Algebras

Release 10.1

The Sage Development Team

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CONTENTS

CHAPTER

ONE

CATALOG OF ALGEBRAS

The `algebras` object may be used to access examples of various algebras currently implemented in Sage. Using tab-completion on this object is an easy way to discover and quickly create the algebras that are available (as listed here).

Let `<tab>` indicate pressing the `Tab` key. So begin by typing `algebras.<tab>` to the see the currently implemented named algebras.

- *algebras.AlternatingCentralExtensionQuantumOnsager*
- *algebras.ArikiKoike*
- *algebras.AskeyWilson*
- `algebras.Blob`
- *algebras.Brauer*
- *algebras.Clifford*
- *algebras.ClusterAlgebra*
- *algebras.CubicHecke*
- *algebras.Descent*
- *algebras.DifferentialWeyl*
- *algebras.DownUp*
- *algebras.Exterior*
- *algebras.FiniteDimensional*
- `algebras.FQSym`
- *algebras.Free*
- *algebras.FreeZinbiel*
- *algebras.FreePreLie*
- *algebras.FreeDendriform*
- `algebras.FSym`
- *algebras.GradedCommutative*
- *algebras.Group*
- *algebras.GrossmanLarson*
- *algebras.Hall*

- *algebras.Incidence*
- *algebras.IwahoriHecke*
- *algebras.Moebius*
- *algebras.Jordan*
- *algebras.Lie*
- algebras.MalvenutoReutenauer
- *algebras.NilCoxeter*
- *algebras.Octonion*
- *algebras.OrlikTerao*
- *algebras.OrlikSolomon*
- *algebras.QuantumClifford*
- *algebras.QuantumGL*
- *algebras.QuantumMatrixCoordinate*
- algebras.QSym
- *algebras.Partition*
- *algebras.PlanarPartition*
- *algebras.qCommutingPolynomials*
- *algebras.qCommutingLaurentPolynomials*
- *algebras.QuantumGroup*
- *algebras.Quaternion*
- *algebras.RationalCherednik*
- *algebras.Schur*
- *algebras.Shuffle*
- *algebras.Steenrod*
- *algebras.TemperleyLieb*
- *algebras.Tensor*
- algebras.WQSym
- *algebras.Yangian*
- *algebras.YokonumaHecke*

CHAPTER TWO

QUANTUM GROUPS

2.1 Alternating Central Extension Quantum Onsager Algebra

AUTHORS:

- Travis Scrimshaw (2021-03): Initial version

class sage.algebras.quantum_groups.ace_quantum_onsager.**ACEQuantumOnsagerAlgebra**(*R*, *q*)

Bases: CombinatorialFreeModule

The alternating central extension of the q-Onsager algebra.

The *alternating central extension* $\mathcal{A}_q$ of the q-Onsager algebra O_q is a current algebra of O_q introduced by Baseilhac and Koizumi [BK2005]. A presentation was given by Baseilhac and Shigechi [BS2010], which was then reformulated in terms of currents in [Ter2021] and then used to prove that the generators form a PBW basis.

Note: This is only for the q-Onsager algebra with parameter $c = q^{-1}(q - q^{-1})^2$.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: AG = A.algebra_generators()
```

We construct the generators $\mathcal{G}_3$, $\mathcal{W}_{-5}$, $\mathcal{W}_2$, and $\widetilde{\mathcal{G}}_4$ and perform some computations:

```
sage: G3 = AG[0,3]
sage: Wm5 = AG[1,-5]
sage: W2 = AG[1,2]
sage: Gt4 = AG[2,4]
sage: [G3, Wm5, W2, Gt4]
[G[3], W[-5], W[2], Gt[4]]
sage: Gt4 * G3
G[3]*Gt[4] + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[-6]*W[1]
 + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[-5]*W[2]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[-4]*W[1]
 + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[-4]*W[3]
 + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[-3]*W[-2]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[-3]*W[2]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[-2]*W[5]
 + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[-1]*W[4]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[-1]*W[6]
```

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```
 + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[0]*W[5]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[0]*W[7]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[3]*W[4]
sage: Wm5 * G3
((q^2-1)/q^2)*G[1]*W[-7] + ((-q^2+1)/q^2)*G[1]*W[7]
 + ((q^2-1)/q^2)*G[2]*W[-6] + ((-q^2+1)/q^2)*G[2]*W[6] + G[3]*W[-5]
 + ((-q^2+1)/q^2)*G[6]*W[-2] + ((q^2-1)/q^2)*G[6]*W[2]
 + ((-q^2+1)/q^2)*G[7]*W[-1] + ((q^2-1)/q^2)*G[7]*W[1]
 + ((-q^2+1)/q^2)*G[8]*W[0] + ((-q^8+2*q^4-1)/q^5)*W[-8]
 + ((q^8-2*q^4+1)/q^5)*W[8]
sage: W2 * G3
(q^2-1)*G[1]*W[-2] + (-q^2+1)*G[1]*W[4] + (-q^2+1)*G[3]*W[0]
 + q^2*G[3]*W[2] + (q^2-1)*G[4]*W[1] + ((-q^8+2*q^4-1)/q^3)*W[-3]
 + ((q^8-2*q^4+1)/q^3)*W[5]
sage: W2 * Wm5
(q^4/(q^8+2*q^6-2*q^2-1))*G[1]*Gt[6] + (-q^4/(q^8+2*q^6-2*q^2-1))*G[6]*Gt[1]
 + W[-5]*W[2] + (q/(q^2+1))*G[7] + (-q/(q^2+1))*Gt[7]
sage: Gt4 * Wm5
((q^2-1)/q^2)*W[-8]*Gt[1] + ((q^2-1)/q^2)*W[-7]*Gt[2]
 + ((q^2-1)/q^2)*W[-6]*Gt[3] + W[-5]*Gt[4] + ((-q^2+1)/q^2)*W[-3]*Gt[6]
 + ((-q^2+1)/q^2)*W[-2]*Gt[7] + ((-q^2+1)/q^2)*W[-1]*Gt[8]
 + ((-q^2+1)/q^2)*W[0]*Gt[9] + ((q^2-1)/q^2)*W[1]*Gt[8]
 + ((q^2-1)/q^2)*W[2]*Gt[7] + ((q^2-1)/q^2)*W[3]*Gt[6]
 + ((-q^2+1)/q^2)*W[6]*Gt[3] + ((-q^2+1)/q^2)*W[7]*Gt[2]
 + ((-q^2+1)/q^2)*W[8]*Gt[1] + ((-q^8+2*q^4-1)/q^5)*W[-9]
 + ((q^8-2*q^4+1)/q^5)*W[9]
sage: Gt4 * W2
(q^2-1)*W[-3]*Gt[1] + (-q^2+1)*W[0]*Gt[4] + (q^2-1)*W[1]*Gt[5]
 + q^2*W[2]*Gt[4] + (-q^2+1)*W[5]*Gt[1] + ((-q^8+2*q^4-1)/q^3)*W[-4]
 + ((q^8-2*q^4+1)/q^3)*W[6]
```

REFERENCES:

- [BK2005]
- [BS2010]
- [Ter2021]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: A.algebra_generators()
Lazy family (generator map(i))_{i in Disjoint union of
 Family (Positive integers, Integer Ring, Positive integers)}
```

dagger()

The antiautomorphism †.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: G = A.algebra_generators()
sage: x = A.an_element()^2
sage: A.dagger(A.dagger(x)) == x
True
sage: A.dagger(G[1,-1] * G[1,1]) == A.dagger(G[1,1]) * A.dagger(G[1,-1])
True
sage: A.dagger(G[0,2] * G[1,3]) == A.dagger(G[1,3]) * A.dagger(G[0,2])
True
sage: A.dagger(G[2,2] * G[1,3]) == A.dagger(G[1,3]) * A.dagger(G[2,2])
True
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: G = A.algebra_generators()
sage: A.degree_on_basis(G[0,1].leading_support())
2
sage: A.degree_on_basis(G[0,2].leading_support())
4
sage: A.degree_on_basis(G[1,-1].leading_support())
3
sage: A.degree_on_basis(G[1,0].leading_support())
1
sage: A.degree_on_basis(G[1,1].leading_support())
1
sage: A.degree_on_basis(G[2,1].leading_support())
2
sage: A.degree_on_basis(G[2,2].leading_support())
4
sage: [x.degree() for x in A.some_elements()]
[1, 5, 3, 1, 5, 2, 4, 2, 4]
```

gens()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: A.algebra_generators()
Lazy family (generator map(i))_{i in Disjoint union of
 Family (Positive integers, Integer Ring, Positive integers)}
```

one_basis()

Return the basis element indexing 1.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: ob = A.one_basis(); ob
1
```

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```
sage: ob.parent()
Free abelian monoid indexed by Disjoint union of
 Family (Positive integers, Integer Ring, Positive integers)
```

product_on_basis(*lhs*, *rhs*)

Return the product of the two basis elements `lhs` and `rhs`.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: G = A.algebra_generators()
sage: q = A.q()
sage: rho = -(q^2 - q^-2)^2
```

We verify the PBW ordering:

```
sage: G[0,1] * G[1,1]  # indirect doctest
G[1]*W[1]
sage: G[1,1] * G[0,1]
q^2*G[1]*W[1] + ((-q^8+2*q^4-1)/q^3)*W[0] + ((q^8-2*q^4+1)/q^3)*W[2]
sage: G[1,-1] * G[1,1]
W[-1]*W[1]
sage: G[1,1] * G[1,-1]
W[-1]*W[1] + (q/(q^2+1))*G[2] + (-q/(q^2+1))*Gt[2]
sage: G[1,1] * G[2,1]
W[1]*Gt[1]
sage: G[2,1] * G[1,1]
q^2*W[1]*Gt[1] + ((-q^8+2*q^4-1)/q^3)*W[0] + ((q^8-2*q^4+1)/q^3)*W[2]
sage: G[0,1] * G[2,1]
G[1]*Gt[1]
sage: G[2,1] * G[0,1]
G[1]*Gt[1] + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[-1]*W[1]
 + ((-q^12+3*q^8-3*q^4+1)/q^6)*W[0]^2
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[0]*W[2]
 + ((q^12-3*q^8+3*q^4-1)/q^6)*W[1]^2
```

We verify some of the defining relations (see Equations (3-14) in [Ter2021]), which are used to construct the PBW basis:

```
sage: G[0,1] * G[0,2] == G[0,2] * G[0,1]
True
sage: G[1,-1] * G[1,-2] == G[1,-2] * G[1,-1]
True
sage: G[1,1] * G[1,2] == G[1,2] * G[1,1]
True
sage: G[2,1] * G[2,2] == G[2,2] * G[2,1]
True
sage: G[1,0] * G[1,2] - G[1,2] * G[1,0] == G[1,-1] * G[1,1] - G[1,1] * G[1,-1]
True
sage: G[1,0] * G[1,2] - G[1,2] * G[1,0] == (G[2,2] - G[0,2]) / (q + ~q)
True
sage: q * G[1,0] * G[0,2] - ~q * G[0,2] * G[1,0] == q * G[2,2] * G[1,0] - ~q *
↪G[1,0] * G[2,2]
```

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```
True
sage: q * G[1,0] * G[0,2] - ~q * G[0,2] * G[1,0] == rho * G[1,-2] - rho * G[1,2]
True
sage: q * G[0,2] * G[1,1] - ~q * G[1,1] * G[0,2] == q * G[1,1] * G[2,2] - ~q *
→G[2,2] * G[1,1]
True
sage: q * G[0,2] * G[1,1] - ~q * G[1,1] * G[0,2] == rho * G[1,3] - rho * G[1,-1]
True
sage: G[1,-2] * G[1,2] - G[1,2] * G[1,-2] == G[1,-1] * G[1,3] - G[1,3] * G[1,-1]
True
sage: G[1,-2] * G[0,2] - G[0,2] * G[1,-2] == G[1,-1] * G[0,3] - G[0,3] * G[1,-1]
True
sage: G[1,1] * G[0,2] - G[0,2] * G[1,1] == G[1,2] * G[0,1] - G[0,1] * G[1,2]
True
sage: G[1,-2] * G[2,2] - G[2,2] * G[1,-2] == G[1,-1] * G[2,3] - G[2,3] * G[1,-1]
True
sage: G[1,1] * G[2,2] - G[2,2] * G[1,1] == G[1,2] * G[2,1] - G[2,1] * G[1,2]
True
sage: G[0,1] * G[2,2] - G[2,2] * G[0,1] == G[0,2] * G[2,1] - G[2,1] * G[0,2]
True
```

q()

Return the parameter q of `self`.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: A.q()
q
```

quantum_onsager_pbw_generator(*i*)

Return the image of the PBW generator of the q-Onsager algebra in `self`.

INPUT:

- `i` – a pair `(k, m)` such that
  - `k=0` and `m` is an integer
  - `k=1` and `m` is a positive integer

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: A.quantum_onsager_pbw_generator((0,0))
W[1]
sage: A.quantum_onsager_pbw_generator((0,1))
(q^3/(q^4-1))*W[1]*Gt[1] - q^2*W[0] + (q^2+1)*W[2]
sage: A.quantum_onsager_pbw_generator((0,2))
(q^6/(q^8-2*q^4+1))*W[1]*Gt[1]^2 + (-q^5/(q^4-1))*W[0]*Gt[1]
 + (q^3/(q^2-1))*W[1]*Gt[2] + (q^3/(q^2-1))*W[2]*Gt[1]
 + (-q^4-q^2)*W[-1] - q^2*W[1] + (q^4+2*q^2+1)*W[3]
sage: A.quantum_onsager_pbw_generator((0,-1))
W[0]
sage: A.quantum_onsager_pbw_generator((0,-2))
```

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```
(q/(q^4-1))*W[0]*Gt[1] + ((q^2+1)/q^2)*W[-1] - 1/q^2*W[1]
sage: A.quantum_onsager_pbw_generator((0,-3))
(q^2/(q^8-2*q^4+1))*W[0]*Gt[1]^2 + (1/(q^3-q))*W[-1]*Gt[1]
 + (1/(q^3-q))*W[0]*Gt[2] - (1/(q^5-q))*W[1]*Gt[1]
 + ((q^4+2*q^2+1)/q^4)*W[-2] - 1/q^2*W[0] + ((-q^2-1)/q^4)*W[2]
sage: A.quantum_onsager_pbw_generator((1,1))
((-q^2+1)/q^2)*W[0]*W[1] + (1/(q^3+q))*G[1] - (1/(q^3+q))*Gt[1]
sage: A.quantum_onsager_pbw_generator((1,2))
-1/q*W[0]*W[1]*Gt[1] + (1/(q^6+q^4-q^2-1))*G[1]*Gt[1]
 + ((-q^4+1)/q^4)*W[-1]*W[1] + (q^2-1)*W[0]^2
 + ((-q^4+1)/q^2)*W[0]*W[2] + ((q^2-1)/q^4)*W[1]^2
 - (1/(q^6+q^4-q^2-1))*Gt[1]^2 + 1/q^3*G[2] - 1/q^3*Gt[2]
```

`sigma()`

The automorphism σ.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: G = A.algebra_generators()
sage: x = A.an_element()^2
sage: A.sigma(A.sigma(x)) == x
True
sage: A.sigma(G[1,-1] * G[1,1]) == A.sigma(G[1,-1]) * A.sigma(G[1,1])
True
sage: A.sigma(G[0,2] * G[1,3]) == A.sigma(G[0,2]) * A.sigma(G[1,3])
True
```

`some_elements()`

Return some elements of `self`.

EXAMPLES:

```
sage: A = algebras.AlternatingCentralExtensionQuantumOnsager(QQ)
sage: A.some_elements()
[W[0], W[3], W[-1], W[1], W[-2], G[1], G[2], Gt[1], Gt[2]]
```

2.2 Fock Space

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version

class `sage.algebras.quantum_groups.fock_space.`**`FockSpace`**(*n*, *multicharge*, *q*, *base_ring*)

Bases: `Parent`, `UniqueRepresentation`

The (fermionic) Fock space of $U_q(\widehat{\mathfrak{sl}}_n)$ with multicharge $(\gamma_1, \ldots, \gamma_m)$.

Fix a positive integer $n > 1$ and fix a sequence $\gamma = (\gamma_1, \ldots, \gamma_m)$, where $\gamma_i \in \mathbf{Z}/n\mathbf{Z}$. *(fermionic) Fock space* $\mathcal{F}$ with multicharge γ is a $U_q(\widehat{\mathfrak{gl}}_n)$-representation with a basis $\{|\lambda\rangle\}$, where λ is a partition tuple of level m. By considering $\mathcal{F}$ as a $U_q(\widehat{\mathfrak{sl}}_n)$-representation, it is not irreducible, but the submodule generated by $|\emptyset^m\rangle$ is isomorphic to the highest weight module $V(\mu)$, where the highest weight $\mu = \sum_i \Lambda_{\gamma_i}$.

Let $R_i(\lambda)$ and $A_i(\lambda)$ be the set of removable and addable, respectively, i-cells of λ, where an i-cell is a cell of residue i (i.e., content modulo n). The action of $U_q(\widehat{\mathfrak{sl}}_n)$ is given as follows:

$$\begin{aligned} e_i|\lambda\rangle &= \sum_{c \in R_i(\lambda)} q^{M_i(\lambda,c)} |\lambda + c\rangle, \\ f_i|\lambda\rangle &= \sum_{c \in A_i(\lambda)} q^{N_i(\lambda,c)} |\lambda - c\rangle, \\ q^{h_i}|\lambda\rangle &= q^{N_i(\lambda)}|\lambda\rangle, \\ q^{d}|\lambda\rangle &= q^{-N^{(0)}(\lambda)}|\lambda\rangle, \end{aligned}$$

where

- $M_i(\lambda, c)$ (resp. $N_i(\lambda, c)$) is the number of removable (resp. addable) i-cells of λ below (resp. above) c minus the number of addable (resp. removable) i-cells of λ below (resp. above) c,
- $N_i(\lambda)$ is the number of addable i-cells minus the number of removable i-cells, and
- $N^{(0)}(\lambda)$ is the total number of 0-cells of λ.

Another interpretation of Fock space is as a semi-infinite wedge product (which each factor we can think of as fermions). This allows a description of the $U_q(\widehat{\mathfrak{gl}}_n)$ action, as well as an explicit description of the bar involution. In particular, the bar involution is the unique semi-linear map satisfying

- $q \mapsto q^{-1}$,
- $\overline{|\emptyset\rangle} = |\emptyset\rangle$, and
- $\overline{f_i|\lambda\rangle} = f_i\overline{|\lambda\rangle}$.

We then define the *canonical basis* or *(lower) global crystal basis* as the unique basis of $\mathcal{F}$ such that

- $\overline{G(\lambda)} = G(\lambda)$,
- $G(\lambda) \equiv |\lambda\rangle \mod q\mathbf{Z}[q]$.

It is also known that this basis is upper unitriangular with respect to dominance order and that both the natural basis and the canonical basis of $\mathcal{F}$ are $\mathbf{Z}$-graded by $|\lambda|$. Additionally, the transition matrices $(d_{\lambda,\nu})_{\lambda,\nu \vdash n}$ given by

$$G(\nu) = \sum_{\lambda \vdash |\nu|} d_{\lambda,\nu} |\lambda\rangle$$

described the decomposition matrices of the Hecke algebras when restricting to $V(\mu)$ [Ariki1996].

To go between the canonical basis and the natural basis, for level 1 Fock space, we follow the LLT algorithm [LLT1996]. Indeed, we first construct an basis $\{A(\nu)\}$ that is an approximation to the lower global crystal basis, in the sense that it is bar-invariant, and then use Gaussian elimination to construct the lower global crystal basis. For higher level Fock space, we follow [Fayers2010], where the higher level is considered as a tensor product space of the corresponding level 1 Fock spaces.

There are three bases currently implemented:

- The natural basis: `F`.
- The approximation basis that comes from LLT(-type) algorithms: `A`.
- The lower global crystal basis: `G`.

Todo:

- Implement the approximation and lower global crystal bases on all partition tuples.

- Implement the bar involution.
- Implement the full $U_q(\widehat{\mathfrak{gl}})$-action.

INPUT:

- `n` – the value n
- `multicharge` – (default: `[0]`) the multicharge
- `q` – (optional) the parameter q
- `base_ring` – (optional) the base ring containing `q`

EXAMPLES:

We start by constructing the natural basis and doing some computations:

```
sage: Fock = FockSpace(3)
sage: F = Fock.natural()
sage: u = F.highest_weight_vector()
sage: u.f(0,2,(1,2),0)
|2, 2, 1> + q*|2, 1, 1, 1>
sage: u.f(0,2,(1,2),0,2)
|3, 2, 1> + q*|3, 1, 1, 1> + q*|2, 2, 2> + q^2*|2, 1, 1, 1, 1>
sage: x = u.f(0,2,(1,2),0,2)
sage: [x.h(i) for i in range(3)]
[q*|3, 2, 1> + q^2*|3, 1, 1, 1> + q^2*|2, 2, 2> + q^3*|2, 1, 1, 1, 1>,
 |3, 2, 1> + q*|3, 1, 1, 1> + q*|2, 2, 2> + q^2*|2, 1, 1, 1, 1>,
 |3, 2, 1> + q*|3, 1, 1, 1> + q*|2, 2, 2> + q^2*|2, 1, 1, 1, 1>]
sage: [x.h_inverse(i) for i in range(3)]
[1/q*|3, 2, 1> + |3, 1, 1, 1> + |2, 2, 2> + q*|2, 1, 1, 1, 1>,
 |3, 2, 1> + q*|3, 1, 1, 1> + q*|2, 2, 2> + q^2*|2, 1, 1, 1, 1>,
 |3, 2, 1> + q*|3, 1, 1, 1> + q*|2, 2, 2> + q^2*|2, 1, 1, 1, 1>]
sage: x.d()
1/q^2*|3, 2, 1> + 1/q*|3, 1, 1, 1> + 1/q*|2, 2, 2> + |2, 1, 1, 1, 1>
```

Next, we construct the approximation and lower global crystal bases and convert to the natural basis:

```
sage: A = Fock.A()
sage: G = Fock.G()
sage: F(A[4,2,2,1])
|4, 2, 2, 1> + q*|4, 2, 1, 1, 1>
sage: F(G[4,2,2,1])
|4, 2, 2, 1> + q*|4, 2, 1, 1, 1>
sage: F(A[7,3,2,1,1])
|7, 3, 2, 1, 1> + q*|7, 2, 2, 2, 1> + q^2*|7, 2, 2, 1, 1, 1>
 + q*|6, 3, 3, 1, 1> + q^2*|6, 2, 2, 2, 2> + q^3*|6, 2, 2, 1, 1, 1, 1>
 + q*|5, 5, 2, 1, 1> + q^2*|5, 4, 3, 1, 1> + (q^2+1)*|4, 4, 3, 2, 1>
 + (q^3+q)*|4, 4, 3, 1, 1, 1> + (q^3+q)*|4, 4, 2, 2, 2>
 + (q^4+q^2)*|4, 4, 2, 1, 1, 1, 1> + q*|4, 3, 3, 3, 1>
 + q^2*|4, 3, 2, 1, 1, 1, 1, 1> + q^2*|4, 2, 2, 2, 2, 2>
 + q^3*|4, 2, 2, 2, 1, 1, 1, 1> + q^2*|3, 3, 3, 3, 2>
 + q^3*|3, 3, 3, 1, 1, 1, 1, 1> + q^3*|3, 2, 2, 2, 2, 2, 1>
 + q^4*|3, 2, 2, 2, 2, 1, 1, 1>
sage: F(G[7,3,2,1,1])
```

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```
|7, 3, 2, 1, 1> + q*|7, 2, 2, 2, 1> + q^2*|7, 2, 2, 1, 1, 1>
 + q*|6, 3, 3, 1, 1> + q^2*|6, 2, 2, 2, 2>
 + q^3*|6, 2, 2, 1, 1, 1, 1> + q*|5, 5, 2, 1, 1>
 + q^2*|5, 4, 3, 1, 1> + q^2*|4, 4, 3, 2, 1>
 + q^3*|4, 4, 3, 1, 1, 1> + q^3*|4, 4, 2, 2, 2>
 + q^4*|4, 4, 2, 1, 1, 1, 1>
sage: A(F(G[7,3,2,1,1]))
A[7, 3, 2, 1, 1] - A[4, 4, 3, 2, 1]
sage: G(F(A[7,3,2,1,1]))
G[7, 3, 2, 1, 1] + G[4, 4, 3, 2, 1]
sage: A(F(G[8,4,3,2,2,1]))
A[8, 4, 3, 2, 2, 1] - A[6, 4, 4, 2, 2, 1, 1] - A[5, 5, 4, 3, 2, 1]
 + ((-q^2-1)/q)*A[5, 4, 4, 3, 2, 1, 1]
sage: G(F(A[8,4,3,2,2,1]))
G[8, 4, 3, 2, 2, 1] + G[6, 4, 4, 2, 2, 1, 1] + G[5, 5, 4, 3, 2, 1]
 + ((q^2+1)/q)*G[5, 4, 4, 3, 2, 1, 1]
```

We can also construct higher level Fock spaces and perform similar computations:

```
sage: Fock = FockSpace(3, [1,0])
sage: F = Fock.natural()
sage: A = Fock.A()
sage: G = Fock.G()
sage: F(G[[2,1],[4,1,1]])
|[2, 1], [4, 1, 1]> + q*|[2, 1], [3, 2, 1]>
 + q^2*|[2, 1], [3, 1, 1, 1]> + q^2*|[2], [4, 2, 1]>
 + q^3*|[2], [4, 1, 1, 1]> + q^4*|[2], [3, 2, 1, 1]>
 + q*|[1, 1, 1], [4, 1, 1]> + q^2*|[1, 1, 1], [3, 2, 1]>
 + q^3*|[1, 1, 1], [3, 1, 1, 1]> + q^2*|[1, 1], [3, 2, 2]>
 + q^3*|[1, 1], [3, 1, 1, 1, 1]> + q^3*|[1], [4, 2, 2]>
 + q^4*|[1], [4, 1, 1, 1, 1]> + q^4*|[1], [3, 2, 2, 1]>
 + q^5*|[1], [3, 2, 1, 1, 1]>
sage: A(F(G[[2,1],[4,1,1]]))
A([2, 1], [4, 1, 1]) - A([2], [4, 2, 1])
sage: G(F(A[[2,1],[4,1,1]]))
G([2, 1], [4, 1, 1]) + G([2], [4, 2, 1])
```

For level 0, the truncated Fock space of [GW1999] is implemented. This can be used to improve the speed of the computation of the lower global crystal basis, provided the truncation is not too small:

```
sage: FS = FockSpace(2)
sage: F = FS.natural()
sage: G = FS.G()
sage: FS3 = FockSpace(2, truncated=3)
sage: F3 = FS3.natural()
sage: G3 = FS3.G()
sage: F(G[6,2,1])
|6, 2, 1> + q*|5, 3, 1> + q^2*|5, 2, 2> + q^3*|5, 2, 1, 1>
 + q*|4, 2, 1, 1, 1> + q^2*|3, 3, 1, 1, 1> + q^3*|3, 2, 2, 1, 1>
 + q^4*|3, 2, 1, 1, 1, 1>
sage: F3(G3[6,2,1])
|6, 2, 1> + q*|5, 3, 1> + q^2*|5, 2, 2>
```

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```
sage: FS5 = FockSpace(2, truncated=5)
sage: F5 = FS5.natural()
sage: G5 = FS5.G()
sage: F5(G5[6,2,1])
|6, 2, 1> + q*|5, 3, 1> + q^2*|5, 2, 2> + q^3*|5, 2, 1, 1>
 + q*|4, 2, 1, 1, 1> + q^2*|3, 3, 1, 1, 1> + q^3*|3, 2, 2, 1, 1>
```

REFERENCES:

- [Ariki1996]
- [LLT1996]
- [Fayers2010]
- [GW1999]

class A(*F*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The A basis of the Fock space which is the approximation of the lower global crystal basis.

The approximation basis A is a basis that is constructed from the highest weight element by applying divided difference operators using the ladder construction of [LLT1996] and [GW1999]. Thus, this basis is bar invariant and upper unitriangular (using dominance order on partitions) when expressed in the natural basis. This basis is then converted to the lower global crystal basis by using Gaussian elimination.

EXAMPLES:

We construct Example 6.5 and 6.7 in [LLT1996]:

```
sage: FS = FockSpace(2)
sage: F = FS.natural()
sage: G = FS.G()
sage: A = FS.A()
sage: F(A[5])
|5> + |3, 2> + 2*q*|3, 1, 1> + q^2*|2, 2, 1> + q^2*|1, 1, 1, 1, 1>
sage: F(A[4,1])
|4, 1> + q*|2, 1, 1, 1>
sage: F(A[3,2])
|3, 2> + q*|3, 1, 1> + q^2*|2, 2, 1>
sage: F(G[5])
|5> + q*|3, 1, 1> + q^2*|1, 1, 1, 1, 1>
```

We construct the examples in Section 5.1 of [Fayers2010]:

```
sage: FS = FockSpace(2, [0, 0])
sage: F = FS.natural()
sage: A = FS.A()
sage: F(A[[2,1],[1]])
|[2, 1], [1]> + q*|[2], [2]> + q^2*|[2], [1, 1]> + q^2*|[1, 1], [2]>
 + q^3*|[1, 1], [1, 1]> + q^4*|[1], [2, 1]>
sage: F(A[[4],[]])
|[4], []> + q*|[3, 1], []> + q*|[2, 1, 1], []>
 + (q^2+1)*|[2, 1], [1]> + 2*q*|[2], [2]> + 2*q^2*|[2], [1, 1]>
 + q^2*|[1, 1, 1, 1], []> + 2*q^2*|[1, 1], [2]>
 + 2*q^3*|[1, 1], [1, 1]> + (q^4+q^2)*|[1], [2, 1]>
```

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```
+ q^2*|[], [4]> + q^3*|[], [3, 1]> + q^3*|[], [2, 1, 1]>
+ q^4*|[], [1, 1, 1, 1]>
```

options = Current options for FockSpace - display: ket

class F(F)

Bases: `CombinatorialFreeModule`, `BindableClass`

The natural basis of the Fock space.

This is the basis indexed by partitions. This has an action of the quantum group $U_q(\widehat{\mathfrak{sl}}_n)$ described in *FockSpace*.

EXAMPLES:

We construct the natural basis and perform some computations:

```
sage: F = FockSpace(4).natural()
sage: q = F.q()
sage: u = F.highest_weight_vector()
sage: u
|>
sage: u.f(0,1,2)
|3>
sage: u.f(0,1,3)
|2, 1>
sage: u.f(0,1,2,0)
0
sage: u.f(0,1,3,2)
|3, 1> + q*|2, 1, 1>
sage: u.f(0,1,2,3)
|4> + q*|3, 1>
sage: u.f(0,1,3,2,2,0)
((q^2+1)/q)*|3, 2, 1>
sage: x = (q^4 * u + u.f(0,1,3,(2,2)))
sage: x
|3, 1, 1> + q^4*|>
sage: x.f(0,1,3)
|4, 3, 1> + q*|4, 2, 1, 1> + q*|3, 3, 2>
 + q^2*|3, 2, 2, 1> + q^4*|2, 1>
sage: x.h_inverse(2)
q^2*|3, 1, 1> + q^4*|>
sage: x.h_inverse(0)
1/q*|3, 1, 1> + q^3*|>
sage: x.d()
1/q*|3, 1, 1> + q^4*|>
sage: x.e(2)
|3, 1> + q*|2, 1, 1>
```

class Element

Bases: `IndexedFreeModuleElement`

An element in the Fock space.

d()

Apply the action of d on `self`.

EXAMPLES:

```
sage: F = FockSpace(2)
sage: F.highest_weight_vector().d()
|>
sage: F[2,1,1].d()
1/q^2*|2, 1, 1>
sage: F[5,3,3,1,1,1].d()
1/q^7*|5, 3, 3, 1, 1, 1>

sage: F = FockSpace(4, [2,0,1])
sage: F.highest_weight_vector().d()
|[], [], []>
sage: F[[2,1],[1],[2]].d()
1/q*|[2, 1], [1], [2]>
sage: F[[4,2,2,1],[1],[5,2]].d()
1/q^5*|[4, 2, 2, 1], [1], [5, 2]>
```

e(**data*)

Apply the action of the divided difference operator $e_i^{(p)}$ on `self`.

INPUT:

- `*data` – a list of indices or pairs (i, p)

EXAMPLES:

```
sage: F = FockSpace(2)
sage: F[2,1,1].e(1)
1/q*|1, 1, 1>
sage: F[2,1,1].e(0)
|2, 1>
sage: F[2,1,1].e(0).e(1)
|2> + q*|1, 1>
sage: F[2,1,1].e(0).e(1).e(1)
((q^2+1)/q)*|1>
sage: F[2,1,1].e(0).e((1, 2))
|1>
sage: F[2,1,1].e(0, 1, 1, 1)
0
sage: F[2,1,1].e(0, (1, 3))
0
sage: F[2,1,1].e(0, (1,2), 0)
|>
sage: F[2,1,1].e(1, 0, 1, 0)
1/q*|>

sage: F = FockSpace(4, [2, 0, 1])
sage: F[[2,1],[1],[2]]
|[2, 1], [1], [2]>
sage: F[[2,1],[1],[2]].e(2)
|[2, 1], [1], [1]>
sage: F[[2,1],[1],[2]].e(1)
```

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```
1/q*|[2], [1], [2]>
sage: F[[2,1],[1],[2]].e(0)
1/q*|[2, 1], [], [2]>
sage: F[[2,1],[1],[2]].e(3)
1/q^2*|[1, 1], [1], [2]>
sage: F[[2,1],[1],[2]].e(3, 2, 1)
1/q^2*|[1, 1], [1], []> + 1/q^2*|[1], [1], [1]>
sage: F[[2,1],[1],[2]].e(3, 2, 1, 0, 1, 2)
2/q^3*|[], [], []>
```

f(**data*)

Apply the action of the divided difference operator $f_i^{(p)}$ on `self`.

INPUT:

- `*data` – a list of indices or pairs (i, p)

EXAMPLES:

```
sage: F = FockSpace(2)
sage: mg = F.highest_weight_vector()
sage: mg.f(0)
|1>
sage: mg.f(0).f(1)
|2> + q*|1, 1>
sage: mg.f(0).f(0)
0
sage: mg.f((0, 2))
0
sage: mg.f(0, 1, 1)
((q^2+1)/q)*|2, 1>
sage: mg.f(0, (1, 2))
|2, 1>
sage: mg.f(0, 1, 0)
|3> + q*|1, 1, 1>

sage: F = FockSpace(4, [2, 0, 1])
sage: mg = F.highest_weight_vector()
sage: mg.f(0)
|[], [1], []>
sage: mg.f(2)
|[1], [], []>
sage: mg.f(1)
|[], [], [1]>
sage: mg.f(1, 0)
|[], [1], [1]> + q*|[], [], [1, 1]>
sage: mg.f(0, 1)
|[], [2], []> + q*|[], [1], [1]>
sage: mg.f(0, 1, 3)
|[], [2, 1], []> + q*|[], [1, 1], [1]>
sage: mg.f(3)
0
```

h(**data*)

Apply the action of h_i on `self`.

EXAMPLES:

```
sage: F = FockSpace(2)
sage: F[2,1,1].h(0)
q*|2, 1, 1>
sage: F[2,1,1].h(1)
|2, 1, 1>
sage: F[2,1,1].h(0, 0)
q^2*|2, 1, 1>

sage: F = FockSpace(4, [2,0,1])
sage: elt = F[[2,1],[1],[2]]
sage: elt.h(0)
q^2*|[2, 1], [1], [2]>
sage: elt.h(1)
|[2, 1], [1], [2]>
sage: elt.h(2)
|[2, 1], [1], [2]>
sage: elt.h(3)
q*|[2, 1], [1], [2]>
```

h_inverse(**data*)

Apply the action of h_i^{-1} on `self`.

EXAMPLES:

```
sage: F = FockSpace(2)
sage: F[2,1,1].h_inverse(0)
1/q*|2, 1, 1>
sage: F[2,1,1].h_inverse(1)
|2, 1, 1>
sage: F[2,1,1].h_inverse(0, 0)
1/q^2*|2, 1, 1>

sage: F = FockSpace(4, [2,0,1])
sage: elt = F[[2,1],[1],[2]]
sage: elt.h_inverse(0)
1/q^2*|[2, 1], [1], [2]>
sage: elt.h_inverse(1)
|[2, 1], [1], [2]>
sage: elt.h_inverse(2)
|[2, 1], [1], [2]>
sage: elt.h_inverse(3)
1/q*|[2, 1], [1], [2]>
```

options = Current options for FockSpace - display: ket

class G(*F*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The lower global crystal basis living inside of Fock space.

EXAMPLES:

We construct some of the tables/entries given in Section 10 of [LLT1996]. For $\widehat{\mathfrak{sl}}_2$:

```
sage: FS = FockSpace(2)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[2])
|2> + q*|1, 1>
sage: F(G[3])
|3> + q*|1, 1, 1>
sage: F(G[2,1])
|2, 1>
sage: F(G[4])
|4> + q*|3, 1> + q*|2, 1, 1> + q^2*|1, 1, 1, 1>
sage: F(G[3,1])
|3, 1> + q*|2, 2> + q^2*|2, 1, 1>
sage: F(G[5])
|5> + q*|3, 1, 1> + q^2*|1, 1, 1, 1, 1>
sage: F(G[4,2])
|4, 2> + q*|4, 1, 1> + q*|3, 3> + q^2*|3, 1, 1, 1>
 + q^2*|2, 2, 2> + q^3*|2, 2, 1, 1>
sage: F(G[4,2,1])
|4, 2, 1> + q*|3, 3, 1> + q^2*|3, 2, 2> + q^3*|3, 2, 1, 1>
sage: F(G[6,2])
|6, 2> + q*|6, 1, 1> + q*|5, 3> + q^2*|5, 1, 1, 1> + q*|4, 3, 1>
 + q^2*|4, 2, 2> + (q^3+q)*|4, 2, 1, 1> + q^2*|4, 1, 1, 1, 1>
 + q^2*|3, 3, 1, 1> + q^3*|3, 2, 2, 1> + q^3*|3, 1, 1, 1, 1, 1>
 + q^3*|2, 2, 2, 1, 1> + q^4*|2, 2, 1, 1, 1, 1>
sage: F(G[5,3,1])
|5, 3, 1> + q*|5, 2, 2> + q^2*|5, 2, 1, 1> + q*|4, 4, 1>
 + q^2*|4, 2, 1, 1, 1> + q^2*|3, 3, 3> + q^3*|3, 3, 1, 1, 1>
 + q^3*|3, 2, 2, 2> + q^4*|3, 2, 2, 1, 1>
sage: F(G[4,3,2,1])
|4, 3, 2, 1>
sage: F(G[7,2,1])
|7, 2, 1> + q*|5, 2, 1, 1, 1> + q^2*|3, 2, 1, 1, 1, 1, 1>
sage: F(G[10,1])
|10, 1> + q*|8, 1, 1, 1> + q^2*|6, 1, 1, 1, 1, 1>
 + q^3*|4, 1, 1, 1, 1, 1, 1, 1>
 + q^4*|2, 1, 1, 1, 1, 1, 1, 1, 1, 1>
sage: F(G[6,3,2])
|6, 3, 2> + q*|6, 3, 1, 1> + q^2*|6, 2, 2, 1> + q^3*|5, 3, 2, 1>
 + q*|4, 3, 2, 1, 1> + q^2*|4, 3, 1, 1, 1, 1>
 + q^3*|4, 2, 2, 1, 1, 1> + q^4*|3, 3, 2, 1, 1, 1>
sage: F(G[5,3,2,1])
|5, 3, 2, 1> + q*|4, 4, 2, 1> + q^2*|4, 3, 3, 1>
 + q^3*|4, 3, 2, 2> + q^4*|4, 3, 2, 1, 1>
```

For $\widehat{\mathfrak{sl}}_3$:

```
sage: FS = FockSpace(3)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[2])
|2>
sage: F(G[1,1])
```

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```
|1, 1>
sage: F(G[3])
|3> + q*|2, 1>
sage: F(G[2,1])
|2, 1> + q*|1, 1, 1>
sage: F(G[4])
|4> + q*|2, 2>
sage: F(G[3,1])
|3, 1>
sage: F(G[2,2])
|2, 2> + q*|1, 1, 1, 1>
sage: F(G[2,1,1])
|2, 1, 1>
sage: F(G[5])
|5> + q*|2, 2, 1>
sage: F(G[2,2,1])
|2, 2, 1> + q*|2, 1, 1, 1>
sage: F(G[4,1,1])
|4, 1, 1> + q*|3, 2, 1> + q^2*|3, 1, 1, 1>
sage: F(G[5,2])
|5, 2> + q*|4, 3> + q^2*|4, 2, 1>
sage: F(G[8])
|8> + q*|5, 2, 1> + q*|3, 3, 1, 1> + q^2*|2, 2, 2, 2>
sage: F(G[7,2])
|7, 2> + q*|4, 2, 2, 1>
sage: F(G[6,2,2])
|6, 2, 2> + q*|6, 1, 1, 1, 1> + q*|4, 4, 2> + q^2*|3, 3, 2, 1, 1>
```

For $\widehat{\mathfrak{sl}}_4$:

```
sage: FS = FockSpace(4)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[4])
|4> + q*|3, 1>
sage: F(G[3,1])
|3, 1> + q*|2, 1, 1>
sage: F(G[2,2])
|2, 2>
sage: F(G[2,1,1])
|2, 1, 1> + q*|1, 1, 1, 1>
sage: F(G[3,2])
|3, 2> + q*|2, 2, 1>
sage: F(G[2,2,2])
|2, 2, 2> + q*|1, 1, 1, 1, 1, 1>
sage: F(G[6,1])
|6, 1> + q*|4, 3>
sage: F(G[3,2,2,1])
|3, 2, 2, 1> + q*|3, 1, 1, 1, 1, 1> + q*|2, 2, 2, 2>
 + q^2*|2, 1, 1, 1, 1, 1, 1>
sage: F(G[7,2])
|7, 2> + q*|6, 2, 1> + q*|5, 4> + q^2*|5, 3, 1>
```

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```
sage: F(G[5,2,2,1])
|5, 2, 2, 1> + q*|5, 1, 1, 1, 1, 1> + q*|4, 2, 2, 1, 1>
 + q^2*|4, 2, 1, 1, 1, 1>
```

We construct the examples in Section 5.1 of [Fayers2010]:

```
sage: FS = FockSpace(2, [0, 0])
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[[2,1],[1]])
|[2, 1], [1]> + q*|[2], [2]> + q^2*|[2], [1, 1]>
 + q^2*|[1, 1], [2]> + q^3*|[1, 1], [1, 1]> + q^4*|[1], [2, 1]>
sage: F(G[[4],[]])
|[4], []> + q*|[3, 1], []> + q*|[2, 1, 1], []> + q^2*|[2, 1], [1]>
 + q*|[2], [2]> + q^2*|[2], [1, 1]> + q^2*|[1, 1, 1, 1], []>
 + q^2*|[1, 1], [2]> + q^3*|[1, 1], [1, 1]> + q^2*|[1], [2, 1]>
 + q^2*|[], [4]> + q^3*|[], [3, 1]> + q^3*|[], [2, 1, 1]>
 + q^4*|[], [1, 1, 1, 1]>
```

options = Current options for FockSpace - display: ket

a_realization()

Return a realization of `self`.

EXAMPLES:

```
sage: FS = FockSpace(2)
sage: FS.a_realization()
Fock space of rank 2 of multicharge (0,) over
 Fraction Field of Univariate Polynomial Ring in q over Integer Ring
 in the natural basis
```

approximation

alias of *A*

canonical

alias of *G*

highest_weight_vector()

Return the module generator of `self` in the natural basis.

EXAMPLES:

```
sage: FS = FockSpace(2)
sage: FS.highest_weight_vector()
|>
sage: FS = FockSpace(4, [2, 0, 1])
sage: FS.highest_weight_vector()
|[], [], []>
```

inject_shorthands(*verbose=True*)

Import standard shorthands into the global namespace.

INPUT:

- `verbose` – boolean (default `True`) if `True`, prints the defined shorthands

EXAMPLES:

```
sage: FS = FockSpace(4)
sage: FS.inject_shorthands()
Injecting A as shorthand for Fock space of rank 4
 of multicharge (0,) over Fraction Field
 of Univariate Polynomial Ring in q over Integer Ring
 in the approximation basis
Injecting F as shorthand for Fock space of rank 4
 of multicharge (0,) over Fraction Field
 of Univariate Polynomial Ring in q over Integer Ring
 in the natural basis
Injecting G as shorthand for Fock space of rank 4
 of multicharge (0,) over Fraction Field
 of Univariate Polynomial Ring in q over Integer Ring
 in the lower global crystal basis
```

lower_global_crystal

alias of *G*

multicharge()

Return the multicharge of `self`.

EXAMPLES:

```
sage: F = FockSpace(2)
sage: F.multicharge()
(0,)

sage: F = FockSpace(4, [2, 0, 1])
sage: F.multicharge()
(2, 0, 1)
```

natural

alias of *F*

options = Current options for FockSpace - display: ket

q()

Return the parameter q of `self`.

EXAMPLES:

```
sage: F = FockSpace(2)
sage: F.q()
q

sage: F = FockSpace(2, q=-1)
sage: F.q()
-1
```

class sage.algebras.quantum_groups.fock_space.**FockSpaceBases**(*base*)

Bases: `Category_realization_of_parent`

The category of bases of a (truncated) Fock space.

class ParentMethods

Bases: object

highest_weight_vector()

Return the highest weight vector of self.

EXAMPLES:

```
sage: FS = FockSpace(2)
sage: F = FS.natural()
sage: F.highest_weight_vector()
|>
sage: A = FS.A()
sage: A.highest_weight_vector()
A[]
sage: G = FS.G()
sage: G.highest_weight_vector()
G[]
```

multicharge()

Return the multicharge of self.

EXAMPLES:

```
sage: FS = FockSpace(4)
sage: A = FS.A()
sage: A.multicharge()
(0,)

sage: FS = FockSpace(4, [1,0,2])
sage: G = FS.G()
sage: G.multicharge()
(1, 0, 2)
```

q()

Return the parameter q of self.

EXAMPLES:

```
sage: FS = FockSpace(2)
sage: A = FS.A()
sage: A.q()
q

sage: FS = FockSpace(2, q=-1)
sage: G = FS.G()
sage: G.q()
-1
```

some_elements()

Return some elements of self.

EXAMPLES:

```
sage: F = FockSpace(3).natural()
sage: F.some_elements()[::13]
[3*|2> + 2*|1> + 2*|>,
 |5>,
 |3, 1, 1, 1>,
 |3, 2, 2>,
 |5, 1, 1, 1>,
 |2, 2, 1, 1, 1, 1>,
 |5, 2, 1, 1>,
 |3, 2, 1, 1, 1, 1>]

sage: F = FockSpace(3, [0,1]).natural()
sage: F.some_elements()[::13]
[2*|[1], []> + 4*|[], [1]> + |[], []>,
 |[1, 1], [1]>,
 |[1, 1, 1], [1]>,
 |[5], []>,
 |[3], [1, 1]>,
 |[1], [2, 2]>,
 |[4, 1, 1], []>,
 |[2, 1, 1, 1], [1]>]
```

super_categories()

The super categories of `self`.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.fock_space import FockSpaceBases
sage: F = FockSpace(2)
sage: bases = FockSpaceBases(F)
sage: bases.super_categories()
[Category of vector spaces with basis over Fraction Field
  of Univariate Polynomial Ring in q over Integer Ring,
 Category of realizations of Fock space of rank 2 of multicharge (0,)
  over Fraction Field of Univariate Polynomial Ring in q over Integer Ring]
```

`sage.algebras.quantum_groups.fock_space.`**`FockSpaceOptions`**`(*get_value, **set_value)`

Sets and displays the global options for elements of the Fock space classes. If no parameters are set, then the function returns a copy of the options dictionary.

The `options` to Fock space can be accessed as the method *`FockSpaceOptions`* of *`FockSpace`* and related parent classes.

OPTIONS:

- `display` – (default: `ket`) Specifies how terms of the natural basis of Fock space should be printed
  - `ket` – displayed as a ket in bra-ket notation
  - `list` – displayed as a list

EXAMPLES:

```
sage: FS = FockSpace(4)
sage: F = FS.natural()
sage: x = F.an_element()
```

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```
sage: y = x.f(3,2,2,0,1)
sage: y
((3*q^2+3)/q)*|3, 3, 1> + (3*q^2+3)*|3, 2, 1, 1>
sage: Partitions.options.display = 'diagram'
sage: y
((3*q^2+3)/q)*|3, 3, 1> + (3*q^2+3)*|3, 2, 1, 1>
sage: ascii_art(y)
((3*q^2+3)/q)*|***\  + (3*q^2+3)*|***\
              |*** >             |**  \
              |*  /              |*   /
                                 |*  /
sage: FockSpace.options.display = 'list'
sage: ascii_art(y)
((3*q^2+3)/q)*F    + (3*q^2+3)*F
               ***              ***
               ***              **
               *                *
                                *
sage: Partitions.options.display = 'compact_high'
sage: y
((3*q^2+3)/q)*F3^2,1 + (3*q^2+3)*F3,2,1^2

sage: Partitions.options._reset()
sage: FockSpace.options._reset()
```

See GlobalOptions for more features of these options.

class sage.algebras.quantum_groups.fock_space.**FockSpaceTruncated**(*n*, *k*, *q*, *base_ring*)

Bases: *FockSpace*

This is the Fock space given by partitions of length no more than k.

This can be formed as the quotient $\mathcal{F}/\mathcal{F}_k$, where $\mathcal{F}_k$ is the submodule spanned by all diagrams of length (strictly) more than k.

We have three bases:

- The natural basis indexed by truncated n-regular partitions: *F*.
- The approximation basis that comes from LLT(-type) algorithms: *A*.
- The lower global crystal basis: *G*.

See also:

FockSpace

EXAMPLES:

```
sage: F = FockSpace(2, truncated=2)
sage: mg = F.highest_weight_vector()
sage: mg.f(0)
|1>
sage: mg.f(0).f(1)
|2> + q*|1, 1>
sage: mg.f(0).f(1).f(0)
|3>
```

Compare this to the full Fock space:

```
sage: F = FockSpace(2)
sage: mg = F.highest_weight_vector()
sage: mg.f(0).f(1).f(0)
|3> + q*|1, 1, 1>
```

REFERENCES:

- [GW1999]

class A(*F*, *algorithm='GW'*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The A basis of the Fock space, which is the approximation basis of the lower global crystal basis.

INPUT:

- `algorithm` – (default `'GW'`) the algorithm to use when computing this basis in the Fock space; the possible values are:
  - `'GW'` – use the algorithm given by Goodman and Wenzl in [GW1999]
  - `'LLT'` – use the LLT algorithm given in [LLT1996]

Note: The bases produced by the two algorithms are not the same in general.

EXAMPLES:

```
sage: FS = FockSpace(5, truncated=4)
sage: F = FS.natural()
sage: A = FS.A()
```

We demonstrate that they are different bases, but both algorithms still compute the basis G:

```
sage: A2 = FS.A('LLT')
sage: G = FS.G()
sage: F(A[12,9])
|12, 9> + q*|12, 4, 4, 1> + q*|8, 8, 5> + (q^2+1)*|8, 8, 4, 1>
sage: F(A2[12,9])
|12, 9> + q*|12, 4, 4, 1> + q*|8, 8, 5> + (q^2+2)*|8, 8, 4, 1>
sage: G._G_to_fock_basis(Partition([12,9]), 'GW')
|12, 9> + q*|12, 4, 4, 1> + q*|8, 8, 5> + q^2*|8, 8, 4, 1>
sage: G._G_to_fock_basis(Partition([12,9]), 'LLT')
|12, 9> + q*|12, 4, 4, 1> + q*|8, 8, 5> + q^2*|8, 8, 4, 1>
```

options = Current options for FockSpace - display: ket

class F(*F*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The natural basis of the truncated Fock space.

This is the natural basis of the full Fock space projected onto the truncated Fock space. It inherits the $U_q(\widehat{\widehat{sl}}_n)$-action from the action on the full Fock space.

EXAMPLES:

```
sage: FS = FockSpace(4)
sage: F = FS.natural()
sage: FS3 = FockSpace(4, truncated=3)
sage: F3 = FS3.natural()
sage: u = F.highest_weight_vector()
sage: u3 = F3.highest_weight_vector()

sage: u3.f(0,3,2,1)
|2, 1, 1>
sage: u.f(0,3,2,1)
|2, 1, 1> + q*|1, 1, 1, 1>

sage: u.f(0,3,2,1,1)
((q^2+1)/q)*|2, 1, 1, 1>
sage: u3.f(0,3,2,1,1)
0
```

class Element

Bases: *Element*

An element in the truncated Fock space.

options = Current options for FockSpace - display: ket

class G(*F*)

Bases: CombinatorialFreeModule, BindableClass

The lower global crystal basis living inside of a truncated Fock space.

EXAMPLES:

```
sage: FS = FockSpace(4, truncated=2)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[3,1])
|3, 1>
sage: F(G[6,2])
|6, 2> + q*|5, 3>
sage: F(G[14])
|14> + q*|11, 3>

sage: FS = FockSpace(3, truncated=4)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[4,1])
|4, 1> + q*|3, 2>
sage: F(G[4,2,2])
|4, 2, 2> + q*|3, 2, 2, 1>
```

We check against the tables in [LLT1996] (after truncating):

```
sage: FS = FockSpace(3, truncated=3)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[10])
```

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```
|10> + q*|8, 2> + q*|7, 2, 1>
sage: F(G[6,4])
|6, 4> + q*|6, 2, 2> + q^2*|4, 4, 2>
sage: F(G[5,5])
|5, 5> + q*|4, 3, 3>

sage: FS = FockSpace(4, truncated=3)
sage: F = FS.natural()
sage: G = FS.G()
sage: F(G[3,3,1])
|3, 3, 1>
sage: F(G[3,2,2])
|3, 2, 2>
sage: F(G[7])
|7> + q*|3, 3, 1>
```

options = Current options for FockSpace - display: ket

approximation

alias of *A*

canonical

alias of *G*

lower_global_crystal

alias of *G*

natural

alias of *F*

2.3 q-Numbers

Note: These are the quantum group q-analogs, not the usual q-analogs typically used in combinatorics (see `sage.combinat.q_analogues`).

`sage.algebras.quantum_groups.q_numbers.`**`q_binomial`**`(n, k, q=None)`

Return the q-binomial coefficient.

Let $[n]_q!$ denote the q-factorial of n given by *sage.algebras.quantum_groups.q_numbers.q_factorial()*. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! \cdot [k]_q!}.$$

INPUT:

- `n, k` – the nonnegative integers n and k defined above
- `q` – (default: $q \in \mathbf{Z}[q, q^{-1}]$) the parameter q (should be invertible)

If `q` is unspecified, then it is taken to be the generator q for a Laurent polynomial ring over the integers.

Note: This is not the "usual" q-binomial but a variant useful for quantum groups. For the version used in combinatorics, see sage.combinat.q_analogues.

Warning: This method uses division by q-factorials. If $[k]_q!$ or $[n-k]_q!$ are zero-divisors, or division is not implemented in the ring containing q, then it will not work.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.q_numbers import q_binomial
sage: q_binomial(2, 1)
q^-1 + q
sage: q_binomial(2, 0)
1
sage: q_binomial(4, 1)
q^-3 + q^-1 + q + q^3
sage: q_binomial(4, 3)
q^-3 + q^-1 + q + q^3
```

sage.algebras.quantum_groups.q_numbers.**q_factorial**(*n*, *q=None*)

Return the q-analog of the factorial $n!$.

The q-factorial is defined by:

$$[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q,$$

where $[n]_q$ denotes the q-integer defined in *sage.algebras.quantum_groups.q_numbers.q_int()*.

INPUT:

- n – the nonnegative integer n defined above
- q – (default: $q \in \mathbf{Z}[q, q^{-1}]$) the parameter q (should be invertible)

If q is unspecified, then it defaults to using the generator q for a Laurent polynomial ring over the integers.

Note: This is not the "usual" q-factorial but a variant useful for quantum groups. For the version used in combinatorics, see sage.combinat.q_analogues.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.q_numbers import q_factorial
sage: q_factorial(3)
q^-3 + 2*q^-1 + 2*q + q^3
sage: p = LaurentPolynomialRing(QQ, 'q').gen()
sage: q_factorial(3, p)
q^-3 + 2*q^-1 + 2*q + q^3
sage: p = ZZ['p'].gen()
sage: q_factorial(3, p)
(p^6 + 2*p^4 + 2*p^2 + 1)/p^3
```

The q-analog of $n!$ is only defined for n a nonnegative integer (github issue #11411):

```
sage: q_factorial(-2)
Traceback (most recent call last):
...
ValueError: argument (-2) must be a nonnegative integer
```

sage.algebras.quantum_groups.q_numbers.**q_int**(*n*, *q=None*)

Return the q-analog of the nonnegative integer n.

The q-analog of the nonnegative integer n is given by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$

INPUT:

- n – the nonnegative integer n defined above
- q – (default: $q \in \mathbf{Z}[q, q^{-1}]$) the parameter q (should be invertible)

If q is unspecified, then it defaults to using the generator q for a Laurent polynomial ring over the integers.

Note: This is not the "usual" q-analog of n (or q-integer) but a variant useful for quantum groups. For the version used in combinatorics, see sage.combinat.q_analogues.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.q_numbers import q_int
sage: q_int(2)
q^-1 + q
sage: q_int(3)
q^-2 + 1 + q^2
sage: q_int(5)
q^-4 + q^-2 + 1 + q^2 + q^4
sage: q_int(5, 1)
5
```

2.4 Quantum Group Representations

AUTHORS:

- Travis Scrimshaw (2018): initial version

class sage.algebras.quantum_groups.representations.**AdjointRepresentation**(*R*, *C*, *q*)

Bases: *CyclicRepresentation*

An (generalized) adjoint representation of a quantum group.

We define an *(generalized) adjoint representation* V of a quantum group U_q to be a cyclic U_q-module with a weight space decomposition $V = \bigoplus_\mu V_\mu$ such that $\dim V_\mu \leq 1$ unless $\mu = 0$. Moreover, we require that there exists a basis $\{y_j | j \in J\}$ for V_0 such that $e_i y_j = 0$ for all $j \neq i \in I$.

For a base ring R, we construct an adjoint representation from its (combinatorial) crystal B by $V = R\{v_b | b \in B\}$

with

$$\begin{aligned}
e_i v_b &= \begin{cases} v_{e_i b}/[\varphi_i(e_i b)]_{q_i}, & \text{if } \operatorname{wt}(b) \neq 0, \\ v_{e_i b} + \sum_{j \neq i} [-A_{ij}]_{q_i}/[2]_{q_i} v_{y_j} & \text{otherwise} \end{cases} \\
f_i v_b &= \begin{cases} v_{f_i b}/[\varepsilon_i(f_i b)]_{q_i}, & \text{if } \operatorname{wt}(b) \neq 0, \\ v_{f_i b} + \sum_{j \neq i} [-A_{ij}]_{q_i}/[2]_{q_i} v_{y_j} & \text{otherwise} \end{cases} \\
K_i v_b &= q^{\langle h_i, \operatorname{wt}(b) \rangle} v_b,
\end{aligned}$$

where $(A_{ij})_{i,j \in I}$ is the Cartan matrix, and we consider $v_0 := 0$.

INPUT:

- `C` – the crystal corresponding to the representation
- `R` – the base ring
- `q` – (default: the generator of `R`) the parameter q of the quantum group

Warning: This assumes that q is generic.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import AdjointRepresentation
sage: R = ZZ['q'].fraction_field()
sage: C = crystals.Tableaux(['D',4], shape=[1,1])
sage: V = AdjointRepresentation(R, C)
sage: V
V((1, 1, 0, 0))
sage: v = V.an_element(); v
2*B[[[1], [2]]] + 2*B[[[1], [3]]] + 3*B[[[2], [3]]]
sage: v.e(2)
2*B[[[1], [2]]]
sage: v.f(2)
2*B[[[1], [3]]]
sage: v.f(4)
2*B[[[1], [-4]]] + 3*B[[[2], [-4]]]
sage: v.K(3)
2*B[[[1], [2]]] + 2*q*B[[[1], [3]]] + 3*q*B[[[2], [3]]]
sage: v.K(2,-2)
2/q^2*B[[[1], [2]]] + 2*q^2*B[[[1], [3]]] + 3*B[[[2], [3]]]

sage: La = RootSystem(['F',4,1]).weight_space().fundamental_weights()
sage: K = crystals.ProjectedLevelZeroLSPaths(La[4])
sage: A = AdjointRepresentation(R, K)
sage: A
V(-Lambda[0] + Lambda[4])
```

Sort the summands uniformly in Python 2 and Python 3:

```
sage: A.print_options(sorting_key=lambda x: str(x))
sage: v = A.an_element(); v
2*B[(-Lambda[0] + Lambda[3] - Lambda[4],)]
 + 2*B[(-Lambda[0] + Lambda[4],)]
```

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```
 + 3*B[(Lambda[0] - Lambda[1] + Lambda[4],)]
sage: v.e(0)
2*B[(Lambda[0] - Lambda[1] + Lambda[3] - Lambda[4],)]
 + 2*B[(Lambda[0] - Lambda[1] + Lambda[4],)]
sage: v.f(0)
3*B[(-Lambda[0] + Lambda[4],)]
```

REFERENCES:

- [OS2018]

e_on_basis(i, b)

Return the action of e_i on the basis element indexed by b.

INPUT:

- i – an element of the index set
- b – an element of basis keys

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: K = crystals.KirillovReshetikhin(['D',3,2], 1,1)
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, K)
sage: mg0 = K.module_generators[0]; mg0
[]
sage: mg1 = K.module_generators[1]; mg1
[[1]]
sage: V.e_on_basis(0, mg0)
((q^2+1)/q)*B[[[-1]]]
sage: V.e_on_basis(0, mg1)
B[[]]
sage: V.e_on_basis(1, mg0)
0
sage: V.e_on_basis(1, mg1)
0
sage: V.e_on_basis(2, mg0)
0
sage: V.e_on_basis(2, mg1)
0

sage: K = crystals.KirillovReshetikhin(['D',4,3], 1,1)
sage: V = AdjointRepresentation(R, K)
sage: V.e_on_basis(0, K.module_generator())
B[[]] + (q/(q^2+1))*B[[[0]]]
```

f_on_basis(i, b)

Return the action of f_i on the basis element indexed by b.

INPUT:

- i – an element of the index set
- b – an element of basis keys

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: K = crystals.KirillovReshetikhin(['D',3,2], 1,1)
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, K)
sage: mg0 = K.module_generators[0]; mg0
[]
sage: mg1 = K.module_generators[1]; mg1
[[1]]
sage: V.f_on_basis(0, mg0)
((q^2+1)/q)*B[[[1]]]
sage: V.f_on_basis(0, mg1)
0
sage: V.f_on_basis(1, mg0)
0
sage: V.f_on_basis(1, mg1)
B[[[2]]]
sage: V.f_on_basis(2, mg0)
0
sage: V.f_on_basis(2, mg1)
0

sage: K = crystals.KirillovReshetikhin(['D',4,3], 1,1)
sage: V = AdjointRepresentation(R, K)
sage: lw = K.module_generator().to_lowest_weight([1,2])[0]
sage: V.f_on_basis(0, lw)
B[[]] + (q/(q^2+1))*B[[[0]]]
```

class sage.algebras.quantum_groups.representations.**CyclicRepresentation**(*R*, *C*, *q*)

Bases: *QuantumGroupRepresentation*

A cyclic quantum group representation that is indexed by either a highest weight crystal or Kirillov-Reshetikhin crystal.

The crystal `C` must either allow `C.module_generator()`, otherwise it is assumed to be generated by `C.module_generators[0]`.

This is meant as an abstract base class for `AdjointRepresentation` and `MinusculeRepresentation`.

module_generator()

Return the module generator of `self`.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: C = crystals.Tableaux(['G',2], shape=[1,1])
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, C)
sage: V.module_generator()
B[[[1], [2]]]

sage: K = crystals.KirillovReshetikhin(['D',4,2], 1,1)
sage: A = AdjointRepresentation(R, K)
```

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```
sage: A.module_generator()
B[[[1]]]
```

class sage.algebras.quantum_groups.representations.**MinusculeRepresentation**(R, C, q)

Bases: *CyclicRepresentation*

A minuscule representation of a quantum group.

A quantum group representation V is *minuscule* if it is cyclic, there is a weight space decomposition $V = \bigoplus_\mu V_\mu$ with $\dim V_\mu \leq 1$, and $e_i^2 V = 0$ and $f_i^2 V = 0$.

For a base ring R, we construct a minuscule representation from its (combinatorial) crystal B by $V = R\{v_b | b \in B\}$ with $e_i v_b = v_{e_i b}$, $f_i v_b = v_{f_i b}$, and $K_i v_b = q^{\langle h_i, \mathrm{wt}(b) \rangle} v_b$, where we consider $v_0 := 0$.

INPUT:

- C – the crystal corresponding to the representation
- R – the base ring
- q – (default: the generator of R) the parameter q of the quantum group

Warning: This assumes that q is generic.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: R = ZZ['q'].fraction_field()
sage: C = crystals.Tableaux(['B',3], shape=[1/2,1/2,1/2])
sage: V = MinusculeRepresentation(R, C)
sage: V
V((1/2, 1/2, 1/2))
sage: v = V.an_element(); v
2*B[[+++, []]] + 2*B[[++-, []]] + 3*B[[+-+, []]]
sage: v.e(3)
2*B[[+++, []]]
sage: v.f(1)
3*B[[-++, []]]
sage: v.f(3)
2*B[[++-, []]] + 3*B[[+--, []]]
sage: v.K(2)
2*B[[+++, []]] + 2*q^2*B[[++-, []]] + 3/q^2*B[[+-+, []]]
sage: v.K(3, -2)
2/q^2*B[[+++, []]] + 2*q^2*B[[++-, []]] + 3/q^2*B[[+-+, []]]

sage: K = crystals.KirillovReshetikhin(['D',4,2], 3,1)
sage: A = MinusculeRepresentation(R, K)
sage: A
V(-Lambda[0] + Lambda[3])
sage: v = A.an_element(); v
2*B[[+++, []]] + 2*B[[++-, []]] + 3*B[[+-+, []]]
sage: v.f(0)
0
```

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```
sage: v.e(0)
2*B[[-++, []]] + 2*B[[-+-, []]] + 3*B[[--+, []]]
```

REFERENCES:

- [OS2018]

e_on_basis(*i*, *b*)

Return the action of e_i on the basis element indexed by b.

INPUT:

- i – an element of the index set
- b – an element of basis keys

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['A',3], shape=[1,1])
sage: R = ZZ['q'].fraction_field()
sage: V = MinusculeRepresentation(R, C)
sage: lw = C.lowest_weight_vectors()[0]
sage: V.e_on_basis(1, lw)
0
sage: V.e_on_basis(2, lw)
B[[[2], [4]]]
sage: V.e_on_basis(3, lw)
0
sage: hw = C.highest_weight_vectors()[0]
sage: all(V.e_on_basis(i, hw) == V.zero() for i in V.index_set())
True
```

f_on_basis(*i*, *b*)

Return the action of f_i on the basis element indexed by b.

