + 90*t^3 - 270*t^2 + 405*t - 243),
 (-1/3*t + 2/3)/(t^2 - t + 1), 2/(t^2 + 2)]
sage: sum(parts) == q
True
sage: q = 2*t / (t + 3)^2
sage: q.partial_fraction_decomposition()
(0, [2/(t + 3), -6/(t^2 + 6*t + 9)])
sage: for p in q.partial_fraction_decomposition()[1]:
....:     print(p.factor())
(2) * (t + 3)^-1
(-6) * (t + 3)^-2
sage: q.partial_fraction_decomposition(decompose_powers=False)
(0, [2*t/(t^2 + 6*t + 9)])
```

We can decompose over a given algebraic extension:

```
sage: R.<x> = QQ[sqrt(2)][]                                                    #␣
↪needs sage.rings.number_field sage.symbolic
sage: r = 1/(x^4+1)                                                            #␣
↪needs sage.rings.number_field sage.symbolic
sage: r.partial_fraction_decomposition()                                       #␣
↪needs sage.rings.number_field sage.symbolic
(0,
 [(-1/4*sqrt2*x + 1/2)/(x^2 - sqrt2*x + 1),
  (1/4*sqrt2*x + 1/2)/(x^2 + sqrt2*x + 1)])

sage: R.<x> = QQ[I][]  # of QQ[sqrt(-1)]                                       #␣
↪needs sage.rings.number_field sage.symbolic
sage: r =  1/(x^4+1)                                                           #␣
↪needs sage.rings.number_field sage.symbolic
sage: r.partial_fraction_decomposition()                                       #␣
↪needs sage.rings.number_field sage.symbolic
(0, [(-1/2*I)/(x^2 - I), 1/2*I/(x^2 + I)])
```

We can also ask Sage to find the least extension where the denominator factors in linear terms:

```
sage: # needs sage.rings.number_field
sage: R.<x> = QQ[]
sage: r = 1/(x^4+2)
sage: N = r.denominator().splitting_field('a'); N
Number Field in a with defining polynomial x^8 - 8*x^6 + 28*x^4 + 16*x^2␣
↪+ 36
sage: R1.<x1> = N[]
sage: r1 = 1/(x1^4+2)
sage: r1.partial_fraction_decomposition()
(0,
 [(-1/224*a^6 + 13/448*a^4 - 5/56*a^2 - 25/224)/(x1 - 1/28*a^6 + 13/56*a^
↪4 - 5/7*a^2 - 25/28),
  (1/224*a^6 - 13/448*a^4 + 5/56*a^2 + 25/224)/(x1 + 1/28*a^6 - 13/56*a^4␣
↪+ 5/7*a^2 + 25/28),
  (-5/1344*a^7 + 43/1344*a^5 - 85/672*a^3 - 31/672*a)/(x1 - 5/168*a^7 +␣
↪43/168*a^5 - 85/84*a^3 - 31/84*a),
  (5/1344*a^7 - 43/1344*a^5 + 85/672*a^3 + 31/672*a)/(x1 + 5/168*a^7 - 43/
↪168*a^5 + 85/84*a^3 + 31/84*a)])
```

Or we may work directly over an algebraically closed field:

```
sage: R.<x> = QQbar[]                                                  #␣
↪needs sage.rings.number_field
sage: r =  1/(x^4+1)                                                   #␣
↪needs sage.rings.number_field
sage: r.partial_fraction_decomposition()                               #␣
↪needs sage.rings.number_field
(0,
 [(-0.1767766952966369? - 0.1767766952966369?*I)/(x - 0.7071067811865475?␣
↪- 0.7071067811865475?*I),
  (-0.1767766952966369? + 0.1767766952966369?*I)/(x - 0.7071067811865475?␣
↪+ 0.7071067811865475?*I),
  (0.1767766952966369? - 0.1767766952966369?*I)/(x + 0.7071067811865475? -
↪ 0.7071067811865475?*I),
  (0.1767766952966369? + 0.1767766952966369?*I)/(x + 0.7071067811865475?␣
↪+ 0.7071067811865475?*I)])
```

We do the best we can over inexact fields:

```
sage: # needs sage.rings.number_field sage.rings.real_mpfr
sage: R.<x> = RealField(20)[]
sage: q = 1/(x^2 + x + 2)^2 + 1/(x-1); q
(x^4 + 2.0000*x^3
  + 5.0000*x^2 + 5.0000*x + 3.0000)/(x^5 + x^4 + 3.0000*x^3 - x^2 - 4.
↪0000)
sage: whole, parts = q.partial_fraction_decomposition(); parts
[1.0000/(x - 1.0000),
 1.0000/(x^4 + 2.0000*x^3 + 5.0000*x^2 + 4.0000*x + 4.0000)]
sage: sum(parts)
(x^4 + 2.0000*x^3
  + 5.0000*x^2 + 5.0000*x + 3.0000)/(x^5 + x^4 + 3.0000*x^3 - x^2 - 4.
↪0000)
```

xgcd(*other*)

Return a triple `(g,s,t)` of elements of that field such that `g` is the greatest common divisor of `self` and `other` and `g = s*self + t*other`.

Note: In a field, the greatest common divisor is not very informative, as it is only determined up to a unit. But in the fraction field of an integral domain that provides both xgcd and lcm, it is possible to be a bit more specific and define the gcd uniquely up to a unit of the base ring (rather than in the fraction field).

EXAMPLES:

```
sage: QQ(3).xgcd(QQ(2))
(1, 1, -1)
sage: QQ(3).xgcd(QQ(1/2))
(1/2, 0, 1)
sage: QQ(1/3).xgcd(QQ(2))
(1/3, 1, 0)
sage: QQ(3/2).xgcd(QQ(5/2))
(1/2, 2, -1)

sage: R.<x> = QQ['x']
sage: p = (1+x)^3*(1+2*x^2)/(1-x^5)
sage: q = (1+x)^2*(1+3*x^2)/(1-x^4)
sage: factor(p)                                                              #␣
↪needs sage.libs.pari
(-2) * (x - 1)^-1 * (x + 1)^3 * (x^2 + 1/2) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(q)                                                              #␣
↪needs sage.libs.pari
(-3) * (x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^2 + 1/3)
sage: g, s, t = xgcd(p, q)
sage: g
(x + 1)/(x^7 + x^5 - x^2 - 1)
sage: g == s*p + t*q
True
```

An example without a well defined gcd or xgcd on its base ring:

```
sage: # needs sage.rings.number_field
sage: K = QuadraticField(5)
sage: O = K.maximal_order()
sage: R = PolynomialRing(O, 'x')
sage: F = R.fraction_field()
sage: x = F.gen(0)
sage: x.gcd(x+1)
1
sage: x.xgcd(x+1)
(1, 1/x, 0)
sage: zero = F.zero()
sage: zero.gcd(x)
1
sage: zero.xgcd(x)
(1, 0, 1/x)
sage: zero.xgcd(zero)
(0, 0, 0)
```

class ParentMethods

Bases: `object`

super_categories()

EXAMPLES:

```
sage: QuotientFields().super_categories()
[Category of fields]
```

4.131 Quantum Group Representations

AUTHORS:

- Travis Scrimshaw (2018): initial version

class sage.categories.quantum_group_representations.**QuantumGroupRepresentations**(*base*, *name=None*)

Bases: *Category_module*

The category of quantum group representations.

class ParentMethods

Bases: object

cartan_type()

Return the Cartan type of self.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['C',4], shape=[1])
sage: R = ZZ['q'].fraction_field()
sage: V = MinusculeRepresentation(R, C)
sage: V.cartan_type()
['C', 4]
```

index_set()

Return the index set of self.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['C',4], shape=[1])
sage: R = ZZ['q'].fraction_field()
sage: V = MinusculeRepresentation(R, C)
sage: V.index_set()
(1, 2, 3, 4)
```

q()

Return the quantum parameter q of self.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['C',4], shape=[1])
sage: R = ZZ['q'].fraction_field()
```

(continues on next page)

(continued from previous page)

```
sage: V = MinusculeRepresentation(R, C)
sage: V.q()
q
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of quantum group representations constructed by tensor product of quantum group representations.

Warning: We use the reversed coproduct in order to match the tensor product rule on crystals.

class ParentMethods

Bases: `object`

cartan_type()

Return the Cartan type of `self`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['C',2], shape=[1])
sage: R = ZZ['q'].fraction_field()
sage: V = MinusculeRepresentation(R, C)
sage: T = tensor([V,V])
sage: T.cartan_type()
['C', 2]
```

extra_super_categories()

EXAMPLES:

```
sage: from sage.categories.quantum_group_representations import␣
↪QuantumGroupRepresentations
sage: Cat = QuantumGroupRepresentations(ZZ['q'].fraction_field())
sage: Cat.TensorProducts().extra_super_categories()
[Category of quantum group representations over
 Fraction Field of Univariate Polynomial Ring in q over Integer Ring]
```

class WithBasis(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of quantum group representations with a distinguished basis.

class ElementMethods

Bases: `object`

K(*i*, *power=1*)

Return the action of K_i on `self` to the power `power`.

INPUT:
- `i` – an element of the index set
- `power` – (default: 1) the power of K_i

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: K = crystals.KirillovReshetikhin(['D',4,2], 1,1)
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, K)
sage: v = V.an_element(); v
2*B[[]] + 2*B[[[1]]] + 3*B[[[2]]]
sage: v.K(0)
2*B[[]] + 2/q^2*B[[[1]]] + 3*B[[[2]]]
sage: v.K(1)
2*B[[]] + 2*q^2*B[[[1]]] + 3/q^2*B[[[2]]]
sage: v.K(1, 2)
2*B[[]] + 2*q^4*B[[[1]]] + 3/q^4*B[[[2]]]
sage: v.K(1, -1)
2*B[[]] + 2/q^2*B[[[1]]] + 3*q^2*B[[[2]]]
```

e (*i*)

Return the action of e_i on `self`.

INPUT:

- `i` – an element of the index set

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: C = crystals.Tableaux(['G',2], shape=[1,1])
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, C)
sage: v = V.an_element(); v
2*B[[[1], [2]]] + 2*B[[[1], [3]]] + 3*B[[[2], [3]]]
sage: v.e(1)
((3*q^4+3*q^2+3)/q^2)*B[[[1], [3]]]
sage: v.e(2)
2*B[[[1], [2]]]
```

f (*i*)

Return the action of f_i on `self`.

INPUT:

- `i` – an element of the index set

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, K)
sage: v = V.an_element(); v
2*B[[]] + 2*B[[[1], [2]]] + 3*B[[[1], [3]]]
sage: v.f(0)
((2*q^2+2)/q)*B[[[1], [2]]]
sage: v.f(1)
3*B[[[2], [3]]]
sage: v.f(2)
```

(continues on next page)

(continued from previous page)

```
2*B[[[1], [3]]]
sage: v.f(3)
3*B[[[1], [4]]]
sage: v.f(4)
3*B[[[1], [-4]]]
```

class ParentMethods

Bases: `object`

tensor(**factors*)

Return the tensor product of `self` with the representations `factors`.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import (
....:     MinusculeRepresentation, AdjointRepresentation)
sage: R = ZZ['q'].fraction_field()
sage: CM = crystals.Tableaux(['D',4], shape=[1])
sage: CA = crystals.Tableaux(['D',4], shape=[1,1])
sage: V = MinusculeRepresentation(R, CM)
sage: V.tensor(V, V)
V((1, 0, 0, 0)) # V((1, 0, 0, 0)) # V((1, 0, 0, 0))
sage: A = MinusculeRepresentation(R, CA)
sage: V.tensor(A)
V((1, 0, 0, 0)) # V((1, 1, 0, 0))
sage: B = crystals.Tableaux(['A',2], shape=[1])
sage: W = MinusculeRepresentation(R, B)
sage: tensor([W,V])
Traceback (most recent call last):
...
ValueError: all factors must be of the same Cartan type
sage: tensor([V,A,W])
Traceback (most recent call last):
...
ValueError: all factors must be of the same Cartan type
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of quantum group representations with a distinguished basis constructed by tensor product of quantum group representations with a distinguished basis.

class ParentMethods

Bases: `object`

K_on_basis(*i*, *b*, *power=1*)

Return the action of K_i on the basis element indexed by `b` to the power `power`.

INPUT:

- `i` – an element of the index set
- `b` – an element of basis keys
- `power` – (default: 1) the power of K_i

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import (
```

(continues on next page)

(continued from previous page)

```
....:     MinusculeRepresentation, AdjointRepresentation)
sage: R = ZZ['q'].fraction_field()
sage: CM = crystals.Tableaux(['A',2], shape=[1])
sage: VM = MinusculeRepresentation(R, CM)
sage: CA = crystals.Tableaux(['A',2], shape=[2,1])
sage: VA = AdjointRepresentation(R, CA)
sage: v = tensor([sum(VM.basis()), VA.module_generator()]); v
B[[[1]]] # B[[[1, 1], [2]]]
 + B[[[2]]] # B[[[1, 1], [2]]]
 + B[[[3]]] # B[[[1, 1], [2]]]
sage: v.K(1)  # indirect doctest
q^2*B[[[1]]] # B[[[1, 1], [2]]]
 + B[[[2]]] # B[[[1, 1], [2]]]
 + q*B[[[3]]] # B[[[1, 1], [2]]]
sage: v.K(2, -1)  # indirect doctest
1/q*B[[[1]]] # B[[[1, 1], [2]]]
 + 1/q^2*B[[[2]]] # B[[[1, 1], [2]]]
 + B[[[3]]] # B[[[1, 1], [2]]]
```

e_on_basis (i, b)

Return the action of e_i on the basis element indexed by `b`.

INPUT:

- `i` – an element of the index set
- `b` – an element of basis keys

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import (
....:     MinusculeRepresentation, AdjointRepresentation)
sage: R = ZZ['q'].fraction_field()
sage: CM = crystals.Tableaux(['D',4], shape=[1])
sage: VM = MinusculeRepresentation(R, CM)
sage: CA = crystals.Tableaux(['D',4], shape=[1,1])
sage: VA = AdjointRepresentation(R, CA)
sage: v = tensor([VM.an_element(), VA.an_element()]); v
4*B[[[1]]] # B[[[1], [2]]] + 4*B[[[1]]] # B[[[1], [3]]]
 + 6*B[[[1]]] # B[[[2], [3]]] + 4*B[[[2]]] # B[[[1], [2]]]
 + 4*B[[[2]]] # B[[[1], [3]]] + 6*B[[[2]]] # B[[[2], [3]]]
 + 6*B[[[3]]] # B[[[1], [2]]] + 6*B[[[3]]] # B[[[1], [3]]]
 + 9*B[[[3]]] # B[[[2], [3]]]
sage: v.e(1)  # indirect doctest
4*B[[[1]]] # B[[[1], [2]]]
 + ((4*q+6)/q)*B[[[1]]] # B[[[1], [3]]]
 + 6*B[[[1]]] # B[[[2], [3]]]
 + 6*q*B[[[2]]] # B[[[1], [3]]]
 + 9*B[[[3]]] # B[[[1], [3]]]
sage: v.e(2)  # indirect doctest
4*B[[[1]]] # B[[[1], [2]]]
 + ((6*q+4)/q)*B[[[2]]] # B[[[1], [2]]]
 + 6*B[[[2]]] # B[[[1], [3]]]
 + 9*B[[[2]]] # B[[[2], [3]]]
 + 6*q*B[[[3]]] # B[[[1], [2]]]
sage: v.e(3)  # indirect doctest
0
sage: v.e(4)  # indirect doctest
0
```

f_on_basis (i, b)

Return the action of f_i on the basis element indexed by b.

INPUT:

- i – an element of the index set
- b – an element of basis keys

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: from sage.algebras.quantum_groups.representations import (
....:     MinusculeRepresentation, AdjointRepresentation)
sage: R = ZZ['q'].fraction_field()
sage: KM = crystals.KirillovReshetikhin(['B',3,1], 3,1)
sage: VM = MinusculeRepresentation(R, KM)
sage: KA = crystals.KirillovReshetikhin(['B',3,1], 2,1)
sage: VA = AdjointRepresentation(R, KA)
sage: v = tensor([VM.an_element(), VA.an_element()]); v
4*B[[+++, []]] # B[[]] + 4*B[[+++, []]] # B[[[1], [2]]]
 + 6*B[[+++, []]] # B[[[1], [3]]] + 4*B[[++-, []]] # B[[]]
 + 4*B[[++-, []]] # B[[[1], [2]]]
 + 6*B[[++-, []]] # B[[[1], [3]]] + 6*B[[+-+, []]] # B[[]]
 + 6*B[[+-+, []]] # B[[[1], [2]]]
 + 9*B[[+-+, []]] # B[[[1], [3]]]
sage: v.f(0)  # indirect doctest
((4*q^4+4)/q^2)*B[[+++, []]] # B[[[1], [2]]]
 + ((4*q^4+4)/q^2)*B[[++-, []]] # B[[[1], [2]]]
 + ((6*q^4+6)/q^2)*B[[+-+, []]] # B[[[1], [2]]]
sage: v.f(1)  # indirect doctest
6*B[[+++, []]] # B[[[2], [3]]]
 + 6*B[[++-, []]] # B[[[2], [3]]]
 + 9*B[[+-+, []]] # B[[[2], [3]]]
 + 6*B[[-++, []]] # B[[]]
 + 6*B[[-++, []]] # B[[[1], [2]]]
 + 9*q^2*B[[-++, []]] # B[[[1], [3]]]
sage: v.f(2)  # indirect doctest
4*B[[+++, []]] # B[[[1], [3]]]
 + 4*B[[++-, []]] # B[[[1], [3]]]
 + 4*B[[+-+, []]] # B[[]]
 + 4*q^2*B[[+-+, []]] # B[[[1], [2]]]
 + ((6*q^2+6)/q^2)*B[[+-+, []]] # B[[[1], [3]]]
sage: v.f(3)  # indirect doctest
6*B[[+++, []]] # B[[[1], [0]]]
 + 4*B[[++-, []]] # B[[]]
 + 4*B[[++-, []]] # B[[[1], [2]]]
 + 6*q^2*B[[++-, []]] # B[[[1], [3]]]
 + 6*B[[++-, []]] # B[[[1], [0]]]
 + 9*B[[+-+, []]] # B[[[1], [0]]]
 + 6*B[[+--, []]] # B[[]]
 + 6*B[[+--, []]] # B[[[1], [2]]]
 + 9*q^2*B[[+--, []]] # B[[[1], [3]]]
```

extra_super_categories ()

EXAMPLES:

```
sage: from sage.categories.quantum_group_representations import␣
↪QuantumGroupRepresentations
sage: Cat = QuantumGroupRepresentations(ZZ['q'].fraction_field())
sage: Cat.WithBasis().TensorProducts().extra_super_categories()
```

(continues on next page)

(continued from previous page)

```
[Category of quantum group representations with basis over
  Fraction Field of Univariate Polynomial Ring in q over Integer Ring]
```

example()

Return an example of a quantum group representation as per *Category.example*.

EXAMPLES:

```
sage: from sage.categories.quantum_group_representations import␣
↪QuantumGroupRepresentations
sage: Cat = QuantumGroupRepresentations(ZZ['q'].fraction_field())
sage: Cat.example()                                                   #␣
↪needs sage.combinat sage.modules
V((2, 1, 0))
```

super_categories()

Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.quantum_group_representations import␣
↪QuantumGroupRepresentations
sage: QuantumGroupRepresentations(ZZ['q'].fraction_field()).super_categories()
[Category of vector spaces over
  Fraction Field of Univariate Polynomial Ring in q over Integer Ring]
```

4.132 Regular Crystals

class sage.categories.regular_crystals.**RegularCrystals**

Bases: *Category_singleton*

The category of regular crystals.

A crystal is called *regular* if every vertex b satisfies

$$\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq 0\} \quad \text{and} \quad \varphi_i(b) = \max\{k \mid f_i^k(b) \neq 0\}.$$

Note: Regular crystals are sometimes referred to as *normal*. When only one of the conditions (on either φ_i or ε_i) holds, these crystals are sometimes called *seminormal* or *semiregular*.

EXAMPLES:

```
sage: C = RegularCrystals()
sage: C
Category of regular crystals
sage: C.super_categories()
[Category of crystals]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

class ElementMethods

Bases: `object`

demazure_operator_simple(*i*, *ring=None*)

Return the Demazure operator D_i applied to `self`.

INPUT:

- `i` – an element of the index set of the underlying crystal
- `ring` – (default: `QQ`) a ring

OUTPUT:

An element of the `ring`-free module indexed by the underlying crystal.

Let $r = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$, then $D_i(b)$ is defined as follows:

- If $r \geq 0$, this returns the sum of the elements obtained from `self` by application of f_i^k for $0 \leq k \leq r$.
- If $r < 0$, this returns the opposite of the sum of the elements obtained by application of e_i^k for $0 < k < -r$.

REFERENCES:

- [Li1995]
- [Ka1993]

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,2],[2]])
sage: t.demazure_operator_simple(2)
B[[[1, 2], [2]]] + B[[[1, 3], [2]]] + B[[[1, 3], [3]]]
sage: t.demazure_operator_simple(2).parent()
Algebra of The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]]
        over Integer Ring

sage: t.demazure_operator_simple(1)
0

sage: K = crystals.KirillovReshetikhin(['A',2,1],2,1)
sage: t = K(rows=[[3],[2]])
sage: t.demazure_operator_simple(0)
B[[[1, 2]]] + B[[[2, 3]]]
```

dual_equivalence_class(*index_set=None*)

Return the dual equivalence class indexed by `index_set` of `self`.

The dual equivalence class of an element $b \in B$ is the set of all elements of B reachable from b via sequences of i-elementary dual equivalence relations (i.e., i-elementary dual equivalence transformations and their inverses) for i in the index set of B.

For this to be well-defined, the element b has to be of weight 0 with respect to I; that is, we need to have $\varepsilon_j(b) = \varphi_j(b)$ for all $j \in I$.

See [As2008]. See also `dual_equivalence_graph()` for a definition of i-elementary dual equivalence transformations.

INPUT:

- `index_set` – (optional) the index set I (default: the whole index set of the crystal); this has to be a subset of the index set of the crystal (as a list or tuple)

OUTPUT:

The dual equivalence class of `self` indexed by the subset `index_set`. This class is returned as an undirected edge-colored multigraph. The color of an edge is the index i of the dual equivalence relation it encodes.

See also:

- *dual_equivalence_graph()*
- sage.combinat.partition.Partition.dual_equivalence_graph()

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape=[2,2])
sage: G = T(2,1,4,3).dual_equivalence_class()
sage: G.edges(sort=True)
[([[1, 3], [2, 4]], [[1, 2], [3, 4]], 2),
 ([[1, 3], [2, 4]], [[1, 2], [3, 4]], 3)]
sage: T = crystals.Tableaux(['A',4], shape=[3,2])
sage: G = T(2,1,4,3,5).dual_equivalence_class()
sage: G.edges(sort=True)
[([[1, 3, 5], [2, 4]], [[1, 3, 4], [2, 5]], 4),
 ([[1, 3, 5], [2, 4]], [[1, 2, 5], [3, 4]], 2),
 ([[1, 3, 5], [2, 4]], [[1, 2, 5], [3, 4]], 3),
 ([[1, 3, 4], [2, 5]], [[1, 2, 4], [3, 5]], 2),
 ([[1, 2, 4], [3, 5]], [[1, 2, 3], [4, 5]], 3),
 ([[1, 2, 4], [3, 5]], [[1, 2, 3], [4, 5]], 4)]
```

epsilon(*i*)

Return ε_i of self.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).epsilon(1)
0
sage: C(2).epsilon(1)
1
```

phi(*i*)

Return φ_i of self.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).phi(1)
1
sage: C(2).phi(1)
0
```

stembridgeDel_depth(*i*, *j*)

Return the difference in the j-depth of self and f_i of self, where i and j are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The i-depth of a crystal node x is $\varepsilon_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,1],[2]])
sage: t.stembridgeDel_depth(1,2)
0
sage: s = T(rows=[[1,3],[3]])
sage: s.stembridgeDel_depth(1,2)
-1
```

stembridgeDel_rise (i, j)

Return the difference in the j-rise of self and f_i of self, where i and j are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The i-rise of a crystal node x is $\varphi_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,1],[2]])
sage: t.stembridgeDel_rise(1,2)
-1
sage: s = T(rows=[[1,3],[3]])
sage: s.stembridgeDel_rise(1,2)
0
```

stembridgeDelta_depth (i, j)

Return the difference in the j-depth of self and e_i of self, where i and j are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The i-depth of a crystal node x is $-\varepsilon_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,2],[2]])
sage: t.stembridgeDelta_depth(1,2)
0
sage: s = T(rows=[[2,3],[3]])
sage: s.stembridgeDelta_depth(1,2)
-1
```

stembridgeDelta_rise (i, j)

Return the difference in the j-rise of self and e_i of self, where i and j are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The i-rise of a crystal node x is $\varphi_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,2],[2]])
sage: t.stembridgeDelta_rise(1,2)
-1
sage: s = T(rows=[[2,3],[3]])
sage: s.stembridgeDelta_rise(1,2)
0
```

stembridgeTriple (i, j)

Let A be the Cartan matrix of the crystal, x a crystal element, and let i and j be in the index set of the crystal. Further, set b=stembridgeDelta_depth(x,i,j), and c=stembridgeDelta_rise(x,i,j)). If x.e(i) is non-empty, this function returns the triple (A_{ij}, b, c); otherwise it returns None. By the Stembridge local characterization of crystal bases, one should have $A_{ij} = b + c$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,1],[2]])
sage: t.stembridgeTriple(1,2)
sage: s = T(rows=[[1,2],[2]])
sage: s.stembridgeTriple(1,2)
(-1, 0, -1)

sage: T = crystals.Tableaux(['B',2], shape=[2,1])
sage: t = T(rows=[[1,2],[2]])
sage: t.stembridgeTriple(1,2)
(-2, 0, -2)
sage: s = T(rows=[[-1,-1],[0]])
sage: s.stembridgeTriple(1,2)
(-2, -2, 0)
sage: u = T(rows=[[0,2],[1]])
sage: u.stembridgeTriple(1,2)
(-2, -1, -1)
```

weight()

Return the weight of this crystal element.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).weight()
(1, 0, 0, 0, 0, 0)
```

class MorphismMethods

Bases: `object`

is_isomorphism()

Check if `self` is a crystal isomorphism, which is true if and only if this is a strict embedding with the same number of connected components.

EXAMPLES:

```
sage: A21 = RootSystem(['A',2,1])
sage: La = A21.weight_space(extended=True).fundamental_weights()
sage: B = crystals.LSPaths(La[0])
sage: La = A21.weight_lattice(extended=True).fundamental_weights()
sage: C = crystals.GeneralizedYoungWalls(2, La[0])
sage: H = Hom(B, C)
sage: from sage.categories.highest_weight_crystals import␣
↪HighestWeightCrystalMorphism
sage: class Psi(HighestWeightCrystalMorphism):
....:     def is_strict(self):
....:         return True
sage: psi = Psi(H, C.module_generators); psi
['A', 2, 1] Crystal morphism:
  From: The crystal of LS paths of type ['A', 2, 1] and weight Lambda[0]
  To:   Highest weight crystal of generalized Young walls
        of Cartan type ['A', 2, 1] and highest weight Lambda[0]
  Defn: (Lambda[0],) |--> []
sage: psi.is_isomorphism()
True
```

class ParentMethods

Bases: `object`

demazure_operator(*element*, *reduced_word*)

Returns the application of Demazure operators D_i for i from `reduced_word` on `element`.

INPUT:

- `element` – an element of a free module indexed by the underlying crystal
- `reduced_word` – a reduced word of the Weyl group of the same type as the underlying crystal

OUTPUT:

- an element of the free module indexed by the underlying crystal

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = CombinatorialFreeModule(QQ, T)
sage: t = T.highest_weight_vector()
sage: b = 2*C(t)
sage: T.demazure_operator(b,[1,2,1])
2*B[[[1, 1], [2]]] + 2*B[[[1, 2], [2]]] + 2*B[[[1, 3], [2]]]
+ 2*B[[[1, 1], [3]]] + 2*B[[[1, 2], [3]]] + 2*B[[[1, 3], [3]]]
+ 2*B[[[2, 2], [3]]] + 2*B[[[2, 3], [3]]]
```

The Demazure operator is idempotent:

```
sage: T = crystals.Tableaux("A1", shape=[4])
sage: C = CombinatorialFreeModule(QQ, T)
sage: b = C(T.module_generators[0]); b
B[[[1, 1, 1, 1]]]
sage: e = T.demazure_operator(b,[1]); e
B[[[1, 1, 1, 1]]] + B[[[1, 1, 1, 2]]] + B[[[1, 1, 2, 2]]]
+ B[[[1, 2, 2, 2]]] + B[[[2, 2, 2, 2]]]
sage: e == T.demazure_operator(e,[1])
True

sage: all(T.demazure_operator(T.demazure_operator(C(t),[1]),[1])
....:      == T.demazure_operator(C(t),[1]) for t in T)
True
```

demazure_subcrystal(*element*, *reduced_word*, *only_support=True*)

Return the subcrystal corresponding to the application of Demazure operators D_i for i from `reduced_word` on `element`.

INPUT:

- `element` – an element of a free module indexed by the underlying crystal
- `reduced_word` – a reduced word of the Weyl group of the same type as the underlying crystal
- `only_support` – (default: `True`) only include arrows corresponding to the support of `reduced_word`

OUTPUT:

- the Demazure subcrystal

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T.highest_weight_vector()
sage: S = T.demazure_subcrystal(t, [1,2])
sage: list(S)
[[[1, 1], [2]], [[1, 2], [2]], [[1, 1], [3]],
 [[1, 2], [3]], [[2, 2], [3]]]
sage: S = T.demazure_subcrystal(t, [2,1])
sage: list(S)
```

(continues on next page)

(continued from previous page)

```
[[[1, 1], [2]], [[1, 2], [2]], [[1, 1], [3]],
 [[1, 3], [2]], [[1, 3], [3]]]
```

We construct an example where we don't only want the arrows indicated by the support of the reduced word:

```
sage: K = crystals.KirillovReshetikhin(['A',1,1], 1, 2)
sage: mg = K.module_generator()
sage: S = K.demazure_subcrystal(mg, [1])
sage: S.digraph().edges(sort=True)
[([[1, 1]], [[1, 2]], 1), ([[1, 2]], [[2, 2]], 1)]
sage: S = K.demazure_subcrystal(mg, [1], only_support=False)
sage: S.digraph().edges(sort=True)
[([[1, 1]], [[1, 2]], 1),
 ([[1, 2]], [[1, 1]], 0),
 ([[1, 2]], [[2, 2]], 1),
 ([[2, 2]], [[1, 2]], 0)]
```

dual_equivalence_graph(*X=None*, *index_set=None*, *directed=True*)

Return the dual equivalence graph indexed by `index_set` on the subset `X` of `self`.

Let $b \in B$ be an element of weight 0, so $\varepsilon_j(b) = \varphi_j(b)$ for all $j \in I$, where I is the indexing set. We say b' is an i-elementary dual equivalence transformation of b (where $i \in I$) if

- $\varepsilon_i(b) = 1$ and $\varepsilon_{i-1}(b) = 0$, and
- $b' = f_{i-1} f_i e_{i-1} e_i b$.

We can do the inverse procedure by interchanging i and $i - 1$ above.

Note: If the index set is not an ordered interval, we let $i - 1$ mean the index appearing before i in I.

This definition comes from [As2008] Section 4 (where our $\varphi_j(b)$ and $\varepsilon_j(b)$ are denoted by $\epsilon(b, j)$ and $-\delta(b, j)$, respectively).

The dual equivalence graph of B is defined to be the colored graph whose vertices are the elements of B of weight 0, and whose edges of color i (for $i \in I$) connect pairs $\{b, b'\}$ such that b' is an i-elementary dual equivalence transformation of b.

Note: This dual equivalence graph is a generalization of $\mathcal{G}(\mathcal{X})$ in [As2008] Section 4 except we do not require $\varepsilon_i(b) = 0, 1$ for all i.

This definition can be generalized by choosing a subset X of the set of all vertices of B of weight 0, and restricting the dual equivalence graph to the vertex set X.

INPUT:

- `X` – (optional) the vertex set X (default: the whole set of vertices of `self` of weight 0)
- `index_set` – (optional) the index set I (default: the whole index set of `self`); this has to be a subset of the index set of `self` (as a list or tuple)
- `directed` – (default: `True`) whether to have the dual equivalence graph be directed, where the head of an edge $b - b'$ is b and the tail is $b' = f_{i-1} f_i e_{i-1} e_i b$)

See also:

`sage.combinat.partition.Partition.dual_equivalence_graph()`

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape=[2,2])
sage: G = T.dual_equivalence_graph()
sage: G.edges(sort=True)
[([[1, 3], [2, 4]], [[1, 2], [3, 4]], 2),
 ([[1, 2], [3, 4]], [[1, 3], [2, 4]], 3)]
sage: T = crystals.Tableaux(['A',4], shape=[3,2])
sage: G = T.dual_equivalence_graph()
sage: G.edges(sort=True)
[([[1, 3, 5], [2, 4]], [[1, 3, 4], [2, 5]], 4),
 ([[1, 3, 5], [2, 4]], [[1, 2, 5], [3, 4]], 2),
 ([[1, 3, 4], [2, 5]], [[1, 2, 4], [3, 5]], 2),
 ([[1, 2, 5], [3, 4]], [[1, 3, 5], [2, 4]], 3),
 ([[1, 2, 4], [3, 5]], [[1, 2, 3], [4, 5]], 3),
 ([[1, 2, 3], [4, 5]], [[1, 2, 4], [3, 5]], 4)]

sage: T = crystals.Tableaux(['A',4], shape=[3,1])
sage: G = T.dual_equivalence_graph(index_set=[1,2,3])
sage: G.vertices(sort=True)
[[[1, 3, 4], [2]], [[1, 2, 4], [3]], [[1, 2, 3], [4]]]
sage: G.edges(sort=True)
[([[1, 3, 4], [2]], [[1, 2, 4], [3]], 2),
 ([[1, 2, 4], [3]], [[1, 2, 3], [4]], 3)]
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of regular crystals constructed by tensor product of regular crystals.

extra_super_categories()

EXAMPLES:

```
sage: RegularCrystals().TensorProducts().extra_super_categories()
[Category of regular crystals]
```

additional_structure()

Return `None`.

Indeed, the category of regular crystals defines no new structure: it only relates ε_a and φ_a to e_a and f_a respectively.

See also:

Category.additional_structure()

Todo: Should this category be a `CategoryWithAxiom`?

EXAMPLES:

```
sage: RegularCrystals().additional_structure()
```

example(*n=3*)

Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = RegularCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

super_categories()

EXAMPLES:

```
sage: RegularCrystals().super_categories()
[Category of crystals]
```

4.133 Regular Supercrystals

class sage.categories.regular_supercrystals.**RegularSuperCrystals**

Bases: *Category_singleton*

The category of crystals for super Lie algebras.

EXAMPLES:

```
sage: from sage.categories.regular_supercrystals import RegularSuperCrystals
sage: C = RegularSuperCrystals()
sage: C
Category of regular super crystals
sage: C.super_categories()
[Category of finite super crystals]
```

Parents in this category should implement the following methods:

- either an attribute `_cartan_type` or a method `cartan_type`
- `module_generators`: a list (or container) of distinct elements that generate the crystal using f_i and e_i

Furthermore, their elements `x` should implement the following methods:

- `x.e(i)` (returning $e_i(x)$)
- `x.f(i)` (returning $f_i(x)$)
- `x.weight()` (returning $\mathrm{wt}(x)$)

EXAMPLES:

```
sage: from sage.misc.abstract_method import abstract_methods_of_class
sage: from sage.categories.regular_supercrystals import RegularSuperCrystals
sage: abstract_methods_of_class(RegularSuperCrystals().element_class)
{'optional': [], 'required': ['e', 'f', 'weight']}
```

class ElementMethods

Bases: `object`

epsilon(i)

Return ε_i of `self`.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',[1,2]], shape=[2,1])
sage: c = C.an_element(); c
[[-2, -2], [-1]]
sage: c.epsilon(2)
0
sage: c.epsilon(0)
0
```

(continues on next page)

(continued from previous page)

```
sage: c.epsilon(-1)
0
```

phi(*i*)

Return φ_i of `self`.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',[1,2]], shape=[2,1])
sage: c = C.an_element(); c
[[-2, -2], [-1]]
sage: c.phi(1)
0
sage: c.phi(2)
0
sage: c.phi(0)
1
```

class TensorProducts(*category*, **args*)

Bases: `TensorProductsCategory`

The category of regular crystals constructed by tensor product of regular crystals.

extra_super_categories()

EXAMPLES:

```
sage: from sage.categories.regular_supercrystals import␣
↪RegularSuperCrystals
sage: RegularSuperCrystals().TensorProducts().extra_super_categories()
[Category of regular super crystals]
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.regular_supercrystals import RegularSuperCrystals
sage: C = RegularSuperCrystals()
sage: C.super_categories()
[Category of finite super crystals]
```

4.134 Right modules

class sage.categories.right_modules.**RightModules**(*base*, *name=None*)

Bases: `Category_over_base_ring`

The category of right modules right modules over an rng (ring not necessarily with unit), i.e. an abelian group with right multiplication by elements of the rng

EXAMPLES:

```
sage: RightModules(QQ)
Category of right modules over Rational Field
sage: RightModules(QQ).super_categories()
[Category of commutative additive groups]
```

class ElementMethods

Bases: `object`

class ParentMethods

Bases: `object`

super_categories()

EXAMPLES:

```
sage: RightModules(QQ).super_categories()
[Category of commutative additive groups]
```

4.135 Ring ideals

class sage.categories.ring_ideals.**RingIdeals**(*R*)

Bases: *Category_ideal*

The category of two-sided ideals in a fixed ring.

EXAMPLES:

```
sage: Ideals(Integers(200))
Category of ring ideals in Ring of integers modulo 200
sage: C = Ideals(IntegerRing()); C
Category of ring ideals in Integer Ring
sage: I = C([8,12,18])
sage: I
Principal ideal (2) of Integer Ring
```

See also: *CommutativeRingIdeals*.

Todo:

- If useful, implement `RingLeftIdeals` and `RingRightIdeals` of which `RingIdeals` would be a subcategory.
- Make `RingIdeals(R)`, return `CommutativeRingIdeals(R)` when `R` is commutative.

super_categories()

EXAMPLES:

```
sage: RingIdeals(ZZ).super_categories()
[Category of modules over Integer Ring]
sage: RingIdeals(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

4.136 Rings

class sage.categories.rings.**Rings**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of rings

Associative rings with unit, not necessarily commutative

EXAMPLES:

```
sage: Rings()
Category of rings
sage: sorted(Rings().super_categories(), key=str)
[Category of rngs, Category of semirings]

sage: sorted(Rings().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Unital']

sage: Rings() is (CommutativeAdditiveGroups() & Monoids()).Distributive()
True
sage: Rings() is Rngs().Unital()
True
sage: Rings() is Semirings().AdditiveInverse()
True
```

Todo: (see github issue #sage_trac/wiki/CategoriesRoadMap)

- Make Rings() into a subcategory or alias of Algebras(ZZ);
- A parent P in the category `Rings()` should automatically be in the category `Algebras(P)`.

Commutative

alias of *CommutativeRings*

Division

alias of *DivisionRings*

class ElementMethods

Bases: `object`

inverse_of_unit()

Return the inverse of this element if it is a unit.

OUTPUT:

An element in the same ring as this element.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: S = R.quo(x^2 + x + 1)                                             #␣
↪needs sage.libs.pari
sage: S(1).inverse_of_unit()                                             #␣
↪needs sage.libs.pari
1
```

This method fails when the element is not a unit:

```
sage: 2.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: inverse does not exist
```

The inverse returned is in the same ring as this element:

```
sage: a = -1
sage: a.parent()
Integer Ring
sage: a.inverse_of_unit().parent()
Integer Ring
```

Note that this is often not the case when computing inverses in other ways:

```
sage: (~a).parent()
Rational Field
sage: (1/a).parent()
Rational Field
```

is_unit()

Return whether this element is a unit in the ring.

Note: This is a generic implementation for (non-commutative) rings which only works for the one element, its additive inverse, and the zero element. Most rings should provide a more specialized implementation.

EXAMPLES:

```
sage: # needs sage.modules
sage: MS = MatrixSpace(ZZ, 2)
sage: MS.one().is_unit()
True
sage: MS.zero().is_unit()
False
sage: MS([1,2,3,4]).is_unit()
False
```

class MorphismMethods

Bases: object

extend_to_fraction_field()

Return the extension of this morphism to fraction fields of the domain and the codomain.

EXAMPLES:

```
sage: S.<x> = QQ[]
sage: f = S.hom([x + 1]); f
Ring endomorphism of Univariate Polynomial Ring in x over Rational Field
    Defn: x |--> x + 1

sage: g = f.extend_to_fraction_field(); g                              #␣
↪needs sage.libs.singular
Ring endomorphism of Fraction Field of Univariate Polynomial Ring in x
```

(continues on next page)

(continued from previous page)

```
 over Rational Field
    Defn: x |--> x + 1
sage: g(x)                                                              #␣
↪needs sage.libs.singular
x + 1
sage: g(1/x)                                                            #␣
↪needs sage.libs.singular
1/(x + 1)
```

If this morphism is not injective, it does not extend to the fraction field and an error is raised:

```
sage: f = GF(5).coerce_map_from(ZZ)
sage: f.extend_to_fraction_field()
Traceback (most recent call last):
...
ValueError: the morphism is not injective
```

is_injective()

Return whether or not this morphism is injective.

EXAMPLES:

```
sage: # needs sage.libs.singular
sage: R.<x,y> = QQ[]
sage: R.hom([x, y^2], R).is_injective()
True
sage: R.hom([x, x^2], R).is_injective()
False
sage: S.<u,v> = R.quotient(x^3*y)
sage: R.hom([v, u], S).is_injective()
False
sage: S.hom([-u, v], S).is_injective()
True
sage: S.cover().is_injective()
False
```

If the domain is a field, the homomorphism is injective:

```
sage: K.<x> = FunctionField(QQ)
sage: L.<y> = FunctionField(QQ)
sage: f = K.hom([y]); f
Function Field morphism:
  From: Rational function field in x over Rational Field
  To:   Rational function field in y over Rational Field
  Defn: x |--> y
sage: f.is_injective()
True
```

Unless the codomain is the zero ring:

```
sage: codomain = Integers(1)
sage: f = QQ.hom([Zmod(1)(0)], check=False)
sage: f.is_injective()
False
```

Homomorphism from rings of characteristic zero to rings of positive characteristic can not be injective:

```
sage: R.<x> = ZZ[]
sage: f = R.hom([GF(3)(1)]); f
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Finite Field of size 3
  Defn: x |--> 1
sage: f.is_injective()
False
```

A morphism whose domain is an order in a number field is injective if the codomain has characteristic zero:

```
sage: K.<x> = FunctionField(QQ)
sage: f = ZZ.hom(K); f
Composite map:
  From: Integer Ring
  To:   Rational function field in x over Rational Field
  Defn:   Conversion via FractionFieldElement_1poly_field map:
          From: Integer Ring
          To:   Fraction Field of Univariate Polynomial Ring in x
                over Rational Field
        then
          Isomorphism:
          From: Fraction Field of Univariate Polynomial Ring in x
                over Rational Field
          To:   Rational function field in x over Rational Field
sage: f.is_injective()
True
```

A coercion to the fraction field is injective:

```
sage: R = ZpFM(3)                                                           #␣
↪needs sage.rings.padics
sage: R.fraction_field().coerce_map_from(R).is_injective()
True
```

NoZeroDivisors

alias of *Domains*

class ParentMethods

Bases: `object`

bracket (*x*, *y*)

Return the Lie bracket $[x, y] = xy - yx$ of x and y.

INPUT:

- `x`, `y` – elements of `self`

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: F = AlgebrasWithBasis(QQ).example()
sage: F
An example of an algebra with basis:
 the free algebra on the generators ('a', 'b', 'c') over Rational Field
sage: a, b, c = F.algebra_generators()
sage: F.bracket(a, b)
B[word: ab] - B[word: ba]
```

This measures the default of commutation between x and y. F endowed with the bracket operation is a Lie algebra; in particular, it satisfies Jacobi's identity:

```
sage: (F.bracket(F.bracket(a,b), c) + F.bracket(F.bracket(b,c), a)       #␣
↪needs sage.combinat sage.modules
....:  + F.bracket(F.bracket(c,a), b))
0
```

characteristic()

Return the characteristic of this ring.

EXAMPLES:

```
sage: QQ.characteristic()
0
sage: GF(19).characteristic()
19
sage: Integers(8).characteristic()
8
sage: Zp(5).characteristic()                                              #␣
↪needs sage.rings.padics
0
```

free_module(*base=None*, *basis=None*, *map=True*)

Return a free module V over the specified subring together with maps to and from V.

The default implementation only supports the case that the base ring is the ring itself.

INPUT:

- `base` – a subring R so that this ring is isomorphic to a finite-rank free R-module V
- `basis` – (optional) a basis for this ring over the base
- `map` – boolean (default `True`), whether to return R-linear maps to and from V

OUTPUT:

- A finite-rank free R-module V
- An R-module isomorphism from V to this ring (only included if `map` is `True`)
- An R-module isomorphism from this ring to V (only included if `map` is `True`)

EXAMPLES:

```
sage: # needs sage.modules
sage: R.<x> = QQ[[]]
sage: V, from_V, to_V = R.free_module(R)
sage: v = to_V(1 + x); v
(1 + x)
sage: from_V(v)
1 + x
sage: W, from_W, to_W = R.free_module(R, basis=(1 - x))
sage: W is V
True
sage: w = to_W(1 + x); w
(1 - x^2)
sage: from_W(w)
1 + x + O(x^20)
```

ideal(**args*, ***kwds*)

Create an ideal of this ring.

Note: The code is copied from the base class `Ring`. This is because there are rings that do not inherit from that class, such as matrix algebras. See github issue #7797.

INPUT:

- An element or a list/tuple/sequence of elements.
- `coerce` (optional bool, default `True`): First coerce the elements into this ring.
- `side`, optional string, one of `"twosided"` (default), `"left"`, `"right"`: determines whether the resulting ideal is twosided, a left ideal or a right ideal.

EXAMPLES:

```
sage: # needs sage.modules
sage: MS = MatrixSpace(QQ, 2, 2)
sage: isinstance(MS, Ring)
False
sage: MS in Rings()
True
sage: MS.ideal(2)
Twosided Ideal
(
  [2 0]
  [0 2]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: MS.ideal([MS.0, MS.1], side='right')
Right Ideal
(
  [1 0]
  [0 0],

  [0 1]
  [0 0]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

ideal_monoid()

The monoid of the ideals of this ring.