INPUT:

- i – an element of the index set
- b – an element of basis keys

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['A',3], shape=[1,1])
sage: R = ZZ['q'].fraction_field()
sage: V = MinusculeRepresentation(R, C)
sage: hw = C.highest_weight_vectors()[0]
sage: V.f_on_basis(1, hw)
0
sage: V.f_on_basis(2, hw)
B[[[1], [3]]]
sage: V.f_on_basis(3, hw)
0
```

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```
sage: lw = C.lowest_weight_vectors()[0]
sage: all(V.f_on_basis(i, lw) == V.zero() for i in V.index_set())
True
```

class sage.algebras.quantum_groups.representations.**QuantumGroupRepresentation**(*R*, *C*, *q*)

Bases: CombinatorialFreeModule

A representation of a quantum group whose basis is indexed by the corresponding (combinatorial) crystal.

INPUT:

- C – the crystal corresponding to the representation
- R – the base ring
- q – (default: the generator of R) the parameter q of the quantum group

K_on_basis(*i*, *b*, *power=1*)

Return the action of K_i on the basis element indexed by b to the power power.

INPUT:

- i – an element of the index set
- b – an element of basis keys
- power – (default: 1) the power of K_i

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪MinusculeRepresentation
sage: C = crystals.Tableaux(['A',3], shape=[1,1])
sage: R = ZZ['q'].fraction_field()
sage: V = MinusculeRepresentation(R, C)
sage: [[V.K_on_basis(i, b) for i in V.index_set()] for b in C]
[[B[[[1], [2]]], q*B[[[1], [2]]], B[[[1], [2]]]],
 [q*B[[[1], [3]]], 1/q*B[[[1], [3]]], q*B[[[1], [3]]]],
 [1/q*B[[[2], [3]]], B[[[2], [3]]], q*B[[[2], [3]]]],
 [q*B[[[1], [4]]], B[[[1], [4]]], 1/q*B[[[1], [4]]]],
 [1/q*B[[[2], [4]]], q*B[[[2], [4]]], 1/q*B[[[2], [4]]]],
 [B[[[3], [4]]], 1/q*B[[[3], [4]]], B[[[3], [4]]]]]
sage: [[V.K_on_basis(i, b, -1) for i in V.index_set()] for b in C]
[[B[[[1], [2]]], 1/q*B[[[1], [2]]], B[[[1], [2]]]],
 [1/q*B[[[1], [3]]], q*B[[[1], [3]]], 1/q*B[[[1], [3]]]],
 [q*B[[[2], [3]]], B[[[2], [3]]], 1/q*B[[[2], [3]]]],
 [1/q*B[[[1], [4]]], B[[[1], [4]]], q*B[[[1], [4]]]],
 [q*B[[[2], [4]]], 1/q*B[[[2], [4]]], q*B[[[2], [4]]]],
 [B[[[3], [4]]], q*B[[[3], [4]]], B[[[3], [4]]]]]
```

cartan_type()

Return the Cartan type of self.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.representations import␣
↪AdjointRepresentation
sage: C = crystals.Tableaux(['C',3], shape=[1])
```

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```
sage: R = ZZ['q'].fraction_field()
sage: V = AdjointRepresentation(R, C)
sage: V.cartan_type()
['C', 3]
```

CHAPTER

THREE

FREE ASSOCIATIVE ALGEBRAS AND QUOTIENTS

3.1 Free algebras

AUTHORS:

- David Kohel (2005-09)
- William Stein (2006-11-01): add all doctests; implemented many things.
- Simon King (2011-04): Put free algebras into the category framework. Reimplement free algebra constructor, using a `UniqueFactory` for handling different implementations of free algebras. Allow degree weights for free algebras in letterplace implementation.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: F.base_ring()
Integer Ring
sage: G = FreeAlgebra(F, 2, 'm,n'); G
Free Algebra on 2 generators (m, n) over Free Algebra on 3 generators (x, y, z) over␣
↪Integer Ring
sage: G.base_ring()
Free Algebra on 3 generators (x, y, z) over Integer Ring
```

The above free algebra is based on a generic implementation. By github issue #7797, there is a different implementation *FreeAlgebra_letterplace* based on Singular's letterplace rings. It is currently restricted to weighted homogeneous elements and is therefore not the default. But the arithmetic is much faster than in the generic implementation. Moreover, we can compute Groebner bases with degree bound for its two-sided ideals, and thus provide ideal containment tests:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: F
Free Associative Unital Algebra on 3 generators (x, y, z) over Rational Field
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: I.groebner_basis(degbound=4)
Twosided Ideal (x*y + y*z,
    x*x - y*x - y*y - y*z,
    y*y*y - y*y*z + y*z*y - y*z*z,
    y*y*x + y*y*z + y*z*x + y*z*z,
    y*y*z*y - y*y*z*z + y*z*z*y - y*z*z*z,
    y*z*y*y - y*z*y*z + y*z*z*y - y*z*z*z,
    y*y*z*x + y*y*z*z + y*z*z*x + y*z*z*z,
    y*z*y*x + y*z*y*z + y*z*z*x + y*z*z*z) of Free Associative Unital
```

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```
    Algebra on 3 generators (x, y, z) over Rational Field
sage: y*z*y*y*z*z + 2*y*z*y*z*z*x + y*z*y*z*z*z - y*z*z*y*z*x + y*z*z*z*z*x in I
True
```

Positive integral degree weights for the letterplace implementation was introduced in github issue #7797:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: x.degree()
2
sage: y.degree()
1
sage: z.degree()
3
sage: I = F*[x*y-y*x, x^2+2*y*z, (x*y)^2-z^2]*F
sage: Q.<a,b,c> = F.quo(I)
sage: TestSuite(Q).run()
sage: a^2*b^2
c*c
```

class sage.algebras.free_algebra.**FreeAlgebraFactory**

Bases: `UniqueFactory`

A constructor of free algebras.

See *free_algebra* for examples and corner cases.

EXAMPLES:

```
sage: FreeAlgebra(GF(5),3,'x')
Free Algebra on 3 generators (x0, x1, x2) over Finite Field of size 5
sage: F.<x,y,z> = FreeAlgebra(GF(5),3)
sage: (x+y+z)^2
x^2 + x*y + x*z + y*x + y^2 + y*z + z*x + z*y + z^2
sage: FreeAlgebra(GF(5),3, 'xx, zba, Y')
Free Algebra on 3 generators (xx, zba, Y) over Finite Field of size 5
sage: FreeAlgebra(GF(5),3, 'abc')
Free Algebra on 3 generators (a, b, c) over Finite Field of size 5
sage: FreeAlgebra(GF(5),1, 'z')
Free Algebra on 1 generators (z,) over Finite Field of size 5
sage: FreeAlgebra(GF(5),1, ['alpha'])
Free Algebra on 1 generators (alpha,) over Finite Field of size 5
sage: FreeAlgebra(FreeAlgebra(ZZ,1,'a'), 2, 'x')
Free Algebra on 2 generators (x0, x1) over Free Algebra on 1 generators (a,) over␣
↪Integer Ring
```

Free algebras are globally unique:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: G = FreeAlgebra(ZZ,3,'x,y,z')
sage: F is G
True
sage: F.<x,y,z> = FreeAlgebra(GF(5),3)  # indirect doctest
sage: F is loads(dumps(F))
True
```

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```
sage: F is FreeAlgebra(GF(5),['x','y','z'])
True
sage: copy(F) is F is loads(dumps(F))
True
sage: TestSuite(F).run()
```

By github issue #7797, we provide a different implementation of free algebras, based on Singular's "letterplace rings". Our letterplace wrapper allows for choosing positive integral degree weights for the generators of the free algebra. However, only (weighted) homogeneous elements are supported. Of course, isomorphic algebras in different implementations are not identical:

```
sage: G = FreeAlgebra(GF(5),['x','y','z'], implementation='letterplace')
sage: F == G
False
sage: G is FreeAlgebra(GF(5),['x','y','z'], implementation='letterplace')
True
sage: copy(G) is G is loads(dumps(G))
True
sage: TestSuite(G).run()
```

```
sage: H = FreeAlgebra(GF(5),['x','y','z'], implementation='letterplace', degrees=[1,
↪2,3])
sage: F != H != G
True
sage: H is FreeAlgebra(GF(5),['x','y','z'], implementation='letterplace',␣
↪degrees=[1,2,3])
True
sage: copy(H) is H is loads(dumps(H))
True
sage: TestSuite(H).run()
```

Free algebras commute with their base ring.

```
sage: K.<a,b> = FreeAlgebra(QQ,2)
sage: K.is_commutative()
False
sage: L.<c> = FreeAlgebra(K,1)
sage: L.is_commutative()
False
sage: s = a*b^2 * c^3; s
a*b^2*c^3
sage: parent(s)
Free Algebra on 1 generators (c,) over Free Algebra on 2 generators (a, b) over␣
↪Rational Field
sage: c^3 * a * b^2
a*b^2*c^3
```

create_key(*base_ring*, *arg1=None*, *arg2=None*, *sparse=None*, *order=None*, *names=None*, *name=None*, *implementation=None*, *degrees=None*)

Create the key under which a free algebra is stored.

create_object(*version*, *key*)

Construct the free algebra that belongs to a unique key.

NOTE:

Of course, that method should not be called directly, since it does not use the cache of free algebras.

class sage.algebras.free_algebra.**FreeAlgebra_generic**(*R*, *n*, *names*)

Bases: CombinatorialFreeModule, Algebra

The free algebra on n generators over a base ring.

INPUT:

- R – a ring
- n – an integer
- names – the generator names

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, 3); F
Free Algebra on 3 generators (x, y, z) over Rational Field
sage: mul(F.gens())
x*y*z
sage: mul([ F.gen(i%3) for i in range(12) ])
x*y*z*x*y*z*x*y*z*x*y*z
sage: mul([ F.gen(i%3) for i in range(12) ]) + mul([ F.gen(i%2) for i in range(12)␣
↪])
x*y*x*y*x*y*x*y*x*y*x*y + x*y*z*x*y*z*x*y*z*x*y*z
sage: (2 + x*z + x^2)^2 + (x - y)^2
4 + 5*x^2 - x*y + 4*x*z - y*x + y^2 + x^4 + x^3*z + x*z*x^2 + x*z*x*z
```

Element

alias of *FreeAlgebraElement*

algebra_generators()

Return the algebra generators of self.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: F.algebra_generators()
Finite family {'x': x, 'y': y, 'z': z}
```

g_algebra(*relations*, *names=None*, *order='degrevlex'*, *check=True*)

The G-Algebra derived from this algebra by relations.

By default is assumed, that two variables commute.

Todo:

- Coercion doesn't work yet, there is some cheating about assumptions
- The optional argument check controls checking the degeneracy conditions. Furthermore, the default values interfere with non-degeneracy conditions.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ,3)
sage: G = A.g_algebra({y*x: -x*y})
sage: (x,y,z) = G.gens()
sage: x*y
x*y
sage: y*x
-x*y
sage: z*x
x*z
sage: (x,y,z) = A.gens()
sage: G = A.g_algebra({y*x: -x*y+1})
sage: (x,y,z) = G.gens()
sage: y*x
-x*y + 1
sage: (x,y,z) = A.gens()
sage: G = A.g_algebra({y*x: -x*y+z})
sage: (x,y,z) = G.gens()
sage: y*x
-x*y + z
```

gen(*i*)

The `i`-th generator of the algebra.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: F.gen(0)
x
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: F.gens()
(x, y, z)
```

is_commutative()

Return True if this free algebra is commutative.

EXAMPLES:

```
sage: R.<x> = FreeAlgebra(QQ,1)
sage: R.is_commutative()
True
sage: R.<x,y> = FreeAlgebra(QQ,2)
sage: R.is_commutative()
False
```

is_field(*proof=True*)

Return True if this Free Algebra is a field, which is only if the base ring is a field and there are no generators

EXAMPLES:

```
sage: A = FreeAlgebra(QQ,0,'')
sage: A.is_field()
True
sage: A = FreeAlgebra(QQ,1,'x')
sage: A.is_field()
False
```

lie_polynomial(*w*)

Return the Lie polynomial associated to the Lyndon word `w`. If `w` is not Lyndon, then return the product of Lie polynomials of the Lyndon factorization of `w`.

Given a Lyndon word w, the Lie polynomial L_w is defined recursively by $L_w = [L_u, L_v]$, where $w = uv$ is the `standard factorization` of w, and $L_w = w$ when w is a single letter.

INPUT:

- `w` – a word or an element of the free monoid

EXAMPLES:

```
sage: F = FreeAlgebra(QQ, 3, 'x,y,z')
sage: M.<x,y,z> = FreeMonoid(3)
sage: F.lie_polynomial(x*y)
x*y - y*x
sage: F.lie_polynomial(y*x)
y*x
sage: F.lie_polynomial(x^2*y*x)
x^2*y*x - 2*x*y*x^2 + y*x^3
sage: F.lie_polynomial(y*z*x*z*x*z)
y*z*x*z*x*z - y*z*x*z^2*x - y*z^2*x^2*z + y*z^2*x*z*x
 - z*y*x*z*x*z + z*y*x*z^2*x + z*y*z*x^2*z - z*y*z*x*z*x
```

monoid()

The free monoid of generators of the algebra.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: F.monoid()
Free monoid on 3 generators (x, y, z)
```

ngens()

The number of generators of the algebra.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: F.ngens()
3
```

one_basis()

Return the index of the basis element 1.

EXAMPLES:

```
sage: F = FreeAlgebra(QQ, 2, 'x,y')
sage: F.one_basis()
1
sage: F.one_basis().parent()
Free monoid on 2 generators (x, y)
```

pbw_basis()

Return the Poincaré-Birkhoff-Witt (PBW) basis of `self`.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(QQ, 2)
sage: F.poincare_birkhoff_witt_basis()
The Poincare-Birkhoff-Witt basis of Free Algebra on 2 generators (x, y) over␣
↪Rational Field
```

pbw_element(*elt*)

Return the element `elt` in the Poincaré-Birkhoff-Witt basis.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(QQ, 2)
sage: F.pbw_element(x*y - y*x + 2)
2*PBW[1] + PBW[x*y]
sage: F.pbw_element(F.one())
PBW[1]
sage: F.pbw_element(x*y*x + x^3*y)
PBW[x*y]*PBW[x] + PBW[y]*PBW[x]^2 + PBW[x^3*y]
 + 3*PBW[x^2*y]*PBW[x] + 3*PBW[x*y]*PBW[x]^2 + PBW[y]*PBW[x]^3
```

poincare_birkhoff_witt_basis()

Return the Poincaré-Birkhoff-Witt (PBW) basis of `self`.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(QQ, 2)
sage: F.poincare_birkhoff_witt_basis()
The Poincare-Birkhoff-Witt basis of Free Algebra on 2 generators (x, y) over␣
↪Rational Field
```

product_on_basis(*x*, *y*)

Return the product of the basis elements indexed by `x` and `y`.

EXAMPLES:

```
sage: F = FreeAlgebra(ZZ,3,'x,y,z')
sage: I = F.basis().keys()
sage: x,y,z = I.gens()
sage: F.product_on_basis(x*y, z*y)
x*y*z*y
```

quo(*mons*, *mats=None*, *names=None*, ***args*)

Return a quotient algebra.

The quotient algebra is defined via the action of a free algebra A on a (finitely generated) free module. The input for the quotient algebra is a list of monomials (in the underlying monoid for A) which form a free basis for the module of A, and a list of matrices, which give the action of the free generators of A on this monomial basis.

EXAMPLES:

Here is the quaternion algebra defined in terms of three generators:

```
sage: n = 3
sage: A = FreeAlgebra(QQ,n,'i')
sage: F = A.monoid()
sage: i, j, k = F.gens()
sage: mons = [ F(1), i, j, k ]
sage: M = MatrixSpace(QQ,4)
sage: mats = [M([0,1,0,0, -1,0,0,0, 0,0,0,-1, 0,0,1,0]),  M([0,0,1,0, 0,0,0,1, -
↪1,0,0,0, 0,-1,0,0]),  M([0,0,0,1, 0,0,-1,0, 0,1,0,0, -1,0,0,0]) ]
sage: H.<i,j,k> = A.quotient(mons, mats); H
Free algebra quotient on 3 generators ('i', 'j', 'k') and dimension 4 over␣
↪Rational Field
```

quotient(*mons*, *mats=None*, *names=None*, ***args*)

Return a quotient algebra.

The quotient algebra is defined via the action of a free algebra A on a (finitely generated) free module. The input for the quotient algebra is a list of monomials (in the underlying monoid for A) which form a free basis for the module of A, and a list of matrices, which give the action of the free generators of A on this monomial basis.

EXAMPLES:

Here is the quaternion algebra defined in terms of three generators:

```
sage: n = 3
sage: A = FreeAlgebra(QQ,n,'i')
sage: F = A.monoid()
sage: i, j, k = F.gens()
sage: mons = [ F(1), i, j, k ]
sage: M = MatrixSpace(QQ,4)
sage: mats = [M([0,1,0,0, -1,0,0,0, 0,0,0,-1, 0,0,1,0]),  M([0,0,1,0, 0,0,0,1, -
↪1,0,0,0, 0,-1,0,0]),  M([0,0,0,1, 0,0,-1,0, 0,1,0,0, -1,0,0,0]) ]
sage: H.<i,j,k> = A.quotient(mons, mats); H
Free algebra quotient on 3 generators ('i', 'j', 'k') and dimension 4 over␣
↪Rational Field
```

class sage.algebras.free_algebra.**PBWBasisOfFreeAlgebra**(*alg*)

Bases: CombinatorialFreeModule

The Poincaré-Birkhoff-Witt basis of the free algebra.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(QQ, 2)
sage: PBW = F.pbw_basis()
sage: px, py = PBW.gens()
sage: px * py
PBW[x*y] + PBW[y]*PBW[x]
sage: py * px
PBW[y]*PBW[x]
```

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```
sage: px * py^3 * px - 2*px * py
-2*PBW[x*y] - 2*PBW[y]*PBW[x] + PBW[x*y^3]*PBW[x]
 + 3*PBW[y]*PBW[x*y^2]*PBW[x] + 3*PBW[y]^2*PBW[x*y]*PBW[x]
 + PBW[y]^3*PBW[x]^2
```

We can convert between the two bases:

```
sage: p = PBW(x*y - y*x + 2); p
2*PBW[1] + PBW[x*y]
sage: F(p)
2 + x*y - y*x
sage: f = F.pbw_element(x*y*x + x^3*y + x + 3)
sage: F(PBW(f)) == f
True
sage: p = px*py + py^4*px^2
sage: F(p)
x*y + y^4*x^2
sage: PBW(F(p)) == p
True
```

Note that multiplication in the PBW basis agrees with multiplication as monomials:

```
sage: F(px * py^3 * px - 2*px * py) == x*y^3*x - 2*x*y
True
```

We verify Examples 1 and 2 in [MR1989]:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: PBW = F.pbw_basis()
sage: PBW(x*y*z)
PBW[x*y*z] + PBW[x*z*y] + PBW[y]*PBW[x*z] + PBW[y*z]*PBW[x]
 + PBW[z]*PBW[x*y] + PBW[z]*PBW[y]*PBW[x]
sage: PBW(x*y*y*x)
PBW[x*y^2]*PBW[x] + 2*PBW[y]*PBW[x*y]*PBW[x] + PBW[y]^2*PBW[x]^2
```

class Element

Bases: `IndexedFreeModuleElement`

expand()

Expand `self` in the monomials of the free algebra.

EXAMPLES:

```
sage: F = FreeAlgebra(QQ, 2, 'x,y')
sage: PBW = F.pbw_basis()
sage: x,y = F.monoid().gens()
sage: f = PBW(x^2*y) + PBW(x) + PBW(y^4*x)
sage: f.expand()
x + x^2*y - 2*x*y*x + y*x^2 + y^4*x
```

algebra_generators()

Return the generators of `self` as an algebra.

EXAMPLES:

```
sage: PBW = FreeAlgebra(QQ, 2, 'x,y').pbw_basis()
sage: gens = PBW.algebra_generators(); gens
(PBW[x], PBW[y])
sage: all(g.parent() is PBW for g in gens)
True
```

expansion(*t*)

Return the expansion of the element `t` of the Poincaré-Birkhoff-Witt basis in the monomials of the free algebra.

EXAMPLES:

```
sage: F = FreeAlgebra(QQ, 2, 'x,y')
sage: PBW = F.pbw_basis()
sage: x,y = F.monoid().gens()
sage: PBW.expansion(PBW(x*y))
x*y - y*x
sage: PBW.expansion(PBW.one())
1
sage: PBW.expansion(PBW(x*y*x) + 2*PBW(x) + 3)
3 + 2*x + x*y*x - y*x^2
```

free_algebra()

Return the associated free algebra of `self`.

EXAMPLES:

```
sage: PBW = FreeAlgebra(QQ, 2, 'x,y').pbw_basis()
sage: PBW.free_algebra()
Free Algebra on 2 generators (x, y) over Rational Field
```

gen(*i*)

Return the `i`-th generator of `self`.

EXAMPLES:

```
sage: PBW = FreeAlgebra(QQ, 2, 'x,y').pbw_basis()
sage: PBW.gen(0)
PBW[x]
sage: PBW.gen(1)
PBW[y]
```

gens()

Return the generators of `self` as an algebra.

EXAMPLES:

```
sage: PBW = FreeAlgebra(QQ, 2, 'x,y').pbw_basis()
sage: gens = PBW.algebra_generators(); gens
(PBW[x], PBW[y])
sage: all(g.parent() is PBW for g in gens)
True
```

one_basis()

Return the index of the basis element for 1.

EXAMPLES:

```
sage: PBW = FreeAlgebra(QQ, 2, 'x,y').pbw_basis()
sage: PBW.one_basis()
1
sage: PBW.one_basis().parent()
Free monoid on 2 generators (x, y)
```

product(*u*, *v*)

Return the product of two elements u and v.

EXAMPLES:

```
sage: F = FreeAlgebra(QQ, 2, 'x,y')
sage: PBW = F.pbw_basis()
sage: x, y = PBW.gens()
sage: PBW.product(x, y)
PBW[x*y] + PBW[y]*PBW[x]
sage: PBW.product(y, x)
PBW[y]*PBW[x]
sage: PBW.product(y^2*x, x*y*x)
PBW[y]^2*PBW[x^2*y]*PBW[x] + 2*PBW[y]^2*PBW[x*y]*PBW[x]^2 + PBW[y]^3*PBW[x]^3
```

sage.algebras.free_algebra.**is_FreeAlgebra**(*x*)

Return True if x is a free algebra; otherwise, return False.

EXAMPLES:

```
sage: from sage.algebras.free_algebra import is_FreeAlgebra
sage: is_FreeAlgebra(5)
False
sage: is_FreeAlgebra(ZZ)
False
sage: is_FreeAlgebra(FreeAlgebra(ZZ,100,'x'))
True
sage: is_FreeAlgebra(FreeAlgebra(ZZ,10,'x',implementation='letterplace'))
True
sage: is_FreeAlgebra(FreeAlgebra(ZZ,10,'x',implementation='letterplace',␣
↪degrees=list(range(1,11))))
True
```

3.2 Free algebra elements

AUTHORS:

- David Kohel (2005-09)

class sage.algebras.free_algebra_element.**FreeAlgebraElement**(*A*, *x*)

Bases: `IndexedFreeModuleElement`, `AlgebraElement`

A free algebra element.

to_pbw_basis()

Return `self` in the Poincaré-Birkhoff-Witt (PBW) basis.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(ZZ, 3)
sage: p = x^2*y + 3*y*x + 2
sage: p.to_pbw_basis()
2*PBW[1] + 3*PBW[y]*PBW[x] + PBW[x^2*y]
 + 2*PBW[x*y]*PBW[x] + PBW[y]*PBW[x]^2
```

variables()

Return the variables used in `self`.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(ZZ,3)
sage: elt = x + x*y + x^3*y
sage: elt.variables()
[x, y]
sage: elt = x + x^2 - x^4
sage: elt.variables()
[x]
sage: elt = x + z*y + z*x
sage: elt.variables()
[x, y, z]
```

3.3 Free associative unital algebras, implemented via Singular's letterplace rings

AUTHOR:

- Simon King (2011-03-21): github issue #7797

With this implementation, Groebner bases out to a degree bound and normal forms can be computed for twosided weighted homogeneous ideals of free algebras. For now, all computations are restricted to weighted homogeneous elements, i.e., other elements cannot be created by arithmetic operations.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: F
Free Associative Unital Algebra on 3 generators (x, y, z) over Rational Field
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: I
Twosided Ideal (x*y + y*z, x*x + x*y - y*x - y*y) of Free Associative Unital Algebra on␣
↪3 generators (x, y, z) over Rational Field
sage: x*(x*I.0-I.1*y+I.0*y)-I.1*y*z
x*y*x*y + x*y*y*y - x*y*y*z + x*y*z*y + y*x*y*z + y*y*y*z
sage: x^2*I.0-x*I.1*y+x*I.0*y-I.1*y*z in I
True
```

The preceding containment test is based on the computation of Groebner bases with degree bound:

```
sage: I.groebner_basis(degbound=4)
Twosided Ideal (x*y + y*z,
```

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```
    x*x - y*x - y*y - y*z,
    y*y*y - y*y*z + y*z*y - y*z*z,
    y*y*x + y*y*z + y*z*x + y*z*z,
    y*y*z*y - y*y*z*z + y*z*z*y - y*z*z*z,
    y*z*y*y - y*z*y*z + y*z*z*y - y*z*z*z,
    y*y*z*x + y*y*z*z + y*z*z*x + y*z*z*z,
    y*z*y*x + y*z*y*z + y*z*z*x + y*z*z*z) of Free Associative Unital
    Algebra on 3 generators (x, y, z) over Rational Field
```

When reducing an element by I, the original generators are chosen:

```
sage: (y*z*y*y).reduce(I)
y*z*y*y
```

However, there is a method for computing the normal form of an element, which is the same as reduction by the Groebner basis out to the degree of that element:

```
sage: (y*z*y*y).normal_form(I)
y*z*y*z - y*z*z*y + y*z*z*z
sage: (y*z*y*y).reduce(I.groebner_basis(4))
y*z*y*z - y*z*z*y + y*z*z*z
```

The default term order derives from the degree reverse lexicographic order on the commutative version of the free algebra:

```
sage: F.commutative_ring().term_order()
Degree reverse lexicographic term order
```

A different term order can be chosen, and of course may yield a different normal form:

```
sage: L.<a,b,c> = FreeAlgebra(QQ, implementation='letterplace', order='lex')
sage: L.commutative_ring().term_order()
Lexicographic term order
sage: J = L*[a*b+b*c,a^2+a*b-b*c-c^2]*L
sage: J.groebner_basis(4)
Twosided Ideal (2*b*c*b - b*c*c + c*c*b,
    a*b + b*c,
    -a*c*c + 2*b*c*a + 2*b*c*c + c*c*a,
    a*c*c*b - 2*b*c*c*b + b*c*c*c,
    a*a - 2*b*c - c*c,
    a*c*c*a - 2*b*c*c*a - 4*b*c*c*c - c*c*c*c) of Free Associative Unital
    Algebra on 3 generators (a, b, c) over Rational Field
sage: (b*c*b*b).normal_form(J)
1/2*b*c*c*b - 1/2*c*c*b*b
```

Here is an example with degree weights:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[1,2,3])
sage: (x*y+z).degree()
3
```

Todo: The computation of Groebner bases only works for global term orderings, and all elements must be weighted homogeneous with respect to positive integral degree weights. It is ongoing work in Singular to lift these restrictions.

We support coercion from the letterplace wrapper to the corresponding generic implementation of a free algebra (*FreeAlgebra_generic*), but there is no coercion in the opposite direction, since the generic implementation also comprises non-homogeneous elements.

We also do not support coercion from a subalgebra, or between free algebras with different term orderings, yet.

class sage.algebras.letterplace.free_algebra_letterplace.**FreeAlgebra_letterplace**

Bases: Algebra

Finitely generated free algebra, with arithmetic restricted to weighted homogeneous elements.

Note: The restriction to weighted homogeneous elements should be lifted as soon as the restriction to homogeneous elements is lifted in Singular's "Letterplace algebras".

EXAMPLES:

```
sage: K.<z> = GF(25)
sage: F.<a,b,c> = FreeAlgebra(K, implementation='letterplace')
sage: F
Free Associative Unital Algebra on 3 generators (a, b, c) over Finite Field in z of␣
↪size 5^2
sage: P = F.commutative_ring()
sage: P
Multivariate Polynomial Ring in a, b, c over Finite Field in z of size 5^2
```

We can do arithmetic as usual, as long as we stay (weighted) homogeneous:

```
sage: (z*a+(z+1)*b+2*c)^2
(z + 3)*a*a + (2*z + 3)*a*b + (2*z)*a*c + (2*z + 3)*b*a + (3*z + 4)*b*b + (2*z +␣
↪2)*b*c + (2*z)*c*a + (2*z + 2)*c*b - c*c
sage: a+1
Traceback (most recent call last):
...
ArithmeticError: can only add elements of the same weighted degree
```

commutative_ring()

Return the commutative version of this free algebra.

NOTE:

This commutative ring is used as a unique key of the free algebra.

EXAMPLES:

```
sage: K.<z> = GF(25)
sage: F.<a,b,c> = FreeAlgebra(K, implementation='letterplace')
sage: F
Free Associative Unital Algebra on 3 generators (a, b, c) over Finite Field in␣
↪z of size 5^2
sage: F.commutative_ring()
Multivariate Polynomial Ring in a, b, c over Finite Field in z of size 5^2
sage: FreeAlgebra(F.commutative_ring()) is F
True
```

current_ring()

Return the commutative ring that is used to emulate the non-commutative multiplication out to the current degree.

EXAMPLES:

```
sage: F.<a,b,c> = FreeAlgebra(QQ, implementation='letterplace')
sage: F.current_ring()
Multivariate Polynomial Ring in a, b, c over Rational Field
sage: a*b
a*b
sage: F.current_ring()
Multivariate Polynomial Ring in a, b, c, a_1, b_1, c_1 over Rational Field
sage: F.set_degbound(3)
sage: F.current_ring()
Multivariate Polynomial Ring in a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 over␣
↪Rational Field
```

degbound()

Return the degree bound that is currently used.

Note: When multiplying two elements of this free algebra, the degree bound will be dynamically adapted. It can also be set by *set_degbound()*.

EXAMPLES:

In order to avoid we get a free algebras from the cache that was created in another doctest and has a different degree bound, we choose a base ring that does not appear in other tests:

```
sage: F.<x,y,z> = FreeAlgebra(ZZ, implementation='letterplace')
sage: F.degbound()
1
sage: x*y
x*y
sage: F.degbound()
2
sage: F.set_degbound(4)
sage: F.degbound()
4
```

gen(i)

Return the i-th generator.

INPUT:

i – an integer.

OUTPUT:

Generator number i.

EXAMPLES:

```
sage: F.<a,b,c> = FreeAlgebra(QQ, implementation='letterplace')
sage: F.1 is F.1  # indirect doctest
True
```

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```
sage: F.gen(2)
c
```

generator_degrees()

ideal_monoid()

Return the monoid of ideals of this free algebra.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(GF(2), implementation='letterplace')
sage: F.ideal_monoid()
Monoid of ideals of Free Associative Unital Algebra on 2 generators (x, y) over␣
↪Finite Field of size 2
sage: F.ideal_monoid() is F.ideal_monoid()
True
```

is_commutative()

Tell whether this algebra is commutative, i.e., whether the generator number is one.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: F.is_commutative()
False
sage: FreeAlgebra(QQ, implementation='letterplace', names=['x']).is_
↪commutative()
True
```

is_field(*proof=True*)

Tell whether this free algebra is a field.

Note: This would only be the case in the degenerate case of no generators. But such an example cannot be constructed in this implementation.

ngens()

Return the number of generators.

EXAMPLES:

```
sage: F.<a,b,c> = FreeAlgebra(QQ, implementation='letterplace')
sage: F.ngens()
3
```

set_degbound(*d*)

Increase the degree bound that is currently in place.

Note: The degree bound cannot be decreased.

EXAMPLES:

In order to avoid we get a free algebras from the cache that was created in another doctest and has a different degree bound, we choose a base ring that does not appear in other tests:

```
sage: F.<x,y,z> = FreeAlgebra(GF(251), implementation='letterplace')
sage: F.degbound()
1
sage: x*y
x*y
sage: F.degbound()
2
sage: F.set_degbound(4)
sage: F.degbound()
4
sage: F.set_degbound(2)
sage: F.degbound()
4
```

term_order_of_block()

Return the term order that is used for the commutative version of this free algebra.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: F.term_order_of_block()
Degree reverse lexicographic term order
sage: L.<a,b,c> = FreeAlgebra(QQ, implementation='letterplace',order='lex')
sage: L.term_order_of_block()
Lexicographic term order
```

class `sage.algebras.letterplace.free_algebra_letterplace.`**FreeAlgebra_letterplace_libsingular**

Bases: `object`

Internally used wrapper around a Singular Letterplace polynomial ring.

`sage.algebras.letterplace.free_algebra_letterplace.`**freeAlgebra**(*ring=None*, *interruptible=True*, *attributes=None*, **args*)

This function is an automatically generated C wrapper around the Singular function 'freeAlgebra'.

This wrapper takes care of converting Sage datatypes to Singular datatypes and vice versa. In addition to whatever parameters the underlying Singular function accepts when called, this function also accepts the following keyword parameters:

INPUT:

- `args` – a list of arguments
- `ring` – a multivariate polynomial ring
- `interruptible` – if `True` pressing `Ctrl` + `C` during the execution of this function will interrupt the computation (default: `True`)
- `attributes` – a dictionary of optional Singular attributes assigned to Singular objects (default: `None`)

If `ring` is not specified, it is guessed from the given arguments. If this is not possible, then a dummy ring, univariate polynomial ring over `QQ`, is used.

EXAMPLES:

```
sage: groebner = sage.libs.singular.function_factory.ff.groebner
sage: P.<x, y> = PolynomialRing(QQ)
sage: I = P.ideal(x^2-y, y+x)
sage: groebner(I)
[x + y, y^2 - y]
sage: triangL = sage.libs.singular.function_factory.ff.triang__lib.triangL
sage: P.<x1, x2> = PolynomialRing(QQ, order='lex')
sage: f1 = 1/2*((x1^2 + 2*x1 - 4)*x2^2 + 2*(x1^2 + x1)*x2 + x1^2)
sage: f2 = 1/2*((x1^2 + 2*x1 + 1)*x2^2 + 2*(x1^2 + x1)*x2 - 4*x1^2)
sage: I = Ideal(Ideal(f1,f2).groebner_basis()[::-1])
sage: triangL(I, attributes={I:{'isSB':1}})
[[x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5],
 [x2, x1^2],
 [x2, x1^2],
 [x2, x1^2]]
```

The Singular documentation for 'freeAlgebra' is given below.

```
Singular documentation not found
```

3.4 Weighted homogeneous elements of free algebras, in letterplace implementation

AUTHOR:

- Simon King (2011-03-23): Github issue github issue #7797

class sage.algebras.letterplace.free_algebra_element_letterplace.**FreeAlgebraElement_letterplace**

Bases: AlgebraElement

Weighted homogeneous elements of a free associative unital algebra (letterplace implementation)

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: x+y
x + y
sage: x*y !=y*x
True
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: (y^3).reduce(I)
y*y*y
sage: (y^3).normal_form(I)
y*y*z - y*z*y + y*z*z
```

Here is an example with nontrivial degree weights:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: I = F*[x*y-y*x, x^2+2*y*z, (x*y)^2-z^2]*F
sage: x.degree()
2
```

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```
sage: y.degree()
1
sage: z.degree()
3
sage: (x*y)^3
x*y*x*y*x*y
sage: ((x*y)^3).normal_form(I)
z*z*y*x
sage: ((x*y)^3).degree()
9
```

degree()

Return the degree of this element.

Note: Generators may have a positive integral degree weight. All elements must be weighted homogeneous.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: ((x+y+z)^3).degree()
3
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: ((x*y+z)^3).degree()
9
```

lc()

The leading coefficient of this free algebra element, as element of the base ring.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: ((2*x+3*y-4*z)^2*(5*y+6*z)).lc()
20
sage: ((2*x+3*y-4*z)^2*(5*y+6*z)).lc().parent() is F.base()
True
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: ((2*x*y+z)^2).lc()
4
```

letterplace_polynomial()

Return the commutative polynomial that is used internally to represent this free algebra element.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: ((x+y-z)^2).letterplace_polynomial()
x*x_1 + x*y_1 - x*z_1 + y*x_1 + y*y_1 - y*z_1 - z*x_1 - z*y_1 + z*z_1
```

If degree weights are used, the letterplace polynomial is homogenized by slack variables:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: ((x*y+z)^2).letterplace_polynomial()
x*x__1*y_2*x_3*x__4*y_5 + x*x__1*y_2*z_3*x__4*x__5 + z*x__1*x__2*x_3*x__4*y_5 +
→z*x__1*x__2*z_3*x__4*x__5
```

lm()

The leading monomial of this free algebra element.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: ((2*x+3*y-4*z)^2*(5*y+6*z)).lm()
x*x*y
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: ((2*x*y+z)^2).lm()
x*y*x*y
```

lm_divides(p)

Tell whether or not the leading monomial of self divides the leading monomial of another element.

Note: A free algebra element p divides another one q if there are free algebra elements s and t such that $spt = q$.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: ((2*x*y+z)^2*z).lm()
x*y*x*y*z
sage: (y*x*y-y^4).lm()
y*x*y
sage: (y*x*y-y^4).lm_divides((2*x*y+z)^2*z)
True
```

lt()

The leading term (monomial times coefficient) of this free algebra element.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: ((2*x+3*y-4*z)^2*(5*y+6*z)).lt()
20*x*x*y
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[2,1,3])
sage: ((2*x*y+z)^2).lt()
4*x*y*x*y
```

normal_form(I)

Return the normal form of this element with respect to a twosided weighted homogeneous ideal.

INPUT:

A twosided homogeneous ideal I of the parent F of this element, x.

OUTPUT:

The normal form of x wrt. I.

Note: The normal form is computed by reduction with respect to a Groebnerbasis of I with degree bound $deg(x)$.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: (x^5).normal_form(I)
-y*z*z*z*x - y*z*z*z*y - y*z*z*z*z
```

We verify two basic properties of normal forms: The difference of an element and its normal form is contained in the ideal, and if two elements of the free algebra differ by an element of the ideal then they have the same normal form:

```
sage: x^5 - (x^5).normal_form(I) in I
True
sage: (x^5+x*I.0*y*z-3*z^2*I.1*y).normal_form(I) == (x^5).normal_form(I)
True
```

Here is an example with non-trivial degree weights:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace', degrees=[1,2,3])
sage: I = F*[x*y-y*x+z, y^2+2*x*z, (x*y)^2-z^2]*F
sage: ((x*y)^3).normal_form(I)
z*z*y*x - z*z*z
sage: (x*y)^3-((x*y)^3).normal_form(I) in I
True
sage: ((x*y)^3+2*z*I.0*z+y*I.1*z-x*I.2*y).normal_form(I) == ((x*y)^3).normal_
↪form(I)
True
```

reduce(G)

Reduce this element by a list of elements or by a twosided weighted homogeneous ideal.

INPUT:

Either a list or tuple of weighted homogeneous elements of the free algebra, or an ideal of the free algebra, or an ideal in the commutative polynomial ring that is currently used to implement the multiplication in the free algebra.

OUTPUT:

The twosided reduction of this element by the argument.

Note: This may not be the normal form of this element, unless the argument is a twosided Groebner basis up to the degree of this element.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: p = y^2*z*y^2+y*z*y*z*y
```

We compute the letterplace version of the Groebner basis of I with degree bound 4:

```
sage: G = F._reductor_(I.groebner_basis(4).gens(),4)
sage: G.ring() is F.current_ring()
True
```

Since the element p is of degree 5, it is no surprise that its reductions with respect to the original generators of I (of degree 2), or with respect to G (Groebner basis with degree bound 4), or with respect to the Groebner basis with degree bound 5 (which yields its normal form) are pairwise different:

```
sage: p.reduce(I)
y*y*z*y*y + y*z*y*z*y
sage: p.reduce(G)
y*y*z*z*y + y*z*y*z*y - y*z*z*y*y + y*z*z*z*y
sage: p.normal_form(I)
y*y*z*z*z + y*z*y*z*z - y*z*z*y*z + y*z*z*z*z
sage: p.reduce(I) != p.reduce(G) != p.normal_form(I) != p.reduce(I)
True
```

`sage.algebras.letterplace.free_algebra_element_letterplace.poly_reduce`(*ring=None*, *interruptible=True*, *attributes=None*, **args*)

This function is an automatically generated C wrapper around the Singular function 'NF'.

This wrapper takes care of converting Sage datatypes to Singular datatypes and vice versa. In addition to whatever parameters the underlying Singular function accepts when called, this function also accepts the following keyword parameters:

INPUT:

- `args` – a list of arguments
- `ring` – a multivariate polynomial ring
- `interruptible` – if `True` pressing `Ctrl` + `C` during the execution of this function will interrupt the computation (default: `True`)
- `attributes` – a dictionary of optional Singular attributes assigned to Singular objects (default: `None`)

If `ring` is not specified, it is guessed from the given arguments. If this is not possible, then a dummy ring, univariate polynomial ring over `QQ`, is used.

EXAMPLES:

```
sage: groebner = sage.libs.singular.function_factory.ff.groebner
sage: P.<x, y> = PolynomialRing(QQ)
sage: I = P.ideal(x^2-y, y+x)
sage: groebner(I)
[x + y, y^2 - y]
sage: triangL = sage.libs.singular.function_factory.ff.triang__lib.triangL
sage: P.<x1, x2> = PolynomialRing(QQ, order='lex')
sage: f1 = 1/2*((x1^2 + 2*x1 - 4)*x2^2 + 2*(x1^2 + x1)*x2 + x1^2)
sage: f2 = 1/2*((x1^2 + 2*x1 + 1)*x2^2 + 2*(x1^2 + x1)*x2 - 4*x1^2)
sage: I = Ideal(Ideal(f1,f2).groebner_basis()[::-1])
sage: triangL(I, attributes={I:{'isSB':1}})
[[x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5],
 [x2, x1^2],
```

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```
[x2, x1^2],
[x2, x1^2]]
```

The Singular documentation for 'NF' is given below.

```
5.1.129 reduce
--------------

`*Syntax:*'
     `reduce (' poly_expression`,' ideal_expression `)'
     `reduce (' poly_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' poly_expression`,' poly_expression`,' ideal_expression
     `)'
     `reduce (' vector_expression`,' ideal_expression `)'
     `reduce (' vector_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' vector_expression`,' module_expression `)'
     `reduce (' vector_expression`,' module_expression`,'
     int_expression `)'
     `reduce (' vector_expression`,' poly_expression`,'
     module_expression `)'
     `reduce (' ideal_expression`,' ideal_expression `)'
     `reduce (' ideal_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' ideal_expression`,' matrix_expression`,'
     ideal_expression `)'
     `reduce (' module_expression`,' ideal_expression `)'
     `reduce (' module_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' module_expression`,' module_expression `)'
     `reduce (' module_expression`,' module_expression`,'
     int_expression `)'
     `reduce (' module_expression`,' matrix_expression`,'
     module_expression `)'
     `reduce (' poly/vector/ideal/module`,' ideal/module`,' int`,'
     intvec `)'
     `reduce (' ideal`,' matrix`,' ideal`,' int `)'
     `reduce (' poly`,' poly`,' ideal`,' int `)'
     `reduce (' poly`,' poly`,' ideal`,' int`,' intvec `)'

`*Type:*'
     the type of the first argument

`*Purpose:*'
     reduces a polynomial, vector, ideal  or module to its normal form
     with respect to an ideal or module represented by a standard basis.
     Returns 0 if and only if the polynomial (resp. vector, ideal,
     module) is an element (resp. subideal, submodule) of the ideal
     (resp. module).  The result may have no meaning if the second
     argument is not a standard basis.
     The third (optional) argument of type int modifies the behavior:
```

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```
        * 0 default

        * 1 consider only the leading term and do no tail reduction.

        * 2 tail reduction:n the local/mixed ordering case: reduce also
          with bad ecart

        * 4 reduce without division, return possibly a non-zero
          constant multiple of the remainder

     If a second argument `u' of type poly or matrix is given, the
     first argument `p' is replaced by `p/u'.  This works only for zero
     dimensional ideals (resp. modules) in the third argument and
     gives, even in a local ring, a reduced normal form which is the
     projection to the quotient by the ideal (resp. module).  One may
     give a degree bound in the fourth argument with respect to a
     weight vector in the fifth argument in order have a finite
     computation.  If some of the weights are zero, the procedure may
     not terminate!

`*Note_*'
     The commands `reduce' and `NF' are synonymous.

`*Example:*'
          ring r1 = 0,(z,y,x),ds;
          poly s1=2x5y+7x2y4+3x2yz3;
          poly s2=1x2y2z2+3z8;
          poly s3=4xy5+2x2y2z3+11x10;
          ideal i=s1,s2,s3;
          ideal j=std(i);
          reduce(3z3yx2+7y4x2+yx5+z12y2x2,j);
        ==> -yx5+2401/81y14x2+2744/81y11x5+392/27y8x8+224/81y5x11+16/81y2x14
          reduce(3z3yx2+7y4x2+yx5+z12y2x2,j,1);
        ==> -yx5+z12y2x2
          // 4 arguments:
          ring rs=0,x,ds;
          // normalform of 1/(1+x) w.r.t. (x3) up to degree 5
          reduce(poly(1),1+x,ideal(x3),5);
        ==> // ** _ is no standard basis
        ==> 1-x+x2

* Menu:

See
* division::
* ideal::
* module::
* poly operations::
* std::
* vector::
```

3.5 Homogeneous ideals of free algebras

For twosided ideals and when the base ring is a field, this implementation also provides Groebner bases and ideal containment tests.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: F
Free Associative Unital Algebra on 3 generators (x, y, z) over Rational Field
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: I
Twosided Ideal (x*y + y*z, x*x + x*y - y*x - y*y) of Free Associative Unital Algebra on
↪3 generators (x, y, z) over Rational Field
```

One can compute Groebner bases out to a finite degree, can compute normal forms and can test containment in the ideal:

```
sage: I.groebner_basis(degbound=3)
Twosided Ideal (x*y + y*z,
    x*x - y*x - y*y - y*z,
    y*y*y - y*y*z + y*z*y - y*z*z,
    y*y*x + y*y*z + y*z*x + y*z*z) of Free Associative Unital Algebra
    on 3 generators (x, y, z) over Rational Field
sage: (x*y*z*y*x).normal_form(I)
y*z*z*y*z + y*z*z*z*x + y*z*z*z*z
sage: x*y*z*y*x - (x*y*z*y*x).normal_form(I) in I
True
```

AUTHOR:

- Simon King (2011-03-22): See github issue #7797.

class sage.algebras.letterplace.letterplace_ideal.**LetterplaceIdeal**(*ring*, *gens*, *coerce=True*, *side='twosided'*)

Bases: `Ideal_nc`

Graded homogeneous ideals in free algebras.

In the two-sided case over a field, one can compute Groebner bases up to a degree bound, normal forms of graded homogeneous elements of the free algebra, and ideal containment.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: I
Twosided Ideal (x*y + y*z, x*x + x*y - y*x - y*y) of Free Associative Unital
↪Algebra on 3 generators (x, y, z) over Rational Field
sage: I.groebner_basis(2)
Twosided Ideal (x*y + y*z, x*x - y*x - y*y - y*z) of Free Associative Unital
↪Algebra on 3 generators (x, y, z) over Rational Field
sage: I.groebner_basis(4)
Twosided Ideal (x*y + y*z,
    x*x - y*x - y*y - y*z,
    y*y*y - y*y*z + y*z*y - y*z*z,
```

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```
    y*y*x + y*y*z + y*z*x + y*z*z,
    y*y*z*y - y*y*z*z + y*z*z*y - y*z*z*z,
    y*z*y*y - y*z*y*z + y*z*z*y - y*z*z*z,
    y*y*z*x + y*y*z*z + y*z*z*x + y*z*z*z,
    y*z*y*x + y*z*y*z + y*z*z*x + y*z*z*z) of Free Associative Unital
    Algebra on 3 generators (x, y, z) over Rational Field
```

Groebner bases are cached. If one has computed a Groebner basis out to a high degree then it will also be returned if a Groebner basis with a lower degree bound is requested:

```
sage: I.groebner_basis(2) is I.groebner_basis(4)
True
```

Of course, the normal form of any element has to satisfy the following:

```
sage: x*y*z*y*x - (x*y*z*y*x).normal_form(I) in I
True
```

Left and right ideals can be constructed, but only twosided ideals provide Groebner bases:

```
sage: JL = F*[x*y+y*z,x^2+x*y-y*x-y^2]; JL
Left Ideal (x*y + y*z, x*x + x*y - y*x - y*y) of Free Associative Unital Algebra on
↪3 generators (x, y, z) over Rational Field
sage: JR = [x*y+y*z,x^2+x*y-y*x-y^2]*F; JR
Right Ideal (x*y + y*z, x*x + x*y - y*x - y*y) of Free Associative Unital Algebra
↪on 3 generators (x, y, z) over Rational Field
sage: JR.groebner_basis(2)
Traceback (most recent call last):
...
TypeError: This ideal is not two-sided. We can only compute two-sided Groebner bases
sage: JL.groebner_basis(2)
Traceback (most recent call last):
...
TypeError: This ideal is not two-sided. We can only compute two-sided Groebner bases
```

Also, it is currently not possible to compute a Groebner basis when the base ring is not a field:

```
sage: FZ.<a,b,c> = FreeAlgebra(ZZ, implementation='letterplace')
sage: J = FZ*[a^3-b^3]*FZ
sage: J.groebner_basis(2)
Traceback (most recent call last):
...
TypeError: Currently, we can only compute Groebner bases if the ring of
↪coefficients is a field
```

The letterplace implementation of free algebras also provides integral degree weights for the generators, and we can compute Groebner bases for twosided graded homogeneous ideals:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace',degrees=[1,2,3])
sage: I = F*[x*y+z-y*x,x*y*z-x^6+y^3]*F
sage: I.groebner_basis(Infinity)
Twosided Ideal (x*y - y*x + z,
x*x*x*x*x*x - y*x*z - y*y*y + z*z,
```

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```
x*z*z - y*x*x*z + y*x*z*x + y*y*z + y*z*y + z*x*z + z*y*y - z*z*x,
x*x*x*x*x*z + x*x*x*x*z*x + x*x*x*z*x*x + x*x*z*x*x*x + x*z*x*x*x*x +
y*x*z*y - y*y*x*z + y*z*z + z*x*x*x*x*x - z*z*y,
x*x*x*x*z*y*y + x*x*x*z*y*y*x - x*x*x*z*y*z - x*x*z*y*x*z + x*x*z*y*y*x*x +
x*x*z*y*y*y - x*x*z*y*z*x - x*z*y*x*x*z - x*z*y*x*z*x +
x*z*y*y*x*x*x + 2*x*z*y*y*y*x - 2*x*z*y*y*z - x*z*y*z*x*x -
x*z*y*z*y + y*x*z*x*x*x*x*x - 4*y*x*z*x*x*z - 4*y*x*z*x*z*x +
4*y*x*z*y*x*x*x + 3*y*x*z*y*y*x - 4*y*x*z*y*z + y*y*x*x*x*x*z +
y*y*x*x*x*z*x - 3*y*y*x*x*z*x*x - y*y*x*x*z*y +
5*y*y*x*z*x*x*x + 4*y*y*x*z*y*x - 4*y*y*y*x*x*z +
4*y*y*y*x*z*x + 3*y*y*y*y*z + 4*y*y*y*z*x*x + 6*y*y*y*z*y +
y*y*z*x*x*x*x + y*y*z*x*z + 7*y*y*z*y*x*x + 7*y*y*z*y*y -
7*y*y*z*z*x - y*z*x*x*x*z - y*z*x*x*z*x + 3*y*z*x*z*x*x +
y*z*x*z*y + y*z*y*x*x*x*x - 3*y*z*y*x*z + 7*y*z*y*y*x*x +
3*y*z*y*y*y - 3*y*z*y*z*x - 5*y*z*z*x*x*x - 4*y*z*z*y*x +
4*y*z*z*z - z*y*x*x*x*z - z*y*x*x*z*x - z*y*x*z*x*x -
z*y*x*z*y + z*y*y*x*x*x*x - 3*z*y*y*x*z + 3*z*y*y*y*x*x +
z*y*y*y*y - 3*z*y*y*z*x - z*y*z*x*x*x - 2*z*y*z*y*x +
2*z*y*z*z - z*z*x*x*x*x*x + 4*z*z*x*x*z + 4*z*z*x*z*x -
4*z*z*y*x*x*x - 3*z*z*y*y*x + 4*z*z*y*z + 4*z*z*z*x*x +
2*z*z*z*y)
of Free Associative Unital Algebra on 3 generators (x, y, z) over Rational Field
```

Again, we can compute normal forms:

```
sage: (z*I.0-I.1).normal_form(I)
0
sage: (z*I.0-x*y*z).normal_form(I)
-y*x*z + z*z
```

groebner_basis(*degbound=None*)

Twosided Groebner basis with degree bound.

INPUT:

- degbound (optional integer, or Infinity): If it is provided, a Groebner basis at least out to that degree is returned. By default, the current degree bound of the underlying ring is used.

ASSUMPTIONS:

Currently, we can only compute Groebner bases for twosided ideals, and the ring of coefficients must be a field. A $TypeError$ is raised if one of these conditions is violated.

Note:

- The result is cached. The same Groebner basis is returned if a smaller degree bound than the known one is requested.
- If the degree bound Infinity is requested, it is attempted to compute a complete Groebner basis. But we cannot guarantee that the computation will terminate, since not all twosided homogeneous ideals of a free algebra have a finite Groebner basis.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
```

Since F was cached and since its degree bound cannot be decreased, it may happen that, as a side effect of other tests, it already has a degree bound bigger than 3. So, we cannot test against the output of `I.groebner_basis()`:

```
sage: F.set_degbound(3)
sage: I.groebner_basis()   # not tested
Twosided Ideal (y*y*y - y*y*z + y*z*y - y*z*z, y*y*x + y*y*z + y*z*x + y*z*z,␣
↪x*y + y*z, x*x - y*x - y*y - y*z) of Free Associative Unital Algebra on 3␣
↪generators (x, y, z) over Rational Field
sage: I.groebner_basis(4)
Twosided Ideal (x*y + y*z,
    x*x - y*x - y*y - y*z,
    y*y*y - y*y*z + y*z*y - y*z*z,
    y*y*x + y*y*z + y*z*x + y*z*z,
    y*y*z*y - y*y*z*z + y*z*z*y - y*z*z*z,
    y*z*y*y - y*z*y*z + y*z*z*y - y*z*z*z,
    y*y*z*x + y*y*z*z + y*z*z*x + y*z*z*z,
    y*z*y*x + y*z*y*z + y*z*z*x + y*z*z*z) of Free Associative
    Unital Algebra on 3 generators (x, y, z) over Rational Field
sage: I.groebner_basis(2) is I.groebner_basis(4)
True
sage: G = I.groebner_basis(4)
sage: G.groebner_basis(3) is G
True
```

If a finite complete Groebner basis exists, we can compute it as follows:

```
sage: I = F*[x*y-y*x,x*z-z*x,y*z-z*y,x^2*y-z^3,x*y^2+z*x^2]*F
sage: I.groebner_basis(Infinity)
Twosided Ideal (-y*z + z*y,
    -x*z + z*x,
    -x*y + y*x,
    x*x*z + x*y*y,
    x*x*y - z*z*z,
    x*x*x*z + y*z*z*z,
    x*z*z*z*z + y*y*z*z*z) of Free Associative Unital Algebra
    on 3 generators (x, y, z) over Rational Field
```

Since the commutators of the generators are contained in the ideal, we can verify the above result by a computation in a polynomial ring in negative lexicographic order:

```
sage: P.<c,b,a> = PolynomialRing(QQ,order='neglex')
sage: J = P*[a^2*b-c^3,a*b^2+c*a^2]
sage: J.groebner_basis()
[b*a^2 - c^3, b^2*a + c*a^2, c*a^3 + c^3*b, c^3*b^2 + c^4*a]
```

Apparently, the results are compatible, by sending a to x, b to y and c to z.

reduce(G)

Reduction of this ideal by another ideal, or normal form of an algebra element with respect to this ideal.

INPUT:

- G: A list or tuple of elements, an ideal, the ambient algebra, or a single element.

OUTPUT:

- The normal form of G with respect to this ideal, if G is an element of the algebra.
- The reduction of this ideal by the elements resp. generators of G, if G is a list, tuple or ideal.
- The zero ideal, if G is the algebra containing this ideal.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ, implementation='letterplace')
sage: I = F*[x*y+y*z,x^2+x*y-y*x-y^2]*F
sage: I.reduce(F)
Twosided Ideal (0) of Free Associative Unital Algebra on 3 generators (x, y, z)␣
↪over Rational Field
sage: I.reduce(x^3)
-y*z*x - y*z*y - y*z*z
sage: I.reduce([x*y])
Twosided Ideal (y*z, x*x - y*x - y*y) of Free Associative Unital Algebra on 3␣
↪generators (x, y, z) over Rational Field
sage: I.reduce(F*[x^2+x*y,y^2+y*z]*F)
Twosided Ideal (x*y + y*z, -y*x + y*z) of Free Associative Unital Algebra on 3␣
↪generators (x, y, z) over Rational Field
```

sage.algebras.letterplace.letterplace_ideal.**poly_reduce**(*ring=None*, *interruptible=True*, *attributes=None*, **args*)

This function is an automatically generated C wrapper around the Singular function 'NF'.

This wrapper takes care of converting Sage datatypes to Singular datatypes and vice versa. In addition to whatever parameters the underlying Singular function accepts when called, this function also accepts the following keyword parameters:

INPUT:

- args – a list of arguments
- ring – a multivariate polynomial ring
- interruptible – if True pressing Ctrl + C during the execution of this function will interrupt the computation (default: True)
- attributes – a dictionary of optional Singular attributes assigned to Singular objects (default: None)

If ring is not specified, it is guessed from the given arguments. If this is not possible, then a dummy ring, univariate polynomial ring over QQ, is used.

EXAMPLES:

```
sage: groebner = sage.libs.singular.function_factory.ff.groebner
sage: P.<x, y> = PolynomialRing(QQ)
sage: I = P.ideal(x^2-y, y+x)
sage: groebner(I)
[x + y, y^2 - y]
sage: triangL = sage.libs.singular.function_factory.ff.triang__lib.triangL
sage: P.<x1, x2> = PolynomialRing(QQ, order='lex')
sage: f1 = 1/2*((x1^2 + 2*x1 - 4)*x2^2 + 2*(x1^2 + x1)*x2 + x1^2)
sage: f2 = 1/2*((x1^2 + 2*x1 + 1)*x2^2 + 2*(x1^2 + x1)*x2 - 4*x1^2)
sage: I = Ideal(Ideal(f1,f2).groebner_basis()[::-1])
```

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```
sage: triangL(I, attributes={I:{'isSB':1}})
[[x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5],
 [x2, x1^2],
 [x2, x1^2],
 [x2, x1^2]]
```

The Singular documentation for 'NF' is given below.

```
5.1.129 reduce
--------------

`*Syntax:*'
     `reduce (' poly_expression`,' ideal_expression `)'
     `reduce (' poly_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' poly_expression`,' poly_expression`,' ideal_expression
     `)'
     `reduce (' vector_expression`,' ideal_expression `)'
     `reduce (' vector_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' vector_expression`,' module_expression `)'
     `reduce (' vector_expression`,' module_expression`,'
     int_expression `)'
     `reduce (' vector_expression`,' poly_expression`,'
     module_expression `)'
     `reduce (' ideal_expression`,' ideal_expression `)'
     `reduce (' ideal_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' ideal_expression`,' matrix_expression`,'
     ideal_expression `)'
     `reduce (' module_expression`,' ideal_expression `)'
     `reduce (' module_expression`,' ideal_expression`,' int_expression
     `)'
     `reduce (' module_expression`,' module_expression `)'
     `reduce (' module_expression`,' module_expression`,'
     int_expression `)'
     `reduce (' module_expression`,' matrix_expression`,'
     module_expression `)'
     `reduce (' poly/vector/ideal/module`,' ideal/module`,' int`,'
     intvec `)'
     `reduce (' ideal`,' matrix`,' ideal`,' int `)'
     `reduce (' poly`,' poly`,' ideal`,' int `)'
     `reduce (' poly`,' poly`,' ideal`,' int`,' intvec `)'

`*Type:*'
     the type of the first argument

`*Purpose:*'
     reduces a polynomial, vector, ideal  or module to its normal form
     with respect to an ideal or module represented by a standard basis.
     Returns 0 if and only if the polynomial (resp. vector, ideal,
     module) is an element (resp. subideal, submodule) of the ideal
```

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```
    (resp. module).  The result may have no meaning if the second
    argument is not a standard basis.
    The third (optional) argument of type int modifies the behavior:
       * 0 default

       * 1 consider only the leading term and do no tail reduction.

       * 2 tail reduction:n the local/mixed ordering case: reduce also
         with bad ecart

       * 4 reduce without division, return possibly a non-zero
         constant multiple of the remainder

    If a second argument `u' of type poly or matrix is given, the
    first argument `p' is replaced by `p/u'.  This works only for zero
    dimensional ideals (resp. modules) in the third argument and
    gives, even in a local ring, a reduced normal form which is the
    projection to the quotient by the ideal (resp. module).  One may
    give a degree bound in the fourth argument with respect to a
    weight vector in the fifth argument in order have a finite
    computation.  If some of the weights are zero, the procedure may
    not terminate!

`*Note_*'
    The commands `reduce' and `NF' are synonymous.

`*Example:*'
              ring r1 = 0,(z,y,x),ds;
              poly s1=2x5y+7x2y4+3x2yz3;
              poly s2=1x2y2z2+3z8;
              poly s3=4xy5+2x2y2z3+11x10;
              ideal i=s1,s2,s3;
              ideal j=std(i);
              reduce(3z3yx2+7y4x2+yx5+z12y2x2,j);
            ==> -yx5+2401/81y14x2+2744/81y11x5+392/27y8x8+224/81y5x11+16/81y2x14
              reduce(3z3yx2+7y4x2+yx5+z12y2x2,j,1);
            ==> -yx5+z12y2x2
              // 4 arguments:
              ring rs=0,x,ds;
              // normalform of 1/(1+x) w.r.t. (x3) up to degree 5
              reduce(poly(1),1+x,ideal(x3),5);
            ==> // ** _ is no standard basis
            ==> 1-x+x2

* Menu:

See
* division::
* ideal::
* module::
* poly operations::
* std::
```

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```
* vector::
```

sage.algebras.letterplace.letterplace_ideal.**singular_twostd**(*ring=None*, *interruptible=True*, *attributes=None*, **args*)

This function is an automatically generated C wrapper around the Singular function 'twostd'.

This wrapper takes care of converting Sage datatypes to Singular datatypes and vice versa. In addition to whatever parameters the underlying Singular function accepts when called, this function also accepts the following keyword parameters:

INPUT:

- `args` – a list of arguments
- `ring` – a multivariate polynomial ring
- `interruptible` – if `True` pressing `Ctrl` + `C` during the execution of this function will interrupt the computation (default: `True`)
- `attributes` – a dictionary of optional Singular attributes assigned to Singular objects (default: `None`)

If `ring` is not specified, it is guessed from the given arguments. If this is not possible, then a dummy ring, univariate polynomial ring over `QQ`, is used.

EXAMPLES:

```
sage: groebner = sage.libs.singular.function_factory.ff.groebner
sage: P.<x, y> = PolynomialRing(QQ)
sage: I = P.ideal(x^2-y, y+x)
sage: groebner(I)
[x + y, y^2 - y]
sage: triangL = sage.libs.singular.function_factory.ff.triang__lib.triangL
sage: P.<x1, x2> = PolynomialRing(QQ, order='lex')
sage: f1 = 1/2*((x1^2 + 2*x1 - 4)*x2^2 + 2*(x1^2 + x1)*x2 + x1^2)
sage: f2 = 1/2*((x1^2 + 2*x1 + 1)*x2^2 + 2*(x1^2 + x1)*x2 - 4*x1^2)
sage: I = Ideal(Ideal(f1,f2).groebner_basis()[::-1])
sage: triangL(I, attributes={I:{'isSB':1}})
[[x2^4 + 4*x2^3 - 6*x2^2 - 20*x2 + 5, 8*x1 - x2^3 + x2^2 + 13*x2 - 5],
 [x2, x1^2],
 [x2, x1^2],
 [x2, x1^2]]
```

The Singular documentation for 'twostd' is given below.

```
5.1.153 system
--------------

`*Syntax:*'
     `system (' string_expression `)'
     `system (' string_expression`,' expression `)'

`*Type:*'
     depends on the desired function, may be none

`*Purpose:*'
     interface to internal data and the operating system. The
```

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```
    string_expression determines the command to execute. Some commands
    require an additional argument (second form) where the type of the
    argument depends on the command. See below for a list of all
    possible commands.

`*Note_*'
    Not all functions work on every platform.

`*Functions:*'

   `system("alarm",' int `)'
        abort the Singular process after computing for that many
        seconds (system+user cpu time).

   `system("absFact",' poly `)'
        absolute factorization of the polynomial (from a polynomial
        ring over a transzedental extension) Returns a list of the
        ideal of the factors, intvec of multiplicities, ideal of
        minimal polynomials and the bumber of factors.

   `system("blackbox")'
        list all blackbox data types.

   `system("browsers");'
        returns a string about available help browsers.  *Note The
        online help system::.

   `system("bracket",' poly, poly `)'
        returns the Lie bracket [p,q].

   `system("complexNearZero",' number_expression `)'
        checks for a small value for floating point numbers

   `system("contributors")'
        returns names of people who contributed to the SINGULAR
        kernel as string.

   `system("content",p)'
        returns p/content(p) for poly/vector

   `system("cpu")'
        returns the number of cpus as int (for creating multiple
        threads/processes).  (see `system("--cpus")').

   `system("denom_list")'
        returns the list of denominators (number) which occurred in
        the latest std computationi(s).  Is reset to the empty list
        at ring changes or by this system call.

   `system("eigenvals",' matrix `)'
        returns the list of the eigenvalues of the matrix (as ideal,
        intvec).  (see `system("hessenberg")').
```

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```
`system("env",' ring `)'
      returns the enveloping algebra (i.e. R tensor R^opp) See
      `system("opp")'.

`system("executable",' string `)'
      returns the path of the command given as argument or the
      empty string (for: not found) See `system("Singular")'.  See
      `system("getenv","PATH")'.

`system("getenv",' string_expression`)'
      returns the value of the shell environment variable given as
      the second argument. The return type is string.

`system("getPrecDigits")'
      returns the precision for floating point numbers

`system("gmsnf",' ideal, ideal, matrix,int, int `)'
      Gauss-Manin system: for gmspoly.lib, gmssing.lib

`system("HC")'
      returns the degree of the "highest corner" from the last std
      computation (or 0).

`system("hessenberg",' matrix `)'
      returns the Hessenberg matrix (via QR algorithm).

`system("install",' s1, s2, p3, i4 `)'
      install a new method p3 for s2 for the newstruct type s1.  s2
      must be a reserved operator with i4 operands (i4 may be
      1,2,3; use 4 for more than 3 or a varying number of arguments)
      See *Note Commands for user defined types::.

`system("LLL",' B `)'
      B must be a matrix or an intmat.  Interface to NTLs LLL
      (Exact Arithmetic Variant over ZZ).  Returns the same type as
      the input.
      B is an m x n matrix, viewed as m rows of n-vectors.  m may
      be less than, equal to, or greater than n, and the rows need
      not be linearly independent.  B is transformed into an
      LLL-reduced basis.  The first m-rank(B) rows of B are zero.
      More specifically, elementary row transformations are
      performed on B so that the non-zero rows of new-B form an
      LLL-reduced basis for the lattice spanned by the rows of
      old-B.

`system("nblocks")' or `system("nblocks",' ring_name `)'
      returns the number of blocks of the given ring, or of the
      current basering, if no second argument is given. The return
      type is int.

`system("nc_hilb",' ideal, int, [,...] `)'
```

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```
        internal support for ncHilb.lib, return nothing

   `system("neworder",' ideal `)'
        string of the ring variables in an heurically good order for
        `char_series'

   `system("newstruct")'
        list all newstruct data types.

   `system("opp",' ring `)'
        returns the opposite ring.

   `system("oppose",' ring R, poly p `)'
        returns the opposite polynomial of p from R.

   `system("pcvLAddL",' list, list `)'
        `system("pcvPMulL",' poly, list `)'
        `system("pcvMinDeg",' poly `)'
        `system("pcvP2CV",' list, int, int `)'
        `system("pcvCV2P",' list, int, int `)'
        `system("pcvDim",' int, int `)'
        `system("pcvBasis",' int, int `)' internal for mondromy.lib

   `system("pid")'
        returns the process number as int (for creating unique names).

   `system("random")' or `system("random",' int `)'
        returns or sets the seed of the random generator.

   `system("reduce_bound",' poly, ideal, int `)'
        or `system("reduce_bound",' ideal, ideal, int `)'
        or `system("reduce_bound",' vector, module, int `)'
        or `system("reduce_bound",' module, module, int `)' returns
        the normalform of the first argument wrt. the second up to
        the given degree bound (wrt. total degree)

   `system("reserve",' int `)'
        reserve a port and listen with the given backlog.  (see
        `system("reservedLink")').

   `system("reservedLink")'
        accept a connect at the reserved port and return a
        (write-only) link to it.  (see `system("reserve")').

   `system("rref",' matrix ` )'
        return a reduced row echelon form of the constant matrix M
        (see `system("rref")').

   `system("semaphore",' string, int `)'
        operations for semaphores: string may be `"init"', `"exists"',
        `"acquire"', `"try_acquire"', `"release"', `"get_value"', and
        int is the number of the semaphore.  Returns -2 for wrong
```

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```
      command, -1 for error or the result of the command.

`system("semic",' list, list `)'
      or `system("semic",' list, list, int `)' computes from list
      of spectrum numbers and list of spectrum numbers the
      semicontinuity index (qh, if 3rd argument is 1).

`system("setenv",'string_expression, string_expression`)'
      sets the shell environment variable given as the second
      argument to the value given as the third argument. Returns
      the third argument. Might not be available on all platforms.

`system("sh"', string_expression `)'
      shell escape, returns the return code of the shell as int.
      The string is sent literally to the shell.

`system("shrinktest",' poly, i2 `)'
      internal for shift algebra (with i2 variables): shrink the
      poly

`system("Singular")'
      returns the absolute (path) name of the running SINGULAR as
      string.

`system("SingularBin")'
      returns the absolute path name of directory of the running
      SINGULAR as string (ending in /)

`system("SingularLib")'
      returns the colon separated library search path name as
      string.

`system("spadd",' list, list `)'
      or `system("spadd",' list, list, int `)' computes from list
      of spectrum numbers and list of spectrum numbers the sum of
      the lists.

`system("spectrum",' poly `)'
      or  `system("spectrum",' poly, int `)'

`system("spmul",' list, int `)'
      or `system("spmul",' list, list, int `)' computes from list
      of spectrum numbers the multiple of it.

`system("std_syz",' module, int `)'
      compute a partial groebner base of a module, stop after the
      given column

`system("tensorModuleMult",' int, module `)'
      internal for sheafcoh.lib (see id_TensorModuleMult)

`system("twostd",' ideal `)'
```

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```
        returns the two-sided standard basis of the two-sided ideal.

   `system("uname")'
        returns a string identifying the architecture for which
        SINGULAR was compiled.

   `system("verifyGB",' ideal_expression/module_expression `)'
        checks, if an ideal/module is a Groebner base

   `system("version")'
        returns the version number of SINGULAR as int.  (Version
        a-b-c-d returns a*1000+b*100+c*10+d)

   `system("with")'
        without an argument: returns a string describing the current
        version of SINGULAR, its build options, the used path names
        and other configurations
        with a string argument: test for that feature and return an
        int.

   `system("--cpus")'
        returns the number of available cpu cores as int (for using
        multiple cores).  (see `system("cpu")').

   `system("'-`")'
        prints the values of all options.

   `system("'-long_option_name`")'
        returns the value of the (command-line) option
        long_option_name. The type of the returned value is either
        string or int.  *Note Command line options::, for more info.

   `system("'-long_option_name`",' expression`)'
        sets the value of the (command-line) option long_option_name
        to the value given by the expression. Type of the expression
        must be string, or int. *Note Command line options::, for
        more info. Among others, this can be used for setting the
        seed of the random number generator, the used help browser,
        the minimal display time, or the timer resolution.

`*Example:*'
          // a listing of the current directory:
          system("sh","ls");
          // execute a shell, return to SINGULAR with exit:
          system("sh","sh");
          string unique_name="/tmp/xx"+string(system("pid"));
          unique_name;
          ==> /tmp/xx4711
          system("uname")
          ==> ix86-Linux
          system("getenv","PATH");
          ==> /bin:/usr/bin:/usr/local/bin
```

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```
system("Singular");
==> /usr/local/bin/Singular
// report value of all options
system("--");
==> // --batch           0
==> // --execute
==> // --sdb             0
==> // --echo            1
==> // --profile         0
==> // --quiet           1
==> // --sort            0
==> // --random          12345678
==> // --no-tty          1
==> // --user-option
==> // --allow-net       0
==> // --browser
==> // --cntrlc
==> // --emacs           0
==> // --log
==> // --no-stdlib       0
==> // --no-rc           1
==> // --no-warn         0
==> // --no-out          0
==> // --no-shell        0
==> // --min-time        "0.5"
==> // --cpus            4
==> // --threads         4
==> // --flint-threads   1
==> // --MPport
==> // --MPhost
==> // --link
==> // --ticks-per-sec   1
// set minimal display time to 0.02 seconds
system("--min-time", "0.02");
// set timer resolution to 0.01 seconds
system("--ticks-per-sec", 100);
// re-seed random number generator
system("--random", 12345678);
// allow your web browser to access HTML pages from the net
system("--allow-net", 1);
// and set help browser to firefox
system("--browser", "firefox");
==> // ** No help browser 'firefox' available.
==> // ** Setting help browser to 'dummy'.
```

3.6 Finite dimensional free algebra quotients

REMARK:

This implementation only works for finite dimensional quotients, since a list of basis monomials and the multiplication matrices need to be explicitly provided.

The homogeneous part of a quotient of a free algebra over a field by a finitely generated homogeneous twosided ideal is available in a different implementation. See `free_algebra_letterplace` and `quotient_ring`.

class `sage.algebras.free_algebra_quotient.FreeAlgebraQuotient`(*A*, *mons*, *mats*, *names*)

Bases: `UniqueRepresentation`, `Algebra`, `object`

Return a quotient algebra defined via the action of a free algebra A on a (finitely generated) free module.

The input for the quotient algebra is a list of monomials (in the underlying monoid for A) which form a free basis for the module of A, and a list of matrices, which give the action of the free generators of A on this monomial basis.

EXAMPLES:

Quaternion algebra defined in terms of three generators:

```
sage: n = 3
sage: A = FreeAlgebra(QQ,n,'i')
sage: F = A.monoid()
sage: i, j, k = F.gens()
sage: mons = [ F(1), i, j, k ]
sage: M = MatrixSpace(QQ,4)
sage: mats = [M([0,1,0,0, -1,0,0,0, 0,0,0,-1, 0,0,1,0]),  M([0,0,1,0, 0,0,0,1, -1,0,
↪0,0, 0,-1,0,0]),  M([0,0,0,1, 0,0,-1,0, 0,1,0,0, -1,0,0,0]) ]
sage: H3.<i,j,k> = FreeAlgebraQuotient(A,mons,mats)
sage: x = 1 + i + j + k
sage: x
1 + i + j + k
sage: x**128
-170141183460469231731687303715884105728 +
↪170141183460469231731687303715884105728*i +
↪170141183460469231731687303715884105728*j +
↪170141183460469231731687303715884105728*k
```

Same algebra defined in terms of two generators, with some penalty on already slow arithmetic.

```
sage: n = 2
sage: A = FreeAlgebra(QQ,n,'x')
sage: F = A.monoid()
sage: i, j = F.gens()
sage: mons = [ F(1), i, j, i*j ]
sage: r = len(mons)
sage: M = MatrixSpace(QQ,r)
sage: mats = [M([0,1,0,0, -1,0,0,0, 0,0,0,-1, 0,0,1,0]), M([0,0,1,0, 0,0,0,1, -1,0,
↪0,0, 0,-1,0,0]) ]
sage: H2.<i,j> = A.quotient(mons,mats)
sage: k = i*j
sage: x = 1 + i + j + k
sage: x
```

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```
1 + i + j + i*j
sage: x**128
-170141183460469231731687303715884105728 +␣
↪170141183460469231731687303715884105728*i +␣
↪170141183460469231731687303715884105728*j +␣
↪170141183460469231731687303715884105728*i*j
```

Element

alias of *FreeAlgebraQuotientElement*

dimension()

The rank of the algebra (as a free module).

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].dimension()
4
```

free_algebra()

The free algebra generating the algebra.

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].free_algebra()
Free Algebra on 3 generators (i0, i1, i2) over Rational Field
```

gen(*i*)

The i-th generator of the algebra.

EXAMPLES:

```
sage: H, (i,j,k) = sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)
sage: H.gen(0)
i
sage: H.gen(2)
k
```

An IndexError is raised if an invalid generator is requested:

```
sage: H.gen(3)
Traceback (most recent call last):
...
IndexError: argument i (= 3) must be between 0 and 2
```

Negative indexing into the generators is not supported:

```
sage: H.gen(-1)
Traceback (most recent call last):
...
IndexError: argument i (= -1) must be between 0 and 2
```

matrix_action()

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].matrix_
↪action()
(
[ 0  1  0  0]  [ 0  0  1  0]  [ 0  0  0  1]
[-1  0  0  0]  [ 0  0  0  1]  [ 0  0 -1  0]
[ 0  0  0 -1]  [-1  0  0  0]  [ 0  1  0  0]
[ 0  0  1  0], [ 0 -1  0  0], [-1  0  0  0]
)
```

module()

The free module of the algebra.

EXAMPLES:

```
sage: H = sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0]; H
Free algebra quotient on 3 generators ('i', 'j', 'k') and dimension 4 over␣
↪Rational Field
sage: H.module()
Vector space of dimension 4 over Rational Field
```

monoid()

The free monoid of generators of the algebra.

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].monoid()
Free monoid on 3 generators (i0, i1, i2)
```

monomial_basis()

The free monoid of generators of the algebra as elements of a free monoid.

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].monomial_
↪basis()
(1, i0, i1, i2)
```

ngens()

The number of generators of the algebra.

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].ngens()
3
```

rank()

The rank of the algebra (as a free module).

EXAMPLES:

```
sage: sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)[0].rank()
4
```

sage.algebras.free_algebra_quotient.**hamilton_quatalg**(*R*)

Hamilton quaternion algebra over the commutative ring R, constructed as a free algebra quotient.

INPUT:

- R – a commutative ring

OUTPUT:

- Q – quaternion algebra
- gens – generators for Q

EXAMPLES:

```
sage: H, (i,j,k) = sage.algebras.free_algebra_quotient.hamilton_quatalg(ZZ)
sage: H
Free algebra quotient on 3 generators ('i', 'j', 'k') and dimension 4 over Integer␣
↪Ring
sage: i^2
-1
sage: i in H
True
```

Note that there is another vastly more efficient models for quaternion algebras in Sage; the one here is mainly for testing purposes:

```
sage: R.<i,j,k> = QuaternionAlgebra(QQ,-1,-1)  # much fast than the above
```

3.7 Free algebra quotient elements

AUTHORS:

- William Stein (2011-11-19): improved doctest coverage to 100%
- David Kohel (2005-09): initial version

class sage.algebras.free_algebra_quotient_element.**FreeAlgebraQuotientElement**(*A*, *x*)

Bases: AlgebraElement

Create the element x of the FreeAlgebraQuotient A.

EXAMPLES:

```
sage: H, (i,j,k) = sage.algebras.free_algebra_quotient.hamilton_quatalg(ZZ)
sage: sage.algebras.free_algebra_quotient.FreeAlgebraQuotientElement(H, i)
i
sage: a = sage.algebras.free_algebra_quotient.FreeAlgebraQuotientElement(H, 1); a
1
sage: a in H
True
```

vector()

Return underlying vector representation of this element.

EXAMPLES:

```
sage: H, (i,j,k) = sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)
sage: ((2/3)*i - j).vector()
(0, 2/3, -1, 0)
```

`sage.algebras.free_algebra_quotient_element.`**`is_FreeAlgebraQuotientElement`**(*x*)

EXAMPLES:

```
sage: H, (i,j,k) = sage.algebras.free_algebra_quotient.hamilton_quatalg(QQ)
sage: sage.algebras.free_algebra_quotient_element.is_FreeAlgebraQuotientElement(i)
True
```

Of course this is testing the data type:

```
sage: sage.algebras.free_algebra_quotient_element.is_FreeAlgebraQuotientElement(1)
False
sage: sage.algebras.free_algebra_quotient_element.is_
↪FreeAlgebraQuotientElement(H(1))
True
```

3.8 Tensor Algebras

AUTHORS:

- Travis Scrimshaw (2014-01-24): Initial version

Todo:

- Coerce to/from free algebra.

class `sage.algebras.tensor_algebra.`**`BaseRingLift`**

Bases: `Morphism`

Morphism $R \to T(M)$ which identifies the base ring R of a tensor algebra $T(M)$ with the 0-th graded part of $T(M)$.

class `sage.algebras.tensor_algebra.`**`TensorAlgebra`**(*M*, *prefix='T'*, *category=None*, ***options*)

Bases: `CombinatorialFreeModule`

The tensor algebra $T(M)$ of a module M.

Let $\{b_i\}_{i \in I}$ be a basis of the R-module M. Then the tensor algebra $T(M)$ of M is an associative R-algebra, with a basis consisting of all tensors of the form $b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_n}$ for nonnegative integers n and n-tuples $(i_1, i_2, \ldots, i_n) \in I^n$. The product of $T(M)$ is given by

$$(b_{i_1} \otimes \cdots \otimes b_{i_m}) \cdot (b_{j_1} \otimes \cdots \otimes b_{j_n}) = b_{i_1} \otimes \cdots \otimes b_{i_m} \otimes b_{j_1} \otimes \cdots \otimes b_{j_n}.$$

As an algebra, it is generated by the basis vectors b_i of M. It is an $\mathbf{N}$-graded R-algebra, with the degree of each b_i being 1.

It also has a Hopf algebra structure: The comultiplication is the unique algebra morphism $\delta : T(M) \to T(M) \otimes T(M)$ defined by:

$$\delta(b_i) = b_i \otimes 1 + 1 \otimes b_i$$

(where the $\otimes$ symbol here forms tensors in $T(M) \otimes T(M)$, not inside $T(M)$ itself). The counit is the unique algebra morphism $T(M) \to R$ sending each b_i to 0. Its antipode S satisfies

$$S(b_{i_1} \otimes \cdots \otimes b_{i_m}) = (-1)^m (b_{i_m} \otimes \cdots \otimes b_{i_1}).$$

This is a connected graded cocommutative Hopf algebra.

REFERENCES:

- Wikipedia article Tensor_algebra

See also:

TensorAlgebra

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: TA.dimension()
+Infinity
sage: TA.base_ring()
Rational Field
sage: TA.algebra_generators()
Finite family {'a': B['a'], 'b': B['b'], 'c': B['c']}
```

algebra_generators()

Return the generators of this algebra.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: TA.algebra_generators()
Finite family {'a': B['a'], 'b': B['b'], 'c': B['c']}
sage: m = SymmetricFunctions(QQ).m()
sage: Tm = TensorAlgebra(m)
sage: Tm.algebra_generators()
Lazy family (generator(i))_{i in Partitions}
```

antipode_on_basis(*m*)

Return the antipode of the simple tensor indexed by `m`.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: s = TA(['a','b','c']).leading_support()
sage: TA.antipode_on_basis(s)
-B['c'] # B['b'] # B['a']
sage: t = TA(['a', 'b', 'b', 'b']).leading_support()
sage: TA.antipode_on_basis(t)
B['b'] # B['b'] # B['b'] # B['a']
```

base_module()

Return the base module of `self`.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: TA.base_module() is C
True
```

construction()

Return the functorial construction of `self`.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(ZZ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: f, M = TA.construction()
sage: M == C
True
sage: f(M) == TA
True
```

coproduct_on_basis(*m*)

Return the coproduct of the simple tensor indexed by `m`.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C, tensor_symbol="(X)")
sage: TA.coproduct_on_basis(TA.one_basis())
1 # 1
sage: I = TA.indices()
sage: ca = TA.coproduct_on_basis(I.gen('a')); ca
1 # B['a'] + B['a'] # 1
sage: s = TA(['a','b','c']).leading_support()
sage: cp = TA.coproduct_on_basis(s); cp
1 # B['a'](X)B['b'](X)B['c'] + B['a'] # B['b'](X)B['c']
 + B['a'](X)B['b'] # B['c'] + B['a'](X)B['b'](X)B['c'] # 1
 + B['a'](X)B['c'] # B['b'] + B['b'] # B['a'](X)B['c']
 + B['b'](X)B['c'] # B['a'] + B['c'] # B['a'](X)B['b']
```

We check that $\Delta(a \otimes b \otimes c) = \Delta(a)\Delta(b)\Delta(c)$:

```
sage: cb = TA.coproduct_on_basis(I.gen('b'))
sage: cc = TA.coproduct_on_basis(I.gen('c'))
sage: cp == ca * cb * cc
True
```

counit(*x*)

Return the counit of `x`.

INPUT:

- `x` – an element of `self`

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
```

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```
sage: x = TA(['a','b','c'])
sage: TA.counit(x)
0
sage: TA.counit(x + 3)
3
```

degree_on_basis(*m*)

Return the degree of the simple tensor `m`, which is its length (thought of as an element in the free monoid).

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: s = TA(['a','b','c']).leading_support(); s
F['a']*F['b']*F['c']
sage: TA.degree_on_basis(s)
3
```

gens()

Return the generators of this algebra.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: TA.algebra_generators()
Finite family {'a': B['a'], 'b': B['b'], 'c': B['c']}
sage: m = SymmetricFunctions(QQ).m()
sage: Tm = TensorAlgebra(m)
sage: Tm.algebra_generators()
Lazy family (generator(i))_{i in Partitions}
```

one_basis()

Return the empty word, which indexes the 1 of this algebra.

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: TA.one_basis()
1
sage: TA.one_basis().parent()
Free monoid indexed by {'a', 'b', 'c'}
sage: m = SymmetricFunctions(QQ).m()
sage: Tm = TensorAlgebra(m)
sage: Tm.one_basis()
1
sage: Tm.one_basis().parent()
Free monoid indexed by Partitions
```

product_on_basis(*a*, *b*)

Return the product of the basis elements indexed by `a` and `b`, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

INPUT:

- a, b – basis indices

EXAMPLES:

```
sage: C = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: TA = TensorAlgebra(C)
sage: I = TA.indices()
sage: g = I.gens()
sage: TA.product_on_basis(g['a']*g['b'], g['a']*g['c'])
B['a'] # B['b'] # B['a'] # B['c']
```

class sage.algebras.tensor_algebra.**TensorAlgebraFunctor**(*base*)

Bases: ConstructionFunctor

The tensor algebra functor.

Let R be a unital ring. Let V_R and A_R be the categories of R-modules and R-algebras respectively. The functor $T : V_R \to A_R$ sends an R-module M to the tensor algebra $T(M)$. The functor T is left-adjoint to the forgetful functor $F : A_R \to V_R$.

INPUT:

- base – the base R

rank = 20

CHAPTER

FOUR

FINITE DIMENSIONAL ALGEBRAS

4.1 Finite-Dimensional Algebras

class sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra.**FiniteDimensionalAlgebra**(*k*, *table*, *names*, *category*)

Bases: UniqueRepresentation, Algebra

Create a finite-dimensional k-algebra from a multiplication table.

INPUT:

- `k` – a field
- `table` – a list of matrices
- `names` – (default: `'e'`) string; names for the basis elements
- `assume_associative` – (default: `False`) boolean; if `True`, then the category is set to `category.Associative()` and methods requiring associativity assume this
- `category` – (default: `MagmaticAlgebras(k).FiniteDimensional().WithBasis()`) the category to which this algebra belongs

The list `table` must have the following form: there exists a finite-dimensional k-algebra of degree n with basis $(e_1, \ldots, e_n)$ such that the i-th element of `table` is the matrix of right multiplication by e_i with respect to the basis $(e_1, \ldots, e_n)$.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0, 1],
↪ [0, 0]])])
sage: A
Finite-dimensional algebra of degree 2 over Finite Field of size 3
sage: TestSuite(A).run()

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B
Finite-dimensional algebra of degree 3 over Rational Field
```

Element

alias of *FiniteDimensionalAlgebraElement*

base_extend(*F*)

Return `self` base changed to the field `F`.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebra(GF(2), [Matrix([1])])
sage: k.<y> = GF(4)
sage: C.base_extend(k)
Finite-dimensional algebra of degree 1 over Finite Field in y of size 2^2
```

basis()

Return a list of the basis elements of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.basis()
Family (e0, e1)
```

cardinality()

Return the cardinality of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(7), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [2, 3]])])
sage: A.cardinality()
49

sage: B = FiniteDimensionalAlgebra(RR, [Matrix([[1, 0], [0, 1]]), Matrix([[0,␣
↪1], [2, 3]])])
sage: B.cardinality()
+Infinity

sage: C = FiniteDimensionalAlgebra(RR, [])
sage: C.cardinality()
1
```

degree()

Return the number of generators of `self`, i.e., the degree of `self` over its base field.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.ngens()
2
```

from_base_ring(*x*)

gen(*i*)

Return the i-th basis element of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.gen(0)
e0
```

ideal(*gens=None*, *given_by_matrix=False*, *side=None*)

Return the right ideal of `self` generated by `gens`.

INPUT:

- `A` – a *FiniteDimensionalAlgebra*
- `gens` – (default: None) - either an element of `A` or a list of elements of `A`, given as vectors, matrices, or FiniteDimensionalAlgebraElements. If `given_by_matrix` is `True`, then `gens` should instead be a matrix whose rows form a basis of an ideal of `A`.
- `given_by_matrix` – boolean (default: `False`) - if `True`, no checking is done
- `side` – ignored but necessary for coercions

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.ideal(A([1,1]))
Ideal (e0 + e1) of Finite-dimensional algebra of degree 2 over Finite Field of␣
↪size 3
```

is_associative()

Return `True` if `self` is associative.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],[-
↪1,0]])])
sage: A.is_associative()
True

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,1]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,1], [0,0,0], [1,0,0]])])
sage: B.is_associative()
False

sage: e = B.basis()
sage: (e[1]*e[2])*e[2]==e[1]*(e[2]*e[2])
False
```

is_commutative()

Return `True` if `self` is commutative.

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B.is_commutative()
True

sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,0,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,1,0], [0,0,1]])])
sage: C.is_commutative()
False
```

is_finite()

Return `True` if the cardinality of `self` is finite.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(7), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [2, 3]])])
sage: A.is_finite()
True

sage: B = FiniteDimensionalAlgebra(RR, [Matrix([[1, 0], [0, 1]]), Matrix([[0,␣
↪1], [2, 3]])])
sage: B.is_finite()
False

sage: C = FiniteDimensionalAlgebra(RR, [])
sage: C.is_finite()
True
```

is_unitary()

Return `True` if `self` has a two-sided multiplicative identity element.

> **Warning:** This uses linear algebra; thus expect wrong results when the base ring is not a field.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [])
sage: A.is_unitary()
True

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[-1,0]])])
sage: B.is_unitary()
True

sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[0,0], [0,0]]), Matrix([[0,0],␣
↪[0,0]])])
sage: C.is_unitary()
False

sage: D = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[1,0],␣
↪[0,1]])])
```

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```
sage: D.is_unitary()
False

sage: E = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0],[1,0]]), Matrix([[0,1],[0,
↪1]])])
sage: E.is_unitary()
False

sage: F = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,1]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,1], [0,0,0], [1,0,0]])])
sage: F.is_unitary()
True

sage: G = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,1]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,1,0], [0,0,0], [1,0,0]])])
sage: G.is_unitary()  # Unique right identity, but no left identity.
False
```

is_zero()

Return True if self is the zero ring.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [])
sage: A.is_zero()
True

sage: B = FiniteDimensionalAlgebra(GF(7), [Matrix([0])])
sage: B.is_zero()
False
```

left_table()

Return the list of matrices for left multiplication by the basis elements.

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],[-
↪1,0]])])
sage: T = B.left_table(); T
(
[1 0]  [ 0  1]
[0 1], [-1  0]
)
```

We check immutability:

```
sage: T[0] = "vandalized by h4xx0r"
Traceback (most recent call last):
...
TypeError: 'tuple' object does not support item assignment
sage: T[1][0] = [13, 37]
Traceback (most recent call last):
...
```

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```
ValueError: matrix is immutable; please change a copy instead
 (i.e., use copy(M) to change a copy of M).
```

maximal_ideal()

Compute the maximal ideal of the local algebra `self`.

Note: `self` must be unitary, commutative, associative and local (have a unique maximal ideal).

OUTPUT:

- *FiniteDimensionalAlgebraIdeal*; the unique maximal ideal of `self`. If `self` is not a local algebra, a `ValueError` is raised.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.maximal_ideal()
Ideal (0, e1) of Finite-dimensional algebra of degree 2 over Finite Field of␣
↪size 3

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B.maximal_ideal()
Traceback (most recent call last):
...
ValueError: algebra is not local
```

maximal_ideals()

Return a list consisting of all maximal ideals of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.maximal_ideals()
[Ideal (e1) of Finite-dimensional algebra of degree 2 over Finite Field of size␣
↪3]

sage: B = FiniteDimensionalAlgebra(QQ, [])
sage: B.maximal_ideals()
[]
```

ngens()

Return the number of generators of `self`, i.e., the degree of `self` over its base field.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.ngens()
2
```

one()

Return the multiplicative identity element of `self`, if it exists.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [])
sage: A.one()
0

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[-1,0]])])
sage: B.one()
e0

sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[0,0], [0,0]]), Matrix([[0,0],␣
↪[0,0]])])
sage: C.one()
Traceback (most recent call last):
...
TypeError: algebra is not unitary

sage: D = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,1]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,1], [0,0,0], [1,0,0]])])
sage: D.one()
e0

sage: E = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,1]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,1,0], [0,0,0], [1,0,0]])])
sage: E.one()
Traceback (most recent call last):
...
TypeError: algebra is not unitary
```

primary_decomposition()

Return the primary decomposition of `self`.

Note: `self` must be unitary, commutative and associative.

OUTPUT:

- a list consisting of the quotient maps `self` -> A, with A running through the primary factors of `self`

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.primary_decomposition()
[Morphism from Finite-dimensional algebra of degree 2 over Finite Field of size␣
↪3 to Finite-dimensional algebra of degree 2 over Finite Field of size 3 given␣
↪by matrix [1 0]
[0 1]]

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
```

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```
sage: B.primary_decomposition()
[Morphism from Finite-dimensional algebra of degree 3 over Rational Field to␣
↪Finite-dimensional algebra of degree 1 over Rational Field given by matrix [0]
[0]
[1], Morphism from Finite-dimensional algebra of degree 3 over Rational Field␣
↪to Finite-dimensional algebra of degree 2 over Rational Field given by matrix␣
↪[1 0]
[0 1]
[0 0]]
```

quotient_map(*ideal*)

Return the quotient of `self` by `ideal`.

INPUT:

- `ideal` – a `FiniteDimensionalAlgebraIdeal`

OUTPUT:

- *FiniteDimensionalAlgebraMorphism*; the quotient homomorphism

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: q0 = A.quotient_map(A.zero_ideal())
sage: q0
Morphism from Finite-dimensional algebra of degree 2 over Finite Field of size␣
↪3 to Finite-dimensional algebra of degree 2 over Finite Field of size 3 given␣
↪by matrix
[1 0]
[0 1]
sage: q1 = A.quotient_map(A.ideal(A.gen(1)))
sage: q1
Morphism from Finite-dimensional algebra of degree 2 over Finite Field of size␣
↪3 to Finite-dimensional algebra of degree 1 over Finite Field of size 3 given␣
↪by matrix
[1]
[0]
```

random_element(**args*, ***kwargs*)

Return a random element of `self`.

Optional input parameters are propagated to the `random_element` method of the underlying `VectorSpace`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.random_element()  # random
e0 + 2*e1

sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
```

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```
sage: B.random_element(num_bound=1000)  # random
215/981*e0 + 709/953*e1 + 931/264*e2
```

table()

Return the multiplication table of `self`, as a list of matrices for right multiplication by the basis elements.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: A.table()
(
[1 0]  [0 1]
[0 1], [0 0]
)
```

4.2 Elements of Finite Algebras

class sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra_element.**FiniteDimensionalAlgebraElement**

Bases: `AlgebraElement`

Create an element of a *FiniteDimensionalAlgebra* using a multiplication table.

INPUT:

- `A` – a *FiniteDimensionalAlgebra* which will be the parent
- `elt` – vector, matrix or element of the base field (default: `None`)
- `check` – boolean (default: `True`); if `False` and `elt` is a matrix, assume that it is known to be the matrix of an element

If `elt` is a vector or a matrix consisting of a single row, it is interpreted as a vector of coordinates with respect to the given basis of `A`. If `elt` is a square matrix, it is interpreted as a multiplication matrix with respect to this basis.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[0,0]])])
sage: A(17)
2*e0
sage: A([1,1])
e0 + e1
```

characteristic_polynomial()

Return the characteristic polynomial of `self`.

Note: This function just returns the characteristic polynomial of the matrix of right multiplication by `self`. This may not be a very meaningful invariant if the algebra is not unitary and associative.

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B(0).characteristic_polynomial()
x^3
sage: b = B.random_element()
sage: f = b.characteristic_polynomial(); f  # random
x^3 - 8*x^2 + 16*x
sage: f(b) == 0
True
```

inverse()

Return the two-sided multiplicative inverse of `self`, if it exists.

This assumes that the algebra to which `self` belongs is associative.

Note: If an element of a finite-dimensional unitary associative algebra over a field admits a left inverse, then this is the unique left inverse, and it is also a right inverse.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[-1,0]])])
sage: C([1,2]).inverse()
1/5*e0 - 2/5*e1
```

is_invertible()

Return `True` if `self` has a two-sided multiplicative inverse.

This assumes that the algebra to which `self` belongs is associative.

Note: If an element of a unitary finite-dimensional algebra over a field admits a left inverse, then this is the unique left inverse, and it is also a right inverse.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[-1,0]])])
sage: C([1,2]).is_invertible()
True
sage: C(0).is_invertible()
False
```

is_nilpotent()

Return `True` if `self` is nilpotent.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[0,0]])])
sage: C([1,0]).is_nilpotent()
False
sage: C([0,1]).is_nilpotent()
```

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```
True

sage: A = FiniteDimensionalAlgebra(QQ, [Matrix([0])])
sage: A([1]).is_nilpotent()
True
```

is_zerodivisor()

Return True if self is a left or right zero-divisor.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[0,0]])])
sage: C([1,0]).is_zerodivisor()
False
sage: C([0,1]).is_zerodivisor()
True
```

left_matrix()

Return the matrix for multiplication by self from the left.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,0,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,1,0], [0,0,1]])])
sage: C([1,2,0]).left_matrix()
[1 0 0]
[0 1 0]
[0 2 0]
```

matrix()

Return the matrix for multiplication by self from the right.

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B(5).matrix()
[5 0 0]
[0 5 0]
[0 0 5]
```

minimal_polynomial()

Return the minimal polynomial of self.

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B(0).minimal_polynomial()
x
sage: b = B.random_element()
sage: f = b.minimal_polynomial(); f  # random
x^3 + 1/2*x^2 - 7/16*x + 1/16
```

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```
sage: f(b) == 0
True
```

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- `copy` – ignored

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0], [0,1]]), Matrix([[0,1],␣
↪[-1,0]])])
sage: elt = B(Matrix([[1,1], [-1,1]]))
sage: elt.monomial_coefficients()
{0: 1, 1: 1}
```

vector()

Return `self` as a vector.

EXAMPLES:

```
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1,0,0], [0,1,0], [0,0,0]]),␣
↪Matrix([[0,1,0], [0,0,0], [0,0,0]]), Matrix([[0,0,0], [0,0,0], [0,0,1]])])
sage: B(5).vector()
(5, 0, 5)
```

sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra_element.**unpickle_FiniteDimensional**

Helper for unpickling of finite dimensional algebra elements.

4.3 Ideals of Finite Algebras

class sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra_ideal.**FiniteDimensionalAlgeb**

Bases: `Ideal_generic`

An ideal of a *FiniteDimensionalAlgebra*.

INPUT:

- `A` – a finite-dimensional algebra
- `gens` – the generators of this ideal
- `given_by_matrix` – (default: `False`) whether the basis matrix is given by `gens`

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0, 1],
↪ [0, 0]])])
sage: A.ideal(A([0,1]))
Ideal (e1) of Finite-dimensional algebra of degree 2 over Finite Field of size 3
```

basis_matrix()

Return the echelonized matrix whose rows form a basis of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: I = A.ideal(A([1,1]))
sage: I.basis_matrix()
[1 0]
[0 1]
```

vector_space()

Return `self` as a vector space.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(GF(3), [Matrix([[1, 0], [0, 1]]), Matrix([[0,
↪ 1], [0, 0]])])
sage: I = A.ideal(A([1,1]))
sage: I.vector_space()
Vector space of degree 2 and dimension 2 over Finite Field of size 3
Basis matrix:
[1 0]
[0 1]
```

4.4 Morphisms Between Finite Algebras

class sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra_morphism.**FiniteDimensionalAl**

Bases: `RingHomset_generic`

Set of morphisms between two finite-dimensional algebras.

zero()

Construct the zero morphism of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [Matrix([1])])
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([[1, 0], [0, 1]]), Matrix([[0,␣
↪1], [0, 0]])])
sage: H = Hom(A, B)
sage: H.zero()
```

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```
Morphism from Finite-dimensional algebra of degree 1 over Rational Field to
 Finite-dimensional algebra of degree 2 over Rational Field given by matrix
[0 0]
```

class sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra_morphism.**FiniteDimensionalAl**

Bases: RingHomomorphism_im_gens

Create a morphism between two *finite-dimensional algebras*.

INPUT:

- parent – the parent homset
- f – matrix of the underlying k-linear map
- unitary – boolean (default: True); if True and check is also True, raise a ValueError unless A and B are unitary and f respects unit elements
- check – boolean (default: True); check whether the given k-linear map really defines a (not necessarily unitary) k-algebra homomorphism

The algebras A and B must be defined over the same base field.

EXAMPLES:

```
sage: from sage.algebras.finite_dimensional_algebras.finite_dimensional_algebra_
↪morphism import FiniteDimensionalAlgebraMorphism
sage: A = FiniteDimensionalAlgebra(QQ, [Matrix([[1, 0], [0, 1]]), Matrix([[0, 1],␣
↪[0, 0]])])
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([1])])
sage: H = Hom(A, B)
sage: f = H(Matrix([[1], [0]]))
sage: f.domain() is A
True
sage: f.codomain() is B
True
sage: f(A.basis()[0])
e
sage: f(A.basis()[1])
0
```

Todo: An example illustrating unitary flag.

inverse_image(*I*)

Return the inverse image of I under self.

INPUT:

- I – FiniteDimensionalAlgebraIdeal, an ideal of self.codomain()

OUTPUT:

– FiniteDimensionalAlgebraIdeal, the inverse image of I under self.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [Matrix([[1, 0], [0, 1]]), Matrix([[0,␣
↪1], [0, 0]])])
sage: I = A.maximal_ideal()
sage: q = A.quotient_map(I)
sage: B = q.codomain()
sage: q.inverse_image(B.zero_ideal()) == I
True
```

matrix()

Return the matrix of `self`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebra(QQ, [Matrix([[1, 0], [0, 1]]), Matrix([[0,␣
↪1], [0, 0]])])
sage: B = FiniteDimensionalAlgebra(QQ, [Matrix([1])])
sage: M = Matrix([[1], [0]])
sage: H = Hom(A, B)
sage: f = H(M)
sage: f.matrix() == M
True
```

CHAPTER

FIVE

NAMED ASSOCIATIVE ALGEBRAS

5.1 Affine nilTemperley Lieb Algebra of type A

class sage.algebras.affine_nil_temperley_lieb.**AffineNilTemperleyLiebTypeA**(*n*, *R=Integer Ring*, *prefix='a'*)

Bases: CombinatorialFreeModule

Construct the affine nilTemperley Lieb algebra of type $A_{n-1}^{(1)}$ as used in [Pos2005].

INPUT:

- n – a positive integer

The affine nilTemperley Lieb algebra is generated by a_i for $i = 0, 1, \ldots, n-1$ subject to the relations $a_i a_i = a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = 0$ and $a_i a_j = a_j a_i$ for $i - j \not\equiv \pm 1$, where the indices are taken modulo n.

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(4)
sage: a = A.algebra_generators(); a
Finite family {0: a0, 1: a1, 2: a2, 3: a3}
sage: a[1]*a[2]*a[0] == a[1]*a[0]*a[2]
True
sage: a[0]*a[3]*a[0]
0
sage: A.an_element()
2*a0 + 1 + 3*a1 + a0*a1*a2*a3
```

algebra_generator(*i*)

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(3)
sage: A.algebra_generator(1)
a1
sage: A = AffineNilTemperleyLiebTypeA(3, prefix = 't')
sage: A.algebra_generator(1)
t1
```

algebra_generators()

Return the generators a_i for $i = 0, 1, 2, \ldots, n-1$.

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(3)
sage: a = A.algebra_generators();a
Finite family {0: a0, 1: a1, 2: a2}
sage: a[1]
a1
```

has_no_braid_relation(*w*, *i*)

Assuming that w contains no relations of the form s_i^2 or $s_i s_{i+1} s_i$ or $s_i s_{i-1} s_i$, tests whether $w s_i$ contains terms of this form.

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(5)
sage: W = A.weyl_group()
sage: s=W.simple_reflections()
sage: A.has_no_braid_relation(s[2]*s[1]*s[0]*s[4]*s[3],0)
False
sage: A.has_no_braid_relation(s[2]*s[1]*s[0]*s[4]*s[3],2)
True
sage: A.has_no_braid_relation(s[4],2)
True
```

index_set()

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(3)
sage: A.index_set()
(0, 1, 2)
```

one_basis()

Return the unit of the underlying Weyl group, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(3)
sage: A.one_basis()
[1 0 0]
[0 1 0]
[0 0 1]
sage: A.one_basis() == A.weyl_group().one()
True
sage: A.one()
1
```

product_on_basis(*w*, *w1*)

Return $a_w a_{w1}$, where w and $w1$ are in the Weyl group assuming that w does not contain any braid relations.

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(5)
sage: W = A.weyl_group()
sage: s = W.simple_reflections()
sage: [A.product_on_basis(s[1],x) for x in s]
```

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```
[a1*a0, 0, a1*a2, a3*a1, a4*a1]

sage: a = A.algebra_generators()
sage: x = a[1] * a[2]
sage: x
a1*a2
sage: x * a[1]
0
sage: x * a[2]
0
sage: x * a[0]
a1*a2*a0

sage: [x * a[1] for x in a]
[a0*a1, 0, a2*a1, a3*a1, a4*a1]

sage: w = s[1]*s[2]*s[1]
sage: A.product_on_basis(w,s[1])
Traceback (most recent call last):
...
AssertionError
```

weyl_group()

EXAMPLES:

```
sage: A = AffineNilTemperleyLiebTypeA(3)
sage: A.weyl_group()
Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root space)
```

5.2 Askey-Wilson Algebras

AUTHORS:

- Travis Scrimshaw (2018-08): initial version

class sage.algebras.askey_wilson.**AlgebraMorphism**(*domain*, *on_generators*, *position=0*, *codomain=None*, *category=None*)

Bases: ModuleMorphismByLinearity

An algebra morphism of the Askey-Wilson algebra defined by the images of the generators.

class sage.algebras.askey_wilson.**AskeyWilsonAlgebra**(R, q)

Bases: CombinatorialFreeModule

The (universal) Askey-Wilson algebra.

Let R be a commutative ring. The *universal Askey-Wilson* algebra is an associative unital algebra Δ_q over $R[q, q^-1]$ given by the generators $A, B, C, \alpha, \beta, \gamma$ that satisfy the following relations:

$$(q - q^{-1})\alpha = (q^2 - q^{-2})A + qBC - q^{-1}CB,$$
$$(q - q^{-1})\beta = (q^2 - q^{-2})B + qCA - q^{-1}AC,$$
$$(q - q^{-1})\gamma = (q^2 - q^{-2})C + qAB - q^{-1}BA.$$

The universal Askey-Wilson contains a `Casimir element` Ω, and the elements α, β, γ, Ω generate the center of Δ_q, which is isomorphic to the polynomial ring $(R[q, q^-1])[\alpha, \beta, \gamma, \Omega]$ (assuming q is not a root of unity). Furthermore, the relations imply that Δ_q has a basis given by monomials $A^i B^j C^k \alpha^r \beta^s \gamma^t$, where $i, j, k, r, s, t \in \mathbf{Z}_{\geq 0}$.

The universal Askey-Wilson algebra also admits a faithful action of $PSL_2(\mathbf{Z})$ given by the automorphisms ρ (`permutation_automorphism()`):

$$A \mapsto B \mapsto C \mapsto A, \qquad \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha.$$

and σ (`reflection_automorphism()`):

$$A \mapsto B \mapsto A, C \mapsto C + \frac{AB - BA}{q - q^{-1}}, \qquad \alpha \mapsto \beta \mapsto \alpha, \gamma \mapsto \gamma.$$

Note that $\rho^3 = \sigma^2 = 1$ and

$$\sigma(C) = C - qAB - (1 + q^2)C + q\gamma = C - qAB - q^2 C + q\gamma.$$

The Askey-Wilson $AW_q(a, b, c)$ algebra is a specialization of the universal Askey-Wilson algebra by $\alpha = a$, beta = b\`, $\gamma = c$, where $a, b, c \in R$. $AW_q(a, b, c)$ was first introduced by [Zhedanov1991] to describe the Askey-Wilson polynomials. The Askey-Wilson algebra has a central extension of Δ_q.

INPUT:

- `R` – a commutative ring
- `q` – (optional) the parameter q; must be invertible in `R`

If `q` is not specified, then `R` is taken to be the base ring of a Laurent polynomial ring with variable q. Otherwise the element `q` must be an element of `R`.

Note: No check is performed to ensure `q` is not a root of unity, which may lead to violations of the results in [Terwilliger2011].

EXAMPLES:

We create the universal Askey-Wilson algebra and check the defining relations:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.inject_variables()
Defining A, B, C, a, b, g
sage: q = AW.q()
sage: (q^2-q^-2)*A + q*B*C - q^-1*C*B == (q-q^-1)*a
True
sage: (q^2-q^-2)*B + q*C*A - q^-1*A*C == (q-q^-1)*b
True
sage: (q^2-q^-2)*C + q*A*B - q^-1*B*A == (q-q^-1)*g
True
```

Next, we perform some computations:

```
sage: C * A
(q^-2)*A*C + (q^-3-q)*B - (q^-2-1)*b
sage: B^2 * g^2 * A
q^4*A*B^2*g^2 - (q^-1-q^7)*B*C*g^2 + (1-q^4)*B*g^3
 + (1-2*q^4+q^8)*A*g^2 - (q-q^3-q^5+q^7)*a*g^2
```

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```
sage: (B^3 - A) * (C^2 + q*A*B)
q^7*A*B^4 + B^3*C^2 - (q^2-q^14)*B^3*C + (q-q^7)*B^3*g - q*A^2*B
 + (3*q^3-4*q^7+q^19)*A*B^2 - A*C^2 - (1-q^6-q^8+q^14)*B^2*a
 - (q^-2-3*q^6+3*q^14-q^22)*B*C
 + (q^-1+q-3*q^3-q^5+2*q^7-q^9+q^13+q^15-q^19)*B*g
 + (2*q^-1-6*q^3+5*q^7-2*q^19+q^23)*A
 - (2-2*q^2-4*q^4+4*q^6+q^8-q^10+q^12-q^14+q^16-q^18-q^20+q^22)*a
```

We check the elements α, β, and γ are in the center:

```
sage: all(x * gen == gen * x for gen in AW.algebra_generators() for x in [a,b,g])
True
```

We verify that the `Casimir element` is in the center:

```
sage: Omega = AW.casimir_element()
sage: all(x * Omega == Omega * x for x in [A,B,C])
True

sage: x = AW.an_element()
sage: O2 = Omega^2
sage: x * O2 == O2 * x
True
```

We prove Lemma 2.1 in [Terwilliger2011]:

```
sage: (q^2-q^-2) * C == (q-q^-1) * g - (q*A*B - q^-1*B*A)
True
sage: (q-q^-1) * (q^2-q^-2) * a == (B^2*A - (q^2+q^-2)*B*A*B + A*B^2
....:                               + (q^2-q^-2)^2*A + (q-q^-1)^2*B*g)
True
sage: (q-q^-1) * (q^2-q^-2) * b == (A^2*B - (q^2+q^-2)*A*B*A + B*A^2
....:                               + (q^2-q^-2)^2*B + (q-q^-1)^2*A*g)
True
```

We prove Theorem 2.2 in [Terwilliger2011]:

```
sage: q3 = q^-2 + 1 + q^2
sage: A^3*B - q3*A^2*B*A + q3*A*B*A^2 - B*A^3 == -(q^2-q^-2)^2 * (A*B - B*A)
True
sage: B^3*A - q3*B^2*A*B + q3*B*A*B^2 - A*B^3 == -(q^2-q^-2)^2 * (B*A - A*B)
True
sage: (A^2*B^2 - B^2*A^2 + (q^2+q^-2)*(B*A*B*A-A*B*A*B)
....:  == -(q^1-q^-1)^2 * (A*B - B*A) * g)
True
```

We construct an Askey-Wilson algebra over $\mathbf{F}_5$ at $q = 2$:

```
sage: AW = algebras.AskeyWilson(GF(5), q=2)
sage: A,B,C,a,b,g = AW.algebra_generators()
sage: q = AW.q()
sage: Omega = AW.casimir_element()
```

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```
sage: B * A
4*A*B + 2*g
sage: C * A
4*A*C + 2*b
sage: C * B
4*B*C + 2*a
sage: Omega^2
A^2*B^2*C^2 + A^3*B*C + A*B^3*C + A*B*C^3 + A^4 + 4*A^3*a
 + 2*A^2*B^2 + A^2*B*b + 2*A^2*C^2 + 4*A^2*C*g + 4*A^2*a^2
 + 4*A*B^2*a + 4*A*C^2*a + B^4 + B^3*b + 2*B^2*C^2 + 4*B^2*C*g
 + 4*B^2*b^2 + B*C^2*b + C^4 + 4*C^3*g + 4*C^2*g^2 + 2*a*b*g

sage: (q^2-q^-2)*A + q*B*C - q^-1*C*B == (q-q^-1)*a
True
sage: (q^2-q^-2)*B + q*C*A - q^-1*A*C == (q-q^-1)*b
True
sage: (q^2-q^-2)*C + q*A*B - q^-1*B*A == (q-q^-1)*g
True
sage: all(x * Omega == Omega * x for x in [A,B,C])
True
```

REFERENCES:

- [Terwilliger2011]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: G = AW.algebra_generators()
sage: G['A']
A
sage: G['a']
a
sage: list(G)
[A, B, C, a, b, g]
```

an_element()

Return an element of `self`.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
```

casimir_element()

Return the Casimir element of `self`.

The Casimir element of the Askey-Wilson algebra Δ_q is

$$\Omega = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma.$$

The center $Z(\Delta_q)$ is generated by α, β, γ, and Ω.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.casimir_element()
q*A*B*C + q^2*A^2 - q*A*a + (q^-2)*B^2 - (q^-1)*B*b + q^2*C^2 - q*C*g
```

We check that the Casimir element is in the center:

```
sage: Omega = AW.casimir_element()
sage: all(Omega * gen == gen * Omega for gen in AW.algebra_generators())
True
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.gens()
(A, B, C, a, b, g)
```

loop_representation()

Return the map π from `self` to 2×2 matrices over $R[\lambda, \lambda^{-1}]$, where F is the fraction field of the base ring of `self`.

Let AW be the Askey-Wilson algebra over R, and let F be the fraction field of R. Let M be the space of 2×2 matrices over $F[\lambda, \lambda^{-1}]$. Consider the following elements of M:

$$\mathcal{A} = \begin{pmatrix} \lambda & 1-\lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda - 1 & \lambda \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1 & \lambda - 1 \\ 1-\lambda^{-1} & \lambda + \lambda^{-1} - 1 \end{pmatrix}.$$

From Lemma 3.11 of [Terwilliger2011], we define a representation $\pi : AW \to M$ by

$$A \mapsto q\mathcal{A} + q^{-1}\mathcal{A}^{-1}, \qquad B \mapsto q\mathcal{B} + q^{-1}\mathcal{B}^{-1}, \qquad C \mapsto q\mathcal{C} + q^{-1}\mathcal{C}^{-1},$$

$$\alpha, \beta, \gamma \mapsto \nu I,$$

where $\nu = (q^2 + q^-2)(\lambda + \lambda^{-1}) + (\lambda + \lambda^{-1})^2$.

We call this representation the *loop representation* as it is a representation using the loop group $SL_2(F[\lambda, \lambda^{-1}])$.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: q = AW.q()
sage: pi = AW.loop_representation()
sage: A,B,C,a,b,g = [pi(gen) for gen in AW.algebra_generators()]
sage: A
[                1/q*lambda^-1 + q*lambda ((-q^2 + 1)/q)*lambda^-1 + ((q^2 - 1)/
↪q)]
[                                       0                 q*lambda^-1 + 1/
↪q*lambda]
sage: B
[             q*lambda^-1 + 1/q*lambda                                     0]
[((-q^2 + 1)/q) + ((q^2 - 1)/q)*lambda             1/q*lambda^-1 + q*lambda]
sage: C
```

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```
[1/q*lambda^-1 + ((q^2 - 1)/q) + 1/q*lambda      ((q^2 - 1)/q) + ((-q^2 + 1)/
↪q)*lambda]
[  ((q^2 - 1)/q)*lambda^-1 + ((-q^2 + 1)/q)    q*lambda^-1 + ((-q^2 + 1)/q) +␣
↪q*lambda]
sage: a
[lambda^-2 + ((q^4 + 1)/q^2)*lambda^-1 + 2 + ((q^4 + 1)/q^2)*lambda + lambda^2 ␣
↪                                                                            0]
[                                                                            0␣
↪lambda^-2 + ((q^4 + 1)/q^2)*lambda^-1 + 2 + ((q^4 + 1)/q^2)*lambda + lambda^2]
sage: a == b
True
sage: a == g
True

sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: x = pi(AW.an_element())
sage: y = (q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: x == y
True
```

We check the defining relations of the Askey-Wilson algebra:

```
sage: A + (q*B*C - q^-1*C*B) / (q^2 - q^-2) == a / (q + q^-1)
True
sage: B + (q*C*A - q^-1*A*C) / (q^2 - q^-2) == b / (q + q^-1)
True
sage: C + (q*A*B - q^-1*B*A) / (q^2 - q^-2) == g / (q + q^-1)
True
```

We check Lemma 3.12 in [Terwilliger2011]:

```
sage: M = pi.codomain()
sage: la = M.base_ring().gen()
sage: p = M([[0,-1],[1,1]])
sage: s = M([[0,1],[la,0]])
sage: rho = AW.rho()
sage: sigma = AW.sigma()
sage: all(p*pi(gen)*~p == pi(rho(gen)) for gen in AW.algebra_generators())
True
sage: all(s*pi(gen)*~s == pi(sigma(gen)) for gen in AW.algebra_generators())
True
```

one_basis()

Return the index of the basis element 1 of `self`.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.one_basis()
(0, 0, 0, 0, 0, 0)
```

permutation_automorphism()

Return the permutation automorphism ρ of `self`.

We define the automorphism ρ by

$$A \mapsto B \mapsto C \mapsto A, \qquad \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha.$$

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: rho = AW.permutation_automorphism()
sage: [rho(gen) for gen in AW.algebra_generators()]
[B, C, A, b, g, a]

sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: rho(AW.an_element())
(q^-3+3+2*q+q^2)*a^3*b*g + q^5*A^2*B*g + 3*q^2*C*a*b^2
 - (q^-2-q^6)*A*C*g + (q-q^5)*A*g^2 - (q^-3-2*q+q^5)*B*g
 + (q^-2-1-q^2+q^4)*b*g + B

sage: r3 = rho * rho * rho
sage: [r3(gen) for gen in AW.algebra_generators()]
[A, B, C, a, b, g]
sage: r3(AW.an_element()) == AW.an_element()
True
```

pi()

Return the map π from `self` to 2×2 matrices over $R[\lambda, \lambda^{-1}]$, where F is the fraction field of the base ring of `self`.

Let AW be the Askey-Wilson algebra over R, and let F be the fraction field of R. Let M be the space of 2×2 matrices over $F[\lambda, \lambda^{-1}]$. Consider the following elements of M:

$$\mathcal{A} = \begin{pmatrix} \lambda & 1-\lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda-1 & \lambda \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1 & \lambda-1 \\ 1-\lambda^{-1} & \lambda+\lambda^{-1}-1 \end{pmatrix}.$$

From Lemma 3.11 of [Terwilliger2011], we define a representation $\pi : AW \to M$ by

$$A \mapsto q\mathcal{A} + q^{-1}\mathcal{A}^{-1}, \qquad B \mapsto q\mathcal{B} + q^{-1}\mathcal{B}^{-1}, \qquad C \mapsto q\mathcal{C} + q^{-1}\mathcal{C}^{-1},$$

$$\alpha, \beta, \gamma \mapsto \nu I,$$

where $\nu = (q^2 + q^-2)(\lambda + \lambda^{-1}) + (\lambda + \lambda^{-1})^2$.

We call this representation the *loop representation* as it is a representation using the loop group $SL_2(F[\lambda, \lambda^{-1}])$.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: q = AW.q()
sage: pi = AW.loop_representation()
sage: A,B,C,a,b,g = [pi(gen) for gen in AW.algebra_generators()]
sage: A
[                1/q*lambda^-1 + q*lambda ((-q^2 + 1)/q)*lambda^-1 + ((q^2 - 1)/
↪q)]
[                                       0                 q*lambda^-1 + 1/
↪q*lambda]
```

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```
sage: B
[          q*lambda^-1 + 1/q*lambda                                  0]
[((-q^2 + 1)/q) + ((q^2 - 1)/q)*lambda            1/q*lambda^-1 + q*lambda]
sage: C
[1/q*lambda^-1 + ((q^2 - 1)/q) + 1/q*lambda      ((q^2 - 1)/q) + ((-q^2 + 1)/
↪q)*lambda]
[  ((q^2 - 1)/q)*lambda^-1 + ((-q^2 + 1)/q)    q*lambda^-1 + ((-q^2 + 1)/q) +␣
↪q*lambda]
sage: a
[lambda^-2 + ((q^4 + 1)/q^2)*lambda^-1 + 2 + ((q^4 + 1)/q^2)*lambda + lambda^2 ␣
↪                                                                             0]
[                                                                             0␣
↪lambda^-2 + ((q^4 + 1)/q^2)*lambda^-1 + 2 + ((q^4 + 1)/q^2)*lambda + lambda^2]
sage: a == b
True
sage: a == g
True

sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: x = pi(AW.an_element())
sage: y = (q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: x == y
True
```

We check the defining relations of the Askey-Wilson algebra:

```
sage: A + (q*B*C - q^-1*C*B) / (q^2 - q^-2) == a / (q + q^-1)
True
sage: B + (q*C*A - q^-1*A*C) / (q^2 - q^-2) == b / (q + q^-1)
True
sage: C + (q*A*B - q^-1*B*A) / (q^2 - q^-2) == g / (q + q^-1)
True
```

We check Lemma 3.12 in [Terwilliger2011]:

```
sage: M = pi.codomain()
sage: la = M.base_ring().gen()
sage: p = M([[0,-1],[1,1]])
sage: s = M([[0,1],[la,0]])
sage: rho = AW.rho()
sage: sigma = AW.sigma()
sage: all(p*pi(gen)*~p == pi(rho(gen)) for gen in AW.algebra_generators())
True
sage: all(s*pi(gen)*~s == pi(sigma(gen)) for gen in AW.algebra_generators())
True
```

product_on_basis(*x*, *y*)

Return the product of the basis elements indexed by x and y.

INPUT:

- x, y – tuple of length 6

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.product_on_basis((0,0,0,0,0,0), (3,5,2,0,12,3))
A^3*B^5*C^2*b^12*g^3
sage: AW.product_on_basis((0,0,0,5,3,5), (3,5,2,0,12,3))
A^3*B^5*C^2*a^5*b^15*g^8
sage: AW.product_on_basis((7,0,0,5,3,5), (0,5,2,0,12,3))
A^7*B^5*C^2*a^5*b^15*g^8
sage: AW.product_on_basis((7,3,0,5,3,5), (0,2,2,0,12,3))
A^7*B^5*C^2*a^5*b^15*g^8
sage: AW.product_on_basis((0,1,0,5,3,5), (2,0,0,0,5,3))
q^4*A^2*B*a^5*b^8*g^8 - (q^-3-q^5)*A*C*a^5*b^8*g^8
 + (1-q^4)*A*a^5*b^8*g^9 - (q^-4-2+q^4)*B*a^5*b^8*g^8
 + (q^-3-q^-1-q+q^3)*a^5*b^9*g^8
sage: AW.product_on_basis((0,2,1,0,2,0), (1,1,0,2,1,0))
q^4*A*B^3*C*a^2*b^3 - (q^5-q^9)*A^2*B^2*a^2*b^3
 + (q^2-q^4)*A*B^2*a^3*b^3 + (q^-3-q)*B^4*a^2*b^3
 - (q^-2-1)*B^3*a^2*b^4 - (q-q^9)*B^2*C^2*a^2*b^3
 + (1-q^4)*B^2*C*a^2*b^3*g + (q^-4+2-5*q^4+2*q^12)*A*B*C*a^2*b^3
 - (q^-1+q-2*q^3-2*q^5+q^7+q^9)*A*B*a^2*b^3*g
 - (q^-3-q^3-2*q^5+q^7+q^9)*B*C*a^3*b^3
 + (q^-2-1-q^2+q^4)*B*a^3*b^3*g
 - (q^-3-2*q+2*q^9-q^13)*A^2*a^2*b^3
 + (2*q^-2-2-3*q^2+3*q^4+q^10-q^12)*A*a^3*b^3
 + (q^-7-2*q^-3+2*q^5-q^9)*B^2*a^2*b^3
 - (q^-6-q^-4-q^-2+1-q^2+q^4+q^6-q^8)*B*a^2*b^4
 - (q^-7-q^-3-2*q+2*q^5+q^9-q^13)*C^2*a^2*b^3
 + (q^-6-3-2*q^2+5*q^4-q^8+q^10-q^12)*C*a^2*b^3*g
 - (q^-1-2*q+2*q^5-q^7)*a^4*b^3
 - (q^-3-q^-1-2*q+2*q^3+q^5-q^7)*a^2*b^3*g^2
```

q()

Return the parameter q of `self`.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: q = AW.q()
sage: q
q
sage: q.parent()
Univariate Laurent Polynomial Ring in q over Rational Field
```

reflection_automorphism()

Return the reflection automorphism σ of `self`.

We define the automorphism σ by

$$A \mapsto B \mapsto A, \qquad C \mapsto C + \frac{AB - BA}{q - q^{-1}} = C - qAB - (1 + q^2)C + q\gamma,$$

$$\alpha \mapsto \beta \mapsto \alpha, \gamma \mapsto \gamma.$$

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: sigma = AW.reflection_automorphism()
sage: [sigma(gen) for gen in AW.algebra_generators()]
[B, A, -q*A*B - q^2*C + q*g, b, a, g]

sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: sigma(AW.an_element())
q^9*A^2*B^3*a + (q^10+q^14)*A*B^2*C*a - (q^7+q^9)*A*B^2*a*g
 + (q^-3+3+2*q+q^2)*a*b*g^3 + (q-3*q^9+q^13+q^17)*A^2*B*a
 - (q^2-q^6-q^8+q^14)*A*B*a^2 + 3*q^2*A*b^2*g + (q^5-q^9)*B^3*a
 - (q^6-q^8)*B^2*a*b + q^13*B*C^2*a - 2*q^10*B*C*a*g + q^7*B*a*g^2
 + (q^2-2*q^10+q^18)*A*C*a - (q-q^7-2*q^9+2*q^11-q^15+q^17)*A*a*g
 - (q^3-q^7-q^9+q^13)*C*a^2 + (q^2-q^6-2*q^8+2*q^10)*a^2*g
 + (q-3*q^5+3*q^9-q^13)*B*a - (q^2-q^4-2*q^6+2*q^8+q^10-q^12)*a*b + B

sage: s2 = sigma * sigma
sage: [s2(gen) for gen in AW.algebra_generators()]
[A, B, C, a, b, g]
sage: s2(AW.an_element()) == AW.an_element()
True
```

rho()

Return the permutation automorphism ρ of `self`.

We define the automorphism ρ by

$$A \mapsto B \mapsto C \mapsto A, \qquad \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha.$$

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: rho = AW.permutation_automorphism()
sage: [rho(gen) for gen in AW.algebra_generators()]
[B, C, A, b, g, a]

sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: rho(AW.an_element())
(q^-3+3+2*q+q^2)*a^3*b*g + q^5*A^2*B*g + 3*q^2*C*a*b^2
 - (q^-2-q^6)*A*C*g + (q-q^5)*A*g^2 - (q^-3-2*q+q^5)*B*g
 + (q^-2-1-q^2+q^4)*b*g + B

sage: r3 = rho * rho * rho
sage: [r3(gen) for gen in AW.algebra_generators()]
[A, B, C, a, b, g]
sage: r3(AW.an_element()) == AW.an_element()
True
```

sigma()

Return the reflection automorphism σ of `self`.

We define the automorphism σ by

$$A \mapsto B \mapsto A, \qquad C \mapsto C + \frac{AB - BA}{q - q^{-1}} = C - qAB - (1 + q^2)C + q\gamma,$$

$$\alpha \mapsto \beta \mapsto \alpha, \gamma \mapsto \gamma.$$

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: sigma = AW.reflection_automorphism()
sage: [sigma(gen) for gen in AW.algebra_generators()]
[B, A, -q*A*B - q^2*C + q*g, b, a, g]

sage: AW.an_element()
(q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A
sage: sigma(AW.an_element())
q^9*A^2*B^3*a + (q^10+q^14)*A*B^2*C*a - (q^7+q^9)*A*B^2*a*g
 + (q^-3+3+2*q+q^2)*a*b*g^3 + (q-3*q^9+q^13+q^17)*A^2*B*a
 - (q^2-q^6-q^8+q^14)*A*B*a^2 + 3*q^2*A*b^2*g + (q^5-q^9)*B^3*a
 - (q^6-q^8)*B^2*a*b + q^13*B*C^2*a - 2*q^10*B*C*a*g + q^7*B*a*g^2
 + (q^2-2*q^10+q^18)*A*C*a - (q-q^7-2*q^9+2*q^11-q^15+q^17)*A*a*g
 - (q^3-q^7-q^9+q^13)*C*a^2 + (q^2-q^6-2*q^8+2*q^10)*a^2*g
 + (q-3*q^5+3*q^9-q^13)*B*a - (q^2-q^4-2*q^6+2*q^8+q^10-q^12)*a*b + B

sage: s2 = sigma * sigma
sage: [s2(gen) for gen in AW.algebra_generators()]
[A, B, C, a, b, g]
sage: s2(AW.an_element()) == AW.an_element()
True
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: AW = algebras.AskeyWilson(QQ)
sage: AW.some_elements()
(A, B, C, a, b, g, 1,
 (q^-3+3+2*q+q^2)*a*b*g^3 + q*A*C^2*b + 3*q^2*B*a^2*g + A,
 q*A*B*C + q^2*A^2 - q*A*a + (q^-2)*B^2 - (q^-1)*B*b + q^2*C^2 - q*C*g)
```

5.3 Diagram and Partition Algebras

AUTHORS:

- Mike Hansen (2007): Initial version
- Stephen Doty, Aaron Lauve, George H. Seelinger (2012): Implementation of partition, Brauer, Temperley–Lieb, and ideal partition algebras
- Stephen Doty, Aaron Lauve, George H. Seelinger (2015): Implementation of `*Diagram` classes and other methods to improve diagram algebras.
- Mike Zabrocki (2018): Implementation of individual element diagram classes
- Aaron Lauve, Mike Zabrocki (2018): Implementation of orbit basis for Partition algebra.

class `sage.combinat.diagram_algebras.`**AbstractPartitionDiagram**(*parent*, *d*, *check=True*)

Bases: `AbstractSetPartition`

Abstract base class for partition diagrams.

This class represents a single partition diagram, that is used as a basis key for a diagram algebra element. A partition diagram should be a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$. Each such set partition is regarded as a graph on nodes $\{1, \ldots, k, -1, \ldots, -k\}$ arranged in two rows, with nodes $1, \ldots, k$ in the top row from left to right and with nodes $-1, \ldots, -k$ in the bottom row from left to right, and an edge connecting two nodes if and only if the nodes lie in the same subset of the set partition.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: pd1 = da.AbstractPartitionDiagram(pd, [[1,2],[-1,-2]])
sage: pd2 = da.AbstractPartitionDiagram(pd, [[1,2],[-1,-2]])
sage: pd1
{{-2, -1}, {1, 2}}
sage: pd1 == pd2
True
sage: pd1 == [[1,2],[-1,-2]]
True
sage: pd1 == ((-2,-1),(2,1))
True
sage: pd1 == SetPartition([[1,2],[-1,-2]])
True
sage: pd3 = da.AbstractPartitionDiagram(pd, [[1,-2],[-1,2]])
sage: pd1 == pd3
False
sage: pd4 = da.AbstractPartitionDiagram(pd, [[1,2],[3,4]])
Traceback (most recent call last):
...
ValueError: {{1, 2}, {3, 4}} does not represent two rows of vertices of order 2
```

base_diagram()

Return the underlying implementation of the diagram.

OUTPUT:

- tuple of tuples of integers

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: pd([[1,2],[-1,-2]]).base_diagram() == ((-2,-1),(1,2))
True
```

check()

Check the validity of the input for the diagram.

compose(*other*, *check=True*)

Compose `self` with `other`.