Note: The code is copied from the base class of rings. This is since there are rings that do not inherit from that class, such as matrix algebras. See github issue #7797.

EXAMPLES:

```
sage: # needs sage.modules
sage: MS = MatrixSpace(QQ, 2, 2)
sage: isinstance(MS, Ring)
False
sage: MS in Rings()
True
sage: MS.ideal_monoid()
Monoid of ideals of Full MatrixSpace of 2 by 2 dense matrices
over Rational Field
```

Note that the monoid is cached:

```
sage: MS.ideal_monoid() is MS.ideal_monoid()                          #␣
↪needs sage.modules
```

(continues on next page)

(continued from previous page)

```
True
```

is_commutative()

Return whether the ring is commutative.

The answer is `True` only if the category is a sub-category of `CommutativeRings`.

It is recommended to use instead `R in Rings().Commutative()`.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -1, -1)                              ␣
↪  # needs sage.combinat sage.modules
sage: Q.is_commutative()                                                     ␣
↪  # needs sage.combinat sage.modules
False
```

is_ring()

Return `True`, since this in an object of the category of rings.

EXAMPLES:

```
sage: Parent(QQ,category=Rings()).is_ring()
True
```

is_zero()

Return `True` if this is the zero ring.

EXAMPLES:

```
sage: Integers(1).is_zero()
True
sage: Integers(2).is_zero()
False
sage: QQ.is_zero()
False
sage: R.<x> = ZZ[]
sage: R.quo(1).is_zero()
True
sage: R.<x> = GF(101)[]
sage: R.quo(77).is_zero()
True
sage: R.quo(x^2 + 1).is_zero()                                              #␣
↪needs sage.libs.pari
False
```

quo(*I*, *names=None*, ***kwds*)

Quotient of a ring by a two-sided ideal.

Note: This is a synonym for *quotient()*.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ, 2)                                               #␣
↪needs sage.modules
```

(continues on next page)

(continued from previous page)

```
sage: I = MS * MS.gens() * MS                                               #␣
↪needs sage.modules
```

MS is not an instance of Ring.

However it is an instance of the parent class of the category of rings. The quotient method is inherited from there:

```
sage: isinstance(MS, sage.rings.ring.Ring)                                  #␣
↪needs sage.modules
False
sage: isinstance(MS, Rings().parent_class)                                  #␣
↪needs sage.modules
True
sage: MS.quo(I, names=['a','b','c','d'])                                    #␣
↪needs sage.modules
Quotient of Full MatrixSpace of 2 by 2 dense matrices
 over Rational Field by the ideal
(
  [1 0]
  [0 0],

  [0 1]
  [0 0],

  [0 0]
  [1 0],

  [0 0]
  [0 1]
)
```

A test with a subclass of Ring:

```
sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S.<a,b> = R.quo((x^2, y))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
 by the ideal (x^2, y)
sage: S.gens()
(a, 0)
sage: a == b
False
```

quotient(*I*, *names=None*, ***kwds*)

Quotient of a ring by a two-sided ideal.

INPUT:

- I – A twosided ideal of this ring.
- names – (optional) names of the generators of the quotient (if there are multiple generators, you can specify a single character string and the generators are named in sequence starting with 0).
- further named arguments that may be passed to the quotient ring constructor.

EXAMPLES:

Usually, a ring inherits a method sage.rings.ring.Ring.quotient(). So, we need a bit of effort to make the following example work with the category framework:

```
sage: # needs sage.combinat sage.modules
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: from sage.rings.noncommutative_ideals import Ideal_nc
sage: from itertools import product
sage: class PowerIdeal(Ideal_nc):
....:  def __init__(self, R, n):
....:      self._power = n
....:      Ideal_nc.__init__(self, R, [R.prod(m)
....:                                  for m in product(R.gens(),␣
↪repeat=n)])
....:  def reduce(self, x):
....:      R = self.ring()
....:      return add([c*R(m) for m, c in x
....:                  if len(m) < self._power], R(0))
sage: I = PowerIdeal(F, 3)
sage: Q = Rings().parent_class.quotient(F, I); Q
Quotient of Free Algebra on 3 generators (x, y, z) over Rational Field
 by the ideal (x^3, x^2*y, x^2*z, x*y*x, x*y^2, x*y*z, x*z*x,
               x*z*y, x*z^2, y*x^2, y*x*y, y*x*z, y^2*x, y^3,
               y^2*z, y*z*x, y*z*y, y*z^2, z*x^2, z*x*y, z*x*z,
               z*y*x, z*y^2, z*y*z, z^2*x, z^2*y, z^3)
sage: Q.0
xbar
sage: Q.1
ybar
sage: Q.2
zbar
sage: Q.0*Q.1
xbar*ybar
sage: Q.0*Q.1*Q.0
0
```

An example with polynomial rings:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: S = R.quotient(I, 'a')
sage: S.gens()
(a,)

sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: S.<a,b> = R.quotient((x^2, y))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
 by the ideal (x^2, y)
sage: S.gens()
(a, 0)
sage: a == b
False
```

quotient_ring(*I*, *names=None*, ***kwds*)

Quotient of a ring by a two-sided ideal.

Note: This is a synonym for `quotient()`.

INPUT:

- I – an ideal of R
- names – (optional) names of the generators of the quotient. (If there are multiple generators, you can specify a single character string and the generators are named in sequence starting with 0.)
- further named arguments that may be passed to the quotient ring constructor.

OUTPUT:

- R/I – the quotient ring of R by the ideal I

EXAMPLES:

```
sage: MS = MatrixSpace(QQ, 2)                                              #␣
↪needs sage.modules
sage: I = MS * MS.gens() * MS                                              #␣
↪needs sage.modules
```

MS is not an instance of Ring, but it is an instance of the parent class of the category of rings. The quotient method is inherited from there:

```
sage: isinstance(MS, sage.rings.ring.Ring)                                 #␣
↪needs sage.modules
False
sage: isinstance(MS, Rings().parent_class)                                 #␣
↪needs sage.modules
True
sage: MS.quotient_ring(I, names=['a','b','c','d'])                         #␣
↪needs sage.modules
Quotient of Full MatrixSpace of 2 by 2 dense matrices
 over Rational Field by the ideal
(
  [1 0]
  [0 0],

  [0 1]
  [0 0],

  [0 0]
  [1 0],

  [0 0]
  [0 1]
)
```

A test with a subclass of Ring:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: I = R.ideal([4 + 3*x + x^2, 1 + x^2])
sage: S = R.quotient_ring(I, 'a')
sage: S.gens()
(a,)

sage: # needs sage.libs.singular
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: S.<a,b> = R.quotient_ring((x^2, y))
sage: S
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
 by the ideal (x^2, y)
sage: S.gens()
(a, 0)
```

(continues on next page)

(continued from previous page)

```
sage: a == b
False
```

class SubcategoryMethods

Bases: `object`

Division()

Return the full subcategory of the division objects of `self`.

A ring satisfies the *division axiom* if all non-zero elements have multiplicative inverses.

EXAMPLES:

```
sage: Rings().Division()
Category of division rings
sage: Rings().Commutative().Division()
Category of fields
```

NoZeroDivisors()

Return the full subcategory of the objects of `self` having no nonzero zero divisors.

A *zero divisor* in a ring R is an element $x \in R$ such that there exists a nonzero element $y \in R$ such that $x \cdot y = 0$ or $y \cdot x = 0$ (see Wikipedia article Zero_divisor).

EXAMPLES:

```
sage: Rings().NoZeroDivisors()
Category of domains
```

4.137 Rngs

class sage.categories.rngs.**Rngs**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of rngs.

An *rng* $(S, +, *)$ is similar to a ring but not necessarily unital. In other words, it is a combination of a commutative additive group $(S, +)$ and a multiplicative semigroup $(S, *)$, where $*$ distributes over $+$.

EXAMPLES:

```
sage: C = Rngs(); C
Category of rngs
sage: sorted(C.super_categories(), key=str)
[Category of associative additive commutative additive associative additive␣
↪unital distributive magmas and additive magmas,
 Category of commutative additive groups]

sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive']

sage: C is (CommutativeAdditiveGroups() & Semigroups()).Distributive()
True
sage: C.Unital()
Category of rings
```

Unital

alias of *Rings*

4.138 R-trivial semigroups

class sage.categories.r_trivial_semigroups.**RTrivialSemigroups**(*base_category*)

Bases: *CategoryWithAxiom*

Commutative_extra_super_categories()

Implement the fact that a commutative R-trivial semigroup is J-trivial.

EXAMPLES:

```
sage: Semigroups().RTrivial().Commutative_extra_super_categories()
[Category of j trivial semigroups]
```

extra_super_categories()

Implement the fact that a R-trivial semigroup is H-trivial.

EXAMPLES:

```
sage: Semigroups().RTrivial().extra_super_categories()
[Category of h trivial semigroups]
```

4.139 Schemes

class sage.categories.schemes.**AbelianVarieties**(*base*)

Bases: *Schemes_over_base*

The category of abelian varieties over a given field.

EXAMPLES:

```
sage: AbelianVarieties(QQ)
Category of abelian varieties over Rational Field
sage: AbelianVarieties(ZZ)
Traceback (most recent call last):
...
ValueError: category of abelian varieties is only defined over fields
```

class Homsets(*category*, **args*)

Bases: *HomsetsCategory*

Overloaded `Homsets` class to register the homset as an additive abelian group.

EXAMPLES:

```
sage: AbelianVarieties(QQ).Homsets().is_
↪subcategory(CommutativeAdditiveGroups())
True
```

class Endset(*base_category*)

Bases: *CategoryWithAxiom*

Overloaded `Endset` class to register the endset as a ring.

sage: AbelianVarieties(QQ).Endsets().is_subcategory(Rings()) True

extra_super_categories()

Register the endset as a ring.

EXAMPLES:

```
sage: End(EllipticCurve(j=1)) in Rings()
True
```

extra_super_categories()

Register the homset as an additive abelian group.

EXAMPLES:

```
sage: Hom(EllipticCurve(j=1), EllipticCurve(j=2)) in␣
↪CommutativeAdditiveGroups()
True
```

super_categories()

EXAMPLES:

```
sage: AbelianVarieties(QQ).super_categories()
[Category of schemes over Rational Field,
 Category of commutative additive groups]
```

class `sage.categories.schemes.`**Schemes**

Bases: *Category*

The category of all schemes.

EXAMPLES:

```
sage: Schemes()
Category of schemes
```

`Schemes` can also be used to construct the category of schemes over a given base:

```
sage: Schemes(Spec(ZZ))
Category of schemes over Integer Ring

sage: Schemes(ZZ)
Category of schemes over Integer Ring
```

Todo: Make `Schemes()` a singleton category (and remove *Schemes* from the workaround in `category_types.Category_over_base._test_category_over_bases()`).

This is currently incompatible with the dispatching below.

super_categories()

EXAMPLES:

```
sage: Schemes().super_categories()
[Category of sets]
```

class sage.categories.schemes.**Schemes_over_base**(*base*, *name=None*)

Bases: *Category_over_base*

The category of schemes over a given base scheme.

EXAMPLES:

```
sage: Schemes(Spec(ZZ))
Category of schemes over Integer Ring
```

base_scheme()

EXAMPLES:

```
sage: Schemes(Spec(ZZ)).base_scheme()
Spectrum of Integer Ring
```

super_categories()

EXAMPLES:

```
sage: Schemes(Spec(ZZ)).super_categories()
[Category of schemes]
```

4.140 Semigroups

class sage.categories.semigroups.**Semigroups**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of (multiplicative) semigroups.

A *semigroup* is an associative *magma*, that is a set endowed with a multiplicative binary operation $*$ which is associative (see Wikipedia article Semigroup).

The operation $*$ is not required to have a neutral element. A semigroup for which such an element exists is a *monoid*.

EXAMPLES:

```
sage: C = Semigroups(); C
Category of semigroups
sage: C.super_categories()
[Category of magmas]
sage: C.all_super_categories()
[Category of semigroups, Category of magmas,
 Category of sets, Category of sets with partial maps, Category of objects]
sage: C.axioms()
frozenset({'Associative'})
sage: C.example()
An example of a semigroup: the left zero semigroup
```

class Algebras(*category*, **args*)

Bases: *AlgebrasCategory*

class ParentMethods

Bases: `object`

algebra_generators()

The generators of this algebra, as per *MagmaticAlgebras.ParentMethods.algebra_generators()*.

They correspond to the generators of the semigroup.

EXAMPLES:

```
sage: M = FiniteSemigroups().example(); M
An example of a finite semigroup:
the left regular band generated by ('a', 'b', 'c', 'd')
sage: M.semigroup_generators()
Family ('a', 'b', 'c', 'd')
sage: M.algebra(ZZ).algebra_generators()                              #␣
↪needs sage.modules
Family (B['a'], B['b'], B['c'], B['d'])
```

gen(*i=0*)

Return the `i`-th generator of `self`.

EXAMPLES:

```
sage: A = GL(3, GF(7)).algebra(ZZ)                                    #␣
↪needs sage.modules
sage: A.gen(0)                                                        #␣
↪needs sage.groups sage.libs.pari sage.modules
[3 0 0]
[0 1 0]
[0 0 1]
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: a, b = SL2Z.algebra(ZZ).gens(); a, b                            #␣
↪needs sage.groups sage.modules
([ 0 -1]
 [ 1  0],
 [1 1]
 [0 1])
sage: 2*a + b                                                         #␣
↪needs sage.groups sage.modules
2*[ 0 -1]
  [ 1  0]
+
[1 1]
[0 1]
```

ngens()

Return the number of generators of `self`.

EXAMPLES:

```
sage: SL2Z.algebra(ZZ).ngens()                                          #␣
↪needs sage.groups sage.modules
2
sage: DihedralGroup(4).algebra(RR).ngens()                              #␣
↪needs sage.groups sage.modules
2
```

product_on_basis(*g1*, *g2*)

Product, on basis elements, as per *MagmaticAlgebras.WithBasis.ParentMethods.product_on_basis()*.

The product of two basis elements is induced by the product of the corresponding elements of the group.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(); S
An example of a finite semigroup:
 the left regular band generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)                                                 #␣
↪needs sage.modules
sage: a, b, c, d = A.algebra_generators()                               #␣
↪needs sage.modules
sage: a * b + b * d * c * d                                             #␣
↪needs sage.modules
B['ab'] + B['bdc']
```

regular_representation(*side='left'*)

Return the regular representation of `self`.

INPUT:

- `side` – (default: `"left"`) whether this is the `"left"` or `"right"` regular representation

EXAMPLES:

```
sage: # needs sage.groups
sage: G = groups.permutation.Dihedral(4)
sage: A = G.algebra(QQ)                                                 #␣
↪needs sage.modules
sage: V = A.regular_representation()                                    #␣
↪needs sage.modules
sage: V == G.regular_representation(QQ)                                 #␣
↪needs sage.modules
True
```

trivial_representation(*side='twosided'*)

Return the trivial representation of `self`.

INPUT:

- `side` – ignored

EXAMPLES:

```
sage: # needs sage.groups
sage: G = groups.permutation.Dihedral(4)
sage: A = G.algebra(QQ)                                                 #␣
↪needs sage.modules
sage: V = A.trivial_representation()                                    #␣
↪needs sage.modules
```

(continues on next page)

(continued from previous page)

```
sage: V == G.trivial_representation(QQ)                                  #␣
↪needs sage.modules
True
```

extra_super_categories()

Implement the fact that the algebra of a semigroup is indeed a (not necessarily unital) algebra.

EXAMPLES:

```
sage: Semigroups().Algebras(QQ).extra_super_categories()
[Category of semigroups]
sage: Semigroups().Algebras(QQ).super_categories()
[Category of associative algebras over Rational Field,
 Category of magma algebras over Rational Field]
```

Aperiodic

alias of `AperiodicSemigroups`

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

extra_super_categories()

Implement the fact that a Cartesian product of semigroups is a semigroup.

EXAMPLES:

```
sage: Semigroups().CartesianProducts().extra_super_categories()
[Category of semigroups]
sage: Semigroups().CartesianProducts().super_categories()
[Category of semigroups, Category of Cartesian products of magmas]
```

class ElementMethods

Bases: `object`

Finite

alias of `FiniteSemigroups`

FinitelyGeneratedAsMagma

alias of `FinitelyGeneratedSemigroups`

HTrivial

alias of `HTrivialSemigroups`

JTrivial

alias of `JTrivialSemigroups`

LTrivial

alias of `LTrivialSemigroups`

class ParentMethods

Bases: `object`

cayley_graph(*side='right'*, *simple=False*, *elements=None*, *generators=None*, *connecting_set=None*)

Return the Cayley graph for this finite semigroup.

INPUT:

- `side` – “left”, “right”, or “twosided”: the side on which the generators act (default:”right”)

- `simple` – boolean (default:False): if True, returns a simple graph (no loops, no labels, no multiple edges)
- `generators` – a list, tuple, or family of elements of `self` (default: `self.semigroup_generators()`)
- `connecting_set` – alias for `generators`; deprecated
- `elements` – a list (or iterable) of elements of `self`

OUTPUT:

- `DiGraph`

EXAMPLES:

We start with the (right) Cayley graphs of some classical groups:

```
sage: # needs sage.graphs sage.groups
sage: D4 = DihedralGroup(4); D4
Dihedral group of order 8 as a permutation group
sage: G = D4.cayley_graph()
sage: show(G, color_by_label=True, edge_labels=True)                    #␣
↪needs sage.plot
sage: A5 = AlternatingGroup(5); A5
Alternating group of order 5!/2 as a permutation group
sage: G = A5.cayley_graph()
sage: G.show3d(color_by_label=True, edge_size=0.01,                     #␣
↪needs sage.plot
....:          edge_size2=0.02, vertex_size=0.03)
sage: G.show3d(vertex_size=0.03,        # long time (less than a minute),␣
↪needs sage.plot
....:          edge_size=0.01, edge_size2=0.02,
....:          vertex_colors={(1,1,1): G.vertices(sort=True)},
....:          bgcolor=(0,0,0), color_by_label=True,
....:          xres=700, yres=700, iterations=200)
sage: G.num_edges()
120

sage: # needs sage.combinat sage.graphs sage.groups
sage: w = WeylGroup(['A', 3])
sage: d = w.cayley_graph(); d
Digraph on 24 vertices
sage: d.show3d(color_by_label=True, edge_size=0.01, vertex_size=0.03)   #␣
↪needs sage.plot
```

Alternative generators may be specified:

```
sage: # needs sage.graphs sage.groups
sage: G = A5.cayley_graph(generators=[A5.gens()[0]])
sage: G.num_edges()
60
sage: g = PermutationGroup([(i + 1, j + 1)
....:                       for i in range(5)
....:                       for j in range(5) if j != i])
sage: g.cayley_graph(generators=[(1,2), (2,3)])
Digraph on 120 vertices
```

If `elements` is specified, then only the subgraph induced and those elements is returned. Here we use it to display the Cayley graph of the free monoid truncated on the elements of length at most 3:

```
sage: # needs sage.combinat sage.graphs
sage: M = Monoids().example(); M
An example of a monoid:
```

(continues on next page)

(continued from previous page)

```
 the free monoid generated by ('a', 'b', 'c', 'd')
sage: elements = [M.prod(w)
....:             for w in sum((list(Words(M.semigroup_generators(), k))
....:                          for k in range(4)), [])]
sage: G = M.cayley_graph(elements=elements)
sage: G.num_verts(), G.num_edges()
(85, 84)
sage: G.show3d(color_by_label=True, edge_size=0.001, vertex_size=0.01)   #␣
↪needs sage.plot
```

We now illustrate the `side` and `simple` options on a semigroup:

```
sage: S = FiniteSemigroups().example(alphabet=('a', 'b'))
sage: g = S.cayley_graph(simple=True)                                      #␣
↪needs sage.graphs
sage: g.vertices(sort=True)                                                #␣
↪needs sage.graphs
['a', 'ab', 'b', 'ba']
sage: g.edges(sort=True)                                                   #␣
↪needs sage.graphs
[('a', 'ab', None), ('b', 'ba', None)]
```

```
sage: g = S.cayley_graph(side="left", simple=True)                         #␣
↪needs sage.graphs
sage: g.vertices(sort=True)                                                #␣
↪needs sage.graphs
['a', 'ab', 'b', 'ba']
sage: g.edges(sort=True)                                                   #␣
↪needs sage.graphs
[('a', 'ba', None), ('ab', 'ba', None), ('b', 'ab', None),
('ba', 'ab', None)]
```

```
sage: g = S.cayley_graph(side="twosided", simple=True)                     #␣
↪needs sage.graphs
sage: g.vertices(sort=True)                                                #␣
↪needs sage.graphs
['a', 'ab', 'b', 'ba']
sage: g.edges(sort=True)                                                   #␣
↪needs sage.graphs
[('a', 'ab', None), ('a', 'ba', None), ('ab', 'ba', None),
('b', 'ab', None), ('b', 'ba', None), ('ba', 'ab', None)]
```

```
sage: g = S.cayley_graph(side="twosided")                                  #␣
↪needs sage.graphs
sage: g.vertices(sort=True)                                                #␣
↪needs sage.graphs
['a', 'ab', 'b', 'ba']
sage: g.edges(sort=True)                                                   #␣
↪needs sage.graphs
[('a', 'a', (0, 'left')), ('a', 'a', (0, 'right')), ('a', 'ab', (1, 'right
↪')), ('a', 'ba', (1, 'left')), ('ab', 'ab', (0, 'left')), ('ab', 'ab',␣
↪(0, 'right')), ('ab', 'ab', (1, 'right')), ('ab', 'ba', (1, 'left')), (
↪'b', 'ab', (0, 'left')), ('b', 'b', (1, 'left')), ('b', 'b', (1, 'right
↪')), ('b', 'ba', (0, 'right')), ('ba', 'ab', (0, 'left')), ('ba', 'ba',␣
↪(0, 'right')), ('ba', 'ba', (1, 'left')), ('ba', 'ba', (1, 'right'))]
```

```
sage: s1 = SymmetricGroup(1); s = s1.cayley_graph()                         #␣
↪needs sage.graphs sage.groups
sage: s.vertices(sort=False)                                                #␣
↪needs sage.graphs sage.groups
[()]
```

Todo:

- Add more options for constructing subgraphs of the Cayley graph, handling the standard use cases when exploring large/infinite semigroups (a predicate, generators of an ideal, a maximal length in term of the generators)
- Specify good default layout/plot/latex options in the graph
- Generalize to combinatorial modules with module generators / operators

AUTHORS:

- Bobby Moretti (2007-08-10)
- Robert Miller (2008-05-01): editing
- Nicolas M. Thiery (2008-12): extension to semigroups, `side`, `simple`, and `elements` options, …

magma_generators()

An alias for *semigroup_generators()*.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c',
↪ 'd')
sage: S.magma_generators()
Family ('a', 'b', 'c', 'd')
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

prod(*args*)

Return the product of the list of elements `args` inside `self`.

EXAMPLES:

```
sage: S = Semigroups().example("free")
sage: S.prod([S('a'), S('b'), S('c')])
'abc'
sage: S.prod([])
Traceback (most recent call last):
...
AssertionError: Cannot compute an empty product in a semigroup
```

regular_representation(*base_ring=None*, *side='left'*)

Return the regular representation of `self` over `base_ring`.

- `side` – (default: `"left"`) whether this is the `"left"` or `"right"` regular representation

EXAMPLES:

```
sage: G = groups.permutation.Dihedral(4)                                    #␣
↪needs sage.groups
sage: G.regular_representation()                                           #␣
↪needs sage.groups
```

(continues on next page)

(continued from previous page)

```
Left Regular Representation of Dihedral group of order 8
 as a permutation group over Integer Ring
```

semigroup_generators()

Return distinguished semigroup generators for `self`.

OUTPUT: a family

This method is optional.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c',
↪ 'd')
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

subsemigroup(*generators*, *one=None*, *category=None*)

Return the multiplicative subsemigroup generated by `generators`.

INPUT:

- `generators` – a finite family of elements of `self`, or a list, iterable, … that can be converted into one (see `Family`).
- `one` – a unit for the subsemigroup, or `None`.
- `category` – a category

This implementation lazily constructs all the elements of the semigroup, and the right Cayley graph relations between them, and uses the latter as an automaton.

See `AutomaticSemigroup` for details.

EXAMPLES:

```
sage: R = IntegerModRing(15)
sage: M = R.subsemigroup([R(3), R(5)]); M                                     #␣
↪needs sage.combinat
A subsemigroup of (Ring of integers modulo 15) with 2 generators
sage: M.list()                                                                #␣
↪needs sage.combinat
[3, 5, 9, 0, 10, 12, 6]
```

By default, M is just in the category of subsemigroups:

```
sage: M in Semigroups().Subobjects()                                          #␣
↪needs sage.combinat
True
```

In the following example, we specify that M is a submonoid of the finite monoid R (it shares the same unit), and a group by itself:

```
sage: M = R.subsemigroup([R(-1)],                                             #␣
↪needs sage.combinat
....:     category=Monoids().Finite().Subobjects() & Groups()); M
A submonoid of (Ring of integers modulo 15) with 1 generators
sage: M.list()                                                                #␣
↪needs sage.combinat
[1, 14]
```

(continues on next page)

(continued from previous page)

```
sage: M.one()                                                          #␣
↪needs sage.combinat
1
```

In the following example, M is a group; however, its unit does not coincide with that of R, so M is only a subsemigroup, and we need to specify its unit explicitly:

```
sage: M = R.subsemigroup([R(5)],                                       #␣
↪needs sage.combinat
....:     category=Semigroups().Finite().Subobjects() & Groups()); M
Traceback (most recent call last):
...
ValueError: For a monoid which is just a subsemigroup,
the unit should be specified

sage: # needs sage.groups
sage: M = R.subsemigroup([R(5)], one=R(10),
....:     category=Semigroups().Finite().Subobjects() & Groups()); M
A subsemigroup of (Ring of integers modulo 15) with 1 generators
sage: M in Groups()
True
sage: M.list()
[10, 5]
sage: M.one()
10
```

Todo:

- Fix the failure in TESTS by providing a default implementation of `__invert__` for finite groups (or even finite monoids).
- Provide a default implementation of `one` for a finite monoid, so that we would not need to specify it explicitly?

trivial_representation(*base_ring=None*, *side='twosided'*)

Return the trivial representation of `self` over `base_ring`.

INPUT:

- `base_ring` – (optional) the base ring; the default is $\mathbf{Z}$
- `side` – ignored

EXAMPLES:

```
sage: G = groups.permutation.Dihedral(4)                               #␣
↪needs sage.groups
sage: G.trivial_representation()                                       #␣
↪needs sage.groups
Trivial representation of Dihedral group of order 8
 as a permutation group over Integer Ring
```

class Quotients(*category*, **args*)

Bases: `QuotientsCategory`

class ParentMethods

Bases: `object`

semigroup_generators()

Return semigroup generators for `self` by retracting the semigroup generators of the ambient semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().Quotients().example().semigroup_
↪generators() # todo: not implemented
```

example()

Return an example of quotient of a semigroup, as per *Category.example()*.

EXAMPLES:

```
sage: Semigroups().Quotients().example()
An example of a (sub)quotient semigroup: a quotient of the left zero␣
↪semigroup
```

RTrivial

alias of *RTrivialSemigroups*

class SubcategoryMethods

Bases: `object`

Aperiodic()

Return the full subcategory of the aperiodic objects of `self`.

A (multiplicative) *semigroup* S is *aperiodic* if for any element $s \in S$, the sequence $s, s^2, s^3, \ldots$ eventually stabilizes.

In terms of variety, this can be described by the equation $s^\omega s = s$.

EXAMPLES:

```
sage: Semigroups().Aperiodic()
Category of aperiodic semigroups
```

An aperiodic semigroup is H-trivial:

```
sage: Semigroups().Aperiodic().axioms()
frozenset({'Aperiodic', 'Associative', 'HTrivial'})
```

In the finite case, the two notions coincide:

```
sage: Semigroups().Aperiodic().Finite() is Semigroups().HTrivial().
↪Finite()
True
```

See also:

- Wikipedia article Aperiodic_semigroup
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.Aperiodic*

HTrivial()

Return the full subcategory of the H-trivial objects of `self`.

Let S be (multiplicative) *semigroup*. Two elements of S are in the same H-class if they are in the same L-class and in the same R-class.

The semigroup S is *H-trivial* if all its H-classes are trivial (that is of cardinality 1).

EXAMPLES:

```
sage: C = Semigroups().HTrivial(); C
Category of h trivial semigroups
sage: Semigroups().HTrivial().Finite().example()
NotImplemented
```

See also:

- Wikipedia article Green%27s_relations
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.Aperiodic*

JTrivial()

Return the full subcategory of the J-trivial objects of `self`.

Let S be (multiplicative) *semigroup*. The *J-preorder* $\leq_J$ on S is defined by:

$$x \leq_J y \qquad \iff \qquad x \in SyS$$

The *J-classes* are the equivalence classes for the associated equivalence relation. The semigroup S is *J-trivial* if all its J-classes are trivial (that is of cardinality 1), or equivalently if the J-preorder is in fact a partial order.

EXAMPLES:

```
sage: C = Semigroups().JTrivial(); C
Category of j trivial semigroups
```

A semigroup is J-trivial if and only if it is L-trivial and R-trivial:

```
sage: sorted(C.axioms())
['Associative', 'HTrivial', 'JTrivial', 'LTrivial', 'RTrivial']
sage: Semigroups().LTrivial().RTrivial()
Category of j trivial semigroups
```

For a commutative semigroup, all three axioms are equivalent:

```
sage: Semigroups().Commutative().LTrivial()
Category of commutative j trivial semigroups
sage: Semigroups().Commutative().RTrivial()
Category of commutative j trivial semigroups
```

See also:

- Wikipedia article Green%27s_relations
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.HTrivial*

LTrivial()

Return the full subcategory of the L-trivial objects of `self`.

Let S be (multiplicative) *semigroup*. The *L-preorder* $\leq_L$ on S is defined by:

$$x \leq_L y \qquad \iff \qquad x \in Sy$$

The *L-classes* are the equivalence classes for the associated equivalence relation. The semigroup S is *L-trivial* if all its L-classes are trivial (that is of cardinality 1), or equivalently if the L-preorder is in fact a partial order.

EXAMPLES:

```
sage: C = Semigroups().LTrivial(); C
Category of l trivial semigroups
```

A L-trivial semigroup is H-trivial:

```
sage: sorted(C.axioms())
['Associative', 'HTrivial', 'LTrivial']
```

See also:

- Wikipedia article Green%27s_relations
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.HTrivial*

RTrivial()

Return the full subcategory of the R-trivial objects of `self`.

Let S be (multiplicative) *semigroup*. The *R-preorder* $\leq_R$ on S is defined by:

$$x \leq_R y \qquad \iff \qquad x \in yS$$

The *R-classes* are the equivalence classes for the associated equivalence relation. The semigroup S is *R-trivial* if all its R-classes are trivial (that is of cardinality 1), or equivalently if the R-preorder is in fact a partial order.

EXAMPLES:

```
sage: C = Semigroups().RTrivial(); C
Category of r trivial semigroups
```

An R-trivial semigroup is H-trivial:

```
sage: sorted(C.axioms())
['Associative', 'HTrivial', 'RTrivial']
```

See also:

- Wikipedia article Green%27s_relations
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.HTrivial*

class Subquotients(*category*, **args*)

Bases: *SubquotientsCategory*

The category of subquotient semi-groups.

EXAMPLES:

```
sage: Semigroups().Subquotients().all_super_categories()
[Category of subquotients of semigroups,
 Category of semigroups,
 Category of subquotients of magmas,
 Category of magmas,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

[Category of subquotients of semigroups,
 Category of semigroups,
 Category of subquotients of magmas,
 Category of magmas,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

example()

Returns an example of subquotient of a semigroup, as per *Category.example()*.

EXAMPLES:

```
sage: Semigroups().Subquotients().example()
An example of a (sub)quotient semigroup: a quotient of the left zero␣
↪semigroup
```

Unital

alias of *Monoids*

example(*choice='leftzero'*, ***kwds*)

Returns an example of a semigroup, as per *Category.example()*.

INPUT:

- `choice` – str (default: 'leftzero'). Can be either 'leftzero' for the left zero semigroup, or 'free' for the free semigroup.
- `**kwds` – keyword arguments passed onto the constructor for the chosen semigroup.

EXAMPLES:

```
sage: Semigroups().example(choice='leftzero')
An example of a semigroup: the left zero semigroup
sage: Semigroups().example(choice='free')
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c', 'd
↪')
sage: Semigroups().example(choice='free', alphabet=('a','b'))
An example of a semigroup: the free semigroup generated by ('a', 'b')
```

4.141 Semirngs

class sage.categories.semirings.**Semirings**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

The category of semirings.

A semiring $(S, +, *)$ is similar to a ring, but without the requirement that each element must have an additive inverse. In other words, it is a combination of a commutative additive monoid $(S, +)$ and a multiplicative monoid $(S, *)$, where $*$ distributes over $+$.

See also:

Wikipedia article Semiring

EXAMPLES:

```
sage: Semirings()
Category of semirings
sage: Semirings().super_categories()
[Category of associative additive commutative additive associative additive␣
↪unital distributive magmas and additive magmas,
 Category of monoids]

sage: sorted(Semirings().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveUnital', 'Associative',
↪'Distributive', 'Unital']

sage: Semirings() is (CommutativeAdditiveMonoids() & Monoids()).Distributive()
True

sage: Semirings().AdditiveInverse()
Category of rings
```

4.142 Semisimple Algebras

class sage.categories.semisimple_algebras.**SemisimpleAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

The category of semisimple algebras over a given base ring.

EXAMPLES:

```
sage: from sage.categories.semisimple_algebras import SemisimpleAlgebras
sage: C = SemisimpleAlgebras(QQ); C
Category of semisimple algebras over Rational Field
```

This category is best constructed as:

```
sage: D = Algebras(QQ).Semisimple(); D
Category of semisimple algebras over Rational Field
sage: D is C
True

sage: C.super_categories()
[Category of algebras over Rational Field]
```

Typically, finite group algebras are semisimple:

```
sage: DihedralGroup(5).algebra(QQ) in SemisimpleAlgebras                   #␣
↪needs sage.groups
True
```

Unless the characteristic of the field divides the order of the group:

```
sage: DihedralGroup(5).algebra(IntegerModRing(5)) in SemisimpleAlgebras    #␣
↪needs sage.groups
False

sage: DihedralGroup(5).algebra(IntegerModRing(7)) in SemisimpleAlgebras    #␣
↪needs sage.groups
True
```

See also:

Wikipedia article Semisimple_algebra

class FiniteDimensional(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

WithBasis

alias of `FiniteDimensionalSemisimpleAlgebrasWithBasis`

class ParentMethods

Bases: `object`

radical_basis(***keywords*)

Return a basis of the Jacobson radical of this algebra.

- `keywords` – for compatibility; ignored.

OUTPUT: the empty list since this algebra is semisimple.

EXAMPLES:

```
sage: A = SymmetricGroup(4).algebra(QQ)                                    #␣
↪needs sage.groups
sage: A.radical_basis()                                                    #␣
↪needs sage.groups
()
```

super_categories()

EXAMPLES:

```
sage: Algebras(QQ).Semisimple().super_categories()
[Category of algebras over Rational Field]
```

4.143 Sets

exception sage.categories.sets_cat.**EmptySetError**

Bases: ValueError

Exception raised when some operation can't be performed on the empty set.

EXAMPLES:

```
sage: def first_element(st):
....:  if not st: raise EmptySetError("no elements")
....:  else: return st[0]
sage: first_element(Set((1,2,3)))
1
sage: first_element(Set([]))
Traceback (most recent call last):
...
EmptySetError: no elements
```

class sage.categories.sets_cat.**Sets**

Bases: *Category_singleton*

The category of sets.

The base category for collections of elements with = (equality).

This is also the category whose objects are all parents.

EXAMPLES:

```
sage: Sets()
Category of sets
sage: Sets().super_categories()
[Category of sets with partial maps]
sage: Sets().all_super_categories()
[Category of sets, Category of sets with partial maps, Category of objects]
```

Let us consider an example of set:

```
sage: P = Sets().example("inherits")
sage: P
Set of prime numbers
```

See `P??` for the code.

P is in the category of sets:

```
sage: P.category()
Category of sets
```

and therefore gets its methods from the following classes:

```
sage: for cl in P.__class__.mro(): print(cl)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Abstract'>
<class 'sage.structure.unique_representation.UniqueRepresentation'>
<class 'sage.structure.unique_representation.CachedRepresentation'>
<class 'sage.misc.fast_methods.WithEqualityById'>
```

(continues on next page)

(continued from previous page)

```
<class 'sage.structure.parent.Parent'>
<class 'sage.structure.category_object.CategoryObject'>
<class 'sage.structure.sage_object.SageObject'>
<class 'sage.categories.sets_cat.Sets.parent_class'>
<class 'sage.categories.sets_with_partial_maps.SetsWithPartialMaps.parent_class'>
<class 'sage.categories.objects.Objects.parent_class'>
<... 'object'>
```

We run some generic checks on P:

```
sage: TestSuite(P).run(verbose=True)                                      #␣
↪needs sage.libs.pari
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

Now, we manipulate some elements of P:

```
sage: P.an_element()
47
sage: x = P(3)
sage: x.parent()
Set of prime numbers
sage: x in P, 4 in P
(True, False)
sage: x.is_prime()
True
```

They get their methods from the following classes:

```
sage: for cl in x.__class__.mro(): print(cl)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category.
↪element_class'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits.Element'>
<class 'sage.rings.integer.IntegerWrapper'>
<class 'sage.rings.integer.Integer'>
<class 'sage.structure.element.EuclideanDomainElement'>
<class 'sage.structure.element.PrincipalIdealDomainElement'>
<class 'sage.structure.element.DedekindDomainElement'>
```

(continues on next page)

(continued from previous page)

```
<class 'sage.structure.element.IntegralDomainElement'>
<class 'sage.structure.element.CommutativeRingElement'>
<class 'sage.structure.element.RingElement'>
<class 'sage.structure.element.ModuleElement'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Abstract.Element'>
<class 'sage.structure.element.Element'>
<class 'sage.structure.sage_object.SageObject'>
<class 'sage.categories.sets_cat.Sets.element_class'>
<class 'sage.categories.sets_with_partial_maps.SetsWithPartialMaps.element_class'>
<class 'sage.categories.objects.Objects.element_class'>
<... 'object'>
```

FIXME: Objects.element_class is not very meaningful …

class Algebras(*category*, **args*)

Bases: `AlgebrasCategory`

class ParentMethods

Bases: `object`

construction()

Return the functorial construction of `self`.

EXAMPLES:

```
sage: A = GroupAlgebra(KleinFourGroup(), QQ)                                  #␣
↪needs sage.groups sage.modules
sage: F, arg = A.construction(); F, arg                                       #␣
↪needs sage.groups sage.modules
(GroupAlgebraFunctor, Rational Field)
sage: F(arg) is A                                                             #␣
↪needs sage.groups sage.modules
True
```

This also works for structures such as monoid algebras (see github issue #27937):

```
sage: A = FreeAbelianMonoid('x,y').algebra(QQ)                                #␣
↪needs sage.groups sage.modules
sage: F, arg = A.construction(); F, arg                                       #␣
↪needs sage.groups sage.modules
(The algebra functorial construction,
 Free abelian monoid on 2 generators (x, y))
sage: F(arg) is A                                                             #␣
↪needs sage.groups sage.modules
True
```

extra_super_categories()

EXAMPLES:

```
sage: Sets().Algebras(ZZ).super_categories()
[Category of modules with basis over Integer Ring]

sage: Sets().Algebras(QQ).extra_super_categories()
[Category of vector spaces with basis over Rational Field]

sage: Sets().example().algebra(ZZ).categories()                                 #␣
↪needs sage.modules
```

(continues on next page)

(continued from previous page)

```
[Category of set algebras over Integer Ring,
 Category of modules with basis over Integer Ring,
 ...
 Category of objects]
```

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example()
sage: C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: C.category()
Category of Cartesian products of sets
sage: C.categories()
[Category of Cartesian products of sets, Category of sets,
 Category of sets with partial maps,
 Category of objects]
sage: TestSuite(C).run()
```

class ElementMethods

Bases: `object`

cartesian_factors()

Return the Cartesian factors of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: H = CombinatorialFreeModule(ZZ, [4,7]); H.rename("H")
sage: S = cartesian_product([F, G, H])
sage: x = (S.monomial((0,4)) + 2 * S.monomial((0,5))
....:      + 3 * S.monomial((1,6)) + 4 * S.monomial((2,4))
....:      + 5 * S.monomial((2,7)))
sage: x.cartesian_factors()
(B[4] + 2*B[5], 3*B[6], 4*B[4] + 5*B[7])
sage: [s.parent() for s in x.cartesian_factors()]
[F, G, H]
sage: S.zero().cartesian_factors()
(0, 0, 0)
sage: [s.parent() for s in S.zero().cartesian_factors()]
[F, G, H]
```

cartesian_projection(*i*)

Return the projection of `self` onto the i-th factor of the Cartesian product.

INPUT:

- `i` – the index of a factor of the Cartesian product

EXAMPLES:

```
sage: # needs sage.modules
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: S = cartesian_product([F, G])
sage: x = (S.monomial((0,4)) + 2 * S.monomial((0,5))
....:      + 3 * S.monomial((1,6)))
sage: x.cartesian_projection(0)
B[4] + 2*B[5]
sage: x.cartesian_projection(1)
3*B[6]
```

class ParentMethods

Bases: `object`

an_element()

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: C.an_element()
(47, 42, 1)
```

cardinality()

Return the cardinality of self.

EXAMPLES:

```
sage: E = FiniteEnumeratedSet([1,2,3])
sage: C = cartesian_product([E, SymmetricGroup(4)])                        #␣
↪needs sage.groups
sage: C.cardinality()                                                      #␣
↪needs sage.groups
72

sage: E = FiniteEnumeratedSet([])
sage: C = cartesian_product([E, ZZ, QQ])
sage: C.cardinality()
0

sage: C = cartesian_product([ZZ, QQ])
sage: C.cardinality()
+Infinity

sage: cartesian_product([GF(5), Permutations(10)]).cardinality()
18144000
sage: cartesian_product([GF(71)]*20).cardinality() == 71**20
True
```

cartesian_factors()

Return the Cartesian factors of `self`.

EXAMPLES:

```
sage: cartesian_product([QQ, ZZ, ZZ]).cartesian_factors()
(Rational Field, Integer Ring, Integer Ring)
```

cartesian_projection(*i*)

Return the natural projection onto the i-th Cartesian factor of `self`.

INPUT:

- `i` – the index of a Cartesian factor of `self`

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: x = C.an_element(); x
(47, 42, 1)
sage: pi = C.cartesian_projection(1)
sage: pi(x)
42
```

construction()

The construction functor and the list of Cartesian factors.

EXAMPLES:

```
sage: C = cartesian_product([QQ, ZZ, ZZ])
sage: C.construction()
(The cartesian_product functorial construction,
(Rational Field, Integer Ring, Integer Ring))
```

is_empty()

Return whether this set is empty.

EXAMPLES:

```
sage: S1 = FiniteEnumeratedSet([1,2,3])
sage: S2 = Set([])
sage: cartesian_product([S1,ZZ]).is_empty()
False
sage: cartesian_product([S1,S2,S1]).is_empty()
True
```

is_finite()

Return whether this set is finite.

EXAMPLES:

```
sage: E = FiniteEnumeratedSet([1,2,3])
sage: C = cartesian_product([E, SymmetricGroup(4)])                    #␣
↪needs sage.groups
sage: C.is_finite()                                                    #␣
↪needs sage.groups
True

sage: cartesian_product([ZZ,ZZ]).is_finite()
False
sage: cartesian_product([ZZ, Set(), ZZ]).is_finite()
True
```

random_element(**args*)

Return a random element of this Cartesian product.

The extra arguments are passed down to each of the factors of the Cartesian product.

EXAMPLES:

```
sage: C = cartesian_product([Permutations(10)]*5)
sage: C.random_element()           # random
([2, 9, 4, 7, 1, 8, 6, 10, 5, 3],
 [8, 6, 5, 7, 1, 4, 9, 3, 10, 2],
 [5, 10, 3, 8, 2, 9, 1, 4, 7, 6],
 [9, 6, 10, 3, 2, 1, 5, 8, 7, 4],
 [8, 5, 2, 9, 10, 3, 7, 1, 4, 6])

sage: C = cartesian_product([ZZ]*10)
sage: c1 = C.random_element()
sage: c1                    # random
(3, 1, 4, 1, 1, -3, 0, -4, -17, 2)
sage: c2 = C.random_element(4,7)
sage: c2                    # random
(6, 5, 6, 4, 5, 6, 6, 4, 5, 5)
sage: all(4 <= i < 7 for i in c2)
True
```

example()

EXAMPLES:

```
sage: Sets().CartesianProducts().example()
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
```

extra_super_categories()

A Cartesian product of sets is a set.

EXAMPLES:

```
sage: Sets().CartesianProducts().extra_super_categories()
[Category of sets]
sage: Sets().CartesianProducts().super_categories()
[Category of sets]
```

class ElementMethods

Bases: `object`

cartesian_product(**elements*)

Return the Cartesian product of its arguments, as an element of the Cartesian product of the parents of those elements.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example()                                                    #␣
↪needs sage.combinat sage.modules
sage: a, b, c = A.algebra_generators()                                   #␣
↪needs sage.combinat sage.modules
sage: a.cartesian_product(b, c)                                          #␣
↪needs sage.combinat sage.modules
B[(0, word: a)] + B[(1, word: b)] + B[(2, word: c)]
```

FIXME: is this a policy that we want to enforce on all parents?

Enumerated

alias of *EnumeratedSets*

Facade

alias of *FacadeSets*

Finite

alias of *FiniteSets*

class Infinite(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class ParentMethods

Bases: object

cardinality()

Count the elements of the enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.cardinality()
+Infinity
```

is_empty()

Return whether this set is empty.

Since this set is infinite this always returns False.

EXAMPLES:

```
sage: C = InfiniteEnumeratedSets().example()
sage: C.is_empty()
False
```

is_finite()

Return whether this set is finite.

Since this set is infinite this always returns False.

EXAMPLES:

```
sage: C = InfiniteEnumeratedSets().example()
sage: C.is_finite()
False
```

class IsomorphicObjects(*category*, **args*)

Bases: *IsomorphicObjectsCategory*

A category for isomorphic objects of sets.

EXAMPLES:

```
sage: Sets().IsomorphicObjects()
Category of isomorphic objects of sets
sage: Sets().IsomorphicObjects().all_super_categories()
[Category of isomorphic objects of sets,
 Category of subobjects of sets, Category of quotients of sets,
 Category of subquotients of sets,
```

(continues on next page)

(continued from previous page)

```
Category of sets,
Category of sets with partial maps,
Category of objects]
```

class ParentMethods

Bases: `object`

Metric

alias of *MetricSpaces*

class MorphismMethods

Bases: `object`

image(*domain_subset=None*)

Return the image of the domain or of `domain_subset`.

EXAMPLES:

```
sage: # needs sage.combinat
sage: P = Partitions(6)
sage: H = Hom(P, ZZ)
sage: f = H(ZZ.sum)
sage: X = f.image()                                                         #␣
↪needs sage.libs.flint
sage: list(X)                                                               #␣
↪needs sage.libs.flint
[6]
```

is_injective()

Return whether this map is injective.

EXAMPLES:

```
sage: f = ZZ.hom(GF(3)); f
Natural morphism:
  From: Integer Ring
  To:   Finite Field of size 3
sage: f.is_injective()
False
```

class ParentMethods

Bases: `object`

CartesianProduct

alias of `CartesianProduct`

algebra(*base_ring*, *category=None*, ***kwds*)

Return the algebra of `self` over `base_ring`.

INPUT:

- `self` – a parent S
- `base_ring` – a ring K
- `category` – a super category of the category of S, or `None`

This returns the space of formal linear combinations of elements of S with coefficients in K, endowed with whatever structure can be induced from that of S. See the documentation of *sage.categories.algebra_functor* for details.

EXAMPLES:

If S is a `group`, the result is its group algebra KS:

```
sage: # needs sage.groups sage.modules
sage: S = DihedralGroup(4); S
Dihedral group of order 8 as a permutation group
sage: A = S.algebra(QQ); A
Algebra of Dihedral group of order 8 as a permutation group
 over Rational Field
sage: A.category()
Category of finite group algebras over Rational Field
sage: a = A.an_element(); a
() + (1,3) + 2*(1,3)(2,4) + 3*(1,4,3,2)
```

This space is endowed with an algebra structure, obtained by extending by bilinearity the multiplication of G to a multiplication on RG:

```
sage: a * a                                                              #␣
↪needs sage.groups sage.modules
6*() + 4*(2,4) + 3*(1,2)(3,4) + 12*(1,2,3,4) + 2*(1,3)
 + 13*(1,3)(2,4) + 6*(1,4,3,2) + 3*(1,4)(2,3)
```

If S is a `monoid`, the result is its monoid algebra KS:

```
sage: S = Monoids().example(); S
An example of a monoid:
 the free monoid generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ); A                                               #␣
↪needs sage.modules
Algebra of
 An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
 over Rational Field
sage: A.category()                                                       #␣
↪needs sage.modules
Category of monoid algebras over Rational Field
```

Similarly, we can construct algebras for additive magmas, monoids, and groups.

One may specify for which category one takes the algebra; here we build the algebra of the additive group GF_3:

```
sage: # needs sage.modules
sage: from sage.categories.additive_groups import AdditiveGroups
sage: S = GF(7)
sage: A = S.algebra(QQ, category=AdditiveGroups()); A
Algebra of Finite Field of size 7 over Rational Field
sage: A.category()
Category of finite dimensional additive group algebras
         over Rational Field
sage: a = A(S(1))
sage: a
1
sage: 1 + a * a * a
0 + 3
```

Note that the `category` keyword needs to be fed with the structure on S to be used, not the induced structure on the result.

an_element()

Return a (preferably typical) element of this parent.

This is used both for illustration and testing purposes. If the set `self` is empty, *an_element()* should raise the exception *EmptySetError*.

This default implementation calls `_an_element_()` and caches the result. Any parent should implement either *an_element()* or `_an_element_()`.

EXAMPLES:

```
sage: CDF.an_element()
↪# needs sage.rings.complex_double
1.0*I
sage: ZZ[['t']].an_element()
t
```

cartesian_product(**parents*, ***kwargs*)

Return the Cartesian product of the parents.

INPUT:

- `parents` – a list (or other iterable) of parents.
- `category` – (default: `None`) the category the Cartesian product belongs to. If `None` is passed, then `category_from_parents()` is used to determine the category.
- `extra_category` – (default: `None`) a category that is added to the Cartesian product in addition to the categories obtained from the parents.
- other keyword arguments will passed on to the class used for this Cartesian product (see also `CartesianProduct`).

OUTPUT:

The Cartesian product.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example(); A.rename("A")                                      #␣
↪needs sage.combinat sage.modules
sage: A.cartesian_product(A, A)                                           #␣
↪needs sage.combinat sage.modules
A (+) A (+) A
sage: ZZ.cartesian_product(GF(2), FiniteEnumeratedSet([1,2,3]))
The Cartesian product of (Integer Ring,
                          Finite Field of size 2, {1, 2, 3})

sage: C = ZZ.cartesian_product(A); C                                      #␣
↪needs sage.combinat sage.modules
The Cartesian product of (Integer Ring, A)
```

construction()

Return a pair `(functor, parent)` such that `functor(parent)` returns `self`. If `self` does not have a functorial construction, return `None`.

EXAMPLES:

```
sage: QQ.construction()
(FractionField, Integer Ring)
sage: f, R = QQ['x'].construction()
sage: f
```

(continues on next page)

(continued from previous page)

```
Poly[x]
sage: R
Rational Field
sage: f(R)
Univariate Polynomial Ring in x over Rational Field
```

is_parent_of(*element*)

Return whether `self` is the parent of `element`.

INPUT:

- `element` – any object

EXAMPLES:

```
sage: S = ZZ
sage: S.is_parent_of(1)
True
sage: S.is_parent_of(2/1)
False
```

This method differs from `__contains__()` because it does not attempt any coercion:

```
sage: 2/1 in S, S.is_parent_of(2/1)
(True, False)
sage: int(1) in S, S.is_parent_of(int(1))
(True, False)
```

some_elements()

Return a list (or iterable) of elements of `self`.

This is typically used for running generic tests (see `TestSuite`).

This default implementation calls `an_element()`.

EXAMPLES:

```
sage: S = Sets().example(); S
Set of prime numbers (basic implementation)
sage: S.an_element()
47
sage: S.some_elements()
[47]
sage: S = Set([])
sage: list(S.some_elements())
[]
```

This method should return an iterable, *not* an iterator.

class Quotients(*category*, **args*)

Bases: `QuotientsCategory`

A category for quotients of sets.

See also:

`Sets().Quotients()`

EXAMPLES:

```
sage: Sets().Quotients()
Category of quotients of sets
sage: Sets().Quotients().all_super_categories()
[Category of quotients of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

class ParentMethods

Bases: `object`

class Realizations(*category*, **args*)

Bases: *RealizationsCategory*

class ParentMethods

Bases: `object`

realization_of()

Return the parent this is a realization of.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                          #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: In = A.In(); In                                                     #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: In.realization_of()                                                 #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
```

class SubcategoryMethods

Bases: `object`

Algebras(*base_ring*)

Return the category of objects constructed as algebras of objects of `self` over `base_ring`.

INPUT:

- `base_ring` – a ring

See *Sets.ParentMethods.algebra()* for the precise meaning in Sage of the *algebra of an object*.

EXAMPLES:

```
sage: Monoids().Algebras(QQ)
Category of monoid algebras over Rational Field

sage: Groups().Algebras(QQ)
Category of group algebras over Rational Field

sage: AdditiveMagmas().AdditiveAssociative().Algebras(QQ)
Category of additive semigroup algebras over Rational Field

sage: Monoids().Algebras(Rings())
Category of monoid algebras over Category of rings
```

See also:

- *algebra_functor.AlgebrasCategory*
- *CovariantFunctorialConstruction*

CartesianProducts()

Return the full subcategory of the objects of `self` constructed as Cartesian products.

See also:

- *cartesian_product.CartesianProductFunctor*
- `RegressiveCovariantFunctorialConstruction`

EXAMPLES:

```
sage: Sets().CartesianProducts()
Category of Cartesian products of sets
sage: Semigroups().CartesianProducts()
Category of Cartesian products of semigroups
sage: EuclideanDomains().CartesianProducts()
Category of Cartesian products of commutative rings
```

Enumerated()

Return the full subcategory of the enumerated objects of `self`.

An enumerated object can be iterated to get its elements.

EXAMPLES:

```
sage: Sets().Enumerated()
Category of enumerated sets
sage: Rings().Finite().Enumerated()
Category of finite enumerated rings
sage: Rings().Infinite().Enumerated()
Category of infinite enumerated rings
```

Facade()

Return the full subcategory of the facade objects of `self`.

What is a facade set?

Recall that, in Sage, *sets are modelled by *parents**, and their elements know which distinguished set they belong to. For example, the ring of integers $\mathbf{Z}$ is modelled by the parent `ZZ`, and integers know that they belong to this set:

```
sage: ZZ
Integer Ring
sage: 42.parent()
Integer Ring
```

Sometimes, it is convenient to represent the elements of a parent `P` by elements of some other parent. For example, the elements of the set of prime numbers are represented by plain integers:

```
sage: Primes()
Set of all prime numbers: 2, 3, 5, 7, ...
sage: p = Primes().an_element(); p
43
```

(continues on next page)

(continued from previous page)

```
sage: p.parent()
Integer Ring
```

In this case, `P` is called a *facade set*.

This feature is advertised through the category of P:

```
sage: Primes().category()
Category of facade infinite enumerated sets
sage: Sets().Facade()
Category of facade sets
```

Typical use cases include modeling a subset of an existing parent:

```
sage: Set([4,6,9])                    # random
{4, 6, 9}
sage: Sets().Facade().example()
An example of facade set: the monoid of positive integers
```

or the union of several parents:

```
sage: Sets().Facade().example("union")
An example of a facade set: the integers completed by +-infinity
```

or endowing an existing parent with more (or less!) structure:

```
sage: Posets().example("facade")
An example of a facade poset: the positive integers ordered by␣
↪divisibility
```

Let us investigate in detail a close variant of this last example: let P be set of divisors of 12 partially ordered by divisibility. There are two options for representing its elements:

1. as plain integers:

   ```
   sage: P = Poset((divisors(12), attrcall("divides")), facade=True)    ␣
   ↪ # needs sage.graphs
   ```

2. as integers, modified to be aware that their parent is P:

   ```
   sage: Q = Poset((divisors(12), attrcall("divides")), facade=False)   ␣
   ↪ # needs sage.graphs
   ```

The advantage of option 1. is that one needs not do conversions back and forth between P and $\mathbf{Z}$. The disadvantage is that this introduces an ambiguity when writing $2 < 3$: does this compare 2 and 3 w.r.t. the natural order on integers or w.r.t. divisibility?:

```
sage: 2 < 3
True
```

To raise this ambiguity, one needs to explicitly specify the underlying poset as in $2 <_P 3$:

```
sage: P = Posets().example("facade")
sage: P.lt(2,3)
False
```

On the other hand, with option 2. and once constructed, the elements know unambiguously how to compare themselves:

```
sage: Q(2) < Q(3)                                                          #␣
↪needs sage.graphs
False
sage: Q(2) < Q(6)                                                          #␣
↪needs sage.graphs
True
```

Beware that `P(2)` is still the integer 2. Therefore `P(2) < P(3)` still compares 2 and 3 as integers!:

```
sage: P(2) < P(3)
True
```

In short P being a facade parent is one of the programmatic counterparts (with e.g. coercions) of the usual mathematical idiom: "for ease of notation, we identify an element of P with the corresponding integer". Too many identifications lead to confusion; the lack thereof leads to heavy, if not obfuscated, notations. Finding the right balance is an art, and even though there are common guidelines, it is ultimately up to the writer to choose which identifications to do. This is no different in code.

See also:

The following examples illustrate various ways to implement subsets like the set of prime numbers; look at their code for details:

```
sage: Sets().example("facade")
Set of prime numbers (facade implementation)
sage: Sets().example("inherits")
Set of prime numbers
sage: Sets().example("wrapper")
Set of prime numbers (wrapper implementation)
```

Specifications

A parent which is a facade must either:

- call `Parent.__init__()` using the `facade` parameter to specify a parent, or tuple thereof.
- overload the method `facade_for()`.

Note: The concept of facade parents was originally introduced in the computer algebra system MuPAD.

Finite()

Return the full subcategory of the finite objects of `self`.

EXAMPLES:

```
sage: Sets().Finite()
Category of finite sets
sage: Rings().Finite()
Category of finite rings
```

Infinite()

Return the full subcategory of the infinite objects of `self`.

EXAMPLES:

```
sage: Sets().Infinite()
Category of infinite sets
sage: Rings().Infinite()
Category of infinite rings
```

IsomorphicObjects()

Return the full subcategory of the objects of `self` constructed by isomorphism.

Given a concrete category `As()` (i.e. a subcategory of `Sets()`), `As().IsomorphicObjects()` returns the category of objects of `As()` endowed with a distinguished description as the image of some other object of `As()` by an isomorphism in this category.

See *Subquotients()* for background.

EXAMPLES:

In the following example, A is defined as the image by $x \mapsto x^2$ of the finite set $B = \{1, 2, 3\}$:

```
sage: A = FiniteEnumeratedSets().IsomorphicObjects().example(); A
The image by some isomorphism of An example of a finite enumerated set:
↪{1,2,3}
```

Since B is a finite enumerated set, so is A:

```
sage: A in FiniteEnumeratedSets()
True
sage: A.cardinality()
3
sage: A.list()
[1, 4, 9]
```

The isomorphism from B to A is available as:

```
sage: A.retract(3)
9
```

and its inverse as:

```
sage: A.lift(9)
3
```

It often is natural to declare those morphisms as coercions so that one can do `A(b)` and `B(a)` to go back and forth between A and B (TODO: refer to a category example where the maps are declared as a coercion). This is not done by default. Indeed, in many cases one only wants to transport part of the structure of B to A. Assume for example, that one wants to construct the set of integers $B = ZZ$, endowed with `max` as addition, and `+` as multiplication instead of the usual `+` and `*`. One can construct A as isomorphic to B as an infinite enumerated set. However A is *not* isomorphic to B as a ring; for example, for $a \in A$ and $a \in B$, the expressions $a + A(b)$ and $B(a) + b$ give completely different results; hence we would not want the expression $a + b$ to be implicitly resolved to any one of above two, as the coercion mechanism would do.

Coercions also cannot be used with facade parents (see *Sets.Facade*) like in the example above.

We now look at a category of isomorphic objects:

```
sage: C = Sets().IsomorphicObjects(); C
Category of isomorphic objects of sets
```

(continues on next page)

(continued from previous page)

```
sage: C.super_categories()
[Category of subobjects of sets, Category of quotients of sets]

sage: C.all_super_categories()
[Category of isomorphic objects of sets,
 Category of subobjects of sets,
 Category of quotients of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Unless something specific about isomorphic objects is implemented for this category, one actually get an optimized super category:

```
sage: C = Semigroups().IsomorphicObjects(); C
Join of Category of quotients of semigroups
    and Category of isomorphic objects of sets
```

See also:

- *`Subquotients()`* for background
- *`isomorphic_objects.IsomorphicObjectsCategory`*
- `RegressiveCovariantFunctorialConstruction`

`Metric``()`

Return the subcategory of the metric objects of `self`.

`Quotients``()`

Return the full subcategory of the objects of `self` constructed as quotients.

Given a concrete category `As()` (i.e. a subcategory of `Sets()`), `As().Quotients()` returns the category of objects of `As()` endowed with a distinguished description as quotient (in fact homomorphic image) of some other object of `As()`.

Implementing an object of `As().Quotients()` is done in the same way as for `As().Subquotients()`; namely by providing an ambient space and a lift and a retract map. See *`Subquotients()`* for detailed instructions.

See also:

- *`Subquotients()`* for background
- *`quotients.QuotientsCategory`*
- `RegressiveCovariantFunctorialConstruction`

EXAMPLES:

```
sage: C = Semigroups().Quotients(); C
Category of quotients of semigroups
sage: C.super_categories()
[Category of subquotients of semigroups, Category of quotients of sets]
sage: C.all_super_categories()
[Category of quotients of semigroups,
 Category of subquotients of semigroups,
 Category of semigroups,
 Category of subquotients of magmas,
 Category of magmas,
 Category of quotients of sets,
```

(continues on next page)

(continued from previous page)

```
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

The caller is responsible for checking that the given category admits a well defined category of quotients:

```
sage: EuclideanDomains().Quotients()
Join of Category of euclidean domains
    and Category of subquotients of monoids
    and Category of quotients of semigroups
```

Subobjects()

Return the full subcategory of the objects of `self` constructed as subobjects.

Given a concrete category `As()` (i.e. a subcategory of `Sets()`), `As().Subobjects()` returns the category of objects of `As()` endowed with a distinguished embedding into some other object of `As()`.

Implementing an object of `As().Subobjects()` is done in the same way as for `As().Subquotients()`; namely by providing an ambient space and a lift and a retract map. In the case of a trivial embedding, the two maps will typically be identity maps that just change the parent of their argument. See *Subquotients()* for detailed instructions.

See also:

- *Subquotients()* for background
- *subobjects.SubobjectsCategory*
- `RegressiveCovariantFunctorialConstruction`

EXAMPLES:

```
sage: C = Sets().Subobjects(); C
Category of subobjects of sets

sage: C.super_categories()
[Category of subquotients of sets]

sage: C.all_super_categories()
[Category of subobjects of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Unless something specific about subobjects is implemented for this category, one actually gets an optimized super category:

```
sage: C = Semigroups().Subobjects(); C
Join of Category of subquotients of semigroups
    and Category of subobjects of sets
```

The caller is responsible for checking that the given category admits a well defined category of subobjects.

Subquotients()

Return the full subcategory of the objects of `self` constructed as subquotients.

Given a concrete category `self == As()` (i.e. a subcategory of `Sets()`), `As().Subquotients()` returns the category of objects of `As()` endowed with a distinguished description as subquotient of some other object of `As()`.

EXAMPLES:

```
sage: Monoids().Subquotients()
Category of subquotients of monoids
```

A parent A in `As()` is further in `As().Subquotients()` if there is a distinguished parent B in `As()`, called the *ambient set*, a subobject B' of B, and a pair of maps:

$$l : A \to B' \text{ and } r : B' \to A$$

called respectively the *lifting map* and *retract map* such that $r \circ l$ is the identity of A and r is a morphism in `As()`.

Todo: Draw the typical commutative diagram.

It follows that, for each operation *op* of the category, we have some property like:

$$op_A(e) = r(op_B(l(e))), \text{ for all } e \in A$$

This allows for implementing the operations on A from those on B.

The two most common use cases are:

- *homomorphic images* (or *quotients*), when $B' = B$, r is an homomorphism from B to A (typically a canonical quotient map), and l a section of it (not necessarily a homomorphism); see `Quotients()`;
- *subobjects* (up to an isomorphism), when l is an embedding from A into B; in this case, B' is typically isomorphic to A through the inverse isomorphisms r and l; see `Subobjects()`;

Note:

- The usual definition of "subquotient" (Wikipedia article Subquotient) does not involve the lifting map l. This map is required in Sage's context to make the definition constructive. It is only used in computations and does not affect their results. This is relatively harmless since the category is a concrete category (i.e., its objects are sets and its morphisms are set maps).
- In mathematics, especially in the context of quotients, the retract map r is often referred to as a *projection map* instead.
- Since B' is not specified explicitly, it is possible to abuse the framework with situations where B' is not quite a subobject and r not quite a morphism, as long as the lifting and retract maps can be used as above to compute all the operations in A. Use at your own risk!

Assumptions:

- For any category `As()`, `As().Subquotients()` is a subcategory of `As()`.

  Example: a subquotient of a group is a group (e.g., a left or right quotient of a group by a non-normal subgroup is not in this category).
- This construction is covariant: if `As()` is a subcategory of `Bs()`, then `As().Subquotients()` is a subcategory of `Bs().Subquotients()`.

  Example: if A is a subquotient of B in the category of groups, then it is also a subquotient of B in the category of monoids.
- If the user (or a program) calls `As().Subquotients()`, then it is assumed that subquotients are well defined in this category. This is not checked, and probably never will be. Note that, if a category `As()` does not specify anything about its subquotients, then its subquotient category looks like this:

```
sage: EuclideanDomains().Subquotients()
Join of Category of euclidean domains
    and Category of subquotients of monoids
```

Interface: the ambient set B of A is given by `A.ambient()`. The subset B' needs not be specified, so the retract map is handled as a partial map from B to A.

The lifting and retract map are implemented respectively as methods `A.lift(a)` and `A.retract(b)`. As a shorthand for the former, one can use alternatively `a.lift()`:

```
sage: S = Semigroups().Subquotients().example(); S
An example of a (sub)quotient semigroup: a quotient of the left zero␣
↪semigroup
sage: S.ambient()
An example of a semigroup: the left zero semigroup
sage: S(3).lift().parent()
An example of a semigroup: the left zero semigroup
sage: S(3) * S(1) == S.retract( S(3).lift() * S(1).lift() )
True
```

See `S?` for more.

Todo: use a more interesting example, like $\mathbf{Z}/n\mathbf{Z}$.

See also:

- *`Quotients()`*, *`Subobjects()`*, *`IsomorphicObjects()`*
- *`subquotients.SubquotientsCategory`*
- `RegressiveCovariantFunctorialConstruction`

`Topological()`

Return the subcategory of the topological objects of `self`.

class `Subobjects`(*category*, **args*)

Bases: *`SubobjectsCategory`*

A category for subobjects of sets.

See also:

`Sets().Subobjects()`

EXAMPLES:

```
sage: Sets().Subobjects()
Category of subobjects of sets
sage: Sets().Subobjects().all_super_categories()
[Category of subobjects of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

class `ParentMethods`

Bases: `object`

class `Subquotients`(*category*, **args*)

Bases: *`SubquotientsCategory`*

A category for subquotients of sets.

See also:

`Sets().Subquotients()`

EXAMPLES:

```
sage: Sets().Subquotients()
Category of subquotients of sets
sage: Sets().Subquotients().all_super_categories()
[Category of subquotients of sets, Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

class ElementMethods

Bases: `object`

lift()

Lift `self` to the ambient space for its parent.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: s = S.an_element()
sage: s, s.parent()
(42, An example of a (sub)quotient semigroup:
      a quotient of the left zero semigroup)
sage: S.lift(s), S.lift(s).parent()
(42, An example of a semigroup: the left zero semigroup)
sage: s.lift(), s.lift().parent()
(42, An example of a semigroup: the left zero semigroup)
```

class ParentMethods

Bases: `object`

ambient()

Return the ambient space for `self`.

EXAMPLES:

```
sage: Semigroups().Subquotients().example().ambient()
An example of a semigroup: the left zero semigroup
```

See also:

`Sets.SubcategoryMethods.Subquotients()` for the specifications and *`lift()`* and *`retract()`*.

lift(*x*)

Lift x to the ambient space for `self`.

INPUT:

- `x` – an element of `self`

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: s = S.an_element()
sage: s, s.parent()
```

(continues on next page)

(continued from previous page)

```
(42, An example of a (sub)quotient semigroup:
      a quotient of the left zero semigroup)
sage: S.lift(s), S.lift(s).parent()
(42, An example of a semigroup: the left zero semigroup)
sage: s.lift(), s.lift().parent()
(42, An example of a semigroup: the left zero semigroup)
```

See also:

Sets.SubcategoryMethods.Subquotients for the specifications, *ambient()*, *retract()*, and also *Sets.Subquotients.ElementMethods.lift()*.

retract(*x*)

Retract x to self.

INPUT:
- x – an element of the ambient space for self

See also:

Sets.SubcategoryMethods.Subquotients for the specifications, *ambient()*, *retract()*, and also Sets.Subquotients.ElementMethods.retract().

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: s = S.ambient().an_element()
sage: s, s.parent()
(42, An example of a semigroup: the left zero semigroup)
sage: S.retract(s), S.retract(s).parent()
(42, An example of a (sub)quotient semigroup:
      a quotient of the left zero semigroup)
```

Topological

alias of *TopologicalSpaces*

class WithRealizations(*category*, **args*)

Bases: *WithRealizationsCategory*

class ParentMethods

Bases: object

class Realizations(*parent_with_realization*)

Bases: *Category_realization_of_parent*

super_categories()

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                    #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.Realizations().super_categories()                           #␣
↪needs sage.modules
[Category of realizations of sets]
```

a_realization()

Return a realization of self.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                            #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.a_realization()                                                     #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
 in the Fundamental basis
```

facade_for()

Return the parents `self` is a facade for, that is the realizations of `self`

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                            #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.facade_for()                                                        #␣
↪needs sage.modules
[The subset algebra of {1, 2, 3} over Rational Field in the␣
↪Fundamental basis,
 The subset algebra of {1, 2, 3} over Rational Field in the In basis,
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis]

sage: # needs sage.combinat sage.modules
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: f = A.F().an_element(); f
F[{}] + 2*F[{1}] + 3*F[{2}] + F[{1, 2}]
sage: i = A.In().an_element(); i
In[{}] + 2*In[{1}] + 3*In[{2}] + In[{1, 2}]
sage: o = A.Out().an_element(); o
Out[{}] + 2*Out[{1}] + 3*Out[{2}] + Out[{1, 2}]
sage: f in A, i in A, o in A
(True, True, True)
```

inject_shorthands(*shorthands=None*, *verbose=True*)

Import standard shorthands into the global namespace.

INPUT:

- `shorthands` – a list (or iterable) of strings (default: `self._shorthands`) or `"all"` (for `self._shorthands_all`)
- **verbose – boolean (default True);**
  whether to print the defined shorthands

EXAMPLES:

When computing with a set with multiple realizations, like `SymmetricFunctions` or *`SubsetAlgebra`*, it is convenient to define shorthands for the various realizations, but cumbersome to do it by hand:

```
sage: S = SymmetricFunctions(ZZ); S                                         #␣
↪needs sage.combinat sage.modules
Symmetric Functions over Integer Ring
sage: s = S.s(); s                                                          #␣
↪needs sage.combinat sage.modules
Symmetric Functions over Integer Ring in the Schur basis
sage: e = S.e(); e                                                          #␣
↪needs sage.combinat sage.modules
Symmetric Functions over Integer Ring in the elementary basis
```

This method automates the process:

```
sage: # needs sage.combinat sage.modules
sage: S.inject_shorthands()
Defining e as shorthand for
 Symmetric Functions over Integer Ring in the elementary basis
Defining f as shorthand for
 Symmetric Functions over Integer Ring in the forgotten basis
Defining h as shorthand for
 Symmetric Functions over Integer Ring in the homogeneous basis
Defining m as shorthand for
 Symmetric Functions over Integer Ring in the monomial basis
Defining p as shorthand for
 Symmetric Functions over Integer Ring in the powersum basis
Defining s as shorthand for
 Symmetric Functions over Integer Ring in the Schur basis
sage: s[1] + e[2] * p[1,1] + 2*h[3] + m[2,1]
s[1] - 2*s[1, 1, 1] + s[1, 1, 1, 1] + s[2, 1]
+ 2*s[2, 1, 1] + s[2, 2] + 2*s[3] + s[3, 1]
sage: e
Symmetric Functions over Integer Ring in the elementary basis
sage: p
Symmetric Functions over Integer Ring in the powersum basis
sage: s
Symmetric Functions over Integer Ring in the Schur basis
```

Sometimes, like for symmetric functions, one can request for all shorthands to be defined, including less common ones:

```
sage: S.inject_shorthands("all")                                          #␣
↪needs sage.combinat sage.modules
Defining e as shorthand for
 Symmetric Functions over Integer Ring in the elementary basis
Defining f as shorthand for
 Symmetric Functions over Integer Ring in the forgotten basis
Defining h as shorthand for
 Symmetric Functions over Integer Ring in the homogeneous basis
Defining ht as shorthand for
 Symmetric Functions over Integer Ring in the
  induced trivial symmetric group character basis
Defining m as shorthand for
 Symmetric Functions over Integer Ring in the monomial basis
Defining o as shorthand for
 Symmetric Functions over Integer Ring in the orthogonal basis
Defining p as shorthand for
 Symmetric Functions over Integer Ring in the powersum basis
Defining s as shorthand for
 Symmetric Functions over Integer Ring in the Schur basis
Defining sp as shorthand for
 Symmetric Functions over Integer Ring in the symplectic basis
Defining st as shorthand for
 Symmetric Functions over Integer Ring in the
  irreducible symmetric group character basis
Defining w as shorthand for
 Symmetric Functions over Integer Ring in the Witt basis
```

The messages can be silenced by setting `verbose=False`:

```
sage: # needs sage.combinat sage.modules
sage: Q = QuasiSymmetricFunctions(ZZ)
sage: Q.inject_shorthands(verbose=False)
sage: F[1,2,1] + 5*M[1,3] + F[2]^2
5*F[1, 1, 1, 1] - 5*F[1, 1, 2] - 3*F[1, 2, 1] + 6*F[1, 3] +
2*F[2, 2] + F[3, 1] + F[4]
sage: F
Quasisymmetric functions over the Integer Ring in the
 Fundamental basis
sage: M
Quasisymmetric functions over the Integer Ring in the
 Monomial basis
```

One can also just import a subset of the shorthands:

```
sage: # needs sage.combinat sage.modules
sage: SQ = SymmetricFunctions(QQ)
sage: SQ.inject_shorthands(['p', 's'], verbose=False)
sage: p
Symmetric Functions over Rational Field in the powersum basis
sage: s
Symmetric Functions over Rational Field in the Schur basis
```

Note that e is left unchanged:

```
sage: e                                                              #␣
↪needs sage.combinat sage.modules
Symmetric Functions over Integer Ring in the elementary basis
```

realizations()

Return all the realizations of self that self is aware of.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                     #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.realizations()                                               #␣
↪needs sage.modules
[The subset algebra of {1, 2, 3} over Rational Field in the␣
↪Fundamental basis,
 The subset algebra of {1, 2, 3} over Rational Field in the In basis,
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis]
```

Note: Constructing a parent P in the category A.Realizations() automatically adds P to this list by calling A._register_realization(A)

example(*base_ring=None*, *set=None*)

Return an example of set with multiple realizations, as per *Category.example()*.

EXAMPLES:

```
sage: Sets().WithRealizations().example()                              #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
```

(continues on next page)

(continued from previous page)

```
sage: Sets().WithRealizations().example(ZZ, Set([1,2]))                    #␣
↪needs sage.modules
The subset algebra of {1, 2} over Integer Ring
```

extra_super_categories()

A set with multiple realizations is a facade parent.

EXAMPLES:

```
sage: Sets().WithRealizations().extra_super_categories()
[Category of facade sets]
sage: Sets().WithRealizations().super_categories()
[Category of facade sets]
```

example(*choice=None*)

Return examples of objects of `Sets()`, as per *Category.example()*.

EXAMPLES:

```
sage: Sets().example()
Set of prime numbers (basic implementation)

sage: Sets().example("inherits")
Set of prime numbers

sage: Sets().example("facade")
Set of prime numbers (facade implementation)

sage: Sets().example("wrapper")
Set of prime numbers (wrapper implementation)
```

super_categories()

We include SetsWithPartialMaps between Sets and Objects so that we can define morphisms between sets that are only partially defined. This is also to have the Homset constructor not complain that SetsWithPartialMaps is not a supercategory of Fields, for example.

EXAMPLES:

```
sage: Sets().super_categories()
[Category of sets with partial maps]
```

sage.categories.sets_cat.**print_compare**(*x*, *y*)

Helper method used in `Sets.ParentMethods._test_elements_eq_symmetric()`, `Sets.ParentMethods._test_elements_eq_tranisitive()`.

INPUT:

- `x` – an element
- `y` – an element

EXAMPLES:

```
sage: from sage.categories.sets_cat import print_compare
sage: print_compare(1,2)
1 != 2
```

(continues on next page)

(continued from previous page)

```
sage: print_compare(1,1)
1 == 1
```

4.144 Sets With a Grading

class sage.categories.sets_with_grading.**SetsWithGrading**

Bases: *Category*

The category of sets with a grading.

A *set with a grading* is a set S equipped with a grading by some other set I (by default the set $\mathbf{N}$ of the non-negative integers):

$$S = \biguplus_{i \in I} S_i$$

where the *graded components* S_i are (usually finite) sets. The *grading* function maps each element s of S to its *grade* i, so that $s \in S_i$.

From implementation point of view, if the graded set is enumerated then each graded component should be enumerated (there is a check in the method `_test_graded_components()`). The contrary needs not be true.

To implement this category, a parent must either implement *graded_component()* or *subset()*. If only *subset()* is implemented, the first argument must be the grading for compatibility with *graded_component()*. Additionally either the parent must implement *grading()* or its elements must implement a method `grade()`. See the example *sage.categories.examples.sets_with_grading.NonNegativeIntegers*.

Finally, if the graded set is enumerated (see *EnumeratedSets*) then each graded component should be enumerated. The contrary needs not be true.

EXAMPLES:

A typical example of a set with a grading is the set of non-negative integers graded by themselves:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.category()
Category of facade infinite sets with grading
sage: N.grading_set()
Non negative integers
```

The *grading function* is given by `N.grading`:

```
sage: N.grading(4)
4
```

The graded component N_i is the set with one element i:

```
sage: N.graded_component(grade=5)
{5}
sage: N.graded_component(grade=42)
{42}
```

Here are some information about this category:

```
sage: SetsWithGrading()
Category of sets with grading
sage: SetsWithGrading().super_categories()
[Category of sets]
sage: SetsWithGrading().all_super_categories()
[Category of sets with grading,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Todo:

- This should be moved to `Sets().WithGrading()`.
- Should the grading set be a parameter for this category?
- Does the enumeration need to be compatible with the grading? Be careful that the fact that graded components are allowed to be finite or infinite make the answer complicated.

class ParentMethods

Bases: `object`

generating_series()

Default implementation for generating series.

OUTPUT:

A series, indexed by the grading set.

EXAMPLES:

```
   sage: N = SetsWithGrading().example(); N
   Non negative integers
   sage: N.generating_series()
   1/(-z + 1)

   sage: Permutations().generating_series()                              ␣
↪  # needs sage.combinat
   1 + z + 2*z^2 + 6*z^3 + 24*z^4 + 120*z^5 + 720*z^6 + O(z^7)

.. TODO::

    - Very likely, this should always return a lazy power series.
```

graded_component(*grade*)

Return the graded component of `self` with grade `grade`.

The default implementation just calls the method *`subset()`* with the first argument `grade`.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.graded_component(3)
{3}
```

grading(*elt*)

Return the grading of the element `elt` of `self`.

This default implementation calls `elt.grade()`.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.grading(4)
4
```

grading_set()

Return the set `self` is graded by. By default, this is the set of non-negative integers.

EXAMPLES:

```
sage: SetsWithGrading().example().grading_set()
Non negative integers
```

subset(**args*, ***options*)

Return the subset of `self` described by the given parameters.

See also:

-*graded_component()*

EXAMPLES:

```
sage: W = WeightedIntegerVectors([3,2,1]); W                              #␣
↪needs sage.combinat
Integer vectors weighted by [3, 2, 1]
sage: W.subset(4)                                                         #␣
↪needs sage.combinat
Integer vectors of 4 weighted by [3, 2, 1]
```

super_categories()

EXAMPLES:

```
sage: SetsWithGrading().super_categories()
[Category of sets]
```

4.145 SetsWithPartialMaps

class sage.categories.sets_with_partial_maps.**SetsWithPartialMaps**

Bases: *Category_singleton*

The category whose objects are sets and whose morphisms are maps that are allowed to raise a `ValueError` on some inputs.

This category is equivalent to the category of pointed sets, via the equivalence sending an object X to X union {error}, a morphism f to the morphism of pointed sets that sends x to f(x) if f does not raise an error on x, or to error if it does.

EXAMPLES:

```
sage: SetsWithPartialMaps()
Category of sets with partial maps

sage: SetsWithPartialMaps().super_categories()
[Category of objects]
```

super_categories()

EXAMPLES:

```
sage: SetsWithPartialMaps().super_categories()
[Category of objects]
```

4.146 Shephard Groups

class sage.categories.shephard_groups.**ShephardGroups**

Bases: *Category_singleton*

The category of Shephard groups.

EXAMPLES:

```
sage: from sage.categories.shephard_groups import ShephardGroups
sage: C = ShephardGroups(); C
Category of shephard groups
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.shephard_groups import ShephardGroups
sage: ShephardGroups().super_categories()
[Category of finite generalized Coxeter groups]
```

4.147 Simplicial Complexes

class sage.categories.simplicial_complexes.**SimplicialComplexes**

Bases: *Category_singleton*

The category of abstract simplicial complexes.

An abstract simplicial complex A is a collection of sets X such that:

- $\emptyset \in A$,
- if $X \subset Y \in A$, then $X \in A$.

Todo: Implement the category of simplicial complexes considered as *CW complexes* and rename this to the category of AbstractSimplicialComplexes with appropriate functors.

EXAMPLES:

```
sage: from sage.categories.simplicial_complexes import SimplicialComplexes
sage: C = SimplicialComplexes(); C
Category of simplicial complexes
```

class Connected(*base_category*)

Bases: `CategoryWithAxiom`

The category of connected simplicial complexes.

EXAMPLES:

```
sage: from sage.categories.simplicial_complexes import SimplicialComplexes
sage: C = SimplicialComplexes().Connected()
sage: TestSuite(C).run()
```

class Finite(*base_category*)

Bases: `CategoryWithAxiom`

Category of finite simplicial complexes.

class ParentMethods

Bases: `object`

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: S = SimplicialComplex([[1,3,4], [1,2],[2,5],[4,5]])          #␣
↪needs sage.graphs
sage: S.dimension()                                                  #␣
↪needs sage.graphs
2
```

class ParentMethods

Bases: `object`

faces()

Return the faces of `self`.

EXAMPLES:

```
sage: S = SimplicialComplex([[1,3,4], [1,2],[2,5],[4,5]])            #␣
↪needs sage.graphs
sage: S.faces()                                                        #␣
↪needs sage.graphs
{-1: {()},
 0: {(1,), (2,), (3,), (4,), (5,)},
 1: {(1, 2), (1, 3), (1, 4), (2, 5), (3, 4), (4, 5)},
 2: {(1, 3, 4)}}
```

facets()

Return the facets of `self`.

EXAMPLES:

```
sage: S = SimplicialComplex([[1,3,4], [1,2],[2,5],[4,5]])                #␣
↪needs sage.graphs
sage: sorted(S.facets())                                                  #␣
↪needs sage.graphs
[(1, 2), (1, 3, 4), (2, 5), (4, 5)]
```

class SubcategoryMethods

Bases: `object`

Connected()

Return the full subcategory of the connected objects of `self`.