The composition of two diagrams X and Y is given by placing X on top of Y and removing all loops.

OUTPUT:

A tuple where the first entry is the composite diagram and the second entry is how many loop were removed.

Note: This is not really meant to be called directly, but it works to call it this way if desired.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: pd([[1,2],[-1,-2]]).compose(pd([[1,2],[-1,-2]]))
({{-2, -1}, {1, 2}}, 1)
```

count_blocks_of_size(*n*)

Count the number of blocks of a given size.

INPUT:

- n – a positive integer

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import PartitionDiagram
sage: pd = PartitionDiagram([[1,-3,-5],[2,4],[3,-1,-2],[5],[-4]])
sage: pd.count_blocks_of_size(1)
2
sage: pd.count_blocks_of_size(2)
1
sage: pd.count_blocks_of_size(3)
2
```

diagram()

Return the underlying implementation of the diagram.

OUTPUT:

- tuple of tuples of integers

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: pd([[1,2],[-1,-2]]).base_diagram() == ((-2,-1),(1,2))
True
```

dual()

Return the dual diagram of `self` by flipping it top-to-bottom.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import PartitionDiagram
sage: D = PartitionDiagram([[1,-1],[2,-2,-3],[3]])
sage: D.dual()
{{-3}, {-2, 2, 3}, {-1, 1}}
```

is_planar()

Test if the diagram `self` is planar.

A diagram element is planar if the graph of the nodes is planar.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import BrauerDiagram
sage: BrauerDiagram([[1,-2],[2,-1]]).is_planar()
False
sage: BrauerDiagram([[1,-1],[2,-2]]).is_planar()
True
```

order()

Return the maximum entry in the diagram element.

A diagram element will be a partition of the set $\{-1, -2, \ldots, -k, 1, 2, \ldots, k\}$. The order of the diagram element is the value k.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import PartitionDiagram
sage: PartitionDiagram([[1,-1],[2,-2,-3],[3]]).order()
3
sage: PartitionDiagram([[1,-1]]).order()
1
sage: PartitionDiagram([[1,-3,-5],[2,4],[3,-1,-2],[5],[-4]]).order()
5
```

propagating_number()

Return the propagating number of the diagram.

The propagating number is the number of blocks with both a positive and negative number.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: d1 = pd([[1,-2],[2,-1]])
sage: d1.propagating_number()
2
sage: d2 = pd([[1,2],[-2,-1]])
sage: d2.propagating_number()
0
```

set_partition()

Return the underlying implementation of the diagram as a set of sets.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: X = pd([[1,2],[-1,-2]]).set_partition(); X
{{-2, -1}, {1, 2}}
sage: X.parent()
Set partitions
```

class sage.combinat.diagram_algebras.**AbstractPartitionDiagrams**(*order*, *category=None*)

Bases: Parent, UniqueRepresentation

This is an abstract base class for partition diagrams.

The primary use of this class is to serve as basis keys for diagram algebras, but diagrams also have properties in their own right. Furthermore, this class is meant to be extended to create more efficient contains methods.

INPUT:

- `order` – integer or integer $+1/2$; the order of the diagrams
- `category` – (default: `FiniteEnumeratedSets()`); the category

All concrete classes should implement attributes

- `_name` – the name of the class
- `_diagram_func` – an iterator function that takes the order as its only input

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.PartitionDiagrams(2)
sage: pd
Partition diagrams of order 2
sage: pd.an_element() in pd
True
sage: elm = pd([[1,2],[-1,-2]])
sage: elm in pd
True
```

`Element`

alias of *`AbstractPartitionDiagram`*

class `sage.combinat.diagram_algebras.`**`BrauerAlgebra`**(*k*, *q*, *base_ring*, *prefix*)

Bases: *`SubPartitionAlgebra`*, *`UnitDiagramMixin`*

A Brauer algebra.

The Brauer algebra of rank k is an algebra with basis indexed by the collection of set partitions of $\{1, \ldots, k, -1, \ldots, -k\}$ with block size 2.

This algebra is a subalgebra of the partition algebra. For more information, see *`PartitionAlgebra`*.

INPUT:

- `k` – rank of the algebra
- `q` – the deformation parameter q

OPTIONAL ARGUMENTS:

- `base_ring` – (default `None`) a ring containing `q`; if `None` then just takes the parent of `q`
- `prefix` – (default `"B"`) a label for the basis elements

EXAMPLES:

We now define the Brauer algebra of rank 2 with parameter `x` over $\mathbf{Z}$:

```
sage: R.<x> = ZZ[]
sage: B = BrauerAlgebra(2, x, R)
sage: B
Brauer Algebra of rank 2 with parameter x
 over Univariate Polynomial Ring in x over Integer Ring
sage: B.basis()
Lazy family (Term map from Brauer diagrams of order 2 to Brauer Algebra
 of rank 2 with parameter x over Univariate Polynomial Ring in x
 over Integer Ring(i))_{i in Brauer diagrams of order 2}
```

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```
sage: B.basis().keys()
Brauer diagrams of order 2
sage: B.basis().keys()([[-2, 1], [2, -1]])
{{-2, 1}, {-1, 2}}
sage: b = B.basis().list(); b
[B{{-2, -1}, {1, 2}}, B{{-2, 1}, {-1, 2}}, B{{-2, 2}, {-1, 1}}]
sage: b[0]
B{{-2, -1}, {1, 2}}
sage: b[0]^2
x*B{{-2, -1}, {1, 2}}
sage: b[0]^5
x^4*B{{-2, -1}, {1, 2}}
```

Note, also that since the symmetric group algebra is contained in the Brauer algebra, there is also a conversion between the two.

```
sage: R.<x> = ZZ[]
sage: B = BrauerAlgebra(2, x, R)
sage: S = SymmetricGroupAlgebra(R, 2)
sage: S([2,1])*B([[1,-1],[2,-2]])
B{{-2, 1}, {-1, 2}}
```

jucys_murphy(j)

Return the j-th generalized Jucys-Murphy element of `self`.

The j-th Jucys-Murphy element of a Brauer algebra is simply the j-th Jucys-Murphy element of the symmetric group algebra with an extra $(z-1)/2$ term, where z is the parameter of the Brauer algebra.

REFERENCES:

EXAMPLES:

```
sage: z = var('z')                                                    #␣
↪optional - sage.symbolic
sage: B = BrauerAlgebra(3,z)                                          #␣
↪optional - sage.symbolic
sage: B.jucys_murphy(1)                                               #␣
↪optional - sage.symbolic
(1/2*z-1/2)*B{{-3, 3}, {-2, 2}, {-1, 1}}
sage: B.jucys_murphy(3)                                               #␣
↪optional - sage.symbolic
-B{{-3, -2}, {-1, 1}, {2, 3}} - B{{-3, -1}, {-2, 2}, {1, 3}}
 + B{{-3, 1}, {-2, 2}, {-1, 3}} + B{{-3, 2}, {-2, 3}, {-1, 1}}
 + (1/2*z-1/2)*B{{-3, 3}, {-2, 2}, {-1, 1}}
```

options = Current options for Brauer diagram - display: normal

class sage.combinat.diagram_algebras.**BrauerDiagram**(*parent*, *d*, *check=True*)

Bases: *AbstractPartitionDiagram*

A Brauer diagram.

A Brauer diagram for an integer k is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$ with block size 2.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(2)
sage: bd1 = bd([[1,2],[-1,-2]])
sage: bd2 = bd([[1,2,-1,-2]])
Traceback (most recent call last):
...
ValueError: all blocks of {{-2, -1, 1, 2}} must be of size 2
```

bijection_on_free_nodes(*two_line=False*)

Return the induced bijection - as a list of $(x, f(x))$ values - from the free nodes on the top at the Brauer diagram to the free nodes at the bottom of `self`.

OUTPUT:

If `two_line` is `True`, then the output is the induced bijection as a two-row list `(inputs, outputs)`.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-2,-3],[3,-1]])
sage: elm.bijection_on_free_nodes()
[[3, -1]]
sage: elm2 = bd([[1,-2],[2,-3],[3,-1]])
sage: elm2.bijection_on_free_nodes(two_line=True)
[[1, 2, 3], [-2, -3, -1]]
```

check()

Check the validity of the input for `self`.

involution_permutation_triple(*curt=True*)

Return the involution permutation triple of `self`.

From Graham-Lehrer (see *BrauerDiagrams*), a Brauer diagram is a triple (D_1, D_2, π), where:

- D_1 is a partition of the top nodes;
- D_2 is a partition of the bottom nodes;
- π is the induced permutation on the free nodes.

INPUT:

- `curt` – (default: `True`) if `True`, then return bijection on free nodes as a one-line notation (standardized to look like a permutation), else, return the honest mapping, a list of pairs $(i, -j)$ describing the bijection on free nodes

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-2,-3],[3,-1]])
sage: elm.involution_permutation_triple()
([(1, 2)], [(-3, -2)], [1])
sage: elm.involution_permutation_triple(curt=False)
([(1, 2)], [(-3, -2)], [[3, -1]])
```

is_elementary_symmetric()

Check if is elementary symmetric.

Let (D_1, D_2, π) be the Graham-Lehrer representation of the Brauer diagram d. We say d is *elementary symmetric* if $D_1 = D_2$ and π is the identity.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-1,-2],[3,-3]])
sage: elm.is_elementary_symmetric()
True
sage: elm2 = bd([[1,2],[-1,-3],[3,-2]])
sage: elm2.is_elementary_symmetric()
False
```

options = Current options for Brauer diagram - display: normal

perm()

Return the induced bijection on the free nodes of `self` in one-line notation, re-indexed and treated as a permutation.

See also:

bijection_on_free_nodes()

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-2,-3],[3,-1]])
sage: elm.perm()
[1]
```

class sage.combinat.diagram_algebras.**BrauerDiagrams**(*order*, *category=None*)

Bases: *AbstractPartitionDiagrams*

This class represents all Brauer diagrams of integer or integer $+1/2$ order. For more information on Brauer diagrams, see *BrauerAlgebra*.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(2); bd
Brauer diagrams of order 2
sage: bd.list()
[{{-2, -1}, {1, 2}}, {{-2, 1}, {-1, 2}}, {{-2, 2}, {-1, 1}}]

sage: bd = da.BrauerDiagrams(5/2); bd
Brauer diagrams of order 5/2
sage: bd.list()
[{{-3, 3}, {-2, -1}, {1, 2}},
 {{-3, 3}, {-2, 1}, {-1, 2}},
 {{-3, 3}, {-2, 2}, {-1, 1}}]
```

Element

alias of *BrauerDiagram*

cardinality()

Return the cardinality of `self`.

The number of Brauer diagrams of integer order k is $(2k-1)!!$.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: bd.cardinality()
15

sage: bd = da.BrauerDiagrams(7/2)
sage: bd.cardinality()
15
```

from_involution_permutation_triple(*D1_D2_pi*)

Construct a Brauer diagram of `self` from an involution permutation triple.

A Brauer diagram can be represented as a triple where the first entry is a list of arcs on the top row of the diagram, the second entry is a list of arcs on the bottom row of the diagram, and the third entry is a permutation on the remaining nodes. This triple is called the *involution permutation triple*. For more information, see [GL1996].

INPUT:

- `D1_D2_pi`– a list or tuple where the first entry is a list of arcs on the top of the diagram, the second entry is a list of arcs on the bottom of the diagram, and the third entry is a permutation on the free nodes.

REFERENCES:

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(4)
sage: bd.from_involution_permutation_triple([[[1,2]],[[3,4]],[2,1]])
{{-4, -3}, {-2, 3}, {-1, 4}, {1, 2}}
```

options = Current options for Brauer diagram - display: normal

symmetric_diagrams(*l=None*, *perm=None*)

Return the list of Brauer diagrams with symmetric placement of l arcs, and with free nodes permuted according to $perm$.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(4)
sage: bd.symmetric_diagrams(l=1, perm=[2,1])
[{{-4, -2}, {-3, 1}, {-1, 3}, {2, 4}},
 {{-4, -3}, {-2, 1}, {-1, 2}, {3, 4}},
 {{-4, -1}, {-3, 2}, {-2, 3}, {1, 4}},
 {{-4, 2}, {-3, -1}, {-2, 4}, {1, 3}},
```

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```
{{-4, 3}, {-3, 4}, {-2, -1}, {1, 2}},
{{-4, 1}, {-3, -2}, {-1, 4}, {2, 3}}]
```

class sage.combinat.diagram_algebras.**DiagramAlgebra**(*k*, *q*, *base_ring*, *prefix*, *diagrams*, *category=None*)

Bases: CombinatorialFreeModule

Abstract class for diagram algebras and is not designed to be used directly.

class Element

Bases: IndexedFreeModuleElement

An element of a diagram algebra.

This subclass provides a few additional methods for partition algebra elements. Most element methods are already implemented elsewhere.

diagram()

Return the underlying diagram of self if self is a basis element. Raises an error if self is not a basis element.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: P = PartitionAlgebra(2, x, R)
sage: elt = 3*P([[1,2],[-2,-1]])
sage: elt.diagram()
{{-2, -1}, {1, 2}}
```

diagrams()

Return the diagrams in the support of self.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: P = PartitionAlgebra(2, x, R)
sage: elt = 3*P([[1,2],[-2,-1]]) + P([[1,2],[-2], [-1]])
sage: sorted(elt.diagrams(), key=str)
[{{-2, -1}, {1, 2}}, {{-2}, {-1}, {1, 2}}]
```

order()

Return the order of self.

The order of a partition algebra is defined as half of the number of nodes in the diagrams.

EXAMPLES:

```
sage: q = var('q')                                                      #␣
↪optional - sage.symbolic
sage: PA = PartitionAlgebra(2, q)                                       #␣
↪optional - sage.symbolic
sage: PA.order()                                                        #␣
↪optional - sage.symbolic
2
```

set_partitions()

Return the collection of underlying set partitions indexing the basis elements of a given diagram algebra.

Todo: Is this really necessary? deprecate?

class sage.combinat.diagram_algebras.**DiagramBasis**(*k*, *q*, *base_ring*, *prefix*, *diagrams*, *category=None*)

Bases: *DiagramAlgebra*

Abstract base class for diagram algebras in the diagram basis.

product_on_basis(*d1*, *d2*)

Return the product $D_{d_1} D_{d_2}$ by two basis diagrams.

class sage.combinat.diagram_algebras.**IdealDiagram**(*parent*, *d*, *check=True*)

Bases: *AbstractPartitionDiagram*

The element class for a ideal diagram.

An ideal diagram for an integer k is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$ where the propagating number is strictly smaller than the order.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import IdealDiagrams as IDs
sage: IDs(2)
Ideal diagrams of order 2
sage: IDs(2).list()
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2}, {-1, 1, 2}},
 {{-2, -1}, {1, 2}},
 {{-2}, {-1}, {1, 2}},
 {{-2, -1, 1}, {2}},
 {{-2, 1}, {-1}, {2}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}},
 {{-2}, {-1, 1}, {2}},
 {{-2}, {-1, 2}, {1}},
 {{-2, -1}, {1}, {2}},
 {{-2}, {-1}, {1}, {2}}]

sage: from sage.combinat.diagram_algebras import PartitionDiagrams as PDs
sage: PDs(4).cardinality() == factorial(4) + IDs(4).cardinality()
True
```

check()

Check the validity of the input for `self`.

class sage.combinat.diagram_algebras.**IdealDiagrams**(*order*, *category=None*)

Bases: *AbstractPartitionDiagrams*

All "ideal" diagrams of integer or integer $+1/2$ order.

If k is an integer then an ideal diagram of order k is a partition diagram of order k with propagating number less than k.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: id = da.IdealDiagrams(3)
sage: id.an_element() in id
True
sage: id.cardinality() == len(id.list())
True
sage: da.IdealDiagrams(3/2).list()
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}}]
```

Element

alias of *IdealDiagram*

class sage.combinat.diagram_algebras.**OrbitBasis**(*alg*)

Bases: *DiagramAlgebra*

The orbit basis of the partition algebra.

Let D_π represent the diagram basis element indexed by the partition π, then (see equations (2.14), (2.17) and (2.18) of [BH2017])

$$D_\pi = \sum_{\tau \geq \pi} O_\tau,$$

where the sum is over all partitions τ which are coarser than π and O_τ is the orbit basis element indexed by the partition τ.

If $\mu_{2k}(\pi, \tau)$ represents the Moebius function of the partition lattice, then

$$O_\pi = \sum_{\tau \geq \pi} \mu_{2k}(\pi, \tau) D_\tau.$$

If τ is a partition of ℓ blocks and the i^{th} block of τ is a union of b_i blocks of π, then

$$\mu_{2k}(\pi, \tau) = \prod_{i=1}^{\ell} (-1)^{b_i - 1} (b_i - 1)!.$$

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P2 = PartitionAlgebra(2, x, R)
sage: O2 = P2.orbit_basis(); O2
Orbit basis of Partition Algebra of rank 2 with parameter x over
 Univariate Polynomial Ring in x over Rational Field
sage: oa = O2([[1],[-1],[2,-2]]); ob = O2([[-1,-2,2],[1]]); oa, ob
(O{{-2, 2}, {-1}, {1}}, O{{-2, -1, 2}, {1}})
sage: oa * ob
(x-2)*O{{-2, -1, 2}, {1}}
```

We can convert between the two bases:

```
sage: pa = P2(oa); pa
2*P{{-2, -1, 1, 2}} - P{{-2, -1, 2}, {1}} - P{{-2, 1, 2}, {-1}}
```

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```
 + P{{-2, 2}, {-1}, {1}} - P{{-2, 2}, {-1, 1}}
sage: pa * ob
(-x+2)*P{{-2, -1, 1, 2}} + (x-2)*P{{-2, -1, 2}, {1}}
sage: _ == pa * P2(ob)
True
sage: O2(pa * ob)
(x-2)*O{{-2, -1, 2}, {1}}
```

Note that the unit in the orbit basis is not a single diagram, in contrast to the natural diagram basis:

```
sage: P2.one()
P{{-2, 2}, {-1, 1}}
sage: O2.one()
O{{-2, -1, 1, 2}} + O{{-2, 2}, {-1, 1}}
sage: O2.one() == P2.one()
True
```

class Element

Bases: *Element*

to_diagram_basis()

Expand `self` in the natural diagram basis of the partition algebra.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(2, x, R)
sage: O = P.orbit_basis()
sage: elt = O.an_element(); elt
3*O{{-2}, {-1, 1, 2}} + 2*O{{-2, -1, 1, 2}} + 2*O{{-2, 1, 2}, {-1}}
sage: elt.to_diagram_basis()
3*P{{-2}, {-1, 1, 2}} - 3*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
sage: pp = P.an_element(); pp
3*P{{-2}, {-1, 1, 2}} + 2*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
sage: op = pp.to_orbit_basis(); op
3*O{{-2}, {-1, 1, 2}} + 7*O{{-2, -1, 1, 2}} + 2*O{{-2, 1, 2}, {-1}}
sage: pp == op.to_diagram_basis()
True
```

diagram_basis()

Return the associated partition algebra of `self` in the diagram basis.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: O2 = PartitionAlgebra(2, x, R).orbit_basis()
sage: P2 = O2.diagram_basis(); P2
Partition Algebra of rank 2 with parameter x over Univariate
Polynomial Ring in x over Rational Field
sage: o2 = O2.an_element(); o2
3*O{{-2}, {-1, 1, 2}} + 2*O{{-2, -1, 1, 2}} + 2*O{{-2, 1, 2}, {-1}}
sage: P2(o2)
3*P{{-2}, {-1, 1, 2}} - 3*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
```

one()

Return the element 1 of the partition algebra in the orbit basis.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P2 = PartitionAlgebra(2, x, R)
sage: O2 = P2.orbit_basis()
sage: O2.one()
O{{-2, -1, 1, 2}} + O{{-2, 2}, {-1, 1}}
```

product_on_basis(*d1*, *d2*)

Return the product $O_{d_1} O_{d_2}$ of two elements in the orbit basis `self`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: OP = PartitionAlgebra(2, x, R).orbit_basis()
sage: SP = OP.basis().keys(); sp = SP([[-2, -1, 1, 2]])
sage: OP.product_on_basis(sp, sp)
O{{-2, -1, 1, 2}}
sage: o1 = OP.one(); o2 = OP([]); o3 = OP.an_element()
sage: o2 == o1
False
sage: o1 * o1 == o1
True
sage: o3 * o1 == o1 * o3 and o3 * o1 == o3
True
sage: o4 = (3*OP([[-2, -1, 1], [2]]) + 2*OP([[-2, -1, 1, 2]])
....:       + 2*OP([[-2, -1, 2], [1]]))
sage: o4 * o4
6*O{{-2, -1, 1}, {2}} + 4*O{{-2, -1, 1, 2}} + 4*O{{-2, -1, 2}, {1}}
```

We compute Examples 4.5 in [BH2017]:

```
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(3,x); O = P.orbit_basis()
sage: O[[1,2,3],[-1,-2,-3]] * O[[1,2,3],[-1,-2,-3]]
(x-2)*O{{-3, -2, -1}, {1, 2, 3}} + (x-1)*O{{-3, -2, -1, 1, 2, 3}}

sage: P = PartitionAlgebra(4,x); O = P.orbit_basis()
sage: O[[1],[-1],[2,3],[4,-2],[-3,-4]] * O[[1],[2,-2],[3,4],[-1,-3],[-4]]
(x^2-11*x+30)*O{{-4}, {-3, -1}, {-2, 4}, {1}, {2, 3}}
 + (x^2-9*x+20)*O{{-4}, {-3, -1, 1}, {-2, 4}, {2, 3}}
 + (x^2-9*x+20)*O{{-4}, {-3, -1, 2, 3}, {-2, 4}, {1}}
 + (x^2-9*x+20)*O{{-4, 1}, {-3, -1}, {-2, 4}, {2, 3}}
 + (x^2-7*x+12)*O{{-4, 1}, {-3, -1, 2, 3}, {-2, 4}}
 + (x^2-9*x+20)*O{{-4, 2, 3}, {-3, -1}, {-2, 4}, {1}}
 + (x^2-7*x+12)*O{{-4, 2, 3}, {-3, -1, 1}, {-2, 4}}

sage: O[[1,-1],[2,-2],[3],[4,-3],[-4]] * O[[1,-2],[2],[3,-1],[4],[-3],[-4]]
(x-6)*O{{-4}, {-3}, {-2, 1}, {-1, 4}, {2}, {3}}
 + (x-5)*O{{-4}, {-3, 3}, {-2, 1}, {-1, 4}, {2}}
 + (x-5)*O{{-4, 3}, {-3}, {-2, 1}, {-1, 4}, {2}}
```

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```
sage: P = PartitionAlgebra(6,x); O = P.orbit_basis()
sage: (O[[1,-2,-3],[2,4],[3,5,-6],[6],[-1],[-4,-5]]
....:  * O[[1,-2],[2,3],[4],[5],[6,-4,-5,-6],[-1,-3]])
0

sage: (O[[1,-2],[2,-3],[3,5],[4,-5],[6,-4],[-1],[-6]]
....:  * O[[1,-2],[2,-1],[3,-4],[4,-6],[5,-3],[6,-5]])
O{{-6, 6}, {-5}, {-4, 2}, {-3, 4}, {-2}, {-1, 1}, {3, 5}}
```

REFERENCES:

- [BH2017]

class sage.combinat.diagram_algebras.**PartitionAlgebra**(*k*, *q*, *base_ring*, *prefix*)

Bases: *DiagramBasis*, *UnitDiagramMixin*

A partition algebra.

A partition algebra of rank k over a given ground ring R is an algebra with (R-module) basis indexed by the collection of set partitions of $\{1,\dots,k,-1,\dots,-k\}$. Each such set partition can be represented by a graph on nodes $\{1,\dots,k,-1,\dots,-k\}$ arranged in two rows, with nodes $1,\dots,k$ in the top row from left to right and with nodes $-1,\dots,-k$ in the bottom row from left to right, and edges drawn such that the connected components of the graph are precisely the parts of the set partition. (This choice of edges is often not unique, and so there are often many graphs representing one and the same set partition; the representation nevertheless is useful and vivid. We often speak of "diagrams" to mean graphs up to such equivalence of choices of edges; of course, we could just as well speak of set partitions.)

There is not just one partition algebra of given rank over a given ground ring, but rather a whole family of them, indexed by the elements of R. More precisely, for every $q \in R$, the partition algebra of rank k over R with parameter q is defined to be the R-algebra with basis the collection of all set partitions of $\{1,\dots,k,-1,\dots,-k\}$, where the product of two basis elements is given by the rule

$$a \cdot b = q^N (a \circ b),$$

where $a \circ b$ is the composite set partition obtained by placing the diagram (i.e., graph) of a above the diagram of b, identifying the bottom row nodes of a with the top row nodes of b, and omitting any closed "loops" in the middle. The number N is the number of connected components formed by the omitted loops.

The parameter q is a deformation parameter. Taking $q = 1$ produces the semigroup algebra (over the base ring) of the partition monoid, in which the product of two set partitions is simply given by their composition.

The partition algebra is regarded as an example of a "diagram algebra" due to the fact that its natural basis is given by certain graphs often called diagrams.

There are a number of predefined elements for the partition algebra. We define the cup/cap pair by *a()*. The simple transpositions are denoted *s()*. Finally, we define elements *e()*, where if $i = (2r+1)/2$, then `e(i)` contains the blocks $\{r+1\}$ and $\{-r-1\}$ and if $i \in \mathbf{Z}$, then e_i contains the block $\{-i,-i-1,i,i+1\}$, with all other blocks being $\{-j,j\}$. So we have:

```
sage: P = PartitionAlgebra(4, 0)
sage: P.a(2)
P{{-4, 4}, {-3, -2}, {-1, 1}, {2, 3}}
sage: P.e(3/2)
P{{-4, 4}, {-3, 3}, {-2}, {-1, 1}, {2}}
sage: P.e(2)
```

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```
P{{-4, 4}, {-3, -2, 2, 3}, {-1, 1}}
sage: P.e(5/2)
P{{-4, 4}, {-3}, {-2, 2}, {-1, 1}, {3}}
sage: P.s(2)
P{{-4, 4}, {-3, 2}, {-2, 3}, {-1, 1}}
```

An excellent reference for partition algebras and their various subalgebras (Brauer algebra, Temperley–Lieb algebra, etc) is the paper [HR2005].

INPUT:

- `k` – rank of the algebra
- `q` – the deformation parameter q

OPTIONAL ARGUMENTS:

- `base_ring` – (default `None`) a ring containing `q`; if `None`, then Sage automatically chooses the parent of `q`
- `prefix` – (default `"P"`) a label for the basis elements

EXAMPLES:

The following shorthand simultaneously defines the univariate polynomial ring over the rationals as well as the variable `x`:

```
sage: R.<x> = PolynomialRing(QQ)
sage: R
Univariate Polynomial Ring in x over Rational Field
sage: x
x
sage: x.parent() is R
True
```

We now define the partition algebra of rank 2 with parameter `x` over $\mathbf{Z}$ in the usual (diagram) basis:

```
sage: R.<x> = ZZ[]
sage: A2 = PartitionAlgebra(2, x, R)
sage: A2
Partition Algebra of rank 2 with parameter x
 over Univariate Polynomial Ring in x over Integer Ring
sage: A2.basis().keys()
Partition diagrams of order 2
sage: A2.basis().keys()([[-2, 1, 2], [-1]])
{{-2, 1, 2}, {-1}}
sage: A2.basis().list()
[P{{-2, -1, 1, 2}}, P{{-2, 1, 2}, {-1}},
 P{{-2}, {-1, 1, 2}}, P{{-2, -1}, {1, 2}},
 P{{-2}, {-1}, {1, 2}}, P{{-2, -1, 1}, {2}},
 P{{-2, 1}, {-1, 2}}, P{{-2, 1}, {-1}, {2}},
 P{{-2, 2}, {-1, 1}}, P{{-2, -1, 2}, {1}},
 P{{-2, 2}, {-1}, {1}}, P{{-2}, {-1, 1}, {2}},
 P{{-2}, {-1, 2}, {1}}, P{{-2, -1}, {1}, {2}},
 P{{-2}, {-1}, {1}, {2}}]
sage: E = A2([[1,2],[-2,-1]]); E
P{{-2, -1}, {1, 2}}
```

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```
sage: E in A2.basis().list()
True
sage: E^2
x*P{{-2, -1}, {1, 2}}
sage: E^5
x^4*P{{-2, -1}, {1, 2}}
sage: (A2([[2,-2],[-1,1]]) - 2*A2([[1,2],[-1,-2]]))^2
(4*x-4)*P{{-2, -1}, {1, 2}} + P{{-2, 2}, {-1, 1}}
```

Next, we construct an element:

```
sage: a2 = A2.an_element(); a2
3*P{{-2}, {-1, 1, 2}} + 2*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
```

There is a natural embedding into partition algebras on more elements, by adding identity strands:

```
sage: A4 = PartitionAlgebra(4, x, R)
sage: A4(a2)
3*P{{-4, 4}, {-3, 3}, {-2}, {-1, 1, 2}}
 + 2*P{{-4, 4}, {-3, 3}, {-2, -1, 1, 2}}
 + 2*P{{-4, 4}, {-3, 3}, {-2, 1, 2}, {-1}}
```

Thus, the empty partition corresponds to the identity:

```
sage: A4([])
P{{-4, 4}, {-3, 3}, {-2, 2}, {-1, 1}}
sage: A4(5)
5*P{{-4, 4}, {-3, 3}, {-2, 2}, {-1, 1}}
```

The group algebra of the symmetric group is a subalgebra:

```
sage: S3 = SymmetricGroupAlgebra(ZZ, 3)
sage: s3 = S3.an_element(); s3
[1, 2, 3] + 2*[1, 3, 2] + 3*[2, 1, 3] + [3, 1, 2]
sage: A4(s3)
P{{-4, 4}, {-3, 1}, {-2, 3}, {-1, 2}}
 + 2*P{{-4, 4}, {-3, 2}, {-2, 3}, {-1, 1}}
 + 3*P{{-4, 4}, {-3, 3}, {-2, 1}, {-1, 2}}
 + P{{-4, 4}, {-3, 3}, {-2, 2}, {-1, 1}}
sage: A4([2,1])
P{{-4, 4}, {-3, 3}, {-2, 1}, {-1, 2}}
```

Be careful not to confuse the embedding of the group algebra of the symmetric group with the embedding of partial set partitions. The latter are embedded by adding the parts $\{i, -i\}$ if possible, and singletons sets for the remaining parts:

```
sage: A4([[2,1]])
P{{-4, 4}, {-3, 3}, {-2}, {-1}, {1, 2}}
sage: A4([[-1,3],[-2,-3,1]])
P{{-4, 4}, {-3, -2, 1}, {-1, 3}, {2}}
```

Another subalgebra is the Brauer algebra, which has perfect matchings as basis elements. The group algebra of the symmetric group is in fact a subalgebra of the Brauer algebra:

```
sage: B3 = BrauerAlgebra(3, x, R)
sage: b3 = B3(s3); b3
B{{-3, 1}, {-2, 3}, {-1, 2}} + 2*B{{-3, 2}, {-2, 3}, {-1, 1}}
 + 3*B{{-3, 3}, {-2, 1}, {-1, 2}} + B{{-3, 3}, {-2, 2}, {-1, 1}}
```

An important basis of the partition algebra is the *orbit basis*:

```
sage: O2 = A2.orbit_basis()
sage: o2 = O2([[1,2],[-1,-2]]) + O2([[1,2,-1,-2]]); o2
O{{-2, -1}, {1, 2}} + O{{-2, -1, 1, 2}}
```

The diagram basis element corresponds to the sum of all orbit basis elements indexed by coarser set partitions:

```
sage: A2(o2)
P{{-2, -1}, {1, 2}}
```

We can convert back from the orbit basis to the diagram basis:

```
sage: o2 = O2.an_element(); o2
3*O{{-2}, {-1, 1, 2}} + 2*O{{-2, -1, 1, 2}} + 2*O{{-2, 1, 2}, {-1}}
sage: A2(o2)
3*P{{-2}, {-1, 1, 2}} - 3*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
```

One can work with partition algebras using a symbol for the parameter, leaving the base ring unspecified. This implies that the underlying base ring is Sage's symbolic ring.

```
sage: q = var('q')                                                      #␣
↪optional - sage.symbolic
sage: PA = PartitionAlgebra(2, q); PA                                   #␣
↪optional - sage.symbolic
Partition Algebra of rank 2 with parameter q over Symbolic Ring
sage: PA([[1,2],[-2,-1]])^2 == q*PA([[1,2],[-2,-1]])                    #␣
↪optional - sage.symbolic
True
sage: ((PA([[2, -2], [1, -1]]) - 2*PA([[-2, -1], [1, 2]]))^2            #␣
↪optional - sage.symbolic
....:    == (4*q-4)*PA([[1, 2], [-2, -1]]) + PA([[2, -2], [1, -1]]))
True
```

The identity element of the partition algebra is the set partition $\{\{1,-1\},\{2,-2\},\ldots,\{k,-k\}\}$:

```
sage: P = PA.basis().list()                                             #␣
↪optional - sage.symbolic
sage: PA.one()                                                          #␣
↪optional - sage.symbolic
P{{-2, 2}, {-1, 1}}
sage: PA.one() * P[7] == P[7]                                           #␣
↪optional - sage.symbolic
True
sage: P[7] * PA.one() == P[7]                                           #␣
↪optional - sage.symbolic
True
```

We now give some further examples of the use of the other arguments. One may wish to "specialize" the parameter to a chosen element of the base ring:

```
sage: R.<q> = RR[]
sage: PA = PartitionAlgebra(2, q, R, prefix='B')
sage: PA
Partition Algebra of rank 2 with parameter q over
 Univariate Polynomial Ring in q over Real Field with 53 bits of precision
sage: PA([[1,2],[-1,-2]])
1.00000000000000*B{{-2, -1}, {1, 2}}
sage: PA = PartitionAlgebra(2, 5, base_ring=ZZ, prefix='B')
sage: PA
Partition Algebra of rank 2 with parameter 5 over Integer Ring
sage: ((PA([[2, -2], [1, -1]]) - 2*PA([[-2, -1], [1, 2]]))^2
....:   == 16*PA([[-2, -1], [1, 2]]) + PA([[2, -2], [1, -1]]))
True
```

Symmetric group algebra elements and elements from other subalgebras of the partition algebra (e.g., `BrauerAlgebra` and `TemperleyLiebAlgebra`) can also be coerced into the partition algebra:

```
sage: S = SymmetricGroupAlgebra(SR, 2)                                          #␣
↪optional - sage.symbolic
sage: B = BrauerAlgebra(2, x, SR)                                               #␣
↪optional - sage.symbolic
sage: A = PartitionAlgebra(2, x, SR)                                            #␣
↪optional - sage.symbolic
sage: S([2,1]) * A([[1,-1],[2,-2]])                                             #␣
↪optional - sage.symbolic
P{{-2, 1}, {-1, 2}}
sage: B([[-1,-2],[2,1]]) * A([[1],[-1],[2,-2]])                                 #␣
↪optional - sage.symbolic
P{{-2}, {-1}, {1, 2}}
sage: A([[1],[-1],[2,-2]]) * B([[-1,-2],[2,1]])                                 #␣
↪optional - sage.symbolic
P{{-2, -1}, {1}, {2}}
```

The same is true if the elements come from a subalgebra of a partition algebra of smaller order, or if they are defined over a different base ring:

```
sage: R = FractionField(ZZ['q']); q = R.gen()
sage: S = SymmetricGroupAlgebra(ZZ, 2)
sage: B = BrauerAlgebra(2, q, ZZ[q])
sage: A = PartitionAlgebra(3, q, R)
sage: S([2,1]) * A([[1,-1],[2,-3],[3,-2]])
P{{-3, 1}, {-2, 3}, {-1, 2}}
sage: A(B([[-1,-2],[2,1]]))
P{{-3, 3}, {-2, -1}, {1, 2}}
```

class Element

Bases: *Element*

dual()

Return the dual of `self`.

The dual of an element in the partition algebra is formed by taking the dual of each diagram in the support.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(2, x, R)
sage: elt = P.an_element(); elt
3*P{{-2}, {-1, 1, 2}} + 2*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
sage: elt.dual()
3*P{{-2, -1, 1}, {2}} + 2*P{{-2, -1, 1, 2}} + 2*P{{-2, -1, 2}, {1}}
```

to_orbit_basis()

Return `self` in the orbit basis of the associated partition algebra.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(2, x, R)
sage: pp = P.an_element();
sage: pp.to_orbit_basis()
3*O{{-2}, {-1, 1, 2}} + 7*O{{-2, -1, 1, 2}} + 2*O{{-2, 1, 2}, {-1}}
sage: pp = (3*P([[-2], [-1, 1, 2]]) + 2*P([[-2, -1, 1, 2]])
....:       + 2*P([[-2, 1, 2], [-1]])); pp
3*P{{-2}, {-1, 1, 2}} + 2*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
sage: pp.to_orbit_basis()
3*O{{-2}, {-1, 1, 2}} + 7*O{{-2, -1, 1, 2}} + 2*O{{-2, 1, 2}, {-1}}
```

L(i)

Return the `i`-th Jucys-Murphy element L_i from [Eny2012].

INPUT:

- `i` – a half integer between 1/2 and k

ALGORITHM:

We use the recursive definition for L_{2i} given in [Cre2020]. See also [Eny2012] and [Eny2013].

Note: $L_{1/2}$ and L_1 differs from [HR2005].

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.jucys_murphy_element(1/2)
0
sage: P3.jucys_murphy_element(1)
P{{-3, 3}, {-2, 2}, {-1}, {1}}
sage: P3.jucys_murphy_element(2)
P{{-3, 3}, {-2}, {-1, 1}, {2}} - P{{-3, 3}, {-2}, {-1, 1, 2}}
 + P{{-3, 3}, {-2, -1}, {1, 2}} - P{{-3, 3}, {-2, -1, 1}, {2}}
 + P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.jucys_murphy_element(3/2)
n*P{{-3, 3}, {-2, -1, 1, 2}} - P{{-3, 3}, {-2, -1, 2}, {1}}
 - P{{-3, 3}, {-2, 1, 2}, {-1}} + P{{-3, 3}, {-2, 2}, {-1, 1}}
sage: P3.L(3/2) * P3.L(2) == P3.L(2) * P3.L(3/2)
True
```

We test the relations in Lemma 2.2.3(2) in [Cre2020] (v1):

```
sage: k = 4
sage: R.<n> = QQ[]
sage: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)
True
sage: all(x * y == y * x for x in L for y in L)  # long time
True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k)]
sage: gens += [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
True
```

Also the relations in Lemma 2.2.3(3) in [Cre2020] (v1):

```
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....:     == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,k))
True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....:     == (P.L(i/2) + P.L((i+1)/2)) * P.e(i/2)
....:     == n * P.e(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,k))
True
sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,k))
True
sage: all(P.sigma((2*i+1)/2) * P.e((2*i+1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,k))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,k))
True
```

The same tests for a half integer partition algebra:

```
sage: k = 9/2
sage: R.<n> = QQ[]
sage: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)
True
sage: all(x * y == y * x for x in L for y in L)  # long time
True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k-1/2)]
sage: gens += [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
True
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....:     == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,floor(k)))
```

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```
True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....:     == (P.L(i/2) + P.L((i+1)/2)) * P.e(i/2)
....:     == n * P.e(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,ceil(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,ceil(k)))
True
sage: all(P.sigma((2*i+1)/2) * P.e((2*i+1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,floor(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,floor(k)))
True
```

a(i)

Return the element a_i in `self`.

The element a_i is the cap and cup at $(i, i + 1)$, so it contains the blocks $\{i, i + 1\}$, $\{-i, -i - 1\}$. Other blocks are of the form $\{-j, j\}$.

INPUT:

- `i` – an integer between 1 and $k - 1$

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.a(1)
P{{-3, 3}, {-2, -1}, {1, 2}}
sage: P3.a(2)
P{{-3, -2}, {-1, 1}, {2, 3}}

sage: P3 = PartitionAlgebra(5/2, n)
sage: P3.a(1)
P{{-3, 3}, {-2, -1}, {1, 2}}
sage: P3.a(2)
Traceback (most recent call last):
...
ValueError: i must be an integer between 1 and 1
```

e(i)

Return the element e_i in `self`.

If $i = (2r + 1)/2$, then e_i contains the blocks $\{r + 1\}$ and $\{-r - 1\}$. If $i \in \mathbf{Z}$, then e_i contains the block $\{-i, -i - 1, i, i + 1\}$. Other blocks are of the form $\{-j, j\}$.

INPUT:

- `i` – a half integer between 1/2 and $k - 1/2$

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.e(1)
P{{-3, 3}, {-2, -1, 1, 2}}
sage: P3.e(2)
P{{-3, -2, 2, 3}, {-1, 1}}
sage: P3.e(1/2)
P{{-3, 3}, {-2, 2}, {-1}, {1}}
sage: P3.e(5/2)
P{{-3}, {-2, 2}, {-1, 1}, {3}}
sage: P3.e(0)
Traceback (most recent call last):
...
ValueError: i must be an (half) integer between 1/2 and 5/2
sage: P3.e(3)
Traceback (most recent call last):
...
ValueError: i must be an (half) integer between 1/2 and 5/2

sage: P2h = PartitionAlgebra(5/2,n)
sage: [P2h.e(k/2) for k in range(1,5)]
[P{{-3, 3}, {-2, 2}, {-1}, {1}},
 P{{-3, 3}, {-2, -1, 1, 2}},
 P{{-3, 3}, {-2}, {-1, 1}, {2}},
 P{{-3, -2, 2, 3}, {-1, 1}}]
```

generator_a(i)

Return the element a_i in `self`.

The element a_i is the cap and cup at $(i, i+1)$, so it contains the blocks $\{i, i+1\}$, $\{-i, -i-1\}$. Other blocks are of the form $\{-j, j\}$.

INPUT:

- `i` – an integer between 1 and $k-1$

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.a(1)
P{{-3, 3}, {-2, -1}, {1, 2}}
sage: P3.a(2)
P{{-3, -2}, {-1, 1}, {2, 3}}

sage: P3 = PartitionAlgebra(5/2, n)
sage: P3.a(1)
P{{-3, 3}, {-2, -1}, {1, 2}}
sage: P3.a(2)
Traceback (most recent call last):
...
ValueError: i must be an integer between 1 and 1
```

generator_e(i)

Return the element e_i in `self`.

If $i = (2r + 1)/2$, then e_i contains the blocks $\{r + 1\}$ and $\{-r - 1\}$. If $i \in \mathbf{Z}$, then e_i contains the block $\{-i, -i - 1, i, i + 1\}$. Other blocks are of the form $\{-j, j\}$.

INPUT:

- `i` – a half integer between 1/2 and $k - 1/2$

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.e(1)
P{{-3, 3}, {-2, -1, 1, 2}}
sage: P3.e(2)
P{{-3, -2, 2, 3}, {-1, 1}}
sage: P3.e(1/2)
P{{-3, 3}, {-2, 2}, {-1}, {1}}
sage: P3.e(5/2)
P{{-3}, {-2, 2}, {-1, 1}, {3}}
sage: P3.e(0)
Traceback (most recent call last):
...
ValueError: i must be an (half) integer between 1/2 and 5/2
sage: P3.e(3)
Traceback (most recent call last):
...
ValueError: i must be an (half) integer between 1/2 and 5/2

sage: P2h = PartitionAlgebra(5/2,n)
sage: [P2h.e(k/2) for k in range(1,5)]
[P{{-3, 3}, {-2, 2}, {-1}, {1}},
 P{{-3, 3}, {-2, -1, 1, 2}},
 P{{-3, 3}, {-2}, {-1, 1}, {2}},
 P{{-3, -2, 2, 3}, {-1, 1}}]
```

generator_s(i)

Return the `i`-th simple transposition s_i in `self`.

Borrowing the notation from the symmetric group, the i-th simple transposition s_i has blocks of the form $\{-i, i + 1\}$, $\{-i - 1, i\}$. Other blocks are of the form $\{-j, j\}$.

INPUT:

- `i` – an integer between 1 and $k - 1$

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.s(1)
P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.s(2)
P{{-3, 2}, {-2, 3}, {-1, 1}}

sage: R.<n> = ZZ[]
sage: P2h = PartitionAlgebra(5/2,n)
```

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```
sage: P2h.s(1)
P{{-3, 3}, {-2, 1}, {-1, 2}}
```

jucys_murphy_element(*i*)

Return the i-th Jucys-Murphy element L_i from [Eny2012].

INPUT:

- i – a half integer between 1/2 and k

ALGORITHM:

We use the recursive definition for L_{2i} given in [Cre2020]. See also [Eny2012] and [Eny2013].

Note: $L_{1/2}$ and L_1 differs from [HR2005].

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.jucys_murphy_element(1/2)
0
sage: P3.jucys_murphy_element(1)
P{{-3, 3}, {-2, 2}, {-1}, {1}}
sage: P3.jucys_murphy_element(2)
P{{-3, 3}, {-2}, {-1, 1}, {2}} - P{{-3, 3}, {-2}, {-1, 1, 2}}
 + P{{-3, 3}, {-2, -1}, {1, 2}} - P{{-3, 3}, {-2, -1, 1}, {2}}
 + P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.jucys_murphy_element(3/2)
n*P{{-3, 3}, {-2, -1, 1, 2}} - P{{-3, 3}, {-2, -1, 2}, {1}}
 - P{{-3, 3}, {-2, 1, 2}, {-1}} + P{{-3, 3}, {-2, 2}, {-1, 1}}
sage: P3.L(3/2) * P3.L(2) == P3.L(2) * P3.L(3/2)
True
```

We test the relations in Lemma 2.2.3(2) in [Cre2020] (v1):

```
sage: k = 4
sage: R.<n> = QQ[]
sage: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)
True
sage: all(x * y == y * x for x in L for y in L)  # long time
True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k)]
sage: gens += [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
True
```

Also the relations in Lemma 2.2.3(3) in [Cre2020] (v1):

```
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....:     == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,k))
True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....:     == (P.L(i/2) + P.L((i+1)/2)) * P.e(i/2)
....:     == n * P.e(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,k))
True
sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,k))
True
sage: all(P.sigma((2*i+1)/2) * P.e((2*i+1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,k))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,k))
True
```

The same tests for a half integer partition algebra:

```
sage: k = 9/2
sage: R.<n> = QQ[]
sage: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)
True
sage: all(x * y == y * x for x in L for y in L)  # long time
True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k-1/2)]
sage: gens += [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
True
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....:     == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,floor(k)))
True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....:     == (P.L(i/2) + P.L((i+1)/2)) * P.e(i/2)
....:     == n * P.e(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,ceil(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,ceil(k)))
True
sage: all(P.sigma((2*i+1)/2) * P.e((2*i+1)/2) * P.e(2*i/2)
....:     == P.L(2*i/2) * P.e(2*i/2) for i in range(1,floor(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:     == P.e(2*i/2) * P.L(2*i/2) for i in range(1,floor(k)))
```

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```
True
```

orbit_basis()

Return the orbit basis of `self`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P2 = PartitionAlgebra(2, x, R)
sage: O2 = P2.orbit_basis(); O2
Orbit basis of Partition Algebra of rank 2 with parameter x over
 Univariate Polynomial Ring in x over Rational Field
sage: pp = 7 * P2[{-1}, {-2, 1, 2}] - 2 * P2[{-2}, {-1, 1}, {2}]; pp
-2*P{{-2}, {-1, 1}, {2}} + 7*P{{-2, 1, 2}, {-1}}
sage: op = pp.to_orbit_basis(); op
-2*O{{-2}, {-1, 1}, {2}} - 2*O{{-2}, {-1, 1, 2}}
 - 2*O{{-2, -1, 1}, {2}} + 5*O{{-2, -1, 1, 2}}
 + 7*O{{-2, 1, 2}, {-1}} - 2*O{{-2, 2}, {-1, 1}}
sage: op == O2(op)
True
sage: pp * op.leading_term()
4*P{{-2}, {-1, 1}, {2}} - 4*P{{-2, -1, 1}, {2}}
 + 14*P{{-2, -1, 1, 2}} - 14*P{{-2, 1, 2}, {-1}}
```

s(i)

Return the `i`-th simple transposition s_i in `self`.

Borrowing the notation from the symmetric group, the i-th simple transposition s_i has blocks of the form $\{-i, i+1\}$, $\{-i-1, i\}$. Other blocks are of the form $\{-j, j\}$.

INPUT:

- `i` – an integer between 1 and $k-1$

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.s(1)
P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.s(2)
P{{-3, 2}, {-2, 3}, {-1, 1}}

sage: R.<n> = ZZ[]
sage: P2h = PartitionAlgebra(5/2,n)
sage: P2h.s(1)
P{{-3, 3}, {-2, 1}, {-1, 2}}
```

sigma(i)

Return the element σ_i from [Eny2012] of `self`.

INPUT:

- `i` – a half integer between 1/2 and $k-1/2$

Note: In [Cre2020] and [Eny2013], these are the elements σ_{2i}.

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.sigma(1)
P{{-3, 3}, {-2, 2}, {-1, 1}}
sage: P3.sigma(3/2)
P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.sigma(2)
-P{{-3, -1, 1, 3}, {-2, 2}} + P{{-3, -1, 3}, {-2, 1, 2}}
 + P{{-3, 1, 3}, {-2, -1, 2}} - P{{-3, 3}, {-2, -1, 1, 2}}
 + P{{-3, 3}, {-2, 2}, {-1, 1}}
sage: P3.sigma(5/2)
-P{{-3, -1, 1, 2}, {-2, 3}} + P{{-3, -1, 2}, {-2, 1, 3}}
 + P{{-3, 1, 2}, {-2, -1, 3}} - P{{-3, 2}, {-2, -1, 1, 3}}
 + P{{-3, 2}, {-2, 3}, {-1, 1}}
```

We test the relations in Lemma 2.2.3(1) in [Cre2020] (v1):

```
sage: k = 4
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(k, x)
sage: all(P.sigma(i/2).dual() == P.sigma(i/2)
....:     for i in range(1,2*k))
True
sage: all(P.sigma(i)*P.sigma(i+1/2) == P.sigma(i+1/2)*P.sigma(i) == P.s(i)
....:     for i in range(1,floor(k)))
True
sage: all(P.sigma(i)*P.e(i) == P.e(i)*P.sigma(i) == P.e(i)
....:     for i in range(1,floor(k)))
True
sage: all(P.sigma(i+1/2)*P.e(i) == P.e(i)*P.sigma(i+1/2) == P.e(i)
....:     for i in range(1,floor(k)))
True

sage: k = 9/2
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(k, x)
sage: all(P.sigma(i/2).dual() == P.sigma(i/2)
....:     for i in range(1,2*k-1))
True
sage: all(P.sigma(i)*P.sigma(i+1/2) == P.sigma(i+1/2)*P.sigma(i) == P.s(i)
....:     for i in range(1,k-1/2))
True
sage: all(P.sigma(i)*P.e(i) == P.e(i)*P.sigma(i) == P.e(i)
....:     for i in range(1,floor(k)))
True
sage: all(P.sigma(i+1/2)*P.e(i) == P.e(i)*P.sigma(i+1/2) == P.e(i)
....:     for i in range(1,floor(k)))
True
```

class sage.combinat.diagram_algebras.**PartitionDiagram**(*parent*, *d*, *check=True*)

Bases: *AbstractPartitionDiagram*

The element class for a partition diagram.

A partition diagram for an integer k is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import PartitionDiagram, PartitionDiagrams
sage: PartitionDiagrams(1)
Partition diagrams of order 1
sage: PartitionDiagrams(1).list()
[{{-1, 1}}, {{-1}, {1}}]
sage: PartitionDiagram([[1,-1]])
{{-1, 1}}
sage: PartitionDiagram(((1,-2),(2,-1))).parent()
Partition diagrams of order 2
```

class sage.combinat.diagram_algebras.**PartitionDiagrams**(*order*, *category=None*)

Bases: *AbstractPartitionDiagrams*

This class represents all partition diagrams of integer or integer $+1/2$ order.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.PartitionDiagrams(1); pd
Partition diagrams of order 1
sage: pd.list()
[{{-1, 1}}, {{-1}, {1}}]

sage: pd = da.PartitionDiagrams(3/2); pd
Partition diagrams of order 3/2
sage: pd.list()
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}}]
```

Element

alias of *PartitionDiagram*

cardinality()

The cardinality of partition diagrams of half-integer order n is the $2n$-th Bell number.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.PartitionDiagrams(3)
sage: pd.cardinality()
203

sage: pd = da.PartitionDiagrams(7/2)
sage: pd.cardinality()
877
```

class sage.combinat.diagram_algebras.**PlanarAlgebra**(*k*, *q*, *base_ring*, *prefix*)

Bases: *SubPartitionAlgebra*, *UnitDiagramMixin*

A planar algebra.

The planar algebra of rank k is an algebra with basis indexed by the collection of all planar set partitions of $\{1, \ldots, k, -1, \ldots, -k\}$.

This algebra is thus a subalgebra of the partition algebra. For more information, see *PartitionAlgebra*.

INPUT:

- k – rank of the algebra
- q – the deformation parameter q

OPTIONAL ARGUMENTS:

- base_ring – (default None) a ring containing q; if None then just takes the parent of q
- prefix – (default "Pl") a label for the basis elements

EXAMPLES:

We define the planar algebra of rank 2 with parameter x over $\mathbf{Z}$:

```
sage: R.<x> = ZZ[]
sage: Pl = PlanarAlgebra(2, x, R); Pl
Planar Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over␣
↪Integer Ring
sage: Pl.basis().keys()
Planar diagrams of order 2
sage: Pl.basis().keys()([[-1, 1], [2, -2]])
{{-2, 2}, {-1, 1}}
sage: Pl.basis().list()
[Pl{{-2}, {-1}, {1, 2}},
 Pl{{-2}, {-1}, {1}, {2}},
 Pl{{-2, 1}, {-1}, {2}},
 Pl{{-2, 2}, {-1}, {1}},
 Pl{{-2, 1, 2}, {-1}},
 Pl{{-2, 2}, {-1, 1}},
 Pl{{-2}, {-1, 1}, {2}},
 Pl{{-2}, {-1, 2}, {1}},
 Pl{{-2}, {-1, 1, 2}},
 Pl{{-2, -1}, {1, 2}},
 Pl{{-2, -1}, {1}, {2}},
 Pl{{-2, -1, 1}, {2}},
 Pl{{-2, -1, 2}, {1}},
 Pl{{-2, -1, 1, 2}}]
sage: E = Pl([[1,2],[-1,-2]])
sage: E^2 == x*E
True
sage: E^5 == x^4*E
True
```

class sage.combinat.diagram_algebras.**PlanarDiagram**(*parent*, *d*, *check=True*)

Bases: *AbstractPartitionDiagram*

The element class for a planar diagram.

A planar diagram for an integer k is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$ so that the diagram is non-crossing.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import PlanarDiagrams
sage: PlanarDiagrams(2)
Planar diagrams of order 2
sage: PlanarDiagrams(2).list()
[{{-2}, {-1}, {1, 2}},
 {{-2}, {-1}, {1}, {2}},
 {{-2, 1}, {-1}, {2}},
 {{-2, 2}, {-1}, {1}},
 {{-2, 1, 2}, {-1}},
 {{-2, 2}, {-1, 1}},
 {{-2}, {-1, 1}, {2}},
 {{-2}, {-1, 2}, {1}},
 {{-2}, {-1, 1, 2}},
 {{-2, -1}, {1, 2}},
 {{-2, -1}, {1}, {2}},
 {{-2, -1, 1}, {2}},
 {{-2, -1, 2}, {1}},
 {{-2, -1, 1, 2}}]
```

check()

Check the validity of the input for `self`.

class sage.combinat.diagram_algebras.**PlanarDiagrams**(*order*, *category=None*)

Bases: *AbstractPartitionDiagrams*

All planar diagrams of integer or integer $+1/2$ order.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pld = da.PlanarDiagrams(1); pld
Planar diagrams of order 1
sage: pld.list()
[{{-1, 1}}, {{-1}, {1}}]

sage: pld = da.PlanarDiagrams(3/2); pld
Planar diagrams of order 3/2
sage: pld.list()
[{{-2, 1, 2}, {-1}},
 {{-2, 2}, {-1}, {1}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, -1, 1, 2}}]
```

Element

alias of *PlanarDiagram*

cardinality()

Return the cardinality of `self`.

The number of all planar diagrams of order k is the $2k$-th Catalan number.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pld = da.PlanarDiagrams(3)
sage: pld.cardinality()
132
```

class sage.combinat.diagram_algebras.**PropagatingIdeal**(*k*, *q*, *base_ring*, *prefix*)

Bases: *SubPartitionAlgebra*

A propagating ideal.

The propagating ideal of rank k is a non-unital algebra with basis indexed by the collection of ideal set partitions of $\{1, \ldots, k, -1, \ldots, -k\}$. We say a set partition is *ideal* if its propagating number is less than k.

This algebra is a non-unital subalgebra and an ideal of the partition algebra. For more information, see *PartitionAlgebra*.

EXAMPLES:

We now define the propagating ideal of rank 2 with parameter x over $\mathbf{Z}$:

```
sage: R.<x> = QQ[]
sage: I = PropagatingIdeal(2, x, R); I
Propagating Ideal of rank 2 with parameter x
 over Univariate Polynomial Ring in x over Rational Field
sage: I.basis().keys()
Ideal diagrams of order 2
sage: I.basis().list()
[I{{-2, -1, 1, 2}},
 I{{-2, 1, 2}, {-1}},
 I{{-2}, {-1, 1, 2}},
 I{{-2, -1}, {1, 2}},
 I{{-2}, {-1}, {1, 2}},
 I{{-2, -1, 1}, {2}},
 I{{-2, 1}, {-1}, {2}},
 I{{-2, -1, 2}, {1}},
 I{{-2, 2}, {-1}, {1}},
 I{{-2}, {-1, 1}, {2}},
 I{{-2}, {-1, 2}, {1}},
 I{{-2, -1}, {1}, {2}},
 I{{-2}, {-1}, {1}, {2}}]
sage: E = I([[1,2],[-1,-2]])
sage: E^2 == x*E
True
sage: E^5 == x^4*E
True
```

class Element

Bases: *Element*

An element of a propagating ideal.

We need to take care of exponents since we are not unital.

class sage.combinat.diagram_algebras.**SubPartitionAlgebra**(*k*, *q*, *base_ring*, *prefix*, *diagrams*, *category=None*)

Bases: *DiagramBasis*

A subalgebra of the partition algebra in the diagram basis indexed by a subset of the diagrams.

class Element

Bases: *Element*

to_orbit_basis()

Return `self` in the orbit basis of the associated ambient partition algebra.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: B = BrauerAlgebra(2, x, R)
sage: bb = B([[-2, -1], [1, 2]]); bb
B{{-2, -1}, {1, 2}}
sage: bb.to_orbit_basis()
O{{-2, -1}, {1, 2}} + O{{-2, -1, 1, 2}}
```

ambient()

Return the partition algebra `self` is a sub-algebra of.

EXAMPLES:

```
sage: x = var('x')                                                    #␣
↪optional - sage.symbolic
sage: BA = BrauerAlgebra(2, x)                                        #␣
↪optional - sage.symbolic
sage: BA.ambient()                                                    #␣
↪optional - sage.symbolic
Partition Algebra of rank 2 with parameter x over Symbolic Ring
```

lift()

Return the lift map from diagram subalgebra to the ambient space.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: BA = BrauerAlgebra(2, x, R)
sage: E = BA([[1,2],[-1,-2]])
sage: lifted = BA.lift(E); lifted
B{{-2, -1}, {1, 2}}
sage: lifted.parent() is BA.ambient()
True
```

retract(*x*)

Retract an appropriate partition algebra element to the corresponding element in the partition subalgebra.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: BA = BrauerAlgebra(2, x, R)
sage: PA = BA.ambient()
sage: E = PA([[1,2], [-1,-2]])
sage: BA.retract(E) in BA
True
```

sage.combinat.diagram_algebras.**TL_diagram_ascii_art**(*diagram*, *use_unicode=False*, *blobs=[]*)

Return ascii art for a Temperley-Lieb diagram `diagram`.

INPUT:

- `diagram` – a list of pairs of matchings of the set $\{-1, \ldots, -n, 1, \ldots, n\}$
- `use_unicode` – (default: `False`): whether or not to use unicode art instead of ascii art
- `blobs` – (optional) a list of matchings with blobs on them

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import TL_diagram_ascii_art
sage: TL = [(-15,-12), (-14,-13), (-11,15), (-10,14), (-9,-6),
....:       (-8,-7), (-5,-4), (-3,1), (-2,-1), (2,3), (4,5),
....:       (6,11), (7, 8), (9,10), (12,13)]
sage: TL_diagram_ascii_art(TL, use_unicode=False)
 o o o o o o o o o o o o o o o
 | `-` `-` | `-` `-` | `-` | |
 |         `---------`     | |
 |                 .-------` |
 `---.             | .-------`
     |     .-----. | | .-----.
 .-. | .-. | .-. | | | | .-. |
 o o o o o o o o o o o o o o o
sage: TL_diagram_ascii_art(TL, use_unicode=True)
 ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
 | +-+ +-+ | +-+ +-+ | +-+ | |
 |         +---+     | |
 |                 +---+ |
 +-+             | +---+
     |     +--+ | | +--+
 +-+ | +-+ | +-+ | | | | +-+ |
 ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

sage: TL = [(-20,-9), (-19,-10), (-18,-11), (-17,-16), (-15,-12), (2,3),
....:       (-14,-13), (-8,16), (-7,7), (-6,6), (-5,1), (-4,-3), (-2,-1),
....:       (4,5), (8,15), (9,10), (11,14), (12,13), (17,20), (18,19)]
sage: TL_diagram_ascii_art(TL, use_unicode=False, blobs=[(-2,-1), (-5,1)])
 o o o o o o o o o o o o o o o o o o o o
 | `-` `-` | | | `-` | `-` | | | | `-` |
 |         | | |     `-----` | | `-----`
 |         | | `-------------` |
 `---0---. | | .---------------`
         | | | | .---------------------.
         | | | | | .-----------------. |
         | | | | | | .-------------. | |
         | | | | | | | .-----.     | | |
 .0. .-. | | | | | | | | .-. | .-. | | |
 o o o o o o o o o o o o o o o o o o o o
sage: TL_diagram_ascii_art(TL, use_unicode=True, blobs=[(-2,-1), (-5,1)])
 ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
 | +-+ +-+ | | | +-+ | +-+ | | | | +-+ |
 |         | | |     +--+ | | +--+
 |         | | +------+ |
```

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```
+-●-+ | | +------+
        | | | | +--------+
        | | | | | +-------+ |
        | | | | | | +------+ | |
        | | | | | | | +--+     | | |
+●+ +-+ | | | | | | | | +-+ | +-+ | | |
o o o o o o o o o o o o o o o o o o o o
```

class sage.combinat.diagram_algebras.**TemperleyLiebAlgebra**(*k*, *q*, *base_ring*, *prefix*)

Bases: *SubPartitionAlgebra*, *UnitDiagramMixin*

A Temperley–Lieb algebra.

The Temperley–Lieb algebra of rank k is an algebra with basis indexed by the collection of planar set partitions of $\{1, \ldots, k, -1, \ldots, -k\}$ with block size 2.

This algebra is thus a subalgebra of the partition algebra. For more information, see *PartitionAlgebra*.

INPUT:

- `k` – rank of the algebra
- `q` – the deformation parameter q

OPTIONAL ARGUMENTS:

- `base_ring` – (default `None`) a ring containing `q`; if `None` then just takes the parent of `q`
- `prefix` – (default `"T"`) a label for the basis elements

EXAMPLES:

We define the Temperley–Lieb algebra of rank 2 with parameter x over $\mathbf{Z}$:

```
sage: R.<x> = ZZ[]
sage: T = TemperleyLiebAlgebra(2, x, R); T
Temperley-Lieb Algebra of rank 2 with parameter x
 over Univariate Polynomial Ring in x over Integer Ring
sage: T.basis()
Lazy family (Term map from Temperley Lieb diagrams of order 2
 to Temperley-Lieb Algebra of rank 2 with parameter x over
 Univariate Polynomial Ring in x over Integer
 Ring(i))_{i in Temperley Lieb diagrams of order 2}
sage: T.basis().keys()
Temperley Lieb diagrams of order 2
sage: T.basis().keys()([[-1, 1], [2, -2]])
{{-2, 2}, {-1, 1}}
sage: b = T.basis().list(); b
[T{{-2, -1}, {1, 2}}, T{{-2, 2}, {-1, 1}}]
sage: b[0]
T{{-2, -1}, {1, 2}}
sage: b[0]^2 == x*b[0]
True
sage: b[0]^5 == x^4*b[0]
True
```

class sage.combinat.diagram_algebras.**TemperleyLiebDiagram**(*parent*, *d*, *check=True*)

Bases: *AbstractPartitionDiagram*

The element class for a Temperley-Lieb diagram.

A Temperley-Lieb diagram for an integer k is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$ so that the blocks are all of size 2 and the diagram is planar.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import TemperleyLiebDiagrams
sage: TemperleyLiebDiagrams(2)
Temperley Lieb diagrams of order 2
sage: TemperleyLiebDiagrams(2).list()
[{{-2, -1}, {1, 2}}, {{-2, 2}, {-1, 1}}]
```

check()

Check the validity of the input for `self`.

class sage.combinat.diagram_algebras.**TemperleyLiebDiagrams**(*order*, *category=None*)

Bases: *AbstractPartitionDiagrams*

All Temperley-Lieb diagrams of integer or integer $+1/2$ order.

For more information on Temperley-Lieb diagrams, see *TemperleyLiebAlgebra*.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: td = da.TemperleyLiebDiagrams(3); td
Temperley Lieb diagrams of order 3
sage: td.list()
[{{-3, 3}, {-2, -1}, {1, 2}},
 {{-3, 1}, {-2, -1}, {2, 3}},
 {{-3, -2}, {-1, 1}, {2, 3}},
 {{-3, -2}, {-1, 3}, {1, 2}},
 {{-3, 3}, {-2, 2}, {-1, 1}}]

sage: td = da.TemperleyLiebDiagrams(5/2); td
Temperley Lieb diagrams of order 5/2
sage: td.list()
[{{-3, 3}, {-2, -1}, {1, 2}}, {{-3, 3}, {-2, 2}, {-1, 1}}]
```

Element

alias of *TemperleyLiebDiagram*

cardinality()

Return the cardinality of `self`.

The number of Temperley–Lieb diagrams of integer order k is the k-th Catalan number.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: td = da.TemperleyLiebDiagrams(3)
sage: td.cardinality()
5
```

class sage.combinat.diagram_algebras.**UnitDiagramMixin**

Bases: `object`

Mixin class for diagram algebras that have the unit indexed by the *identity_set_partition()*.

one_basis()

The following constructs the identity element of `self`.