EXAMPLES:

```
sage: from sage.categories.simplicial_complexes import SimplicialComplexes
sage: SimplicialComplexes().Connected()
Category of connected simplicial complexes
```

super_categories()

Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.simplicial_complexes import SimplicialComplexes
sage: SimplicialComplexes().super_categories()
[Category of sets]
```

4.148 Simplicial Sets

class sage.categories.simplicial_sets.**SimplicialSets**

Bases: *Category_singleton*

The category of simplicial sets.

A simplicial set X is a collection of sets X_i, indexed by the non-negative integers, together with maps

$$\begin{aligned} d_i : X_n \to X_{n-1}, \quad 0 \leq i \leq n \quad \text{(face maps)} \\ s_j : X_n \to X_{n+1}, \quad 0 \leq j \leq n \quad \text{(degeneracy maps)} \end{aligned}$$

satisfying the *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\ d_i s_j &= s_{j-1} d_i \quad \text{if } i < j \\ d_j s_j &= 1 = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1 \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j \end{aligned}$$

Morphisms are sequences of maps $f_i : X_i \to Y_i$ which commute with the face and degeneracy maps.

EXAMPLES:

```
sage: from sage.categories.simplicial_sets import SimplicialSets
sage: C = SimplicialSets(); C
Category of simplicial sets
```

class Finite(*base_category*)

Bases: `CategoryWithAxiom`

Category of finite simplicial sets.

The objects are simplicial sets with finitely many non-degenerate simplices.

class Homsets(*category*, **args*)

Bases: `HomsetsCategory`

class Endset(*base_category*)

Bases: `CategoryWithAxiom`

class ParentMethods

Bases: `object`

one()

Return the identity morphism in $\mathrm{Hom}(S, S)$.

EXAMPLES:

```
sage: T = simplicial_sets.Torus()                                   #␣
↪needs sage.graphs
sage: Hom(T, T).identity()                                          #␣
↪needs sage.graphs
Simplicial set endomorphism of Torus
  Defn: Identity map
```

class ParentMethods

Bases: `object`

is_finite()

Return `True` if this simplicial set is finite, i.e., has a finite number of nondegenerate simplices.

EXAMPLES:

```
sage: simplicial_sets.Torus().is_finite()                                #␣
↪needs sage.graphs
True
sage: C5 = groups.misc.MultiplicativeAbelian([5])                        #␣
↪needs sage.graphs sage.groups
sage: simplicial_sets.ClassifyingSpace(C5).is_finite()                   #␣
↪needs sage.graphs sage.groups
False
```

is_pointed()

Return `True` if this simplicial set is pointed, i.e., has a base point.

EXAMPLES:

```
sage: # needs sage.graphs
sage: from sage.topology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0)
sage: w = AbstractSimplex(0)
sage: e = AbstractSimplex(1)
sage: X = SimplicialSet({e: (v, w)})
sage: Y = SimplicialSet({e: (v, w)}, base_point=w)
sage: X.is_pointed()
```

(continues on next page)

(continued from previous page)

```
False
sage: Y.is_pointed()
True
```

set_base_point(*point*)

Return a copy of this simplicial set in which the base point is set to `point`.

INPUT:

- `point` – a 0-simplex in this simplicial set

EXAMPLES:

```
sage: # needs sage.graphs
sage: from sage.topology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0, name='v_0')
sage: w = AbstractSimplex(0, name='w_0')
sage: e = AbstractSimplex(1)
sage: X = SimplicialSet({e: (v, w)})
sage: Y = SimplicialSet({e: (v, w)}, base_point=w)
sage: Y.base_point()
w_0
sage: X_star = X.set_base_point(w)
sage: X_star.base_point()
w_0
sage: Y_star = Y.set_base_point(v)
sage: Y_star.base_point()
v_0
```

class Pointed(*base_category*)

Bases: *CategoryWithAxiom*

class Finite(*base_category*)

Bases: *CategoryWithAxiom*

class ParentMethods

Bases: `object`

fat_wedge(*n*)

Return the n-th fat wedge of this pointed simplicial set.

This is the subcomplex of the n-fold product X^n consisting of those points in which at least one factor is the base point. Thus when $n = 2$, this is the wedge of the simplicial set with itself, but when n is larger, the fat wedge is larger than the n-fold wedge.

EXAMPLES:

```
sage: # needs sage.graphs
sage: S1 = simplicial_sets.Sphere(1)
sage: S1.fat_wedge(0)
Point
sage: S1.fat_wedge(1)
S^1
sage: S1.fat_wedge(2).fundamental_group()                          #␣
↪needs sage.groups
Finitely presented group < e0, e1 |  >
sage: S1.fat_wedge(4).homology()                                   #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.modules
{0: 0, 1: Z x Z x Z x Z, 2: Z^6, 3: Z x Z x Z x Z}
```

smash_product(**others*)

Return the smash product of this simplicial set with `others`.

INPUT:

- `others` – one or several simplicial sets

EXAMPLES:

```
sage: # needs sage.graphs sage.groups
sage: S1 = simplicial_sets.Sphere(1)
sage: RP2 = simplicial_sets.RealProjectiveSpace(2)
sage: X = S1.smash_product(RP2)
sage: X.homology(base_ring=GF(2))                                        #␣
↪needs sage.modules
{0: Vector space of dimension 0 over Finite Field of size 2,
 1: Vector space of dimension 0 over Finite Field of size 2,
 2: Vector space of dimension 1 over Finite Field of size 2,
 3: Vector space of dimension 1 over Finite Field of size 2}

sage: T = S1.product(S1)                                                 #␣
↪needs sage.graphs sage.groups
sage: X = T.smash_product(S1)                                            #␣
↪needs sage.graphs sage.groups
sage: X.homology(reduced=False)                                          #␣
↪needs sage.graphs sage.groups sage.modules
{0: Z, 1: 0, 2: Z x Z, 3: Z}
```

unset_base_point()

Return a copy of this simplicial set in which the base point has been forgotten.

EXAMPLES:

```
sage: # needs sage.graphs
sage: from sage.topology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0, name='v_0')
sage: w = AbstractSimplex(0, name='w_0')
sage: e = AbstractSimplex(1)
sage: Y = SimplicialSet({e: (v, w)}, base_point=w)
sage: Y.is_pointed()
True
sage: Y.base_point()
w_0
sage: Z = Y.unset_base_point()
sage: Z.is_pointed()
False
```

class ParentMethods

Bases: `object`

base_point()

Return this simplicial set's base point

EXAMPLES:

```
sage: # needs sage.graphs
sage: from sage.topology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0, name='*')
sage: e = AbstractSimplex(1)
sage: S1 = SimplicialSet({e: (v, v)}, base_point=v)
sage: S1.is_pointed()
True
sage: S1.base_point()
*
```

base_point_map(*domain=None*)

Return a map from a one-point space to this one, with image the base point.

This raises an error if this simplicial set does not have a base point.

INPUT:

- `domain` – optional, default `None`. Use this to specify a particular one-point space as the domain. The default behavior is to use the `sage.topology.simplicial_set.Point()` function to use a standard one-point space.

EXAMPLES:

```
sage: # needs sage.graphs
sage: T = simplicial_sets.Torus()
sage: f = T.base_point_map(); f
Simplicial set morphism:
  From: Point
  To:   Torus
  Defn: Constant map at (v_0, v_0)
sage: S3 = simplicial_sets.Sphere(3)
sage: g = S3.base_point_map()
sage: f.domain() == g.domain()
True
sage: RP3 = simplicial_sets.RealProjectiveSpace(3)                   #␣
↪needs sage.groups
sage: temp = simplicial_sets.Simplex(0)
sage: pt = temp.set_base_point(temp.n_cells(0)[0])
sage: h = RP3.base_point_map(domain=pt)                              #␣
↪needs sage.groups
sage: f.domain() == h.domain()                                       #␣
↪needs sage.groups
False

sage: C5 = groups.misc.MultiplicativeAbelian([5])                    #␣
↪needs sage.graphs sage.groups
sage: BC5 = simplicial_sets.ClassifyingSpace(C5)                     #␣
↪needs sage.graphs sage.groups
sage: BC5.base_point_map()                                           #␣
↪needs sage.graphs sage.groups
Simplicial set morphism:
  From: Point
  To:   Classifying space of Multiplicative Abelian group isomorphic␣
↪to C5
  Defn: Constant map at 1
```

connectivity(*max_dim=None*)

Return the connectivity of this pointed simplicial set.

INPUT:

- `max_dim` – specify a maximum dimension through which to check. This is required if this simplicial set is simply connected and not finite.

The dimension of the first nonzero homotopy group. If simply connected, this is the same as the dimension of the first nonzero homology group.

Warning: See the warning for the *is_simply_connected()* method.

The connectivity of a contractible space is `+Infinity`.

EXAMPLES:

```
sage: # needs sage.graphs sage.groups
sage: simplicial_sets.Sphere(3).connectivity()
2
sage: simplicial_sets.Sphere(0).connectivity()
-1
sage: K = simplicial_sets.Simplex(4)
sage: K = K.set_base_point(K.n_cells(0)[0])
sage: K.connectivity()
+Infinity
sage: X = simplicial_sets.Torus().suspension(2)
sage: X.connectivity()
2

sage: C2 = groups.misc.MultiplicativeAbelian([2])                    #␣
↪needs sage.graphs sage.groups
sage: BC2 = simplicial_sets.ClassifyingSpace(C2)                     #␣
↪needs sage.graphs sage.groups
sage: BC2.connectivity()                                             #␣
↪needs sage.graphs sage.groups
0
```

cover(*character*)

Return the cover of the simplicial set associated to a character of the fundamental group.

The character is represented by a dictionary, that assigns an element of a finite group to each non-degenerate 1-dimensional cell. It should correspond to an epimorphism from the fundamental group.

INPUT:

- `character` – a dictionary

EXAMPLES:

```
sage: # needs sage.graphs sage.groups
sage: S1 = simplicial_sets.Sphere(1)
sage: W = S1.wedge(S1)
sage: G = CyclicPermutationGroup(3)
sage: (a, b) = W.n_cells(1)
sage: C = W.cover({a : G.gen(0), b : G.gen(0)^2})
sage: C.face_data()
{(*, ()): None,
 (*, (1,2,3)): None,
 (*, (1,3,2)): None,
 (sigma_1, ()): ((*, (1,2,3)), (*, ())),
 (sigma_1, ()): ((*, (1,3,2)), (*, ())),
 (sigma_1, (1,2,3)): ((*, (1,3,2)), (*, (1,2,3))),
```

(continues on next page)

(continued from previous page)

```
 (sigma_1, (1,2,3)): ((*, ()), (*, (1,2,3))),
 (sigma_1, (1,3,2)): ((*, ()), (*, (1,3,2))),
 (sigma_1, (1,3,2)): ((*, (1,2,3)), (*, (1,3,2)))}
sage: C.homology(1)                                                    #␣
↪needs sage.modules
Z x Z x Z x Z
sage: C.fundamental_group()
Finitely presented group < e0, e1, e2, e3 |  >
```

covering_map(*character*)

Return the covering map associated to a character.

The character is represented by a dictionary that assigns an element of a finite group to each non-degenerate 1-dimensional cell. It should correspond to an epimorphism from the fundamental group.

INPUT:

- `character` – a dictionary

EXAMPLES:

```
sage: # needs sage.graphs sage.groups
sage: S1 = simplicial_sets.Sphere(1)
sage: W = S1.wedge(S1)
sage: G = CyclicPermutationGroup(3)
sage: a, b = W.n_cells(1)
sage: C = W.covering_map({a : G.gen(0), b : G.one()}); C
Simplicial set morphism:
  From: Simplicial set with 9 non-degenerate simplices
  To:   Wedge: (S^1 v S^1)
  Defn: [(*, ()), (*, (1,2,3)), (*, (1,3,2)), (sigma_1, ()),
         (sigma_1, ()), (sigma_1, (1,2,3)), (sigma_1, (1,2,3)),
         (sigma_1, (1,3,2)), (sigma_1, (1,3,2))]
        --> [*, *, *, sigma_1, sigma_1, sigma_1, sigma_1, sigma_1,␣
↪sigma_1]
sage: C.domain()
Simplicial set with 9 non-degenerate simplices
sage: C.domain().face_data()
{(*, ()): None,
 (*, (1,2,3)): None,
 (*, (1,3,2)): None,
 (sigma_1, ()): ((*, (1,2,3)), (*, ())),
 (sigma_1, ()): ((*, ()), (*, ())),
 (sigma_1, (1,2,3)): ((*, (1,3,2)), (*, (1,2,3))),
 (sigma_1, (1,2,3)): ((*, (1,2,3)), (*, (1,2,3))),
 (sigma_1, (1,3,2)): ((*, ()), (*, (1,3,2))),
 (sigma_1, (1,3,2)): ((*, (1,3,2)), (*, (1,3,2)))}
```

fundamental_group(*simplify=True*)

Return the fundamental group of this pointed simplicial set.

INPUT:

- `simplify` (bool, optional `True`) – if `False`, then return a presentation of the group in terms of generators and relations. If `True`, the default, simplify as much as GAP is able to.

Algorithm: we compute the edge-path group – see Section 19 of [Kan1958] and Wikipedia article Fundamental_group. Choose a spanning tree for the connected component of the 1-skeleton containing the base point, and then the group's generators are given by the non-degenerate edges. There are two types of relations: $e = 1$ if e is in the spanning tree, and for every 2-simplex, if its faces are e_0, e_1, and e_2, then we impose the relation $e_0 e_1^{-1} e_2 = 1$, where we first set $e_i = 1$ if e_i is degenerate.

EXAMPLES:

```
sage: S1 = simplicial_sets.Sphere(1)                                    #␣
↪needs sage.graphs
sage: eight = S1.wedge(S1)                                              #␣
↪needs sage.graphs
sage: eight.fundamental_group()   # free group on 2 generators         #␣
↪needs sage.graphs sage.groups
Finitely presented group < e0, e1 |  >
```

The fundamental group of a disjoint union of course depends on the choice of base point:

```
sage: T = simplicial_sets.Torus()                                       #␣
↪needs sage.graphs
sage: K = simplicial_sets.KleinBottle()                                 #␣
↪needs sage.graphs
sage: X = T.disjoint_union(K)                                           #␣
↪needs sage.graphs

sage: # needs sage.graphs
sage: X_0 = X.set_base_point(X.n_cells(0)[0])
sage: X_0.fundamental_group().is_abelian()                              #␣
↪needs sage.groups
True
sage: X_1 = X.set_base_point(X.n_cells(0)[1])
sage: X_1.fundamental_group().is_abelian()                              #␣
↪needs sage.groups
False

sage: RP3 = simplicial_sets.RealProjectiveSpace(3)                      #␣
↪needs sage.graphs sage.groups
sage: RP3.fundamental_group()                                           #␣
↪needs sage.graphs sage.groups
Finitely presented group < e | e^2 >
```

Compute the fundamental group of some classifying spaces:

```
sage: C5 = groups.misc.MultiplicativeAbelian([5])                       #␣
↪needs sage.graphs sage.groups
sage: BC5 = C5.nerve()                                                  #␣
↪needs sage.graphs sage.groups
sage: BC5.fundamental_group()                                           #␣
↪needs sage.graphs sage.groups
Finitely presented group < e0 | e0^5 >

sage: # needs sage.graphs sage.groups
sage: Sigma3 = groups.permutation.Symmetric(3)
sage: BSigma3 = Sigma3.nerve()
sage: pi = BSigma3.fundamental_group(); pi
Finitely presented group < e1, e2 | e2^2, e1^3, (e2*e1)^2 >
sage: pi.order()
6
sage: pi.is_abelian()
False
```

The sphere has a trivial fundamental group:

```
sage: S2 = simplicial_sets.Sphere(2)                                    #␣
↪needs sage.graphs
sage: S2.fundamental_group()                                            #␣
↪needs sage.graphs sage.groups
Finitely presented group <  |  >
```

is_simply_connected()

Return `True` if this pointed simplicial set is simply connected.

> **Warning:** Determining simple connectivity is not always possible, because it requires determining when a group, as given by generators and relations, is trivial. So this conceivably may give a false negative in some cases.

EXAMPLES:

```
sage: # needs sage.graphs sage.groups
sage: T = simplicial_sets.Torus()
sage: T.is_simply_connected()
False
sage: T.suspension().is_simply_connected()
True
sage: simplicial_sets.KleinBottle().is_simply_connected()
False

sage: # needs sage.graphs
sage: S2 = simplicial_sets.Sphere(2)
sage: S3 = simplicial_sets.Sphere(3)
sage: (S2.wedge(S3)).is_simply_connected()                              #␣
↪needs sage.groups
True
sage: X = S2.disjoint_union(S3)
sage: X = X.set_base_point(X.n_cells(0)[0])
sage: X.is_simply_connected()
False

sage: C3 = groups.misc.MultiplicativeAbelian([3])                       #␣
↪needs sage.graphs sage.groups
sage: BC3 = simplicial_sets.ClassifyingSpace(C3)                        #␣
↪needs sage.graphs sage.groups
sage: BC3.is_simply_connected()                                         #␣
↪needs sage.graphs sage.groups
False
```

twisted_chain_complex(*twisting_operator=None*, *dimensions=None*, *augmented=False*, *cochain=False*, *verbose=False*, *subcomplex=None*, *check=False*)

Return the normalized chain complex twisted by some operator.

A twisting operator is a map from the set of simplices to some algebra. The differentials are then twisted by this operator.

INPUT:

- `twisting_operator` – a dictionary, associating the twist of each simplex. If it is not given, the canonical one (associated to the laurent polynomial ring abelianization of the fundamental group, ignoring torsion) is used.
- `dimensions` – if `None`, compute the chain complex in all dimensions. If a list or tuple of

integers, compute the chain complex in those dimensions, setting the chain groups in all other dimensions to zero.

- `augmented` (optional, default `False`) – if `True`, return the augmented chain complex (that is, include a class in dimension -1 corresponding to the empty cell).
- `cochain` (optional, default `False`) – if `True`, return the cochain complex (that is, the dual of the chain complex).
- `verbose` (optional, default `False`) – ignored.
- `subcomplex` (optional, default `None`) – if present, compute the chain complex relative to this subcomplex.
- `check` (optional, default `False`) – If `True`, make sure that the chain complex is actually a chain complex: the differentials are composable and their product is zero.

The normalized chain complex of a simplicial set is isomorphic to the chain complex obtained by modding out by degenerate simplices, and the latter is what is actually constructed here.

EXAMPLES:

```
sage: W = simplicial_sets.Sphere(1).wedge(simplicial_sets.Sphere(2))
sage: W.nondegenerate_simplices()
[*, sigma_1, sigma_2]
sage: s1 = W.nondegenerate_simplices()[1]
sage: L.<t> = LaurentPolynomialRing(QQ)
sage: tw = {s1:t}
sage: ChC = W.twisted_chain_complex(tw)
sage: ChC.differential(1)
[-1 + t]
sage: ChC.differential(2)
[0]
```

```
sage: X = simplicial_sets.Torus()
sage: C = X.twisted_chain_complex()
sage: C.differential(1)
[      f3 - 1 f2*f3^-1 - 1        f2 - 1]
sage: C.differential(2)
[       1 f2*f3^-1]
[      f3        1]
[      -1       -1]
sage: C.differential(3)
[]
```

```
sage: Y = simplicial_sets.RealProjectiveSpace(2)
sage: C = Y.twisted_chain_complex()
sage: C.differential(1)
[-1 + F1]
sage: C.differential(2)
[1 + F1]
sage: C.differential(3)
[]
```

twisted_homology(*n*, *reduced=False*)

The n-th twisted homology module of the simplicial set with respect to the abelianization of the fundamental_group.

It is a module over a polynomial ring, including relations to make some variables the multiplicative inverses of others.

INPUT:

- `n` - a positive integer.

- reduced - (default: False) if set to True, the presentation matrix will be reduced.

EXAMPLES:

```
sage: X = simplicial_sets.Sphere(1).wedge(simplicial_sets.Sphere(2))
sage: X.twisted_homology(1)
Quotient module by Submodule of Ambient free module of rank 0 over the␣
↪integral domain Multivariate Polynomial Ring in f1, f1inv over␣
↪Integer Ring
Generated by the rows of the matrix:
[]
sage: X.twisted_homology(2)
Quotient module by Submodule of Ambient free module of rank 1 over the␣
↪integral domain Multivariate Polynomial Ring in f1, f1inv over␣
↪Integer Ring
Generated by the rows of the matrix:
[f1*f1inv - 1]
```

```
sage: Y = simplicial_sets.Torus()
sage: Y.twisted_homology(1)
Quotient module by Submodule of Ambient free module of rank 5 over the␣
↪integral domain Multivariate Polynomial Ring in f2, f2inv, f3, f3inv␣
↪over Integer Ring
Generated by the rows of the matrix:
[           1            0            0            0            0]
[           0            1            0            0            0]
[           0            0            1            0            0]
[           0            0            0            1            0]
[           0            0            0            0            1]
[f2*f2inv - 1            0            0            0            0]
[           0 f2*f2inv - 1            0            0            0]
[           0            0 f2*f2inv - 1            0            0]
[           0            0            0 f2*f2inv - 1            0]
[           0            0            0            0 f2*f2inv - 1]
[f3*f3inv - 1            0            0            0            0]
[           0 f3*f3inv - 1            0            0            0]
[           0            0 f3*f3inv - 1            0            0]
[           0            0            0 f3*f3inv - 1            0]
[           0            0            0            0 f3*f3inv - 1]
sage: Y.twisted_homology(2)
Quotient module by Submodule of Ambient free module of rank 0 over the␣
↪integral domain Multivariate Polynomial Ring in f2, f2inv, f3, f3inv␣
↪over Integer Ring
Generated by the rows of the matrix:
[]
sage: Y.twisted_homology(1, reduced=True)
Quotient module by Submodule of Ambient free module of rank 5 over the␣
↪integral domain Multivariate Polynomial Ring in f2, f2inv, f3, f3inv␣
↪over Integer Ring
Generated by the rows of the matrix:
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
```

universal_cover()

Return the universal cover of the simplicial set. The fundamental group must be finite in order to ensure that the universal cover is a simplicial set of finite type.

EXAMPLES:

```
sage: # needs sage.groups
sage: RP3 = simplicial_sets.RealProjectiveSpace(3)
sage: C = RP3.universal_cover(); C
Simplicial set with 8 non-degenerate simplices
sage: C.face_data()
{(1, 1): None,
 (1, e): None,
 (f, 1): ((1, e), (1, 1)),
 (f, e): ((1, 1), (1, e)),
 (f * f, 1): ((f, e), s_0 (1, 1), (f, 1)),
 (f * f, e): ((f, 1), s_0 (1, e), (f, e)),
 (f * f * f, 1): ((f * f, e), s_0 (f, 1), s_1 (f, 1), (f * f, 1)),
 (f * f * f, e): ((f * f, 1), s_0 (f, e), s_1 (f, e), (f * f, e))}
sage: C.fundamental_group()
Finitely presented group <  |  >
```

universal_cover_map()

Return the universal covering map of the simplicial set.

It requires the fundamental group to be finite.

EXAMPLES:

```
sage: RP2 = simplicial_sets.RealProjectiveSpace(2)                       #␣
↪needs sage.groups
sage: phi = RP2.universal_cover_map(); phi                               #␣
↪needs sage.groups
Simplicial set morphism:
  From: Simplicial set with 6 non-degenerate simplices
  To:   RP^2
  Defn: [(1, 1), (1, e), (f, 1), (f, e), (f * f, 1), (f * f, e)]
        --> [1, 1, f, f, f * f, f * f]
sage: phi.domain().face_data()                                           #␣
↪needs sage.groups
    {(1, 1): None,
     (1, e): None,
     (f, 1): ((1, e), (1, 1)),
     (f, e): ((1, 1), (1, e)),
     (f * f, 1): ((f, e), s_0 (1, 1), (f, 1)),
     (f * f, e): ((f, 1), s_0 (1, e), (f, e))}
```

class SubcategoryMethods

Bases: `object`

Pointed()

A simplicial set is *pointed* if it has a distinguished base point.

EXAMPLES:

```
sage: from sage.categories.simplicial_sets import SimplicialSets
sage: SimplicialSets().Pointed().Finite()
Category of finite pointed simplicial sets
sage: SimplicialSets().Finite().Pointed()
Category of finite pointed simplicial sets
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.simplicial_sets import SimplicialSets
sage: SimplicialSets().super_categories()
[Category of sets]
```

4.149 Super Algebras

class sage.categories.super_algebras.**SuperAlgebras**(*base_category*)

Bases: *SuperModulesCategory*

The category of super algebras.

An R-*super algebra* is an R-super module A endowed with an R-algebra structure satisfying

$$A_0 A_0 \subseteq A_0, \qquad A_0 A_1 \subseteq A_1, \qquad A_1 A_0 \subseteq A_1, \qquad A_1 A_1 \subseteq A_0$$

and $1 \in A_0$.

EXAMPLES:

```
sage: Algebras(ZZ).Super()
Category of super algebras over Integer Ring
```

class ParentMethods

Bases: object

graded_algebra()

Return the associated graded algebra to self.

Warning: Because a super module M is naturally $\mathbf{Z}/2\mathbf{Z}$-graded, and graded modules have a natural filtration induced by the grading, if M has a different filtration, then the associated graded module $\operatorname{gr} M \neq M$. This is most apparent with super algebras, such as the differential Weyl algebra, and the multiplication may not coincide.

tensor(**parents*, ***kwargs*)

Return the tensor product of the parents.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A.<x,y,z> = ExteriorAlgebra(ZZ); A.rename("A")
sage: T = A.tensor(A,A); T
A # A # A
sage: T in Algebras(ZZ).Graded().SignedTensorProducts()
True
sage: T in Algebras(ZZ).Graded().TensorProducts()
False
sage: A.rename(None)
```

This also works when the other elements do not have a signed tensor product (github issue #31266):

```
sage: # needs sage.combinat sage.modules
sage: a = SteenrodAlgebra(3).an_element()
sage: M = CombinatorialFreeModule(GF(3), ['s', 't', 'u'])
```

(continues on next page)

(continued from previous page)

```
sage: s = M.basis()['s']
sage: tensor([a, s])                                                    #␣
↪needs sage.rings.finite_rings
2*Q_1 Q_3 P(2,1) # B['s']
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

extra_super_categories()

EXAMPLES:

```
sage: Coalgebras(QQ).Graded().SignedTensorProducts().extra_super_
↪categories()
[Category of graded coalgebras over Rational Field]
sage: Coalgebras(QQ).Graded().SignedTensorProducts().super_categories()
[Category of graded coalgebras over Rational Field]
```

Meaning: a signed tensor product of coalgebras is a coalgebra

class SubcategoryMethods

Bases: `object`

Supercommutative()

Return the full subcategory of the supercommutative objects of `self`.

A super algebra M is *supercommutative* if, for all homogeneous $x, y \in M$,

$$x \cdot y = (-1)^{|x||y|} y \cdot x.$$

REFERENCES:

Wikipedia article Supercommutative_algebra

EXAMPLES:

```
sage: Algebras(ZZ).Super().Supercommutative()
Category of supercommutative algebras over Integer Ring
sage: Algebras(ZZ).Super().WithBasis().Supercommutative()
Category of supercommutative algebras with basis over Integer Ring
```

Supercommutative

alias of *SupercommutativeAlgebras*

extra_super_categories()

EXAMPLES:

```
sage: Algebras(ZZ).Super().super_categories() # indirect doctest
[Category of graded algebras over Integer Ring,
 Category of super modules over Integer Ring]
```

4.150 Super algebras with basis

class sage.categories.super_algebras_with_basis.**SuperAlgebrasWithBasis**(*base_category*)

Bases: *SuperModulesCategory*

The category of super algebras with a distinguished basis

EXAMPLES:

```
sage: C = Algebras(ZZ).WithBasis().Super(); C
Category of super algebras with basis over Integer Ring
```

class ElementMethods

Bases: object

supercommutator(*x*)

Return the supercommutator of self and x.

Let A be a superalgebra. The *supercommutator* of homogeneous elements $x, y \in A$ is defined by

$$[x, y\} = xy - (-1)^{|x||y|} yx$$

and extended to all elements by linearity.

EXAMPLES:

```
sage: # needs sage.modules
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: a = x*y - z
sage: b = x - y + y*z
sage: a.supercommutator(b)
-5*x*y + 8*x*z - 2*y*z - 6*x + 12*y - 5*z
sage: a.supercommutator(Cl.one())
0
sage: Cl.one().supercommutator(a)
0
sage: Cl.zero().supercommutator(a)
0
sage: a.supercommutator(Cl.zero())
0

sage: # needs sage.modules
sage: Q = QuadraticForm(ZZ, 2, [-1,1,-3])
sage: Cl.<x,y> = CliffordAlgebra(Q)
sage: [a.supercommutator(b) for a in Cl.basis() for b in Cl.basis()]
[0, 0, 0, 0, 0, -2, 1, -x - 2*y, 0, 1,
 -6, 6*x + y, 0, x + 2*y, -6*x - y, 0]
sage: [a*b-b*a for a in Cl.basis() for b in Cl.basis()]
[0, 0, 0, 0, 0, 0, 2*x*y - 1, -x - 2*y, 0,
 -2*x*y + 1, 0, 6*x + y, 0, x + 2*y, -6*x - y, 0]
```

Exterior algebras inherit from Clifford algebras, so supercommutators work as well. We verify the exterior algebra is supercommutative:

```
sage: E.<x,y,z,w> = ExteriorAlgebra(QQ)                                     #␣
↪needs sage.modules
sage: all(b1.supercommutator(b2) == 0                                       #␣
↪needs sage.modules
....:     for b1 in E.basis() for b2 in E.basis())
True
```

class ParentMethods

Bases: `object`

graded_algebra()

Return the associated graded module to `self`.

See `AssociatedGradedAlgebra` for the definition and the properties of this.

See also:

`graded_algebra()`

EXAMPLES:

```
sage: W.<x,y> = algebras.DifferentialWeyl(QQ)                               #␣
↪needs sage.modules
sage: W.graded_algebra()                                                    #␣
↪needs sage.combinat sage.modules
Graded Algebra of Differential Weyl algebra of
 polynomials in x, y over Rational Field
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

The category of super algebras with basis constructed by tensor product of super algebras with basis.

extra_super_categories()

EXAMPLES:

```
sage: Algebras(QQ).Super().SignedTensorProducts().extra_super_categories()
[Category of super algebras over Rational Field]
sage: Algebras(QQ).Super().SignedTensorProducts().super_categories()
[Category of signed tensor products of graded algebras over Rational␣
↪Field,
 Category of super algebras over Rational Field]
```

Meaning: a signed tensor product of super algebras is a super algebra

extra_super_categories()

EXAMPLES:

```
sage: C = Algebras(ZZ).WithBasis().Super()
sage: sorted(C.super_categories(), key=str) # indirect doctest
[Category of graded algebras with basis over Integer Ring,
 Category of super algebras over Integer Ring,
 Category of super modules with basis over Integer Ring]
```

4.151 Super Hopf algebras with basis

class sage.categories.super_hopf_algebras_with_basis.**SuperHopfAlgebrasWithBasis**(*base_category*)

Bases: *SuperModulesCategory*

The category of super Hopf algebras with a distinguished basis.

EXAMPLES:

```
sage: C = HopfAlgebras(ZZ).WithBasis().Super(); C
Category of super Hopf algebras with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of super Hopf algebras over Integer Ring,
 Category of super algebras with basis over Integer Ring,
 Category of super coalgebras with basis over Integer Ring]
```

class ParentMethods

Bases: object

antipode()

The antipode of this Hopf algebra.

If antipode_basis() is available, this constructs the antipode morphism from self to self by extending it by linearity. Otherwise, self.antipode_by_coercion() is used, if available.

EXAMPLES:

```
sage: A = SteenrodAlgebra(7)                                                #␣
↪needs sage.combinat sage.modules
sage: a = A.an_element()                                                    #␣
↪needs sage.combinat sage.modules
sage: a, A.antipode(a)                                                      #␣
↪needs sage.combinat sage.modules
(6 Q_1 Q_3 P(2,1), Q_1 Q_3 P(2,1))
```

4.152 Super Lie Conformal Algebras

AUTHORS:

- Reimundo Heluani (2019-10-05): Initial implementation.

class sage.categories.super_lie_conformal_algebras.**SuperLieConformalAlgebras**(*base_category*)

Bases: *SuperModulesCategory*

The category of super Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(AA).Super()                                        #␣
↪needs sage.rings.number_field
Category of super Lie conformal algebras over Algebraic Real Field
```

Notice that we can force to have a *purely even* super Lie conformal algebra:

```
sage: bosondict = {('a','a'): {1:{('K',0):1}}}
sage: R = LieConformalAlgebra(QQ, bosondict, names=('a',),                      #␣
↪needs sage.combinat sage.modules
....:                        central_elements=('K',), super=True)
sage: [g.is_even_odd() for g in R.gens()]                                       #␣
↪needs sage.combinat sage.modules
[0, 0]
```

class ElementMethods

Bases: object

is_even_odd()

Return 0 if this element is *even* and 1 if it is *odd*.

EXAMPLES:

```
sage: R = lie_conformal_algebras.NeveuSchwarz(QQ)                               #␣
↪needs sage.combinat sage.modules
sage: R.inject_variables()                                                      #␣
↪needs sage.combinat sage.modules
Defining L, G, C
sage: G.is_even_odd()                                                           #␣
↪needs sage.combinat sage.modules
1
```

class Graded(*base_category*)

Bases: *GradedModulesCategory*

The category of H-graded super Lie conformal algebras.

EXAMPLES:

```
sage: LieConformalAlgebras(AA).Super().Graded()                                 #␣
↪needs sage.rings.number_field
Category of H-graded super Lie conformal algebras over Algebraic Real Field
```

class ParentMethods

Bases: object

example()

An example parent in this category.

EXAMPLES:

```
sage: LieConformalAlgebras(QQ).Super().example()                                #␣
↪needs sage.combinat sage.modules
The Neveu-Schwarz super Lie conformal algebra over Rational Field
```

extra_super_categories()

The extra super categories of self.

EXAMPLES:

```
sage: LieConformalAlgebras(QQ).Super().super_categories()
[Category of super modules over Rational Field,
 Category of Lambda bracket algebras over Rational Field]
```

4.153 Super modules

class sage.categories.super_modules.**SuperModules**(*base_category*)

Bases: *SuperModulesCategory*

The category of super modules.

An R-*super module* (where R is a ring) is an R-module M equipped with a decomposition $M = M_0 \oplus M_1$ into two R-submodules M_0 and M_1 (called the *even part* and the *odd part* of M, respectively).

Thus, an R-super module automatically becomes a $\mathbf{Z}/2\mathbf{Z}$-graded R-module, with M_0 being the degree-0 component and M_1 being the degree-1 component.

EXAMPLES:

```
sage: Modules(ZZ).Super()
Category of super modules over Integer Ring
sage: Modules(ZZ).Super().super_categories()
[Category of graded modules over Integer Ring]
```

The category of super modules defines the super structure which shall be preserved by morphisms:

```
sage: Modules(ZZ).Super().additional_structure()
Category of super modules over Integer Ring
```

class ElementMethods

Bases: object

is_even()

Return if `self` is an even element.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: cat = Algebras(QQ).WithBasis().Super()
sage: C = CombinatorialFreeModule(QQ, Partitions(), category=cat)
sage: C.degree_on_basis = sum
sage: C.basis()[2,2,1].is_even()
False
sage: C.basis()[2,2].is_even()
True
```

is_even_odd()

Return `0` if `self` is an even element or `1` if an odd element.

Note: The default implementation assumes that the even/odd is determined by the parity of `degree()`.

Overwrite this method if the even/odd behavior is desired to be independent.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: cat = Algebras(QQ).WithBasis().Super()
sage: C = CombinatorialFreeModule(QQ, Partitions(), category=cat)
sage: C.degree_on_basis = sum
sage: C.basis()[2,2,1].is_even_odd()
```

(continues on next page)

(continued from previous page)

```
1
sage: C.basis()[2,2].is_even_odd()
0
```

is_odd()

Return if `self` is an odd element.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: cat = Algebras(QQ).WithBasis().Super()
sage: C = CombinatorialFreeModule(QQ, Partitions(), category=cat)
sage: C.degree_on_basis = sum
sage: C.basis()[2,2,1].is_odd()
True
sage: C.basis()[2,2].is_odd()
False
```

class ParentMethods

Bases: `object`

extra_super_categories()

Adds *VectorSpaces* to the super categories of `self` if the base ring is a field.

EXAMPLES:

```
sage: Modules(QQ).Super().extra_super_categories()
[Category of vector spaces over Rational Field]
sage: Modules(ZZ).Super().extra_super_categories()
[]
```

This makes sure that `Modules(QQ).Super()` returns an instance of *SuperModules* and not a join category of an instance of this class and of `VectorSpaces(QQ)`:

```
sage: type(Modules(QQ).Super())
<class 'sage.categories.super_modules.SuperModules_with_category'>
```

Todo: Get rid of this workaround once there is a more systematic approach for the alias `Modules(QQ)` -> `VectorSpaces(QQ)`. Probably the latter should be a category with axiom, and covariant constructions should play well with axioms.

super_categories()

EXAMPLES:

```
sage: Modules(ZZ).Super().super_categories()
[Category of graded modules over Integer Ring]
```

Nota bene:

```
sage: Modules(QQ).Super()
Category of super modules over Rational Field
sage: Modules(QQ).Super().super_categories()
[Category of graded modules over Rational Field]
```

class sage.categories.super_modules.**SuperModulesCategory**(*base_category*)

Bases: *CovariantConstructionCategory*, *Category_over_base_ring*

EXAMPLES:

```
sage: C = Algebras(QQ).Super()
sage: C
Category of super algebras over Rational Field
sage: C.base_category()
Category of algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of graded algebras over Rational Field,
 Category of super modules over Rational Field]

sage: AlgebrasWithBasis(QQ).Super().base_ring()
Rational Field
sage: HopfAlgebrasWithBasis(QQ).Super().base_ring()
Rational Field
```

classmethod default_super_categories(*category*, **args*)

Return the default super categories of $F_{Cat}(A, B, ...)$ for $A, B, ...$ parents in Cat.

INPUT:

- `cls` – the category class for the functor F
- `category` – a category Cat
- `*args` – further arguments for the functor

OUTPUT:

A join category.

This implements the property that subcategories constructed by the set of whitelisted axioms is a subcategory.

EXAMPLES:

```
sage: HopfAlgebras(ZZ).WithBasis().FiniteDimensional().Super()   # indirect␣
↪doctest
Category of finite dimensional super Hopf algebras with basis over Integer␣
↪Ring
```

4.154 Super modules with basis

class sage.categories.super_modules_with_basis.**SuperModulesWithBasis**(*base_category*)

Bases: *SuperModulesCategory*

The category of super modules with a distinguished basis.

An *R-super module with a distinguished basis* is an R-super module equipped with an R-module basis whose elements are homogeneous.

EXAMPLES:

```
sage: C = GradedModulesWithBasis(QQ); C
Category of graded vector spaces with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of filtered vector spaces with basis over Rational Field,
 Category of graded modules with basis over Rational Field,
 Category of graded vector spaces over Rational Field]
sage: C is ModulesWithBasis(QQ).Graded()
True
```

class ElementMethods

Bases: `object`

even_component()

Return the even component of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x*y + x - 3*y + 4
sage: a.even_component()
x*y + 4
```

is_even_odd()

Return `0` if `self` is an even element and `1` if `self` is an odd element.

EXAMPLES:

```
sage: # needs sage.modules
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x + y
sage: a.is_even_odd()
1
sage: a = x*y + 4
sage: a.is_even_odd()
0
sage: a = x + 4
sage: a.is_even_odd()
Traceback (most recent call last):
...
ValueError: element is not homogeneous

sage: E.<x,y> = ExteriorAlgebra(QQ)                                     #␣
↪needs sage.modules
sage: (x*y).is_even_odd()                                               #␣
↪needs sage.modules
0
```

is_super_homogeneous()

Return whether this element is homogeneous, in the sense of a super module (i.e., is even or odd).

EXAMPLES:

```
sage: # needs sage.modules
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
```

(continues on next page)

(continued from previous page)

```
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x + y
sage: a.is_super_homogeneous()
True
sage: a = x*y + 4
sage: a.is_super_homogeneous()
True
sage: a = x*y + x - 3*y + 4
sage: a.is_super_homogeneous()
False
```

The exterior algebra has a $\mathbf{Z}$ grading, which induces the $\mathbf{Z}/2\mathbf{Z}$ grading. However the definition of homogeneous elements differs because of the different gradings:

```
sage: # needs sage.combinat sage.modules
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: a = x*y + 4
sage: a.is_super_homogeneous()
True
sage: a.is_homogeneous()
False
```

odd_component()

Return the odd component of `self`.

EXAMPLES:

```
sage: # needs sage.modules
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x*y + x - 3*y + 4
sage: a.odd_component()
x - 3*y
```

class ParentMethods

Bases: `object`

4.155 Supercommutative Algebras

class sage.categories.supercommutative_algebras.**SupercommutativeAlgebras**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of supercommutative algebras.

An R-*supercommutative algebra* is an R-super algebra $A = A_0 \oplus A_1$ endowed with an R-super algebra structure satisfying:

$$x_0 x'_0 = x'_0 x_0, \qquad x_1 x'_1 = -x'_1 x_1, \qquad x_0 x_1 = x_1 x_0,$$

for all $x_0, x'_0 \in A_0$ and $x_1, x'_1 \in A_1$.

EXAMPLES:

```
sage: Algebras(ZZ).Supercommutative()
Category of supercommutative algebras over Integer Ring
```

class SignedTensorProducts(*category*, **args*)

Bases: *SignedTensorProductsCategory*

extra_super_categories()

Return the extra super categories of `self`.

A signed tensor product of supercommutative algebras is a supercommutative algebra.

EXAMPLES:

```
sage: C = Algebras(ZZ).Supercommutative().SignedTensorProducts()
sage: C.extra_super_categories()
[Category of supercommutative algebras over Integer Ring]
```

class WithBasis(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class ParentMethods

Bases: `object`

4.156 Supercrystals

class `sage.categories.supercrystals.`**SuperCrystals**

Bases: *Category_singleton*

class Finite(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class ElementMethods

Bases: `object`

is_genuine_highest_weight(*index_set=None*)

Return whether `self` is a genuine highest weight element.

INPUT:

- `index_set` – (optional) the index set of the (sub)crystal on which to check

EXAMPLES:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: for b in B.highest_weight_vectors():
....:     print("{} {}".format(b, b.is_genuine_highest_weight()))
[[-2, -2, -2], [-1, -1], [1]] True
[[-2, -2, -2], [-1, 2], [1]] False
[[-2, -2, 2], [-1, -1], [1]] False
sage: [b for b in B if b.is_genuine_highest_weight([-1,0])]
[[[-2, -2, -2], [-1, -1], [1]],
 [[-2, -2, -2], [-1, -1], [2]],
 [[-2, -2, -2], [-1, 2], [2]],
 [[-2, -2, 2], [-1, -1], [2]],
 [[-2, -2, 2], [-1, 2], [2]],
 [[-2, -2, -2], [-1, 2], [1]],
 [[-2, -2, 2], [-1, -1], [1]],
 [[-2, -2, 2], [-1, 2], [1]]]
```

is_genuine_lowest_weight(*index_set=None*)

Return whether `self` is a genuine lowest weight element.

INPUT:

- `index_set` – (optional) the index set of the (sub)crystal on which to check

EXAMPLES:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: for b in sorted(B.lowest_weight_vectors()):
....:     print("{} {}".format(b, b.is_genuine_lowest_weight()))
[[-2, 1, 2], [-1, 2], [1]] False
[[-2, 1, 2], [-1, 2], [2]] False
[[-1, 1, 2], [1, 2], [2]] True
sage: [b for b in B if b.is_genuine_lowest_weight([-1,0])]
[[[-2, -1, 1], [-1, 1], [1]],
 [[-2, -1, 1], [-1, 1], [2]],
 [[-2, 1, 2], [-1, 1], [2]],
 [[-2, 1, 2], [-1, 1], [1]],
 [[-1, -1, 1], [1, 2], [2]],
 [[-1, -1, 1], [1, 2], [1]],
 [[-1, 1, 2], [1, 2], [2]],
 [[-1, 1, 2], [1, 2], [1]]]
```

class ParentMethods

Bases: `object`

character()

Return the character of `self`.

Todo: Once the *WeylCharacterRing* is implemented, make this consistent with the implementation in `sage.categories.classical_crystals.ClassicalCrystals.ParentMethods.character()`.

EXAMPLES:

```
sage: B = crystals.Letters(['A',[1,2]])
sage: B.character()
B[(1, 0, 0, 0, 0)] + B[(0, 1, 0, 0, 0)] + B[(0, 0, 1, 0, 0)]
 + B[(0, 0, 0, 1, 0)] + B[(0, 0, 0, 0, 1)]
```

connected_components()

Return the connected components of `self` as subcrystals.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.connected_components()
[Subcrystal of The crystal of letters for type ['A', [1, 2]]]

sage: T = B.tensor(B)
sage: T.connected_components()
[Subcrystal of Full tensor product of the crystals
  [The crystal of letters for type ['A', [1, 2]],
   The crystal of letters for type ['A', [1, 2]]],
 Subcrystal of Full tensor product of the crystals
```

(continues on next page)

(continued from previous page)

```
 [The crystal of letters for type ['A', [1, 2]],
  The crystal of letters for type ['A', [1, 2]]]]
```

connected_components_generators()

Return the tuple of genuine highest weight elements of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.genuine_highest_weight_vectors()
(-2,)

sage: T = B.tensor(B)
sage: T.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
sage: s1, s2 = T.connected_components()
sage: s = s1 + s2
sage: s.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
```

digraph(*index_set=None*)

Return the `DiGraph` associated to `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,3]])
sage: G = B.digraph(); G
Multi-digraph on 6 vertices
sage: Q = crystals.Letters(['Q',3])
sage: G = Q.digraph(); G
Multi-digraph on 3 vertices
sage: G.edges(sort=True)
[(1, 2, -1), (1, 2, 1), (2, 3, -2), (2, 3, 2)]
```

The edges of the crystal graph are by default colored using blue for edge 1, red for edge 2, green for edge 3, and dashed with the corresponding color for barred edges. Edge 0 is dotted black:

```
sage: view(G)   # optional - dot2tex graphviz, not tested (opens␣
↪external window)
```

genuine_highest_weight_vectors()

Return the tuple of genuine highest weight elements of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.genuine_highest_weight_vectors()
(-2,)

sage: T = B.tensor(B)
sage: T.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
sage: s1, s2 = T.connected_components()
sage: s = s1 + s2
sage: s.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
```

genuine_lowest_weight_vectors()

Return the tuple of genuine lowest weight elements of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.genuine_lowest_weight_vectors()
(3,)

sage: T = B.tensor(B)
sage: T.genuine_lowest_weight_vectors()
([3, 3], [3, 2])
sage: s1, s2 = T.connected_components()
sage: s = s1 + s2
sage: s.genuine_lowest_weight_vectors()
([3, 3], [3, 2])
```

highest_weight_vectors()

Return the highest weight vectors of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.highest_weight_vectors()
(-2,)

sage: T = B.tensor(B)
sage: T.highest_weight_vectors()
([-2, -2], [-2, -1])
```

We give an example from [BKK2000] that has fake highest weight vectors:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: B.highest_weight_vectors()
([[-2, -2, -2], [-1, -1], [1]],
 [[-2, -2, -2], [-1, 2], [1]],
 [[-2, -2, 2], [-1, -1], [1]])
sage: B.genuine_highest_weight_vectors()
([[-2, -2, -2], [-1, -1], [1]],)
```

lowest_weight_vectors()

Return the lowest weight vectors of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.lowest_weight_vectors()
(3,)

sage: T = B.tensor(B)
sage: sorted(T.lowest_weight_vectors())
[[3, 2], [3, 3]]
```

We give an example from [BKK2000] that has fake lowest weight vectors:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: sorted(B.lowest_weight_vectors())
[[[-2, 1, 2], [-1, 2], [1]],
```

(continues on next page)

(continued from previous page)

```
 [[-2, 1, 2], [-1, 2], [2]],
 [[-1, 1, 2], [1, 2], [2]]]
sage: B.genuine_lowest_weight_vectors()
([[-1, 1, 2], [1, 2], [2]],)
```

class ParentMethods

Bases: `object`

tensor(**crystals*, ***options*)

Return the tensor product of `self` with the crystals `B`.

EXAMPLES:

```
sage: B = crystals.Letters(['A',[1,2]])
sage: C = crystals.Tableaux(['A',[1,2]], shape = [2,1])
sage: T = C.tensor(B); T
Full tensor product of the crystals
 [Crystal of BKK tableaux of shape [2, 1] of gl(2|3),
  The crystal of letters for type ['A', [1, 2]]]
sage: S = B.tensor(C); S
Full tensor product of the crystals
 [The crystal of letters for type ['A', [1, 2]],
  Crystal of BKK tableaux of shape [2, 1] of gl(2|3)]
sage: G = T.digraph()
sage: H = S.digraph()
sage: G.is_isomorphic(H, edge_labels= True)
True
```

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

The category of regular crystals constructed by tensor product of regular crystals.

extra_super_categories()

EXAMPLES:

```
sage: from sage.categories.supercrystals import SuperCrystals
sage: SuperCrystals().TensorProducts().extra_super_categories()
[Category of super crystals]
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.supercrystals import SuperCrystals
sage: C = SuperCrystals()
sage: C.super_categories()
[Category of crystals]
```

4.157 Topological Spaces

class sage.categories.topological_spaces.**TopologicalSpaces**(*category*, **args*)

Bases: *TopologicalSpacesCategory*

The category of topological spaces.

EXAMPLES:

```
sage: Sets().Topological()
Category of topological spaces
sage: Sets().Topological().super_categories()
[Category of sets]
```

The category of topological spaces defines the topological structure, which shall be preserved by morphisms:

```
sage: Sets().Topological().additional_structure()
Category of topological spaces
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that a (finite) Cartesian product of topological spaces is a topological space.

EXAMPLES:

```
sage: from sage.categories.topological_spaces import TopologicalSpaces
sage: C = TopologicalSpaces().CartesianProducts()
sage: C.extra_super_categories()
[Category of topological spaces]
sage: C.super_categories()
[Category of Cartesian products of sets, Category of topological spaces]
sage: C.axioms()
frozenset()
```

class Compact(*base_category*)

Bases: *CategoryWithAxiom*

The category of compact topological spaces.

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that a (finite) Cartesian product of compact topological spaces is compact.

EXAMPLES:

```
sage: from sage.categories.topological_spaces import TopologicalSpaces
sage: C = TopologicalSpaces().Compact().CartesianProducts()
sage: C.extra_super_categories()
[Category of compact topological spaces]
sage: C.super_categories()
[Category of Cartesian products of topological spaces,
 Category of compact topological spaces]
sage: C.axioms()
frozenset({'Compact'})
```

class Connected(*base_category*)

Bases: *CategoryWithAxiom*

The category of connected topological spaces.

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

Implement the fact that a (finite) Cartesian product of connected topological spaces is connected.

EXAMPLES:

```
sage: from sage.categories.topological_spaces import TopologicalSpaces
sage: C = TopologicalSpaces().Connected().CartesianProducts()
sage: C.extra_super_categories()
[Category of connected topological spaces]
sage: C.super_categories()
[Category of Cartesian products of topological spaces,
 Category of connected topological spaces]
sage: C.axioms()
frozenset({'Connected'})
```

class SubcategoryMethods

Bases: `object`

Compact()

Return the subcategory of the compact objects of `self`.

EXAMPLES:

```
sage: Sets().Topological().Compact()
Category of compact topological spaces
```

Connected()

Return the full subcategory of the connected objects of `self`.

EXAMPLES:

```
sage: Sets().Topological().Connected()
Category of connected topological spaces
```

class sage.categories.topological_spaces.**TopologicalSpacesCategory**(*category*, **args*)

Bases: *RegressiveCovariantConstructionCategory*

4.158 Kac-Moody Algebras With Triangular Decomposition Basis

AUTHORS:

- Travis Scrimshaw (07-15-2017): Initial implementation

class sage.categories.triangular_kac_moody_algebras.**TriangularKacMoodyAlgebras**(*base*, *name=None*)

Bases: *Category_over_base_ring*

Category of Kac-Moody algebras with a distinguished basis that respects the triangular decomposition.

We require that the grading group is the root lattice of the appropriate Cartan type.

class ElementMethods

Bases: `object`

part()

Return whether the element `v` is in the lower, zero, or upper part of `self`.

OUTPUT:

-1 if `v` is in the lower part, 0 if in the zero part, or 1 if in the upper part

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = LieAlgebra(QQ, cartan_type="F4")
sage: L.inject_variables()
Defining e1, e2, e3, e4, f1, f2, f3, f4, h1, h2, h3, h4
sage: e1.part()
1
sage: f4.part()
-1
sage: (h2 + h3).part()
0
sage: (f1.bracket(f2) + 4*f4).part()
-1
sage: (e1 + f1).part()
Traceback (most recent call last):
...
ValueError: element is not in one part
```

class ParentMethods

Bases: `object`

e(*i=None*)

Return the generators e of `self`.

INPUT:

- `i` – (optional) if specified, return just the generator e_i

EXAMPLES:

```
sage: L = lie_algebras.so(QQ, 5)                                            #␣
↪needs sage.combinat sage.modules
sage: L.e()                                                                 #␣
↪needs sage.combinat sage.modules
Finite family {1: E[alpha[1]], 2: E[alpha[2]]}
sage: L.e(1)                                                                #␣
↪needs sage.combinat sage.modules
E[alpha[1]]
```

f(*i=None*)

Return the generators f of `self`.

INPUT:

- `i` – (optional) if specified, return just the generator f_i

EXAMPLES:

```
sage: L = lie_algebras.so(QQ, 5)                                            #␣
↪needs sage.combinat sage.modules
sage: L.f()                                                                 #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.combinat sage.modules
Finite family {1: E[-alpha[1]], 2: E[-alpha[2]]}
sage: L.f(1)                                                         #␣
↪needs sage.combinat sage.modules
E[-alpha[1]]
```

verma_module(*la*, *basis_key=None*, ***kwds*)

Return the Verma module with highest weight `la` over `self`.

INPUT:

- `basis_key` – (optional) a key function for the indexing set of the basis elements of `self`

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: L = lie_algebras.sl(QQ, 3)
sage: P = L.cartan_type().root_system().weight_lattice()
sage: La = P.fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: M
Verma module with highest weight Lambda[1] + Lambda[2]
 of Lie algebra of ['A', 2] in the Chevalley basis
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.triangular_kac_moody_algebras import␣
↪TriangularKacMoodyAlgebras
sage: TriangularKacMoodyAlgebras(QQ).super_categories()
[Join of Category of graded Lie algebras with basis over Rational Field
     and Category of kac moody algebras over Rational Field]
```

4.159 Unique factorization domains

class sage.categories.unique_factorization_domains.**UniqueFactorizationDomains**

Bases: *Category_singleton*

The category of (constructive) unique factorization domains.

In a constructive unique factorization domain we can constructively factor members into a product of a finite number of irreducible elements.

EXAMPLES:

```
sage: UniqueFactorizationDomains()
Category of unique factorization domains
sage: UniqueFactorizationDomains().super_categories()
[Category of gcd domains]
```

class ElementMethods

Bases: `object`

radical(**args*, ***kwds*)

Return the radical of this element, i.e. the product of its irreducible factors.

This default implementation calls `squarefree_decomposition` if available, and `factor` otherwise.

See also:

squarefree_part()

EXAMPLES:

```
sage: Pol.<x> = QQ[]
sage: (x^2*(x-1)^3).radical()
x^2 - x
sage: pol = 37 * (x-1)^3 * (x-2)^2 * (x-1/3)^7 * (x-3/7)
sage: pol.radical()
37*x^4 - 2923/21*x^3 + 1147/7*x^2 - 1517/21*x + 74/7

sage: Integer(10).radical()
10
sage: Integer(-100).radical()
10
sage: Integer(0).radical()
Traceback (most recent call last):
...
ArithmeticError: radical of 0 is not defined
```

The next example shows how to compute the radical of a number, assuming no prime > 100000 has exponent > 1 in the factorization:

```
sage: n = 2^1000-1; n / radical(n, limit=100000)
125
```

squarefree_part()

Return the square-free part of this element, i.e. the product of its irreducible factors appearing with odd multiplicity.

This default implementation calls `squarefree_decomposition`.

See also:

radical()

EXAMPLES:

```
sage: Pol.<x> = QQ[]
sage: (x^2*(x-1)^3).squarefree_part()
x - 1
sage: pol = 37 * (x-1)^3 * (x-2)^2 * (x-1/3)^7 * (x-3/7)
sage: pol.squarefree_part()
37*x^3 - 1369/21*x^2 + 703/21*x - 37/7
```

class ParentMethods

Bases: `object`

is_unique_factorization_domain(*proof=True*)

Return True, since this in an object of the category of unique factorization domains.

EXAMPLES:

```
sage: UFD = UniqueFactorizationDomains()
sage: Parent(QQ, category=UFD).is_unique_factorization_domain()
True
```

additional_structure()

Return whether `self` is a structure category.

See also:

Category.additional_structure()

The category of unique factorization domains does not define additional structure: a ring morphism between unique factorization domains is a unique factorization domain morphism.

EXAMPLES:

```
sage: UniqueFactorizationDomains().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: UniqueFactorizationDomains().super_categories()
[Category of gcd domains]
```

4.160 Unital algebras

class sage.categories.unital_algebras.**UnitalAlgebras**(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of non-associative algebras over a given base ring.

A non-associative algebra over a ring R is a module over R which is also a unital magma.

Warning: Until github issue #15043 is implemented, *Algebras* is the category of associative unital algebras; thus, unlike the name suggests, *UnitalAlgebras* is not a subcategory of *Algebras* but of *MagmaticAlgebras*.

EXAMPLES:

```
sage: from sage.categories.unital_algebras import UnitalAlgebras
sage: C = UnitalAlgebras(ZZ); C
Category of unital algebras over Integer Ring
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

The category of unital algebras constructed as Cartesian products of unital algebras.

This construction gives the direct product of algebras. See discussion on:

- http://groups.google.fr/group/sage-devel/browse_thread/thread/35a72b1d0a2fc77a/348f42ae77a66d16#348f42ae77a66d16
- Wikipedia article Direct_product

class ParentMethods

Bases: object

one()

Return the multiplicative unit element.

EXAMPLES:

```
sage: S2 = simplicial_complexes.Sphere(2)
sage: H = S2.cohomology_ring(QQ)
sage: C = cartesian_product([H, H])
sage: one = C.one()
sage: one
B[(0, (0, 0))] + B[(1, (0, 0))]
sage: one == one * one
True
sage: all(b == b * one for b in C.basis())
True
```

extra_super_categories()

A Cartesian product of algebras is endowed with a natural unital algebra structure.

EXAMPLES:

```
sage: from sage.categories.unital_algebras import UnitalAlgebras
sage: C = UnitalAlgebras(QQ).CartesianProducts()
sage: C.extra_super_categories()
[Category of unital algebras over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of distributive magmas and additive␣
↪magmas,
 Category of Cartesian products of unital magmas,
 Category of Cartesian products of vector spaces over Rational Field,
 Category of unital algebras over Rational Field]
```

class ParentMethods

Bases: object

from_base_ring(*r*)

Return the canonical embedding of r into self.

INPUT:

- r – an element of self.base_ring()

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A                          #␣
↪needs sage.combinat sage.modules
An example of an algebra with basis:
 the free algebra on the generators ('a', 'b', 'c') over Rational Field
sage: A.from_base_ring(1)                                              #␣
↪needs sage.combinat sage.modules
B[word: ]
```

class WithBasis(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class ParentMethods

Bases: object

from_base_ring()

from_base_ring_from_one_basis(*r*)

Implement the canonical embedding from the ground ring.

INPUT:

- `r` – an element of the coefficient ring

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.from_base_ring_from_one_basis(3)
3*B[word: ]
sage: A.from_base_ring(3)
3*B[word: ]
sage: A(3)
3*B[word: ]
```

one()

Return the multiplicative unit element.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()                                   #␣
↪needs sage.combinat sage.modules
sage: A.one_basis()                                                         #␣
↪needs sage.combinat sage.modules
word:
sage: A.one()                                                               #␣
↪needs sage.combinat sage.modules
B[word: ]
```

one_basis()

When the one of an algebra with basis is an element of this basis, this optional method can return the index of this element. This is used to provide a default implementation of *one()*, and an optimized default implementation of *from_base_ring()*.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
word:
sage: A.one()
B[word: ]
sage: A.from_base_ring(4)
4*B[word: ]
```

one_from_one_basis()

Return the one of the algebra, as per `Monoids.ParentMethods.one()`

By default, this is implemented from *one_basis()*, if available.

EXAMPLES:

```
sage: # needs sage.combinat sage.modules
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
```

(continues on next page)

(continued from previous page)

```
word:
sage: A.one_from_one_basis()
B[word: ]
sage: A.one()
B[word: ]
```

4.161 Vector Bundles

class sage.categories.vector_bundles.**VectorBundles**(*base_space*, *base_field*, *name=None*)

Bases: *Category_over_base_ring*

The category of vector bundles over any base space and base field.

See also:

TopologicalVectorBundle

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: from sage.categories.vector_bundles import VectorBundles
sage: C = VectorBundles(M, RR); C
Category of vector bundles over Real Field with 53 bits of precision
 with base space 2-dimensional topological manifold M
sage: C.super_categories()
[Category of topological spaces]
```

class Differentiable(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of differentiable vector bundles.

A differentiable vector bundle is a differentiable manifold with differentiable surjective projection on a differentiable base space.

class Smooth(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

The category of smooth vector bundles.

A smooth vector bundle is a smooth manifold with smooth surjective projection on a smooth base space.

class SubcategoryMethods

Bases: object

Differentiable()

Return the subcategory of the differentiable objects of self.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: from sage.categories.vector_bundles import VectorBundles
sage: VectorBundles(M, RR).Differentiable()
Category of differentiable vector bundles over Real Field with
 53 bits of precision with base space 2-dimensional
 differentiable manifold M
```

Smooth()

Return the subcategory of the smooth objects of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: from sage.categories.vector_bundles import VectorBundles
sage: VectorBundles(M, RR).Smooth()
Category of smooth vector bundles over Real Field with 53 bits
 of precision with base space 2-dimensional differentiable
 manifold M
```

base_space()

Return the base space of this category.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='top')
sage: from sage.categories.vector_bundles import VectorBundles
sage: VectorBundles(M, RR).base_space()
2-dimensional topological manifold M
```

super_categories()

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: from sage.categories.vector_bundles import VectorBundles
sage: VectorBundles(M, RR).super_categories()
[Category of topological spaces]
```

4.162 Vector Spaces

class sage.categories.vector_spaces.**VectorSpaces**(*K*)

Bases: *Category_module*

The category of (abstract) vector spaces over a given field

??? with an embedding in an ambient vector space ???

EXAMPLES:

```
sage: VectorSpaces(QQ)
Category of vector spaces over Rational Field
sage: VectorSpaces(QQ).super_categories()
[Category of modules over Rational Field]
```

class CartesianProducts(*category*, **args*)

Bases: *CartesianProductsCategory*

extra_super_categories()

The category of vector spaces is closed under Cartesian products:

```
sage: C = VectorSpaces(QQ)
sage: C.CartesianProducts()
Category of Cartesian products of vector spaces over Rational Field
```

(continues on next page)

(continued from previous page)

```
sage: C in C.CartesianProducts().super_categories()
True
```

class DualObjects(*category*, **args*)

Bases: *DualObjectsCategory*

extra_super_categories()

Returns the dual category

EXAMPLES:

The category of algebras over the Rational Field is dual to the category of coalgebras over the same field:

```
sage: C = VectorSpaces(QQ)
sage: C.dual()
Category of duals of vector spaces over Rational Field
sage: C.dual().super_categories() # indirect doctest
[Category of vector spaces over Rational Field]
```

class ElementMethods

Bases: `object`

class Filtered(*base_category*)

Bases: *FilteredModulesCategory*

Category of filtered vector spaces.

class FiniteDimensional(*base_category*)

Bases: *CategoryWithAxiom_over_base_ring*

class TensorProducts(*category*, **args*)

Bases: *TensorProductsCategory*

extra_super_categories()

Implement the fact that a (finite) tensor product of finite dimensional vector spaces is a finite dimensional vector space.

EXAMPLES:

```
sage: VectorSpaces(QQ).FiniteDimensional().TensorProducts().extra_
↪super_categories()
[Category of finite dimensional vector spaces over Rational Field]
sage: VectorSpaces(QQ).FiniteDimensional().TensorProducts().
↪FiniteDimensional()
Category of tensor products of finite dimensional vector spaces over␣
↪Rational Field
```

class Graded(*base_category*)

Bases: *GradedModulesCategory*

Category of graded vector spaces.

class ParentMethods

Bases: `object`

dimension()

Return the dimension of this vector space.

EXAMPLES:

```
sage: M = FreeModule(FiniteField(19), 100)                              #␣
↪needs sage.modules
sage: W = M.submodule([M.gen(50)])                                      #␣
↪needs sage.modules
sage: W.dimension()                                                     #␣
↪needs sage.modules
1

sage: M = FiniteRankFreeModule(QQ, 3)                                   #␣
↪needs sage.modules
sage: M.dimension()                                                     #␣
↪needs sage.modules
3
sage: M.tensor_module(1, 2).dimension()                                 #␣
↪needs sage.modules
27
```

class TensorProducts(*category*, **args*)

Bases: `TensorProductsCategory`

extra_super_categories()

The category of vector spaces is closed under tensor products:

```
sage: C = VectorSpaces(QQ)
sage: C.TensorProducts()
Category of tensor products of vector spaces over Rational Field
sage: C in C.TensorProducts().super_categories()
True
```

class WithBasis(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

class CartesianProducts(*category*, **args*)

Bases: `CartesianProductsCategory`

extra_super_categories()

The category of vector spaces with basis is closed under Cartesian products:

```
sage: C = VectorSpaces(QQ).WithBasis()
sage: C.CartesianProducts()
Category of Cartesian products of vector spaces with basis over␣
↪Rational Field
sage: C in C.CartesianProducts().super_categories()
True
```

class Filtered(*base_category*)

Bases: `FilteredModulesCategory`

Category of filtered vector spaces with basis.

example(*base_ring=None*)

Return an example of a graded vector space with basis, as per `Category.example()`.

EXAMPLES:

```
sage: Modules(QQ).WithBasis().Graded().example()                       #␣
↪needs sage.combinat sage.modules
```

(continues on next page)

(continued from previous page)

```
An example of a graded module with basis:
 the free module on partitions over Rational Field
```

class FiniteDimensional(*base_category*)

Bases: `CategoryWithAxiom_over_base_ring`

class TensorProducts(*category*, *args*)

Bases: `TensorProductsCategory`

extra_super_categories()

Implement the fact that a (finite) tensor product of finite dimensional vector spaces is a finite dimensional vector space.

EXAMPLES:

```
sage: VectorSpaces(QQ).WithBasis().FiniteDimensional().
↪TensorProducts().extra_super_categories()
[Category of finite dimensional vector spaces with basis over␣
↪Rational Field]
sage: VectorSpaces(QQ).WithBasis().FiniteDimensional().
↪TensorProducts().FiniteDimensional()
Category of tensor products of finite dimensional vector spaces with␣
↪basis over Rational Field
```

class Graded(*base_category*)

Bases: `GradedModulesCategory`

Category of graded vector spaces with basis.

example(*base_ring=None*)

Return an example of a graded vector space with basis, as per `Category.example()`.

EXAMPLES:

```
sage: Modules(QQ).WithBasis().Graded().example()                    #␣
↪needs sage.combinat sage.modules
An example of a graded module with basis:
 the free module on partitions over Rational Field
```

class TensorProducts(*category*, *args*)

Bases: `TensorProductsCategory`

extra_super_categories()

The category of vector spaces with basis is closed under tensor products:

```
sage: C = VectorSpaces(QQ).WithBasis()
sage: C.TensorProducts()
Category of tensor products of vector spaces with basis over Rational␣
↪Field
sage: C in C.TensorProducts().super_categories()
True
```

is_abelian()

Return whether this category is abelian.

This is always `True` since the base ring is a field.

EXAMPLES:

```
sage: VectorSpaces(QQ).WithBasis().is_abelian()
True
```

additional_structure()

Return None.

Indeed, the category of vector spaces defines no additional structure: a bimodule morphism between two vector spaces is a vector space morphism.

See also:

Category.additional_structure()

Todo: Should this category be a CategoryWithAxiom?

EXAMPLES:

```
sage: VectorSpaces(QQ).additional_structure()
```

base_field()

Returns the base field over which the vector spaces of this category are all defined.

EXAMPLES:

```
sage: VectorSpaces(QQ).base_field()
Rational Field
```

super_categories()

EXAMPLES:

```
sage: VectorSpaces(QQ).super_categories()
[Category of modules over Rational Field]
```

4.163 Weyl Groups

class sage.categories.weyl_groups.**WeylGroups**

Bases: *Category_singleton*

The category of Weyl groups

See the Wikipedia page of Weyl Groups.

EXAMPLES:

```
sage: WeylGroups()
Category of Weyl groups
sage: WeylGroups().super_categories()
[Category of Coxeter groups]
```

Here are some examples:

```
sage: WeylGroups().example()            # todo: not implemented
sage: FiniteWeylGroups().example()
The symmetric group on {0, ..., 3}
```

(continues on next page)

(continued from previous page)

```
sage: AffineWeylGroups().example()      # todo: not implemented
sage: WeylGroup(["B", 3])
Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space)
```

This one will eventually be also in this category:

```
sage: SymmetricGroup(4)
Symmetric group of order 4! as a permutation group
```

class ElementMethods

Bases: `object`

bruhat_lower_covers_coroots()

Return all 2-tuples (`v`, α) where `v` is covered by `self` and α is the positive coroot such that `self` = `v` s_α where s_α is the reflection orthogonal to α.

ALGORITHM:

See *bruhat_lower_covers()* and *bruhat_lower_covers_reflections()* for Coxeter groups.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.bruhat_lower_covers_coroots()
[(s1*s2*s1, alphacheck[1] + alphacheck[2] + alphacheck[3]),
 (s3*s2*s1, alphacheck[2]), (s3*s1*s2, alphacheck[1])]
```

bruhat_upper_covers_coroots()

Returns all 2-tuples (`v`, α) where `v` is covers `self` and α is the positive coroot such that `self` = `v` s_α where s_α is the reflection orthogonal to α.

ALGORITHM:

See *bruhat_upper_covers()* and *bruhat_upper_covers_reflections()* for Coxeter groups.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.bruhat_upper_covers_coroots()
[(s1*s2*s3*s2*s1, alphacheck[3]),
 (s2*s3*s1*s2*s1, alphacheck[2] + alphacheck[3]),
 (s3*s4*s1*s2*s1, alphacheck[4]),
 (s4*s3*s1*s2*s1, alphacheck[1] + alphacheck[2] + alphacheck[3] +␣
↪alphacheck[4])]
```

inversion_arrangement(*side='right'*)

Return the inversion hyperplane arrangement of `self`.

INPUT:

- `side` – `'right'` (default) or `'left'`

OUTPUT:

A (central) hyperplane arrangement whose hyperplanes correspond to the inversions of `self` given as roots.

The side parameter determines on which side to compute the inversions.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([1, 2, 3, 1, 2])
sage: A = w.inversion_arrangement(); A
Arrangement of 5 hyperplanes of dimension 3 and rank 3
sage: A.hyperplanes()
(Hyperplane 0*a1 + 0*a2 + a3 + 0,
 Hyperplane 0*a1 + a2 + 0*a3 + 0,
 Hyperplane 0*a1 + a2 + a3 + 0,
 Hyperplane a1 + a2 + 0*a3 + 0,
 Hyperplane a1 + a2 + a3 + 0)
```

The identity element gives the empty arrangement:

```
sage: W = WeylGroup(['A',3])
sage: W.one().inversion_arrangement()
Empty hyperplane arrangement of dimension 3
```

inversions(*side='right'*, *inversion_type='reflections'*)

Return the set of inversions of self.

INPUT:

- side – 'right' (default) or 'left'
- inversion_type – 'reflections' (default), 'roots', or 'coroots'

OUTPUT:

For reflections, the set of reflections r in the Weyl group such that self r < self. For (co)roots, the set of positive (co)roots that are sent by self to negative (co)roots; their associated reflections are described above.

If side is 'left', the inverse Weyl group element is used.

EXAMPLES:

```
sage: W = WeylGroup(['C',2], prefix="s")
sage: w = W.from_reduced_word([1,2])
sage: w.inversions()
[s2, s2*s1*s2]
sage: w.inversions(inversion_type = 'reflections')
[s2, s2*s1*s2]
sage: w.inversions(inversion_type = 'roots')
[alpha[2], alpha[1] + alpha[2]]
sage: w.inversions(inversion_type = 'coroots')
[alphacheck[2], alphacheck[1] + 2*alphacheck[2]]
sage: w.inversions(side = 'left')
[s1, s1*s2*s1]
sage: w.inversions(side = 'left', inversion_type = 'roots')
[alpha[1], 2*alpha[1] + alpha[2]]
sage: w.inversions(side = 'left', inversion_type = 'coroots')
[alphacheck[1], alphacheck[1] + alphacheck[2]]
```

is_pieri_factor()

Returns whether self is a Pieri factor, as used for computing Stanley symmetric functions.

See also:

- *stanley_symmetric_function()*

- *WeylGroups.ParentMethods.pieri_factors()*

EXAMPLES:

```
sage: W = WeylGroup(['A',5,1])
sage: W.from_reduced_word([3,2,5]).is_pieri_factor()
True
sage: W.from_reduced_word([3,2,4,5]).is_pieri_factor()
False

sage: W = WeylGroup(['C',4,1])
sage: W.from_reduced_word([0,2,1]).is_pieri_factor()
True
sage: W.from_reduced_word([0,2,1,0]).is_pieri_factor()
False

sage: W = WeylGroup(['B',3])
sage: W.from_reduced_word([3,2,3]).is_pieri_factor()
False
sage: W.from_reduced_word([2,1,2]).is_pieri_factor()
True
```

left_pieri_factorizations(*max_length=None*)

Returns all factorizations of `self` as uv, where u is a Pieri factor and v is an element of the Weyl group.

See also:

- *WeylGroups.ParentMethods.pieri_factors()*
- `sage.combinat.root_system.pieri_factors`

EXAMPLES:

If we take $w = w_0$ the maximal element of a strict parabolic subgroup of type $A_{n_1} \times \cdots \times A_{n_k}$, then the Pieri factorizations are in correspondence with all Pieri factors, and there are $\prod 2^{n_i}$ of them:

```
sage: W = WeylGroup(['A', 4, 1])
sage: W.from_reduced_word([]).left_pieri_factorizations().cardinality()
1
sage: W.from_reduced_word([1]).left_pieri_factorizations().cardinality()
2
sage: W.from_reduced_word([1,2,1]).left_pieri_factorizations().
↪cardinality()
4
sage: W.from_reduced_word([1,2,3,1,2,1]).left_pieri_factorizations().
↪cardinality()
8

sage: W.from_reduced_word([1,3]).left_pieri_factorizations().cardinality()
4
sage: W.from_reduced_word([1,3,4,3]).left_pieri_factorizations().
↪cardinality()
8

sage: W.from_reduced_word([2,1]).left_pieri_factorizations().cardinality()
3
sage: W.from_reduced_word([1,2]).left_pieri_factorizations().cardinality()
2
sage: [W.from_reduced_word([1,2]).left_pieri_factorizations(max_length=i).
↪cardinality()
```

(continues on next page)

(continued from previous page)

```
....:  for i in [-1, 0, 1, 2]]
[0, 1, 2, 2]

sage: W = WeylGroup(['C',4,1])
sage: w = W.from_reduced_word([0,3,2,1,0])
sage: w.left_pieri_factorizations().cardinality()
7
sage: [(u.reduced_word(),v.reduced_word())
....:  for (u,v) in w.left_pieri_factorizations()]
[([], [3, 2, 0, 1, 0]),
([0], [3, 2, 1, 0]),
([3], [2, 0, 1, 0]),
([3, 0], [2, 1, 0]),
([3, 2], [0, 1, 0]),
([3, 2, 0], [1, 0]),
([3, 2, 0, 1], [0])]

sage: W = WeylGroup(['B',4,1])
sage: W.from_reduced_word([0,2,1,0]).left_pieri_factorizations().
↪cardinality()
6
```

quantum_bruhat_successors(*index_set=None*, *roots=False*, *quantum_only=False*)

Return the successors of `self` in the quantum Bruhat graph on the parabolic quotient of the Weyl group determined by the subset of Dynkin nodes `index_set`.

INPUT:

- `self` – a Weyl group element, which is assumed to be of minimum length in its coset with respect to the parabolic subgroup
- `index_set` – (default: `None`) indicates the set of simple reflections used to generate the parabolic subgroup; the default value indicates that the subgroup is the identity
- `roots` – (default: `False`) if `True`, returns the list of 2-tuples (`w`, α) where `w` is a successor and α is the positive root associated with the successor relation
- `quantum_only` – (default: `False`) if `True`, returns only the quantum successors

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2])
sage: w.quantum_bruhat_successors([1], roots = True)
[(s3, alpha[2]), (s1*s2*s3*s2, alpha[3]),
 (s2*s3*s1*s2, alpha[1] + alpha[2] + alpha[3])]
sage: w.quantum_bruhat_successors([1,3])
[1, s2*s3*s1*s2]
sage: w.quantum_bruhat_successors(roots = True)
[(s3*s1*s2*s1, alpha[1]),
 (s3*s1, alpha[2]),
 (s1*s2*s3*s2, alpha[3]),
 (s2*s3*s1*s2, alpha[1] + alpha[2] + alpha[3])]
sage: w.quantum_bruhat_successors()
[s3*s1*s2*s1, s3*s1, s1*s2*s3*s2, s2*s3*s1*s2]
sage: w.quantum_bruhat_successors(quantum_only = True)
[s3*s1]
sage: w = W.from_reduced_word([2,3])
sage: w.quantum_bruhat_successors([1,3])
Traceback (most recent call last):
...
```

(continues on next page)

(continued from previous page)

```
ValueError: s2*s3 is not of minimum length in its coset
of the parabolic subgroup generated by the reflections (1, 3)
```

reflection_to_coroot()

Return the coroot associated with the reflection `self`.

EXAMPLES:

```
sage: W = WeylGroup(['C',2],prefix="s")
sage: W.from_reduced_word([1,2,1]).reflection_to_coroot()
alphacheck[1] + alphacheck[2]
sage: W.from_reduced_word([1,2]).reflection_to_coroot()
Traceback (most recent call last):
...
ValueError: s1*s2 is not a reflection
sage: W.long_element().reflection_to_coroot()
Traceback (most recent call last):
...
ValueError: s2*s1*s2*s1 is not a reflection
```

reflection_to_root()

Return the root associated with the reflection `self`.

EXAMPLES:

```
sage: W = WeylGroup(['C',2],prefix="s")
sage: W.from_reduced_word([1,2,1]).reflection_to_root()
2*alpha[1] + alpha[2]
sage: W.from_reduced_word([1,2]).reflection_to_root()
Traceback (most recent call last):
...
ValueError: s1*s2 is not a reflection
sage: W.long_element().reflection_to_root()
Traceback (most recent call last):
...
ValueError: s2*s1*s2*s1 is not a reflection
```

stanley_symmetric_function()

Return the affine Stanley symmetric function indexed by `self`.

INPUT:

- `self` – an element w of a Weyl group

Returns the affine Stanley symmetric function indexed by w. Stanley symmetric functions are defined as generating series of the factorizations of w into Pieri factors and weighted by a statistic on Pieri factors.

See also:

- *stanley_symmetric_function_as_polynomial()*
- *WeylGroups.ParentMethods.pieri_factors()*
- `sage.combinat.root_system.pieri_factors`

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1])
sage: W.from_reduced_word([3,1,2,0,3,1,0]).stanley_symmetric_function()
8*m[1, 1, 1, 1, 1, 1, 1] + 4*m[2, 1, 1, 1, 1, 1]
+ 2*m[2, 2, 1, 1, 1] + m[2, 2, 2, 1]
```

(continues on next page)

(continued from previous page)

```
sage: A = AffinePermutationGroup(['A',3,1])
sage: A.from_reduced_word([3,1,2,0,3,1,0]).stanley_symmetric_function()
8*m[1, 1, 1, 1, 1, 1, 1] + 4*m[2, 1, 1, 1, 1, 1]
+ 2*m[2, 2, 1, 1, 1] + m[2, 2, 2, 1]

sage: W = WeylGroup(['C',3,1])
sage: W.from_reduced_word([0,2,1,0]).stanley_symmetric_function()
32*m[1, 1, 1, 1] + 16*m[2, 1, 1] + 8*m[2, 2] + 4*m[3, 1]

sage: W = WeylGroup(['B',3,1])
sage: W.from_reduced_word([3,2,1]).stanley_symmetric_function()
2*m[1, 1, 1] + m[2, 1] + 1/2*m[3]

sage: W = WeylGroup(['B',4])
sage: w = W.from_reduced_word([3,2,3,1])
sage: w.stanley_symmetric_function()  # long time (6s on sage.math, 2011)
48*m[1, 1, 1, 1] + 24*m[2, 1, 1] + 12*m[2, 2] + 8*m[3, 1] + 2*m[4]

sage: A = AffinePermutationGroup(['A',4,1])
sage: a = A([-2,0,1,4,12])
sage: a.stanley_symmetric_function()
6*m[1, 1, 1, 1, 1, 1, 1, 1] + 5*m[2, 1, 1, 1, 1, 1, 1]
+ 4*m[2, 2, 1, 1, 1, 1] + 3*m[2, 2, 2, 1, 1] + 2*m[2, 2, 2, 2]
+ 4*m[3, 1, 1, 1, 1, 1] + 3*m[3, 2, 1, 1, 1] + 2*m[3, 2, 2, 1]
+ 2*m[3, 3, 1, 1] + m[3, 3, 2] + 3*m[4, 1, 1, 1, 1]
+ 2*m[4, 2, 1, 1] + m[4, 2, 2] + m[4, 3, 1]
```

One more example (github issue #14095):

```
sage: G = SymmetricGroup(4)
sage: w = G.from_reduced_word([3,2,3,1])
sage: w.stanley_symmetric_function()
3*m[1, 1, 1, 1] + 2*m[2, 1, 1] + m[2, 2] + m[3, 1]
```

REFERENCES:

- [BH1994]
- [Lam2008]
- [LSS2009]
- [Pon2010]

stanley_symmetric_function_as_polynomial(*max_length=None*)

Returns a multivariate generating function for the number of factorizations of a Weyl group element into Pieri factors of decreasing length, weighted by a statistic on Pieri factors.

See also:

- *stanley_symmetric_function()*
- *WeylGroups.ParentMethods.pieri_factors()*
- sage.combinat.root_system.pieri_factors

INPUT:

- `self` – an element w of a Weyl group W
- `max_length` – a non negative integer or infinity (default: infinity)

Returns the generating series for the Pieri factorizations $w = u_1 \cdots u_k$, where u_i is a Pieri factor for all i, $l(w) = \sum_{i=1}^{k} l(u_i)$ and `max_length` $\geq l(u_1) \geq \cdots \geq l(u_k)$.

A factorization $u_1 \cdots u_k$ contributes a monomial of the form $\prod_i x_{l(u_i)}$, with coefficient given by

$\prod_i 2^{c(u_i)}$, where c is a type-dependent statistic on Pieri factors, as returned by the method `u[i].stanley_symm_poly_weight()`.

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1])
sage: W.from_reduced_word([]).stanley_symmetric_function_as_polynomial()
1
sage: W.from_reduced_word([1]).stanley_symmetric_function_as_polynomial()
x1
sage: W.from_reduced_word([1,2]).stanley_symmetric_function_as_
→polynomial()
x1^2
sage: W.from_reduced_word([2,1]).stanley_symmetric_function_as_
→polynomial()
x1^2 + x2
sage: W.from_reduced_word([1,2,1]).stanley_symmetric_function_as_
→polynomial()
2*x1^3 + x1*x2
sage: W.from_reduced_word([1,2,1,0]).stanley_symmetric_function_as_
→polynomial()
3*x1^4 + 2*x1^2*x2 + x2^2 + x1*x3
sage: x = W.from_reduced_word([1,2,3,1,2,1,0])
sage: x.stanley_symmetric_function_as_polynomial()  # long time
22*x1^7 + 11*x1^5*x2 + 5*x1^3*x2^2 + 3*x1^4*x3 + 2*x1*x2^3 + x1^2*x2*x3
sage: y = W.from_reduced_word([3,1,2,0,3,1,0])
sage: y.stanley_symmetric_function_as_polynomial()  # long time
8*x1^7 + 4*x1^5*x2 + 2*x1^3*x2^2 + x1*x2^3

sage: W = WeylGroup(['C',3,1])
sage: W.from_reduced_word([0,2,1,0]).stanley_symmetric_function_as_
→polynomial()
32*x1^4 + 16*x1^2*x2 + 8*x2^2 + 4*x1*x3

sage: W = WeylGroup(['B',3,1])
sage: W.from_reduced_word([3,2,1]).stanley_symmetric_function_as_
→polynomial()
2*x1^3 + x1*x2 + 1/2*x3
```

Algorithm: Induction on the left Pieri factors. Note that this induction preserves subsets of W which are stable by taking right factors, and in particular Grassmanian elements.

Finite

alias of *`FiniteWeylGroups`*

class ParentMethods

Bases: `object`

bruhat_cone (*x*, *y*, *side='upper'*, *backend='cdd'*)

Return the (upper or lower) Bruhat cone associated to the interval `[x,y]`.

To a cover relation $v \prec w$ in strong Bruhat order you can assign a positive root β given by the unique reflection s_β such that $s_\beta v = w$.

The upper Bruhat cone of the interval $[x, y]$ is the non-empty, polyhedral cone generated by the roots corresponding to $x \prec a$ for all atoms a in the interval. The lower Bruhat cone of the interval $[x, y]$ is the non-empty, polyhedral cone generated by the roots corresponding to $c \prec y$ for all coatoms c in the interval.

INPUT:

- x - an element in the group W
- y - an element in the group W
- side (default: 'upper') – must be one of the following:
  - 'upper' - return the upper Bruhat cone of the interval [x, y]
  - 'lower' - return the lower Bruhat cone of the interval [x, y]
- backend – string (default: 'cdd'); the backend to use to create the polyhedron

EXAMPLES:

```
sage: W = WeylGroup(['A',2])
sage: x = W.from_reduced_word([1])
sage: y = W.w0
sage: W.bruhat_cone(x, y)
A 2-dimensional polyhedron in QQ^3
 defined as the convex hull of 1 vertex and 2 rays

sage: W = WeylGroup(['E',6])
sage: x = W.one()
sage: y = W.w0
sage: W.bruhat_cone(x, y, side='lower')
A 6-dimensional polyhedron in QQ^8
 defined as the convex hull of 1 vertex and 6 rays
```

REFERENCES:

- [Dy1994]
- [JS2021]

coxeter_matrix()

Return the Coxeter matrix associated to self.

EXAMPLES:

```
sage: G = WeylGroup(['A',3])
sage: G.coxeter_matrix()
[1 3 2]
[3 1 3]
[2 3 1]
```

pieri_factors(**args*, ***keywords*)

Returns the set of Pieri factors in this Weyl group.

For any type, the set of Pieri factors forms a lower ideal in Bruhat order, generated by all the conjugates of some special element of the Weyl group. In type A_n, this special element is $s_n \cdots s_1$, and the conjugates are obtained by rotating around this reduced word.

These are used to compute Stanley symmetric functions.

See also:

- *WeylGroups.ElementMethods.stanley_symmetric_function()*
- sage.combinat.root_system.pieri_factors

EXAMPLES:

```
sage: W = WeylGroup(['A',5,1])
sage: PF = W.pieri_factors()
sage: PF.cardinality()
63

sage: W = WeylGroup(['B',3])
```

(continues on next page)

(continued from previous page)

```
sage: PF = W.pieri_factors()
sage: sorted([w.reduced_word() for w in PF])
[[],
 [1],
 [1, 2],
 [1, 2, 1],
 [1, 2, 3],
 [1, 2, 3, 1],
 [1, 2, 3, 2],
 [1, 2, 3, 2, 1],
 [2],
 [2, 1],
 [2, 3],
 [2, 3, 1],
 [2, 3, 2],
 [2, 3, 2, 1],
 [3],
 [3, 1],
 [3, 1, 2],
 [3, 1, 2, 1],
 [3, 2],
 [3, 2, 1]]
sage: W = WeylGroup(['C',4,1])
sage: PF = W.pieri_factors()
sage: W.from_reduced_word([3,2,0]) in PF
True
```

quantum_bruhat_graph(*index_set=()*)

Return the quantum Bruhat graph of the quotient of the Weyl group by a parabolic subgroup W_J.

INPUT:

- `index_set` – (default: ()) a tuple J of nodes of the Dynkin diagram

By default, the value for `index_set` indicates that the subgroup is trivial and the quotient is the full Weyl group.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: g = W.quantum_bruhat_graph((1,3))
sage: g
Parabolic Quantum Bruhat Graph of Weyl Group of type ['A', 3]
 (as a matrix group acting on the ambient space)
 for nodes (1, 3): Digraph on 6 vertices
sage: g.vertices(sort=True)
[s2*s3*s1*s2, s3*s1*s2, s1*s2, s3*s2, s2, 1]
sage: g.edges(sort=True)
[(s2*s3*s1*s2, s2, alpha[2]),
 (s3*s1*s2, s2*s3*s1*s2, alpha[1] + alpha[2] + alpha[3]),
 (s3*s1*s2, 1, alpha[2]),
 (s1*s2, s3*s1*s2, alpha[2] + alpha[3]),
 (s3*s2, s3*s1*s2, alpha[1] + alpha[2]),
 (s2, s1*s2, alpha[1] + alpha[2]),
 (s2, s3*s2, alpha[2] + alpha[3]),
 (1, s2, alpha[2])]
sage: W = WeylGroup(['A',3,1], prefix="s")
sage: g = W.quantum_bruhat_graph()
Traceback (most recent call last):
```

(continues on next page)

(continued from previous page)

```
...
ValueError: the Cartan type ['A', 3, 1] is not finite
```

additional_structure()

Return None.

Indeed, the category of Weyl groups defines no additional structure: Weyl groups are a special class of Coxeter groups.

See also:

Category.additional_structure()

Todo: Should this category be a CategoryWithAxiom?

EXAMPLES:

```
sage: WeylGroups().additional_structure()
```

super_categories()

EXAMPLES:

```
sage: WeylGroups().super_categories()
[Category of Coxeter groups]
```

4.164 Technical Categories

4.164.1 Facade Sets

For background, see *What is a facade set?*.

class sage.categories.facade_sets.**FacadeSets**(*base_category*)

Bases: *CategoryWithAxiom_singleton*

class ParentMethods

Bases: object

facade_for()

Returns the parents this set is a facade for

This default implementation assumes that self has an attribute _facade_for, typically initialized by Parent.__init__(). If the attribute is not present, the method raises a NotImplementedError.

EXAMPLES:

```
sage: S = Sets().Facade().example(); S
An example of facade set: the monoid of positive integers
sage: S.facade_for()
(Integer Ring,)
```

Check that github issue #13801 is corrected:

```
sage: class A(Parent):
....:     def __init__(self):
....:         Parent.__init__(self, category=Sets(), facade=True)
sage: a = A()
sage: a.facade_for()
Traceback (most recent call last):
...
NotImplementedError: this parent did not specify which parents it is a␣
↪facade for
```

is_parent_of(*element*)

Returns whether `self` is the parent of `element`

INPUT:

- `element` – any object

Since `self` is a facade domain, this actually tests whether the parent of `element` is any of the parent `self` is a facade for.

EXAMPLES:

```
sage: S = Sets().Facade().example(); S
An example of facade set: the monoid of positive integers
sage: S.is_parent_of(1)
True
sage: S.is_parent_of(1/2)
False
```

This method differs from `__contains__()` in two ways. First, this does not take into account the fact that `self` may be a strict subset of the parent(s) it is a facade for:

```
sage: -1 in S, S.is_parent_of(-1)
(False, True)
```

Furthermore, there is no coercion attempted:

```
sage: int(1) in S, S.is_parent_of(int(1))
(True, False)
```

Warning: this implementation does not handle facade parents of facade parents. Is this a feature we want generically?

example(*choice='subset'*)

Returns an example of facade set, as per *Category.example()*.

INPUT:

- `choice` – 'union' or 'subset' (default: 'subset').

EXAMPLES:

```
sage: Sets().Facade().example()
An example of facade set: the monoid of positive integers
sage: Sets().Facade().example(choice='union')
An example of a facade set: the integers completed by +-infinity
sage: Sets().Facade().example(choice='subset')
An example of facade set: the monoid of positive integers
```

CHAPTER

FIVE

FUNCTORIAL CONSTRUCTIONS

5.1 Covariant Functorial Constructions

A *functorial construction* is a collection of functors $(F_{Cat})_{Cat}$ (indexed by a collection of categories) which associate to a sequence of parents $(A, B, ...)$ in a category Cat a parent $F_{Cat}(A, B, ...)$. Typical examples of functorial constructions are `cartesian_product` and `tensor_product`.

The category of $F_{Cat}(A, B, ...)$, which only depends on Cat, is called the (functorial) construction category.

A functorial construction is *(category)-covariant* if for every categories Cat and $SuperCat$, the category of $F_{Cat}(A, B, ...)$ is a subcategory of the category of $F_{SuperCat}(A, B, ...)$ whenever Cat is a subcategory of $SuperCat$. A functorial construction is *(category)-regressive* if the category of $F_{Cat}(A, B, ...)$ is a subcategory of Cat.

The goal of this module is to provide generic support for covariant functorial constructions. In particular, given some parents A, B, …, in respective categories Cat_A, Cat_B, …, it provides tools for calculating the best known category for the parent $F(A, B, ...)$. For examples, knowing that Cartesian products of semigroups (resp. monoids, groups) have a semigroup (resp. monoid, group) structure, and given a group B and two monoids A and C it can calculate that $A \times B \times C$ is naturally endowed with a monoid structure.

See *`CovariantFunctorialConstruction`*, *`CovariantConstructionCategory`* and *`RegressiveCovariantConstructionCategory`* for more details.