It is not called directly; instead one should use `DA.one()` if `DA` is a defined diagram algebra.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(2, x, R)
sage: P.one_basis()
{{-2, 2}, {-1, 1}}
```

sage.combinat.diagram_algebras.**brauer_diagrams**(*k*)

Return a generator of all Brauer diagrams of order `k`.

A Brauer diagram of order k is a partition diagram of order k with block size 2.

INPUT:

- `k` – the order of the Brauer diagrams

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: [SetPartition(p) for p in da.brauer_diagrams(2)]
[{{-2, -1}, {1, 2}}, {{-2, 1}, {-1, 2}}, {{-2, 2}, {-1, 1}}]
sage: [SetPartition(p) for p in da.brauer_diagrams(5/2)]
[{{-3, 3}, {-2, -1}, {1, 2}},
 {{-3, 3}, {-2, 1}, {-1, 2}},
 {{-3, 3}, {-2, 2}, {-1, 1}}]
```

sage.combinat.diagram_algebras.**diagram_latex**(*diagram*, *fill=False*, *edge_options=None*, *edge_additions=None*)

Return latex code for the diagram `diagram` using tikz.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import PartitionDiagrams, diagram_latex
sage: P = PartitionDiagrams(2)
sage: D = P([[1,2],[-2,-1]])
sage: print(diagram_latex(D)) # indirect doctest
\begin{tikzpicture}[scale = 0.5,thick, baseline={(0,-1ex/2)}]
\tikzstyle{vertex} = [shape = circle, minimum size = 7pt, inner sep = 1pt]
\node[vertex] (G--2) at (1.5, -1) [shape = circle, draw] {};
\node[vertex] (G--1) at (0.0, -1) [shape = circle, draw] {};
\node[vertex] (G-1) at (0.0, 1) [shape = circle, draw] {};
\node[vertex] (G-2) at (1.5, 1) [shape = circle, draw] {};
\draw[] (G--2) .. controls +(-0.5, 0.5) and +(0.5, 0.5) .. (G--1);
\draw[] (G-1) .. controls +(0.5, -0.5) and +(-0.5, -0.5) .. (G-2);
\end{tikzpicture}
```

sage.combinat.diagram_algebras.**ideal_diagrams**(*k*)

Return a generator of all "ideal" diagrams of order `k`.

An ideal diagram of order k is a partition diagram of order k with propagating number less than k.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: all_diagrams = da.partition_diagrams(2)
sage: [SetPartition(p) for p in all_diagrams if p not in da.ideal_diagrams(2)]
[{{-2, 1}, {-1, 2}}, {{-2, 2}, {-1, 1}}]

sage: all_diagrams = da.partition_diagrams(3/2)
sage: [SetPartition(p) for p in all_diagrams if p not in da.ideal_diagrams(3/2)]
[{{-2, 2}, {-1, 1}}]
```

sage.combinat.diagram_algebras.**identity_set_partition**(*k*)

Return the identity set partition $\{\{1, -1\}, \ldots, \{k, -k\}\}$.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: SetPartition(da.identity_set_partition(2))
{{-2, 2}, {-1, 1}}
```

sage.combinat.diagram_algebras.**is_planar**(*sp*)

Return `True` if the diagram corresponding to the set partition `sp` is planar; otherwise, return `False`.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: da.is_planar( da.to_set_partition([[1,-2],[2,-1]]))
False
sage: da.is_planar( da.to_set_partition([[1,-1],[2,-2]]))
True
```

sage.combinat.diagram_algebras.**pair_to_graph**(*sp1*, *sp2*)

Return a graph consisting of the disjoint union of the graphs of set partitions `sp1` and `sp2` along with edges joining the bottom row (negative numbers) of `sp1` to the top row (positive numbers) of `sp2`.

The vertices of the graph `sp1` appear in the result as pairs `(k, 1)`, whereas the vertices of the graph `sp2` appear as pairs `(k, 2)`.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: sp1 = da.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = da.to_set_partition([[1,-2],[2,-1]])
sage: g = da.pair_to_graph( sp1, sp2 ); g
Graph on 8 vertices

sage: g.vertices(sort=True)
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]
sage: g.edges(sort=True)
[((-2, 1), (1, 1), None), ((-2, 1), (2, 2), None),
 ((-2, 2), (1, 2), None), ((-1, 1), (1, 2), None),
 ((-1, 1), (2, 1), None), ((-1, 2), (2, 2), None)]
```

Another example which used to be wrong until github issue #15958:

```
sage: sp3 = da.to_set_partition([[1, -1], [2], [-2]])
sage: sp4 = da.to_set_partition([[1], [-1], [2], [-2]])
```

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```
sage: g = da.pair_to_graph( sp3, sp4 ); g
Graph on 8 vertices

sage: g.vertices(sort=True)
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]
sage: g.edges(sort=True)
[((-2, 1), (2, 2), None), ((-1, 1), (1, 1), None),
 ((-1, 1), (1, 2), None)]
```

sage.combinat.diagram_algebras.**partition_diagrams**(k)

Return a generator of all partition diagrams of order k.

A partition diagram of order $k \in \mathbf{Z}$ to is a set partition of $\{1, \ldots, k, -1, \ldots, -k\}$. If we have $k - 1/2 \in ZZ$, then a partition diagram of order $k \in 1/2\mathbf{Z}$ is a set partition of $\{1, \ldots, k + 1/2, -1, \ldots, -(k + 1/2)\}$ with $k + 1/2$ and $-(k + 1/2)$ in the same block. See [HR2005].

INPUT:

- k – the order of the partition diagrams

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: [SetPartition(p) for p in da.partition_diagrams(2)]
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2}, {-1, 1, 2}},
 {{-2, -1}, {1, 2}},
 {{-2}, {-1}, {1, 2}},
 {{-2, -1, 1}, {2}},
 {{-2, 1}, {-1, 2}},
 {{-2, 1}, {-1}, {2}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}},
 {{-2}, {-1, 1}, {2}},
 {{-2}, {-1, 2}, {1}},
 {{-2, -1}, {1}, {2}},
 {{-2}, {-1}, {1}, {2}}]
sage: [SetPartition(p) for p in da.partition_diagrams(3/2)]
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}}]
```

sage.combinat.diagram_algebras.**planar_diagrams**(k)

Return a generator of all planar diagrams of order k.

A planar diagram of order k is a partition diagram of order k that has no crossings.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: all_diagrams = [SetPartition(p) for p in da.partition_diagrams(2)]
```

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```
sage: da2 = [SetPartition(p) for p in da.planar_diagrams(2)]
sage: [p for p in all_diagrams if p not in da2]
[{{-2, 1}, {-1, 2}}]
sage: all_diagrams = [SetPartition(p) for p in da.partition_diagrams(5/2)]
sage: da5o2 = [SetPartition(p) for p in da.planar_diagrams(5/2)]
sage: [p for p in all_diagrams if p not in da5o2]
[{{-3, -1, 3}, {-2, 1, 2}},
 {{-3, -2, 1, 3}, {-1, 2}},
 {{-3, -1, 1, 3}, {-2, 2}},
 {{-3, 1, 3}, {-2, -1, 2}},
 {{-3, 1, 3}, {-2, 2}, {-1}},
 {{-3, 1, 3}, {-2}, {-1, 2}},
 {{-3, -1, 2, 3}, {-2, 1}},
 {{-3, 3}, {-2, 1}, {-1, 2}},
 {{-3, -1, 3}, {-2, 1}, {2}},
 {{-3, -1, 3}, {-2, 2}, {1}}]
```

`sage.combinat.diagram_algebras.`**`planar_partitions_rec`**(X)

Iterate over all planar set partitions of `X` by using a recursive algorithm.

ALGORITHM:

To construct the set partition $\rho = \{\rho_1, \ldots, \rho_k\}$ of $[n]$, we remove the part of the set partition containing the last element of `X`, which, we consider to be $\rho_k = \{i_1, \ldots, i_m\}$ without loss of generality. The remaining parts come from the planar set partitions of $\{1, \ldots, i_1 - 1\}, \{i_1 + 1, \ldots, i_2 - 1\}, \ldots, \{i_m + 1, \ldots, n\}$.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: list(da.planar_partitions_rec([1,2,3]))
[([1, 2], [3]), ([1], [2], [3]), ([2], [1, 3]), ([1], [2, 3]), ([1, 2, 3],)]
```

`sage.combinat.diagram_algebras.`**`propagating_number`**(sp)

Return the propagating number of the set partition `sp`.

The propagating number is the number of blocks with both a positive and negative number.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: sp1 = da.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = da.to_set_partition([[1,2],[-2,-1]])
sage: da.propagating_number(sp1)
2
sage: da.propagating_number(sp2)
0
```

`sage.combinat.diagram_algebras.`**`temperley_lieb_diagrams`**(k)

Return a generator of all Temperley–Lieb diagrams of order `k`.

A Temperley–Lieb diagram of order k is a partition diagram of order k with block size 2 and is planar.

INPUT:

- `k` – the order of the Temperley–Lieb diagrams

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: [SetPartition(p) for p in da.temperley_lieb_diagrams(2)]
[{{-2, -1}, {1, 2}}, {{-2, 2}, {-1, 1}}]
sage: [SetPartition(p) for p in da.temperley_lieb_diagrams(5/2)]
[{{-3, 3}, {-2, -1}, {1, 2}}, {{-3, 3}, {-2, 2}, {-1, 1}}]
```

sage.combinat.diagram_algebras.**to_Brauer_partition**(*l*, *k=None*)

Same as *to_set_partition()* but assumes omitted elements are connected straight through.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: f = lambda sp: SetPartition(da.to_Brauer_partition(sp))
sage: f([[1,2],[-1,-2]]) == SetPartition([[1,2],[-1,-2]])
True
sage: f([[1,3],[-1,-3]]) == SetPartition([[1,3],[-3,-1],[2,-2]])
True
sage: f([[1,-4],[-3,-1],[3,4]]) == SetPartition([[-3,-1],[2,-2],[1,-4],[3,4]])
True
sage: p = SetPartition([[1,2],[-1,-2],[3,-3],[4,-4]])
sage: SetPartition(da.to_Brauer_partition([[1,2],[-1,-2]], k=4)) == p
True
```

sage.combinat.diagram_algebras.**to_graph**(*sp*)

Return a graph representing the set partition sp.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: g = da.to_graph( da.to_set_partition([[1,-2],[2,-1]])); g
Graph on 4 vertices

sage: g.vertices(sort=True)
[-2, -1, 1, 2]
sage: g.edges(sort=True)
[(-2, 1, None), (-1, 2, None)]
```

sage.combinat.diagram_algebras.**to_set_partition**(*l*, *k=None*)

Convert input to a set partition of $\{1, \ldots, k, -1, \ldots, -k\}$

Convert a list of a list of numbers to a set partitions. Each list of numbers in the outer list specifies the numbers contained in one of the blocks in the set partition.

If k is specified, then the set partition will be a set partition of $\{1, \ldots, k, -1, \ldots, -k\}$. Otherwise, k will default to the minimum number needed to contain all of the specified numbers.

INPUT:

- l - a list of lists of integers
- k - integer (optional, default None)

OUTPUT:

- a list of sets

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: f = lambda sp: SetPartition(da.to_set_partition(sp))
sage: f([[1,-1],[2,-2]]) == SetPartition(da.identity_set_partition(2))
True
sage: da.to_set_partition([[1]])
[{1}, {-1}]
sage: da.to_set_partition([[1,-1],[-2,3]],9/2)
[{-1, 1}, {-2, 3}, {2}, {-4, 4}, {-5, 5}, {-3}]
```

5.4 Clifford Algebras

AUTHORS:

- Travis Scrimshaw (2013-09-06): Initial version
- Trevor K. Karn (2022-07-27): Rewrite basis indexing using FrozenBitset

class sage.algebras.clifford_algebra.**CliffordAlgebra**(*Q*, *names*, *category=None*)

Bases: `CombinatorialFreeModule`

The Clifford algebra of a quadratic form.

Let $Q : V \to \mathbf{k}$ denote a quadratic form on a vector space V over a field $\mathbf{k}$. The Clifford algebra $Cl(V, Q)$ is defined as $T(V)/I_Q$ where $T(V)$ is the tensor algebra of V and I_Q is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v)$ for all $v \in V$.

We abuse notation to denote the projection of a pure tensor $x_1 \otimes x_2 \otimes \cdots \otimes x_m \in T(V)$ onto $T(V)/I_Q = Cl(V, Q)$ by $x_1 \wedge x_2 \wedge \cdots \wedge x_m$. This is motivated by the fact that $Cl(V, Q)$ is the exterior algebra $\wedge V$ when $Q = 0$ (one can also think of a Clifford algebra as a quantization of the exterior algebra). See `ExteriorAlgebra` for the concept of an exterior algebra.

From the definition, a basis of $Cl(V, Q)$ is given by monomials of the form

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\},$$

where $n = \dim(V)$ and where $\{e_1, e_2, \cdots, e_n\}$ is any fixed basis of V. Hence

$$\dim(Cl(V, Q)) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Note: The algebra $Cl(V, Q)$ is a $\mathbf{Z}/2\mathbf{Z}$-graded algebra, but not (in general) $\mathbf{Z}$-graded (in a reasonable way).

This construction satisfies the following universal property. Let $i : V \to Cl(V, Q)$ denote the natural inclusion (which is an embedding). Then for every associative $\mathbf{k}$-algebra A and any $\mathbf{k}$-linear map $j : V \to A$ satisfying

$$j(v)^2 = Q(v) \cdot 1_A$$

for all $v \in V$, there exists a unique $\mathbf{k}$-algebra homomorphism $f : Cl(V, Q) \to A$ such that $f \circ i = j$. This property determines the Clifford algebra uniquely up to canonical isomorphism. The inclusion i is commonly used to identify V with a vector subspace of $Cl(V)$.

The Clifford algebra $Cl(V, Q)$ is a $\mathbf{Z}_2$-graded algebra (where $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$); this grading is determined by placing all elements of V in degree 1. It is also an $\mathbf{N}$-filtered algebra, with the filtration too being defined by placing all elements of V in degree 1. The `degree()` gives the $\mathbf{N}$-*filtration* degree, and to get the super degree use instead `is_even_odd()`.

The Clifford algebra also can be considered as a covariant functor from the category of vector spaces equipped with quadratic forms to the category of algebras. In fact, if (V, Q) and (W, R) are two vector spaces endowed with quadratic forms, and if $g : W \to V$ is a linear map preserving the quadratic form, then we can define an algebra morphism $Cl(g) : Cl(W, R) \to Cl(V, Q)$ by requiring that it send every $w \in W$ to $g(w) \in V$. Since the quadratic form R on W is uniquely determined by the quadratic form Q on V (due to the assumption that g preserves the quadratic form), this fact can be rewritten as follows: If (V, Q) is a vector space with a quadratic form, and W is another vector space, and $\phi : W \to V$ is any linear map, then we obtain an algebra morphism $Cl(\phi) : Cl(W, \phi(Q)) \to Cl(V, Q)$ where $\phi(Q) = \phi^T \cdot Q \cdot \phi$ (we consider ϕ as a matrix) is the quadratic form Q pulled back to W. In fact, the map ϕ preserves the quadratic form because of

$$\phi(Q)(x) = x^T \cdot \phi^T \cdot Q \cdot \phi \cdot x = (\phi \cdot x)^T \cdot Q \cdot (\phi \cdot x) = Q(\phi(x)).$$

Hence we have $\phi(w)^2 = Q(\phi(w)) = \phi(Q)(w)$ for all $w \in W$.

REFERENCES:

- Wikipedia article Clifford_algebra

INPUT:

- `Q` – a quadratic form
- `names` – (default: `'e'`) the generator names

EXAMPLES:

To create a Clifford algebra, all one needs to do is specify a quadratic form:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl = CliffordAlgebra(Q)
sage: Cl
The Clifford algebra of the Quadratic form in 3 variables
 over Integer Ring with coefficients:
[ 1 2 3 ]
[ * 4 5 ]
[ * * 6 ]
```

We can also explicitly name the generators. In this example, the Clifford algebra we construct is an exterior algebra (since we choose the quadratic form to be zero):

```
sage: Q = QuadraticForm(ZZ, 4, [0]*10)
sage: Cl.<a,b,c,d> = CliffordAlgebra(Q)
sage: a*d
a*d
sage: d*c*b*a + a + 4*b*c
a*b*c*d + 4*b*c + a
```

`Element`

alias of `CliffordAlgebraElement`

`algebra_generators()`

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.algebra_generators()
Finite family {'x': x, 'y': y, 'z': z}
```

center_basis()

Return a list of elements which correspond to a basis for the center of `self`.

This assumes that the ground ring can be used to compute the kernel of a matrix.

See also:

supercenter_basis(), http://math.stackexchange.com/questions/129183/center-of-clifford-algebra-depending-on-the-parity-of-dim-v

Todo: Deprecate this in favor of a method called *center()* once subalgebras are properly implemented in Sage.

EXAMPLES:

```
sage: Q = QuadraticForm(QQ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Z = Cl.center_basis(); Z
(1, -2/5*x*y*z + x - 3/5*y + 2/5*z)
sage: all(z*b - b*z == 0 for z in Z for b in Cl.basis())
True

sage: Q = QuadraticForm(QQ, 3, [1,-2,-3, 4, 2, 1])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Z = Cl.center_basis(); Z
(1, -x*y*z + x + 3/2*y - z)
sage: all(z*b - b*z == 0 for z in Z for b in Cl.basis())
True

sage: Q = QuadraticForm(QQ, 2, [1,-2,-3])
sage: Cl.<x,y> = CliffordAlgebra(Q)
sage: Cl.center_basis()
(1,)

sage: Q = QuadraticForm(QQ, 2, [-1,1,-3])
sage: Cl.<x,y> = CliffordAlgebra(Q)
sage: Cl.center_basis()
(1,)
```

A degenerate case:

```
sage: Q = QuadraticForm(QQ, 3, [4,4,-4,1,-2,1])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.center_basis()
(1, x*y*z + x - 2*y - 2*z, x*y + x*z - 2*y*z)
```

The most degenerate case (the exterior algebra):

```
sage: Q = QuadraticForm(QQ, 3)
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.center_basis()
(1, x*y, x*z, y*z, x*y*z)
```

degree_on_basis(m)

Return the degree of the monomial indexed by m.

We are considering the Clifford algebra to be $\mathbf{N}$-filtered, and the degree of the monomial m is the length of m.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.degree_on_basis((0,))
1
sage: Cl.degree_on_basis((0,1))
2
```

dimension()

Return the rank of self as a free module.

Let V be a free R-module of rank n; then, $Cl(V, Q)$ is a free R-module of rank 2^n.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.dimension()
8
```

free_module()

Return the underlying free module V of self.

This is the free module on which the quadratic form that was used to construct self is defined.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.free_module()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

gen(i)

Return the i-th standard generator of the algebra self.

This is the i-th basis vector of the vector space on which the quadratic form defining self is defined, regarded as an element of self.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: [Cl.gen(i) for i in range(3)]
[x, y, z]
```

gens()

Return the generators of `self` (as an algebra).

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.gens()
(x, y, z)
```

graded_algebra()

Return the associated graded algebra of `self`.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.graded_algebra()
The exterior algebra of rank 3 over Integer Ring
```

is_commutative()

Check if `self` is a commutative algebra.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.is_commutative()
False
```

lift_isometry(*m*, *names=None*)

Lift an invertible isometry `m` of the quadratic form of `self` to a Clifford algebra morphism.

Given an invertible linear map $m : V \to W$ (here represented by a matrix acting on column vectors), this method returns the algebra morphism $Cl(m)$ from $Cl(V, Q)$ to $Cl(W, m^{-1}(Q))$, where $Cl(V, Q)$ is the Clifford algebra `self` and where $m^{-1}(Q)$ is the pullback of the quadratic form Q to W along the inverse map $m^{-1} : W \to V$. See the documentation of `CliffordAlgebra` for how this pullback and the morphism $Cl(m)$ are defined.

INPUT:

- `m` – an isometry of the quadratic form of `self`
- `names` – (default: `'e'`) the names of the generators of the Clifford algebra of the codomain of (the map represented by) `m`

OUTPUT:

The algebra morphism $Cl(m)$ from `self` to $Cl(W, m^{-1}(Q))$.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: m = matrix([[1,1,2],[0,1,1],[0,0,1]])
sage: phi = Cl.lift_isometry(m, 'abc')
sage: phi(x)
a
```

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```
sage: phi(y)
a + b
sage: phi(x*y)
a*b + 1
sage: phi(x) * phi(y)
a*b + 1
sage: phi(z*y)
a*b - a*c - b*c
sage: phi(z) * phi(y)
a*b - a*c - b*c
sage: phi(x + z) * phi(y + z) == phi((x + z) * (y + z))
True
```

lift_module_morphism(*m*, *names=None*)

Lift the matrix m to an algebra morphism of Clifford algebras.

Given a linear map $m : W \to V$ (here represented by a matrix acting on column vectors), this method returns the algebra morphism $Cl(m) : Cl(W, m(Q)) \to Cl(V, Q)$, where $Cl(V, Q)$ is the Clifford algebra self and where $m(Q)$ is the pullback of the quadratic form Q to W. See the documentation of *CliffordAlgebra* for how this pullback and the morphism $Cl(m)$ are defined.

Note: This is a map into self.

INPUT:

- m – a matrix
- names – (default: 'e') the names of the generators of the Clifford algebra of the domain of (the map represented by) m

OUTPUT:

The algebra morphism $Cl(m)$ from $Cl(W, m(Q))$ to self.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: m = matrix([[1,-1,-1],[0,1,-1],[1,1,1]])
sage: phi = Cl.lift_module_morphism(m, 'abc')
sage: phi
Generic morphism:
  From: The Clifford algebra of the Quadratic form in 3 variables over Integer␣
↪Ring with coefficients:
[ 10 17 3 ]
[ * 11 0 ]
[ * * 5 ]
  To:   The Clifford algebra of the Quadratic form in 3 variables over Integer␣
↪Ring with coefficients:
[ 1 2 3 ]
[ * 4 5 ]
[ * * 6 ]
sage: a,b,c = phi.domain().gens()
sage: phi(a)
```

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```
x + z
sage: phi(b)
-x + y + z
sage: phi(c)
-x - y + z
sage: phi(a + 3*b)
-2*x + 3*y + 4*z
sage: phi(a) + 3*phi(b)
-2*x + 3*y + 4*z
sage: phi(a*b)
x*y + 2*x*z - y*z + 7
sage: phi(b*a)
-x*y - 2*x*z + y*z + 10
sage: phi(a*b + c)
x*y + 2*x*z - y*z - x - y + z + 7
sage: phi(a*b) + phi(c)
x*y + 2*x*z - y*z - x - y + z + 7
```

We check that the map is an algebra morphism:

```
sage: phi(a)*phi(b)
x*y + 2*x*z - y*z + 7
sage: phi(a*b)
x*y + 2*x*z - y*z + 7
sage: phi(a*a)
10
sage: phi(a)*phi(a)
10
sage: phi(b*a)
-x*y - 2*x*z + y*z + 10
sage: phi(b) * phi(a)
-x*y - 2*x*z + y*z + 10
sage: phi((a + b)*(a + c)) == phi(a + b) * phi(a + c)
True
```

We can also lift arbitrary linear maps:

```
sage: m = matrix([[1,1],[0,1],[1,1]])
sage: phi = Cl.lift_module_morphism(m, 'ab')
sage: a,b = phi.domain().gens()
sage: phi(a)
x + z
sage: phi(b)
x + y + z
sage: phi(a*b)
x*y - y*z + 15
sage: phi(a)*phi(b)
x*y - y*z + 15
sage: phi(b*a)
-x*y + y*z + 12
sage: phi(b)*phi(a)
-x*y + y*z + 12
```

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```
sage: m = matrix([[1,1,1,2], [0,1,1,1], [0,1,1,1]])
sage: phi = Cl.lift_module_morphism(m, 'abcd')
sage: a,b,c,d = phi.domain().gens()
sage: phi(a)
x
sage: phi(b)
x + y + z
sage: phi(c)
x + y + z
sage: phi(d)
2*x + y + z
sage: phi(a*b*c + d*a)
-x*y - x*z + 21*x + 7
sage: phi(a*b*c*d)
21*x*y + 21*x*z + 42
```

ngens()

Return the number of algebra generators of `self`.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.ngens()
3
```

one_basis()

Return the basis index of the element 1. The element 1 is indexed by the emptyset, which is represented by the `sage.data_structures.bitset.Bitset` `0`.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.one_basis()
0
```

pseudoscalar()

Return the unit pseudoscalar of `self`.

Given the basis $e_1, e_2, \ldots, e_n$ of the underlying R-module, the unit pseudoscalar is defined as $e_1 \cdot e_2 \cdots e_n$.

This depends on the choice of basis.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.pseudoscalar()
x*y*z

sage: Q = QuadraticForm(ZZ, 0, [])
sage: Cl = CliffordAlgebra(Q)
```

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```
sage: Cl.pseudoscalar()
1
```

REFERENCES:

- Wikipedia article Classification_of_Clifford_algebras#Unit_pseudoscalar

quadratic_form()

Return the quadratic form of `self`.

This is the quadratic form used to define `self`. The quadratic form on `self` is yet to be implemented.

EXAMPLES:

```
sage: Q = QuadraticForm(ZZ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.quadratic_form()
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 2 3 ]
[ * 4 5 ]
[ * * 6 ]
```

supercenter_basis()

Return a list of elements which correspond to a basis for the supercenter of `self`.

This assumes that the ground ring can be used to compute the kernel of a matrix.

See also:

center_basis(), http://math.stackexchange.com/questions/129183/center-of-clifford-algebra-depending-on-the-parity-of

Todo: Deprecate this in favor of a method called *supercenter()* once subalgebras are properly implemented in Sage.

EXAMPLES:

```
sage: Q = QuadraticForm(QQ, 3, [1,2,3,4,5,6])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: SZ = Cl.supercenter_basis(); SZ
(1,)
sage: all(z.supercommutator(b) == 0 for z in SZ for b in Cl.basis())
True

sage: Q = QuadraticForm(QQ, 3, [1,-2,-3, 4, 2, 1])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.supercenter_basis()
(1,)

sage: Q = QuadraticForm(QQ, 2, [1,-2,-3])
sage: Cl.<x,y> = CliffordAlgebra(Q)
sage: Cl.supercenter_basis()
(1,)

sage: Q = QuadraticForm(QQ, 2, [-1,1,-3])
```

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```
sage: Cl.<x,y> = CliffordAlgebra(Q)
sage: Cl.supercenter_basis()
(1,)
```

Singular vectors of a quadratic form generate in the supercenter:

```
sage: Q = QuadraticForm(QQ, 3, [1/2,-2,4,256/249,3,-185/8])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.supercenter_basis()
(1, x + 249/322*y + 22/161*z)

sage: Q = QuadraticForm(QQ, 3, [4,4,-4,1,-2,1])
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.supercenter_basis()
(1, x + 2*z, y + z, x*y + x*z - 2*y*z)
```

The most degenerate case:

```
sage: Q = QuadraticForm(QQ, 3)
sage: Cl.<x,y,z> = CliffordAlgebra(Q)
sage: Cl.supercenter_basis()
(1, x, y, z, x*y, x*z, y*z, x*y*z)
```

class `sage.algebras.clifford_algebra.`**`CliffordAlgebraIndices`**(*Qdim*)

Bases: `UniqueRepresentation`, `Parent`

A facade parent for the indices of Clifford algebra. Users should not create instances of this class directly.

`cardinality()`

Return the cardinality of `self`.

EXAMPLES:

```
sage: from sage.algebras.clifford_algebra import CliffordAlgebraIndices
sage: idx = CliffordAlgebraIndices(7)
sage: idx.cardinality() == 2^7
True
sage: len(idx) == 2^7
True
```

class `sage.algebras.clifford_algebra.`**`ExteriorAlgebra`**(*R*, *names*)

Bases: *`CliffordAlgebra`*

An exterior algebra of a free module over a commutative ring.

Let V be a module over a commutative ring R. The exterior algebra (or Grassmann algebra) $\Lambda(V)$ of V is defined as the quotient of the tensor algebra $T(V)$ of V modulo the two-sided ideal generated by all tensors of the form $x \otimes x$ with $x \in V$. The multiplication on $\Lambda(V)$ is denoted by $\wedge$ (so $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is the projection of $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ onto $\Lambda(V)$) and called the "exterior product" or "wedge product".

If V is a rank-n free R-module with a basis $\{e_1, \ldots, e_n\}$, then $\Lambda(V)$ is the R-algebra noncommutatively generated by the n generators $e_1, \ldots, e_n$ subject to the relations $e_i^2 = 0$ for all i, and $e_i e_j = -e_j e_i$ for all $i < j$. As an R-module, $\Lambda(V)$ then has a basis $(\bigwedge_{i \in I} e_i)$ with I ranging over the subsets of $\{1, 2, \ldots, n\}$ (where $\bigwedge_{i \in I} e_i$ is the wedge product of e_i for i running through all elements of I from smallest to largest), and hence is free of rank 2^n.

The exterior algebra of an R-module V can also be realized as the Clifford algebra of V for the quadratic form Q given by $Q(v) = 0$ for all vectors $v \in V$. See *CliffordAlgebra* for the notion of a Clifford algebra.

The exterior algebra of an R-module V is a connected $\mathbf{Z}$-graded Hopf superalgebra. It is commutative in the super sense (i.e., the odd elements anticommute and square to 0).

This class implements the exterior algebra $\Lambda(R^n)$ for n a nonnegative integer.

INPUT:

- `R` – the base ring, *or* the free module whose exterior algebra is to be computed
- `names` – a list of strings to name the generators of the exterior algebra; this list can either have one entry only (in which case the generators will be called `e + '0'`, `e + '1'`, ..., `e + 'n-1'`, with `e` being said entry), or have `n` entries (in which case these entries will be used directly as names for the generators)
- `n` – the number of generators, i.e., the rank of the free module whose exterior algebra is to be computed (this doesn't have to be provided if it can be inferred from the rest of the input)

REFERENCES:

- Wikipedia article Exterior_algebra

Element

alias of `ExteriorAlgebraElement`

antipode_on_basis(*m*)

Return the antipode on the basis element indexed by `m`.

Given a basis element ω, the antipode is defined by $S(\omega) = (-1)^{\deg(\omega)}\omega$.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: E.antipode_on_basis(())
1
sage: E.antipode_on_basis((1,))
-y
sage: E.antipode_on_basis((1,2))
y*z
```

boundary(*s_coeff*)

Return the boundary operator ∂ defined by the structure coefficients `s_coeff` of a Lie algebra.

For more on the boundary operator, see *ExteriorAlgebraBoundary*.

INPUT:

- `s_coeff` – a dictionary whose keys are in $I \times I$, where I is the index set of the underlying vector space V, and whose values can be coerced into 1-forms (degree 1 elements) in `E` (usually, these values will just be elements of V)

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: E.boundary({(0,1): z, (1,2): x, (2,0): y})
Boundary endomorphism of The exterior algebra of rank 3 over Rational Field
```

coboundary(*s_coeff*)

Return the coboundary operator d defined by the structure coefficients `s_coeff` of a Lie algebra.

For more on the coboundary operator, see *ExteriorAlgebraCoboundary*.

INPUT:

- s_coeff – a dictionary whose keys are in $I \times I$, where I is the index set of the underlying vector space V, and whose values can be coerced into 1-forms (degree 1 elements) in E (usually, these values will just be elements of V)

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: E.coboundary({(0,1): z, (1,2): x, (2,0): y})
Coboundary endomorphism of The exterior algebra of rank 3 over Rational Field
```

coproduct_on_basis(a)

Return the coproduct on the basis element indexed by a.

The coproduct is defined by

$$\Delta(e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{k=0}^{m} \sum_{\sigma \in Ush_{k,m-k}} (-1)^\sigma (e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(k)}}) \otimes (e_{i_{\sigma(k+1)}} \wedge \cdots \wedge e_{i_{\sigma(m)}}),$$

where $Ush_{k,m-k}$ denotes the set of all $(k, m-k)$-unshuffles (i.e., permutations in S_m which are increasing on the interval $\{1, 2, \ldots, k\}$ and on the interval $\{k+1, k+2, \ldots, k+m\}$).

> **Warning:** This coproduct is a homomorphism of superalgebras, not a homomorphism of algebras!

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: E.coproduct_on_basis((0,))
1 # x + x # 1
sage: E.coproduct_on_basis((0,1))
1 # x*y + x # y - y # x + x*y # 1
sage: E.coproduct_on_basis((0,1,2))
1 # x*y*z + x # y*z - y # x*z + x*y # z
 + z # x*y - x*z # y + y*z # x + x*y*z # 1
```

counit(x)

Return the counit of x.

The counit of an element ω of the exterior algebra is its constant coefficient.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: elt = x*y - 2*x + 3
sage: E.counit(elt)
3
```

degree_on_basis(m)

Return the degree of the monomial indexed by m.

The degree of m in the $\mathbf{Z}$-grading of self is defined to be the length of m.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: E.degree_on_basis(())
0
sage: E.degree_on_basis((0,))
1
sage: E.degree_on_basis((0,1))
2
```

interior_product_on_basis(*a*, *b*)

Return the interior product $\iota_b a$ of a with respect to b.

See `interior_product()` for more information.

In this method, a and b are supposed to be basis elements (see `interior_product()` for a method that computes interior product of arbitrary elements), and to be input as their keys.

This depends on the choice of basis of the vector space whose exterior algebra is `self`.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: k = list(E.basis().keys())
sage: E.interior_product_on_basis(k[1], k[1])
1
sage: E.interior_product_on_basis(k[5], k[1])
z
sage: E.interior_product_on_basis(k[2], k[5])
0
sage: E.interior_product_on_basis(k[5], k[2])
0
sage: E.interior_product_on_basis(k[7], k[5])
-y
```

Check github issue #34694:

```
sage: E = ExteriorAlgebra(SR,'e',3)
sage: E.inject_variables()
Defining e0, e1, e2
sage: a = (e0*e1).interior_product(e0)
sage: a * e0
-e0*e1
```

lift_morphism(*phi*, *names=None*)

Lift the matrix m to an algebra morphism of exterior algebras.

Given a linear map $\phi : V \to W$ (here represented by a matrix acting on column vectors over the base ring of V), this method returns the algebra morphism $\Lambda(\phi) : \Lambda(V) \to \Lambda(W)$. This morphism is defined on generators $v_i \in \Lambda(V)$ by $v_i \mapsto \phi(v_i)$.

Note: This is the map going out of `self` as opposed to `lift_module_morphism()` for general Clifford algebras.

INPUT:

- phi – a linear map ϕ from V to W, encoded as a matrix

- `names` – (default: `'e'`) the names of the generators of the Clifford algebra of the domain of (the map represented by) `phi`

OUTPUT:

The algebra morphism $\Lambda(\phi)$ from `self` to $\Lambda(W)$.

EXAMPLES:

```
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: phi = matrix([[0,1],[1,1],[1,2]]); phi
[0 1]
[1 1]
[1 2]
sage: L = E.lift_morphism(phi, ['a','b','c']); L
Generic morphism:
  From: The exterior algebra of rank 2 over Rational Field
  To:   The exterior algebra of rank 3 over Rational Field
sage: L(x)
b + c
sage: L(y)
a + b + 2*c
sage: L.on_basis()((1,))
a + b + 2*c
sage: p = L(E.one()); p
1
sage: p.parent()
The exterior algebra of rank 3 over Rational Field
sage: L(x*y)
-a*b - a*c + b*c
sage: L(x)*L(y)
-a*b - a*c + b*c
sage: L(x + y)
a + 2*b + 3*c
sage: L(x) + L(y)
a + 2*b + 3*c
sage: L(1/2*x + 2)
1/2*b + 1/2*c + 2
sage: L(E(3))
3

sage: psi = matrix([[1, -3/2]]); psi
[   1 -3/2]
sage: Lp = E.lift_morphism(psi, ['a']); Lp
Generic morphism:
  From: The exterior algebra of rank 2 over Rational Field
  To:   The exterior algebra of rank 1 over Rational Field
sage: Lp(x)
a
sage: Lp(y)
-3/2*a
sage: Lp(x + 2*y + 3)
-2*a + 3
```

lifted_bilinear_form(M)

Return the bilinear form on the exterior algebra `self` $= \Lambda(V)$ which is obtained by lifting the bilinear form f on V given by the matrix `M`.

Let V be a module over a commutative ring R, and let $f : V \times V \to R$ be a bilinear form on V. Then, a bilinear form $\Lambda(f) : \Lambda(V) \times \Lambda(V) \to R$ on $\Lambda(V)$ can be canonically defined as follows: For every $n \in \mathbf{N}$, $m \in \mathbf{N}$, $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m \in V$, we define

$$\Lambda(f)(v_1 \wedge v_2 \wedge \cdots \wedge v_n, w_1 \wedge w_2 \wedge \cdots \wedge w_m) := \begin{cases} 0, & \text{if } n \neq m; \\ \det G, & \text{if } n = m \end{cases},$$

where G is the $n \times m$-matrix whose (i, j)-th entry is $f(v_i, w_j)$. This bilinear form $\Lambda(f)$ is known as the bilinear form on $\Lambda(V)$ obtained by lifting the bilinear form f. Its restriction to the 1-st homogeneous component V of $\Lambda(V)$ is f.

The bilinear form $\Lambda(f)$ is symmetric if f is.

INPUT:

- `M` – a matrix over the same base ring as `self`, whose (i, j)-th entry is $f(e_i, e_j)$, where $(e_1, e_2, \ldots, e_N)$ is the standard basis of the module V for which `self` $= \Lambda(V)$ (so that $N = \dim(V)$), and where f is the bilinear form which is to be lifted.

OUTPUT:

A bivariate function which takes two elements p and q of `self` to $\Lambda(f)(p, q)$.

Note: This takes a bilinear form on V as matrix, and returns a bilinear form on `self` as a function in two arguments. We do not return the bilinear form as a matrix since this matrix can be huge and one often needs just a particular value.

Todo: Implement a class for bilinear forms and rewrite this method to use that class.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: M = Matrix(QQ, [[1, 2, 3], [2, 3, 4], [3, 4, 5]])
sage: Eform = E.lifted_bilinear_form(M)
sage: Eform
Bilinear Form from The exterior algebra of rank 3 over Rational
Field (+) The exterior algebra of rank 3 over Rational Field to
Rational Field
sage: Eform(x*y, y*z)
-1
sage: Eform(x*y, y)
0
sage: Eform(x*(y+z), y*z)
-3
sage: Eform(x*(y+z), y*(z+x))
0
sage: N = Matrix(QQ, [[3, 1, 7], [2, 0, 4], [-1, -3, -1]])
sage: N.determinant()
-8
sage: Eform = E.lifted_bilinear_form(N)
sage: Eform(x, E.one())
```

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```
0
sage: Eform(x, x*z*y)
0
sage: Eform(E.one(), E.one())
1
sage: Eform(E.zero(), E.one())
0
sage: Eform(x, y)
1
sage: Eform(z, y)
-3
sage: Eform(x*z, y*z)
20
sage: Eform(x+x*y+x*y*z, z+z*y+z*y*x)
11
```

Todo: Another way to compute this bilinear form seems to be to map x and y to the appropriate Clifford algebra and there compute $x^t y$, then send the result back to the exterior algebra and return its constant coefficient. Or something like this. Once the maps to the Clifford and back are implemented, check if this is faster.

volume_form()

Return the volume form of `self`.

Given the basis $e_1, e_2, \ldots, e_n$ of the underlying R-module, the volume form is defined as $e_1 \wedge e_2 \wedge \cdots \wedge e_n$.

This depends on the choice of basis.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: E.volume_form()
x*y*z
```

class sage.algebras.clifford_algebra.**ExteriorAlgebraBoundary**(*E*, *s_coeff*)

Bases: *ExteriorAlgebraDifferential*

The boundary ∂ of an exterior algebra $\Lambda(L)$ defined by the structure coefficients of L.

Let L be a Lie algebra. We give the exterior algebra $E = \Lambda(L)$ a chain complex structure by considering a differential $\partial : \Lambda^{k+1}(L) \to \Lambda^k(L)$ defined by

$$\partial(x_1 \wedge x_2 \wedge \cdots \wedge x_{k+1}) = \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{k+1}$$

where $\hat{x}_i$ denotes a missing index. The corresponding homology is the Lie algebra homology.

INPUT:

- `E` – an exterior algebra of a vector space L
- `s_coeff` – a dictionary whose keys are in $I \times I$, where I is the index set of the basis of the vector space L, and whose values can be coerced into 1-forms (degree 1 elements) in `E`; this dictionary will be used to define the Lie algebra structure on L (indeed, the i-th coordinate of the Lie bracket of the j-th and k-th basis vectors of L for $j < k$ is set to be the value at the key (j, k) if this key appears in `s_coeff`, or otherwise the negated of the value at the key (k, j))

Warning: The values of `s_coeff` are supposed to be coercible into 1-forms in `E`; but they can also be dictionaries themselves (in which case they are interpreted as giving the coordinates of vectors in `L`). In the interest of speed, these dictionaries are not sanitized or checked.

Warning: For any two distinct elements i and j of I, the dictionary `s_coeff` must have only one of the pairs (i, j) and (j, i) as a key. This is not checked.

EXAMPLES:

We consider the differential given by Lie algebra given by the cross product $\times$ of $\mathbf{R}^3$:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: par = E.boundary({(0,1): z, (1,2): x, (2,0): y})
sage: par(x)
0
sage: par(x*y)
z
sage: par(x*y*z)
0
sage: par(x+y-y*z+x*y)
-x + z
sage: par(E.zero())
0
```

We check that $\partial \circ \partial = 0$:

```
sage: p2 = par * par
sage: all(p2(b) == 0 for b in E.basis())
True
```

Another example: the Lie algebra $\mathfrak{sl}_2$, which has a basis e, f, h satisfying $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$:

```
sage: E.<e,f,h> = ExteriorAlgebra(QQ)
sage: par = E.boundary({(0,1): h, (2,1): -2*f, (2,0): 2*e})
sage: par(E.zero())
0
sage: par(e)
0
sage: par(e*f)
h
sage: par(f*h)
2*f
sage: par(h*f)
-2*f
sage: C = par.chain_complex(); C
Chain complex with at most 4 nonzero terms over Rational Field
sage: ascii_art(C)
                          [ 0 -2  0]       [0]
                          [ 0  0  2]       [0]
            [0 0 0]       [ 1  0  0]       [0]
```

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```
 0 <-- C_0 <-------- C_1 <----------- C_2 <---- C_3 <-- 0
sage: C.homology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 0 over Rational Field,
 2: Vector space of dimension 0 over Rational Field,
 3: Vector space of dimension 1 over Rational Field}
```

Over the integers:

```
sage: C = par.chain_complex(R=ZZ); C
Chain complex with at most 4 nonzero terms over Integer Ring
sage: ascii_art(C)
                          [ 0 -2  0]        [0]
                          [ 0  0  2]        [0]
            [0 0 0]       [ 1  0  0]        [0]
 0 <-- C_0 <-------- C_1 <----------- C_2 <---- C_3 <-- 0
sage: C.homology()
{0: Z, 1: C2 x C2, 2: 0, 3: Z}
```

REFERENCES:

- Wikipedia article Exterior_algebra#Lie_algebra_homology

chain_complex(*R=None*)

Return the chain complex over R determined by `self`.

INPUT:

- R – the base ring; the default is the base ring of the exterior algebra

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: par = E.boundary({(0,1): z, (1,2): x, (2,0): y})
sage: C = par.chain_complex(); C
Chain complex with at most 4 nonzero terms over Rational Field
sage: ascii_art(C)
                          [ 0  0  1]        [0]
                          [ 0 -1  0]        [0]
            [0 0 0]       [ 1  0  0]        [0]
 0 <-- C_0 <-------- C_1 <----------- C_2 <---- C_3 <-- 0
```

class sage.algebras.clifford_algebra.**ExteriorAlgebraCoboundary**(*E*, *s_coeff*)

Bases: `ExteriorAlgebraDifferential`

The coboundary d of an exterior algebra $\Lambda(L)$ defined by the structure coefficients of a Lie algebra L.

Let L be a Lie algebra. We endow its exterior algebra $E = \Lambda(L)$ with a cochain complex structure by considering a differential $d : \Lambda^k(L) \to \Lambda^{k+1}(L)$ defined by

$$dx_i = \sum_{j<k} s_{jk}^i x_j x_k,$$

where $(x_1, x_2, \ldots, x_n)$ is a basis of L, and where s_{jk}^i is the x_i-coordinate of the Lie bracket $[x_j, x_k]$.

The corresponding cohomology is the Lie algebra cohomology of L.

This can also be thought of as the exterior derivative, in which case the resulting cohomology is the de Rham cohomology of a manifold whose exterior algebra of differential forms is E.

INPUT:

- E – an exterior algebra of a vector space L
- s_coeff – a dictionary whose keys are in $I \times I$, where I is the index set of the basis of the vector space L, and whose values can be coerced into 1-forms (degree 1 elements) in E; this dictionary will be used to define the Lie algebra structure on L (indeed, the i-th coordinate of the Lie bracket of the j-th and k-th basis vectors of L for $j < k$ is set to be the value at the key (j, k) if this key appears in s_coeff, or otherwise the negated of the value at the key (k, j))

> **Warning:** For any two distinct elements i and j of I, the dictionary s_coeff must have only one of the pairs (i, j) and (j, i) as a key. This is not checked.

EXAMPLES:

We consider the differential coming from the Lie algebra given by the cross product $\times$ of $\mathbf{R}^3$:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: d = E.coboundary({(0,1): z, (1,2): x, (0, 2): -y})
sage: d(x)
y*z
sage: d(y)
-x*z
sage: d(x+y-y*z)
-x*z + y*z
sage: d(x*y)
0
sage: d(E.one())
0
sage: d(E.zero())
0
```

We check that $d \circ d = 0$:

```
sage: d2 = d * d
sage: all(d2(b) == 0 for b in E.basis())
True
```

Another example: the Lie algebra $\mathfrak{sl}_2$, which has a basis e, f, h satisfying $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$:

```
sage: E.<e,f,h> = ExteriorAlgebra(QQ)
sage: d = E.coboundary({(0,1): h, (2,1): -2*f, (2,0): 2*e})
sage: d(E.zero())
0
sage: d(e)
-2*e*h
sage: d(f)
2*f*h
sage: d(h)
e*f
```

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```
sage: d(e*f)
0
sage: d(f*h)
0
sage: d(e*h)
0
sage: C = d.chain_complex(); C
Chain complex with at most 4 nonzero terms over Rational Field
sage: ascii_art(C)
                          [ 0  0  1]       [0]
                          [-2  0  0]       [0]
            [0 0 0]       [ 0  2  0]       [0]
 0 <-- C_3 <-------- C_2 <----------- C_1 <---- C_0 <-- 0
sage: C.homology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 0 over Rational Field,
 2: Vector space of dimension 0 over Rational Field,
 3: Vector space of dimension 1 over Rational Field}
```

Over the integers:

```
sage: C = d.chain_complex(R=ZZ); C
Chain complex with at most 4 nonzero terms over Integer Ring
sage: ascii_art(C)
                          [ 0  0  1]       [0]
                          [-2  0  0]       [0]
            [0 0 0]       [ 0  2  0]       [0]
 0 <-- C_3 <-------- C_2 <----------- C_1 <---- C_0 <-- 0
sage: C.homology()
{0: Z, 1: 0, 2: C2 x C2, 3: Z}
```

REFERENCES:

- Wikipedia article Exterior_algebra#Differential_geometry

chain_complex(*R=None*)

Return the chain complex over R determined by `self`.

INPUT:

- R – the base ring; the default is the base ring of the exterior algebra

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: d = E.coboundary({(0,1): z, (1,2): x, (2,0): y})
sage: C = d.chain_complex(); C
Chain complex with at most 4 nonzero terms over Rational Field
sage: ascii_art(C)
                          [ 0  0  1]       [0]
                          [ 0 -1  0]       [0]
            [0 0 0]       [ 1  0  0]       [0]
 0 <-- C_3 <-------- C_2 <----------- C_1 <---- C_0 <-- 0
```

class sage.algebras.clifford_algebra.**ExteriorAlgebraDifferential**(*E*, *s_coeff*)

Bases: `ModuleMorphismByLinearity`, `UniqueRepresentation`

Internal class to store the data of a boundary or coboundary of an exterior algebra $\Lambda(L)$ defined by the structure coefficients of a Lie algebra L.

See *ExteriorAlgebraBoundary* and *ExteriorAlgebraCoboundary* for the actual classes, which inherit from this.

> **Warning:** This is not a general class for differentials on the exterior algebra.

homology(*deg=None*, ***kwds*)

Return the homology determined by `self`.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: par = E.boundary({(0,1): z, (1,2): x, (2,0): y})
sage: par.homology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 0 over Rational Field,
 2: Vector space of dimension 0 over Rational Field,
 3: Vector space of dimension 1 over Rational Field}
sage: d = E.coboundary({(0,1): z, (1,2): x, (2,0): y})
sage: d.homology()
{0: Vector space of dimension 1 over Rational Field,
 1: Vector space of dimension 0 over Rational Field,
 2: Vector space of dimension 0 over Rational Field,
 3: Vector space of dimension 1 over Rational Field}
```

class sage.algebras.clifford_algebra.**ExteriorAlgebraIdeal**(*ring*, *gens*, *coerce=True*, *side='twosided'*)

Bases: `Ideal_nc`

An ideal of the exterior algebra.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: I = E.ideal(x*y); I
Twosided Ideal (x*y) of The exterior algebra of rank 3 over Rational Field
```

We can also use it to build a quotient:

```
sage: Q = E.quotient(I); Q
Quotient of The exterior algebra of rank 3 over Rational Field by the ideal (x*y)
sage: Q.inject_variables()
Defining xbar, ybar, zbar
sage: xbar * ybar
0
```

groebner_basis(*term_order=None*, *reduced=True*)

Return the (reduced) Gröbner basis of `self`.

INPUT:

- `term_order` – the term order used to compute the Gröbner basis; must be one of the following:
  - `"neglex"` – (default) negative (read right-to-left) lex order
  - `"degrevlex"` – degree reverse lex order
  - `"deglex"` – degree lex order
- `reduced` – (default: `True`) whether or not to return the reduced Gröbner basis

EXAMPLES:

We compute an example:

```
sage: E.<a,b,c,d,e> = ExteriorAlgebra(QQ)
sage: rels = [c*d*e - b*d*e + b*c*e - b*c*d,
....:          c*d*e - a*d*e + a*c*e - a*c*d,
....:          b*d*e - a*d*e + a*b*e - a*b*d,
....:          b*c*e - a*c*e + a*b*e - a*b*c,
....:          b*c*d - a*c*d + a*b*d - a*b*c]
sage: I = E.ideal(rels)
sage: I.groebner_basis()
(-a*b*c + a*b*d - a*c*d + b*c*d,
 -a*b*c + a*b*e - a*c*e + b*c*e,
 -a*b*d + a*b*e - a*d*e + b*d*e,
 -a*c*d + a*c*e - a*d*e + c*d*e)
```

With different term orders:

```
sage: I.groebner_basis("degrevlex")
(b*c*d - b*c*e + b*d*e - c*d*e,
 a*c*d - a*c*e + a*d*e - c*d*e,
 a*b*d - a*b*e + a*d*e - b*d*e,
 a*b*c - a*b*e + a*c*e - b*c*e)

sage: I.groebner_basis("deglex")
(-a*b*c + a*b*d - a*c*d + b*c*d,
 -a*b*c + a*b*e - a*c*e + b*c*e,
 -a*b*d + a*b*e - a*d*e + b*d*e,
 -a*c*d + a*c*e - a*d*e + c*d*e)
```

The example above was computed first using M2, which agrees with the `"degrevlex"` ordering:

```
E = QQ[a..e, SkewCommutative => true]
I = ideal( c*d*e - b*d*e + b*c*e - b*c*d,
            c*d*e - a*d*e + a*c*e - a*c*d,
            b*d*e - a*d*e + a*b*e - a*b*d,
            b*c*e - a*c*e + a*b*e - a*b*c,
            b*c*d - a*c*d + a*b*d - a*b*c)
groebnerBasis(I)

returns:
o3 = | bcd-bce+bde-cde acd-ace+ade-cde abd-abe+ade-bde abc-abe+ace-bce |
```

By default, the Gröbner basis is reduced, but we can get non-reduced Gröber bases (which are not unique):

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: I = E.ideal([x+y*z])
sage: I.groebner_basis(reduced=False)
(x*y, x*z, y*z + x, x*y*z)
sage: I.groebner_basis(reduced=True)
(x*y, x*z, y*z + x)
```

However, if we have already computed a reduced Gröbner basis (with a given term order), then we return that:

```
sage: I = E.ideal([x+y*z])  # A fresh ideal
sage: I.groebner_basis()
(x*y, x*z, y*z + x)
sage: I.groebner_basis(reduced=False)
(x*y, x*z, y*z + x)
```

reduce(*f*)

Reduce `f` modulo `self`.

EXAMPLES:

```
sage: E.<x,y,z> = ExteriorAlgebra(QQ)
sage: I = E.ideal(x*y);
sage: I.reduce(x*y + x*y*z + z)
z
sage: I.reduce(x*y + x + y)
x + y
sage: I.reduce(x*y + x*y*z)
0

sage: E.<a,b,c,d> = ExteriorAlgebra(QQ)
sage: I = E.ideal([a+b*c])
sage: I.reduce(I.gen(0) * d)
0
```

5.5 Cluster algebras

This file constructs cluster algebras using the Parent-Element framework. The implementation mainly utilizes structural theorems from [FZ2007].

The key points being used here are these:

- cluster variables are parametrized by their g-vectors;
- g-vectors (together with c-vectors) provide a self-standing model for the combinatorics behind any cluster algebra;
- each cluster variable in any cluster algebra can be computed, by the separation of additions formula, from its g-vector and F-polynomial.

Accordingly this file provides three classes:

- *ClusterAlgebra*
- *ClusterAlgebraSeed*

- *ClusterAlgebraElement*

ClusterAlgebra, constructed as a subobject of `sage.rings.polynomial.laurent_polynomial_ring.LaurentPolynomialRing_generic`, is the frontend of this implementation. It provides all the algebraic features (like ring morphisms), it computes cluster variables, it is responsible for controlling the exploration of the exchange graph and serves as the repository for all the data recursively computed so far. In particular, all g-vectors and all F-polynomials of known cluster variables as well as a mutation path by which they can be obtained are recorded. In the optic of efficiency, this implementation does not store directly the exchange graph nor the exchange relations. Both of these could be added to *ClusterAlgebra* with minimal effort.

ClusterAlgebraSeed provides the combinatorial backbone for *ClusterAlgebra*. It is an auxiliary class and therefore its instances should **not** be directly created by the user. Rather it should be accessed via *ClusterAlgebra.current_seed()* and *ClusterAlgebra.initial_seed()*. The task of performing current seed mutations is delegated to this class. Seeds are considered equal if they have the same parent cluster algebra and they can be obtained from each other by a permutation of their data (i.e. if they coincide as unlabelled seeds). Cluster algebras whose initial seeds are equal in the above sense are not considered equal but are endowed with coercion maps to each other. More generally, a cluster algebra is endowed with coercion maps from any cluster algebra which is obtained by freezing a collection of initial cluster variables and/or permuting both cluster variables and coefficients.

ClusterAlgebraElement is a thin wrapper around `sage.rings.polynomial.laurent_polynomial.LaurentPolynomial` providing all the functions specific to cluster variables. Elements of a cluster algebra with principal coefficients have special methods and these are grouped in the subclass *PrincipalClusterAlgebraElement*.

One more remark about this implementation. Instances of *ClusterAlgebra* are built by identifying the initial cluster variables with the generators of *ClusterAlgebra.ambient()*. In particular, this forces a specific embedding into the ambient field of rational expressions. In view of this, although cluster algebras themselves are independent of the choice of initial seed, *ClusterAlgebra.mutate_initial()* is forced to return a different instance of *ClusterAlgebra*. At the moment there is no coercion implemented among the two instances but this could in principle be added to *ClusterAlgebra.mutate_initial()*.

REFERENCES:

- [FZ2007]
- [LLZ2014]
- [NZ2012]

AUTHORS:

- Dylan Rupel (2015-06-15): initial version
- Salvatore Stella (2015-06-15): initial version

EXAMPLES:

We begin by creating a simple cluster algebra and printing its initial exchange matrix:

```
sage: A = ClusterAlgebra(['A', 2]); A
A Cluster Algebra with cluster variables x0, x1 and no coefficients over Integer Ring
sage: A.b_matrix()
[ 0  1]
[-1  0]
```

`A` is of finite type so we can explore all its exchange graph:

```
sage: A.explore_to_depth(infinity)
```

and get all its g-vectors, F-polynomials, and cluster variables:

```
sage: sorted(A.g_vectors_so_far())
[(-1, 0), (-1, 1), (0, -1), (0, 1), (1, 0)]
sage: sorted(A.F_polynomials_so_far(), key=str)
[1, 1, u0 + 1, u0*u1 + u0 + 1, u1 + 1]
sage: sorted(A.cluster_variables_so_far(), key=str)
[(x0 + 1)/x1, (x0 + x1 + 1)/(x0*x1), (x1 + 1)/x0, x0, x1]
```

Simple operations among cluster variables behave as expected:

```
sage: s = A.cluster_variable((0, -1)); s
(x0 + 1)/x1
sage: t = A.cluster_variable((-1, 1)); t
(x1 + 1)/x0
sage: t + s
(x0^2 + x1^2 + x0 + x1)/(x0*x1)
sage: _.parent() == A
True
sage: t - s
(-x0^2 + x1^2 - x0 + x1)/(x0*x1)
sage: _.parent() == A
True
sage: t*s
(x0*x1 + x0 + x1 + 1)/(x0*x1)
sage: _.parent() == A
True
sage: t/s
(x1^2 + x1)/(x0^2 + x0)
sage: _.parent() == A
False
```

Division is not guaranteed to yield an element of `A` so it returns an element of `A.ambient().fraction_field()` instead:

```
sage: (t/s).parent() == A.ambient().fraction_field()
True
```

We can compute denominator vectors of any element of `A`:

```
sage: (t*s).d_vector()
(1, 1)
```

Since we are in rank 2 and we do not have coefficients we can compute the greedy element associated to any denominator vector:

```
sage: A.rank() == 2 and A.coefficients() == ()
True
sage: A.greedy_element((1, 1))
(x0 + x1 + 1)/(x0*x1)
sage: _ == t*s
False
```

not surprising since there is no cluster in `A` containing both `t` and `s`:

```
sage: seeds = A.seeds(mutating_F=false)
sage: [ S for S in seeds if (0, -1) in S and (-1, 1) in S ]
[]
```

indeed:

```
sage: A.greedy_element((1, 1)) == A.cluster_variable((-1, 0))
True
```

Disabling F-polynomials in the computation just done was redundant because we already explored the whole exchange graph before. Though in different circumstances it could have saved us considerable time.

g-vectors and F-polynomials can be computed from elements of `A` only if `A` has principal coefficients at the initial seed:

```
sage: (t*s).g_vector()
Traceback (most recent call last):
...
AttributeError: 'ClusterAlgebra_with_category.element_class' object has no attribute 'g_
↪vector'
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.explore_to_depth(infinity)
sage: s = A.cluster_variable((0, -1)); s
(x0*y1 + 1)/x1
sage: t = A.cluster_variable((-1, 1)); t
(x1 + y0)/x0
sage: (t*s).g_vector()
(-1, 0)
sage: (t*s).F_polynomial()
u0*u1 + u0 + u1 + 1
sage: (t*s).is_homogeneous()
True
sage: (t+s).is_homogeneous()
False
sage: (t+s).homogeneous_components()
{(-1, 1): (x1 + y0)/x0, (0, -1): (x0*y1 + 1)/x1}
```

Each cluster algebra is endowed with a reference to a current seed; it could be useful to assign a name to it:

```
sage: A = ClusterAlgebra(['F', 4])
sage: len(A.g_vectors_so_far())
4
sage: A.current_seed()
The initial seed of a Cluster Algebra with cluster variables x0, x1, x2, x3
 and no coefficients over Integer Ring
sage: A.current_seed() == A.initial_seed()
True
sage: S = A.current_seed()
sage: S.b_matrix()
[ 0  1  0  0]
[-1  0 -1  0]
[ 0  2  0  1]
[ 0  0 -1  0]
sage: S.g_matrix()
[1 0 0 0]
```

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```
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: S.cluster_variables()
[x0, x1, x2, x3]
```

and use S to walk around the exchange graph of A:

```
sage: S.mutate(0); S
The seed of a Cluster Algebra with cluster variables x0, x1, x2, x3
 and no coefficients over Integer Ring obtained from the initial
 by mutating in direction 0
sage: S.b_matrix()
[ 0 -1  0  0]
[ 1  0 -1  0]
[ 0  2  0  1]
[ 0  0 -1  0]
sage: S.g_matrix()
[-1  0  0  0]
[ 1  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
sage: S.cluster_variables()
[(x1 + 1)/x0, x1, x2, x3]
sage: S.mutate('sinks'); S
The seed of a Cluster Algebra with cluster variables x0, x1, x2, x3
 and no coefficients over Integer Ring obtained from the initial
 by mutating along the sequence [0, 2]
sage: S.mutate([2, 3, 2, 1, 0]); S
The seed of a Cluster Algebra with cluster variables x0, x1, x2, x3
 and no coefficients over Integer Ring obtained from the initial
 by mutating along the sequence [0, 3, 2, 1, 0]
sage: S.g_vectors()
[(0, 1, -2, 0), (-1, 2, -2, 0), (0, 1, -1, 0), (0, 0, 0, -1)]
sage: S.cluster_variable(3)
(x2 + 1)/x3
```

Walking around by mutating S updates the informations stored in A:

```
sage: len(A.g_vectors_so_far())
10
sage: A.current_seed().path_from_initial_seed()
[0, 3, 2, 1, 0]
sage: A.current_seed() == S
True
```

Starting from A.initial_seed() still records data in A but does not update A.current_seed():

```
sage: S1 = A.initial_seed()
sage: S1.mutate([2, 1, 3])
sage: len(A.g_vectors_so_far())
11
```

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```
sage: S1 == A.current_seed()
False
```

Since `ClusterAlgebra` inherits from `UniqueRepresentation`, computed data is shared across instances:

```
sage: A1 = ClusterAlgebra(['F', 4])
sage: A1 is A
True
sage: len(A1.g_vectors_so_far())
11
```

It can be useful, at times to forget all computed data. Because of `UniqueRepresentation` this cannot be achieved by simply creating a new instance; instead it has to be manually triggered by:

```
sage: A.clear_computed_data()
sage: len(A.g_vectors_so_far())
4
```

Given a cluster algebra `A` we may be looking for a specific cluster variable:

```
sage: A = ClusterAlgebra(['E', 8, 1])
sage: v = (-1, 1, -1, 1, -1, 1, 0, 0, 1)
sage: A.find_g_vector(v, depth=2)
sage: seq = A.find_g_vector(v); seq  # random
[0, 1, 2, 4, 3]
sage: v in A.initial_seed().mutate(seq, inplace=False).g_vectors()
True
```

This also performs mutations of F-polynomials:

```
sage: A.F_polynomial((-1, 1, -1, 1, -1, 1, 0, 0, 1))
u0*u1*u2*u3*u4 + u0*u1*u2*u4 + u0*u2*u3*u4 + u0*u1*u2 + u0*u2*u4
 + u2*u3*u4 + u0*u2 + u0*u4 + u2*u4 + u0 + u2 + u4 + 1
```

which might not be a good idea in algebras that are too big. One workaround is to first disable F-polynomials and then recompute only the desired mutations:

```
sage: A.reset_exploring_iterator(mutating_F=False)  # long time
sage: v = (-1, 1, -2, 2, -1, 1, -1, 1, 1)  # long time
sage: seq = A.find_g_vector(v); seq  # long time random
[1, 0, 2, 6, 5, 4, 3, 8, 1]
sage: S = A.initial_seed().mutate(seq, inplace=False)   # long time
sage: v in S.g_vectors()   # long time
True
sage: A.current_seed().mutate(seq)    # long time
sage: A.F_polynomial((-1, 1, -2, 2, -1, 1, -1, 1, 1))   # long time
u0*u1^2*u2^2*u3*u4*u5*u6*u8 +
...
2*u2 + u4 + u6 + 1
```

We can manually freeze cluster variables and get coercions in between the two algebras:

```
sage: A = ClusterAlgebra(['F', 4]); A
A Cluster Algebra with cluster variables x0, x1, x2, x3 and no coefficients
 over Integer Ring
sage: A1 = ClusterAlgebra(A.b_matrix().matrix_from_columns([0, 1, 2]), coefficient_
↪prefix='x'); A1
A Cluster Algebra with cluster variables x0, x1, x2 and coefficient x3
 over Integer Ring
sage: A.has_coerce_map_from(A1)
True
```

and we also have an immersion of `A.base()` into `A` and of `A` into `A.ambient()`:

```
sage: A.has_coerce_map_from(A.base())
True
sage: A.ambient().has_coerce_map_from(A)
True
```

but there is currently no coercion in between algebras obtained by mutating at the initial seed:

```
sage: A1 = A.mutate_initial(0); A1
A Cluster Algebra with cluster variables x4, x1, x2, x3 and no coefficients
 over Integer Ring
sage: A.b_matrix() == A1.b_matrix()
False
sage: [X.has_coerce_map_from(Y) for X, Y in [(A, A1), (A1, A)]]
[False, False]
```

class `sage.algebras.cluster_algebra.`**`ClusterAlgebra`**(*B*, ***kwargs*)

Bases: `Parent`, `UniqueRepresentation`

A Cluster Algebra.

INPUT:

- `data` – some data defining a cluster algebra; it can be anything that can be parsed by `ClusterQuiver`
- `scalars` – a ring (default $\mathbf{Z}$); the scalars over which the cluster algebra is defined
- `cluster_variable_prefix` – string (default `'x'`); it needs to be a valid variable name
- `cluster_variable_names` – a list of strings; each element needs to be a valid variable name; supersedes `cluster_variable_prefix`
- `coefficient_prefix` – string (default `'y'`); it needs to be a valid variable name.
- `coefficient_names` – a list of strings; each element needs to be a valid variable name; supersedes `cluster_variable_prefix`
- `principal_coefficients` – bool (default `False`); supersedes any coefficient defined by `data`

ALGORITHM:

The implementation is mainly based on [FZ2007] and [NZ2012].