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

class `sage.categories.covariant_functorial_construction.`**`CovariantConstructionCategory`**(*category*, **args*)

Bases: *`FunctorialConstructionCategory`*

Abstract class for categories F_{Cat} obtained through a covariant functorial construction

`additional_structure`()

Return the additional structure defined by `self`.

By default, a functorial construction category `A.F()` defines additional structure if and only if A is the category defining F. The rationale is that, for a subcategory B of A, the fact that $B.F()$ morphisms shall preserve the F-specific structure is already imposed by $A.F()$.

See also:

- *`Category.additional_structure()`*.
- *`is_construction_defined_by_base()`*.

EXAMPLES:

```
sage: Modules(ZZ).Graded().additional_structure()
Category of graded modules over Integer Ring
sage: Algebras(ZZ).Graded().additional_structure()
```

classmethod default_super_categories(*category*, **args*)

Return the default super categories of $F_{Cat}(A, B, ...)$ for $A, B, ...$ parents in Cat.

INPUT:

- `cls` – the category class for the functor F
- `category` – a category Cat
- `*args` – further arguments for the functor

OUTPUT: a (join) category

The default implementation is to return the join of the categories of $F(A, B, ...)$ for $A, B, ...$ in turn in each of the super categories of `category`.

This is implemented as a class method, in order to be able to reconstruct the functorial category associated to each of the super categories of `category`.

EXAMPLES:

Bialgebras are both algebras and coalgebras:

```
sage: Bialgebras(QQ).super_categories()
[Category of algebras over Rational Field,
 Category of coalgebras over Rational Field]
```

Hence tensor products of bialgebras are tensor products of algebras and tensor products of coalgebras:

```
sage: Bialgebras(QQ).TensorProducts().super_categories()
[Category of tensor products of algebras over Rational Field,
 Category of tensor products of coalgebras over Rational Field]
```

Here is how *`default_super_categories()`* was called internally:

```
sage: C = sage.categories.tensor.TensorProductsCategory
sage: C.default_super_categories(Bialgebras(QQ))
Join of Category of tensor products of algebras over Rational Field
    and Category of tensor products of coalgebras over Rational Field
```

We now show a similar example, with the `Algebra` functor which takes a parameter **Q**:

```
sage: FiniteMonoids().super_categories()
[Category of monoids, Category of finite semigroups]
sage: sorted(FiniteMonoids().Algebras(QQ).super_categories(), key=str)
[Category of finite dimensional algebras with basis over Rational Field,
 Category of finite set algebras over Rational Field,
 Category of monoid algebras over Rational Field]
```

Note that neither the category of *finite* semigroup algebras nor that of monoid algebras appear in the result; this is because there is currently nothing specific implemented about them.

Here is how *`default_super_categories()`* was called internally:

```
sage: C = sage.categories.algebra_functor.AlgebrasCategory
sage: C.default_super_categories(FiniteMonoids(), QQ)
Join of Category of finite dimensional algebras with basis over Rational Field
    and Category of monoid algebras over Rational Field
    and Category of finite set algebras over Rational Field
```

is_construction_defined_by_base()

Return whether the construction is defined by the base of `self`.

EXAMPLES:

The graded functorial construction is defined by the modules category. Hence this method returns `True` for graded modules and `False` for other graded xxx categories:

```
sage: Modules(ZZ).Graded().is_construction_defined_by_base()
True
sage: Algebras(QQ).Graded().is_construction_defined_by_base()
False
sage: Modules(ZZ).WithBasis().Graded().is_construction_defined_by_base()
False
```

This is implemented as follows: given the base category A and the construction F of `self`, that is `self=A.F()`, check whether no super category of A has F defined.

Note: Recall that, when A does not implement the construction `F`, a join category is returned. Therefore, in such cases, this method is not available:

```
sage: Bialgebras(QQ).Graded().is_construction_defined_by_base()
Traceback (most recent call last):
...
AttributeError: 'JoinCategory_with_category' object has
no attribute 'is_construction_defined_by_base'
```

class sage.categories.covariant_functorial_construction.**CovariantFunctorialConstruction**

Bases: `UniqueRepresentation`, `SageObject`

An abstract class for construction functors F (eg F = Cartesian product, tensor product, **Q**-algebra, …) such that:

- Each category Cat (eg Cat = `Groups()`) can provide a category F_{Cat} for parents constructed via this functor (e.g. F_{Cat} = `CartesianProductsOf(Groups())`).
- For every category Cat, F_{Cat} is a subcategory of $F_{SuperCat}$ for every super category $SuperCat$ of Cat (the functorial construction is (category)-covariant).
- For parents A, B, …, respectively in the categories Cat_A, Cat_B, …, the category of $F(A, B, ...)$ is F_{Cat} where Cat is the meet of the categories Cat_A, Cat_B, …,.

This covers two slightly different use cases:

- In the first use case, one uses directly the construction functor to create new parents:

```
sage: tensor()   # todo: not implemented (add an example)
```

or even new elements, which indirectly constructs the corresponding parent:

```
sage: tensor(...) # todo: not implemented
```

- In the second use case, one implements a parent, and then put it in the category F_{Cat} to specify supplementary mathematical information about that parent.

The main purpose of this class is to handle automatically the trivial part of the category hierarchy. For example, `CartesianProductsOf(Groups())` is set automatically as a subcategory of `CartesianProductsOf(Monoids())`.

In practice, each subclass of this class should provide the following attributes:

- `_functor_category` - a string which should match the name of the nested category class to be used in each category to specify information and generic operations for elements of this category.
- `_functor_name` - a string which specifies the name of the functor, and also (when relevant) of the method on parents and elements used for calling the construction.

TODO: What syntax do we want for F_{Cat}? For example, for the tensor product construction, which one do we want among (see chat on IRC, on 07/12/2009):

- `tensor(Cat)`
- `tensor((Cat, Cat))`
- `tensor.of((Cat, Cat))`
- `tensor.category_from_categories((Cat, Cat, Cat))`
- `Cat.TensorProducts()`

The syntax `Cat.TensorProducts()` does not supports well multivariate constructions like `tensor.of([Algebras(), HopfAlgebras(), ...])`. Also it forces every category to be (somehow) aware of all the tensorial construction that could apply to it, even those which are only induced from super categories.

Note: for each functorial construction, there probably is one (or several) largest categories on which it applies. For example, the `CartesianProducts()` construction makes only sense for concrete categories, that is subcategories of `Sets()`. Maybe we want to model this one way or the other.

category_from_categories(*categories*)

Return the category of $F(A, B, ...)$ for $A, B, ...$ parents in the given categories.

INPUT:

- `self`: a functor F
- `categories`: a non empty tuple of categories

EXAMPLES:

```
sage: Cat1 = Rings()
sage: Cat2 = Groups()
sage: cartesian_product.category_from_categories((Cat1, Cat1, Cat1))
Join of Category of rings and ...
    and Category of Cartesian products of monoids
    and Category of Cartesian products of commutative additive groups

sage: cartesian_product.category_from_categories((Cat1, Cat2))
Category of Cartesian products of monoids
```

category_from_category(*category*)

Return the category of $F(A, B, ...)$ for $A, B, ...$ parents in `category`.

INPUT:

- `self`: a functor F
- `category`: a category

EXAMPLES:

```
sage: tensor.category_from_category(ModulesWithBasis(QQ))
Category of tensor products of vector spaces with basis over Rational Field
```

# TODO: add support for parametrized functors

`category_from_parents`(*parents*)

Return the category of $F(A, B, ...)$ for $A, B, ...$ parents.

INPUT:

- self: a functor F
- parents: a list (or iterable) of parents.

EXAMPLES:

```
sage: E = CombinatorialFreeModule(QQ, ["a", "b", "c"])                    #␣
↪needs sage.modules
sage: tensor.category_from_parents((E, E, E))                              #␣
↪needs sage.modules
Category of tensor products of
 finite dimensional vector spaces with basis over Rational Field
```

class `sage.categories.covariant_functorial_construction.`**`FunctorialConstructionCategory`**(*cat-e-gory*, **args*

Bases: `Category`

Abstract class for categories F_{Cat} obtained through a functorial construction

`base_category`()

Return the base category of the category `self`.

For any category `B` = F_{Cat} obtained through a functorial construction F, the call `B.base_category()` returns the category Cat.

EXAMPLES:

```
sage: Semigroups().Quotients().base_category()
Category of semigroups
```

classmethod `category_of`(*category*, **args*)

Return the image category of the functor F_{Cat}.

This is the main entry point for constructing the category F_{Cat} of parents $F(A, B, ...)$ constructed from parents $A, B, ...$ in Cat.

INPUT:

- `cls` – the category class for the functorial construction F
- `category` – a category Cat
- `*args` – further arguments for the functor

EXAMPLES:

```
sage: C = sage.categories.tensor.TensorProductsCategory
sage: C.category_of(ModulesWithBasis(QQ))
Category of tensor products of vector spaces with basis over Rational Field

sage: C = sage.categories.algebra_functor.AlgebrasCategory
sage: C.category_of(FiniteMonoids(), QQ)
Join of Category of finite dimensional algebras with basis over Rational Field
    and Category of monoid algebras over Rational Field
    and Category of finite set algebras over Rational Field
```

extra_super_categories()

Return the extra super categories of a construction category.

Default implementation which returns `[]`.

EXAMPLES:

```
sage: Sets().Subquotients().extra_super_categories()
[]
sage: Semigroups().Quotients().extra_super_categories()
[]
```

super_categories()

Return the super categories of a construction category.

EXAMPLES:

```
sage: Sets().Subquotients().super_categories()
[Category of sets]
sage: Semigroups().Quotients().super_categories()
[Category of subquotients of semigroups, Category of quotients of sets]
```

class sage.categories.covariant_functorial_construction.**RegressiveCovariantConstructionCate**

Bases: `CovariantConstructionCategory`

Abstract class for categories F_{Cat} obtained through a regressive covariant functorial construction

classmethod default_super_categories(*category*, **args*)

Return the default super categories of $F_{Cat}(A, B, ...)$ for $A, B, ...$ parents in Cat.

INPUT:

- `cls` – the category class for the functor F
- `category` – a category Cat
- `*args` – further arguments for the functor

OUTPUT:

A join category.

This implements the property that an induced subcategory is a subcategory.

EXAMPLES:

A subquotient of a monoid is a monoid, and a subquotient of semigroup:

```
sage: Monoids().Subquotients().super_categories()
[Category of monoids, Category of subquotients of semigroups]
```

5.2 Cartesian Product Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2008-2010): initial revision and refactorization

class sage.categories.cartesian_product.**CartesianProductFunctor**(*category=None*)

Bases: *CovariantFunctorialConstruction*, *MultivariateConstructionFunctor*

The Cartesian product functor.

EXAMPLES:

```
sage: cartesian_product
The cartesian_product functorial construction
```

`cartesian_product` takes a finite collection of sets, and constructs the Cartesian product of those sets:

```
sage: A = FiniteEnumeratedSet(['a','b','c'])
sage: B = FiniteEnumeratedSet([1,2])
sage: C = cartesian_product([A, B]); C
The Cartesian product of ({'a', 'b', 'c'}, {1, 2})
sage: C.an_element()
('a', 1)
sage: C.list()          # todo: not implemented
[['a', 1], ['a', 2], ['b', 1], ['b', 2], ['c', 1], ['c', 2]]
```

If those sets are endowed with more structure, say they are monoids (hence in the category `Monoids()`), then the result is automatically endowed with its natural monoid structure:

```
sage: M = Monoids().example()
sage: M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.rename('M')
sage: C = cartesian_product([M, ZZ, QQ])
sage: C
The Cartesian product of (M, Integer Ring, Rational Field)
sage: C.an_element()
('abcd', 1, 1/2)
sage: C.an_element()^2
('abcdabcd', 1, 1/4)
sage: C.category()
Category of Cartesian products of monoids

sage: Monoids().CartesianProducts()
Category of Cartesian products of monoids
```

The Cartesian product functor is covariant: if `A` is a subcategory of `B`, then `A.CartesianProducts()` is a subcategory of `B.CartesianProducts()` (see also *CovariantFunctorialConstruction*):

```
sage: C.categories()
[Category of Cartesian products of monoids,
```

(continues on next page)

(continued from previous page)

```
 Category of monoids,
 Category of Cartesian products of semigroups,
 Category of semigroups,
 Category of Cartesian products of unital magmas,
 Category of Cartesian products of magmas,
 Category of unital magmas,
 Category of magmas,
 Category of Cartesian products of sets,
 Category of sets, ...]

[Category of Cartesian products of monoids,
 Category of monoids,
 Category of Cartesian products of semigroups,
 Category of semigroups,
 Category of Cartesian products of magmas,
 Category of unital magmas,
 Category of magmas,
 Category of Cartesian products of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Hence, the role of `Monoids().CartesianProducts()` is solely to provide mathematical information and algorithms which are relevant to Cartesian product of monoids. For example, it specifies that the result is again a monoid, and that its multiplicative unit is the Cartesian product of the units of the underlying sets:

```
sage: C.one()
('', 1, 1)
```

Those are implemented in the nested class *`Monoids.CartesianProducts`* of `Monoids(QQ)`. This nested class is itself a subclass of *`CartesianProductsCategory`*.

`symbol = ' (+) '`

class `sage.categories.cartesian_product.`**`CartesianProductsCategory`**(*category*, **args*)

Bases: *`CovariantConstructionCategory`*

An abstract base class for all `CartesianProducts` categories.

`CartesianProducts`()

Return the category of (finite) Cartesian products of objects of `self`.

By associativity of Cartesian products, this is `self` (a Cartesian product of Cartesian products of A's is a Cartesian product of A's).

EXAMPLES:

```
sage: ModulesWithBasis(QQ).CartesianProducts().CartesianProducts()
Category of Cartesian products of vector spaces with basis over Rational Field
```

`base_ring`()

The base ring of a Cartesian product is the base ring of the underlying category.

EXAMPLES:

```
sage: Algebras(ZZ).CartesianProducts().base_ring()
Integer Ring
```

5.3 Tensor Product Functorial Construction

AUTHORS:

- Nicolas M. Thiéry (2008-2010): initial revision and refactorization

class `sage.categories.tensor.`**`TensorProductFunctor`**

Bases: *`CovariantFunctorialConstruction`*

A singleton class for the tensor functor.

This functor takes a collection of vector spaces (or modules with basis), and constructs the tensor product of those vector spaces. If this vector space is in a subcategory, say that of `Algebras(QQ)`, it is automatically endowed with its natural algebra structure, thanks to the category `Algebras(QQ).TensorProducts()` of tensor products of algebras. For elements, it constructs the natural tensor product element in the corresponding tensor product of their parents.

The tensor functor is covariant: if `A` is a subcategory of `B`, then `A.TensorProducts()` is a subcategory of `B.TensorProducts()` (see also *`CovariantFunctorialConstruction`*). Hence, the role of `Algebras(QQ).TensorProducts()` is solely to provide mathematical information and algorithms which are relevant to tensor product of algebras.

Those are implemented in the nested class *`TensorProducts`* of `Algebras(QQ)`. This nested class is itself a subclass of *`TensorProductsCategory`*.

`symbol = ' # '`

`unicode_symbol = ' � '`

class `sage.categories.tensor.`**`TensorProductsCategory`**(*category*, **args*)

Bases: *`CovariantConstructionCategory`*

An abstract base class for all TensorProducts's categories

`TensorProducts``()`

Returns the category of tensor products of objects of `self`

By associativity of tensor products, this is `self` (a tensor product of tensor products of Cat's is a tensor product of Cat's)

EXAMPLES:

```
sage: ModulesWithBasis(QQ).TensorProducts().TensorProducts()
Category of tensor products of vector spaces with basis over Rational Field
```

`base``()`

The base of a tensor product is the base (usually a ring) of the underlying category.

EXAMPLES:

```
sage: ModulesWithBasis(ZZ).TensorProducts().base()
Integer Ring
```

`sage.categories.tensor.`**`tensor = The tensor functorial construction`**

The tensor product functorial construction

See *`TensorProductFunctor`* for more information

EXAMPLES:

```
sage: tensor
The tensor functorial construction
```

5.4 Signed Tensor Product Functorial Construction

AUTHORS:

- Travis Scrimshaw (2019-07): initial version

class sage.categories.signed_tensor.**SignedTensorProductFunctor**

Bases: *CovariantFunctorialConstruction*

A singleton class for the signed tensor functor.

This functor takes a collection of graded algebras (possibly with basis) and constructs the signed tensor product of those algebras. If this algebra is in a subcategory, say that of `Algebras(QQ).Graded()`, it is automatically endowed with its natural algebra structure, thanks to the category `Algebras(QQ).Graded().SignedTensorProducts()` of signed tensor products of graded algebras. For elements, it constructs the natural tensor product element in the corresponding tensor product of their parents.

The signed tensor functor is covariant: if `A` is a subcategory of `B`, then `A.SignedTensorProducts()` is a subcategory of `B.SignedTensorProducts()` (see also *CovariantFunctorialConstruction*). Hence, the role of `Algebras(QQ).Graded().SignedTensorProducts()` is solely to provide mathematical information and algorithms which are relevant to signed tensor product of graded algebras.

Those are implemented in the nested class `SignedTensorProducts` of `Algebras(QQ).Graded()`. This nested class is itself a subclass of *SignedTensorProductsCategory*.

EXAMPLES:

```
sage: tensor_signed
The signed tensor functorial construction
```

symbol = ' # '

unicode_symbol = ' ⍰ '

class sage.categories.signed_tensor.**SignedTensorProductsCategory**(*category*, **args*)

Bases: *CovariantConstructionCategory*

An abstract base class for all SignedTensorProducts's categories.

SignedTensorProducts()

Return the category of signed tensor products of objects of `self`.

By associativity of signed tensor products, this is `self` (a tensor product of signed tensor products of Cat's is a tensor product of Cat's with the same twisting morphism)

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).Graded().SignedTensorProducts().
↪SignedTensorProducts()
Category of signed tensor products of graded algebras with basis
 over Rational Field
```

base()

The base of a signed tensor product is the base (usually a ring) of the underlying category.

EXAMPLES:

```
sage: AlgebrasWithBasis(ZZ).Graded().SignedTensorProducts().base()
Integer Ring
```

5.5 Dual functorial construction

AUTHORS:

- Nicolas M. Thiery (2009-2010): initial revision

class sage.categories.dual.**DualFunctor**

Bases: *CovariantFunctorialConstruction*

A singleton class for the dual functor

symbol = '^*'

class sage.categories.dual.**DualObjectsCategory**(*category*, **args*)

Bases: *CovariantConstructionCategory*

5.6 Group algebras and beyond: the Algebra functorial construction

5.6.1 Introduction: group algebras

Let G be a group and R be a ring. For example:

```
sage: G = DihedralGroup(3)
sage: R = QQ
```

The *group algebra* $A = RG$ of G over R is the space of formal linear combinations of elements of $group$ with coefficients in R:

```
sage: A = G.algebra(R); A
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
sage: a = A.an_element(); a
() + (1,2) + 3*(1,2,3) + 2*(1,3,2)
```

This space is endowed with an algebra structure, obtained by extending by bilinearity the multiplication of G to a multiplication on RG:

```
sage: A in Algebras
True
sage: a * a
14*() + 5*(2,3) + 2*(1,2) + 10*(1,2,3) + 13*(1,3,2) + 5*(1,3)
```

In particular, the product of two basis elements is induced by the product of the corresponding elements of the group, and the unit of the group algebra is indexed by the unit of the group:

```
sage: (s, t) = A.algebra_generators()
sage: s*t
(1,2)
sage: A.one_basis()
()
sage: A.one()
()
```

For the user convenience and backward compatibility, the group algebra can also be constructed with:

```
sage: GroupAlgebra(G, R)
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
```

Since github issue #18700, both constructions are strictly equivalent:

```
sage: GroupAlgebra(G, R) is G.algebra(R)
True
```

Group algebras are further endowed with a Hopf algebra structure; see below.

5.6.2 Generalizations

The above construction extends to weaker multiplicative structures than groups: magmas, semigroups, monoids. For a monoid S, we obtain the monoid algebra RS, which is defined exactly as above:

```
sage: S = Monoids().example(); S
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ); A
Algebra of An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
        over Rational Field
sage: A.category()
Category of monoid algebras over Rational Field
```

This construction also extends to additive structures: magmas, semigroups, monoids, or groups:

```
sage: S = CommutativeAdditiveMonoids().example(); S
An example of a commutative monoid:
 the free commutative monoid generated by ('a', 'b', 'c', 'd')
sage: U = S.algebra(QQ); U
Algebra of An example of a commutative monoid:
           the free commutative monoid generated by ('a', 'b', 'c', 'd')
        over Rational Field
```

Despite saying “free module”, this is really an algebra, whose multiplication is induced by the addition of elements of S:

```
sage: U in Algebras(QQ)
True
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: U(a) * U(b)
B[a + b]
```

To catter uniformly for the use cases above and some others, for S a set and K a ring, we define in Sage the *algebra of* `$S$` as the K-free module with basis indexed by S, endowed with whatever algebraic structure can be induced from that of S.

Warning: In most use cases, the result is actually an algebra, hence the name of this construction. In other cases this name is misleading:

```
sage: A = Sets().example().algebra(QQ); A
Algebra of Set of prime numbers (basic implementation)
        over Rational Field
sage: A.category()
Category of set algebras over Rational Field
sage: A in Algebras(QQ)
False
```

Suggestions for a uniform, meaningful, and non misleading name are welcome!

To achieve this flexibility, the features are implemented as a *Covariant Functorial Constructions* that is essentially a hierarchy of categories each providing the relevant additional features:

```
sage: A = DihedralGroup(3).algebra(QQ)
sage: A.categories()
[Category of finite group algebras over Rational Field,
 ...
 Category of group algebras over Rational Field,
 ...
 Category of monoid algebras over Rational Field,
 ...
 Category of semigroup algebras over Rational Field,
 ...
 Category of unital magma algebras over Rational Field,
 ...
 Category of magma algebras over Rational Field,
 ...
 Category of set algebras over Rational Field,
 ...]
```

5.6.3 Specifying the algebraic structure

Constructing the algebra of a set endowed with both an additive and a multiplicative structure is ambiguous:

```
sage: Z3 = IntegerModRing(3)
sage: A = Z3.algebra(QQ)
Traceback (most recent call last):
...
TypeError:  `S = Ring of integers modulo 3` is both
 an additive and a multiplicative semigroup.
 Constructing its algebra is ambiguous.
 Please use, e.g., S.algebra(QQ, category=Semigroups())
```

This ambiguity can be resolved using the `category` argument of the construction:

```
sage: A = Z3.algebra(QQ, category=Monoids()); A
Algebra of Ring of integers modulo 3 over Rational Field
sage: A.category()
Category of finite dimensional monoid algebras over Rational Field

sage: A = Z3.algebra(QQ, category=CommutativeAdditiveGroups()); A
Algebra of Ring of integers modulo 3 over Rational Field
```

(continues on next page)

(continued from previous page)

```
sage: A.category()
Category of finite dimensional commutative additive group algebras
 over Rational Field
```

In general, the `category` argument can be used to specify which structure of S shall be extended to KS.

5.6.4 Group algebras, continued

Let us come back to the case of a group algebra $A = RG$. It is endowed with more structure and in particular that of a *Hopf algebra*:

```
sage: G = DihedralGroup(3)
sage: A = G.algebra(R); A
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
sage: A in HopfAlgebras(R).FiniteDimensional().WithBasis()
True
```

The basis elements are *group-like* for the coproduct: $\Delta(g) = g \otimes g$:

```
sage: s
(1,2,3)
sage: s.coproduct()
(1,2,3) # (1,2,3)
```

The counit is the constant function 1 on the basis elements:

```
sage: A = GroupAlgebra(DihedralGroup(6), QQ)
sage: [A.counit(g) for g in A.basis()]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

The antipode is given on basis elements by $\chi(g) = g^{-1}$:

```
sage: A = GroupAlgebra(DihedralGroup(3), QQ)
sage: s
(1,2,3)
sage: s.antipode()
(1,3,2)
```

By Maschke's theorem, for a finite group whose cardinality does not divide the characteristic of the base field, the algebra is semisimple:

```
sage: SymmetricGroup(5).algebra(QQ) in Algebras(QQ).Semisimple()
True
sage: CyclicPermutationGroup(10).algebra(FiniteField(7)) in Algebras.Semisimple
True
sage: CyclicPermutationGroup(10).algebra(FiniteField(5)) in Algebras.Semisimple
False
```

5.6.5 Coercions

Let RS be the algebra of some structure S. Then RS admits the natural coercion from any other algebra $R'S'$ of some structure S', as long as R' coerces into R and S' coerces into S.

For example, since there is a natural inclusion from the dihedral group D_2 of order 4 into the symmetric group S_4 of order 4!, and since there is a natural map from the integers to the rationals, there is a natural map from $\mathbf{Z}[D_2]$ to $\mathbf{Q}[S_4]$:

```
sage: A = DihedralGroup(2).algebra(ZZ)
sage: B = SymmetricGroup(4).algebra(QQ)
sage: a = A.an_element(); a
() + 2*(3,4) + 3*(1,2) + (1,2)(3,4)
sage: b = B.an_element(); b
() + (2,3,4) + 2*(1,3)(2,4) + 3*(1,4)(2,3)
sage: B(a)
() + 2*(3,4) + 3*(1,2) + (1,2)(3,4)
sage: a * b  # a is automatically converted to an element of B
() + 2*(3,4) + 2*(2,3) + (2,3,4) + 3*(1,2) + (1,2)(3,4) + (1,3,2)
 + 3*(1,3,4,2) + 5*(1,3)(2,4) + 13*(1,3,2,4) + 12*(1,4,2,3) + 5*(1,4)(2,3)
sage: parent(a * b)
Symmetric group algebra of order 4 over Rational Field
```

There is no obvious map in the other direction, though:

```
sage: A(b)
Traceback (most recent call last):
...
TypeError: do not know how to make x (= () + (2,3,4) + 2*(1,3)(2,4) + 3*(1,4)(2,3))
 an element of self
 (=Algebra of Dihedral group of order 4 as a permutation group over Integer Ring)
```

If S is a unital (additive) magma, then RS is a unital algebra, and thus admits a coercion from its base ring R and any ring that coerces into R.

```
sage: G = DihedralGroup(2)
sage: A = G.algebra(ZZ)
sage: A(2)
2*()
```

If S is a multiplicative group, then RS admits a coercion from S and from any group which coerce into S:

```
sage: g = DihedralGroup(2).gen(0); g
(3,4)
sage: A(g)
(3,4)
sage: A(2) * g
2*(3,4)
```

Note that there is an ambiguity if S' is a group which coerces into both R and S. For example) if S is the additive group $(\mathbf{Z}, +)$, and $A = RS$ is its group algebra, then the integer 2 can be coerced into A in two ways – via S, or via the base ring R – and *the answers are different*. It that case the coercion to R takes precedence. In particular, if $\mathbf{Z}$ is the ring (or group) of integers, then $\mathbf{Z}$ will coerce to any RS, by sending $\mathbf{Z}$ to R. In generic code, it is therefore recommended to always explicitly use `A.monomial(g)` to convert an element of the group into A.

AUTHORS:

- David Loeffler (2008-08-24): initial version
- Martin Raum (2009-08): update to use new coercion model – see github issue #6670.

- John Palmieri (2011-07): more updates to coercion, categories, etc., group algebras constructed using CombinatorialFreeModule – see github issue #6670.
- Nicolas M. Thiéry (2010-2017), Travis Scrimshaw (2017): generalization to a covariant functorial construction for monoid algebras, and beyond – see e.g. github issue #18700.

class sage.categories.algebra_functor.**AlgebraFunctor**(*base_ring*)

Bases: *CovariantFunctorialConstruction*

For a fixed ring, a functor sending a group/… to the corresponding group/… algebra.

EXAMPLES:

```
sage: from sage.categories.algebra_functor import AlgebraFunctor
sage: F = AlgebraFunctor(QQ); F
The algebra functorial construction
sage: F(DihedralGroup(3))
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
```

base_ring()

Return the base ring for this functor.

EXAMPLES:

```
sage: from sage.categories.algebra_functor import AlgebraFunctor
sage: AlgebraFunctor(QQ).base_ring()
Rational Field
```

class sage.categories.algebra_functor.**AlgebrasCategory**(*category*, **args*)

Bases: *CovariantConstructionCategory*, *Category_over_base_ring*

An abstract base class for categories of monoid algebras, groups algebras, and the like.

See also:

- *Sets.ParentMethods.algebra()*
- *Sets.SubcategoryMethods.Algebras()*
- *CovariantFunctorialConstruction*

INPUT:

- base_ring – a ring

EXAMPLES:

```
sage: C = Groups().Algebras(QQ); C
Category of group algebras over Rational Field
sage: C = Monoids().Algebras(QQ); C
Category of monoid algebras over Rational Field

sage: C._short_name()
'Algebras'
sage: latex(C) # todo: improve that
\mathbf{Algebras}(\mathbf{Monoids})
```

class ParentMethods

Bases: object

coproduct_on_basis(*g*)

Return the coproduct of the element g of the basis.

Each basis element g is group-like. This method is used to compute the coproduct of any element.

EXAMPLES:

```
sage: PF = NonDecreasingParkingFunctions(4)
sage: A = PF.algebra(ZZ); A
Algebra of Non-decreasing parking functions of size 4 over Integer Ring
sage: g = PF.an_element(); g
[1, 1, 1, 1]
sage: A.coproduct_on_basis(g)
B[[1, 1, 1, 1]] # B[[1, 1, 1, 1]]
sage: a = A.an_element(); a
2*B[[1, 1, 1, 1]] + 2*B[[1, 1, 1, 2]] + 3*B[[1, 1, 1, 3]]
sage: a.coproduct()
2*B[[1, 1, 1, 1]] # B[[1, 1, 1, 1]] +
2*B[[1, 1, 1, 2]] # B[[1, 1, 1, 2]] +
3*B[[1, 1, 1, 3]] # B[[1, 1, 1, 3]]
```

class sage.categories.algebra_functor.**GroupAlgebraFunctor**(*group*)

Bases: *ConstructionFunctor*

For a fixed group, a functor sending a commutative ring to the corresponding group algebra.

INPUT:

- group – the group associated to each group algebra under consideration

EXAMPLES:

```
sage: from sage.categories.algebra_functor import GroupAlgebraFunctor
sage: F = GroupAlgebraFunctor(KleinFourGroup()); F
GroupAlgebraFunctor
sage: A = F(QQ); A
Algebra of The Klein 4 group of order 4, as a permutation group over Rational␣
↪Field
```

group()

Return the group which is associated to this functor.

EXAMPLES:

```
sage: from sage.categories.algebra_functor import GroupAlgebraFunctor
sage: GroupAlgebraFunctor(CyclicPermutationGroup(17)).group() ==␣
↪CyclicPermutationGroup(17)
True
```

5.7 Subquotient Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

class sage.categories.subquotients.**SubquotientsCategory**(*category*, **args*)

Bases: *RegressiveCovariantConstructionCategory*

5.8 Quotients Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

class sage.categories.quotients.**QuotientsCategory**(*category*, **args*)

Bases: *RegressiveCovariantConstructionCategory*

classmethod default_super_categories(*category*)

Returns the default super categories of `category.Quotients()`

Mathematical meaning: if A is a quotient of B in the category C, then A is also a subquotient of B in the category C.

INPUT:

- `cls` – the class `QuotientsCategory`
- `category` – a category Cat

OUTPUT: a (join) category

In practice, this returns `category.Subquotients()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category` and `cat.Quotients()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`, which has `cat=Monoids()` as super category. Then, a subgroup of a group G is simultaneously a subquotient of G, a group by itself, and a quotient monoid of `G`:

```
sage: Groups().Quotients().super_categories()
[Category of groups, Category of subquotients of monoids, Category of␣
↪quotients of semigroups]
```

Mind the last item above: there is indeed currently nothing implemented about quotient monoids.

This resulted from the following call:

```
sage: sage.categories.quotients.QuotientsCategory.default_super_
↪categories(Groups())
Join of Category of groups and Category of subquotients of monoids and␣
↪Category of quotients of semigroups
```

5.9 Subobjects Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

class sage.categories.subobjects.**SubobjectsCategory**(*category*, **args*)

Bases: *RegressiveCovariantConstructionCategory*

classmethod default_super_categories(*category*)

Returns the default super categories of `category.Subobjects()`

Mathematical meaning: if A is a subobject of B in the category C, then A is also a subquotient of B in the category C.

INPUT:

- `cls` – the class `SubobjectsCategory`
- `category` – a category Cat

OUTPUT: a (join) category

In practice, this returns `category.Subquotients()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category` and `cat.Subobjects()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`, which has `cat=Monoids()` as super category. Then, a subgroup of a group G is simultaneously a subquotient of G, a group by itself, and a submonoid of G:

```
sage: Groups().Subobjects().super_categories()
[Category of groups, Category of subquotients of monoids, Category of␣
↪subobjects of sets]
```

Mind the last item above: there is indeed currently nothing implemented about submonoids.

This resulted from the following call:

```
sage: sage.categories.subobjects.SubobjectsCategory.default_super_
↪categories(Groups())
Join of Category of groups and Category of subquotients of monoids and␣
↪Category of subobjects of sets
```

5.10 Isomorphic Objects Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

class sage.categories.isomorphic_objects.**IsomorphicObjectsCategory**(*category*, **args*)

Bases: *RegressiveCovariantConstructionCategory*

classmethod default_super_categories(*category*)

Returns the default super categories of `category.IsomorphicObjects()`

Mathematical meaning: if A is the image of B by an isomorphism in the category C, then A is both a subobject of B and a quotient of B in the category C.

INPUT:

- `cls` – the class `IsomorphicObjectsCategory`
- `category` – a category Cat

OUTPUT: a (join) category

In practice, this returns `category.Subobjects()` and `category.Quotients()`, joined together with the result of the method *`RegressiveCovariantConstructionCategory.default_super_categories()`* (that is the join of `category` and `cat.IsomorphicObjects()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`, which has `cat=Monoids()` as super category. Then, the image of a group G' by a group isomorphism is simultaneously a subgroup of G, a subquotient of G, a group by itself, and the image of G by a monoid isomorphism:

```
sage: Groups().IsomorphicObjects().super_categories()
[Category of groups,
 Category of subquotients of monoids,
 Category of quotients of semigroups,
 Category of isomorphic objects of sets]
```

Mind the last item above: there is indeed currently nothing implemented about isomorphic objects of monoids.

This resulted from the following call:

```
sage: sage.categories.isomorphic_objects.IsomorphicObjectsCategory.default_
↪super_categories(Groups())
Join of Category of groups and
Category of subquotients of monoids and
Category of quotients of semigroups and
Category of isomorphic objects of sets
```

5.11 Homset categories

class `sage.categories.homsets.`**`Homsets`**

Bases: *`Category_singleton`*

The category of all homsets.

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets()
Category of homsets
```

This is a subcategory of `Sets()`:

```
sage: Homsets().super_categories()
[Category of sets]
```

By this, we assume that all homsets implemented in Sage are sets, or equivalently that we only implement locally small categories. See Wikipedia article Category_(mathematics).

github issue #17364: every homset category shall be a subcategory of the category of all homsets:

```
sage: Schemes().Homsets().is_subcategory(Homsets())
True
sage: AdditiveMagmas().Homsets().is_subcategory(Homsets())
True
sage: AdditiveMagmas().AdditiveUnital().Homsets().is_subcategory(Homsets())
True
```

This is tested in `HomsetsCategory._test_homsets_category()`.

class Endset (*base_category*)

Bases: *CategoryWithAxiom*

The category of all endomorphism sets.

This category serves too purposes: making sure that the `Endset` axiom is implemented in the category where it's defined, namely `Homsets`, and specifying that `Endsets` are monoids.

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets().Endset()
Category of endsets
```

class ParentMethods

Bases: `object`

is_endomorphism_set ()

Return `True` as `self` is in the category of `Endsets`.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: E = End(P)
sage: E.is_endomorphism_set()
True
```

extra_super_categories ()

Implement the fact that endsets are monoids.

See also:

CategoryWithAxiom.extra_super_categories()

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets().Endset().extra_super_categories()
[Category of monoids]
```

class ParentMethods

Bases: `object`

is_endomorphism_set ()

Return `True` if the domain and codomain of `self` are the same object.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: f.parent().is_endomorphism_set()
False
sage: g = P.hom([2*t])
sage: g.parent().is_endomorphism_set()
True
```

class SubcategoryMethods

Bases: `object`

Endset()

Return the subcategory of the homsets of `self` that are endomorphism sets.

EXAMPLES:

```
sage: Sets().Homsets().Endset()
Category of endsets of sets

sage: Posets().Homsets().Endset()
Category of endsets of posets
```

super_categories()

Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets()
Category of homsets
```

class `sage.categories.homsets.`**HomsetsCategory**(*category*, **args*)

Bases: *FunctorialConstructionCategory*

base()

If this homsets category is subcategory of a category with a base, return that base.

Todo: Is this really useful?

EXAMPLES:

```
sage: ModulesWithBasis(ZZ).Homsets().base()
Integer Ring
```

classmethod default_super_categories(*category*)

Return the default super categories of `category.Homsets()`.

INPUT:

- `cls` – the category class for the functor F
- `category` – a category Cat

OUTPUT: a category

As for the other functorial constructions, if `category` implements a nested `Homsets` class, this method is used in combination with `category.Homsets().extra_super_categories()` to compute the super categories of `category.Homsets()`.

EXAMPLES:

If `category` has one or more full super categories, then the join of their respective homsets category is returned. In this example, this join consists of a single category:

```
sage: from sage.categories.homsets import HomsetsCategory
sage: from sage.categories.additive_groups import AdditiveGroups

sage: C = AdditiveGroups()
sage: C.full_super_categories()
[Category of additive inverse additive unital additive magmas,
 Category of additive monoids]
sage: H = HomsetsCategory.default_super_categories(C); H
Category of homsets of additive monoids
sage: type(H)
<class 'sage.categories.additive_monoids.AdditiveMonoids.Homsets_with_category
↪'>
```

and, given that nothing specific is currently implemented for homsets of additive groups, `H` is directly the category thereof:

```
sage: C.Homsets()
Category of homsets of additive monoids
```

Similarly for rings: a ring homset is just a homset of unital magmas and additive magmas:

```
sage: Rings().Homsets()
Category of homsets of unital magmas and additive unital additive magmas
```

Otherwise, if `category` implements a nested class `Homsets`, this method returns the category of all homsets:

```
sage: AdditiveMagmas.Homsets
<class 'sage.categories.additive_magmas.AdditiveMagmas.Homsets'>
sage: HomsetsCategory.default_super_categories(AdditiveMagmas())
Category of homsets
```

which gives one of the super categories of `category.Homsets()`:

```
sage: AdditiveMagmas().Homsets().super_categories()
[Category of additive magmas, Category of homsets]
sage: AdditiveMagmas().AdditiveUnital().Homsets().super_categories()
[Category of additive unital additive magmas, Category of homsets]
```

the other coming from `category.Homsets().extra_super_categories()`:

```
sage: AdditiveMagmas().Homsets().extra_super_categories()
[Category of additive magmas]
```

Finally, as a last resort, this method returns a stub category modelling the homsets of this category:

```
sage: hasattr(Posets, "Homsets")
False
sage: H = HomsetsCategory.default_super_categories(Posets()); H
Category of homsets of posets
sage: type(H)
<class 'sage.categories.homsets.HomsetsOf_with_category'>
```

(continues on next page)

(continued from previous page)

```
sage: Posets().Homsets()
Category of homsets of posets
```

class sage.categories.homsets.**HomsetsOf**(*category*, **args*)

Bases: *HomsetsCategory*

Default class for homsets of a category.

This is used when a category C defines some additional structure but not a homset category of its own. Indeed, unlike for covariant functorial constructions, we cannot represent the homset category of C by just the join of the homset categories of its super categories.

EXAMPLES:

```
sage: C = (Magmas() & Posets()).Homsets(); C
Category of homsets of magmas and posets
sage: type(C)
<class 'sage.categories.homsets.HomsetsOf_with_category'>
```

super_categories()

Return the super categories of `self`.

A stub homset category admits a single super category, namely the category of all homsets.

EXAMPLES:

```
sage: C = (Magmas() & Posets()).Homsets(); C
Category of homsets of magmas and posets
sage: type(C)
<class 'sage.categories.homsets.HomsetsOf_with_category'>
sage: C.super_categories()
[Category of homsets]
```

5.12 Realizations Covariant Functorial Construction

See also:

- *Sets().WithRealizations* for an introduction to *realizations* and *with realizations*.
- *sage.categories.covariant_functorial_construction* for an introduction to covariant functorial constructions.
- *sage.categories.examples.with_realizations* for an example.

class sage.categories.realizations.**Category_realization_of_parent**(*parent_with_realization*)

Bases: *Category_over_base*, `BindableClass`

An abstract base class for categories of all realizations of a given parent

INPUT:

- `parent_with_realization` – a parent

See also:

Sets().WithRealizations

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A                          #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
```

The role of this base class is to implement some technical goodies, like the binding `A.Realizations()` when a subclass `Realizations` is implemented as a nested class in `A` (see the *code of the example*):

```
sage: C = A.Realizations(); C                                             #␣
↪needs sage.modules
Category of realizations of
 The subset algebra of {1, 2, 3} over Rational Field
```

as well as the name for that category.

`sage.categories.realizations.`**`Realizations`**`(`*self*`)`

Return the category of realizations of the parent `self` or of objects of the category `self`

INPUT:

- `self` – a parent or a concrete category

Note: this *function* is actually inserted as a *method* in the class *Category* (see *Realizations()*). It is defined here for code locality reasons.

EXAMPLES:

The category of realizations of some algebra:

```
sage: Algebras(QQ).Realizations()
Join of Category of algebras over Rational Field
    and Category of realizations of unital magmas
```

The category of realizations of a given algebra:

```
sage: A = Sets().WithRealizations().example(); A                          #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
sage: A.Realizations()                                                    #␣
↪needs sage.modules
Category of realizations of
 The subset algebra of {1, 2, 3} over Rational Field

sage: C = GradedHopfAlgebrasWithBasis(QQ).Realizations(); C
Join of Category of graded Hopf algebras with basis over Rational Field
    and Category of realizations of Hopf algebras over Rational Field
sage: C.super_categories()
[Category of graded Hopf algebras with basis over Rational Field,
 Category of realizations of Hopf algebras over Rational Field]

sage: TestSuite(C).run()
```

See also:

- *Sets().WithRealizations*
- `ClasscallMetaclass`

Todo: Add an optional argument to allow for:

```
sage: Realizations(A, category=Blahs()) # todo: not implemented
```

class `sage.categories.realizations.`**`RealizationsCategory`**(*category*, **args*)

Bases: *`RegressiveCovariantConstructionCategory`*

An abstract base class for all categories of realizations category

Relization are implemented as *`RegressiveCovariantConstructionCategory`*. See there for the documentation of how the various bindings such as `Sets().Realizations()` and `P.Realizations()`, where `P` is a parent, work.

See also:

`Sets().WithRealizations`

5.13 With Realizations Covariant Functorial Construction

See also:

- *`Sets().WithRealizations`* for an introduction to *realizations* and *with realizations*.
- *`sage.categories.covariant_functorial_construction`* for an introduction to covariant functorial constructions.

`sage.categories.with_realizations.`**`WithRealizations`**(*self*)

Return the category of parents in `self` endowed with multiple realizations.

INPUT:

- `self` – a category

See also:

- The documentation and code (*`sage.categories.examples.with_realizations`*) of `Sets().WithRealizations().example()` for more on how to use and implement a parent with several realizations.
- Various use cases:
  - `SymmetricFunctions`
  - `QuasiSymmetricFunctions`
  - `NonCommutativeSymmetricFunctions`
  - `SymmetricFunctionsNonCommutingVariables`
  - `DescentAlgebra`
  - `algebras.Moebius`
  - `IwahoriHeckeAlgebra`
  - `ExtendedAffineWeylGroup`
- The Implementing Algebraic Structures thematic tutorial.
- *`sage.categories.realizations`*

Note: this *function* is actually inserted as a *method* in the class `Category` (see `WithRealizations()`). It is defined here for code locality reasons.