EXAMPLES:

```
sage: B = matrix([(0, 1, 0, 0), (-1, 0, -1, 0), (0, 1, 0, 1), (0, 0, -2, 0), (-1, 0,
↪ 0, 0), (0, -1, 0, 0)])
sage: A = ClusterAlgebra(B); A
```

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```
A Cluster Algebra with cluster variables x0, x1, x2, x3
 and coefficients y0, y1 over Integer Ring
sage: A.gens()
(x0, x1, x2, x3, y0, y1)
sage: A = ClusterAlgebra(['A', 2]); A
A Cluster Algebra with cluster variables x0, x1 and no coefficients
 over Integer Ring
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True); A.gens()
(x0, x1, y0, y1)
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True, coefficient_prefix=
↪'x'); A.gens()
(x0, x1, x2, x3)
sage: A = ClusterAlgebra(['A', 3], principal_coefficients=True, cluster_variable_
↪names=['a', 'b', 'c']); A.gens()
(a, b, c, y0, y1, y2)
sage: A = ClusterAlgebra(['A', 3], principal_coefficients=True, cluster_variable_
↪names=['a', 'b'])
Traceback (most recent call last):
...
ValueError: cluster_variable_names should be an iterable of 3 valid variable names
sage: A = ClusterAlgebra(['A', 3], principal_coefficients=True, coefficient_names=[
↪'a', 'b', 'c']); A.gens()
(x0, x1, x2, a, b, c)
sage: A = ClusterAlgebra(['A', 3], principal_coefficients=True, coefficient_names=[
↪'a', 'b'])
Traceback (most recent call last):
...
ValueError: coefficient_names should be an iterable of 3 valid variable names
```

F_polynomial(*g_vector*)

Return the F-polynomial with g-vector `g_vector` if it has been found.

INPUT:

- `g_vector` – tuple; the g-vector of the F-polynomial to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: A.F_polynomial((-1, 1))
Traceback (most recent call last):
...
KeyError: 'the g-vector (-1, 1) has not been found yet'
sage: A.initial_seed().mutate(0, mutating_F=False)
sage: A.F_polynomial((-1, 1))
Traceback (most recent call last):
...
KeyError: 'the F-polynomial with g-vector (-1, 1) has not been computed yet;
 you can compute it by mutating from the initial seed along the sequence [0]'
sage: A.initial_seed().mutate(0)
sage: A.F_polynomial((-1, 1))
u0 + 1
```

F_polynomials()

Return an iterator producing all the F_polynomials of `self`.

ALGORITHM:

This method does not use the caching framework provided by `self`, but recomputes all the F-polynomials from scratch. On the other hand it stores the results so that other methods like `F_polynomials_so_far()` can access them afterwards.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: len(list(A.F_polynomials()))
9
```

F_polynomials_so_far()

Return a list of the F-polynomials encountered so far.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: A.current_seed().mutate(0)
sage: sorted(A.F_polynomials_so_far(), key=str)
[1, 1, u0 + 1]
```

ambient()

Return the Laurent polynomial ring containing `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.ambient()
Multivariate Laurent Polynomial Ring in x0, x1, y0, y1 over Integer Ring
```

b_matrix()

Return the initial exchange matrix of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.b_matrix()
[ 0  1]
[-1  0]
```

clear_computed_data()

Clear the cache of computed g-vectors and F-polynomials and reset both the current seed and the exploring iterator.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: sorted(A.g_vectors_so_far())
[(0, 1), (1, 0)]
sage: A.current_seed().mutate([1, 0])
```

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```
sage: sorted(A.g_vectors_so_far())
[(-1, 0), (0, -1), (0, 1), (1, 0)]
sage: A.clear_computed_data()
sage: sorted(A.g_vectors_so_far())
[(0, 1), (1, 0)]
```

cluster_fan(*depth=+Infinity*)

Return the cluster fan (the fan of g-vectors) of `self`.

INPUT:

- `depth` – a positive integer or infinity (default `infinity`); the maximum depth at which to compute

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.cluster_fan()
Rational polyhedral fan in 2-d lattice N
```

cluster_variable(*g_vector*)

Return the cluster variable with g-vector `g_vector` if it has been found.

INPUT:

- `g_vector` – tuple; the g-vector of the cluster variable to return

ALGORITHM:

This function computes cluster variables from their g-vectors and F-polynomials using the "separation of additions" formula of Theorem 3.7 in [FZ2007].

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.initial_seed().mutate(0)
sage: A.cluster_variable((-1, 1))
(x1 + 1)/x0
```

cluster_variables()

Return an iterator producing all the cluster variables of `self`.

ALGORITHM:

This method does not use the caching framework provided by `self`, but recomputes all the cluster variables from scratch. On the other hand it stores the results so that other methods like *cluster_variables_so_far()* can access them afterwards.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: len(list(A.cluster_variables()))
9
```

cluster_variables_so_far()

Return a list of the cluster variables encountered so far.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: A.current_seed().mutate(0)
sage: sorted(A.cluster_variables_so_far(), key=str)
[(x1 + 1)/x0, x0, x1]
```

coefficient(*j*)

Return the j-th coefficient of self.

INPUT:

- j – an integer in range(self.parent().rank()); the index of the coefficient to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.coefficient(0)
y0
```

coefficient_names()

Return the list of coefficient names.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: A.coefficient_names()
()
sage: A1 = ClusterAlgebra(['B', 2], principal_coefficients=True)
sage: A1.coefficient_names()
('y0', 'y1')
sage: A2 = ClusterAlgebra(['C', 3], principal_coefficients=True, coefficient_
↪prefix='x')
sage: A2.coefficient_names()
('x3', 'x4', 'x5')
```

coefficients()

Return the list of coefficients of self.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.coefficients()
(y0, y1)
sage: A1 = ClusterAlgebra(['B', 2])
sage: A1.coefficients()
()
```

contains_seed(*seed*)

Test if seed is a seed of self.

INPUT:

- seed – a *ClusterAlgebraSeed*

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True); A
A Cluster Algebra with cluster variables x0, x1 and coefficients y0, y1 over␣
↪Integer Ring
sage: S = copy(A.current_seed())
sage: A.contains_seed(S)
True
```

coxeter_element()

Return the Coxeter element associated to the initial exchange matrix, if acyclic.

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,1,1],[-1,0,1],[-1,-1,0]]))
sage: A.coxeter_element()
[0, 1, 2]
```

Raise an error if the initial exchange matrix is not acyclic:

```
sage: A = ClusterAlgebra(matrix([[0,1,-1],[-1,0,1],[1,-1,0]]))
sage: A.coxeter_element()
Traceback (most recent call last):
...
ValueError: the initial exchange matrix is not acyclic
```

current_seed()

Return the current seed of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: A.current_seed()
The initial seed of a Cluster Algebra with cluster variables x0, x1
 and no coefficients over Integer Ring
```

d_vector_to_g_vector(*d*)

Return the g-vector of an element of `self` having d-vector `d`

INPUT:

- `d` – the d-vector

ALGORITHM:

This method implements the piecewise-linear map
nu_c introduced in Section 9.1 of [ReSt2020].

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,1,1],[-1,0,1],[-1,-1,0]]))
sage: A.d_vector_to_g_vector((1,0,-1))
(-1, 1, 2)
```

euler_matrix()

Return the Euler matrix associated to `self`.

ALGORITHM:

This method returns the matrix of the bilinear form defined in Equation (2.1) of [ReSt2020] .

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,1,1],[-1,0,1],[-1,-1,0]]))
sage: A.euler_matrix()
[ 1  0  0]
[-1  1  0]
[-1 -1  1]
```

Raise an error if the initial exchange matrix is not acyclic:

```
sage: A = ClusterAlgebra(matrix([[0,1,-1],[-1,0,1],[1,-1,0]]))
sage: A.euler_matrix()
Traceback (most recent call last):
...
ValueError: the initial exchange matrix is not acyclic
```

explore_to_depth(*depth*)

Explore the exchange graph of `self` up to distance `depth` from the initial seed.

INPUT:

- `depth` – a positive integer or infinity; the maximum depth at which to stop searching

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 4])
sage: A.explore_to_depth(infinity)
sage: len(A.g_vectors_so_far())
14
```

find_g_vector(*g_vector*, *depth=+Infinity*)

Return a mutation sequence to obtain a seed containing the g-vector `g_vector` from the initial seed.

INPUT:

- `g_vector` – a tuple: the g-vector to find
- `depth` – a positive integer or infinity (default `infinity`); the maximum distance from `self.current_seed` to reach

OUTPUT:

This function returns a list of integers if it can find `g_vector`, otherwise it returns `None`. If the exploring iterator stops, it means that the algebra is of finite type and `g_vector` is not the g-vector of any cluster variable. In this case the function resets the iterator and raises an error.

EXAMPLES:

```
sage: A = ClusterAlgebra(['G', 2], principal_coefficients=True)
sage: A.clear_computed_data()
sage: A.find_g_vector((-2, 3), depth=2)
sage: A.find_g_vector((-2, 3), depth=3)
[0, 1, 0]
sage: A.find_g_vector((1, 1), depth=3)
sage: A.find_g_vector((1, 1), depth=4)
Traceback (most recent call last):
```

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```
...
ValueError: (1, 1) is not the g-vector of any cluster variable of a
 Cluster Algebra with cluster variables x0, x1 and coefficients y0, y1
 over Integer Ring
```

g_vector_to_d_vector(*g*)

Return the d-vector of an element of `self` having g-vector `g`

INPUT:

- `g` – the g-vector

ALGORITHM:

This method implements the inverse of the piecewise-linear map nu_c introduced in Section 9.1 of [ReSt2020].

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,1,1],[-1,0,1],[-1,-1,0]]))
sage: A.g_vector_to_d_vector((-1,1,2))
(1, 0, -1)
```

g_vectors(*mutating_F=True*)

Return an iterator producing all the g-vectors of `self`.

INPUT:

- `mutating_F` – bool (default `True`); whether to compute F-polynomials; disable this for speed considerations

ALGORITHM:

This method does not use the caching framework provided by `self`, but recomputes all the g-vectors from scratch. On the other hand it stores the results so that other methods like *g_vectors_so_far()* can access them afterwards.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: len(list(A.g_vectors()))
9
```

g_vectors_so_far()

Return a list of the g-vectors of cluster variables encountered so far.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: A.current_seed().mutate(0)
sage: sorted(A.g_vectors_so_far())
[(-1, 1), (0, 1), (1, 0)]
```

gens()

Return the list of initial cluster variables and coefficients of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.gens()
(x0, x1, y0, y1)
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True, coefficient_
↪prefix='x')
sage: A.gens()
(x0, x1, x2, x3)
```

greedy_element(*d_vector*)

Return the greedy element with denominator vector `d_vector`.

INPUT:

- `d_vector` – tuple of 2 integers; the denominator vector of the element to compute

ALGORITHM:

This implements greedy elements of a rank 2 cluster algebra using Equation (1.5) from [LLZ2014].

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', [1, 1], 1])
sage: A.greedy_element((1, 1))
(x0^2 + x1^2 + 1)/(x0*x1)
```

initial_cluster_variable(*j*)

Return the `j`-th initial cluster variable of `self`.

INPUT:

- `j` – an integer in `range(self.parent().rank())`; the index of the cluster variable to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.initial_cluster_variable(0)
x0
```

initial_cluster_variable_names()

Return the list of initial cluster variable names.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.initial_cluster_variable_names()
('x0', 'x1')
sage: A1 = ClusterAlgebra(['B', 2], cluster_variable_prefix='a')
sage: A1.initial_cluster_variable_names()
('a0', 'a1')
```

initial_cluster_variables()

Return the list of initial cluster variables of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: A.initial_cluster_variables()
(x0, x1)
```

initial_seed()

Return the initial seed of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.initial_seed()
The initial seed of a Cluster Algebra with cluster variables x0, x1 and no␣
↪coefficients over Integer Ring
```

is_acyclic()

Return `True` if the exchange matrix in the initial seed is acyclic, `False` otherwise.

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,1,1],[-1,0,1],[-1,-1,0]]))
sage: A.is_acyclic()
True
sage: A = ClusterAlgebra(matrix([[0,1,-1],[-1,0,1],[1,-1,0]]))
sage: A.is_acyclic()
False
```

lift(*x*)

Return `x` as an element of *ambient()*.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: x = A.cluster_variable((1, 0))
sage: A.lift(x).parent()
Multivariate Laurent Polynomial Ring in x0, x1, y0, y1 over Integer Ring
```

lower_bound()

Return the lower bound associated to `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['F', 4])
sage: A.lower_bound()
Traceback (most recent call last):
...
NotImplementedError: not implemented yet
```

mutate_initial(*direction*, ***kwargs*)

Return the cluster algebra obtained by mutating `self` at the initial seed.

> **Warning:** This method is significantly slower than *ClusterAlgebraSeed.mutate()*. It is therefore advisable to use the latter for exploration purposes.

INPUT:

- `direction` – in which direction(s) to mutate, it can be:
  - an integer in `range(self.rank())` to mutate in one direction only
  - an iterable of such integers to mutate along a sequence

  - a string "sinks" or "sources" to mutate at all sinks or sources simultaneously

- `mutating_F` – bool (default `True`); whether to compute F-polynomials while mutating

Note: While knowing F-polynomials is essential to computing cluster variables, the process of mutating them is quite slow. If you care only about combinatorial data like g-vectors and c-vectors, setting `mutating_F=False` yields significant benefits in terms of speed.

ALGORITHM:

This function computes data for the new algebra from known data for the old algebra using Equation (4.2) from [NZ2012] for g-vectors, and Equation (6.21) from [FZ2007] for F-polynomials. The exponent h in the formula for F-polynomials is `-min(0, old_g_vect[k])` due to [NZ2012] Proposition 4.2.

EXAMPLES:

```
sage: A = ClusterAlgebra(['F', 4])
sage: A.explore_to_depth(infinity)
sage: B = A.b_matrix()
sage: B.mutate(0)
sage: A1 = ClusterAlgebra(B)
sage: A1.explore_to_depth(infinity)
sage: A2 = A1.mutate_initial(0)
sage: A2._F_poly_dict == A._F_poly_dict
True
```

Check that we did not mess up the original algebra because of `UniqueRepresentation`:

```
sage: A = ClusterAlgebra(['A',2])
sage: A.mutate_initial(0) is A
False
```

A faster example without recomputing F-polynomials:

```
sage: A = ClusterAlgebra(matrix([[0,5],[-5,0]]))
sage: A.mutate_initial([0,1]*10, mutating_F=False)
A Cluster Algebra with cluster variables x20, x21 and no coefficients over
↪Integer Ring
```

Check that github issue #28176 is fixed:

```
sage: A = ClusterAlgebra( matrix(5,[0,1,-1,1,-1]), cluster_variable_names=['p13
↪'], coefficient_names=['p12','p23','p34','p41']); A
A Cluster Algebra with cluster variable p13 and coefficients p12, p23, p34, p41
↪over Integer Ring
sage: A.mutate_initial(0)
A Cluster Algebra with cluster variable x0 and coefficients p12, p23, p34, p41
↪over Integer Ring

sage: A1 = ClusterAlgebra(['A',[2,1],1])
sage: A2 = A1.mutate_initial([0,1,0])
sage: len(A2.g_vectors_so_far()) == len(A2.F_polynomials_so_far())
True
sage: all(parent(f) == A2._U for f in A2.F_polynomials_so_far())
```

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```
True
sage: A2.find_g_vector((0,0,1)) == []
True
```

rank()

Return the rank of `self`, i.e. the number of cluster variables in any seed.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True); A
A Cluster Algebra with cluster variables x0, x1
 and coefficients y0, y1 over Integer Ring
sage: A.rank()
2
```

reset_current_seed()

Reset the value reported by *current_seed()* to *initial_seed()*.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: A.current_seed().mutate([1, 0])
sage: A.current_seed() == A.initial_seed()
False
sage: A.reset_current_seed()
sage: A.current_seed() == A.initial_seed()
True
```

reset_exploring_iterator(*mutating_F=True*)

Reset the iterator used to explore `self`.

INPUT:

- `mutating_F` – bool (default `True`); whether to also compute F-polynomials; disable this for speed considerations

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 4])
sage: A.clear_computed_data()
sage: A.reset_exploring_iterator(mutating_F=False)
sage: A.explore_to_depth(infinity)
sage: len(A.g_vectors_so_far())
14
sage: len(A.F_polynomials_so_far())
4
```

retract(*x*)

Return `x` as an element of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2], principal_coefficients=True)
sage: L = A.ambient()
```

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```
sage: x = L.gen(0)
sage: A.retract(x).parent()
A Cluster Algebra with cluster variables x0, x1 and coefficients y0, y1 over
↪Integer Ring
```

scalars()

Return the ring of scalars over which `self` is defined.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.scalars()
Integer Ring
```

seeds(***kwargs*)

Return an iterator running over seeds of `self`.

INPUT:

- `from_current_seed` – bool (default `False`); whether to start the iterator from *current_seed()* or *initial_seed()*
- `mutating_F` – bool (default `True`); whether to compute F-polynomials also; disable this for speed considerations
- `allowed_directions` – iterable of integers (default `range(self.rank())`); the directions in which to mutate
- `depth` – a positive integer or infinity (default `infinity`); the maximum depth at which to stop searching
- `catch_KeyboardInterrupt` – bool (default `False`); whether to catch `KeyboardInterrupt` and return it rather then raising an exception – this allows the iterator returned by this method to be resumed after being interrupted

ALGORITHM:

This function traverses the exchange graph in a breadth-first search.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 4])
sage: A.clear_computed_data()
sage: seeds = A.seeds(allowed_directions=[3, 0, 1])
sage: _ = list(seeds)
sage: sorted(A.g_vectors_so_far())
[(-1, 0, 0, 0),
 (-1, 1, 0, 0),
 (0, -1, 0, 0),
 (0, 0, 0, -1),
 (0, 0, 0, 1),
 (0, 0, 1, 0),
 (0, 1, 0, 0),
 (1, 0, 0, 0)]
```

set_current_seed(*seed*)

Set the value reported by *current_seed()* to `seed`, if it makes sense.

INPUT:

- seed – a *ClusterAlgebraSeed*

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: A.clear_computed_data()
sage: S = copy(A.current_seed())
sage: S.mutate([0, 1, 0])
sage: A.current_seed() == S
False
sage: A.set_current_seed(S)
sage: A.current_seed() == S
True
sage: A1 = ClusterAlgebra(['B', 2])
sage: A.set_current_seed(A1.initial_seed())
Traceback (most recent call last):
...
ValueError: this is not a seed in this cluster algebra
```

theta_basis_F_polynomial(*g_vector*)

Return the F-polynomial of the element of the theta basis of `self` with g-vector `g_vector`.

INPUT:

- `g_vector` – tuple; the g-vector of the F-polynomial to compute

Warning: Elements of the theta basis do not satisfy a separation of additions formula. See the implementation of `sage.algebras.cluster_algebra.theta_basis_F_polynomial()` for further details.

ALGORITHM:

This method uses the fact that the greedy basis and the theta basis coincide in rank 2 and uses the former defining recursion (Equation (1.5) from [LLZ2014]) to compute.

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,-3],[2,0]]), principal_coefficients=True)
sage: A.theta_basis_F_polynomial((-1,-1))
u0^4*u1 + 4*u0^3*u1 + 6*u0^2*u1 + 4*u0*u1 + u0 + u1 + 1

sage: A = ClusterAlgebra(['F', 4])
sage: A.theta_basis_F_polynomial((1, 0, 0, 0))
Traceback (most recent call last):
...
NotImplementedError: currently only implemented for cluster algebras of rank 2
```

theta_basis_element(*g_vector*)

Return the element of the theta basis of `self` with g-vector `g_vector`.

INPUT:

- `g_vector` – tuple; the g-vector of the element to compute

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,-3],[2,0]]), principal_coefficients=True)
sage: A.theta_basis_element((-1,-1))
(x1^8*y0^4*y1 + 4*x1^6*y0^3*y1 + 6*x1^4*y0^2*y1 + x0^3*x1^2*y0 + 4*x1^2*y0*y1 +
↪x0^3 + y1)/(x0^4*x1)

sage: A = ClusterAlgebra(['F', 4])
sage: A.theta_basis_element((1, 0, 0, 0))
Traceback (most recent call last):
...
NotImplementedError: currently only implemented for cluster algebras of rank 2
```

Note: Elements of the theta basis correspond with the associated cluster monomial only for appropriate coefficient choices. For example:

```
sage: A = ClusterAlgebra(matrix([[0,-1],[1,0],[-1,0]]))
sage: A.theta_basis_element((-1,0))
(x1 + y0)/(x0*y0)
```

while:

```
sage: _ = A.find_g_vector((-1,0));
sage: A.cluster_variable((-1,0))
(x1 + y0)/x0
```

In particular theta basis elements do not satisfy a separation of additions formula.

Warning: Currently only cluster algebras of rank 2 are supported

See also:

`sage.algebras.cluster_algebra.theta_basis_F_polynomial()`

upper_bound()

Return the upper bound associated to `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['F', 4])
sage: A.upper_bound()
Traceback (most recent call last):
...
NotImplementedError: not implemented yet
```

upper_cluster_algebra()

Return the upper cluster algebra associated to `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['F', 4])
sage: A.upper_cluster_algebra()
Traceback (most recent call last):
```

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```
...
NotImplementedError: not implemented yet
```

class sage.algebras.cluster_algebra.**ClusterAlgebraElement**

Bases: ElementWrapper

An element of a cluster algebra.

d_vector()

Return the denominator vector of self as a tuple of integers.

EXAMPLES:

```
sage: A = ClusterAlgebra(['F', 4], principal_coefficients=True)
sage: A.current_seed().mutate([0, 2, 1])
sage: x = A.cluster_variable((-1, 2, -2, 2)) * A.cluster_variable((0, 0, 0,␣
↪1))**2
sage: x.d_vector()
(1, 1, 2, -2)
```

class sage.algebras.cluster_algebra.**ClusterAlgebraSeed**(*B*, *C*, *G*, *parent*, ***kwargs*)

Bases: SageObject

A seed in a Cluster Algebra.

INPUT:

- B – a skew-symmetrizable integer matrix
- C – the matrix of c-vectors of self
- G – the matrix of g-vectors of self
- parent – *ClusterAlgebra*; the algebra to which the seed belongs
- path – list (default []); the mutation sequence from the initial seed of parent to self

> **Warning:** Seeds should **not** be created manually: no test is performed to assert that they are built from consistent data nor that they really are seeds of parent. If you create seeds with inconsistent data all sort of things can go wrong, even `__eq__()` is no longer guaranteed to give correct answers. Use at your own risk.

F_polynomial(*j*)

Return the j-th F-polynomial of self.

INPUT:

- j – an integer in range(self.parent().rank()); the index of the F-polynomial to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.F_polynomial(0)
1
```

F_polynomials()

Return all the F-polynomials of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.F_polynomials()
[1, 1, 1]
```

b_matrix()

Return the exchange matrix of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.b_matrix()
[ 0  1  0]
[-1  0 -1]
[ 0  1  0]
```

c_matrix()

Return the matrix whose columns are the c-vectors of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.c_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

c_vector(*j*)

Return the `j`-th c-vector of `self`.

INPUT:

- `j` – an integer in `range(self.parent().rank())`; the index of the c-vector to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.c_vector(0)
(1, 0, 0)
sage: S.mutate(0)
sage: S.c_vector(0)
(-1, 0, 0)
sage: S.c_vector(1)
(1, 1, 0)
```

c_vectors()

Return all the c-vectors of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.c_vectors()
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]
```

cluster_variable(*j*)

Return the j-th cluster variable of self.

INPUT:

- j – an integer in range(self.parent().rank()); the index of the cluster variable to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.cluster_variable(0)
x0
sage: S.mutate(0)
sage: S.cluster_variable(0)
(x1 + 1)/x0
```

cluster_variables()

Return all the cluster variables of self.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.cluster_variables()
[x0, x1, x2]
```

depth()

Return the length of a mutation sequence from the initial seed
of *parent()* to self.

Warning: This is the length of the mutation sequence returned by *path_from_initial_seed()*, which need not be the shortest possible.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: S1 = A.initial_seed()
sage: S1.mutate([0, 1, 0, 1])
sage: S1.depth()
4
sage: S2 = A.initial_seed()
sage: S2.mutate(1)
sage: S2.depth()
1
sage: S1 == S2
True
```

g_matrix()

Return the matrix whose columns are the g-vectors of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.g_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

g_vector(*j*)

Return the `j`-th g-vector of `self`.

INPUT:

- `j` – an integer in `range(self.parent().rank())`; the index of the g-vector to return

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.g_vector(0)
(1, 0, 0)
```

g_vectors()

Return all the g-vectors of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 3])
sage: S = A.initial_seed()
sage: S.g_vectors()
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]
```

mutate(*direction*, ***kwargs*)

Mutate `self`.

INPUT:

- `direction` – in which direction(s) to mutate, it can be:
  - an integer in `range(self.rank())` to mutate in one direction only
  - an iterable of such integers to mutate along a sequence
  - a string "sinks" or "sources" to mutate at all sinks or sources simultaneously
- `inplace` – bool (default `True`); whether to mutate in place or to return a new object
- `mutating_F` – bool (default `True`); whether to compute F-polynomials while mutating

Note: While knowing F-polynomials is essential to computing cluster variables, the process of mutating them is quite slow. If you care only about combinatorial data like g-vectors and c-vectors, setting `mutating_F=False` yields significant benefits in terms of speed.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: S = A.initial_seed()
sage: S.mutate(0); S
The seed of a Cluster Algebra with cluster variables x0, x1
 and no coefficients over Integer Ring obtained from the initial
 by mutating in direction 0
sage: S.mutate(5)
Traceback (most recent call last):
...
ValueError: cannot mutate in direction 5
```

parent()

Return the parent of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['B', 3])
sage: A.current_seed().parent() == A
True
```

path_from_initial_seed()

Return a mutation sequence from the initial seed of *parent()* to `self`.

> **Warning:** This is the path used to compute `self` and it does not have to be the shortest possible.

EXAMPLES:

```
sage: A = ClusterAlgebra(['A', 2])
sage: S1 = A.initial_seed()
sage: S1.mutate([0, 1, 0, 1])
sage: S1.path_from_initial_seed()
[0, 1, 0, 1]
sage: S2 = A.initial_seed()
sage: S2.mutate(1)
sage: S2.path_from_initial_seed()
[1]
sage: S1 == S2
True
```

class sage.algebras.cluster_algebra.**PrincipalClusterAlgebraElement**

Bases: *ClusterAlgebraElement*

An element in a cluster algebra with principle coefficients.

F_polynomial()

Return the F-polynomial of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['B', 2], principal_coefficients=True)
sage: S = A.initial_seed()
sage: S.mutate([0, 1, 0])
sage: S.cluster_variable(0).F_polynomial() == S.F_polynomial(0)
```

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```
True
sage: sum(A.initial_cluster_variables()).F_polynomial()
Traceback (most recent call last):
...
ValueError: this element does not have a well defined g-vector
```

g_vector()

Return the g-vector of `self`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['B', 2], principal_coefficients=True)
sage: A.cluster_variable((1, 0)).g_vector() == (1, 0)
True
sage: sum(A.initial_cluster_variables()).g_vector()
Traceback (most recent call last):
...
ValueError: this element does not have a well defined g-vector
```

homogeneous_components()

Return a dictionary of the homogeneous components of `self`.

OUTPUT:

A dictionary whose keys are homogeneous degrees and whose values are the summands of `self` of the given degree.

EXAMPLES:

```
sage: A = ClusterAlgebra(['B', 2], principal_coefficients=True)
sage: x = A.cluster_variable((1, 0)) + A.cluster_variable((0, 1))
sage: x.homogeneous_components()
{(0, 1): x1, (1, 0): x0}
```

is_homogeneous()

Return `True` if `self` is a homogeneous element of `self.parent()`.

EXAMPLES:

```
sage: A = ClusterAlgebra(['B', 2], principal_coefficients=True)
sage: A.cluster_variable((1, 0)).is_homogeneous()
True
sage: x = A.cluster_variable((1, 0)) + A.cluster_variable((0, 1))
sage: x.is_homogeneous()
False
```

theta_basis_decomposition()

Return the decomposition of `self` in the theta basis.

OUTPUT:

A dictionary whose keys are the g-vectors and whose values are the coefficients in the decomposition of `self` in the theta basis.

EXAMPLES:

```
sage: A = ClusterAlgebra(matrix([[0,-2],[3,0]]), principal_coefficients=True)
sage: f = (A.theta_basis_element((1,0)) + A.theta_basis_element((0,1)))**2 + A.
↪coefficient(1)* A.theta_basis_element((1,1))
sage: decomposition = f.theta_basis_decomposition()
sage: sum(decomposition[g] * A.theta_basis_element(g) for g in decomposition)␣
↪== f
True
sage: f = A.theta_basis_element((4,-4))*A.theta_basis_element((1,-1))
sage: decomposition =  f.theta_basis_decomposition()
sage: sum(decomposition[g] * A.theta_basis_element(g) for g in decomposition)␣
↪== f
True
```

5.6 Descent Algebras

AUTHORS:

- Travis Scrimshaw (2013-07-28): Initial version

class sage.combinat.descent_algebra.**DescentAlgebra**(*R*, *n*)

Bases: UniqueRepresentation, Parent

Solomon's descent algebra.

The descent algebra Σ_n over a ring R is a subalgebra of the symmetric group algebra RS_n. (The product in the latter algebra is defined by $(pq)(i) = q(p(i))$ for any two permutations p and q in S_n and every $i \in \{1, 2, \ldots, n\}$. The algebra Σ_n inherits this product.)

There are three bases currently implemented for Σ_n:

- the standard basis D_S of (sums of) descent classes, indexed by subsets S of $\{1, 2, \ldots, n-1\}$,
- the subset basis B_p, indexed by compositions p of n,
- the idempotent basis I_p, indexed by compositions p of n, which is used to construct the mutually orthogonal idempotents of the symmetric group algebra.

The idempotent basis is only defined when R is a $\mathbf{Q}$-algebra.

We follow the notations and conventions in [GR1989], apart from the order of multiplication being different from the one used in that article. Schocker's exposition [Sch2004], in turn, uses the same order of multiplication as we are, but has different notations for the bases.

INPUT:

- R – the base ring
- n – a nonnegative integer

REFERENCES:

- [GR1989]
- [At1992]
- [MR1995]
- [Sch2004]

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D(); D
Descent algebra of 4 over Rational Field in the standard basis
sage: B = DA.B(); B
Descent algebra of 4 over Rational Field in the subset basis
sage: I = DA.I(); I
Descent algebra of 4 over Rational Field in the idempotent basis
sage: basis_B = B.basis()
sage: elt = basis_B[Composition([1,2,1])] + 4*basis_B[Composition([1,3])]; elt
B[1, 2, 1] + 4*B[1, 3]
sage: D(elt)
5*D{} + 5*D{1} + D{1, 3} + D{3}
sage: I(elt)
7/6*I[1, 1, 1, 1] + 2*I[1, 1, 2] + 3*I[1, 2, 1] + 4*I[1, 3]
```

As syntactic sugar, one can use the notation `D[i,...,l]` to construct elements of the basis; note that for the empty set one must use `D[[]]` due to Python's syntax:

```
sage: D[[]] + D[2] + 2*D[1,2]
D{} + 2*D{1, 2} + D{2}
```

The same syntax works for the other bases:

```
sage: I[1,2,1] + 3*I[4] + 2*I[3,1]
I[1, 2, 1] + 2*I[3, 1] + 3*I[4]
```

class B(*alg*, *prefix='B'*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The subset basis of a descent algebra (indexed by compositions).

The subset basis $(B_S)_{S\subseteq\{1,2,\ldots,n-1\}}$ of Σ_n is formed by

$$B_S = \sum_{T\subseteq S} D_T,$$

where $(D_S)_{S\subseteq\{1,2,\ldots,n-1\}}$ is the `standard basis`. However it is more natural to index the subset basis by compositions of n under the bijection $\{i_1, i_2, \ldots, i_k\} \mapsto (i_1, i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k)$ (where $i_1 < i_2 < \cdots < i_k$), which is what Sage uses to index the basis.

The basis element B_p is denoted Ξ^p in [Sch2004].

By using compositions of n, the product $B_p B_q$ becomes a sum over the non-negative-integer matrices M with row sum p and column sum q. The summand corresponding to M is B_c, where c is the composition obtained by reading M row-by-row from left-to-right and top-to-bottom and removing all zeroes. This multiplication rule is commonly called "Solomon's Mackey formula".

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: list(B.basis())
[B[1, 1, 1, 1], B[1, 1, 2], B[1, 2, 1], B[1, 3],
 B[2, 1, 1], B[2, 2], B[3, 1], B[4]]
```

one_basis()

Return the identity element which is the composition $[n]$, as per `AlgebrasWithBasis.ParentMethods.one_basis`.

EXAMPLES:

```
sage: DescentAlgebra(QQ, 4).B().one_basis()
[4]
sage: DescentAlgebra(QQ, 0).B().one_basis()
[]

sage: all( U * DescentAlgebra(QQ, 3).B().one() == U
....:      for U in DescentAlgebra(QQ, 3).B().basis() )
True
```

product_on_basis(p, q)

Return $B_p B_q$, where p and q are compositions of n.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: p = Composition([1,2,1])
sage: q = Composition([3,1])
sage: B.product_on_basis(p, q)
B[1, 1, 1, 1] + 2*B[1, 2, 1]
```

to_D_basis(p)

Return B_p as a linear combination of D-basis elements.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: D = DA.D()
sage: list(map(D, B.basis()))  # indirect doctest
[D{} + D{1} + D{1, 2} + D{1, 2, 3}
  + D{1, 3} + D{2} + D{2, 3} + D{3},
 D{} + D{1} + D{1, 2} + D{2},
 D{} + D{1} + D{1, 3} + D{3},
 D{} + D{1},
 D{} + D{2} + D{2, 3} + D{3},
 D{} + D{2},
 D{} + D{3},
 D{}]
```

to_I_basis(p)

Return B_p as a linear combination of I-basis elements.

This is done using the formula

$$B_p = \sum_{q \leq p} \frac{1}{\mathbf{k}!(q,p)} I_q,$$

where $\leq$ is the refinement order and $\mathbf{k}!(q,p)$ is defined as follows: When $q \leq p$, we can write q as a concatenation $q_{(1)} q_{(2)} \cdots q_{(k)}$ with each $q_{(i)}$ being a composition of the i-th entry of p, and then we set $\mathbf{k}!(q,p)$ to be $l(q_{(1)})! l(q_{(2)})! \cdots l(q_{(k)})!$, where $l(r)$ denotes the number of parts of any composition r.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: I = DA.I()
sage: list(map(I, B.basis()))  # indirect doctest
[I[1, 1, 1, 1],
 1/2*I[1, 1, 1, 1] + I[1, 1, 2],
 1/2*I[1, 1, 1, 1] + I[1, 2, 1],
 1/6*I[1, 1, 1, 1] + 1/2*I[1, 1, 2] + 1/2*I[1, 2, 1] + I[1, 3],
 1/2*I[1, 1, 1, 1] + I[2, 1, 1],
 1/4*I[1, 1, 1, 1] + 1/2*I[1, 1, 2] + 1/2*I[2, 1, 1] + I[2, 2],
 1/6*I[1, 1, 1, 1] + 1/2*I[1, 2, 1] + 1/2*I[2, 1, 1] + I[3, 1],
 1/24*I[1, 1, 1, 1] + 1/6*I[1, 1, 2] + 1/6*I[1, 2, 1]
  + 1/2*I[1, 3] + 1/6*I[2, 1, 1] + 1/2*I[2, 2] + 1/2*I[3, 1] + I[4]]
```

to_nsym(*p*)

Return B_p as an element in $NSym$, the non-commutative symmetric functions.

This maps B_p to S_p where S denotes the Complete basis of $NSym$.

EXAMPLES:

```
sage: B = DescentAlgebra(QQ, 4).B()
sage: S = NonCommutativeSymmetricFunctions(QQ).Complete()
sage: list(map(S, B.basis()))  # indirect doctest
[S[1, 1, 1, 1],
 S[1, 1, 2],
 S[1, 2, 1],
 S[1, 3],
 S[2, 1, 1],
 S[2, 2],
 S[3, 1],
 S[4]]
```

class D(*alg*, *prefix='D'*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The standard basis of a descent algebra.

This basis is indexed by $S \subseteq \{1, 2, \ldots, n-1\}$, and the basis vector indexed by S is the sum of all permutations, taken in the symmetric group algebra RS_n, whose descent set is S. We denote this basis vector by D_S.

Occasionally this basis appears in literature but indexed by compositions of n rather than subsets of $\{1, 2, \ldots, n-1\}$. The equivalence between these two indexings is owed to the bijection from the power set of $\{1, 2, \ldots, n-1\}$ to the set of all compositions of n which sends every subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n-1\}$ (with $i_1 < i_2 < \cdots < i_k$) to the composition $(i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$.

The basis element corresponding to a composition p (or to the subset of $\{1, 2, \ldots, n-1\}$) is denoted Δ^p in [Sch2004].

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D()
sage: list(D.basis())
```

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```
[D{}, D{1}, D{2}, D{3}, D{1, 2}, D{1, 3}, D{2, 3}, D{1, 2, 3}]

sage: DA = DescentAlgebra(QQ, 0)
sage: D = DA.D()
sage: list(D.basis())
[D{}]
```

one_basis()

Return the identity element, as per `AlgebrasWithBasis.ParentMethods.one_basis`.

EXAMPLES:

```
sage: DescentAlgebra(QQ, 4).D().one_basis()
()
sage: DescentAlgebra(QQ, 0).D().one_basis()
()

sage: all( U * DescentAlgebra(QQ, 3).D().one() == U
....:      for U in DescentAlgebra(QQ, 3).D().basis() )
True
```

product_on_basis(S, T)

Return $D_S D_T$, where S and T are subsets of $[n-1]$.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D()
sage: D.product_on_basis((1, 3), (2,))
D{} + D{1} + D{1, 2} + 2*D{1, 2, 3} + D{1, 3} + D{2} + D{2, 3} + D{3}
```

to_B_basis(S)

Return D_S as a linear combination of B_p-basis elements.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D()
sage: B = DA.B()
sage: list(map(B, D.basis()))  # indirect doctest
[B[4],
 B[1, 3] - B[4],
 B[2, 2] - B[4],
 B[3, 1] - B[4],
 B[1, 1, 2] - B[1, 3] - B[2, 2] + B[4],
 B[1, 2, 1] - B[1, 3] - B[3, 1] + B[4],
 B[2, 1, 1] - B[2, 2] - B[3, 1] + B[4],
 B[1, 1, 1, 1] - B[1, 1, 2] - B[1, 2, 1] + B[1, 3]
  - B[2, 1, 1] + B[2, 2] + B[3, 1] - B[4]]
```

to_symmetric_group_algebra_on_basis(S)

Return D_S as a linear combination of basis elements in the symmetric group algebra.

EXAMPLES:

```
sage: D = DescentAlgebra(QQ, 4).D()
sage: [D.to_symmetric_group_algebra_on_basis(tuple(b))
....:  for b in Subsets(3)]
[[1, 2, 3, 4],
 [2, 1, 3, 4] + [3, 1, 2, 4] + [4, 1, 2, 3],
 [1, 3, 2, 4] + [1, 4, 2, 3] + [2, 3, 1, 4]
  + [2, 4, 1, 3] + [3, 4, 1, 2],
 [1, 2, 4, 3] + [1, 3, 4, 2] + [2, 3, 4, 1],
 [3, 2, 1, 4] + [4, 2, 1, 3] + [4, 3, 1, 2],
 [2, 1, 4, 3] + [3, 1, 4, 2] + [3, 2, 4, 1]
  + [4, 1, 3, 2] + [4, 2, 3, 1],
 [1, 4, 3, 2] + [2, 4, 3, 1] + [3, 4, 2, 1],
 [4, 3, 2, 1]]
```

class I(*alg*, *prefix='I'*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The idempotent basis of a descent algebra.

The idempotent basis $(I_p)_{p\models n}$ is a basis for Σ_n whenever the ground ring is a $\mathbf{Q}$-algebra. One way to compute it is using the formula (Theorem 3.3 in [GR1989])

$$I_p = \sum_{q\leq p} \frac{(-1)^{l(q)-l(p)}}{\mathbf{k}(q,p)} B_q,$$

where $\leq$ is the refinement order and $l(r)$ denotes the number of parts of any composition r, and where $\mathbf{k}(q,p)$ is defined as follows: When $q \leq p$, we can write q as a concatenation $q_{(1)}q_{(2)}\cdots q_{(k)}$ with each $q_{(i)}$ being a composition of the i-th entry of p, and then we set $\mathbf{k}(q,p)$ to be the product $l(q_{(1)})l(q_{(2)})\cdots l(q_{(k)})$.

Let $\lambda(p)$ denote the partition obtained from a composition p by sorting. This basis is called the idempotent basis since for any q such that $\lambda(p) = \lambda(q)$, we have:

$$I_p I_q = s(\lambda) I_p$$

where λ denotes $\lambda(p) = \lambda(q)$, and where $s(\lambda)$ is the stabilizer of λ in S_n. (This is part of Theorem 4.2 in [GR1989].)

It is also straightforward to compute the idempotents E_λ for the symmetric group algebra by the formula (Theorem 3.2 in [GR1989]):

$$E_\lambda = \frac{1}{k!} \sum_{\lambda(p)=\lambda} I_p.$$

Note: The basis elements are not orthogonal idempotents.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: I = DA.I()
sage: list(I.basis())
[I[1, 1, 1, 1], I[1, 1, 2], I[1, 2, 1], I[1, 3], I[2, 1, 1], I[2, 2], I[3, 1],
↪I[4]]
```

idempotent(*la*)

Return the idempotent corresponding to the partition `la` of n.

EXAMPLES:

```
sage: I = DescentAlgebra(QQ, 4).I()
sage: E = I.idempotent([3,1]); E
1/2*I[1, 3] + 1/2*I[3, 1]
sage: E*E == E
True
sage: E2 = I.idempotent([2,1,1]); E2
1/6*I[1, 1, 2] + 1/6*I[1, 2, 1] + 1/6*I[2, 1, 1]
sage: E2*E2 == E2
True
sage: E*E2 == I.zero()
True
```

one()

Return the identity element, which is $B_{[n]}$, in the I basis.

EXAMPLES:

```
sage: DescentAlgebra(QQ, 4).I().one()
1/24*I[1, 1, 1, 1] + 1/6*I[1, 1, 2] + 1/6*I[1, 2, 1]
 + 1/2*I[1, 3] + 1/6*I[2, 1, 1] + 1/2*I[2, 2]
 + 1/2*I[3, 1] + I[4]
sage: DescentAlgebra(QQ, 0).I().one()
I[]
```

one_basis()

The element 1 is not (generally) a basis vector in the I basis, thus this returns a `TypeError`.

EXAMPLES:

```
sage: DescentAlgebra(QQ, 4).I().one_basis()
Traceback (most recent call last):
...
TypeError: 1 is not a basis element in the I basis
```

product_on_basis(*p*, *q*)

Return $I_p I_q$, where p and q are compositions of n.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: I = DA.I()
sage: p = Composition([1,2,1])
sage: q = Composition([3,1])
sage: I.product_on_basis(p, q)
0
sage: I.product_on_basis(p, p)
2*I[1, 2, 1]
```

to_B_basis(*p*)

Return I_p as a linear combination of B-basis elements.

This is computed using the formula (Theorem 3.3 in [GR1989])

$$I_p = \sum_{q \leq p} \frac{(-1)^{l(q)-l(p)}}{\mathbf{k}(q,p)} B_q,$$

where $\leq$ is the refinement order and $l(r)$ denotes the number of parts of any composition r, and where $\mathbf{k}(q,p)$ is defined as follows: When $q \leq p$, we can write q as a concatenation $q_{(1)} q_{(2)} \cdots q_{(k)}$ with each $q_{(i)}$ being a composition of the i-th entry of p, and then we set $\mathbf{k}(q,p)$ to be $l(q_{(1)}) l(q_{(2)}) \cdots l(q_{(k)})$.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: I = DA.I()
sage: list(map(B, I.basis()))  # indirect doctest
[B[1, 1, 1, 1],
 -1/2*B[1, 1, 1, 1] + B[1, 1, 2],
 -1/2*B[1, 1, 1, 1] + B[1, 2, 1],
 1/3*B[1, 1, 1, 1] - 1/2*B[1, 1, 2] - 1/2*B[1, 2, 1] + B[1, 3],
 -1/2*B[1, 1, 1, 1] + B[2, 1, 1],
 1/4*B[1, 1, 1, 1] - 1/2*B[1, 1, 2] - 1/2*B[2, 1, 1] + B[2, 2],
 1/3*B[1, 1, 1, 1] - 1/2*B[1, 2, 1] - 1/2*B[2, 1, 1] + B[3, 1],
 -1/4*B[1, 1, 1, 1] + 1/3*B[1, 1, 2] + 1/3*B[1, 2, 1]
  - 1/2*B[1, 3] + 1/3*B[2, 1, 1] - 1/2*B[2, 2]
  - 1/2*B[3, 1] + B[4]]
```

a_realization()

Return a particular realization of `self` (the B-basis).

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: DA.a_realization()
Descent algebra of 4 over Rational Field in the subset basis
```

idempotent

alias of *I*

standard

alias of *D*

subset

alias of *B*

class sage.combinat.descent_algebra.DescentAlgebraBases(*base*)

Bases: `Category_realization_of_parent`

The category of bases of a descent algebra.

class ElementMethods

Bases: `object`

to_symmetric_group_algebra()

Return `self` in the symmetric group algebra.

EXAMPLES:

```
sage: B = DescentAlgebra(QQ, 4).B()
sage: B[1,3].to_symmetric_group_algebra()
[1, 2, 3, 4] + [2, 1, 3, 4] + [3, 1, 2, 4] + [4, 1, 2, 3]
sage: I = DescentAlgebra(QQ, 4).I()
sage: elt = I(B[1,3])
sage: elt.to_symmetric_group_algebra()
[1, 2, 3, 4] + [2, 1, 3, 4] + [3, 1, 2, 4] + [4, 1, 2, 3]
```

class ParentMethods

Bases: `object`

is_commutative()

Return whether this descent algebra is commutative.

EXAMPLES:

```
sage: B = DescentAlgebra(QQ, 4).B()
sage: B.is_commutative()
False
sage: B = DescentAlgebra(QQ, 1).B()
sage: B.is_commutative()
True
```

is_field(*proof=True*)

Return whether this descent algebra is a field.

EXAMPLES:

```
sage: B = DescentAlgebra(QQ, 4).B()
sage: B.is_field()
False
sage: B = DescentAlgebra(QQ, 1).B()
sage: B.is_field()
True
```

to_symmetric_group_algebra()

Morphism from `self` to the symmetric group algebra.

EXAMPLES:

```
sage: D = DescentAlgebra(QQ, 4).D()
sage: D.to_symmetric_group_algebra(D[1,3])
[2, 1, 4, 3] + [3, 1, 4, 2] + [3, 2, 4, 1] + [4, 1, 3, 2] + [4, 2, 3, 1]
sage: B = DescentAlgebra(QQ, 4).B()
sage: B.to_symmetric_group_algebra(B[1,2,1])
[1, 2, 3, 4] + [1, 2, 4, 3] + [1, 3, 4, 2] + [2, 1, 3, 4]
 + [2, 1, 4, 3] + [2, 3, 4, 1] + [3, 1, 2, 4] + [3, 1, 4, 2]
 + [3, 2, 4, 1] + [4, 1, 2, 3] + [4, 1, 3, 2] + [4, 2, 3, 1]
```

to_symmetric_group_algebra_on_basis(*S*)

Return the basis element index by S as a linear combination of basis elements in the symmetric group algebra.

EXAMPLES:

```
sage: B = DescentAlgebra(QQ, 3).B()
sage: [B.to_symmetric_group_algebra_on_basis(c)
....:  for c in Compositions(3)]
[[1, 2, 3] + [1, 3, 2] + [2, 1, 3]
  + [2, 3, 1] + [3, 1, 2] + [3, 2, 1],
 [1, 2, 3] + [2, 1, 3] + [3, 1, 2],
 [1, 2, 3] + [1, 3, 2] + [2, 3, 1],
 [1, 2, 3]]
sage: I = DescentAlgebra(QQ, 3).I()
sage: [I.to_symmetric_group_algebra_on_basis(c)
....:  for c in Compositions(3)]
[[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1]
  + [3, 1, 2] + [3, 2, 1],
 1/2*[1, 2, 3] - 1/2*[1, 3, 2] + 1/2*[2, 1, 3]
  - 1/2*[2, 3, 1] + 1/2*[3, 1, 2] - 1/2*[3, 2, 1],
 1/2*[1, 2, 3] + 1/2*[1, 3, 2] - 1/2*[2, 1, 3]
  + 1/2*[2, 3, 1] - 1/2*[3, 1, 2] - 1/2*[3, 2, 1],
 1/3*[1, 2, 3] - 1/6*[1, 3, 2] - 1/6*[2, 1, 3]
  - 1/6*[2, 3, 1] - 1/6*[3, 1, 2] + 1/3*[3, 2, 1]]
```

super_categories()

The super categories of `self`.

EXAMPLES:

```
sage: from sage.combinat.descent_algebra import DescentAlgebraBases
sage: DA = DescentAlgebra(QQ, 4)
sage: bases = DescentAlgebraBases(DA)
sage: bases.super_categories()
[Category of finite dimensional algebras with basis over Rational Field,
 Category of realizations of Descent algebra of 4 over Rational Field]
```

5.7 Down-Up Algebras

AUTHORS:

- Travis Scrimshaw (2023-4): initial version

class sage.algebras.down_up_algebra.**DownUpAlgebra**(*alpha*, *beta*, *gamma*, *base_ring*)

Bases: `CombinatorialFreeModule`

The down-up algebra.

Let R be a commutative ring, and let $\alpha, \beta, \gamma \in R$. The *down-up algebra* is the associative unital algebra $DU(\alpha, \beta, \gamma)$ generated by d, u with relations

$$\begin{aligned} d^2u &= \alpha dud + \beta ud^2 + \gamma d, \\ du^2 &= \alpha udu + \beta u^2 d + \gamma u. \end{aligned}$$

The down-up algebra has a PBW-type basis given by

$$\{u^i (du)^j d^k \mid i, j, k \in \mathbf{Z}_{\geq 0}\}.$$

This algebra originates in the study of posets. For a poset P, we define operators acting on $R[P]$ by

$$d(y) = \sum_x x \qquad u(y) = \sum_z z,$$

where y covers x and z covers y. For r-differential posets we have $du - ud = r1$, and thus it affords a representation of a `Weyl algebra`. This Weyl algebra is obtained as the quotient of $DU(0, 1, 2r)$ by the ideal generated by $du - ud - r$. For a (q, r)-differential poset, we have the d and u operators satisfying

$$d^2u = q(q+1)dud - q^3ud^2 + rd,$$
$$du^2 = q(q+1)udu - q^3u^2d + ru,$$

or $\alpha = q(q+1)$, $\beta = -q^3$, and $\gamma = r$. Specializing $q = -1$ recovers the r-differential poset relation.

Two other noteworthy quotients are:

- the q-Weyl algebra from $DU(0, q^2, q+1)$ by the ideal generated by $du - qud - 1$, and
- the quantum plane $R_q[d, u]$, where $du = qud$, from $DU(2q, -q^2, 0)$ by the ideal generated by $du - qud$.

EXAMPLES:

We begin by constructing the down-up algebra and perform some basic computations:

```
sage: R.<a,b,g> = QQ[]
sage: DU = algebras.DownUp(a, b, g)
sage: d, u = DU.gens()
sage: d * u
(d*u)
sage: u * d
u*d
sage: d^2 * u
b*u*d^2 + a*(d*u)*d + g*d
sage: d * u^2
b*u^2*d + a*u*(d*u) + g*u
```

We verify some examples of Proposition 3.5 in [BR1998], which states that the 0-th degree part is commutative:

```
sage: DU0 = [u^i * (d*u)^j * d^i for i,j in
....:        cartesian_product([range(3), range(3)])]
sage: all(x.degree() == 0 for x in DU0)
True
sage: all(x * y == y * x for x, y in cartesian_product([DU0, DU0]))
True
```

We verify that $DU(2, -1, \gamma)$ can be described as the universal enveloping algebra of the 3-dimensional Lie algebra spanned by x, y, z satisfying $z = [x, y]$, $[x, z] = \gamma x$, and $[z, y] = \gamma y$:

```
sage: R.<g> = QQ[]
sage: L = LieAlgebra(R, {('x','y'): {'z': 1}, ('x','z'): {'x': g}, ('z','y'): {'y':␣
↪g}},
....:                names='x,y,z')
sage: x, y, z = L.basis()
sage: (L[x, y], L[x, z], L[z, y])
(z, g*x, g*y)
sage: x, y, z = L.pbw_basis().gens()
sage: x^2*y - 2*x*y*x + y*x^2 == g*x
```

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```
True
sage: x*y^2 - 2*y*x*y + y^2*x == g*y
True
sage: DU = algebras.DownUp(2, -1, g)
sage: d, u = DU.gens()
sage: d^2*u - 2*d*u*d + u*d^2 == g*d
True
sage: d*u^2 - 2*u*d*u + u^2*d == g*u
True
```

Young's lattice is known to be a differential poset. Thus we can construct a representation of $DU(0, 1, 2)$ on this poset (which gives a proof that Fomin's `growth diagrams` are equivalent to edge local rules or shadow lines construction for `RSK()`):

```
sage: DU = algebras.DownUp(0, 1, 2)
sage: d, u = DU.gens()
sage: d^2*u == 0*d*u*d + 1*u*d*d + 2*d
True
sage: d*u^2 == 0*u*d*u + 1*u*u*d + 2*u
True

sage: YL = CombinatorialFreeModule(DU.base_ring(), Partitions())
sage: def d_action(la):
....:     return YL.sum_of_monomials(la.remove_cell(*c) for c in la.removable_
→cells())
sage: def u_action(la):
....:     return YL.sum_of_monomials(la.add_cell(*c) for c in la.addable_cells())
sage: D = YL.module_morphism(on_basis=d_action, codomain=YL)
sage: U = YL.module_morphism(on_basis=u_action, codomain=YL)
sage: for la in PartitionsInBox(5, 5):
....:     b = YL.basis()[la]
....:     assert (D*D*U)(b) == 0*(D*U*D)(b) + 1*(U*D*D)(b) + 2*D(b)
....:     assert (D*U*U)(b) == 0*(U*D*U)(la) + 1*(U*U*D)(b) + 2*U(b)
....:     assert (D*U)(b) == (U*D)(b) + b  # the Weyl algebra relation
```

Todo: Implement the homogenized version.

REFERENCES:

- [BR1998]
- [CM2000]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: DU = algebras.DownUp(2, 3, 4)
sage: dict(DU.algebra_generators())
{'d': d, 'u': u}
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: R.<a,b,g> = QQ[]
sage: DU = algebras.DownUp(a, b, g)
sage: I = DU.indices()
sage: DU.degree_on_basis(I([0, 3, 2]))
-2
sage: DU.degree_on_basis(I([2, 3, 0]))
2
sage: DU.degree_on_basis(I([2, 0, 3]))
-1
sage: DU.degree_on_basis(I([3, 10, 3]))
0
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: DU = algebras.DownUp(2, 3, 4)
sage: DU.gens()
(d, u)
```

one_basis()

Return the index of the basis element of 1.

EXAMPLES:

```
sage: DU = algebras.DownUp(2, 3, 4)
sage: DU.one_basis()
(0, 0, 0)
```

product_on_basis(*m1*, *m2*)

Return the product of the basis elements indexed by `m1` and `m2`.

EXAMPLES:

```
sage: R.<a,b,g> = QQ[]
sage: DU = algebras.DownUp(a, b, g)
sage: I = DU.indices()
sage: DU.product_on_basis(I([2,0,0]), I([4,0,0]))
u^6
sage: DU.product_on_basis(I([2,0,0]), I([0,4,0]))
u^2*(d*u)^4
sage: DU.product_on_basis(I([2,0,0]), I([0,0,4]))
u^2*d^4
sage: DU.product_on_basis(I([0,2,0]), I([0,4,0]))
(d*u)^6
sage: DU.product_on_basis(I([0,2,0]), I([0,0,4]))
(d*u)^2*d^4
sage: DU.product_on_basis(I([0,0,2]), I([0,0,4]))
d^6
```

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```
sage: DU.product_on_basis(I([5,3,1]), I([1,0,4]))
u^5*(d*u)^4*d^4

sage: DU.product_on_basis(I([0,1,0]), I([1,0,0]))
b*u^2*d + a*u*(d*u) + g*u
sage: DU.product_on_basis(I([0,0,2]), I([1,0,0]))
b*u*d^2 + a*(d*u)*d + g*d
sage: DU.product_on_basis(I([0,0,1]), I([2,0,0]))
b*u^2*d + a*u*(d*u) + g*u
sage: DU.product_on_basis(I([0,0,1]), I([0,1,0]))
b*u*d^2 + a*(d*u)*d + g*d

sage: DU.product_on_basis(I([0,1,0]), I([3,0,0]))
(a^2*b+b^2)*u^4*d + (a^3+2*a*b)*u^3*(d*u) + (a^2*g+a*g+b*g+g)*u^3
sage: DU.product_on_basis(I([1,1,3]), I([0,1,1]))
(a^2*b^2+b^3)*u^3*d^6 + (a^3*b+a*b^2)*u^2*(d*u)*d^5 + (a^2*b*g+b^2*g)*u^2*d^5
 + (a^3+2*a*b)*u*(d*u)^2*d^4 + (a^2*g+a*g+b*g+g)*u*(d*u)*d^4
```

verma_module(*la*)

Return the *Verma module* $V(\lambda)$ of self.

EXAMPLES:

```
sage: R.<a,b,g> = QQ[]
sage: DU = algebras.DownUp(a, b, g)
sage: DU.verma_module(5)
Verma module of weight 5 of Down-Up algebra with parameters (a, b, g)
 over Multivariate Polynomial Ring in a, b, g over Rational Field
```

class sage.algebras.down_up_algebra.**VermaModule**(*DU*, *la*)

Bases: CombinatorialFreeModule

The Verma module $V(\lambda)$ of a down-up algebra.

The Verma module $V(\lambda)$ for the down-up algebra generated by d, u is the span of $\{v_n \mid n \in \mathbf{Z}_{\geq 0}\}$ satisfying the relations

$$d \cdot v_n = \lambda_{n-1} v_{n-1}, \qquad u \cdot v_n = v_{n+1},$$

where $\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma$ and we set $\lambda_0 = \lambda$ and $\lambda_{-1} = 0$.

By Proposition 2.4 in [BR1998], $V(\lambda)$ is simple if and only if $\lambda_n \neq 0$ for all $n \geq 0$. Moreover, a maximal submodule is spanned by $\{v_n \mid n > m\}$, where m is the minimal index such that $\lambda_m = 0$. Moreover, this is unique unless $\gamma = \lambda = 0$.

EXAMPLES:

```
sage: R.<a,b> = QQ[]
sage: DU = algebras.DownUp(0, b, 1)
sage: d, u = DU.gens()
sage: V = DU.verma_module(a)
sage: list(V.weights()[:6])
[a, 1, a*b + 1, b + 1, a*b^2 + b + 1, b^2 + b + 1]
sage: v = V.basis()
sage: d^2 * v[2]
```

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```
a*v[0]
sage: d * (d * v[2])
a*v[0]
```

The weight is computed by looking at the scalars associated to the action of du and ud:

```
sage: d*u * v[3]
(b+1)*v[3]
sage: u*d * v[3]
(a*b+1)*v[3]
sage: v[3].weight()
(b + 1, a*b + 1)
```

An $U(\mathfrak{sl}_2)$ example:

```
sage: DU = algebras.DownUp(2, -1, -2)
sage: d, u = DU.gens()
sage: V = DU.verma_module(5)
sage: list(V.weights()[:10])
[5, 8, 9, 8, 5, 0, -7, -16, -27, -40]
sage: v6 = V.basis()[6]
sage: d * v6
0
sage: [V.basis()[i].weight() for i in range(6)]
[(5, 0), (8, 5), (9, 8), (8, 9), (5, 8), (0, 5)]
```

Note that these are the same $\mathfrak{sl}_2$ weights from the usual construction of the irreducible representation $V(5)$ (but they are different as $\mathfrak{gl}_2$ weights):

```
sage: B = crystals.Tableaux(['A',1], shape=[5])
sage: [b.weight() for b in B]
[(5, 0), (4, 1), (3, 2), (2, 3), (1, 4), (0, 5)]
```

An example with periodic weights (see Theorem 2.13 of [BR1998]):

```
sage: k.<z6> = CyclotomicField(6)
sage: al = z6 + 1
sage: (al - 1)^6 == 1
True
sage: DU = algebras.DownUp(al, 1-al, 0)
sage: V = DU.verma_module(5)
sage: list(V.weights()[:8])
[5, 5*z6 + 5, 10*z6, 10*z6 - 5, 5*z6 - 5, 0, 5, 5*z6 + 5]
```

class Element

Bases: `IndexedFreeModuleElement`

An element of a Verma module of a down-up algebra.

is_weight_vector()

Return if `self` is a weight vector.

EXAMPLES:

```
sage: DU = algebras.DownUp(2, -1, -2)
sage: V = DU.verma_module(5)
sage: V.zero().is_weight_vector()
False
sage: B = V.basis()
sage: [B[i].weight() for i in range(6)]
[(5, 0), (8, 5), (9, 8), (8, 9), (5, 8), (0, 5)]
sage: B[5].is_weight_vector()
True
sage: v = B[0] + B[1]
sage: v.is_weight_vector()
False

sage: DU = algebras.DownUp(2, -1, 0)
sage: V = DU.verma_module(0)
sage: B = V.basis()
sage: v = sum(i*B[i] for i in range(1,5))
sage: v.is_weight_vector()
True
```

weight()

Return the weight of `self`.

For v_n, this is the vector with the pair $(\lambda_n, \lambda_{n-1})$.

EXAMPLES:

```
sage: R.<a,b,g> = QQ[]
sage: DU = algebras.DownUp(a, b, g)
sage: V = DU.verma_module(5)
sage: B = V.basis()
sage: B[0].weight()
(5, 0)
sage: B[1].weight()
(5*a + g, 5)
sage: B[2].weight()
(5*a^2 + a*g + 5*b + g, 5*a + g)

sage: V.zero().weight()
Traceback (most recent call last):
...
ValueError: the zero element does not have well-defined weight
sage: (B[0] + B[1]).weight()
Traceback (most recent call last):
...
ValueError: not a weight vector
```

highest_weight_vector()

Return the highest weight vector of `self` that generates `self` as a down-up module.

EXAMPLES:

```
sage: DU = algebras.DownUp(1, 2, 3)
sage: V = DU.verma_module(5)
```

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```
sage: V.highest_weight_vector()
v[0]
```

weights()

Return the sequence of weights $(\lambda_n)_{n=0}^{\infty}$.

EXAMPLES:

```
sage: R.<a,b,g> = QQ[]
sage: DU = algebras.DownUp(a, b, g)
sage: V = DU.verma_module(5)
sage: V.weights()
lazy list [5, 5*a + g, 5*a^2 + a*g + 5*b + g, ...]

sage: V = DU.verma_module(0)
sage: DU = algebras.DownUp(a, 1-a, 0)
sage: V = DU.verma_module(0)
sage: V.weights()
lazy list [0, 0, 0, ...]
```

We reproduce the Fibonacci numbers example from [BR1998]:

```
sage: R.<la> = QQ[]
sage: DU = algebras.DownUp(1, 1, 0, R)
sage: V = DU.verma_module(la)
sage: list(V.weights()[:11])
[la, la, 2*la, 3*la, 5*la, 8*la, 13*la, 21*la, 34*la, 55*la, 89*la]
```

5.8 Fusion Rings

5.8.1 Fusion Rings

class sage.algebras.fusion_rings.fusion_ring.**FusionRing**(*ct*, *base_ring=Integer Ring*, *prefix=None*, *style='lattice'*, *k=None*, *conjugate=False*, *cyclotomic_order=None*, *fusion_labels=None*, *inject_variables=False*)

Bases: `WeylCharacterRing`

Return the Fusion Ring (Verlinde Algebra) of level `k`.

INPUT:

- `ct` – the Cartan type of a simple (finite-dimensional) Lie algebra
- `k` – a nonnegative integer
- `conjugate` – (default `False`) set `True` to obtain the complex conjugate ring
- `cyclotomic_order` – (default computed depending on `ct` and `k`)
- `fusion_labels` – (default None) either a tuple of strings to use as labels of the basis of simple objects, or a string from which the labels will be constructed

- `inject_variables` – (default `False`): use with `fusion_labels`. If `inject_variables` is `True`, the fusion labels will be variables that can be accessed from the command line

The cyclotomic order is an integer N such that all computations will return elements of the cyclotomic field of N-th roots of unity. Normally you will never need to change this but consider changing it if `root_of_unity()` raises a `ValueError`.

This algebra has a basis (sometimes called *primary fields* but here called *simple objects*) indexed by the weights of level $\leq k$. These arise as the fusion algebras of Wess-Zumino-Witten (WZW) conformal field theories, or as Grothendieck groups of tilting modules for quantum groups at roots of unity. The `FusionRing` class is implemented as a variant of the `WeylCharacterRing`.

REFERENCES:

- [BaKi2001] Chapter 3
- [DFMS1996] Chapter 16
- [EGNO2015] Chapter 8
- [Feingold2004]
- [Fuchs1994]
- [Row2006]
- [Walton1990]
- [Wan2010]

EXAMPLES:

```
sage: A22 = FusionRing("A2", 2)
sage: [f1, f2] = A22.fundamental_weights()
sage: M = [A22(x) for x in [0*f1, 2*f1, 2*f2, f1+f2, f2, f1]]
sage: [M[3] * x for x in M]
[A22(1,1),
 A22(0,1),
 A22(1,0),
 A22(0,0) + A22(1,1),
 A22(0,1) + A22(2,0),
 A22(1,0) + A22(0,2)]
```

You may assign your own labels to the basis elements. In the next example, we create the $SO(5)$ fusion ring of level 2, check the weights of the basis elements, then assign new labels to them while injecting them into the global namespace:

```
sage: B22 = FusionRing("B2", 2)
sage: b = [B22(x) for x in B22.get_order()]; b
[B22(0,0), B22(1,0), B22(0,1), B22(2,0), B22(1,1), B22(0,2)]
sage: [x.weight() for x in b]
[(0, 0), (1, 0), (1/2, 1/2), (2, 0), (3/2, 1/2), (1, 1)]
sage: B22.fusion_labels(['I0', 'Y1', 'X', 'Z', 'Xp', 'Y2'], inject_variables=True)
sage: b = [B22(x) for x in B22.get_order()]; b
[I0, Y1, X, Z, Xp, Y2]
sage: [(x, x.weight()) for x in b]
[(I0, (0, 0)),
 (Y1, (1, 0)),
 (X, (1/2, 1/2)),
```

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```
 (Z, (2, 0)),
 (Xp, (3/2, 1/2)),
 (Y2, (1, 1))]
sage: X * Y1
X + Xp
sage: Z * Z
I0
```

A fixed order of the basis keys is available with *get_order()*. This is the order used by methods such as *s_matrix()*. You may use CombinatorialFreeModule.set_order() to reorder the basis:

```
sage: B22.set_order([x.weight() for x in [I0, Y1, Y2, X, Xp, Z]])
sage: [B22(x) for x in B22.get_order()]
[I0, Y1, Y2, X, Xp, Z]
```

To reset the labels, you may run *fusion_labels()* with no parameter:

```
sage: B22.fusion_labels()
sage: [B22(x) for x in B22.get_order()]
[B22(0,0), B22(1,0), B22(0,2), B22(0,1), B22(1,1), B22(2,0)]
```

To reset the order to the default, simply set it to the list of basis element keys:

```
sage: B22.set_order(B22.basis().keys().list())
sage: [B22(x) for x in B22.get_order()]
[B22(0,0), B22(1,0), B22(0,1), B22(2,0), B22(1,1), B22(0,2)]
```

The fusion ring has a number of methods that reflect its role as the Grothendieck ring of a *modular tensor category* (MTC). These include twist methods *Element.twist()* and *Element.ribbon()* for its elements related to the ribbon structure, and the S-matrix *s_ij()*.

There are two natural normalizations of the S-matrix. Both are explained in Chapter 3 of [BaKi2001]. The one that is computed by the method *s_matrix()*, or whose individual entries are computed by *s_ij()* is denoted $\tilde{s}$ in [BaKi2001]. It is not unitary.

The unitary S-matrix is $s = D^{-1/2}\tilde{s}$ where

$$D = \sum_V d_i(V)^2.$$

The sum is over all simple objects V with $d_i(V)$ the *quantum dimension*. We will call quantity D the *global quantum dimension* and $\sqrt{D}$ the *total quantum order*. They are computed by *global_q_dimension()* and *total_q_order()*. The unitary S-matrix s may be obtained using *s_matrix()* with the option `unitary=True`.

Let us check the Verlinde formula, which is [DFMS1996] (16.3). This famous identity states that

$$N^k_{ij} = \sum_l \frac{s(i,\ell)\, s(j,\ell)\, \overline{s(k,\ell)}}{s(I,\ell)},$$

where N^k_{ij} are the fusion coefficients, i.e. the structure constants of the fusion ring, and `I` is the unit object. The S-matrix has the property that if $i*$ denotes the dual object of i, implemented in Sage as `i.dual()`, then

$$s(i*,j) = s(i,j*) = \overline{s(i,j)}.$$

This is equation (16.5) in [DFMS1996]. Thus with $N_{ijk} = N^{k*}_{ij}$ the Verlinde formula is equivalent to

$$N_{ijk} = \sum_l \frac{s(i,\ell)\, s(j,\ell)\, s(k,\ell)}{s(I,\ell)},$$

In this formula s is the normalized unitary S-matrix denoted s in [BaKi2001]. We may define a function that corresponds to the right-hand side, except using $\tilde{s}$ instead of s:

```
sage: def V(i, j, k):
....:     R = i.parent()
....:     return sum(R.s_ij(i, l) * R.s_ij(j, l) * R.s_ij(k, l) / R.s_ij(R.one(), l)
....:                for l in R.basis())
```

This does not produce `self.N_ijk(i, j, k)` exactly, because of the missing normalization factor. The following code to check the Verlinde formula takes this into account:

```
sage: def test_verlinde(R):
....:     b0 = R.one()
....:     c = R.global_q_dimension()
....:     return all(V(i, j, k) == c * R.N_ijk(i, j, k) for i in R.basis()
....:                for j in R.basis() for k in R.basis())
```

Every fusion ring should pass this test:

```
sage: test_verlinde(FusionRing("A2", 1))
True
sage: test_verlinde(FusionRing("B4", 2)) # long time (.56s)
True
```

As an exercise, the reader may verify the examples in Section 5.3 of [RoStWa2009]. Here we check the example of the Ising modular tensor category, which is related to the Belavin, Polyakov, Zamolodchikov minimal model $M(4,3)$ or to an E_8 coset model. See [DFMS1996] Sections 7.4.2 and 18.4.1. [RoStWa2009] Example 5.3.4 tells us how to construct it as the conjugate of the E_8 level 2 *FusionRing*:

```
sage: I = FusionRing("E8", 2, conjugate=True)
sage: I.fusion_labels(["i0", "p", "s"], inject_variables=True)
sage: b = I.basis().list(); b
[i0, p, s]
sage: Matrix([[x*y for x in b] for y in b]) # long time (.93s)
[    i0      p      s]
[     p     i0      s]
[     s      s i0 + p]
sage: [x.twist() for x in b]
[0, 1, 1/8]
sage: [x.ribbon() for x in b]
[1, -1, zeta128^8]
sage: [I.r_matrix(i, j, k) for (i, j, k) in [(s, s, i0), (p, p, i0), (p, s, s), (s,␣
↪p, s), (s, s, p)]]
[-zeta128^56, -1, -zeta128^32, -zeta128^32, zeta128^24]
sage: I.r_matrix(s, s, i0) == I.root_of_unity(-1/8)
True
sage: I.global_q_dimension()
4
sage: I.total_q_order()
2
sage: [x.q_dimension()^2 for x in b]
[1, 1, 2]
sage: I.s_matrix()
[                        1                         1 -zeta128^48 + zeta128^16]
```

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```
[                        1                         1  zeta128^48 - zeta128^16]
[-zeta128^48 + zeta128^16  zeta128^48 - zeta128^16                        0]
sage: I.s_matrix().apply_map(lambda x:x^2)
[1 1 2]
[1 1 2]
[2 2 0]
```

The term *modular tensor category* refers to the fact that associated with the category there is a projective representation of the modular group $SL(2, \mathbf{Z})$. We recall that this group is generated by

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

subject to the relations $(ST)^3 = S^2$, $S^2T = TS^2$, and $S^4 = I$. Let s be the normalized S-matrix, and t the diagonal matrix whose entries are the twists of the simple objects. Let s the unitary S-matrix and t the matrix of twists, and C the conjugation matrix `conj_matrix()`. Let

$$D_+ = \sum_i d_i^2 \theta_i, \qquad D_- = d_i^2 \theta_i^{-1},$$

where d_i and θ_i are the quantum dimensions and twists of the simple objects. Let c be the Virasoro central charge, a rational number that is computed in `virasoro_central_charge()`. It is known that

$$\sqrt{\frac{D_+}{D_-}} = e^{i\pi c/4}.$$

It is proved in [BaKi2001] Equation (3.1.17) that

$$(st)^3 = e^{i\pi c/4} s^2, \qquad s^2 = C, \qquad C^2 = 1, \qquad Ct = tC.$$

Therefore $S \mapsto s, T \mapsto t$ is a projective representation of $SL(2, \mathbf{Z})$. Let us confirm these identities for the Fibonacci MTC `FusionRing("G2", 1)`:

```
sage: R = FusionRing("G2", 1)
sage: S = R.s_matrix(unitary=True)
sage: T = R.twists_matrix()
sage: C = R.conj_matrix()
sage: c = R.virasoro_central_charge(); c
14/5
sage: (S*T)^3 == R.root_of_unity(c/4) * S^2
True
sage: S^2 == C
True
sage: C*T == T*C
True
```

D_minus(*base_coercion=True*)

Return $\sum d_i^2 \theta_i^{-1}$ where i runs through the simple objects, d_i is the quantum dimension and θ_i is the twist.

This is denoted p_- in [BaKi2001] Chapter 3.

EXAMPLES:

```
sage: E83 = FusionRing("E8", 3, conjugate=True)
sage: [Dp, Dm] = [E83.D_plus(), E83.D_minus()]
sage: Dp*Dm == E83.global_q_dimension()
True
sage: c = E83.virasoro_central_charge(); c
-248/11
sage: Dp*Dm == E83.global_q_dimension()
True
```

D_plus(*base_coercion=True*)

Return $\sum d_i^2\theta_i$ where i runs through the simple objects, d_i is the quantum dimension and θ_i is the twist.

This is denoted p_+ in [BaKi2001] Chapter 3.

EXAMPLES:

```
sage: B31 = FusionRing("B3", 1)
sage: Dp = B31.D_plus(); Dp
2*zeta48^13 - 2*zeta48^5
sage: Dm = B31.D_minus(); Dm
-2*zeta48^3
sage: Dp*Dm == B31.global_q_dimension()
True
sage: c = B31.virasoro_central_charge(); c
7/2
sage: Dp/Dm == B31.root_of_unity(c/2)
True
```

class Element

Bases: `Element`

A class for FusionRing elements.

is_simple_object()

Determine whether `self` is a simple object of the fusion ring.

EXAMPLES:

```
sage: A22 = FusionRing("A2", 2)
sage: x = A22(1, 0); x
A22(1,0)
sage: x.is_simple_object()
True
sage: x^2
A22(0,1) + A22(2,0)
sage: (x^2).is_simple_object()
False
```

q_dimension(*base_coercion=True*)

Return the quantum dimension as an element of the cyclotomic field of the 2ℓ-th roots of unity, where $l = m(k + h^\vee)$ with $m = 1, 2, 3$ depending on whether type is simply, doubly or triply laced, k is the level and $h^\vee$ is the dual Coxeter number.

EXAMPLES:

```
sage: B22 = FusionRing("B2", 2)
sage: [(b.q_dimension())^2 for b in B22.basis()]
[1, 4, 5, 1, 5, 4]
```

ribbon(*base_coercion=True*)

Return the twist or ribbon element of `self`.

If h is the rational number modulo 2 produced by `self.twist()`, this method produces $e^{i\pi h}$.

See also:

An additive version of this is available as *twist()*.

EXAMPLES:

```
sage: F = FusionRing("A1", 3)
sage: [x.twist() for x in F.basis()]
[0, 3/10, 4/5, 3/2]
sage: [x.ribbon(base_coercion=False) for x in F.basis()]
[1, zeta40^6, zeta40^12 - zeta40^8 + zeta40^4 - 1, -zeta40^10]
sage: [F.root_of_unity(x, base_coercion=False) for x in [0, 3/10, 4/5, 3/2]]
[1, zeta40^6, zeta40^12 - zeta40^8 + zeta40^4 - 1, -zeta40^10]
```

twist(*reduced=True*)

Return a rational number h such that $\theta = e^{i\pi h}$ is the twist of `self`. The quantity $e^{i\pi h}$ is also available using *ribbon()*.

This method is only available for simple objects. If λ is the weight of the object, then $h = \langle\lambda, \lambda + 2\rho\rangle$, where ρ is half the sum of the positive roots. As in [Row2006], this requires normalizing the invariant bilinear form so that $\langle\alpha, \alpha\rangle = 2$ for short roots.

INPUT:

- `reduced` – (default: `True`) boolean; if `True` then return the twist reduced modulo 2

EXAMPLES:

```
sage: G21 = FusionRing("G2", 1)
sage: [x.twist() for x in G21.basis()]
[0, 4/5]
sage: [G21.root_of_unity(x.twist()) for x in G21.basis()]
[1, zeta60^14 - zeta60^4]
sage: zeta60 = G21.field().gen()
sage: zeta60^((4/5)*(60/2))
zeta60^14 - zeta60^4

sage: F42 = FusionRing("F4", 2)
sage: [x.twist() for x in F42.basis()]
[0, 18/11, 2/11, 12/11, 4/11]

sage: E62 = FusionRing("E6", 2)
sage: [x.twist() for x in E62.basis()]
[0, 26/21, 12/7, 8/21, 8/21, 26/21, 2/3, 4/7, 2/3]
```

weight()

Return the parametrizing dominant weight in the level k alcove.

This method is only available for basis elements.

EXAMPLES:

```
sage: A21 = FusionRing("A2", 1)
sage: [x.weight() for x in A21.basis().list()]
[(0, 0, 0), (2/3, -1/3, -1/3), (1/3, 1/3, -2/3)]
```

N_ijk(*elt_i*, *elt_j*, *elt_k*)

Return the symmetric fusion coefficient N_{ijk}.

INPUT:

- `elt_i`, `elt_j`, `elt_k` – elements of the fusion basis

This is the same as N_{ij}^{k*}, where N_{ij}^k are the structure coefficients of the ring (see `Nk_ij()`), and $k*$' denotes the dual element. The coefficient N_{ijk} is unchanged under permutations of the three basis vectors.

EXAMPLES:

```
sage: G23 = FusionRing("G2", 3)
sage: G23.fusion_labels("g")
sage: b = G23.basis().list(); b
[g0, g1, g2, g3, g4, g5]
sage: [(x, y, z) for x in b for y in b for z in b if G23.N_ijk(x, y, z) > 1]
[(g3, g3, g3), (g3, g3, g4), (g3, g4, g3), (g4, g3, g3)]
sage: all(G23.N_ijk(x, y, z)==G23.N_ijk(y, z, x) for x in b for y in b for z in␣
↪b)
True
sage: all(G23.N_ijk(x, y, z)==G23.N_ijk(y, x, z) for x in b for y in b for z in␣
↪b)
True
```

Nk_ij(*elt_i*, *elt_j*, *elt_k*)

Return the fusion coefficient N_{ij}^k.

These are the structure coefficients of the fusion ring, so

$$i * j = \sum_k N_{ij}^k k.$$

EXAMPLES:

```
sage: A22 = FusionRing("A2", 2)
sage: b = A22.basis().list()
sage: all(x*y == sum(A22.Nk_ij(x, y, k)*k for k in b) for x in b for y in b)
True
```

conj_matrix()

Return the conjugation matrix, which is the permutation matrix for the conjugation (dual) operation on basis elements.

EXAMPLES:

```
sage: FusionRing("A2", 1).conj_matrix()
[1 0 0]
[0 0 1]
[0 1 0]
```

field()

Return a cyclotomic field large enough to contain the 2ℓ-th roots of unity, as well as all the S-matrix entries.

EXAMPLES:

```
sage: FusionRing("A2", 2).field()
Cyclotomic Field of order 60 and degree 16
sage: FusionRing("B2", 2).field()
Cyclotomic Field of order 40 and degree 16
```

fusion_l()

Return the product $\ell = m_g(k + h^\vee)$, where m_g denotes the square of the ratio of the lengths of long to short roots of the underlying Lie algebra, k denotes the level of the FusionRing, and $h^\vee$ denotes the dual Coxeter number of the underlying Lie algebra.

This value is used to define the associated root 2ℓ-th of unity $q = e^{i\pi/\ell}$.

EXAMPLES:

```
sage: B22 = FusionRing('B2', 2)
sage: B22.fusion_l()
10
sage: D52 = FusionRing('D5', 2)
sage: D52.fusion_l()
10
```

fusion_labels(*labels=None*, *inject_variables=False*)

Set the labels of the basis.

INPUT:

- `labels` – (default: `None`) a list of strings or string
- `inject_variables` – (default: `False`) if `True`, then inject the variable names into the global namespace; note that this could override objects already defined

If `labels` is a list, the length of the list must equal the number of basis elements. These become the names of the basis elements.

If `labels` is a string, this is treated as a prefix and a list of names is generated.

If `labels` is `None`, then this resets the labels to the default.

EXAMPLES:

```
sage: A13 = FusionRing("A1", 3)
sage: A13.fusion_labels("x")
sage: fb = list(A13.basis()); fb
[x0, x1, x2, x3]
sage: Matrix([[x*y for y in A13.basis()] for x in A13.basis()])
[     x0      x1      x2      x3]
[     x1 x0 + x2 x1 + x3      x2]
[     x2 x1 + x3 x0 + x2      x1]
[     x3      x2      x1      x0]
```

We give an example where the variables are injected into the global namespace:

```
sage: A13.fusion_labels("y", inject_variables=True)
sage: y0
y0
sage: y0.parent() is A13
True
```

We reset the labels to the default:

```
sage: A13.fusion_labels()
sage: fb
[A13(0), A13(1), A13(2), A13(3)]
sage: y0
A13(0)
```

fusion_level()

Return the level k of `self`.

EXAMPLES:

```
sage: B22 = FusionRing('B2', 2)
sage: B22.fusion_level()
2
```

fvars_field()

Return a field containing the `CyclotomicField` computed by *field()* as well as all the F-symbols of the associated `FMatrix` factory object.

This method is only available if `self` is multiplicity-free.

OUTPUT:

Depending on the `CartanType` associated to `self` and whether a call to an F-matrix solver has been made, this method will return the same field as *field()*, a `NumberField()`, or the `QQbar`. See `FMatrix.attempt_number_field_computation()` for more details.

Before running an F-matrix solver, the output of this method matches that of *field()*. However, the output may change upon successfully computing F-symbols. Requesting braid generators triggers a call to `FMatrix.find_orthogonal_solution()`, so the output of this method may change after such a computation.

By default, the output of methods like *r_matrix()*, *s_matrix()*, *twists_matrix()*, etc. will lie in the `fvars_field`, unless the `base_coercion` option is set to `False`.

This method does not trigger a solver run.

EXAMPLES:

```
sage: A13 = FusionRing("A1", 3, fusion_labels="a", inject_variables=True)
sage: A13.fvars_field()
Cyclotomic Field of order 40 and degree 16
sage: A13.field()
Cyclotomic Field of order 40 and degree 16
sage: a2**4
2*a0 + 3*a2
sage: comp_basis, sig = A13.get_braid_generators(a2, a2, 3, verbose=False)    #␣
↪long time (<3s)
sage: A13.fvars_field()                                                       #␣
```

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```
↪long time
Number Field in a with defining polynomial y^32 - ... - 500*y^2 + 25
sage: a2.q_dimension().parent()                                          #␣
↪long time
Number Field in a with defining polynomial y^32 - ... - 500*y^2 + 25
sage: A13.field()
Cyclotomic Field of order 40 and degree 16
```

In some cases, the `NumberField.optimized_representation()` may be used to obtain a better defining polynomial for the computed `NumberField()`.

gens_satisfy_braid_gp_rels(*sig*)

Return `True` if the matrices in the list `sig` satisfy the braid relations.

This if n is the cardinality of `sig`, this confirms that these matrices define a representation of the Artin braid group on $n + 1$ strands. Tests correctness of *get_braid_generators()*.

EXAMPLES:

```
sage: F41 = FusionRing("F4", 1, fusion_labels="f", inject_variables=True)
sage: f1*f1
f0 + f1
sage: comp, sig = F41.get_braid_generators(f1, f0, 4, verbose=False)
sage: F41.gens_satisfy_braid_gp_rels(sig)
True
```

get_braid_generators(*fusing_anyon*, *total_charge_anyon*, *n_strands*, *checkpoint=False*, *save_results=''*, *warm_start=''*, *use_mp=True*, *verbose=True*)

Compute generators of the Artin braid group on `n_strands` strands.

If $a =$ "'$fusing_anyon$' and $b =$ "'$total_charge_anyon$' the generators are endomorphisms of $\mathrm{Hom}(b, a^n)$.

INPUT:

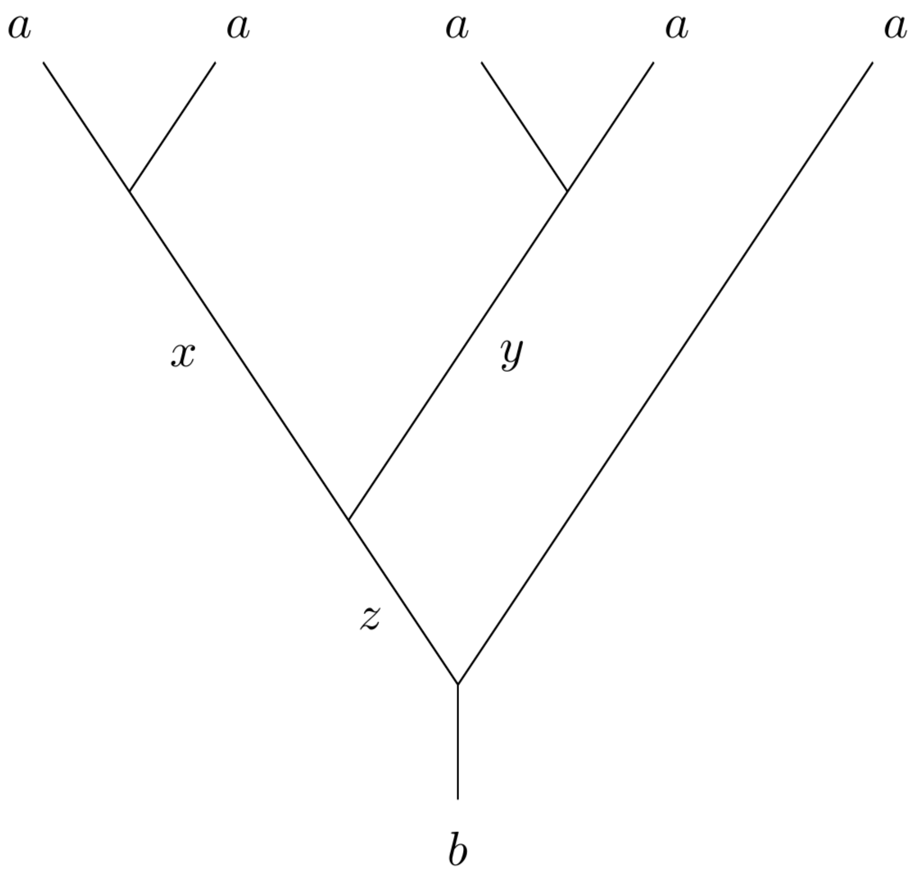
- `fusing_anyon` – a basis element of `self`
- `total_charge_anyon` – a basis element of `self`
- `n_strands` – a positive integer greater than 2
- `checkpoint` – (default: `False`) a boolean indicating whether the F-matrix solver should pickle checkpoints
- `save_results` – (optional) a string indicating the name of a file in which to pickle computed F-symbols for later use
- `warm_start` – (optional) a string indicating the name of a pickled checkpoint file to "warm" start the F-matrix solver. The pickle may be a checkpoint generated by the solver, or a file containing solver results. If all F-symbols are known, we don't run the solver again.
- `use_mp` – (default: `True`) a boolean indicating whether to use multiprocessing to speed up the computation; this is highly recommended. Python 3.8+ is required.
- `verbose` – (default: `True`) boolean indicating whether to be verbose with the computation

For more information on the optional parameters, see `FMatrix.find_orthogonal_solution()`.

Given a simple object in the fusion category, here called `fusing_anyon` allowing the universal R-matrix to act on adjacent pairs in the fusion of `n_strands` copies of `fusing_anyon` produces an action of the braid group. This representation can be decomposed over another anyon, here called `total_charge_anyon`. See [CHW2015].

OUTPUT:

The method outputs a pair of data `(comp_basis, sig)` where `comp_basis` is a list of basis elements of the braid group module, parametrized by a list of fusion ring elements describing a fusion tree. For example with 5 strands the fusion tree is as follows. See `get_computational_basis()` for more information.



`sig` is a list of braid group generators as matrices. In some cases these will be represented as sparse matrices.

In the following example we compute a 5-dimensional braid group representation on 5 strands associated to the spin representation in the modular tensor category $SU(2)_4 \cong SO(3)_2$.

EXAMPLES:

```
sage: A14 = FusionRing("A1", 4)
sage: A14.get_order()
[(0, 0), (1/2, -1/2), (1, -1), (3/2, -3/2), (2, -2)]
sage: A14.fusion_labels(["one", "two", "three", "four", "five"], inject_
↪variables=True)
sage: [A14(x) for x in A14.get_order()]
[one, two, three, four, five]
sage: two ** 5
5*two + 4*four
```

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```
sage: comp_basis, sig = A14.get_braid_generators(two, two, 5, verbose=False) #␣
↪long time
sage: A14.gens_satisfy_braid_gp_rels(sig)                                    #␣
↪long time
True
sage: len(comp_basis) == 5                                                   #␣
↪long time
True
```

get_computational_basis(*a*, *b*, *n_strands*)

Return the so-called computational basis for $\mathrm{Hom}(b, a^n)$.

INPUT:

- a – a basis element
- b – another basis element
- n_strands – the number of strands for a braid group

Let $n =$ n_strands and let k be the greatest integer $\leq n/2$. The braid group acts on $\mathrm{Hom}(b, a^n)$. This action is computed in *get_braid_generators()*. This method returns the computational basis in the form of a list of fusion trees. Each tree is represented by an $(n-2)$-tuple

$$(m_1, \ldots, m_k, l_1, \ldots, l_{k-2})$$

such that each m_j is an irreducible constituent in $a \otimes a$ and

$$\begin{aligned} &b \in l_{k-2} \otimes m_k, \\ &l_{k-2} \in l_{k-3} \otimes m_{k-1}, \\ &\cdots, \\ &l_2 \in l_1 \otimes m_3, \\ &l_1 \in m_1 \otimes m_2, \end{aligned}$$

where $z \in x \otimes y$ means $N_{xy}^z \neq 0$.

As a computational device when n_strands is odd, we pad the vector $(m_1, \ldots, m_k)$ with an additional m_{k+1} equal to a. However, this m_{k+1} does *not* appear in the output of this method.

The following example appears in Section 3.1 of [CW2015].

EXAMPLES:

```
sage: A14 = FusionRing("A1", 4)
sage: A14.get_order()
[(0, 0), (1/2, -1/2), (1, -1), (3/2, -3/2), (2, -2)]
sage: A14.fusion_labels(["zero", "one", "two", "three", "four"], inject_
↪variables=True)
sage: [A14(x) for x in A14.get_order()]
[zero, one, two, three, four]
sage: A14.get_computational_basis(one, two, 4)
[(two, two), (two, zero), (zero, two)]
```

get_fmatrix(**args*, ***kwargs*)

Construct an FMatrix factory to solve the pentagon relations and organize the resulting F-symbols.

EXAMPLES:

```
sage: A15 = FusionRing("A1", 5)
sage: A15.get_fmatrix()
F-Matrix factory for The Fusion Ring of Type A1 and level 5 with Integer Ring␣
→coefficients
```

get_order()

Return the weights of the basis vectors in a fixed order.

You may change the order of the basis using `CombinatorialFreeModule.set_order()`

EXAMPLES:

```
sage: A15 = FusionRing("A1", 5)
sage: w = A15.get_order(); w
[(0, 0), (1/2, -1/2), (1, -1), (3/2, -3/2), (2, -2), (5/2, -5/2)]
sage: A15.set_order([w[k] for k in [0, 4, 1, 3, 5, 2]])
sage: [A15(x) for x in A15.get_order()]
[A15(0), A15(4), A15(1), A15(3), A15(5), A15(2)]
```

> **Warning:** This duplicates *get_order()* from `CombinatorialFreeModule` except the result is *not* cached. Caching of `CombinatorialFreeModule.get_order()` causes inconsistent results after calling `CombinatorialFreeModule.set_order()`.

global_q_dimension(*base_coercion=True*)

Return $\sum d_i^2$, where the sum is over all simple objects and d_i is the quantum dimension.

The global q-dimension is a positive real number.

EXAMPLES:

```
sage: FusionRing("E6", 1).global_q_dimension()
3
```

is_multiplicity_free()

Return `True` if the fusion multiplicities *Nk_ij()* are bounded by 1.

The `FMatrix` is available only for multiplicity free instances of *FusionRing*.

EXAMPLES:

```
sage: [FusionRing(ct, k).is_multiplicity_free() for ct in ("A1", "A2", "B2", "C3
→") for k in (1, 2, 3)]
[True, True, True, True, True, False, True, True, False, True, False, False]
```

r_matrix(*i, j, k, base_coercion=True*)

Return the R-matrix entry corresponding to the subobject `k` in the tensor product of `i` with `j`.

> **Warning:** This method only gives complete information when $N_{ij}^k = 1$ (an important special case). Tables of MTC including R-matrices may be found in Section 5.3 of [RoStWa2009] and in [Bond2007].

The R-matrix is a homomorphism $i \otimes j \rightarrow j \otimes i$. This may be hard to describe since the object $i \otimes j$ may be reducible. However if k is a simple subobject of $i \otimes j$ it is also a subobject of $j \otimes i$. If we fix embeddings $k \rightarrow i \otimes j$, $k \rightarrow j \otimes i$ we may ask for the scalar automorphism of k induced by the R-matrix. This method computes that scalar. It is possible to adjust the set of embeddings $k \to i \otimes j$ (called a *gauge*) so that this scalar equals

$$\pm\sqrt{\frac{\theta_k}{\theta_i\theta_j}}.$$

If $i \neq j$, the gauge may be used to control the sign of the square root. But if $i = j$ then we must be careful about the sign. These cases are computed by a formula of [BDGRTW2019], Proposition 2.3.

EXAMPLES:

```
sage: I = FusionRing("E8", 2, conjugate=True)  # Ising MTC
sage: I.fusion_labels(["i0", "p", "s"], inject_variables=True)
sage: I.r_matrix(s, s, i0) == I.root_of_unity(-1/8)
True
sage: I.r_matrix(p, p, i0)
-1
sage: I.r_matrix(p, s, s) == I.root_of_unity(-1/2)
True
sage: I.r_matrix(s, p, s) == I.root_of_unity(-1/2)
True
sage: I.r_matrix(s, s, p) == I.root_of_unity(3/8)
True
```

root_of_unity(*r*, *base_coercion=True*)

Return $e^{i\pi r}$ as an element of `self.field()` if possible.

INPUT:

- `r` – a rational number

EXAMPLES:

```
sage: A11 = FusionRing("A1", 1)
sage: A11.field()
Cyclotomic Field of order 24 and degree 8
sage: for n in [1..7]:
....:     try:
....:         print(n, A11.root_of_unity(2/n))
....:     except ValueError as err:
....:         print(n, err)
1 1
2 -1
3 zeta24^4 - 1
4 zeta24^6
5 not a root of unity in the field
6 zeta24^4
7 not a root of unity in the field
```

s_ij(*elt_i*, *elt_j*, *base_coercion=True*)

Return the element of the S-matrix of this fusion ring corresponding to the given elements.

This is the unnormalized S-matrix, denoted $\tilde{s}_{ij}$ in [BaKi2001] . To obtain the normalized S-matrix, divide by *global_q_dimension()* or use `S_matrix()` with the option `unitary=True`.

This is computed using the formula

$$s_{i,j} = \frac{1}{\theta_i \theta_j} \sum_k N_{ik}^j d_k \theta_k,$$

where θ_k is the twist and d_k is the quantum dimension. See [Row2006] Equation (2.2) or [EGNO2015] Proposition 8.13.8.

INPUT:

- elt_i, elt_j – elements of the fusion basis

EXAMPLES:

```
sage: G21 = FusionRing("G2", 1)
sage: b = G21.basis()
sage: [G21.s_ij(x, y) for x in b for y in b]
[1, -zeta60^14 + zeta60^6 + zeta60^4, -zeta60^14 + zeta60^6 + zeta60^4, -1]
```

s_ijconj(*elt_i*, *elt_j*, *base_coercion=True*)

Return the conjugate of the element of the S-matrix given by `self.s_ij(elt_i, elt_j, base_coercion=base_coercion)`.

See *s_ij()*.

EXAMPLES:

```
sage: G21 = FusionRing("G2", 1)
sage: b = G21.basis()
sage: [G21.s_ijconj(x, y) for x in b for y in b]
[1, -zeta60^14 + zeta60^6 + zeta60^4, -zeta60^14 + zeta60^6 + zeta60^4, -1]
```

This method works with all possible types of fields returned by `self.fmats.field()`.

s_matrix(*unitary=False*, *base_coercion=True*)

Return the S-matrix of this fusion ring.

OPTIONAL:

- `unitary` – (default: `False`) set to `True` to obtain the unitary S-matrix

Without the `unitary` parameter, this is the matrix denoted $\widetilde{s}$ in [BaKi2001].

EXAMPLES:

```
sage: D91 = FusionRing("D9", 1)
sage: D91.s_matrix()
[          1           1           1           1]
[          1           1          -1          -1]
[          1          -1 -zeta136^34  zeta136^34]
[          1          -1  zeta136^34 -zeta136^34]
sage: S = D91.s_matrix(unitary=True); S
[            1/2             1/2             1/2             1/2]
[            1/2             1/2            -1/2            -1/2]
[            1/2            -1/2 -1/2*zeta136^34  1/2*zeta136^34]
[            1/2            -1/2  1/2*zeta136^34 -1/2*zeta136^34]
sage: S*S.conjugate()
[1 0 0 0]
[0 1 0 0]
```

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```
[0 0 1 0]
[0 0 0 1]
```

some_elements()

Return some elements of self.

EXAMPLES:

```
sage: D41 = FusionRing('D4', 1)
sage: D41.some_elements()
[D41(1,0,0,0), D41(0,0,1,0), D41(0,0,0,1)]
```

test_braid_representation(*max_strands=6*, *anyon=None*)

Check that we can compute valid braid group representations.

INPUT:

- max_strands – (default: 6): maximum number of braid group strands
- anyon – (optional) run this test on this particular simple object

Create a braid group representation using *get_braid_generators()* and confirms the braid relations. This test indirectly partially verifies the correctness of the orthogonal F-matrix solver. If the code were incorrect the method would not be deterministic because the fusing anyon is chosen randomly. (A different choice is made for each number of strands tested.) However the doctest is deterministic since it will always return True. If the anyon parameter is omitted, a random anyon is tested for each number of strands up to max_strands.

EXAMPLES:

```
sage: A21 = FusionRing("A2", 1)
sage: A21.test_braid_representation(max_strands=4)
True
sage: F41 = FusionRing("F4", 1)             # long time
sage: F41.test_braid_representation()       # long time
True
```

total_q_order(*base_coercion=True*)

Return the positive square root of *self.global_q_dimension()* as an element of *self.field()*.

This is implemented as $D_{+}e^{-i\pi c/4}$, where D_+ is *D_plus()* and c is *virasoro_central_charge()*.

EXAMPLES:

```
sage: F = FusionRing("G2", 1)
sage: tqo=F.total_q_order(); tqo
zeta60^15 - zeta60^11 - zeta60^9 + 2*zeta60^3 + zeta60
sage: tqo.is_real_positive()
True
sage: tqo^2 == F.global_q_dimension()
True
```

twists_matrix()

Return a diagonal matrix describing the twist corresponding to each simple object in the FusionRing.

EXAMPLES:

```
sage: B21=FusionRing("B2", 1)
sage: [x.twist() for x in B21.basis().list()]
[0, 1, 5/8]
sage: [B21.root_of_unity(x.twist()) for x in B21.basis().list()]
[1, -1, zeta32^10]
sage: B21.twists_matrix()
[        1         0         0]
[        0        -1         0]
[        0         0 zeta32^10]
```

virasoro_central_charge()

Return the Virasoro central charge of the WZW conformal field theory associated with the Fusion Ring.

If $\mathfrak{g}$ is the corresponding semisimple Lie algebra, this is

$$\frac{k \dim \mathfrak{g}}{k + h^\vee},$$

where k is the level and $h^\vee$ is the dual Coxeter number. See [DFMS1996] Equation (15.61).

Let d_i and θ_i be the quantum dimensions and twists of the simple objects. By Proposition 2.3 in [RoStWa2009], there exists a rational number c such that $D_+/\sqrt{D} = e^{i\pi c/4}$, where $D_+ = \sum d_i^2 \theta_i$ is computed in *D_plus()* and $D = \sum d_i^2 > 0$ is computed by *global_q_dimension()*. Squaring this identity and remembering that $D_+ D_- = D$ gives

$$D_+/D_- = e^{i\pi c/2}.$$

EXAMPLES:

```
sage: R = FusionRing("A1", 2)
sage: c = R.virasoro_central_charge(); c
3/2
sage: Dp = R.D_plus(); Dp
2*zeta32^6
sage: Dm = R.D_minus(); Dm
-2*zeta32^10
sage: Dp / Dm == R.root_of_unity(c/2)
True
```

5.8.2 The F-Matrix of a Fusion Ring

class sage.algebras.fusion_rings.f_matrix.**FMatrix**(*fusion_ring*, *fusion_label='f'*, *var_prefix='fx'*, *inject_variables=False*)

Bases: SageObject

An F-matrix for a *FusionRing*.

INPUT:

- FR – a *FusionRing*
- fusion_label – (optional) a string used to label basis elements of the *FusionRing* associated to self (see *FusionRing.fusion_labels()*)
- var_prefix – (optional) a string indicating the desired prefix for variables denoting F-symbols to be solved

- `inject_variables` – (default: `False`) a boolean indicating whether to inject variables (*FusionRing* basis element labels and F-symbols) into the global namespace

The *FusionRing* or Verlinde algebra is the Grothendieck ring of a modular tensor category [BaKi2001]. Such categories arise in conformal field theory or in the representation theories of affine Lie algebras, or quantum groups at roots of unity. They have applications to low dimensional topology and knot theory, to conformal field theory and to topological quantum computing. The *FusionRing* captures much information about a fusion category, but to complete the picture, the F-matrices or 6j-symbols are needed. For example these are required in order to construct braid group representations. This can be done using the *FusionRing* method *FusionRing.get_braid_generators()*, which uses the F-matrix.

We only undertake to compute the F-matrix if the *FusionRing* is *multiplicity free* meaning that the Fusion coefficients N^{ij}_k are bounded by 1. For Cartan Types X_r and level k, the multiplicity-free cases are given by the following table.

Cartan Type	k
A_1	any
$A_r, r \geq 2$	≤ 2
$B_r, r \geq 2$	≤ 2
C_2	≤ 2
$C_r, r \geq 3$	≤ 1
$D_r, r \geq 4$	≤ 2
G_2, F_4, E_6, E_7	≤ 2
E_8	≤ 3

Beyond this limitation, computation of the F-matrix can involve very large systems of equations. A rule of thumb is that this code can compute the F-matrix for systems with ≤ 14 simple objects (primary fields) on a machine with 16 GB of memory. (Larger examples can be quite time consuming.)

The *FusionRing* and its methods capture much of the structure of the underlying tensor category. But an important aspect that is not encoded in the fusion ring is the associator, which is a homomorphism $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ that requires an additional tool, the F-matrix or 6j-symbol. To specify this, we fix a simple object D and represent the transformation

$$\text{Hom}(D, (A \otimes B) \otimes C) \to \text{Hom}(D, A \otimes (B \otimes C))$$

by a matrix F^{ABC}_D. This depends on a pair of additional simple objects X and Y. Indeed, we can get a basis for $\text{Hom}(D, (A \otimes B) \otimes C)$ indexed by simple objects X in which the corresponding homomorphism factors through $X \otimes C$, and similarly $\text{Hom}(D, A \otimes (B \otimes C))$ has a basis indexed by Y, in which the basis vector factors through $A \otimes Y$.

See [TTWL2009] for an introduction to this topic, [EGNO2015] Section 4.9 for a precise mathematical definition, and [Bond2007] Section 2.5 and [Ab2022] for discussions of how to compute the F-matrix. In addition to [Bond2007], worked out F-matrices may be found in [RoStWa2009] and [CHW2015].

The F-matrix is only determined up to a *gauge*. This is a family of embeddings $C \to A \otimes B$ for simple objects A, B, C such that $\text{Hom}(C, A \otimes B)$ is nonzero. Changing the gauge changes the F-matrix though not in a very essential way. By varying the gauge it is possible to make the F-matrices unitary, or it is possible to make them cyclotomic.

Due to the large number of equations we may fail to find a Groebner basis if there are too many variables.

EXAMPLES:

```
sage: I = FusionRing("E8", 2, conjugate=True)
sage: I.fusion_labels(["i0", "p", "s"], inject_variables=True)
```

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```
sage: f = I.get_fmatrix(inject_variables=True); f
creating variables fx1..fx14
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7, fx8, fx9, fx10, fx11, fx12, fx13
F-Matrix factory for The Fusion Ring of Type E8 and level 2 with Integer Ring␣
↪coefficients
```

We have injected two sets of variables to the global namespace. We created three variables `i0, p, s` to represent the primary fields (simple elements) of the *FusionRing*. Creating the *FMatrix* factory also created variables `fx1, fx2, ..., fx14` in order to solve the hexagon and pentagon equations describing the F-matrix. Since we called *FMatrix* with the parameter `inject_variables=True`, these have been injected into the global namespace. This is not necessary for the code to work but if you want to run the code experimentally you may want access to these variables.

EXAMPLES:

```
sage: f.fmatrix(s, s, s, s)
[fx10 fx11]
[fx12 fx13]
```

The F-matrix has not been computed at this stage, so the F-matrix F^{sss}_s is filled with variables `fx10`, `fx11`, `fx12`, `fx13`. The task is to solve for these.

As explained above The F-matrix $(F^{ABC}_D)_{X,Y}$ two other variables X and Y. We have methods to tell us (depending on A, B, C, D) what the possibilities for these are. In this example with $A = B = C = D = s$ both X and Y are allowed to be i_0 or s.

```
sage: f.f_from(s, s, s, s), f.f_to(s, s, s, s)
([i0, p], [i0, p])
```

The last two statments show that the possible values of X and Y when $A = B = C = D = s$ are i_0 and p.

The F-matrix is computed by solving the so-called pentagon and hexagon equations. The *pentagon equations* reflect the Mac Lane pentagon axiom in the definition of a monoidal category. The hexagon relations reflect the axioms of a *braided monoidal category*, which are constraints on both the F-matrix and on the R-matrix. Optionally, orthogonality constraints may be imposed to obtain an orthogonal F-matrix.

```
sage: sorted(f.get_defining_equations("pentagons"))[1:3]
[fx9*fx12 - fx2*fx13, fx4*fx11 - fx2*fx13]
sage: sorted(f.get_defining_equations("hexagons"))[1:3]
[fx6 - 1, fx2 + 1]
sage: sorted(f.get_orthogonality_constraints())[1:3]
[fx10*fx11 + fx12*fx13, fx10*fx11 + fx12*fx13]
```

There are two methods available to compute an F-matrix. The first, *find_cyclotomic_solution()* uses only the pentagon and hexagon relations. The second, *find_orthogonal_solution()* uses additionally the orthogonality relations. There are some differences that should be kept in mind.

find_cyclotomic_solution() currently works only with smaller examples. For example the *FusionRing* for G_2 at level 2 is too large. When it is available, this method produces an F-matrix whose entries are in the same cyclotomic field as the underlying *FusionRing*.

```
sage: f.find_cyclotomic_solution()
Setting up hexagons and pentagons...
Finding a Groebner basis...
Solving...
```

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```
Fixing the gauge...
adding equation... fx1 - 1
adding equation... fx11 - 1
Done!
```

We now have access to the values of the F-matrix using the methods *fmatrix()* and *fmat()*:

```
sage: f.fmatrix(s, s, s, s)
[(-1/2*zeta128^48 + 1/2*zeta128^16)                                  1]
[                               1/2   (1/2*zeta128^48 - 1/2*zeta128^16)]
sage: f.fmat(s, s, s, s, p, p)
(1/2*zeta128^48 - 1/2*zeta128^16)
```

find_orthogonal_solution() is much more powerful and is capable of handling large cases, sometimes quickly but sometimes (in larger cases) after hours of computation. Its F-matrices are not always in the cyclotomic field that is the base ring of the underlying *FusionRing*, but sometimes in an extension field adjoining some square roots. When this happens, the *FusionRing* is modified, adding an attribute `_basecoer` that is a coercion from the cyclotomic field to the field containing the F-matrix. The field containing the F-matrix is available through *field()*.

```
sage: f = FusionRing("B3", 2).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False, checkpoint=True)     # not tested (~
↪100 s)
sage: all(v in CyclotomicField(56) for v in f.get_fvars().values())  # not tested
True

sage: f = FusionRing("G2", 2).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False) # long time (~11 s)
sage: f.field()                                 # long time
Algebraic Field
```

FR()

Return the *FusionRing* associated to `self`.

EXAMPLES:

```
sage: f = FusionRing("D3", 1).get_fmatrix()
sage: f.FR()
The Fusion Ring of Type D3 and level 1 with Integer Ring coefficients
```

attempt_number_field_computation()

Based on the `CartanType` of `self` and data known on March 17, 2021, determine whether to attempt to find a `NumberField()` containing all the F-symbols.

This method is used by *find_orthogonal_solution()* to determine a field containing all F-symbols. See *field()* and *get_non_cyclotomic_roots()*.

For certain *fusion rings*, the number field computation does not terminate in reasonable time. In these cases, we report F-symbols as elements of the `QQbar`.

EXAMPLES:

```
sage: f = FusionRing("F4", 2).get_fmatrix()
sage: f.attempt_number_field_computation()
```

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```
False
sage: f = FusionRing("G2", 1).get_fmatrix()
sage: f.attempt_number_field_computation()
True
```

Note: In certain cases, F-symbols are found in the associated *FusionRing*'s cyclotomic field and a `NumberField()` computation is not needed. In these cases this method returns `True` but the *find_orthogonal_solution()* solver does *not* undertake a `NumberField()` computation.

certify_pentagons(*use_mp=True*, *verbose=False*)

Obtain a certificate of satisfaction for the pentagon equations, up to floating-point error.

This method converts the computed F-symbols (available through *get_fvars()*) to native Python floats and then checks whether the pentagon equations are satisfied using floating point arithmetic.

When `self.FR().basis()` has many elements, verifying satisfaction of the pentagon relations exactly using *get_defining_equations()* with `option="pentagons"` may take a long time. This method is faster, but it cannot provide mathematical guarantees.

EXAMPLES:

```
sage: f = FusionRing("C3", 1).get_fmatrix()
sage: f.find_orthogonal_solution()        # long time
Computing F-symbols for The Fusion Ring of Type C3 and level 1 with Integer␣
↪Ring coefficients with 71 variables...
Set up 134 hex and orthogonality constraints...
Partitioned 134 equations into 17 components of size:
[12, 12, 6, 6, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1]
Elimination epoch completed... 10 eqns remain in ideal basis
Elimination epoch completed... 0 eqns remain in ideal basis
Hex elim step solved for 51 / 71 variables
Set up 121 reduced pentagons...
Elimination epoch completed... 18 eqns remain in ideal basis
Elimination epoch completed... 5 eqns remain in ideal basis
Pent elim step solved for 64 / 71 variables
Partitioned 5 equations into 1 components of size:
[4]
Elimination epoch completed... 0 eqns remain in ideal basis
Partitioned 6 equations into 6 components of size:
[1, 1, 1, 1, 1, 1]
Computing appropriate NumberField...
sage: f.certify_pentagons()  is None      # not tested (long time ~1.5s, cypari␣
↪issue in doctesting framework)
True
```

clear_equations()

Clear the list of equations to be solved.

EXAMPLES:

```
sage: f = FusionRing("E6", 1).get_fmatrix()
sage: f.get_defining_equations('hexagons', output=False)
```

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```
sage: len(f.ideal_basis)
6
sage: f.clear_equations()
sage: len(f.ideal_basis) == 0
True
```

clear_vars()

Reset the F-symbols.

EXAMPLES:

```
sage: f = FusionRing("C4", 1).get_fmatrix()
sage: fvars = f.get_fvars()
sage: some_key = sorted(fvars)[0]
sage: fvars[some_key]
fx0
sage: fvars[some_key] = 1
sage: f.get_fvars()[some_key]
1
sage: f.clear_vars()
sage: f.get_fvars()[some_key]
fx0
```

equations_graph(*eqns=None*)

Construct a graph corresponding to the given equations.

Every node corresponds to a variable and nodes are connected when the corresponding variables appear together in an equation.

INPUT:

- `eqns` – a list of polynomials

Each polynomial is either an object in the ring returned by *get_poly_ring()* or it is a tuple of pairs representing a polynomial using the internal representation.

If no list of equations is passed, the graph is built from the polynomials in `self.ideal_basis`. In this case the method assumes the internal representation of a polynomial as a tuple of pairs is used.

This method is crucial to *find_orthogonal_solution()*. The hexagon equations, obtained using *get_defining_equations()*, define a disconnected graph that breaks up into many small components. The *find_orthogonal_solution()* solver exploits this when undertaking a Groebner basis computation.

OUTPUT:

A `Graph` object. If a list of polynomial objects was given, the set of nodes in the output graph is the subset polynomial ring generators appearing in the equations.

If the internal representation was used, the set of nodes is the subset of indices corresponding to polynomial ring generators. This option is meant for internal use by the F-matrix solver.

EXAMPLES:

```
sage: f = FusionRing("A3", 1).get_fmatrix()
sage: f.get_poly_ring().ngens()
27
```

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```
sage: he = f.get_defining_equations('hexagons')
sage: graph = f.equations_graph(he)
sage: graph.connected_components_sizes()
[6, 3, 3, 3, 3, 3, 3, 1, 1, 1]
```

f_from(a, b, c, d)

Return the possible x such that there are morphisms $d \to x \otimes c \to (a \otimes b) \otimes c$.

INPUT:

- `a, b, c, d` – basis elements of the associated *FusionRing*

EXAMPLES:

```
sage: fr = FusionRing("A1", 3, fusion_labels="a", inject_variables=True)
sage: f = fr.get_fmatrix()
sage: f.fmatrix(a1, a1, a2, a2)
[fx6 fx7]
[fx8 fx9]
sage: f.f_from(a1, a1, a2, a2)
[a0, a2]
sage: f.f_to(a1, a1, a2, a2)
[a1, a3]
```

f_to(a, b, c, d)

Return the possible y such that there are morphisms $d \to a \otimes y \to a \otimes (b \otimes c)$.

INPUT:

- `a, b, c, d` – basis elements of the associated *FusionRing*

EXAMPLES:

```
sage: b22 = FusionRing("B2", 2)
sage: b22.fusion_labels("b", inject_variables=True)
sage: B = b22.get_fmatrix()
sage: B.fmatrix(b2, b4, b2, b4)
[fx266 fx267 fx268]
[fx269 fx270 fx271]
[fx272 fx273 fx274]
sage: B.f_from(b2, b4, b2, b4)
[b1, b3, b5]
sage: B.f_to(b2, b4, b2, b4)
[b1, b3, b5]
```

field()

Return the base field containing the F-symbols.

When `self` is initialized, the field is set to be the cyclotomic field of the *FusionRing* associated to `self`.

The field may change after running *find_orthogonal_solution()*. At that point, this method could return the associated *FusionRing*'s cyclotomic field, an appropriate `NumberField()` that was computed on the fly by the F-matrix solver, or the `QQbar`.

Depending on the `CartanType` of `self`, the solver may need to compute an extension field containing certain square roots that do not belong to the associated *FusionRing*'s cyclotomic field.

In certain cases we revert to QQbar because the extension field computation does not seem to terminate. See *attempt_number_field_computation()* for more details.

The method *get_non_cyclotomic_roots()* returns a list of roots defining the extension of the *FusionRing*'s cyclotomic field needed to contain all F-symbols.

EXAMPLES:

```
sage: f = FusionRing("G2", 1).get_fmatrix()
sage: f.field()
Cyclotomic Field of order 60 and degree 16
sage: f.find_orthogonal_solution(verbose=False)
sage: f.field()
Number Field in a with defining polynomial y^32 - ... - 22*y^2 + 1
sage: phi = f.get_qqbar_embedding()
sage: [phi(r).n() for r in f.get_non_cyclotomic_roots()]
[-0.786151377757423 - 8.92806368517581e-31*I]
```

Note: Consider using `self.field().optimized_representation()` to obtain an equivalent `NumberField()` with a defining polynomial with smaller coefficients, for a more efficient element representation.

find_cyclotomic_solution(*equations=None*, *algorithm=''*, *verbose=True*, *output=False*)

Solve the hexagon and pentagon relations to evaluate the F-matrix.

This method (omitting the orthogonality constraints) produces output in the cyclotomic field, but it is very limited in the size of examples it can handle: for example, G_2 at level 2 is too large for this method. You may use *find_orthogonal_solution()* to solve much larger examples.

INPUT:

- `equations` – (optional) a set of equations to be solved; defaults to the hexagon and pentagon equations
- `algorithm` – (optional) algorithm to compute Groebner Basis
- `output` – (default: `False`) output a dictionary of F-matrix values; this may be useful to see but may be omitted since this information will be available afterwards via the *fmatrix()* and *fmat()* methods.

EXAMPLES:

```
sage: fr = FusionRing("A2", 1, fusion_labels="a", inject_variables=True)
sage: f = fr.get_fmatrix(inject_variables=True)
creating variables fx1..fx8
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7
sage: f.find_cyclotomic_solution(output=True)
Setting up hexagons and pentagons...
Finding a Groebner basis...
Solving...
Fixing the gauge...
adding equation... fx4 - 1
Done!
{(a2, a2, a2, a0, a1, a1): 1,
 (a2, a2, a1, a2, a1, a0): 1,
 (a2, a1, a2, a2, a0, a0): 1,
 (a2, a1, a1, a1, a0, a2): 1,
 (a1, a2, a2, a2, a0, a1): 1,
```

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```
(a1, a2, a1, a1, a0, a0): 1,
(a1, a1, a2, a1, a2, a0): 1,
(a1, a1, a1, a0, a2, a2): 1}
```

After you successfully run *find_cyclotomic_solution()* you may check the correctness of the F-matrix by running *get_defining_equations()* with `option='hexagons'` and `option='pentagons'`. These should return empty lists of equations.

EXAMPLES:

```
sage: f.get_defining_equations("hexagons")
[]
sage: f.get_defining_equations("pentagons")
[]
```

find_orthogonal_solution(*checkpoint=False*, *save_results=''*, *warm_start=''*, *use_mp=True*, *verbose=True*)

Solve the the hexagon and pentagon relations, along with orthogonality constraints, to evaluate an orthogonal F-matrix.

INPUT:

- `checkpoint` – (default: `False`) a boolean indicating whether the computation should be checkpointed. Depending on the associated `CartanType`, the computation may take hours to complete. For large examples, checkpoints are recommended. This method supports "warm" starting, so the calculation may be resumed from a checkpoint, using the `warm_start` option.

  Checkpoints store necessary state in the pickle file `"fmatrix_solver_checkpoint_" + key + ".pickle"`, where `key` is the result of *get_fr_str()*.

  Checkpoint pickles are automatically deleted when the solver exits a successful run.

- `save_results` – (optional) a string indicating the name of a pickle file in which to store calculated F-symbols for later use.

  If `save_results` is not provided (default), F-matrix results are not stored to file.

  The F-symbols may be saved to file after running the solver using *save_fvars()*.

- `warm_start` – (optional) a string indicating the name of a pickle file containing checkpointed solver state. This file must have been produced by a previous call to the solver using the `checkpoint` option.

  If no file name is provided, the calculation begins from scratch.

- `use_mp` – (default: `True`) a boolean indicating whether to use multiprocessing to speed up calculation. The default value `True` is highly recommended, since parallel processing yields results much more quickly.

- `verbose` – (default: `True`) a boolean indicating whether the solver should print out intermediate progress reports.

OUTPUT:

This method returns `None`. If the solver runs successfully, the results may be accessed through various methods, such as *get_fvars()*, *fmatrix()*, *fmat()*, etc.

EXAMPLES:

```
sage: f = FusionRing("B5", 1).get_fmatrix(fusion_label="b", inject_
↪variables=True)
creating variables fx1..fx14
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7, fx8, fx9, fx10, fx11, fx12,␣
↪fx13
sage: f.find_orthogonal_solution()
Computing F-symbols for The Fusion Ring of Type B5 and level 1 with Integer␣
↪Ring coefficients with 14 variables...
Set up 25 hex and orthogonality constraints...
Partitioned 25 equations into 5 components of size:
[4, 3, 3, 3, 1]
Elimination epoch completed... 0 eqns remain in ideal basis
Hex elim step solved for 10 / 14 variables
Set up 7 reduced pentagons...
Elimination epoch completed... 0 eqns remain in ideal basis
Pent elim step solved for 12 / 14 variables
Partitioned 0 equations into 0 components of size:
[]
Partitioned 2 equations into 2 components of size:
[1, 1]
sage: f.fmatrix(b2, b2, b2, b2)
[ 1/2*zeta80^30 - 1/2*zeta80^10 -1/2*zeta80^30 + 1/2*zeta80^10]
[ 1/2*zeta80^30 - 1/2*zeta80^10  1/2*zeta80^30 - 1/2*zeta80^10]
sage: f.fmat(b2, b2, b2, b2, b0, b1)
-1/2*zeta80^30 + 1/2*zeta80^10
```

Every F-matrix $F^{a,b,c}_d$ is orthogonal and in many cases real. We may use *fmats_are_orthogonal()* and *fvars_are_real()* to obtain correctness certificates.

EXAMPLES:

```
sage: f.fmats_are_orthogonal()
True
```

In any case, the F-symbols are obtained as elements of the associated *FusionRing*'s `Cyclotomic field`, a computed `NumberField()`, or `QQbar`. Currently, the field containing the F-symbols is determined based on the `CartanType` associated to `self`.

See also:

attempt_number_field_computation()

findcases(*output=False*)

Return unknown F-matrix entries.

If run with `output=True`, this returns two dictionaries; otherwise it just returns the number of unknown values.

EXAMPLES:

```
sage: f = FusionRing("G2", 1, fusion_labels=("i0", "t")).get_fmatrix()
sage: f.findcases()
5
sage: f.findcases(output=True)
 ({0: (t, t, t, i0, t, t),
  1: (t, t, t, t, i0, i0),
```

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```
  2: (t, t, t, t, i0, t),
  3: (t, t, t, t, t, i0),
  4: (t, t, t, t, t, t)},
 {(t, t, t, i0, t, t): fx0,
  (t, t, t, t, i0, i0): fx1,
  (t, t, t, t, i0, t): fx2,
  (t, t, t, t, t, i0): fx3,
  (t, t, t, t, t, t): fx4})
```

fmat(*a*, *b*, *c*, *d*, *x*, *y*, *data=True*)

Return the F-Matrix coefficient $(F^{a,b,c}_d)_{x,y}$.

EXAMPLES:

```
sage: fr = FusionRing("G2", 1, fusion_labels=("i0", "t"), inject_variables=True)
sage: f = fr.get_fmatrix()
sage: [f.fmat(t, t, t, t, x, y) for x in fr.basis() for y in fr.basis()]
[fx1, fx2, fx3, fx4]
sage: f.find_cyclotomic_solution(output=True)
Setting up hexagons and pentagons...
Finding a Groebner basis...
Solving...
Fixing the gauge...
adding equation... fx2 - 1
Done!
{(t, t, t, i0, t, t): 1,
 (t, t, t, t, i0, i0): (-zeta60^14 + zeta60^6 + zeta60^4 - 1),
 (t, t, t, t, i0, t): 1,
 (t, t, t, t, t, i0): (-zeta60^14 + zeta60^6 + zeta60^4 - 1),
 (t, t, t, t, t, t): (zeta60^14 - zeta60^6 - zeta60^4 + 1)}
sage: [f.fmat(t, t, t, t, x, y) for x in f._FR.basis() for y in f._FR.basis()]
[(-zeta60^14 + zeta60^6 + zeta60^4 - 1),
 1,
 (-zeta60^14 + zeta60^6 + zeta60^4 - 1),
 (zeta60^14 - zeta60^6 - zeta60^4 + 1)]
```

fmatrix(*a*, *b*, *c*, *d*)

Return the F-Matrix $F^{a,b,c}_d$.

INPUT:

- `a, b, c, d` – basis elements of the associated *FusionRing*

EXAMPLES:

```
sage: fr = FusionRing("A1", 2, fusion_labels="c", inject_variables=True)
sage: f = fr.get_fmatrix(new=True)
sage: f.fmatrix(c1, c1, c1, c1)
[fx0 fx1]
[fx2 fx3]
sage: f.find_cyclotomic_solution(verbose=False);
adding equation... fx4 - 1
adding equation... fx10 - 1
```

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```
sage: f.f_from(c1, c1, c1, c1)
[c0, c2]
sage: f.f_to(c1, c1, c1, c1)
[c0, c2]
sage: f.fmatrix(c1, c1, c1, c1)
[ (1/2*zeta32^12 - 1/2*zeta32^4) (-1/2*zeta32^12 + 1/2*zeta32^4)]
[ (1/2*zeta32^12 - 1/2*zeta32^4)  (1/2*zeta32^12 - 1/2*zeta32^4)]
```

fmats_are_orthogonal()

Verify that all F-matrices are orthogonal.

This method should always return `True` when called after running *find_orthogonal_solution()*.

EXAMPLES:

```
sage: f = FusionRing("D4", 1).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False)
sage: f.fmats_are_orthogonal()
True
```

fvars_are_real()

Test whether all F-symbols are real.

EXAMPLES:

```
sage: f = FusionRing("A1", 3).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False) # long time
sage: f.fvars_are_real()                        # not tested (cypari issue in␣
↪doctesting framework)
True
```

get_coerce_map_from_fr_cyclotomic_field()

Return a coercion map from the associated *FusionRing*'s cyclotomic field into the base field containing all F-symbols (this could be the *FusionRing*'s `Cyclotomic field`, a `NumberField()`, or `QQbar`).

EXAMPLES:

```
sage: f = FusionRing("G2", 1).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False)
sage: f.FR().field()
Cyclotomic Field of order 60 and degree 16
sage: f.field()
Number Field in a with defining polynomial y^32 - ... - 22*y^2 + 1
sage: phi = f.get_coerce_map_from_fr_cyclotomic_field()
sage: phi.domain() == f.FR().field()
True
sage: phi.codomain() == f.field()
True
```

When F-symbols are computed as elements of the associated *FusionRing*'s base `Cyclotomic field`, we have `self.field() == self.FR().field()` and this returns the identity map on `self.field()`.

```
sage: f = FusionRing("A2", 1).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False)
```

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```
sage: phi = f.get_coerce_map_from_fr_cyclotomic_field()
sage: f.field()
Cyclotomic Field of order 48 and degree 16
sage: f.field() == f.FR().field()
True
sage: phi.domain() == f.field()
True
sage: phi.is_identity()
True
```

get_defining_equations(*option*, *output=True*)

Get the equations defining the ideal generated by the hexagon or pentagon relations.

INPUT:

- option – a string determining equations to be set up:
  - 'hexagons' - get equations imposed on the F-matrix by the hexagon relations in the definition of a braided category
  - 'pentagons' - get equations imposed on the F-matrix by the pentagon relations in the definition of a monoidal category
- output – (default: True) a boolean indicating whether results should be returned, where the equations will be polynomials. Otherwise, the constraints are appended to self.ideal_basis. Constraints are stored in the internal tuple representation. The output=False option is meant only for internal use by the F-matrix solver. When computing the hexagon equations with the output=False option, the initial state of the F-symbols is used.

Note: To set up the defining equations using parallel processing, use *start_worker_pool()* to initialize multiple processes *before* calling this method.

EXAMPLES:

```
sage: f = FusionRing("B2", 1).get_fmatrix()
sage: sorted(f.get_defining_equations('hexagons'))
[fx7 + 1,
 fx6 - 1,
 fx2 + 1,
 fx0 - 1,
 fx11*fx12 + (-zeta32^8)*fx13^2 + (zeta32^12)*fx13,
 fx10*fx12 + (-zeta32^8)*fx12*fx13 + (zeta32^4)*fx12,
 fx10*fx11 + (-zeta32^8)*fx11*fx13 + (zeta32^4)*fx11,
 fx10^2 + (-zeta32^8)*fx11*fx12 + (-zeta32^12)*fx10,
 fx4*fx9 + fx7,
 fx3*fx8 - fx6,
 fx1*fx5 + fx2]
sage: pe = f.get_defining_equations('pentagons')
sage: len(pe)
33
```

get_fr_str()

Auto-generate an identifying key for saving results.

EXAMPLES:

```
sage: f = FusionRing("B3", 1).get_fmatrix()
sage: f.get_fr_str()
'B31'
```

get_fvars()

Return a dictionary of F-symbols.

The keys are sextuples (a, b, c, d, x, y) of basis elements of `self.FR()` and the values are the corresponding F-symbols $(F^{a,b,c}_d)_{xy}$.

These values reflect the current state of a solver's computation.

EXAMPLES:

```
sage: f = FusionRing("A2", 1).get_fmatrix(inject_variables=True)
creating variables fx1..fx8
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7
sage: f.get_fvars()[(f1, f1, f1, f0, f2, f2)]
fx0
sage: f.find_orthogonal_solution(verbose=False)
sage: f.get_fvars()[(f1, f1, f1, f0, f2, f2)]
1
```

get_fvars_by_size(*n*, *indices=False*)

Return the set of F-symbols that are entries of an $n \times n$ matrix $F^{a,b,c}_d$.

INPUT:

- n – a positive integer
- `indices` – boolean (default: `False`)

If `indices` is `False` (default), this method returns a set of sextuples (a, b, c, d, x, y) identifying the corresponding F-symbol. Each sextuple is a key in the dictionary returned by *get_fvars()*.

Otherwise the method returns a list of integer indices that internally identify the F-symbols. The `indices=True` option is meant for internal use.

EXAMPLES:

```
sage: f = FusionRing("A2", 2).get_fmatrix(inject_variables=True)
creating variables fx1..fx287
Defining fx0, ..., fx286
sage: f.largest_fmat_size()
2
sage: f.get_fvars_by_size(2)
{(f2, f2, f2, f4, f1, f1),
 (f2, f2, f2, f4, f1, f5),
 ...
 (f4, f4, f4, f4, f4, f0),
 (f4, f4, f4, f4, f4, f4)}
```

get_fvars_in_alg_field()

Return F-symbols as elements of the `QQbar`.

This method uses the embedding defined by *get_qqbar_embedding()* to coerce F-symbols into `QQbar`.

EXAMPLES:

```
sage: fr = FusionRing("G2", 1)
sage: f = fr.get_fmatrix(fusion_label="g", inject_variables=True, new=True)
creating variables fx1..fx5
Defining fx0, fx1, fx2, fx3, fx4
sage: f.find_orthogonal_solution(verbose=False)
sage: f.field()
Number Field in a with defining polynomial y^32 - ... - 22*y^2 + 1
sage: f.get_fvars_in_alg_field()
{(g1, g1, g1, g0, g1, g1): 1,
 (g1, g1, g1, g1, g0, g0): 0.61803399? + 0.?e-8*I,
 (g1, g1, g1, g1, g0, g1): -0.7861514? + 0.?e-8*I,
 (g1, g1, g1, g1, g1, g0): -0.7861514? + 0.?e-8*I,
 (g1, g1, g1, g1, g1, g1): -0.61803399? + 0.?e-8*I}
```

get_non_cyclotomic_roots()

Return a list of roots that define the extension of the associated *FusionRing*'s base `Cyclotomic field`, containing all the F-symbols.

OUTPUT:

The list of non-cyclotomic roots is given as a list of elements of the field returned by *field()*.

If `self.field() == self.FR().field()` then this method returns an empty list.

EXAMPLES:

```
sage: f = FusionRing("E6", 1).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False)
sage: f.field() == f.FR().field()
True
sage: f.get_non_cyclotomic_roots()
[]
sage: f = FusionRing("G2", 1).get_fmatrix()
sage: f.find_orthogonal_solution(verbose=False)
sage: f.field() == f.FR().field()
False
sage: phi = f.get_qqbar_embedding()
sage: [phi(r).n() for r in f.get_non_cyclotomic_roots()]
[-0.786151377757423 - 8.92806368517581e-31*I]
```

When `self.field()` is a `NumberField`, one may use *get_qqbar_embedding()* to embed the resulting values into `QQbar`.

get_orthogonality_constraints(*output=True*)

Get equations imposed on the F-matrix by orthogonality.

INPUT:

- `output` – a boolean

OUTPUT:

If `output=True`, orthogonality constraints are returned as polynomial objects.

Otherwise, the constraints are appended to `self.ideal_basis`. They are stored in the internal tuple representation. The `output=False` option is meant mostly for internal use by the F-matrix solver.

EXAMPLES:

```
sage: f = FusionRing("B4", 1).get_fmatrix()
sage: f.get_orthogonality_constraints()
[fx0^2 - 1,
 fx1^2 - 1,
 fx2^2 - 1,
 fx3^2 - 1,
 fx4^2 - 1,
 fx5^2 - 1,
 fx6^2 - 1,
 fx7^2 - 1,
 fx8^2 - 1,
 fx9^2 - 1,
 fx10^2 + fx12^2 - 1,
 fx10*fx11 + fx12*fx13,
 fx10*fx11 + fx12*fx13,
 fx11^2 + fx13^2 - 1]
```

get_poly_ring()

Return the polynomial ring whose generators denote the desired F-symbols.

EXAMPLES:

```
sage: f = FusionRing("B6", 1).get_fmatrix()
sage: f.get_poly_ring()
Multivariate Polynomial Ring in fx0, ..., fx13 over
 Cyclotomic Field of order 96 and degree 32
```

get_qqbar_embedding()

Return an embedding from the base field containing F-symbols (the associated *FusionRing*'s `Cyclotomic field`, a `NumberField()`, or `QQbar`) into `QQbar`.

This embedding is useful for getting a better sense for the F-symbols, particularly when they are computed as elements of a `NumberField()`. See also *get_non_cyclotomic_roots()*.

EXAMPLES:

```
sage: fr = FusionRing("G2", 1)
sage: f = fr.get_fmatrix(fusion_label="g", inject_variables=True, new=True)
creating variables fx1..fx5
Defining fx0, fx1, fx2, fx3, fx4
sage: f.find_orthogonal_solution()
Computing F-symbols for The Fusion Ring of Type G2 and level 1 with Integer␣
↪Ring coefficients with 5 variables...
Set up 10 hex and orthogonality constraints...
Partitioned 10 equations into 2 components of size:
[4, 1]
Elimination epoch completed... 0 eqns remain in ideal basis
Hex elim step solved for 4 / 5 variables
Set up 0 reduced pentagons...
Pent elim step solved for 4 / 5 variables
Partitioned 0 equations into 0 components of size:
[]
Partitioned 1 equations into 1 components of size:
[1]
```

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```
Computing appropriate NumberField...
sage: phi = f.get_qqbar_embedding()
sage: phi(f.fmat(g1, g1, g1, g1, g1, g1)).n()
-0.618033988749895 + 1.46674215951686e-29*I
```

get_radical_expression()

Return a radical expression of F-symbols.