EXAMPLES:

```
sage: Sets().WithRealizations()
Category of sets with realizations
```

Parent with realizations

Let us now explain the concept of realizations. A *parent with realizations* is a facade parent (see `Sets.Facade`) admitting multiple concrete realizations where its elements are represented. Consider for example an algebra A which admits several natural bases:

```
sage: A = Sets().WithRealizations().example(); A                           #␣
↪needs sage.modules
The subset algebra of {1, 2, 3} over Rational Field
```

For each such basis B one implements a parent P_B which realizes A with its elements represented by expanding them on the basis B:

```
sage: # needs sage.modules
sage: A.F()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: A.Out()
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
sage: A.In()
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: A.an_element()
F[{}] + 2*F[{1}] + 3*F[{2}] + F[{1, 2}]
```

If B and B' are two bases, then the change of basis from B to B' is implemented by a canonical coercion between P_B and $P_{B'}$:

```
sage: # needs sage.combinat sage.modules
sage: F = A.F(); In = A.In(); Out = A.Out()
sage: i = In.an_element(); i
In[{}] + 2*In[{1}] + 3*In[{2}] + In[{1, 2}]
sage: F(i)
7*F[{}] + 3*F[{1}] + 4*F[{2}] + F[{1, 2}]
sage: F.coerce_map_from(Out)
Generic morphism:
  From: The subset algebra of {1, 2, 3} over Rational Field in the Out basis
  To:   The subset algebra of {1, 2, 3} over Rational Field in the Fundamental␣
↪basis
```

allowing for mixed arithmetic:

```
sage: (1 + Out.from_set(1)) * In.from_set(2,3)                             #␣
↪needs sage.combinat sage.modules
Out[{}] + 2*Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}]
+ 2*Out[{1, 3}] + 4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

In our example, there are three realizations:

```
sage: A.realizations()                                                    #␣
↪needs sage.modules
[The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis,
 The subset algebra of {1, 2, 3} over Rational Field in the In basis,
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis]
```

Instead of manually defining the shorthands `F`, `In`, and `Out`, as above one can just do:

```
sage: A.inject_shorthands()                                               #␣
↪needs sage.combinat sage.modules
Defining F as shorthand for
 The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
Defining In as shorthand for
 The subset algebra of {1, 2, 3} over Rational Field in the In basis
Defining Out as shorthand for
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis
```

Rationale

Besides some goodies described below, the role of A is threefold:

- To provide, as illustrated above, a single entry point for the algebra as a whole: documentation, access to its properties and different realizations, etc.
- To provide a natural location for the initialization of the bases and the coercions between, and other methods that are common to all bases.
- To let other objects refer to A while allowing elements to be represented in any of the realizations.

We now illustrate this second point by defining the polynomial ring with coefficients in A:

```
sage: P = A['x']; P                                                       #␣
↪needs sage.modules
Univariate Polynomial Ring in x over
 The subset algebra of {1, 2, 3} over Rational Field
sage: x = P.gen()                                                         #␣
↪needs sage.modules
```

In the following examples, the coefficients turn out to be all represented in the F basis:

```
sage: P.one()                                                             #␣
↪needs sage.modules
F[{}]
sage: (P.an_element() + 1)^2                                              #␣
↪needs sage.modules
F[{}]*x^2 + 2*F[{}]*x + F[{}]
```

However we can create a polynomial with mixed coefficients, and compute with it:

```
sage: p = P([1, In[{1}], Out[{2}] ]); p                                   #␣
↪needs sage.combinat sage.modules
Out[{2}]*x^2 + In[{1}]*x + F[{}]
sage: p^2                                                                 #␣
↪needs sage.combinat sage.modules
Out[{2}]*x^4
+ (-8*In[{}] + 4*In[{1}] + 8*In[{2}] + 4*In[{3}]
   - 4*In[{1, 2}] - 2*In[{1, 3}] - 4*In[{2, 3}] + 2*In[{1, 2, 3}])*x^3
```

(continues on next page)

(continued from previous page)

```
+ (F[{}] + 3*F[{1}] + 2*F[{2}] - 2*F[{1, 2}] - 2*F[{2, 3}] + 2*F[{1, 2, 3}])*x^2
+ (2*F[{}] + 2*F[{1}])*x
+ F[{}]
```

Note how each coefficient involves a single basis which need not be that of the other coefficients. Which basis is used depends on how coercion happened during mixed arithmetic and needs not be deterministic.

One can easily coerce all coefficient to a given basis with:

```
sage: p.map_coefficients(In)                                                    #␣
↪needs sage.combinat sage.modules
(-4*In[{}] + 2*In[{1}] + 4*In[{2}] + 2*In[{3}]
 - 2*In[{1, 2}] - In[{1, 3}] - 2*In[{2, 3}] + In[{1, 2, 3}])*x^2
+ In[{1}]*x + In[{}]
```

Alas, the natural notation for constructing such polynomials does not yet work:

```
sage: In[{1}] * x                                                               #␣
↪needs sage.combinat sage.modules
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *:
'The subset algebra of {1, 2, 3} over Rational Field in the In basis'
and 'Univariate Polynomial Ring in x over
The subset algebra of {1, 2, 3} over Rational Field'
```

The category of realizations of A

The set of all realizations of A, together with the coercion morphisms is a category (whose class inherits from *Category_realization_of_parent*):

```
sage: A.Realizations()                                                          #␣
↪needs sage.modules
Category of realizations of
 The subset algebra of {1, 2, 3} over Rational Field
```

The various parent realizing A belong to this category:

```
sage: A.F() in A.Realizations()                                                 #␣
↪needs sage.modules
True
```

A itself is in the category of algebras with realizations:

```
sage: A in Algebras(QQ).WithRealizations()                                      #␣
↪needs sage.modules
True
```

The (mostly technical) `WithRealizations` categories are the analogs of the `*WithSeveralBases` categories in MuPAD-Combinat. They provide support tools for handling the different realizations and the morphisms between them.

Typically, `VectorSpaces(QQ).FiniteDimensional().WithRealizations()` will eventually be in charge, whenever a coercion $\phi : A \mapsto B$ is registered, to register ϕ^{-1} as coercion $B \mapsto A$ if there is none defined yet. To achieve this, `FiniteDimensionalVectorSpaces` would provide a nested class `WithRealizations` implementing the appropriate logic.

`WithRealizations` is a *regressive covariant functorial construction*. On our example, this simply means that A is automatically in the category of rings with realizations (covariance):

```
sage: A in Rings().WithRealizations()                                    #␣
↪needs sage.modules
True
```

and in the category of algebras (regressiveness):

```
sage: A in Algebras(QQ)                                                  #␣
↪needs sage.modules
True
```

Note: For `C` a category, `C.WithRealizations()` in fact calls `sage.categories.with_realizations.WithRealizations(C)`. The later is responsible for building the hierarchy of the categories with realizations in parallel to that of their base categories, optimizing away those categories that do not provide a `WithRealizations` nested class. See *sage.categories.covariant_functorial_construction* for the technical details.

Note: Design question: currently `WithRealizations` is a regressive construction. That is `self.WithRealizations()` is a subcategory of `self` by default:

```
sage: Algebras(QQ).WithRealizations().super_categories()
[Category of algebras over Rational Field,
 Category of monoids with realizations,
 Category of additive unital additive magmas with realizations]
```

Is this always desirable? For example, `AlgebrasWithBasis(QQ).WithRealizations()` should certainly be a subcategory of `Algebras(QQ)`, but not of `AlgebrasWithBasis(QQ)`. This is because `AlgebrasWithBasis(QQ)` is specifying something about the concrete realization.

---

class `sage.categories.with_realizations.`**`WithRealizationsCategory`**`(`*category*, **args*`)`

Bases: *RegressiveCovariantConstructionCategory*

An abstract base class for all categories of parents with multiple realizations.

See also:

Sets().WithRealizations

The role of this base class is to implement some technical goodies, such as the name for that category.

CHAPTER

SIX

EXAMPLES OF PARENTS USING CATEGORIES

6.1 Examples of algebras with basis

`sage.categories.examples.algebras_with_basis.`**`Example`**

alias of *FreeAlgebra*

class `sage.categories.examples.algebras_with_basis.`**`FreeAlgebra`**`(`*R*, *alphabet=('a', 'b', 'c')*`)`

Bases: `CombinatorialFreeModule`

An example of an algebra with basis: the free algebra

This class illustrates a minimal implementation of an algebra with basis.

`algebra_generators``()`

Return the generators of this algebra, as per *`algebra_generators()`*.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A                                 #␣
↪needs sage.modules
An example of an algebra with basis: the free algebra on the generators ('a',
↪'b', 'c') over Rational Field
sage: A.algebra_generators()                                                 #␣
↪needs sage.modules
Family (B[word: a], B[word: b], B[word: c])
```

`one_basis``()`

Returns the empty word, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES::r

sage: A = AlgebrasWithBasis(QQ).example() # needs sage.modules sage: A.one_basis() # needs sage.modules word: sage: A.one() # needs sage.modules B[word:]

`product_on_basis``(`*w1*, *w2*`)`

Product of basis elements, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

EXAMPLES:

```
sage: # needs sage.modules
sage: A = AlgebrasWithBasis(QQ).example()
sage: Words = A.basis().keys()
```

(continues on next page)

(continued from previous page)

```
sage: A.product_on_basis(Words("acb"), Words("cba"))
B[word: acbcba]
sage: (a,b,c) = A.algebra_generators()
sage: a * (1-b)^2 * c
B[word: abbc] - 2*B[word: abc] + B[word: ac]
```

6.2 Examples of commutative additive monoids

`sage.categories.examples.commutative_additive_monoids.`**`Example`**

alias of *`FreeCommutativeAdditiveMonoid`*

`class` `sage.categories.examples.commutative_additive_monoids.`**`FreeCommutativeAdditiveMonoid`**`(`

Bases: *`FreeCommutativeAdditiveSemigroup`*

An example of a commutative additive monoid: the free commutative monoid

This class illustrates a minimal implementation of a commutative monoid.

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example(); S
An example of a commutative monoid: the free commutative monoid generated by ('a',
↪ 'b', 'c', 'd')

sage: S.category()
Category of commutative additive monoids
```

This is the free semigroup generated by:

```
sage: S.additive_semigroup_generators()
Family (a, b, c, d)
```

with product rule given by $a \times b = a$ for all a, b:

```
sage: (a,b,c,d) = S.additive_semigroup_generators()
```

We conclude by running systematic tests on this commutative monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
```

(continues on next page)

(continued from previous page)

```
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
running ._test_zero() . . . pass
```

class Element(*parent*, *iterable*)

Bases: `Element`

zero()

Returns the zero of this additive monoid, as per `CommutativeAdditiveMonoids.ParentMethods.zero()`.

EXAMPLES:

```
sage: M = CommutativeAdditiveMonoids().example(); M
An example of a commutative monoid: the free commutative monoid generated by (
↪'a', 'b', 'c', 'd')
sage: M.zero()
0
```

6.3 Examples of commutative additive semigroups

sage.categories.examples.commutative_additive_semigroups.**Example**

alias of `FreeCommutativeAdditiveSemigroup`

class sage.categories.examples.commutative_additive_semigroups.**FreeCommutativeAdditiveSemig**

Bases: `UniqueRepresentation`, `Parent`

An example of a commutative additive monoid: the free commutative monoid

This class illustrates a minimal implementation of a commutative additive monoid.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative semigroup: the free commutative semigroup generated␣
↪by ('a', 'b', 'c', 'd')

sage: S.category()
Category of commutative additive semigroups
```

This is the free semigroup generated by:

```
sage: S.additive_semigroup_generators()
Family (a, b, c, d)
```

with product rule given by $a \times b = a$ for all a, b:

```
sage: (a,b,c,d) = S.additive_semigroup_generators()
```

We conclude by running systematic tests on this commutative monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

class Element(*parent*, *iterable*)

Bases: `ElementWrapper`

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: x = F.element_class(F, (('a',4), ('b', 0), ('a', 2), ('c', 1), ('d',␣
↪5)))
sage: x
2*a + c + 5*d
sage: x.value
{'a': 2, 'b': 0, 'c': 1, 'd': 5}
sage: x.parent()
An example of a commutative semigroup: the free commutative semigroup␣
↪generated by ('a', 'b', 'c', 'd')
```

Internally, elements are represented as dense dictionaries which associate to each generator of the monoid its multiplicity. In order to get an element, we wrap the dictionary into an element via `ElementWrapper`:

```
sage: x.value
{'a': 2, 'b': 0, 'c': 1, 'd': 5}
```

additive_semigroup_generators()

Returns the generators of the semigroup.

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: F.additive_semigroup_generators()
Family (a, b, c, d)
```

an_element()

Returns an element of the semigroup.

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: F.an_element()
a + 2*b + 3*c + 4*d
```

summation(*x*, *y*)

Returns the product of `x` and `y` in the semigroup, as per `CommutativeAdditiveSemigroups.ParentMethods.summation()`.

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: (a,b,c,d) = F.additive_semigroup_generators()
sage: F.summation(a,b)
a + b
sage: (a+b) + (a+c)
2*a + b + c
```

6.4 Examples of Coxeter groups

6.5 Example of a crystal

class sage.categories.examples.crystals.**HighestWeightCrystalOfTypeA**(*n=3*)

Bases: `UniqueRepresentation`, `Parent`

An example of a crystal: the highest weight crystal of type A_n of highest weight ω_1.

The purpose of this class is to provide a minimal template for implementing crystals. See `CrystalOfLetters` for a full featured and optimized implementation.

EXAMPLES:

```
sage: C = Crystals().example()
sage: C
Highest weight crystal of type A_3 of highest weight omega_1
sage: C.category()
Category of classical crystals
```

The elements of this crystal are in the set $\{1, \ldots, n+1\}$:

```
sage: C.list()
[1, 2, 3,  4]
sage: C.module_generators[0]
1
```

The crystal operators themselves correspond to the elementary transpositions:

```
sage: b = C.module_generators[0]
sage: b.f(1)
2
sage: b.f(1).e(1) == b
True
```

Only the following basic operations are implemented:

- `cartan_type()` or an attribute _cartan_type
- an attribute module_generators
- *`Element.e()`*
- *`Element.f()`*

All the other usual crystal operations are inherited from the categories; for example:

```
sage: C.cardinality()
4
```

class Element

Bases: `ElementWrapper`

e(*i*)

Returns the action of e_i on `self`.

EXAMPLES:

```
sage: C = Crystals().example(4)
sage: [[c,i,c.e(i)] for i in C.index_set() for c in C if c.e(i) is not␣
↪None]
[[2, 1, 1], [3, 2, 2], [4, 3, 3], [5, 4, 4]]
```

f(*i*)

Returns the action of f_i on `self`.

EXAMPLES:

```
sage: C = Crystals().example(4)
sage: [[c,i,c.f(i)] for i in C.index_set() for c in C if c.f(i) is not␣
↪None]
[[1, 1, 2], [2, 2, 3], [3, 3, 4], [4, 4, 5]]
```

class sage.categories.examples.crystals.**NaiveCrystal**

Bases: `UniqueRepresentation`, `Parent`

This is an example of a “crystal” which does not come from any kind of representation, designed primarily to test the Stembridge local rules with. The crystal has vertices labeled 0 through 5, with 0 the highest weight.

The code here could also possibly be generalized to create a class that automatically builds a crystal from an edge-colored digraph, if someone feels adventurous.

Currently, only the methods `highest_weight_vector()`, `e()`, and `f()` are guaranteed to work.

EXAMPLES:

```
sage: C = Crystals().example(choice='naive')
sage: C.highest_weight_vector()
0
```

class Element

Bases: `ElementWrapper`

e(*i*)

Returns the action of e_i on `self`.

EXAMPLES:

```
sage: C = Crystals().example(choice='naive')
sage: [[c,i,c.e(i)] for i in C.index_set() for c in [C(j) for j in [0..
↪5]] if c.e(i) is not None]
[[1, 1, 0], [2, 1, 1], [3, 1, 2], [5, 1, 3], [4, 2, 0], [5, 2, 4]]
```

f(*i*)

Returns the action of f_i on `self`.

EXAMPLES:

```
sage: C = Crystals().example(choice='naive')
sage: [[c,i,c.f(i)] for i in C.index_set() for c in [C(j) for j in [0..
↪5]] if c.f(i) is not None]
[[0, 1, 1], [1, 1, 2], [2, 1, 3], [3, 1, 5], [0, 2, 4], [4, 2, 5]]
```

6.6 Examples of CW complexes

`sage.categories.examples.cw_complexes.`**Example**

alias of `Surface`

class `sage.categories.examples.cw_complexes.`**Surface**(*bdy=(1, 2, 1, 2)*)

Bases: `UniqueRepresentation`, `Parent`

An example of a CW complex: a (2-dimensional) surface.

This class illustrates a minimal implementation of a CW complex.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example(); X
An example of a CW complex: the surface given by the boundary map (1, 2, 1, 2)

sage: X.category()
Category of finite finite dimensional CW complexes
```

We conclude by running systematic tests on this manifold:

```
sage: TestSuite(X).run()
```

class Element(*parent*, *dim*, *name*)

Bases: Element

A cell in a CW complex.

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: f = X.an_element()
sage: f.dimension()
2
```

an_element()

Return an element of the CW complex, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.an_element()
2-cell f
```

cells()

Return the cells of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: C = X.cells()
sage: sorted((d, C[d]) for d in C.keys())
[(0, (0-cell v,)),
 (1, (0-cell e1, 0-cell e2)),
 (2, (2-cell f,))]
```

6.7 Example of facade set

class sage.categories.examples.facade_sets.**IntegersCompletion**

Bases: UniqueRepresentation, Parent

An example of a facade parent: the set of integers completed with $+-\infty$

This class illustrates a minimal implementation of a facade parent that models the union of several other parents.

EXAMPLES:

```
sage: S = Sets().Facade().example("union"); S
An example of a facade set: the integers completed by +-infinity
```

class sage.categories.examples.facade_sets.**PositiveIntegerMonoid**

Bases: UniqueRepresentation, Parent

An example of a facade parent: the positive integers viewed as a multiplicative monoid

This class illustrates a minimal implementation of a facade parent which models a subset of a set.

EXAMPLES:

```
sage: S = Sets().Facade().example(); S
An example of facade set: the monoid of positive integers
```

6.8 Examples of finite Coxeter groups

class sage.categories.examples.finite_coxeter_groups.**DihedralGroup**(*n=5*)

Bases: UniqueRepresentation, Parent

An example of finite Coxeter group: the n-th dihedral group of order $2n$.

The purpose of this class is to provide a minimal template for implementing finite Coxeter groups. See DihedralGroup for a full featured and optimized implementation.

EXAMPLES:

```
sage: G = FiniteCoxeterGroups().example()
```

This group is generated by two simple reflections s_1 and s_2 subject to the relation $(s_1 s_2)^n = 1$:

```
sage: G.simple_reflections()
Finite family {1: (1,), 2: (2,)}

sage: s1, s2 = G.simple_reflections()
sage: (s1*s2)^5 == G.one()
True
```

An element is represented by its reduced word (a tuple of elements of $self.index_set()$):

```
sage: G.an_element()
(1, 2)

sage: list(G)
[(),
 (1,),
 (2,),
 (1, 2),
 (2, 1),
 (1, 2, 1),
 (2, 1, 2),
 (1, 2, 1, 2),
 (2, 1, 2, 1),
 (1, 2, 1, 2, 1)]
```

This reduced word is unique, except for the longest element where the chosen reduced word is $(1, 2, 1, 2 \dots)$:

```
sage: G.long_element()
(1, 2, 1, 2, 1)
```

class Element

Bases: ElementWrapper

apply_simple_reflection_right (*i*)

Implements CoxeterGroups.ElementMethods.apply_simple_reflection().

EXAMPLES:

```
sage: D5 = FiniteCoxeterGroups().example(5)
sage: [i^2 for i in D5]   # indirect doctest
[(), (), (), (1, 2, 1, 2), (2, 1, 2, 1), (), (), (2, 1), (1, 2), ()]
sage: [i^5 for i in D5]   # indirect doctest
[(), (1,), (2,), (), (), (1, 2, 1), (2, 1, 2), (), (), (1, 2, 1, 2, 1)]
```

has_right_descent (*i*, *positive=False*, *side='right'*)

Implements SemiGroups.ElementMethods.has_right_descent().

EXAMPLES:

```
sage: D6 = FiniteCoxeterGroups().example(6)
sage: s = D6.simple_reflections()
sage: s[1].has_descent(1)
True
sage: s[1].has_descent(1)
True
sage: s[1].has_descent(2)
False
sage: D6.one().has_descent(1)
False
sage: D6.one().has_descent(2)
False
sage: D6.long_element().has_descent(1)
True
sage: D6.long_element().has_descent(2)
True
```

wrapped_class

alias of tuple

coxeter_matrix ()

Return the Coxeter matrix of self.

EXAMPLES:

```
sage: FiniteCoxeterGroups().example(6).coxeter_matrix()
[1 6]
[6 1]
```

degrees ()

Return the degrees of self.

EXAMPLES:

```
sage: FiniteCoxeterGroups().example(6).degrees()
(2, 6)
```

index_set ()

Implements *CoxeterGroups.ParentMethods.index_set()*.

EXAMPLES:

```
sage: D4 = FiniteCoxeterGroups().example(4)
sage: D4.index_set()
(1, 2)
```

one()

Implements `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: D6 = FiniteCoxeterGroups().example(6)
sage: D6.one()
()
```

`sage.categories.examples.finite_coxeter_groups.`**`Example`**

alias of *`DihedralGroup`*

6.9 Example of a finite dimensional algebra with basis

`sage.categories.examples.finite_dimensional_algebras_with_basis.`**`Example`**

alias of *`KroneckerQuiverPathAlgebra`*

class `sage.categories.examples.finite_dimensional_algebras_with_basis.`**`KroneckerQuiverPathAl`**

Bases: `CombinatorialFreeModule`

An example of a finite dimensional algebra with basis: the path algebra of the Kronecker quiver.

This class illustrates a minimal implementation of a finite dimensional algebra with basis. See `sage.quivers.algebra.PathAlgebra` for a full-featured implementation of path algebras.

algebra_generators()

Return algebra generators for this algebra.

See also:

`Algebras.ParentMethods.algebra_generators()`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.algebra_generators()
Finite family {'x': x, 'y': y, 'a': a, 'b': b}
```

one()

Return the unit of this algebra.

See also:

`AlgebrasWithBasis.ParentMethods.one_basis()`

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example()
sage: A.one()
x + y
```

product_on_basis(*w1*, *w2*)

Return the product of the two basis elements indexed by `w1` and `w2`.

See also:

`AlgebrasWithBasis.ParentMethods.product_on_basis()`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example()
```

Here is the multiplication table for the algebra:

```
sage: matrix([[p*q for q in A.basis()] for p in A.basis()])
[x 0 a b]
[0 y 0 0]
[0 a 0 0]
[0 b 0 0]
```

Here we take some products of linear combinations of basis elements:

```
sage: x, y, a, b = A.basis()
sage: a * (1-b)^2 * x
0
sage: x*a + b*y
a + b
sage: x*x
x
sage: x*y
0
sage: x*a*y
a
```

6.10 Examples of a finite dimensional Lie algebra with basis

class `sage.categories.examples.finite_dimensional_lie_algebras_with_basis.`**`AbelianLieAlgebra`**

Bases: `Parent`, `UniqueRepresentation`

An example of a finite dimensional Lie algebra with basis: the abelian Lie algebra.

Let R be a commutative ring, and M an R-module. The *abelian Lie algebra* on M is the R-Lie algebra obtained by endowing M with the trivial Lie bracket ($[a, b] = 0$ for all $a, b \in M$).

This class illustrates a minimal implementation of a finite dimensional Lie algebra with basis.

INPUT:

- `R` – base ring
- `n` – (optional) a nonnegative integer (default: `None`)
- `M` – an R-module (default: the free R-module of rank `n`) to serve as the ground space for the Lie algebra

- `ambient` – (optional) a Lie algebra; if this is set, then the resulting Lie algebra is declared a Lie subalgebra of `ambient`

OUTPUT:

The abelian Lie algebra on M.

class Element

Bases: *Element*

lift()

Return the lift of `self` to the universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 2*a + 2*b + 3*c
sage: elt.lift()                                                          #␣
↪needs sage.combinat
2*b0 + 2*b1 + 3*b2
```

monomial_coefficients(*copy=True*)

Return the monomial coefficients of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 2*a + 2*b + 3*c
sage: elt.monomial_coefficients()
{0: 2, 1: 2, 2: 3}
```

to_vector(*order=None*, *sparse=False*)

Return `self` as a vector in `self.parent().module()`.

See the docstring of the latter method for the meaning of this.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 2*a + 2*b + 3*c
sage: elt.to_vector()
(2, 2, 3)
```

ambient()

Return the ambient Lie algebra of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a+b, b + c])
sage: S.ambient() == L
True
```

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.basis()
Finite family {0: (1, 0, 0), 1: (0, 1, 0), 2: (0, 0, 1)}
```

basis_matrix()

Return the basis matrix of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

from_vector(*v*, *order=None*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u
(1, 0, 0)
sage: parent(u) is L
True
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.gens()
((1, 0, 0), (0, 1, 0), (0, 0, 1))
```

ideal(*gens*)

Return the Lie subalgebra of `self` generated by the elements of the iterable `gens`.

This currently requires the ground ring R to be a field.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.subalgebra([2*a+b, b + c])
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field with
 basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

is_ideal(*A*)

Return if `self` is an ideal of the ambient space `A`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.is_ideal(L)
True
sage: S1 = L.subalgebra([2*a+b, b + c])
sage: S1.is_ideal(L)
True
sage: S2 = L.subalgebra([2*a+b])
sage: S2.is_ideal(S1)
True
sage: S1.is_ideal(S2)
False
```

lie_algebra_generators()

Return the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.basis()
Finite family {0: (1, 0, 0), 1: (0, 1, 0), 2: (0, 0, 1)}
```

module()

Return an R-module which is isomorphic to the underlying R-module of `self`.

See `sage.categories.lie_algebras.LieAlgebras.module()` for an explanation.

In this particular example, this returns the module M that was used to construct `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.module()
Vector space of dimension 3 over Rational Field

sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a+b, b + c])
sage: S.module()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

subalgebra(*gens*)

Return the Lie subalgebra of `self` generated by the elements of the iterable `gens`.

This currently requires the ground ring R to be a field.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.subalgebra([2*a+b, b + c])
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field with
 basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

zero()

Return the zero element.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.zero()
(0, 0, 0)
```

sage.categories.examples.finite_dimensional_lie_algebras_with_basis.**Example**

alias of *AbelianLieAlgebra*

6.11 Examples of finite enumerated sets

class sage.categories.examples.finite_enumerated_sets.**Example**

Bases: UniqueRepresentation, Parent

An example of a finite enumerated set: $\{1, 2, 3\}$

This class provides a minimal implementation of a finite enumerated set.

See FiniteEnumeratedSet for a full featured implementation.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.cardinality()
3
sage: C.list()
[1, 2, 3]
sage: C.an_element()
1
```

This checks that the different methods of the enumerated set C return consistent results:

```
sage: TestSuite(C).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
```

(continues on next page)

(continued from previous page)

```
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

class sage.categories.examples.finite_enumerated_sets.**IsomorphicObjectOfFiniteEnumeratedSet**

Bases: `UniqueRepresentation`, `Parent`

ambient()

Returns the ambient space for `self`, as per *`Sets.Subquotients.ParentMethods.ambient()`*.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().IsomorphicObjects().example(); C
The image by some isomorphism of An example of a finite enumerated set: {1,2,
↪3}
sage: C.ambient()
An example of a finite enumerated set: {1,2,3}
```

lift(*x*)

INPUT:

- `x` – an element of `self`

Lifts `x` to the ambient space for `self`, as per *`Sets.Subquotients.ParentMethods.lift()`*.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().IsomorphicObjects().example(); C
The image by some isomorphism of An example of a finite enumerated set: {1,2,
↪3}
sage: C.lift(9)
3
```

retract(*x*)

INPUT:

- `x` – an element of the ambient space for `self`

Retracts `x` from the ambient space to `self`, as per *Sets.Subquotients.ParentMethods.retract()*.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().IsomorphicObjects().example(); C
The image by some isomorphism of An example of a finite enumerated set: {1,2,
↪3}
sage: C.retract(3)
9
```

6.12 Examples of finite monoids

`sage.categories.examples.finite_monoids.`**`Example`**

alias of *IntegerModMonoid*

`class` `sage.categories.examples.finite_monoids.`**`IntegerModMonoid`**`(n=12)`

Bases: `UniqueRepresentation`, `Parent`

An example of a finite monoid: the integers mod n

This class illustrates a minimal implementation of a finite monoid.

EXAMPLES:

```
sage: S = FiniteMonoids().example(); S
An example of a finite multiplicative monoid: the integers modulo 12

sage: S.category()
Category of finitely generated finite enumerated monoids
```

We conclude by running systematic tests on this monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
```

(continues on next page)

(continued from previous page)

```
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

class Element

Bases: `ElementWrapper`

wrapped_class

alias of `Integer`

an_element()

Returns an element of the monoid, as per *`Sets.ParentMethods.an_element()`*.

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.an_element()
6
```

one()

Return the one of the monoid, as per `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.one()
1
```

product(*x*, *y*)

Return the product of two elements x and y of the monoid, as per `Semigroups.ParentMethods.product()`.

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.product(M(3), M(5))
3
```

semigroup_generators()

Returns a set of generators for `self`, as per *`Semigroups.ParentMethods.semigroup_generators()`*. Currently this returns all integers mod n, which is of course far from optimal!

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.semigroup_generators()
Family (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
```

6.13 Examples of finite semigroups

`sage.categories.examples.finite_semigroups.`**`Example`**

alias of *`LeftRegularBand`*

`class` `sage.categories.examples.finite_semigroups.`**`LeftRegularBand`**(*alphabet=('a', 'b', 'c', 'd')*)

Bases: `UniqueRepresentation`, `Parent`

An example of a finite semigroup

This class provides a minimal implementation of a finite semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(); S
An example of a finite semigroup:
 the left regular band generated by ('a', 'b', 'c', 'd')
```

This is the semigroup generated by:

```
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

such that $x^2 = x$ and $xyx = xy$ for any x and y in S:

```
sage: S('dab')
'dab'
sage: S('dab') * S('acb')
'dabc'
```

It follows that the elements of S are strings without repetitions over the alphabet a, b, c, d:

```
sage: sorted(S.list())
['a', 'ab', 'abc', 'abcd', 'abd', 'abdc', 'ac', 'acb', 'acbd', 'acd',
 'acdb', 'ad', 'adb', 'adbc', 'adc', 'adcb', 'b', 'ba', 'bac',
 'bacd', 'bad', 'badc', 'bc', 'bca', 'bcad', 'bcd', 'bcda', 'bd',
 'bda', 'bdac', 'bdc', 'bdca', 'c', 'ca', 'cab', 'cabd', 'cad',
 'cadb', 'cb', 'cba', 'cbad', 'cbd', 'cbda', 'cd', 'cda', 'cdab',
 'cdb', 'cdba', 'd', 'da', 'dab', 'dabc', 'dac', 'dacb', 'db',
 'dba', 'dbac', 'dbc', 'dbca', 'dc', 'dca', 'dcab', 'dcb', 'dcba']
```

It also follows that there are finitely many of them:

```
sage: S.cardinality()
64
```

Indeed:

```
sage: 4 * ( 1 + 3 * (1 + 2 * (1 + 1)))
64
```

As expected, all the elements of S are idempotents:

```
sage: all( x.is_idempotent() for x in S )
True
```

Now, let us look at the structure of the semigroup:

```
sage: S = FiniteSemigroups().example(alphabet = ('a','b','c'))
sage: S.cayley_graph(side="left", simple=True).plot()                        #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 60 graphics primitives
sage: S.j_transversal_of_idempotents()   # random (arbitrary choice)         #␣
↪needs sage.graphs
['acb', 'ac', 'ab', 'bc', 'a', 'c', 'b']
```

We conclude by running systematic tests on this semigroup:

```
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

class Element

Bases: `ElementWrapper`

wrapped_class

alias of `str`

an_element()

Returns an element of the semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.an_element()
'cdab'

sage: S = FiniteSemigroups().example(("b"))
sage: S.an_element()
'b'
```

product(*x*, *y*)

Returns the product of two elements of the semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S('a') * S('b')
'ab'
sage: S('a') * S('b') * S('a')
'ab'
sage: S('a') * S('a')
'a'
```

semigroup_generators()

Returns the generators of the semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('x','y'))
sage: S.semigroup_generators()
Family ('x', 'y')
```

6.14 Examples of finite Weyl groups

sage.categories.examples.finite_weyl_groups.**Example**

alias of *SymmetricGroup*

class sage.categories.examples.finite_weyl_groups.**SymmetricGroup**(*n=4*)

Bases: UniqueRepresentation, Parent

An example of finite Weyl group: the symmetric group, with elements in list notation.

The purpose of this class is to provide a minimal template for implementing finite Weyl groups. See SymmetricGroup for a full featured and optimized implementation.

EXAMPLES:

```
sage: S = FiniteWeylGroups().example()
sage: S
The symmetric group on {0, ..., 3}
sage: S.category()
Category of finite irreducible Weyl groups
```

The elements of this group are permutations of the set $\{0, \ldots, 3\}$:

```
sage: S.one()
(0, 1, 2, 3)
sage: S.an_element()
(1, 2, 3, 0)
```

The group itself is generated by the elementary transpositions:

```
sage: S.simple_reflections()
Finite family {0: (1, 0, 2, 3), 1: (0, 2, 1, 3), 2: (0, 1, 3, 2)}
```

Only the following basic operations are implemented:

- *one()*
- *product()*

- *simple_reflection()*
- *cartan_type()*
- *Element.has_right_descent()*.

All the other usual Weyl group operations are inherited from the categories:

```
sage: S.cardinality()
24
sage: S.long_element()
(3, 2, 1, 0)
sage: S.cayley_graph(side="left").plot()                                   #␣
↪needs sage.graphs sage.plot
Graphics object consisting of 120 graphics primitives
```

Alternatively, one could have implemented `sage.categories.coxeter_groups.CoxeterGroups.ElementMethods.apply_simple_reflection()` instead of *simple_reflection()* and *product()*. See `CoxeterGroups().example()`.

class Element

Bases: `ElementWrapper`

has_right_descent(*i*)

Implements *CoxeterGroups.ElementMethods.has_right_descent()*.

EXAMPLES:

```
sage: S = FiniteWeylGroups().example()
sage: s = S.simple_reflections()
sage: (s[1] * s[2]).has_descent(2)
True
sage: S._test_has_descent()
```

cartan_type()

Return the Cartan type of `self`.

EXAMPLES:

```
sage: FiniteWeylGroups().example().cartan_type()                           #␣
↪needs sage.modules
['A', 3] relabelled by {1: 0, 2: 1, 3: 2}
```

degrees()

Return the degrees of `self`.

EXAMPLES:

```
sage: W = FiniteWeylGroups().example()
sage: W.degrees()
(2, 3, 4)
```

index_set()

Implements *CoxeterGroups.ParentMethods.index_set()*.

EXAMPLES:

```
sage: FiniteWeylGroups().example().index_set()
[0, 1, 2]
```

one()

Implements `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: FiniteWeylGroups().example().one()
(0, 1, 2, 3)
```

product(x, y)

Implements `Semigroups.ParentMethods.product()`.

EXAMPLES:

```
sage: s = FiniteWeylGroups().example().simple_reflections()
sage: s[1] * s[2]
(0, 2, 3, 1)
```

simple_reflection(i)

Implement `CoxeterGroups.ParentMethods.simple_reflection()` by returning the transposition $(i, i+1)$.

EXAMPLES:

```
sage: FiniteWeylGroups().example().simple_reflection(2)
(0, 1, 3, 2)
```

6.15 Examples of graded connected Hopf algebras with basis

`sage.categories.examples.graded_connected_hopf_algebras_with_basis.`**Example**

alias of *GradedConnectedCombinatorialHopfAlgebraWithPrimitiveGenerator*

class `sage.categories.examples.graded_connected_hopf_algebras_with_basis.`**GradedConnectedCom**

Bases: `CombinatorialFreeModule`

This class illustrates an implementation of a graded Hopf algebra with basis that has one primitive generator of degree 1 and basis elements indexed by non-negative integers.

This Hopf algebra example differs from what topologists refer to as a graded Hopf algebra because the twist operation in the tensor rule satisfies

$$(\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) = \Delta \circ \mu$$

where $\tau(x \otimes y) = y \otimes x$.

coproduct_on_basis(i)

The coproduct of a basis element.

$$\Delta(P_i) = \sum_{j=0}^{i} P_{i-j} \otimes P_j$$

INPUT:

- `i` – a non-negative integer

OUTPUT:

- an element of the tensor square of `self`

degree_on_basis(*i*)

The degree of a non-negative integer is itself

INPUT:

- `i` – a non-negative integer

OUTPUT:

- a non-negative integer

one_basis()

Returns 0, which index the unit of the Hopf algebra.

OUTPUT:

- the non-negative integer 0

EXAMPLES:

```
sage: H = GradedHopfAlgebrasWithBasis(QQ).Connected().example()
sage: H.one_basis()
0
sage: H.one()
P0
```

product_on_basis(*i*, *j*)

The product of two basis elements.

The product of elements of degree `i` and `j` is an element of degree `i+j`.

INPUT:

- `i`, `j` – non-negative integers

OUTPUT:

- a basis element indexed by `i+j`

6.16 Examples of graded modules with basis

sage.categories.examples.graded_modules_with_basis.**Example**

alias of *GradedPartitionModule*

class sage.categories.examples.graded_modules_with_basis.**GradedPartitionModule**(*base_ring*)

Bases: CombinatorialFreeModule

This class illustrates an implementation of a graded module with basis: the free module over partitions.

INPUT:

- `R` – base ring

The implementation involves the following:

- A choice of how to represent elements. In this case, the basis elements are partitions. The algebra is constructed as a `CombinatorialFreeModule` on the set of partitions, so it inherits all of the methods for such objects, and has operations like addition already defined.

```
sage: A = GradedModulesWithBasis(QQ).example()
↪# needs sage.modules
```

- A basis function - this module is graded by the non-negative integers, so there is a function defined in this module, creatively called `basis()`, which takes an integer d as input and returns a family of partitions representing a basis for the algebra in degree d.

```
sage: A.basis(2)
↪# needs sage.modules
Lazy family (Term map from Partitions to An example of a graded module with␣
↪basis: the free module on partitions over Rational Field(i))_{i in␣
↪Partitions of the integer 2}
sage: A.basis(6)[Partition([3,2,1])]
↪# needs sage.modules
P[3, 2, 1]
```

- If the algebra is called `A`, then its basis function is stored as `A.basis`. Thus the function can be used to find a basis for the degree d piece: essentially, just call `A.basis(d)`. More precisely, call `x` for each `x` in `A.basis(d)`.

```
sage: [m for m in A.basis(4)]
↪# needs sage.modules
[P[4], P[3, 1], P[2, 2], P[2, 1, 1], P[1, 1, 1, 1]]
```

- For dealing with basis elements: `degree_on_basis()`, and `_repr_term()`. The first of these defines the degree of any monomial, and then the `degree` method for elements – see the next item – uses it to compute the degree for a linear combination of monomials. The last of these determines the print representation for monomials, which automatically produces the print representation for general elements.

```
sage: A.degree_on_basis(Partition([4,3]))
↪# needs sage.modules
7
sage: A._repr_term(Partition([4,3]))
↪# needs sage.modules
'P[4, 3]'
```

- There is a class for elements, which inherits from `IndexedFreeModuleElement`. An element is determined by a dictionary whose keys are partitions and whose corresponding values are the coefficients. The class implements two things: an `is_homogeneous` method and a `degree` method.

```
sage: p = A.monomial(Partition([3,2,1])); p
↪# needs sage.modules
P[3, 2, 1]
sage: p.is_homogeneous()
↪# needs sage.modules
True
sage: p.degree()
↪# needs sage.modules
6
```

basis(*d=None*)

Return the basis for (the `d`-th homogeneous component of) `self`.

INPUT:

- `d` – (optional, default `None`) nonnegative integer or `None`

OUTPUT:

If `d` is `None`, returns the basis of the module. Otherwise, returns the basis of the homogeneous component of degree `d` (i.e., the subfamily of the basis of the whole module which consists only of the basis vectors lying in $F_d \setminus \bigcup_{i<d} F_i$).

The basis is always returned as a family.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions of the integer 4}
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
```

Checking this method on a filtered algebra. Note that this will typically raise a `NotImplementedError` when this feature is not implemented.

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Free abelian monoid indexed by
 {'x', 'y', 'z'} to An example of a filtered algebra with
 basis: the universal enveloping algebra of Lie algebra
 of RR^3 with cross product over Integer Ring(i))_{i in
 Free abelian monoid indexed by {'x', 'y', 'z'}}
```

An example with a graded algebra:

```
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: E.basis()
Lazy family (Term map from Subsets of {0,1} to
 The exterior algebra of rank 2 over Rational Field(i))_{i in
 Subsets of {0,1}}
```

degree_on_basis(*t*)

The degree of the element determined by the partition `t` in this graded module.

INPUT:

- `t` – the index of an element of the basis of this module, i.e. a partition

OUTPUT: an integer, the degree of the corresponding basis element

EXAMPLES:

```
sage: # needs sage.modules
sage: A = GradedModulesWithBasis(QQ).example()
sage: A.degree_on_basis(Partition((2,1)))
3
sage: A.degree_on_basis(Partition((4,2,1,1,1,1)))
10
sage: type(A.degree_on_basis(Partition((1,1))))
<class 'sage.rings.integer.Integer'>
```

6.17 Examples of graphs

class sage.categories.examples.graphs.**Cycle**(*n=5*)

Bases: UniqueRepresentation, Parent

An example of a graph: the cycle of length n.

This class illustrates a minimal implementation of a graph.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example(); C
An example of a graph: the 5-cycle

sage: C.category()
Category of graphs
```

We conclude by running systematic tests on this graph:

```
sage: TestSuite(C).run()
```

class Element

Bases: ElementWrapper

dimension()

Return the dimension of self.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: e = C.edges()[0]
sage: e.dimension()
2
sage: v = C.vertices()[0]
sage: v.dimension()
1
```

an_element()

Return an element of the graph, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
```

(continues on next page)

(continued from previous page)

```
sage: C.an_element()
0
```

edges()

Return the edges of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.edges()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

vertices()

Return the vertices of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.vertices()
[0, 1, 2, 3, 4]
```

sage.categories.examples.graphs.**Example**

alias of *Cycle*

6.18 Examples of Hopf algebras with basis

class sage.categories.examples.hopf_algebras_with_basis.**MyGroupAlgebra**(R, G)

Bases: `CombinatorialFreeModule`

An example of a Hopf algebra with basis: the group algebra of a group

This class illustrates a minimal implementation of a Hopf algebra with basis.

algebra_generators()

Return the generators of this algebra, as per *algebra_generators()*.

They correspond to the generators of the group.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: A.algebra_generators()
Finite family {(1,2,3): B[(1,2,3)], (1,3): B[(1,3)]}
```

antipode_on_basis(g)

Antipode, on basis elements, as per *HopfAlgebrasWithBasis.ParentMethods.antipode_on_basis()*.

It is given, on basis elements, by $\nu(g) = g^{-1}$

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: A.antipode_on_basis(a)
B[(1,3,2)]
```

coproduct_on_basis(*g*)

Coproduct, on basis elements, as per `HopfAlgebrasWithBasis.ParentMethods.coproduct_on_basis()`.

The basis elements are group like: $\Delta(g) = g \otimes g$.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: A.coproduct_on_basis(a)
B[(1,2,3)] # B[(1,2,3)]
```

counit_on_basis(*g*)

Counit, on basis elements, as per `HopfAlgebrasWithBasis.ParentMethods.counit_on_basis()`.

The counit on the basis elements is 1.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: A.counit_on_basis(a)
1
```

one_basis()

Returns the one of the group, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: A.one_basis()
()
sage: A.one()
B[()]
```

product_on_basis(*g1*, *g2*)

Product, on basis elements, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

The product of two basis elements is induced by the product of the corresponding elements of the group.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: a*b
(1,2)
sage: A.product_on_basis(a, b)
B[(1,2)]
```

6.19 Examples of infinite enumerated sets

`sage.categories.examples.infinite_enumerated_sets.`**`Example`**

alias of *NonNegativeIntegers*

class `sage.categories.examples.infinite_enumerated_sets.`**`NonNegativeIntegers`**

Bases: `UniqueRepresentation`, `Parent`

An example of infinite enumerated set: the non negative integers

This class provides a minimal implementation of an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN
An example of an infinite enumerated set: the non negative integers
sage: NN.cardinality()
+Infinity
sage: NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: NN.element_class
<class 'sage.rings.integer.Integer'>
sage: it = iter(NN)
sage: [next(it), next(it), next(it), next(it), next(it)]
[0, 1, 2, 3, 4]
sage: x = next(it); type(x)
<class 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: x+3
8
sage: NN(15)
15
sage: NN.first()
0
```

This checks that the different methods of NN return consistent results:

```
sage: TestSuite(NN).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
```

(continues on next page)

(continued from previous page)

```
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

Element

alias of `Integer`

an_element()

EXAMPLES:

```
sage: InfiniteEnumeratedSets().example().an_element()
42
```

next(*o*)

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.next(3)
4
```

6.20 Examples of a Lie algebra

`sage.categories.examples.lie_algebras.`**Example**

alias of *`LieAlgebraFromAssociative`*

class `sage.categories.examples.lie_algebras.`**LieAlgebraFromAssociative**(*gens*)

Bases: `Parent`, `UniqueRepresentation`

An example of a Lie algebra: a Lie algebra generated by a set of elements of an associative algebra.

This class illustrates a minimal implementation of a Lie algebra.

Let R be a commutative ring, and A an associative R-algebra. The Lie algebra A (sometimes denoted A^-) is defined to be the R-module A with Lie bracket given by the commutator in A: that is, $[a, b] := ab - ba$ for all $a, b \in A$.

What this class implements is not precisely A^-, however; it is the Lie subalgebra of A^- generated by the elements of the iterable `gens`. This specific implementation does not provide a reasonable containment test (i.e., it does not allow you to check if a given element a of A^- belongs to this Lie subalgebra); it, however, allows computing inside it.

INPUT:

- `gens` – a nonempty iterable consisting of elements of an associative algebra A

OUTPUT:

The Lie subalgebra of A^- generated by the elements of `gens`

EXAMPLES:

We create a model of $\mathfrak{sl}_2$ using matrices:

```
sage: gens = [matrix([[0,1],[0,0]]), matrix([[0,0],[1,0]]), matrix([[1,0],[0,-
↪1]])]
sage: for g in gens:
....:     g.set_immutable()
sage: L = LieAlgebras(QQ).example(gens)
sage: e,f,h = L.lie_algebra_generators()
sage: e.bracket(f) == h
True
sage: h.bracket(e) == 2*e
True
sage: h.bracket(f) == -2*f
True
```

class Element

Bases: `ElementWrapper`

Wrap an element as a Lie algebra element.

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()                                         #␣
↪needs sage.combinat sage.groups
sage: L.lie_algebra_generators()                                            #␣
↪needs sage.combinat sage.groups
Family ([2, 1, 3], [2, 3, 1])
```

zero()

Return the element 0.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()                                         #␣
↪needs sage.combinat sage.groups
sage: L.zero()                                                              #␣
↪needs sage.combinat sage.groups
0
```

6.21 Examples of a Lie algebra with basis

class sage.categories.examples.lie_algebras_with_basis.**AbelianLieAlgebra**(*R*, *gens*)

Bases: `CombinatorialFreeModule`

An example of a Lie algebra: the abelian Lie algebra.

This class illustrates a minimal implementation of a Lie algebra with a distinguished basis.

class Element

Bases: `IndexedFreeModuleElement`

lift()

Return the lift of `self` to the universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: elt = L.an_element()
sage: elt.lift()
3*P[F[2]] + 2*P[F[1]] + 2*P[F[]]
```

bracket_on_basis(*x*, *y*)

Return the Lie bracket on basis elements indexed by `x` and `y`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: L.bracket_on_basis(Partition([4,1]), Partition([2,2,1]))
0
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: L.lie_algebra_generators()
Lazy family (Term map from Partitions to
 An example of a Lie algebra: the abelian Lie algebra on the
 generators indexed by Partitions over Rational
 Field(i))_{i in Partitions}
```

`sage.categories.examples.lie_algebras_with_basis.`**Example**

alias of *AbelianLieAlgebra*

class `sage.categories.examples.lie_algebras_with_basis.`**IndexedPolynomialRing**(*R*, *indices*, ***kwds*)

Bases: `CombinatorialFreeModule`

Polynomial ring whose generators are indexed by an arbitrary set.

Todo: Currently this is just used as the universal enveloping algebra for the example of the abelian Lie algebra. This should be factored out into a more complete class.

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: UEA = L.universal_enveloping_algebra()
sage: UEA.algebra_generators()
Lazy family (algebra generator map(i))_{i in Partitions}
```

one_basis()

Return the index of element 1.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: UEA = L.universal_enveloping_algebra()
sage: UEA.one_basis()
1
sage: UEA.one_basis().parent()
Free abelian monoid indexed by Partitions
```

product_on_basis(*x*, *y*)

Return the product of the monomials indexed by `x` and `y`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: UEA = L.universal_enveloping_algebra()
sage: I = UEA._indices
sage: UEA.product_on_basis(I.an_element(), I.an_element())
P[F[]^4*F[1]^4*F[2]^6]
```

6.22 Examples of magmas

`sage.categories.examples.magmas.`**Example**

alias of `FreeMagma`

class `sage.categories.examples.magmas.`**FreeMagma**(*alphabet=('a', 'b', 'c', 'd')*)

Bases: `UniqueRepresentation`, `Parent`

An example of magma.

The purpose of this class is to provide a minimal template for implementing a magma.

EXAMPLES:

```
sage: M = Magmas().example(); M
An example of a magma: the free magma generated by ('a', 'b', 'c', 'd')
```

This is the free magma generated by:

```
sage: M.magma_generators()
Family ('a', 'b', 'c', 'd')
sage: a, b, c, d = M.magma_generators()
```

and with a non-associative product given by:

```
sage: a * (b * c) * (d * a * b)
'((a*(b*c))*((d*a)*b))'
sage: a * (b * c) == (a * b) * c
False
```

class Element

Bases: `ElementWrapper`

The class for elements of the free magma.

wrapped_class

alias of `str`

an_element()

Return an element of the magma.

EXAMPLES:

```
sage: F = Magmas().example()
sage: F.an_element()
'(((a*b)*c)*d)'
```

magma_generators()

Return the generators of the magma.

EXAMPLES:

```
sage: F = Magmas().example()
sage: F.magma_generators()
Family ('a', 'b', 'c', 'd')
```

product(*x*, *y*)

Return the product of `x` and `y` in the magma, as per *`Magmas.ParentMethods.product()`*.

EXAMPLES:

```
sage: F = Magmas().example()
sage: F('a') * F.an_element()
'(a*(((a*b)*c)*d))'
```

6.23 Examples of manifolds

`sage.categories.examples.manifolds.`**Example**

alias of *`Plane`*

class `sage.categories.examples.manifolds.`**Plane**(*n=3*, *base_ring=None*)

Bases: `UniqueRepresentation`, `Parent`

An example of a manifold: the n-dimensional plane.

This class illustrates a minimal implementation of a manifold.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(QQ).example(); M
An example of a Rational Field manifold: the 3-dimensional plane

sage: M.category()
Category of manifolds over Rational Field
```

We conclude by running systematic tests on this manifold:

```
sage: TestSuite(M).run()
```

Element

alias of `ElementWrapper`

an_element()

Return an element of the manifold, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(QQ).example()
sage: M.an_element()
(0, 0, 0)
```

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(QQ).example()
sage: M.dimension()
3
```

6.24 Examples of monoids

sage.categories.examples.monoids.**Example**

alias of *FreeMonoid*

class sage.categories.examples.monoids.**FreeMonoid**(*alphabet=('a', 'b', 'c', 'd')*)

Bases: *FreeSemigroup*

An example of a monoid: the free monoid

This class illustrates a minimal implementation of a monoid. For a full featured implementation of free monoids, see *FreeMonoid()*.

EXAMPLES:

```
sage: S = Monoids().example(); S
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')

sage: S.category()
Category of monoids
```

This is the free semigroup generated by:

```
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

with product rule given by concatenation of words:

```
sage: S('dab') * S('acb')
'dabacb'
```

and unit given by the empty word:

```
sage: S.one()
''
```

We conclude by running systematic tests on this monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

class Element

Bases: `ElementWrapper`

wrapped_class

alias of `str`

monoid_generators()

Return the generators of this monoid.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.monoid_generators()
Finite family {'a': 'a', 'b': 'b', 'c': 'c', 'd': 'd'}
sage: a,b,c,d = M.monoid_generators()
sage: a*d*c*b
'adcb'
```

one()

Returns the one of the monoid, as per `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
```

(continues on next page)

(continued from previous page)

```
sage: M.one()
''
```

6.25 Examples of posets

class sage.categories.examples.posets.**FiniteSetsOrderedByInclusion**

Bases: UniqueRepresentation, Parent

An example of a poset: finite sets ordered by inclusion

This class provides a minimal implementation of a poset

EXAMPLES:

```
sage: P = Posets().example(); P
An example of a poset: sets ordered by inclusion
```

We conclude by running systematic tests on this poset:

```
sage: TestSuite(P).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

class Element

Bases: ElementWrapper

wrapped_class

alias of Set_object_enumerated

an_element()

Returns an element of this poset

EXAMPLES:

```
sage: B = Posets().example()
sage: B.an_element()
{1, 4, 6}
```

le (*x*, *y*)

Returns whether x is a subset of y

EXAMPLES:

```
sage: P = Posets().example()
sage: P.le( P(Set([1,3])), P(Set([1,2,3])) )
True
sage: P.le( P(Set([1,3])), P(Set([1,3])) )
True
sage: P.le( P(Set([1,2])), P(Set([1,3])) )
False
```

class
sage.categories.examples.posets.**PositiveIntegersOrderedByDivisibilityFacade**

Bases: UniqueRepresentation, Parent

An example of a facade poset: the positive integers ordered by divisibility

This class provides a minimal implementation of a facade poset

EXAMPLES:

```
sage: P = Posets().example("facade"); P
An example of a facade poset: the positive integers ordered by divisibility

sage: P(5)
5
sage: P(0)
Traceback (most recent call last):
...
ValueError: Can't coerce `0` in any parent `An example of a facade poset: the␣
↪positive integers ordered by divisibility` is a facade for

sage: 3 in P
True
sage: 0 in P
False
```

class element_class (*X*, *category=None*)

Bases: Set_object_enumerated, parent_class

A finite enumerated set.

le (*x*, *y*)

Returns whether x is divisible by y

EXAMPLES:

```
sage: P = Posets().example("facade")
sage: P.le(3, 6)
True
sage: P.le(3, 3)
True
```

(continues on next page)

(continued from previous page)

```
sage: P.le(3, 7)
False
```

6.26 Examples of semigroups

class sage.categories.examples.semigroups.**FreeSemigroup**(*alphabet=('a', 'b', 'c', 'd')*)

Bases: UniqueRepresentation, Parent

An example of semigroup.

The purpose of this class is to provide a minimal template for implementing of a semigroup.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c', 'd')
```

This is the free semigroup generated by:

```
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

and with product given by concatenation:

```
sage: S('dab') * S('acb')
'dabacb'
```

class Element

Bases: ElementWrapper

The class for elements of the free semigroup.

wrapped_class

alias of str

an_element()

Returns an element of the semigroup.

EXAMPLES:

```
sage: F = Semigroups().example('free')
sage: F.an_element()
'abcd'
```

product(*x*, *y*)

Returns the product of x and y in the semigroup, as per Semigroups.ParentMethods.product().

EXAMPLES:

```
sage: F = Semigroups().example('free')
sage: F.an_element() * F('a')^5
'abcdaaaaa'
```

semigroup_generators()

Returns the generators of the semigroup.

EXAMPLES:

```
sage: F = Semigroups().example('free')
sage: F.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

class sage.categories.examples.semigroups.**IncompleteSubquotientSemigroup**(*category=None*)

Bases: UniqueRepresentation, Parent

An incompletely implemented subquotient semigroup, for testing purposes

EXAMPLES:

```
sage: S = sage.categories.examples.semigroups.IncompleteSubquotientSemigroup()
sage: S
A subquotient of An example of a semigroup: the left zero semigroup
```

class Element

Bases: ElementWrapper

ambient()

Returns the ambient semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.ambient()
An example of a semigroup: the left zero semigroup
```

class sage.categories.examples.semigroups.**LeftZeroSemigroup**

Bases: UniqueRepresentation, Parent

An example of a semigroup.

This class illustrates a minimal implementation of a semigroup.

EXAMPLES:

```
sage: S = Semigroups().example(); S
An example of a semigroup: the left zero semigroup
```

This is the semigroup that contains all sorts of objects:

```
sage: S.some_elements()
[3, 42, 'a', 3.4, 'raton laveur']
```

with product rule given by $a \times b = a$ for all a, b:

```
sage: S('hello') * S('world')
'hello'
sage: S(3)*S(1)*S(2)
3
sage: S(3)^12312321312321
3
```

class Element

Bases: ElementWrapper

is_idempotent()

Trivial implementation of Semigroups.Element.is_idempotent since all elements of this semigroup are idempotent!

EXAMPLES:

```
sage: S = Semigroups().example()
sage: S.an_element().is_idempotent()
True
sage: S(17).is_idempotent()
True
```

an_element()

Returns an element of the semigroup.

EXAMPLES:

```
sage: Semigroups().example().an_element()
42
```

product(*x*, *y*)

Returns the product of x and y in the semigroup, as per Semigroups.ParentMethods.product().

EXAMPLES:

```
sage: S = Semigroups().example()
sage: S('hello') * S('world')
'hello'
sage: S(3)*S(1)*S(2)
3
```

some_elements()

Returns a list of some elements of the semigroup.

EXAMPLES:

```
sage: Semigroups().example().some_elements()
[3, 42, 'a', 3.4, 'raton laveur']
```

class sage.categories.examples.semigroups.**QuotientOfLeftZeroSemigroup**(*category=None*)

Bases: UniqueRepresentation, Parent

Example of a quotient semigroup

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example(); S
An example of a (sub)quotient semigroup: a quotient of the left zero semigroup
```

This is the quotient of:

```
sage: S.ambient()
An example of a semigroup: the left zero semigroup
```

obtained by setting $x = 42$ for any $x \geq 42$:

```
sage: S(100)
42
sage: S(100) == S(42)
True
```

The product is inherited from the ambient semigroup:

```
sage: S(1)*S(2) == S(1)
True
```

class Element

Bases: `ElementWrapper`

ambient()

Returns the ambient semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.ambient()
An example of a semigroup: the left zero semigroup
```

an_element()

Returns an element of the semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.an_element()
42
```

lift(*x*)

Lift the element `x` into the ambient semigroup.

INPUT:

- `x` – an element of `self`.

OUTPUT:

- an element of `self.ambient()`.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: x = S.an_element(); x
42
sage: S.lift(x)
42
sage: S.lift(x) in S.ambient()
True
sage: y = S.ambient()(100); y
100
sage: S.lift(S(y))
42
```

retract(*x*)

Returns the retract `x` onto an element of this semigroup.

INPUT:

- x – an element of the ambient semigroup (self.ambient()).

OUTPUT:

- an element of self.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: L = S.ambient()
sage: S.retract(L(17))
17
sage: S.retract(L(42))
42
sage: S.retract(L(171))
42
```

some_elements()

Returns a list of some elements of the semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.some_elements()
[1, 2, 3, 8, 42, 42]
```

the_answer()

Returns the Answer to Life, the Universe, and Everything as an element of this semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.the_answer()
42
```

6.27 Examples of semigroups in cython

class sage.categories.examples.semigroups_cython.**IdempotentSemigroups**

Bases: *Category*

class ElementMethods

Bases: object

is_idempotent()

EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import␣
↪LeftZeroSemigroup
sage: S = LeftZeroSemigroup()
sage: S(2).is_idempotent()
True
```

super_categories()

EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import␣
↪IdempotentSemigroups
sage: IdempotentSemigroups().super_categories()
[Category of semigroups]
```

class sage.categories.examples.semigroups_cython.**LeftZeroSemigroup**

Bases: *LeftZeroSemigroup*

An example of semigroup

This class illustrates a minimal implementation of a semi-group where the element class is an extension type, and still gets code from the category. The category itself must be a Python class though.

This is purely a proof of concept. The code obviously needs refactorisation!

Comments:

- one cannot play ugly class surgery tricks (as with _mul_parent). available operations should really be declared to the coercion model!

EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import LeftZeroSemigroup
sage: S = LeftZeroSemigroup(); S
An example of a semigroup: the left zero semigroup
```

This is the semigroup which contains all sort of objects:

```
sage: S.some_elements()
[3, 42, 'a', 3.4, 'raton laveur']
```

with product rule given by $a \times b = a$ for all a, b.

```
sage: S('hello') * S('world')
'hello'

sage: S(3)*S(1)*S(2)
3

sage: S(3)^12312321312321
3

sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
```

(continues on next page)

(continued from previous page)

```
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

That's really the only method which is obtained from the category …

```
sage: S(42).is_idempotent
<bound method IdempotentSemigroups.ElementMethods.is_idempotent of 42>
sage: S(42).is_idempotent()
True

sage: S(42)._pow_int
<bound method IdempotentSemigroups.ElementMethods._pow_int of 42>
sage: S(42)^10
42

sage: S(42).is_idempotent
<bound method IdempotentSemigroups.ElementMethods.is_idempotent of 42>
sage: S(42).is_idempotent()
True
```

Element

alias of *LeftZeroSemigroupElement*

class sage.categories.examples.semigroups_cython.**LeftZeroSemigroupElement**

Bases: Element

EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import LeftZeroSemigroup
sage: S = LeftZeroSemigroup()
sage: x = S(3)
sage: TestSuite(x).run()
```

6.28 Examples of sets

class sage.categories.examples.sets_cat.**PrimeNumbers**

Bases: UniqueRepresentation, Parent

An example of parent in the category of sets: the set of prime numbers.

The elements are represented as plain integers in $\mathbf{Z}$ (facade implementation).

This is a minimal implementations. For more advanced examples of implementations, see also:

```
sage: P = Sets().example("facade")
sage: P = Sets().example("inherits")
sage: P = Sets().example("wrapper")
```

EXAMPLES:

```
sage: P = Sets().example()
sage: P(12)
Traceback (most recent call last):
...
AssertionError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Integer Ring
sage: x = P(13); x
13
sage: type(x)
<class 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: 13 in P
True
sage: 12 in P
False
sage: y = x+1; y
14
sage: type(y)
<class 'sage.rings.integer.Integer'>

sage: TestSuite(P).run(verbose=True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

an_element()

Implements `Sets.ParentMethods.an_element()`.

element_class

alias of `Integer`

class sage.categories.examples.sets_cat.**PrimeNumbers_Abstract**

Bases: `UniqueRepresentation`, `Parent`

This class shows how to write a parent while keeping the choice of the datastructure for the children open. Different class with fixed datastructure will then be constructed by inheriting from `PrimeNumbers_Abstract`.

This is used by:

```
sage: P = Sets().example("facade")
sage: P = Sets().example("inherits")
sage: P = Sets().example("wrapper")
```

class Element

Bases: `Element`

is_prime()

Return whether `self` is a prime number.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: x = P.an_element()
sage: P.an_element().is_prime()
True
```

next()

Return the next prime number.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: p = P.an_element(); p
47
sage: p.next()
53
```

Note: This method is not meant to implement the protocol iterator, and thus not subject to Python 2 vs Python 3 incompatibilities.

an_element()

Implements *Sets.ParentMethods.an_element()*.

next(*i*)

Return the next prime number.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: x = P.next(P.an_element()); x
53
sage: x.parent()
Set of prime numbers
```

some_elements()

Return some prime numbers.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: P.some_elements()
[47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97]
```

class sage.categories.examples.sets_cat.**PrimeNumbers_Facade**

Bases: *PrimeNumbers_Abstract*

An example of parent in the category of sets: the set of prime numbers.

In this alternative implementation, the elements are represented as plain integers in $\mathbf{Z}$ (facade implementation).

EXAMPLES:

```
sage: P = Sets().example("facade")
sage: P(12)
Traceback (most recent call last):
...
ValueError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Integer Ring
sage: x = P(13); x
13
sage: type(x)
<class 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: 13 in P
True
sage: 12 in P
False
sage: y = x+1; y
14
sage: type(y)
<class 'sage.rings.integer.Integer'>

sage: z = P.next(x); z
17
sage: type(z)
<class 'sage.rings.integer.Integer'>
sage: z.parent()
Integer Ring
```

The disadvantage of this implementation is that the elements do not know that they are prime, so that prime testing is slow:

```
sage: pf = Sets().example("facade").an_element()
sage: timeit("pf.is_prime()") #    random
625 loops, best of 3: 4.1 us per loop
```

compared to the other implementations where prime testing is only done if needed during the construction of the element, and later on the elements “know” that they are prime:

```
sage: pw = Sets().example("wrapper").an_element()
sage: timeit("pw.is_prime()")    # random
625 loops, best of 3: 859 ns per loop

sage: pi = Sets().example("inherits").an_element()
sage: timeit("pw.is_prime()")    # random
625 loops, best of 3: 854 ns per loop
```

Note also that the `next` method for the elements does not exist:

```
sage: pf.next()
Traceback (most recent call last):
...
AttributeError: 'sage.rings.integer.Integer' object has no attribute 'next'...
```

unlike in the other implementations:

```
sage: pw.next()
53
sage: pi.next()
53
```

element_class

alias of `Integer`

class sage.categories.examples.sets_cat.**PrimeNumbers_Inherits**

Bases: *PrimeNumbers_Abstract*

An example of parent in the category of sets: the set of prime numbers. In this implementation, the element are stored as object of a new class which inherits from the class Integer (technically `IntegerWrapper`).

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: P
Set of prime numbers
sage: P(12)
Traceback (most recent call last):
...
ValueError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Set of prime numbers
sage: x = P(13); x
13
sage: x.is_prime()
True
sage: type(x)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category.
↪element_class'>
sage: x.parent()
Set of prime numbers
sage: P(13) in P
True
sage: y = x+1; y
14
sage: type(y)
<class 'sage.rings.integer.Integer'>
sage: y.parent()
Integer Ring
sage: type(P(13)+P(17))
<class 'sage.rings.integer.Integer'>
sage: type(P(2)+P(3))
<class 'sage.rings.integer.Integer'>

sage: z = P.next(x); z
17
sage: type(z)
```

(continues on next page)

(continued from previous page)

```
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category.
↪element_class'>
sage: z.parent()
Set of prime numbers

sage: TestSuite(P).run(verbose=True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_construction() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

See also:

```
sage: P = Sets().example("facade")
sage: P = Sets().example("inherits")
sage: P = Sets().example("wrapper")
```

class Element(*parent*, *p*)

Bases: `IntegerWrapper`, *Element*

class sage.categories.examples.sets_cat.**PrimeNumbers_Wrapper**

Bases: *PrimeNumbers_Abstract*

An example of parent in the category of sets: the set of prime numbers.

In this second alternative implementation, the prime integer are stored as a attribute of a sage object by inheriting from *ElementWrapper*. In this case we need to ensure conversion and coercion from this parent and its element to `ZZ` and `Integer`.

EXAMPLES:

```
sage: P = Sets().example("wrapper")
sage: P(12)
Traceback (most recent call last):
...
ValueError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Set of prime numbers (wrapper implementation)
sage: x = P(13); x
```

(continues on next page)

(continued from previous page)

```
13
sage: type(x)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Wrapper_with_category.
↪element_class'>
sage: x.parent()
Set of prime numbers (wrapper implementation)
sage: 13 in P
True
sage: 12 in P
False
sage: y = x+1; y
14
sage: type(y)
<class 'sage.rings.integer.Integer'>

sage: z = P.next(x); z
17
sage: type(z)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Wrapper_with_category.
↪element_class'>
sage: z.parent()
Set of prime numbers (wrapper implementation)
```

class Element

Bases: `ElementWrapper`, *`Element`*

ElementWrapper

alias of `ElementWrapper`

6.29 Example of a set with grading

`sage.categories.examples.sets_with_grading.`**`Example`**

alias of *`NonNegativeIntegers`*

class `sage.categories.examples.sets_with_grading.`**`NonNegativeIntegers`**

Bases: `UniqueRepresentation`, `Parent`

Non negative integers graded by themselves.

EXAMPLES:

```
sage: E = SetsWithGrading().example(); E
Non negative integers
sage: E in Sets().Infinite()
True
sage: E.graded_component(0)
{0}
sage: E.graded_component(100)
{100}
```

an_element()

Return 0.

EXAMPLES:

```
sage: SetsWithGrading().example().an_element()
0
```

generating_series(*var='z'*)

Return $1/(1-z)$.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: f = N.generating_series(); f
1/(-z + 1)
sage: LaurentSeriesRing(ZZ,'z')(f)
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^10 + z^11 + z^12 +␣
↪z^13 + z^14 + z^15 + z^16 + z^17 + z^18 + z^19 + O(z^20)
```

graded_component(*grade*)

Return the component with grade `grade`.

EXAMPLES:

```
sage: N = SetsWithGrading().example()
sage: N.graded_component(65)
{65}
```

grading(*elt*)

Return the grade of `elt`.

EXAMPLES:

```
sage: N = SetsWithGrading().example()
sage: N.grading(10)
10
```

6.30 Examples of parents endowed with multiple realizations

class `sage.categories.examples.with_realizations.`**SubsetAlgebra**(*R*, *S*)

Bases: `UniqueRepresentation`, `Parent`

An example of parent endowed with several realizations

We consider an algebra $A(S)$ whose bases are indexed by the subsets s of a given set S. We consider three natural basis of this algebra: `F`, `In`, and `Out`. In the first basis, the product is given by the union of the indexing sets. That is, for any $s, t \subset S$

$$F_s F_t = F_{s \cup t}$$

The `In` basis and `Out` basis are defined respectively by:

$$In_s = \sum_{t \subset s} F_t \qquad \text{and} \qquad F_s = \sum_{t \supset s} Out_t$$

Each such basis gives a realization of A, where the elements are represented by their expansion in this basis.

This parent, and its code, demonstrate how to implement this algebra and its three realizations, with coercions and mixed arithmetic between them.

See also:

- *Sets().WithRealizations*
- the Implementing Algebraic Structures thematic tutorial.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.base_ring()
Rational Field
```

The three bases of `A`:

```
sage: F   = A.F()  ; F
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: In  = A.In() ; In
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: Out = A.Out(); Out
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
```

One can quickly define all the bases using the following shortcut:

```
sage: A.inject_shorthands()
Defining F as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the Fundamental basis
Defining In as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the In basis
Defining Out as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the Out basis
```

Accessing the basis elements is done with `basis()` method:

```
sage: F.basis().list()
[F[{}], F[{1}], F[{2}], F[{3}], F[{1, 2}], F[{1, 3}], F[{2, 3}], F[{1, 2, 3}]]
```

To access a particular basis element, you can use the `from_set()` method:

```
sage: F.from_set(2,3)
F[{2, 3}]
sage: In.from_set(1,3)
In[{1, 3}]
```

or as a convenient shorthand, one can use the following notation:

```
sage: F[2,3]
F[{2, 3}]
sage: In[1,3]
In[{1, 3}]
```

Some conversions:

```
sage: F(In[2,3])
F[{}] + F[{2}] + F[{3}] + F[{2, 3}]
sage: In(F[2,3])
In[{}] - In[{2}] - In[{3}] + In[{2, 3}]

sage: Out(F[3])
Out[{3}] + Out[{1, 3}] + Out[{2, 3}] + Out[{1, 2, 3}]
```

(continues on next page)

(continued from previous page)

```
sage: F(Out[3])
F[{3}] - F[{1, 3}] - F[{2, 3}] + F[{1, 2, 3}]

sage: Out(In[2,3])
Out[{}] + Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}] + 2*Out[{1, 3}] +␣
↪4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

We can now mix expressions:

```
sage: (1 + Out[1]) * In[2,3]
Out[{}] + 2*Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}] + 2*Out[{1, 3}] +␣
↪4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

class Bases(*parent_with_realization*)

Bases: *Category_realization_of_parent*

The category of the realizations of the subset algebra

class ParentMethods

Bases: `object`

from_set(**args*)

Construct the monomial indexed by the set containing the elements passed as arguments.

EXAMPLES:

```
sage: In = Sets().WithRealizations().example().In(); In
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: In.from_set(2,3)
In[{2, 3}]
```

As a shorthand, one can construct elements using the following notation:

```
sage: In[2,3]
In[{2, 3}]
```

one()

Returns the unit of this algebra.

This default implementation takes the unit in the fundamental basis, and coerces it in `self`.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: In = A.In(); Out = A.Out()
sage: In.one()
In[{}]
sage: Out.one()
Out[{}] + Out[{1}] + Out[{2}] + Out[{3}] + Out[{1, 2}] + Out[{1, 3}] +␣
↪Out[{2, 3}] + Out[{1, 2, 3}]
```

super_categories()

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: C = A.Bases(); C
Category of bases of The subset algebra of {1, 2, 3} over Rational Field
sage: C.super_categories()
[Category of realizations of The subset algebra of {1, 2, 3} over␣
↪Rational Field,
 Join of Category of algebras with basis over Rational Field and
         Category of commutative algebras over Rational Field and
         Category of realizations of unital magmas]
```

F

alias of *Fundamental*

class Fundamental(*A*)

Bases: CombinatorialFreeModule, BindableClass

The Subset algebra, in the fundamental basis

INPUT:

- A – a parent with realization in *SubsetAlgebra*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example()
sage: A.F()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: A.Fundamental()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
```

one()

Return the multiplicative unit element.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()                                  #␣
↪needs sage.combinat sage.modules
sage: A.one_basis()                                                        #␣
↪needs sage.combinat sage.modules
word:
sage: A.one()                                                              #␣
↪needs sage.combinat sage.modules
B[word: ]
```

one_basis()

Returns the index of the basis element which is equal to '1'.

EXAMPLES:

```
sage: F = Sets().WithRealizations().example().F(); F
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental␣
↪basis
sage: F.one_basis()
{}
sage: F.one()
F[{}]
```

product_on_basis(*left*, *right*)

Product of basis elements, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

INPUT:

- `left`, `right` – sets indexing basis elements

EXAMPLES:

```
sage: F = Sets().WithRealizations().example().F(); F
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental␣
↪basis
sage: S = F.basis().keys(); S
Subsets of {1, 2, 3}
sage: F.product_on_basis(S([]), S([]))
F[{}]
sage: F.product_on_basis(S({1}), S({3}))
F[{1, 3}]
sage: F.product_on_basis(S({1,2}), S({2,3}))
F[{1, 2, 3}]
```

class In(*A*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The Subset Algebra, in the `In` basis

INPUT:

- `A` – a parent with realization in *`SubsetAlgebra`*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example()
sage: A.In()
The subset algebra of {1, 2, 3} over Rational Field in the In basis
```

class Out(*A*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The Subset Algebra, in the *Out* basis

INPUT:

- `A` – a parent with realization in *`SubsetAlgebra`*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example()
sage: A.Out()
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
```

a_realization()

Returns the default realization of `self`

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.a_realization()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
```

base_set()

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.base_set()
{1, 2, 3}
```

indices()

The objects that index the basis elements of this algebra.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.indices()
Subsets of {1, 2, 3}
```

indices_key(*x*)

A key function on a set which gives a linear extension of the inclusion order.

INPUT:

- `x` – set

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: sorted(A.indices(), key=A.indices_key)
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
```

supsets(*set*)

Returns all the subsets of S containing `set`

INPUT:

- `set` – a subset of the base set S of `self`

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.supsets(Set((2,)))
[{1, 2, 3}, {2, 3}, {1, 2}, {2}]
```

CHAPTER

SEVEN

INTERNALS

7.1 Specific category classes

This is placed in a separate file from categories.py to avoid circular imports (as morphisms must be very low in the hierarchy with the new coercion model).

class sage.categories.category_types.**AbelianCategory**

Bases: *Category*

is_abelian()

Return `True` as `self` is an abelian category.

EXAMPLES:

```
sage: CommutativeAdditiveGroups().is_abelian()
True
```

class sage.categories.category_types.**Category_ideal**(*ambient*, *name=None*)

Bases: *Category_in_ambient*

classmethod an_instance()

Return an instance of this class.

EXAMPLES:

```
sage: AlgebraIdeals.an_instance()
Category of algebra ideals in Univariate Polynomial Ring in x over Rational␣
↪Field
```

ring()

Return the ambient ring used to describe objects `self`.

EXAMPLES:

```
sage: C = Ideals(IntegerRing())
sage: C.ring()
Integer Ring
```

class sage.categories.category_types.**Category_in_ambient**(*ambient*, *name=None*)

Bases: *Category*

Initialize `self`.

The parameter `name` is ignored.

EXAMPLES:

```
sage: C = Ideals(IntegerRing())
sage: TestSuite(C).run()
```

ambient()

Return the ambient object in which objects of this category are embedded.

EXAMPLES:

```
sage: C = Ideals(IntegerRing())
sage: C.ambient()
Integer Ring
```

class sage.categories.category_types.**Category_module**(*base*, *name=None*)

Bases: *AbelianCategory*, *Category_over_base_ring*

class sage.categories.category_types.**Category_over_base**(*base*, *name=None*)

Bases: *CategoryWithParameters*

A base class for categories over some base object

INPUT:

- base – a category C or an object of such a category

Assumption: the classes for the parents, elements, morphisms, of self should only depend on C. See github issue #11935 for details.

EXAMPLES:

```
sage: Algebras(GF(2)).element_class is Algebras(GF(3)).element_class
True

sage: C = GF(2).category()
sage: Algebras(GF(2)).parent_class is Algebras(C).parent_class
True

sage: C = ZZ.category()
sage: Algebras(ZZ).element_class is Algebras(C).element_class
True
```

classmethod an_instance()

Returns an instance of this class

EXAMPLES:

```
sage: Algebras.an_instance()
Category of algebras over Rational Field
```

base()

Return the base over which elements of this category are defined.

EXAMPLES:

```
sage: C = Algebras(QQ)
sage: C.base()
Rational Field
```

class sage.categories.category_types.**Category_over_base_ring**(*base*, *name=None*)

Bases: *Category_over_base*

Initialize self.

EXAMPLES:

```
sage: C = Algebras(GF(2)); C
Category of algebras over Finite Field of size 2
sage: TestSuite(C).run()
```

base_ring()

Return the base ring over which elements of this category are defined.

EXAMPLES:

```
sage: C = Algebras(GF(2))
sage: C.base_ring()
Finite Field of size 2
```

class sage.categories.category_types.**Elements**(*object*)

Bases: *Category*

The category of all elements of a given parent.

EXAMPLES:

```
sage: a = IntegerRing()(5)
sage: C = a.category(); C
Category of elements of Integer Ring
sage: a in C
True
sage: 2/3 in C
False
sage: loads(C.dumps()) == C
True
```

classmethod an_instance()

Returns an instance of this class

EXAMPLES:

```
sage: Elements.an_instance()
Category of elements of Rational Field
```

object()

EXAMPLES:

```
sage: Elements(ZZ).object()
Integer Ring
```

super_categories()

EXAMPLES:

```
sage: Elements(ZZ).super_categories()
[Category of objects]
```

Todo: Check that this is what we want.

7.2 Singleton categories

class sage.categories.category_singleton.**Category_contains_method_by_parent_class**

Bases: object

Returns whether x is an object in this category.

More specifically, returns True if and only if x has a category which is a subcategory of this one.

EXAMPLES:

```
sage: ZZ in Sets()
True
```

class sage.categories.category_singleton.**Category_singleton**

Bases: *Category*

A base class for implementing singleton category

A *singleton* category is a category whose class takes no parameters like Fields() or Rings(). See also the Singleton design pattern.

This is a subclass of *Category*, with a couple optimizations for singleton categories.

The main purpose is to make the idioms:

```
sage: QQ in Fields()
True
sage: ZZ in Fields()
False
```

as fast as possible, and in particular competitive to calling a constant Python method, in order to foster its systematic use throughout the Sage library. Such tests are time critical, in particular when creating a lot of polynomial rings over small fields like in the elliptic curve code.

EXAMPLES:

```
sage: from sage.categories.category_singleton import Category_singleton
sage: class MyRings(Category):
....:     def super_categories(self): return Rings().super_categories()
sage: class MyRingsSingleton(Category_singleton):
....:     def super_categories(self): return Rings().super_categories()
```

We create three rings. One of them is contained in the usual category of rings, one in the category of "my rings" and the third in the category of "my rings singleton":

```
sage: R = QQ['x,y']
sage: R1 = Parent(category = MyRings())
sage: R2 = Parent(category = MyRingsSingleton())
sage: R in MyRings()
False
sage: R1 in MyRings()
```

(continues on next page)

(continued from previous page)

```
True
sage: R1 in MyRingsSingleton()
False
sage: R2 in MyRings()
False
sage: R2 in MyRingsSingleton()
True
```

One sees that containment tests for the singleton class is a lot faster than for a usual class:

```
sage: # not tested
sage: timeit("R in MyRings()", number=10000)
10000 loops, best of 3: 7.12 µs per loop
sage: timeit("R1 in MyRings()", number=10000)
10000 loops, best of 3: 6.98 µs per loop
sage: timeit("R in MyRingsSingleton()", number=10000)
10000 loops, best of 3: 3.08 µs per loop
sage: timeit("R2 in MyRingsSingleton()", number=10000)
10000 loops, best of 3: 2.99 µs per loop
```

So this is an improvement, but not yet competitive with a pure Cython method:

```
sage: timeit("R.is_ring()", number=10000)                       # not tested
10000 loops, best of 3: 383 ns per loop
```

However, it is competitive with a Python method. Actually it is faster, if one stores the category in a variable:

```
sage: _Rings = Rings()
sage: R3 = Parent(category = _Rings)
sage: R3.is_ring.__module__
'sage.categories.rings'
sage: timeit("R3.is_ring()", number=10000)                      # not tested
10000 loops, best of 3: 2.64 µs per loop
sage: timeit("R3 in Rings()", number=10000)                     # not tested
10000 loops, best of 3: 3.01 µs per loop
sage: timeit("R3 in _Rings", number=10000)                      # not tested
10000 loops, best of 3: 652 ns per loop
```

This might not be easy to further optimize, since the time is consumed in many different spots:

```
sage: timeit("MyRingsSingleton.__classcall__()", number=10000)# not tested
10000 loops, best of 3: 306 ns per loop

sage: X = MyRingsSingleton()
sage: timeit("R in X  ", number=10000)                          # not tested
10000 loops, best of 3: 699 ns per loop

sage: c = MyRingsSingleton().__contains__
sage: timeit("c(R)", number = 10000)                            # not tested
10000 loops, best of 3: 661 ns per loop
```

Warning: A singleton concrete class A should not have a subclass B (necessarily concrete). Otherwise, creating an instance a of A and an instance b of B would break the singleton principle: A would have two instances a and b.

With the current implementation only direct subclasses of *Category_singleton* are supported:

```
sage: class MyRingsSingleton(Category_singleton):
....:     def super_categories(self): return Rings().super_categories()
sage: class Disaster(MyRingsSingleton): pass
sage: Disaster()
Traceback (most recent call last):
...
AssertionError: <class '__main__.Disaster'> is not a direct subclass of <class
↪'sage.categories.category_singleton.Category_singleton'>
```

However, it is acceptable for a direct subclass R of *Category_singleton* to create its unique instance as an instance of a subclass of itself (in which case, its the subclass of R which is concrete, not R itself). This is used for example to plug in extra category code via a dynamic subclass:

```
sage: from sage.categories.category_singleton import Category_singleton
sage: class R(Category_singleton):
....:     def super_categories(self): return [Sets()]
sage: R()
Category of r
sage: R().__class__
<class '__main__.R_with_category'>
sage: R().__class__.mro()
[<class '__main__.R_with_category'>,
 <class '__main__.R'>,
 <class 'sage.categories.category_singleton.Category_singleton'>,
 <class 'sage.categories.category.Category'>,
 <class 'sage.structure.unique_representation.UniqueRepresentation'>,
 <class 'sage.structure.unique_representation.CachedRepresentation'>,
 <class 'sage.misc.fast_methods.WithEqualityById'>,
 <class 'sage.structure.sage_object.SageObject'>,
 <class '__main__.R.subcategory_class'>,
 <class 'sage.categories.sets_cat.Sets.subcategory_class'>,
 <class 'sage.categories.sets_with_partial_maps.SetsWithPartialMaps.
↪subcategory_class'>,
 <class 'sage.categories.objects.Objects.subcategory_class'>,
 <... 'object'>]
sage: R() is R()
True
sage: R() is R().__class__()
True
```

In that case, `R` is an abstract class and has a single concrete subclass, so this does not break the Singleton design pattern.

See also:

Category.__classcall__(), *Category.__init__()*

Note: The `_test_category` test is failing because `MyRingsSingleton()` is not a subcategory of the join of its super categories:

```
sage: C = MyRingsSingleton()
sage: C.super_categories()
[Category of rngs, Category of semirings]
sage: Rngs() & Semirings()
Category of rings
sage: C.is_subcategory(Rings())
False
```

Oh well; it's not really relevant for those tests.

7.3 Fast functions for the category framework

AUTHOR:

- Simon King (initial version)

class sage.categories.category_cy_helper.**AxiomContainer**

Bases: dict

A fast container for axioms.

This is derived from dict. A key is the name of an axiom. The corresponding value is the "rank" of this axiom, that is used to order the axioms in *canonicalize_axioms()*.

EXAMPLES:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: isinstance(all_axioms, sage.categories.category_with_axiom.AxiomContainer)
True
```

add(*axiom*)

Add a new axiom name, of the next rank.

EXAMPLES:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: m = max(all_axioms.values())
sage: all_axioms.add('Awesome')
sage: all_axioms['Awesome'] == m + 1
True
```

To avoid side effects, we remove the added axiom:

```
sage: del all_axioms['Awesome']
```

sage.categories.category_cy_helper.**canonicalize_axioms**(*all_axioms*, *axioms*)

Canonicalize a set of axioms.

INPUT:

- all_axioms – all available axioms
- axioms – a set (or iterable) of axioms

Note: *AxiomContainer* provides a fast container for axioms, and the collection of axioms is stored in *sage.categories.category_with_axiom*. In order to avoid circular imports, we expect that the collection of all axioms is provided as an argument to this auxiliary function.

OUTPUT:

A set of axioms as a tuple sorted according to the order of the tuple all_axioms in *sage.categories.category_with_axiom*.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import canonicalize_axioms, all_
↪axioms
sage: canonicalize_axioms(all_axioms, ["Commutative", "Connected", "WithBasis",
↪"Finite"])
('Finite', 'Connected', 'WithBasis', 'Commutative')
sage: canonicalize_axioms(all_axioms, ["Commutative", "Connected", "Commutative",
↪"WithBasis", "Finite"])
('Finite', 'Connected', 'WithBasis', 'Commutative')
```

sage.categories.category_cy_helper.**category_sort_key**(*category*)

Return `category._cmp_key`.

This helper function is used for sorting lists of categories.

It is semantically equivalent to `operator.attrgetter()` `("_cmp_key")`, but currently faster.

EXAMPLES:

```
sage: from sage.categories.category_cy_helper import category_sort_key
sage: category_sort_key(Rings()) is Rings()._cmp_key
True
```

sage.categories.category_cy_helper.**get_axiom_index**(*all_axioms*, *axiom*)

Helper function: Return the rank of an axiom.

INPUT:

- `all_axioms` – the axiom collection
- `axiom` – string, name of an axiom

EXAMPLES:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: from sage.categories.category_cy_helper import get_axiom_index
sage: get_axiom_index(all_axioms, 'AdditiveCommutative') == all_axioms[
↪'AdditiveCommutative']
True
```

sage.categories.category_cy_helper.**join_as_tuple**(*categories*, *axioms*, *ignore_axioms*)

Helper for *join()*.

INPUT:

- `categories` – tuple of categories to be joined,
- `axioms` – tuple of strings; the names of some supplementary axioms.
- `ignore_axioms` – tuple of pairs `(cat, axiom)`, such that `axiom` will not be applied to `cat`, should `cat` occur in the algorithm.

EXAMPLES:

```
sage: from sage.categories.category_cy_helper import join_as_tuple
sage: T = (Coalgebras(QQ), Sets().Finite(), Algebras(ZZ), SimplicialComplexes())
sage: join_as_tuple(T,(),())
(Category of algebras over Integer Ring,
 Category of finite monoids,
 Category of finite additive groups,
 Category of coalgebras over Rational Field,
```

(continues on next page)

(continued from previous page)

```
 Category of finite simplicial complexes)
sage: join_as_tuple(T,('WithBasis',),())
(Category of algebras with basis over Integer Ring,
 Category of finite monoids,
 Category of coalgebras with basis over Rational Field,
 Category of finite additive groups,
 Category of finite simplicial complexes)
sage: join_as_tuple(T,(),((Monoids(),'Finite'),))
(Category of algebras over Integer Ring,
 Category of finite additive groups,
 Category of coalgebras over Rational Field,
 Category of finite simplicial complexes)
```

7.4 Coercion methods for categories

The purpose of this Cython module is to hold special coercion methods, which are inserted by their respective categories.

7.5 Poor Man's map

class sage.categories.poor_man_map.**PoorManMap**(*function*, *domain=None*, *codomain=None*, *name=None*)

Bases: `SageObject`

A class for maps between sets which are not (yet) modeled by parents

Could possibly disappear when all combinatorial classes / enumerated sets will be parents

INPUT:

- `function` – a callable or an iterable of callables. This represents the underlying function used to implement this map. If it is an iterable, then the callables will be composed to implement this map.
- `domain` – the domain of this map or `None` if the domain is not known or should remain unspecified
- `codomain` – the codomain of this map or `None` if the codomain is not known or should remain unspecified
- `name` – a name for this map or `None` if this map has no particular name

EXAMPLES:

```
sage: from sage.categories.poor_man_map import PoorManMap
sage: f = PoorManMap(factorial, domain=(1, 2, 3), codomain=(1, 2, 6))
sage: f
A map from (1, 2, 3) to (1, 2, 6)
sage: f(3)
6
```

The composition of several functions can be created by passing in a tuple of functions:

```
sage: i = PoorManMap((factorial, sqrt), domain=(1, 4, 9), codomain=(1, 2, 6))
```

However, the same effect can also be achieved by just composing maps:

```
sage: g = PoorManMap(factorial, domain=(1, 2, 3), codomain=(1, 2, 6))
sage: h = PoorManMap(sqrt, domain=(1, 4, 9), codomain=(1, 2, 3))
sage: i == g*h
True
```

codomain()

Returns the codomain of `self`

EXAMPLES:

```
sage: from sage.categories.poor_man_map import PoorManMap
sage: PoorManMap(lambda x: x+1, domain=(1,2,3), codomain=(2,3,4)).codomain()
(2, 3, 4)
```

domain()

Returns the domain of `self`

EXAMPLES:

```
sage: from sage.categories.poor_man_map import PoorManMap
sage: PoorManMap(lambda x: x+1, domain=(1,2,3), codomain=(2,3,4)).domain()
(1, 2, 3)
```

CHAPTER EIGHT

INDICES AND TABLES

PYTHON MODULE INDEX

INDEX