EXAMPLES:

```
sage: f = FusionRing("G2", 1).get_fmatrix()
sage: f.FR().fusion_labels("g", inject_variables=True)
sage: f.find_orthogonal_solution(verbose=False)
sage: radical_fvars = f.get_radical_expression()       # long time (~1.5s)
sage: radical_fvars[g1, g1, g1, g1, g1, g0]            # long time
-sqrt(1/2*sqrt(5) - 1/2)
```

largest_fmat_size()

Get the size of the largest F-matrix F^{abc}_d.

EXAMPLES:

```
sage: f = FusionRing("B3", 2).get_fmatrix()
sage: f.largest_fmat_size()
4
```

load_fvars(*filename*)

Load previously computed F-symbols from a pickle file.

See *save_fvars()* for more information.

EXAMPLES:

```
sage: f = FusionRing("A2", 1).get_fmatrix(new=True)
sage: f.find_orthogonal_solution(verbose=False)
sage: fvars = f.get_fvars()
sage: K = f.field()
sage: filename = f.get_fr_str() + "_solver_results.pickle"
sage: f.save_fvars(filename)
sage: del f
sage: f2 = FusionRing("A2", 1).get_fmatrix(new=True)
sage: f2.load_fvars(filename)
sage: fvars == f2.get_fvars()
True
sage: K == f2.field()
True
sage: os.remove(filename)
```

Note: *save_fvars()*. This method does not work with intermediate checkpoint pickles; it only works with pickles containing *all* F-symbols, i.e. those created by *save_fvars()* and by specifying an optional save_results parameter for *find_orthogonal_solution()*.

save_fvars(*filename*)

Save computed F-symbols for later use.

INPUT:

- filename – a string specifying the name of the pickle file to be used

The current directory is used unless an absolute path to a file in a different directory is provided.

Note: This method should only be used *after* successfully running one of the solvers, e.g. *find_cyclotomic_solution()* or *find_orthogonal_solution()*.

When used in conjunction with *load_fvars()*, this method may be used to restore state of an *FMatrix* object at the end of a successful F-matrix solver run.

EXAMPLES:

```
sage: f = FusionRing("A2", 1).get_fmatrix(new=True)
sage: f.find_orthogonal_solution(verbose=False)
sage: fvars = f.get_fvars()
sage: K = f.field()
sage: filename = f.get_fr_str() + "_solver_results.pickle"
sage: f.save_fvars(filename)
sage: del f
sage: f2 = FusionRing("A2", 1).get_fmatrix(new=True)
sage: f2.load_fvars(filename)
sage: fvars == f2.get_fvars()
True
sage: K == f2.field()
True
sage: os.remove(filename)
```

shutdown_worker_pool()

Shutdown the given worker pool and dispose of shared memory resources created when the pool was set up using *start_worker_pool()*.

Warning: Failure to call this method after using *start_worker_pool()* to create a process pool may result in a memory leak, since shared memory resources outlive the process that created them.

EXAMPLES:

```
sage: f = FusionRing("A1", 3).get_fmatrix(new=True)
sage: f.start_worker_pool()
sage: he = f.get_defining_equations('hexagons')
sage: f.shutdown_worker_pool()
```

start_worker_pool(*processes=None*)

Initialize a multiprocessing worker pool for parallel processing, which may be used e.g. to set up defining equations using *get_defining_equations()*.

This method sets self's pool attribute. The worker pool may be used time and again. Upon initialization, each process in the pool attaches to the necessary shared memory resources.

When you are done using the worker pool, use *shutdown_worker_pool()* to close the pool and properly dispose of shared memory resources.

Note: Python 3.8+ is required, since the `multiprocessing.shared_memory` module must be imported.

INPUT:

- `processes` – an integer indicating the number of workers in the pool; if left unspecified, the number of workers is equals the number of processors available

OUTPUT:

This method returns a boolean indicating whether a worker pool was successfully initialized.

EXAMPLES:

```
sage: f = FusionRing("G2", 1).get_fmatrix(new=True)
sage: f.start_worker_pool()
sage: he = f.get_defining_equations('hexagons')
sage: sorted(he)
[fx0 - 1,
 fx2*fx3 + (zeta60^14 + zeta60^12 - zeta60^6 - zeta60^4 + 1)*fx4^2 + (zeta60^
↪6)*fx4,
 fx1*fx3 + (zeta60^14 + zeta60^12 - zeta60^6 - zeta60^4 + 1)*fx3*fx4 + (zeta60^
↪14 - zeta60^4)*fx3,
 fx1*fx2 + (zeta60^14 + zeta60^12 - zeta60^6 - zeta60^4 + 1)*fx2*fx4 + (zeta60^
↪14 - zeta60^4)*fx2,
 fx1^2 + (zeta60^14 + zeta60^12 - zeta60^6 - zeta60^4 + 1)*fx2*fx3 + (-zeta60^
↪12)*fx1]
sage: pe = f.get_defining_equations('pentagons')
sage: f.shutdown_worker_pool()
```

Warning: This method is needed to initialize the worker pool using the necessary shared memory resources. Simply using the `multiprocessing.Pool` constructor will not work with our class methods.

Warning: Failure to call *`shutdown_worker_pool()`* may result in a memory leak, since shared memory resources outlive the process that created them.

5.8.3 The Fusion Ring of the Drinfeld Double of a Finite Group

class `sage.algebras.fusion_rings.fusion_double.`**`FusionDouble`**(*G*, *prefix='s'*)

Bases: `CombinatorialFreeModule`

The fusion ring corresponding to the Drinfeld double of a finite group.

This is the fusion ring of the modular tensor category of modules over the Drinfeld double of a finite group. Usage is similar to *`FusionRing`*; we refer the reader to that class for more information.

INPUT:

- `G` – a finite group
- `prefix` – (default: `'s'`) a prefix for the names of simple objects
- `inject_varables` – (optional) set to `True` to create variables for the simple objects

REFERENCES:

- [BaKi2001] Chapter 3
- [Mas1995]
- [CHW2015]
- [Goff1999]

EXAMPLES:

```
sage: G = DihedralGroup(5)
sage: H = FusionDouble(G, inject_variables=True)
sage: H.basis()
Finite family {0: s0, 1: s1, 2: s2, 3: s3, 4: s4, 5: s5, 6: s6, 7: s7, 8: s8,
               9: s9, 10: s10, 11: s11, 12: s12, 13: s13, 14: s14, 15: s15}
sage: for x in H.basis():
....:     print ("%s : %s"%(x,x^2))
....:
s0 : s0
s1 : s0
s2 : s0 + s1 + s3
s3 : s0 + s1 + s2
s4 : s0 + s2 + s3 + s6 + s7 + s8 + s9 + s10 + s11 + s12 + s13 + s14 + s15
s5 : s0 + s2 + s3 + s6 + s7 + s8 + s9 + s10 + s11 + s12 + s13 + s14 + s15
s6 : s0 + s1 + s11
s7 : s0 + s1 + s13
s8 : s0 + s1 + s15
s9 : s0 + s1 + s12
s10 : s0 + s1 + s14
s11 : s0 + s1 + s6
s12 : s0 + s1 + s9
s13 : s0 + s1 + s7
s14 : s0 + s1 + s10
s15 : s0 + s1 + s8
sage: s4*s5
s1 + s2 + s3 + s6 + s7 + s8 + s9 + s10 + s11 + s12 + s13 + s14 + s15
sage: s4.ribbon()
1
sage: s5.ribbon()
-1
sage: s8.ribbon()
zeta5^3
```

If the fusion double is multiplicity-free, meaning that the fusion coefficients N_k^{ij} are bounded by 1, then the F-matrix may be computed, by solving the pentagon and hexagon relations as described in [Bond2007] and [Ab2022], just as for `FusionRing`. There is a caveat here, since even if the fusion rules are multiplicity-free, if there are too many F-matrix values to compute, even if many of them are zero, in the current implementation singular cannot create enough variables. At least, this code can compute the F-matrix for the Fusion Double of the symmetric group S_3, duplicating the result of [CHW2015].

```
sage: G1 = SymmetricGroup(3)
sage: H1 = FusionDouble(G1, prefix="u", inject_variables=True)
sage: F = H1.get_fmatrix()
```

The above commands create the F-matrix. You can compute all of the F-matrices with the command:

```
sage: H1.find_orthogonal_solution()  # not tested (10-15 minutes)
```

Individual F-matrices may be computed thus:

```
sage: F.fmatrix(u3, u3, u3, u4)  # not tested
```

See `FMatrix` for more information.

Unfortunately beyond S_3 the number of simple objects is seemingly impractical. Although the *FusionDouble* class and its methods work well for groups of moderate size, the `FMatrix` may not be computable. For the dihedral group of order 8, there are already 22 simple objects, and the F-matrix seems out of reach. The actual limitation is that singular will not create a polynomial ring in more than $2^{15} - 1 = 32767$ symbols, and there are more than this many F-matrix values to be computed for the dihedral group of order 8, so in the current implementation, this FusionRing is out of reach.

It is an open problem to classify the finite groups whose fusion doubles are multiplicity-free. Abelian groups, dihedral groups, dicyclic groups, and all groups of order 16 are multiplicity-free. On the other hand, for groups of order 32, some are multiplicity-free and others are not. These can all be constructed using `SmallPermutationGroup`.

EXAMPLES:

```
sage: G = SmallPermutationGroup(16,9)
sage: F = FusionDouble(G, prefix="b",inject_variables=True)
sage: b13^2 # long time (4s)
b0 + b2 + b4 + b15 + b16 + b17 + b18 + b24 + b26 + b27
```

D_minus(*base_coercion=True*)

Return the positive square root of *self.global_q_dimension()* as an element of *self.field()*.

For the Drinfeld double of a finite group G, this equals the cardinality of G. This is also equal to $\sum d_i^2\theta_i^{\pm 1}$, where i runs through the simple objects, d_i is the quantum dimension, and θ_i is the twist. This sum with θ_i is denoted p_- in [BaKi2001] Chapter 3.

EXAMPLES:

```
sage: FusionDouble(DihedralGroup(7)).total_q_order()
14
```

D_plus(*base_coercion=True*)

Return the positive square root of *self.global_q_dimension()* as an element of *self.field()*.

For the Drinfeld double of a finite group G, this equals the cardinality of G. This is also equal to $\sum d_i^2\theta_i^{\pm 1}$, where i runs through the simple objects, d_i is the quantum dimension, and θ_i is the twist. This sum with θ_i is denoted p_- in [BaKi2001] Chapter 3.

EXAMPLES:

```
sage: FusionDouble(DihedralGroup(7)).total_q_order()
14
```

class Element

Bases: `IndexedFreeModuleElement`

char()

Return the character χ corresponding to `self`.

The data determining a simple object consists of a conjugacy class representative g and an irreducible character χ of the centralizer of g.

See also:

g()

EXAMPLES:

```
sage: G = DihedralGroup(5)
sage: H = FusionDouble(G, prefix="f", inject_variables=True)
sage: f10.g()
(1,2,3,4,5)
sage: f10.char()
Character of Subgroup generated by [(1,2,3,4,5)] of
    (Dihedral group of order 10 as a permutation group)
```

dual()

Return the dual of `self`.

This method is only available for simple objects.

EXAMPLES:

```
sage: G = CyclicPermutationGroup(4)
sage: H = FusionDouble(G, prefix="j")
sage: [x for x in H.basis() if x == x.dual()]
[j0, j1, j8, j9]
```

g()

The data determining a simple object consists of a conjugacy class representative g and an irreducible character χ of the centralizer of g.

Returns the conjugacy class representative of the underlying group corresponding to a simple object. See also *char()*.

EXAMPLES:

```
sage: G = QuaternionGroup()
sage: H = FusionDouble(G, prefix="e", inject_variables=True)
sage: e10.g()
(1,3)(2,4)(5,7)(6,8)
sage: e10.char()
Character of Subgroup generated by [(1,2,3,4)(5,6,7,8), (1,5,3,7)(2,8,4,6)]
    of (Quaternion group of order 8 as a permutation group)
```

is_simple_object()

Determine whether `self` is a simple object (basis element) of the fusion ring.

EXAMPLES:

```
sage: H = FusionDouble(CyclicPermutationGroup(2), prefix="g", inject_
↪variables=True)
sage: [x.is_simple_object() for x in [g0, g1, g0+g1]]
[True, True, False]
```

q_dimension(*base_coercion=True*)

Return the q-dimension of `self`.

This method is only available for simple objects.

EXAMPLES:

```
sage: G = AlternatingGroup(4)
sage: H = FusionDouble(G)
sage: [x.q_dimension() for x in H.basis()]
[1, 1, 1, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4]
sage: sum(x.q_dimension()^2 for x in H.basis()) == G.order()^2
True
```

ribbon(*base_coercion=True*)

The twist or ribbon of the simple object.

EXAMPLES:

```
sage: H = FusionDouble(CyclicPermutationGroup(3))
sage: [i.ribbon() for i in H.basis()]
[1, 1, 1, 1, zeta3, -zeta3 - 1, 1, -zeta3 - 1, zeta3]
```

twist(*reduced=True*)

Return a rational number h such that $\theta = e^{i\pi h}$ is the twist of `self`.

The quantity $e^{i\pi h}$ is also available using `ribbon()`.

This method is only available for simple objects.

EXAMPLES:

```
sage: Q=FusionDouble(CyclicPermutationGroup(3))
sage: [x.twist() for x in Q.basis()]
[0, 0, 0, 0, 2/3, 4/3, 0, 4/3, 2/3]
sage: [x.ribbon() for x in Q.basis()]
[1, 1, 1, 1, zeta3, -zeta3 - 1, 1, -zeta3 - 1, zeta3]
```

N_ijk(*i*, *j*, *k*)

The symmetric invariant of three simple objects.

This is the dimension of

$$Hom(i \otimes j \otimes k, s_0),$$

where s_0 is the unit element (assuming `prefix='s'`). Method of computation is through the Verlinde formula, deducing the values from the known values of the S-matrix.

EXAMPLES:

```
sage: A = FusionDouble(AlternatingGroup(4),prefix="a",inject_variables=True)
sage: [A.N_ijk(a10,a11,x) for x in A.basis()]
[0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0]
```

Nk_ij(*i*, *j*, *k*, *use_characters=False*)

Return the fusion coefficient N^k_{ij}.

INPUT:

- i, j, k – basis elements
- use_characters – (default: False) see the algorithm description below

ALGORITHM:

If use_characters=False, then this is computed using the Verlinde formula:

$$N_{ij}^k = \sum_l \frac{s(i,\ell)\,s(j,\ell)\,\overline{s(k,\ell)}}{s(I,\ell)}.$$

Otherwise we use a character theoretic method to compute the fusion coefficient N_{ij}^k as follows. Each simple object, for example i corresponds to a conjugacy class $\mathcal{C}_i$ of the underlying group G, and an irreducible character χ_i of the centralizer $C(g_i)$ of a fixed representative g_i of $\mathcal{C}_i$. In addition to the fixed representative g_k of the class $\mathcal{C}_i$ and $\mathcal{C}_j$, the formula will make use of variable elements h_i and h_j that are subject to the condition $h_i h_j = g_k$. See [GoMa2010] equation (7).

$$\frac{|\mathcal{C}_k|}{|G|} \sum_{\substack{h_i\in\mathcal{C}_i\\ h_j\in\mathcal{C}_j\\ h_ih_j=g_k}} |C(h_i)\cap C(h_j)|\,\langle \chi_i^{(h_i)}\chi_j^{(h_j)}, \chi_k\rangle_{C(h_i)\cap C(h_j)},$$

where $\chi_i^{(h_i)}$ is the character χ_i of $C(g_i)$ conjugated to a character of $C(h_i)$, when h_i is a conjugate of the fixed representative g_i. More exactly, there exists r_i such that $r_i g_i r_i^{-1} = h_i$, and then $\chi_i^{(h_i)}(x) = \chi_i(r_i^{-1} x r_i)$, and this definition does not depend on the choice of r_i.

Note: This should be functionally equivalent, and testing shows that it is, but it is slower.

EXAMPLES:

```
sage: A = FusionDouble(AlternatingGroup(4),prefix="aa",inject_variables=True)
sage: [A.Nk_ij(aa8,aa10,x) for x in A.basis()]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 1]

sage: B = FusionDouble(CyclicPermutationGroup(2))
sage: all(B.Nk_ij(x,y,z,use_characters=True) == B.Nk_ij(x,y,z)
....:     for x in B.basis() for y in B.basis() for z in B.basis())
True
```

dual(*i*)

Return the dual object i^\ast to i.

The dual is also available as an element method of i.

EXAMPLES:

```
sage: K = FusionDouble(CyclicPermutationGroup(3),prefix="k")
sage: [(x,K.dual(x)) for x in K.basis()]
[(k0, k0),
(k1, k2),
(k2, k1),
(k3, k6),
(k4, k8),
(k5, k7),
(k6, k3),
```

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```
(k7, k5),
(k8, k4)]
sage: all(K.dual(x)==x.dual() for x in K.basis())
True
```

field()

Returns a cyclotomic field large enough to contain the values of R-matrices and twists that can arise for this fusion ring.

EXAMPLES:

```
sage: FusionDouble(SymmetricGroup(3)).field()
Cyclotomic Field of order 24 and degree 8
```

fvars_field()

Return a field containing the `CyclotomicField` computed by *field()* as well as all the F-symbols of the associated `FMatrix` factory object.

This method is only available if `self` is multiplicity-free.

EXAMPLES:

```
sage: FusionDouble(SymmetricGroup(3)).fvars_field()
Cyclotomic Field of order 24 and degree 8
```

get_fmatrix(**args*, ***kwargs*)

Construct an `FMatrix` factory to solve the pentagon and hexagon relations and organize the resulting F-symbols.

EXAMPLES:

```
sage: f = FusionDouble(SymmetricGroup(3)).get_fmatrix(); f
F-Matrix factory for The Fusion Ring of the Drinfeld Double of
    Symmetric group of order 3! as a permutation group
```

global_q_dimension(*base_coercion=True*)

Return the global quantum dimension, which is the sum of the squares of the quantum dimensions of the simple objects. For the Drinfeld double, it is the square of the order of the underlying quantum group.

EXAMPLES:

```
sage: G = SymmetricGroup(4)
sage: H = FusionDouble(G)
sage: H.global_q_dimension()
576
sage: sum(x.q_dimension()^2 for x in H.basis())
576
```

group()

Return the underlying group.

EXAMPLES:

```
sage: FusionDouble(DiCyclicGroup(4)).group()
Dicyclic group of order 16 as a permutation group
```

inject_variables()

Create variables for the simple objects in the global name space.

EXAMPLES:

```
sage: F = FusionDouble(DiCyclicGroup(3), prefix="d")
sage: F.inject_variables()
sage: d0 + d1 + d5
d0 + d1 + d5
```

is_multiplicity_free(*verbose=False*)

Return True if all fusion coefficients are at most 1.

EXAMPLES:

```
sage: FusionDouble(SymmetricGroup(3)).is_multiplicity_free()
True
sage: FusionDouble(SymmetricGroup(4)).is_multiplicity_free()
False

sage: FusionDouble(SymmetricGroup(3)).is_multiplicity_free(True)
Checking multiplicity freeness
True
sage: FusionDouble(SymmetricGroup(4)).is_multiplicity_free(True)
Checking multiplicity freeness
N(s2,s13,s13) = 2
False
```

one_basis()

The unit element of the ring, which is the first basis element.

EXAMPLES:

```
sage: FusionDouble(CyclicPermutationGroup(2), prefix="h").one()
h0
```

product_on_basis(a, b)

Return the product of two basis elements corresponding to keys a and b.

INPUT:

- a`, ``b – keys for the dictionary self._names representing simple objects

EXAMPLES:

```
sage: Q=FusionDouble(SymmetricGroup(3),prefix="q",inject_variables=True)
sage: q3*q4
q1 + q2 + q5 + q6 + q7
sage: Q._names
{0: 'q0', 1: 'q1', 2: 'q2', 3: 'q3', 4: 'q4', 5: 'q5', 6: 'q6', 7: 'q7'}
sage: Q.product_on_basis(3,4)
q1 + q2 + q5 + q6 + q7
```

r_matrix(i, j, k, *base_coercion=True*)

Return the R-matrix entry corresponding to the subobject k in the tensor product of i with j. This method is only correct if the fusion coefficient N_{ij}^k\leq 1. See the *FusionRing* method for more information, including the reason for this caveat, and the algorithm.

EXAMPLES:

```
sage: C = FusionDouble(SymmetricGroup(3),prefix="c",inject_variables=True)
sage: c4*c5
c3 + c4
sage: [C.r_matrix(c4,c5,k) for k in [c3,c4]]
[-zeta24^6, 1]
sage: c6^2
c0 + c1 + c6
sage: [C.r_matrix(c6,c6,k) for k in [c0,c1,c6]]
[zeta3, -zeta3, -zeta3 - 1]
```

root_of_unity(*r*, *base_coercion=True*)

Return $e^{i\pi r}$ as an element of `self.field()` if possible.

INPUT:

- `r` – a rational number

EXAMPLES:

```
sage: H = FusionDouble(DihedralGroup(6))
sage: H.field()
Cyclotomic Field of order 24 and degree 8
sage: for n in [1..7]:
....:     try:
....:         print (n,H.root_of_unity(2/n))
....:     except ValueError as err:
....:         print (n,err)
....:
1 1
2 -1
3 zeta24^4 - 1
4 zeta24^6
5 not a root of unity in the field
6 zeta24^4
7 not a root of unity in the field
```

s_ij(*i*, *j*, *unitary=False*, *base_coercion=True*)

Return the element of the S-matrix of this fusion ring corresponding to the given elements.

Without the unitary option set true, this is the unnormalized S-matrix entry, denoted $\tilde{s}_{ij}$, in [BaKi2001] Chapter 3. The normalized S-matrix entries are denoted s_{ij}.

INPUT:

- `i`, `j`, – a pair of basis elements
- `unitary` – (default: `False`) set to `True` to obtain the unitary S-matrix

EXAMPLES:

```
sage: D = FusionDouble(SymmetricGroup(3), prefix="t", inject_variables=True)
sage: [D.s_ij(t2, x) for x in D.basis()]
[2, 2, 4, 0, 0, -2, -2, -2]
sage: [D.s_ij(t2, x, unitary=True) for x in D.basis()]
[1/3, 1/3, 2/3, 0, 0, -1/3, -1/3, -1/3]
```

s_ijconj(*i*, *j*, *unitary=False*, *base_coercion=True*)

Return the conjugate of the element of the S-matrix given by `self.s_ij(elt_i, elt_j, base_coercion=base_coercion)`.

See also:

s_ij()

EXAMPLES:

```
sage: P=FusionDouble(CyclicPermutationGroup(3),prefix="p",inject_variables=True)
sage: P.s_ij(p1,p3)
zeta3
sage: P.s_ijconj(p1,p3)
-zeta3 - 1
```

s_matrix(*unitary=False*, *base_coercion=True*)

Return the S-matrix of this fusion ring.

OPTIONAL:

- `unitary` – (default: `False`) set to `True` to obtain the unitary S-matrix

Without the `unitary` parameter, this is the matrix denoted $\widetilde{s}$ in [BaKi2001].

EXAMPLES:

```
sage: FusionDouble(SymmetricGroup(3)).s_matrix()
[ 1  1  2  3  3  2  2  2]
[ 1  1  2 -3 -3  2  2  2]
[ 2  2  4  0  0 -2 -2 -2]
[ 3 -3  0  3 -3  0  0  0]
[ 3 -3  0 -3  3  0  0  0]
[ 2  2 -2  0  0  4 -2 -2]
[ 2  2 -2  0  0 -2 -2  4]
[ 2  2 -2  0  0 -2  4 -2]
sage: FusionDouble(SymmetricGroup(3)).s_matrix(unitary=True)
[ 1/6  1/6  1/3  1/2  1/2  1/3  1/3  1/3]
[ 1/6  1/6  1/3 -1/2 -1/2  1/3  1/3  1/3]
[ 1/3  1/3  2/3    0    0 -1/3 -1/3 -1/3]
[ 1/2 -1/2    0  1/2 -1/2    0    0    0]
[ 1/2 -1/2    0 -1/2  1/2    0    0    0]
[ 1/3  1/3 -1/3    0    0  2/3 -1/3 -1/3]
[ 1/3  1/3 -1/3    0    0 -1/3 -1/3  2/3]
[ 1/3  1/3 -1/3    0    0 -1/3  2/3 -1/3]
```

total_q_order(*base_coercion=True*)

Return the positive square root of *self.global_q_dimension()* as an element of *self.field()*.

For the Drinfeld double of a finite group G, this equals the cardinality of G. This is also equal to $\sum d_i^2\theta_i^{\pm 1}$, where i runs through the simple objects, d_i is the quantum dimension, and θ_i is the twist. This sum with θ_i is denoted p_- in [BaKi2001] Chapter 3.

EXAMPLES:

```
sage: FusionDouble(DihedralGroup(7)).total_q_order()
14
```

5.8.4 F-Matrix Backend

Fast F-Matrix Methods

sage.algebras.fusion_rings.fast_parallel_fmats_methods.**executor**(*params*)

Execute a function defined in this module (`sage.algebras.fusion_rings.fast_parallel_fmats_methods`) in a worker process, and supply the factory parameter by constructing a reference to the `FMatrix` object in the worker's memory adress space from its `id`.

INPUT:

- `params` – a tuple `((fn_name, fmats_id), fn_args)` where `fn_name` is the name of the function to be executed, `fmats_id` is the `id` of the `FMatrix` object, and `fn_args` is a tuple containing all arguments to be passed to the function `fn_name`.

Note: When the parent process is forked, each worker gets a copy of every global variable. The virtual memory address of object X in the parent process equals the *virtual* memory address of the copy of object X in each worker, so we may construct references to forked copies of X using an `id` obtained in the parent process.

Fast Fusion Ring Methods for Computing Braid Group Representations

sage.algebras.fusion_rings.fast_parallel_fusion_ring_braid_repn.**executor**(*params*)

Execute a function registered in this module's `mappers` in a worker process, and supply the `FusionRing` parameter by constructing a reference to the FMatrix object in the worker's memory adress space from its `id`.

Note: When the parent process is forked, each worker gets a copy of every global variable. The virtual memory address of object X in the parent process equals the *virtual* memory address of the copy of object X in each worker, so we may construct references to forked copies of X.

Arithmetic Engine for Polynomials as Tuples

sage.algebras.fusion_rings.poly_tup_engine.**apply_coeff_map**(*eq_tup*, *coeff_map*)

Apply `coeff_map` to coefficients.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import apply_coeff_map
sage: sq = lambda x : x**2
sage: R.<x, y, z> = PolynomialRing(ZZ)
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup, _tup_to_
↪poly
sage: _tup_to_poly(apply_coeff_map(poly_to_tup(x + 2*y + 3*z), sq), parent=R)
x + 4*y + 9*z
```

sage.algebras.fusion_rings.poly_tup_engine.**compute_known_powers**(*max_degs*, *val_dict*, *one*)

Pre-compute powers of known values for efficiency when preparing to substitute into a list of polynomials.

INPUT:

- `max_deg` – an `ETuple` indicating the maximal degree of each variable
- `val_dict` – a dictionary of `(var_idx, poly_tup)` key-value pairs

- `poly_tup` – a tuple of `(ETuple, coeff)` pairs reperesenting a multivariate polynomial

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import compute_known_powers
sage: R.<x, y, z> = PolynomialRing(QQ)
sage: polys = [x**3 + 1, x**2*y + z**3, y**2 - 3*y]
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: known_val = { 0 : poly_to_tup(R(-1)), 2 : poly_to_tup(y**2) }
sage: from sage.algebras.fusion_rings.poly_tup_engine import get_variables_degrees
sage: max_deg = get_variables_degrees([poly_to_tup(p) for p in polys], 3)
sage: compute_known_powers(max_deg, known_val, R.base_ring().one())
{0: [(((0, 0, 0), 1),),
     (((0, 0, 0), -1),),
     (((0, 0, 0), 1),),
     (((0, 0, 0), -1),)],
2: [(((0, 0, 0), 1),),
    (((0, 2, 0), 1),),
    (((0, 4, 0), 1),),
    (((0, 6, 0), 1),)]}
```

`sage.algebras.fusion_rings.poly_tup_engine.`**`constant_coeff`**`(`*eq_tup*, *field*`)`

Return the constant coefficient of the polynomial represented by given tuple.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import constant_coeff
sage: from sage.rings.polynomial.polydict import ETuple
sage: poly_tup = ((ETuple([0, 3, 0]), 2), (ETuple([0, 1, 0]), -1), (ETuple([0, 0,␣
↪0]), -2/3))
sage: constant_coeff(poly_tup, QQ)
-2/3
sage: R.<x, y, z> = PolynomialRing(QQ)
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: constant_coeff(poly_to_tup(x**5 + x*y*z - 9), QQ)
-9
```

`sage.algebras.fusion_rings.poly_tup_engine.`**`get_variables_degrees`**`(`*eqns*, *nvars*`)`

Find maximum degrees for each variable in equations.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import get_variables_degrees
sage: R.<x, y, z> = PolynomialRing(QQ)
sage: polys = [x**2 + 1, x*y*z**2 - 4*x*y, x*z**3 - 4/3*y + 1]
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: get_variables_degrees([poly_to_tup(p) for p in polys], 3)
[2, 1, 3]
```

`sage.algebras.fusion_rings.poly_tup_engine.`**`poly_to_tup`**`(`*poly*`)`

Convert a polynomial object into the internal representation as tuple of `(ETuple exp, NumberFieldElement coeff)` pairs.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: R.<x, y> = PolynomialRing(QQ)
sage: poly_to_tup(x**2 + 1)
(((2, 0), 1), ((0, 0), 1))
sage: poly_to_tup(x**2*y**4 - 4/5*x*y**2 + 1/3 * y)
(((2, 4), 1), ((1, 2), -4/5), ((0, 1), 1/3))
```

sage.algebras.fusion_rings.poly_tup_engine.**poly_tup_sortkey**(*eq_tup*)

Return the sortkey of a polynomial represented as a tuple of `(ETuple, coeff)` pairs with respect to the degree lexicographical term order.

Using this key to sort polynomial tuples results in comparing polynomials term by term (we assume the tuple representation is sorted so that the leading term with respect to the degree reverse lexicographical order comes first). For each term, we first compare degrees, then the monomials themselves. Different polynomials can have the same sortkey.

EXAMPLES:

```
sage: F = CyclotomicField(20)
sage: zeta20 = F.gen()
sage: R.<x, y, z> = PolynomialRing(F)
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_tup_sortkey, poly_
↪to_tup
sage: p = (zeta20 + 1)*x^2 + (zeta20^3 + 6)*x*z + (zeta20^2 + 7*zeta20)*z^2 + (2/
↪3*zeta20 + 1/4)*x + y
sage: p1 = poly_to_tup(p); p1
(((2, 0, 0), zeta20 + 1),
 ((1, 0, 1), zeta20^3 + 6),
 ((0, 0, 2), zeta20^2 + 7*zeta20),
 ((1, 0, 0), 2/3*zeta20 + 1/4),
 ((0, 1, 0), 1))
   sage: poly_tup_sortkey(p1)
   (2, 0, 2, 2, 0, 1, -2, 1, 2, -2, 2, 1, 0, 1, 1, -1, 1)
```

sage.algebras.fusion_rings.poly_tup_engine.**resize**(*eq_tup*, *idx_map*, *nvars*)

Return a tuple representing a polynomial in a ring with `len(sorted_vars)` generators.

This method is used for creating polynomial objects with the “right number” of variables for computing Groebner bases of the partitioned equations graph and for adding constraints ensuring certain F-symbols are nonzero.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import resize
sage: from sage.rings.polynomial.polydict import ETuple
sage: K = CyclotomicField(56)
sage: poly_tup = ((ETuple([0, 3, 0, 2]), K(2)), (ETuple([0, 1, 0, 1]), K(-1)),␣
↪(ETuple([0, 0, 0, 0]), K(-2/3)))
sage: idx_map = {1: 0, 3: 1}
sage: resize(poly_tup, idx_map, 2)
(((3, 2), 2), ((1, 1), -1), ((0, 0), -2/3))

sage: R = PolynomialRing(K, 'fx', 20)
sage: R.inject_variables()
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7, fx8, fx9, fx10, fx11, fx12, fx13,␣
↪fx14, fx15, fx16, fx17, fx18, fx19
```

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```
sage: sparse_poly = R(fx0**2 * fx17 + fx3)
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup, _tup_to_
↪poly
sage: S.<x, y, z> = PolynomialRing(K)
sage: _tup_to_poly(resize(poly_to_tup(sparse_poly), {0:0, 3:1, 17:2}, 3), parent=S)
x^2*z + y
```

sage.algebras.fusion_rings.poly_tup_engine.**tup_to_univ_poly**(*eq_tup*, *univ_poly_ring*)

Given a tuple of pairs representing a univariate polynomial and a univariate polynomial ring, return a univariate polynomial object.

Each pair in the tuple is assumed to be of the form `(ETuple, coeff)`, where `coeff` is an element of `univ_poly_ring.base_ring()`.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import tup_to_univ_poly
sage: from sage.rings.polynomial.polydict import ETuple
sage: K = CyclotomicField(56)
sage: poly_tup = ((ETuple([0, 3, 0]), K(2)), (ETuple([0, 1, 0]), K(-1)), (ETuple([0,
↪ 0, 0]), K(-2/3)))
sage: R = K['b']
sage: tup_to_univ_poly(poly_tup, R)
2*b^3 - b - 2/3
```

sage.algebras.fusion_rings.poly_tup_engine.**variables**(*eq_tup*)

Return indices of all variables appearing in eq_tup

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.poly_tup_engine import variables
sage: from sage.rings.polynomial.polydict import ETuple
sage: poly_tup = ((ETuple([0, 3, 0]), 2), (ETuple([0, 1, 0]), -1), (ETuple([0, 0,␣
↪0]), -2/3))
sage: variables(poly_tup)
[1]
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: R.<x, y, z> = PolynomialRing(QQ)
sage: variables(poly_to_tup(x*2*y + y**3 - 4/3*x))
[0, 1]
sage: variables(poly_to_tup(R(1/4)))
[]
```

Shared Memory Managers for F-Symbol Attributes

This module provides an implementation for shared dictionary like state attributes required by the orthogonal F-matrix solver.

Currently, the attributes only work when the base field of the `FMatrix` factory is a cyclotomic field.

class sage.algebras.fusion_rings.shm_managers.**FvarsHandler**

Bases: `object`

A shared memory backed dict-like structure to manage the `_fvars` attribute of an F-matrix.

This structure implements a representation of the F-symbols dictionary using a structured NumPy array backed by a contiguous shared memory object.

The monomial data is stored in the `exp_data` structure. Monomial exponent data is stored contiguously and `ticks` are used to indicate different monomials.

Coefficient data is stored in the `coeff_nums` and `coeff_denom` arrays. The `coeff_denom` array stores the value `d = coeff.denominator()` for each cyclotomic coefficient. The `coeff_nums` array stores the values `c.numerator() * d for c in coeff._coefficients()`, the abridged list representation of the cyclotomic coefficient `coeff`.

Each entry also has a boolean `modified` attribute, indicating whether it has been modified by the parent process. Entry retrieval is cached in each process, so each process must check whether entries have been modified before attempting retrieval.

The parent process should construct this object without a `name` attribute. Children processes use the `name` attribute, accessed via `self.shm.name` to attach to the shared memory block.

Multiprocessing requires Python 3.8+, since we must import the `multiprocessing.shared_memory` module.

INPUT:

- `n_slots` – number of generators of the underlying polynomial ring
- `field` – base field for polynomial ring
- `idx_to_sextuple` – map relating a single integer index to a sextuple of `FusionRing` elements
- `init_data` – a dictionary or *FvarsHandler* object containing known squares for initialization, e.g., from a solver checkpoint
- `use_mp` – an integer indicating the number of child processes used for multiprocessing; if running serially, use 0.
- `pids_name` – the name of a `ShareableList` contaning the process `pid`'s for every process in the pool (including the parent process)
- `name` – the name of a shared memory object (used by child processes for attaching)
- `max_terms` – maximum number of terms in each entry; since we use contiguous C-style memory blocks, the size of the block must be known in advance
- `n_bytes` – the number of bytes that should be allocated for each numerator and each denominator stored by the structure

Note: To properly dispose of shared memory resources, `self.shm.unlink()` must be called before exiting.

Note: If you ever encounter an `OverflowError` when running the `FMatrix.find_orthogonal_solution()` solver, consider increasing the parameter `n_bytes`.

Warning: The current data structure supports up to 2^16 entries, with each monomial in each entry having at most 254 nonzero terms. On average, each of the `max_terms` monomials can have at most 30 terms.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.shm_managers import FvarsHandler
sage: # Create shared data structure
sage: f = FusionRing("A2", 1).get_fmatrix(inject_variables=True, new=True)
creating variables fx1..fx8
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7
sage: f.start_worker_pool()
sage: n_proc = f.pool._processes
sage: pids_name = f._pid_list.shm.name
sage: fvars = FvarsHandler(8, f._field, f._idx_to_sextuple, use_mp=n_proc, pids_
↪name=pids_name)
sage: # In the same shell or in a different shell, attach to fvars
sage: name = fvars.shm.name
sage: fvars2 = FvarsHandler(8, f._field, f._idx_to_sextuple, name=name , use_mp=n_
↪proc, pids_name=pids_name)
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: rhs = tuple((exp, tuple(c._coefficients())) for exp, c in poly_to_tup(fx5**5))
sage: fvars[f2, f1, f2, f2, f0, f0] = rhs
sage: f._tup_to_fpoly(fvars2[f2, f1, f2, f2, f0, f0])
fx5^5
sage: fvars.shm.unlink()
sage: f.shutdown_worker_pool()
```

items()

Iterates through key-value pairs in the data structure as if it were a Python dict.

As in a Python dict, the key-value pairs are yielded in no particular order.

EXAMPLES:

```
sage: f = FusionRing("G2", 1).get_fmatrix(inject_variables=True, new=True)
creating variables fx1..fx5
Defining fx0, fx1, fx2, fx3, fx4
sage: from sage.algebras.fusion_rings.shm_managers import FvarsHandler
sage: shared_fvars = FvarsHandler(5, f._field, f._idx_to_sextuple, init_data=f._
↪fvars)
sage: for sextuple, fvar in shared_fvars.items():
....:     if sextuple == (f1, f1, f1, f1, f1, f1):
....:         f._tup_to_fpoly(fvar)
....:
fx4
```

shm

class sage.algebras.fusion_rings.shm_managers.**KSHandler**

Bases: object

A shared memory backed dict-like structure to manage the _ks attribute of an F-matrix.

This structure implements a representation of the known squares dictionary using a structured NumPy array backed by a contiguous shared memory object.

The structure mimics a dictionary of (idx, known_sq) pairs. Each integer index corresponds to a variable and each known_sq is an element of the F-matrix factory's base cyclotomic field.

Each cyclotomic coefficient is stored as a list of numerators and a list of denominators representing the rational coefficients. The structured array also maintains known attribute that indicates whether the structure contains an entry corresponding to the given index.

The parent process should construct this object without a `name` attribute. Children processes use the `name` attribute, accessed via `self.shm.name` to attach to the shared memory block.

INPUT:

- `n_slots` – the total number of F-symbols
- `field` – F-matrix's base cyclotomic field
- `use_mp` – a boolean indicating whether to construct a shared memory block to back `self`. Requires Python 3.8+, since we must import the `multiprocessing.shared_memory` module.
- `init_data` – a dictionary or *KSHandler* object containing known squares for initialization, e.g., from a solver checkpoint
- **`name` – the name of a shared memory object (used by child processes**
  for attaching)

Note: To properly dispose of shared memory resources, `self.shm.unlink()` must be called before exiting.

Warning: This structure *cannot* modify an entry that has already been set.

EXAMPLES:

```
sage: from sage.algebras.fusion_rings.shm_managers import KSHandler
sage: # Create shared data structure
sage: f = FusionRing("A1", 2).get_fmatrix(inject_variables=True, new=True)
creating variables fx1..fx14
Defining fx0, fx1, fx2, fx3, fx4, fx5, fx6, fx7, fx8, fx9, fx10, fx11, fx12, fx13
sage: n = f._poly_ring.ngens()
sage: f.start_worker_pool()
sage: ks = KSHandler(n, f._field, use_mp=True)
sage: # In the same shell or in a different shell, attach to fvars
sage: name = ks.shm.name
sage: ks2 = KSHandler(n, f._field, name=name, use_mp=True)
sage: from sage.algebras.fusion_rings.poly_tup_engine import poly_to_tup
sage: eqns = [fx1**2 - 4, fx3**2 + f._field.gen()**4 - 1/19*f._field.gen()**2]
sage: ks.update([poly_to_tup(p) for p in eqns])
sage: for idx, sq in ks.items():
....:     print("Index: {}, square: {}".format(idx, sq))
....:
Index: 1, square: 4
Index: 3, square: -zeta32^4 + 1/19*zeta32^2
sage: ks.shm.unlink()
sage: f.shutdown_worker_pool()
```

items()

Iterate through existing entries using Python dict-style syntax.

EXAMPLES:

```
sage: f = FusionRing("A3", 1).get_fmatrix()
sage: f._reset_solver_state()
sage: f.get_orthogonality_constraints(output=False)
```

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```
sage: f._ks.update(f.ideal_basis)
sage: for idx, sq in f._ks.items():
....:     print("Index: {}, sq: {}".format(idx, sq))
....:
Index: 0, sq: 1
Index: 1, sq: 1
Index: 2, sq: 1
Index: 3, sq: 1
Index: 4, sq: 1
...
Index: 25, sq: 1
Index: 26, sq: 1
```

shm

update(*eqns*)

Update `self's shared_memory-backed dictionary of known squares. Keys are variable indices and corresponding values are the squares.

EXAMPLES:

```
sage: f = FusionRing("B5", 1).get_fmatrix()
sage: f._reset_solver_state()
sage: for idx, sq in f._ks.items():
....:     k
....:
sage: f.get_orthogonality_constraints()
[fx0^2 - 1,
 fx1^2 - 1,
 fx2^2 - 1,
 fx3^2 - 1,
 fx4^2 - 1,
 fx5^2 - 1,
 fx6^2 - 1,
 fx7^2 - 1,
 fx8^2 - 1,
 fx9^2 - 1,
 fx10^2 + fx12^2 - 1,
 fx10*fx11 + fx12*fx13,
 fx10*fx11 + fx12*fx13,
 fx11^2 + fx13^2 - 1]
 sage: f.get_orthogonality_constraints(output=False)
 sage: f._ks.update(f.ideal_basis)
 sage: for idx, sq in f._ks.items():
 ....:     print(idx, "-->", sq)
 ....:
 0 --> 1
 1 --> 1
 2 --> 1
 3 --> 1
 4 --> 1
 5 --> 1
 6 --> 1
```

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```
7 --> 1
8 --> 1
9 --> 1
```

Warning: This method assumes every polynomial in `eqns` is *monic*.

`sage.algebras.fusion_rings.shm_managers.make_FvarsHandler`(*n*, *field*, *idx_map*, *init_data*)

Provide pickling / unpickling support for `FvarsHandler`.

`sage.algebras.fusion_rings.shm_managers.make_KSHandler`(*n_slots*, *field*, *init_data*)

Provide pickling / unpickling support for `KSHandler`.

5.9 Hall Algebras

AUTHORS:

- Travis Scrimshaw (2013-10-17): Initial version

class `sage.algebras.hall_algebra.HallAlgebra`(*base_ring*, *q*, *prefix='H'*)

Bases: `CombinatorialFreeModule`

The (classical) Hall algebra.

The *(classical) Hall algebra* over a commutative ring R with a parameter $q \in R$ is defined to be the free R-module with basis (I_λ), where λ runs over all integer partitions. The algebra structure is given by a product defined by

$$I_\mu \cdot I_\lambda = \sum_\nu P^\nu_{\mu,\lambda}(q) I_\nu,$$

where $P^\nu_{\mu,\lambda}$ is a Hall polynomial (see `hall_polynomial()`). The unity of this algebra is $I_\emptyset$.

The (classical) Hall algebra is also known as the Hall-Steinitz algebra.

We can define an R-algebra isomorphism Φ from the R-algebra of symmetric functions (see `SymmetricFunctions`) to the (classical) Hall algebra by sending the r-th elementary symmetric function e_r to $q^{r(r-1)/2} I_{(1^r)}$ for every positive integer r. This isomorphism used to transport the Hopf algebra structure from the R-algebra of symmetric functions to the Hall algebra, thus making the latter a connected graded Hopf algebra. If λ is a partition, then the preimage of the basis element I_λ under this isomorphism is $q^{n(\lambda)} P_\lambda(x; q^{-1})$, where P_λ denotes the λ-th Hall-Littlewood P-function, and where $n(\lambda) = \sum_i (i-1)\lambda_i$.

See section 2.3 in [Sch2006], and sections II.2 and III.3 in [Mac1995] (where our I_λ is called u_λ).

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: H = HallAlgebra(R, q)
sage: H[2,1]*H[1,1]
H[3, 2] + (q+1)*H[3, 1, 1] + (q^2+q)*H[2, 2, 1] + (q^4+q^3+q^2)*H[2, 1, 1, 1]
sage: H[2]*H[2,1]
H[4, 1] + q*H[3, 2] + (q^2-1)*H[3, 1, 1] + (q^3+q^2)*H[2, 2, 1]
sage: H[3]*H[1,1]
H[4, 1] + q^2*H[3, 1, 1]
```

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```
sage: H[3]*H[2,1]
H[5, 1] + q*H[4, 2] + (q^2-1)*H[4, 1, 1] + q^3*H[3, 2, 1]
```

We can rewrite the Hall algebra in terms of monomials of the elements $I_{(1^r)}$:

```
sage: I = H.monomial_basis()
sage: H(I[2,1,1])
H[3, 1] + (q+1)*H[2, 2] + (2*q^2+2*q+1)*H[2, 1, 1]
 + (q^5+2*q^4+3*q^3+3*q^2+2*q+1)*H[1, 1, 1, 1]
sage: I(H[2,1,1])
I[3, 1] + (-q^3-q^2-q-1)*I[4]
```

The isomorphism between the Hall algebra and the symmetric functions described above is implemented as a coercion:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: H = HallAlgebra(R, q)
sage: e = SymmetricFunctions(R).e()
sage: e(H[1,1,1])
1/q^3*e[3]
```

We can also do computations with any special value of `q`, such as 0 or 1 or (most commonly) a prime power. Here is an example using a prime:

```
sage: H = HallAlgebra(ZZ, 2)
sage: H[2,1]*H[1,1]
H[3, 2] + 3*H[3, 1, 1] + 6*H[2, 2, 1] + 28*H[2, 1, 1, 1]
sage: H[3,1]*H[2]
H[5, 1] + H[4, 2] + 6*H[3, 3] + 3*H[4, 1, 1] + 8*H[3, 2, 1]
sage: H[2,1,1]*H[3,1]
H[5, 2, 1] + 2*H[4, 3, 1] + 6*H[4, 2, 2] + 7*H[5, 1, 1, 1]
 + 19*H[4, 2, 1, 1] + 24*H[3, 3, 1, 1] + 48*H[3, 2, 2, 1]
 + 105*H[4, 1, 1, 1, 1] + 224*H[3, 2, 1, 1, 1]
sage: I = H.monomial_basis()
sage: H(I[2,1,1])
H[3, 1] + 3*H[2, 2] + 13*H[2, 1, 1] + 105*H[1, 1, 1, 1]
sage: I(H[2,1,1])
I[3, 1] - 15*I[4]
```

If q is set to 1, the coercion to the symmetric functions sends I_λ to m_λ:

```
sage: H = HallAlgebra(QQ, 1)
sage: H[2,1] * H[2,1]
H[4, 2] + 2*H[3, 3] + 2*H[4, 1, 1] + 2*H[3, 2, 1] + 6*H[2, 2, 2] + 4*H[2, 2, 1, 1]
sage: m = SymmetricFunctions(QQ).m()
sage: m[2,1] * m[2,1]
4*m[2, 2, 1, 1] + 6*m[2, 2, 2] + 2*m[3, 2, 1] + 2*m[3, 3] + 2*m[4, 1, 1] + m[4, 2]
sage: m(H[3,1])
m[3, 1]
```

We can set q to 0 (but should keep in mind that we don't get the Schur functions this way):

```
sage: H = HallAlgebra(QQ, 0)
sage: H[2,1] * H[2,1]
H[4, 2] + H[3, 3] + H[4, 1, 1] - H[3, 2, 1] - H[3, 1, 1, 1]
```

class Element

Bases: `IndexedFreeModuleElement`

scalar(y)

Return the scalar product of `self` and `y`.

The scalar product is given by

$$(I_\lambda, I_\mu) = \delta_{\lambda,\mu} \frac{1}{a_\lambda},$$

where a_λ is given by

$$a_\lambda = q^{|\lambda| + 2n(\lambda)} \prod_k \prod_{i=1}^{l_k} (1 - q^{-i})$$

where $n(\lambda) = \sum_i (i - 1)\lambda_i$ and $\lambda = (1^{l_1}, 2^{l_2}, \ldots, m^{l_m})$.

Note that a_λ can be interpreted as the number of automorphisms of a certain object in a category corresponding to λ. See Lemma 2.8 in [Sch2006] for details.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: H = HallAlgebra(R, q)
sage: H[1].scalar(H[1])
1/(q - 1)
sage: H[2].scalar(H[2])
1/(q^2 - q)
sage: H[2,1].scalar(H[2,1])
1/(q^5 - 2*q^4 + q^3)
sage: H[1,1,1,1].scalar(H[1,1,1,1])
1/(q^16 - q^15 - q^14 + 2*q^11 - q^8 - q^7 + q^6)
sage: H.an_element().scalar(H.an_element())
(4*q^2 + 9)/(q^2 - q)
```

antipode_on_basis(la)

Return the antipode of the basis element indexed by `la`.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: H = HallAlgebra(R, q)
sage: H.antipode_on_basis(Partition([1,1]))
1/q*H[2] + 1/q*H[1, 1]
sage: H.antipode_on_basis(Partition([2]))
-1/q*H[2] + ((q^2-1)/q)*H[1, 1]

sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = HallAlgebra(R, q)
```

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```
sage: H.antipode_on_basis(Partition([1,1]))
(q^-1)*H[2] + (q^-1)*H[1, 1]
sage: H.antipode_on_basis(Partition([2]))
-(q^-1)*H[2] - (q^-1-q)*H[1, 1]
```

coproduct_on_basis(*la*)

Return the coproduct of the basis element indexed by `la`.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: H = HallAlgebra(R, q)
sage: H.coproduct_on_basis(Partition([1,1]))
H[] # H[1, 1] + 1/q*H[1] # H[1] + H[1, 1] # H[]
sage: H.coproduct_on_basis(Partition([2]))
H[] # H[2] + ((q-1)/q)*H[1] # H[1] + H[2] # H[]
sage: H.coproduct_on_basis(Partition([2,1]))
H[] # H[2, 1] + ((q^2-1)/q^2)*H[1] # H[1, 1] + 1/q*H[1] # H[2]
 + ((q^2-1)/q^2)*H[1, 1] # H[1] + 1/q*H[2] # H[1] + H[2, 1] # H[]

sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = HallAlgebra(R, q)
sage: H.coproduct_on_basis(Partition([2]))
H[] # H[2] - (q^-1-1)*H[1] # H[1] + H[2] # H[]
sage: H.coproduct_on_basis(Partition([2,1]))
H[] # H[2, 1] - (q^-2-1)*H[1] # H[1, 1] + (q^-1)*H[1] # H[2]
 - (q^-2-1)*H[1, 1] # H[1] + (q^-1)*H[2] # H[1] + H[2, 1] # H[]
```

counit(*x*)

Return the counit of the element `x`.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: H = HallAlgebra(R, q)
sage: H.counit(H.an_element())
2
```

monomial_basis()

Return the basis of the Hall algebra given by monomials in the $I_{(1^r)}$.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: H = HallAlgebra(R, q)
sage: H.monomial_basis()
Hall algebra with q=q over Univariate Polynomial Ring in q over
 Integer Ring in the monomial basis
```

one_basis()

Return the index of the basis element 1.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: H = HallAlgebra(R, q)
sage: H.one_basis()
[]
```

product_on_basis(*mu*, *la*)

Return the product of the two basis elements indexed by `mu` and `la`.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: H = HallAlgebra(R, q)
sage: H.product_on_basis(Partition([1,1]), Partition([1]))
H[2, 1] + (q^2+q+1)*H[1, 1, 1]
sage: H.product_on_basis(Partition([2,1]), Partition([1,1]))
H[3, 2] + (q+1)*H[3, 1, 1] + (q^2+q)*H[2, 2, 1] + (q^4+q^3+q^2)*H[2, 1, 1, 1]
sage: H.product_on_basis(Partition([3,2]), Partition([2,1]))
H[5, 3] + (q+1)*H[4, 4] + q*H[5, 2, 1] + (2*q^2-1)*H[4, 3, 1]
 + (q^3+q^2)*H[4, 2, 2] + (q^4+q^3)*H[3, 3, 2]
 + (q^4-q^2)*H[4, 2, 1, 1] + (q^5+q^4-q^3-q^2)*H[3, 3, 1, 1]
 + (q^6+q^5)*H[3, 2, 2, 1]
sage: H.product_on_basis(Partition([3,1,1]), Partition([2,1]))
H[5, 2, 1] + q*H[4, 3, 1] + (q^2-1)*H[4, 2, 2]
 + (q^3+q^2)*H[3, 3, 2] + (q^2+q+1)*H[5, 1, 1, 1]
 + (2*q^3+q^2-q-1)*H[4, 2, 1, 1] + (q^4+2*q^3+q^2)*H[3, 3, 1, 1]
 + (q^5+q^4)*H[3, 2, 2, 1] + (q^6+q^5+q^4-q^2-q-1)*H[4, 1, 1, 1, 1]
 + (q^7+q^6+q^5)*H[3, 2, 1, 1, 1]
```

class sage.algebras.hall_algebra.**HallAlgebraMonomials**(*base_ring*, *q*, *prefix='I'*)

Bases: `CombinatorialFreeModule`

The classical Hall algebra given in terms of monomials in the $I_{(1^r)}$.

We first associate a monomial $I_{(1^{r_1})} I_{(1^{r_2})} \cdots I_{(1^{r_k})}$ with the composition $(r_1, r_2, \ldots, r_k)$. However since $I_{(1^r)}$ commutes with $I_{(1^s)}$, the basis is indexed by partitions.

EXAMPLES:

We use the fraction field of $\mathbf{Z}[q]$ for our initial example:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: H = HallAlgebra(R, q)
sage: I = H.monomial_basis()
```

We check that the basis conversions are mutually inverse:

```
sage: all(H(I(H[p])) == H[p] for i in range(7) for p in Partitions(i))
True
sage: all(I(H(I[p])) == I[p] for i in range(7) for p in Partitions(i))
True
```

Since Laurent polynomials are sufficient, we run the same check with the Laurent polynomial ring $\mathbf{Z}[q, q^{-1}]$:

```
sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = HallAlgebra(R, q)
sage: I = H.monomial_basis()
sage: all(H(I(H[p])) == H[p] for i in range(6) for p in Partitions(i)) # long time
True
sage: all(I(H(I[p])) == I[p] for i in range(6) for p in Partitions(i)) # long time
True
```

We can also convert to the symmetric functions. The natural basis corresponds to the Hall-Littlewood basis (up to a renormalization and an inversion of the q parameter), and this basis corresponds to the elementary basis (up to a renormalization):

```
sage: Sym = SymmetricFunctions(R)
sage: e = Sym.e()
sage: e(I[2,1])
(q^-1)*e[2, 1]
sage: e(I[4,2,2,1])
(q^-8)*e[4, 2, 2, 1]
sage: HLP = Sym.hall_littlewood(q).P()
sage: H(I[2,1])
H[2, 1] + (1+q+q^2)*H[1, 1, 1]
sage: HLP(e[2,1])
(1+q+q^2)*HLP[1, 1, 1] + HLP[2, 1]
sage: all( e(H[lam]) == q**-sum([i * x for i, x in enumerate(lam)])
....:          * e(HLP[lam]).map_coefficients(lambda p: p(q**(-1)))
....:      for lam in Partitions(4) )
True
```

We can also do computations using a prime power:

```
sage: H = HallAlgebra(ZZ, 3)
sage: I = H.monomial_basis()
sage: i_elt = I[2,1]*I[1,1]; i_elt
I[2, 1, 1, 1]
sage: H(i_elt)
H[4, 1] + 7*H[3, 2] + 37*H[3, 1, 1] + 136*H[2, 2, 1]
 + 1495*H[2, 1, 1, 1] + 62920*H[1, 1, 1, 1, 1]
```

class Element

Bases: `IndexedFreeModuleElement`

scalar(y)

Return the scalar product of `self` and `y`.

The scalar product is computed by converting into the natural basis.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I[1].scalar(I[1])
1/(q - 1)
sage: I[2].scalar(I[2])
1/(q^4 - q^3 - q^2 + q)
```

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```
sage: I[2,1].scalar(I[2,1])
(2*q + 1)/(q^6 - 2*q^5 + 2*q^3 - q^2)
sage: I[1,1,1,1].scalar(I[1,1,1,1])
24/(q^4 - 4*q^3 + 6*q^2 - 4*q + 1)
sage: I.an_element().scalar(I.an_element())
(4*q^4 - 4*q^2 + 9)/(q^4 - q^3 - q^2 + q)
```

antipode_on_basis(*a*)

Return the antipode of the basis element indexed by a.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.antipode_on_basis(Partition([1]))
-I[1]
sage: I.antipode_on_basis(Partition([2]))
1/q*I[1, 1] - I[2]
sage: I.antipode_on_basis(Partition([2,1]))
-1/q*I[1, 1, 1] + I[2, 1]

sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.antipode_on_basis(Partition([2,1]))
-(q^-1)*I[1, 1, 1] + I[2, 1]
```

coproduct_on_basis(*a*)

Return the coproduct of the basis element indexed by a.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.coproduct_on_basis(Partition([1]))
I[] # I[1] + I[1] # I[]
sage: I.coproduct_on_basis(Partition([2]))
I[] # I[2] + 1/q*I[1] # I[1] + I[2] # I[]
sage: I.coproduct_on_basis(Partition([2,1]))
I[] # I[2, 1] + 1/q*I[1] # I[1, 1] + I[1] # I[2]
 + 1/q*I[1, 1] # I[1] + I[2] # I[1] + I[2, 1] # I[]

sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.coproduct_on_basis(Partition([2,1]))
I[] # I[2, 1] + (q^-1)*I[1] # I[1, 1] + I[1] # I[2]
 + (q^-1)*I[1, 1] # I[1] + I[2] # I[1] + I[2, 1] # I[]
```

counit(*x*)

Return the counit of the element x.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 'q').fraction_field()
sage: q = R.gen()
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.counit(I.an_element())
2
```

one_basis()

Return the index of the basis element 1.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.one_basis()
[]
```

product_on_basis(a, b)

Return the product of the two basis elements indexed by a and b.

EXAMPLES:

```
sage: R.<q> = ZZ[]
sage: I = HallAlgebra(R, q).monomial_basis()
sage: I.product_on_basis(Partition([4,2,1]), Partition([3,2,1]))
I[4, 3, 2, 2, 1, 1]
```

sage.algebras.hall_algebra.**transpose_cmp**(x, y)

Compare partitions x and y in transpose dominance order.

We say partitions μ and λ satisfy $\mu \prec \lambda$ in transpose dominance order if for all $i \geq 1$ we have:

$$l_1 + 2l_2 + \cdots + (i-1)l_{i-1} + i(l_i + l_{i+1} + \cdots) \leq m_1 + 2m_2 + \cdots + (i-1)m_{i-1} + i(m_i + m_{i+1} + \cdots),$$

where l_k denotes the number of appearances of k in λ, and m_k denotes the number of appearances of k in μ.

Equivalently, $\mu \prec \lambda$ if the conjugate of the partition μ dominates the conjugate of the partition λ.

Since this is a partial ordering, we fallback to lex ordering $\mu <_L \lambda$ if we cannot compare in the transpose order.

EXAMPLES:

```
sage: from sage.algebras.hall_algebra import transpose_cmp
sage: transpose_cmp(Partition([4,3,1]), Partition([3,2,2,1]))
-1
sage: transpose_cmp(Partition([2,2,1]), Partition([3,2]))
1
sage: transpose_cmp(Partition([4,1,1]), Partition([4,1,1]))
0
```

5.10 Incidence Algebras

class sage.combinat.posets.incidence_algebras.**IncidenceAlgebra**(*R*, *P*, *prefix='I'*)

Bases: CombinatorialFreeModule

The incidence algebra of a poset.

Let P be a poset and R be a commutative unital associative ring. The *incidence algebra* I_P is the algebra of functions $\alpha\colon P \times P \to R$ such that $\alpha(x, y) = 0$ if $x \not\leq y$ where multiplication is given by convolution:

$$(\alpha * \beta)(x, y) = \sum_{x \leq k \leq y} \alpha(x, k)\beta(k, y).$$

This has a natural basis given by indicator functions for the interval $[a, b]$, i.e. $X_{a,b}(x, y) = \delta_{ax}\delta_{by}$. The incidence algebra is a unital algebra with the identity given by the Kronecker delta $\delta(x, y) = \delta_{xy}$. The Möbius function of P is another element of I_p whose inverse is the ζ function of the poset (so $\zeta(x, y) = 1$ for every interval $[x, y]$).

Todo: Implement the incidence coalgebra.

REFERENCES:

- Wikipedia article Incidence_algebra

class Element

Bases: IndexedFreeModuleElement

An element of an incidence algebra.

is_unit()

Return if `self` is a unit.

EXAMPLES:

```
sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: mu = I.moebius()
sage: mu.is_unit()
True
sage: zeta = I.zeta()
sage: zeta.is_unit()
True
sage: x = mu - I.zeta() + I[2,2]
sage: x.is_unit()
False
sage: y = I.moebius() + I.zeta()
sage: y.is_unit()
True
```

This depends on the base ring:

```
sage: I = P.incidence_algebra(ZZ)
sage: y = I.moebius() + I.zeta()
sage: y.is_unit()
False
```

to_matrix()

Return `self` as a matrix.

We define a matrix $M_{xy} = \alpha(x, y)$ for some element $\alpha \in I_P$ in the incidence algebra I_P and we order the elements $x, y \in P$ by some linear extension of P. This defines an algebra (iso)morphism; in particular, multiplication in the incidence algebra goes to matrix multiplication.

EXAMPLES:

```
sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: I.moebius().to_matrix()
[ 1 -1 -1  1]
[ 0  1  0 -1]
[ 0  0  1 -1]
[ 0  0  0  1]
sage: I.zeta().to_matrix()
[1 1 1 1]
[0 1 0 1]
[0 0 1 1]
[0 0 0 1]
```

delta()

Return the element 1 in `self` (which is the Kronecker delta $\delta(x, y)$).

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.one()
I[0, 0] + I[1, 1] + I[2, 2] + I[3, 3] + I[4, 4] + I[5, 5]
 + I[6, 6] + I[7, 7] + I[8, 8] + I[9, 9] + I[10, 10]
 + I[11, 11] + I[12, 12] + I[13, 13] + I[14, 14] + I[15, 15]
```

moebius()

Return the Möbius function of `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: I.moebius()
I[0, 0] - I[0, 1] - I[0, 2] + I[0, 3] + I[1, 1]
 - I[1, 3] + I[2, 2] - I[2, 3] + I[3, 3]
```

one()

Return the element 1 in `self` (which is the Kronecker delta $\delta(x, y)$).

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.one()
I[0, 0] + I[1, 1] + I[2, 2] + I[3, 3] + I[4, 4] + I[5, 5]
 + I[6, 6] + I[7, 7] + I[8, 8] + I[9, 9] + I[10, 10]
 + I[11, 11] + I[12, 12] + I[13, 13] + I[14, 14] + I[15, 15]
```

poset()

Return the defining poset of `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.poset()
Finite lattice containing 16 elements
sage: I.poset() == P
True
```

product_on_basis(*A*, *B*)

Return the product of basis elements indexed by `A` and `B`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.product_on_basis((1, 3), (3, 11))
I[1, 11]
sage: I.product_on_basis((1, 3), (2, 2))
0
```

reduced_subalgebra(*prefix='R'*)

Return the reduced incidence subalgebra.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.reduced_subalgebra()
Reduced incidence algebra of Finite lattice containing 16 elements
 over Rational Field
```

some_elements()

Return a list of elements of `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(1)
sage: I = P.incidence_algebra(QQ)
sage: Ielts = I.some_elements(); Ielts # random
[2*I[0, 0] + 2*I[0, 1] + 3*I[1, 1],
 I[0, 0] - I[0, 1] + I[1, 1],
 I[0, 0] + I[0, 1] + I[1, 1]]
sage: [a in I for a in Ielts]
[True, True, True]
```

zeta()

Return the ζ function in `self`.

The ζ function on a poset P is given by

$$\zeta(x, y) = \begin{cases} 1 & x \leq y, \\ 0 & x \not\leq y. \end{cases}$$

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.zeta() * I.moebius() == I.one()
True
```

class sage.combinat.posets.incidence_algebras.**ReducedIncidenceAlgebra**(*I*, *prefix='R'*)

Bases: CombinatorialFreeModule

The reduced incidence algebra of a poset.

The reduced incidence algebra R_P is a subalgebra of the incidence algebra I_P where $\alpha(x, y) = \alpha(x', y')$ when $[x, y]$ is isomorphic to $[x', y']$ as posets. Thus the delta, Möbius, and zeta functions are all elements of R_P.

class Element

Bases: IndexedFreeModuleElement

An element of a reduced incidence algebra.

is_unit()

Return if `self` is a unit.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: x = R.an_element()
sage: x.is_unit()
True
```

lift()

Return the lift of `self` to the ambient space.

EXAMPLES:

```
sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: R = I.reduced_subalgebra()
sage: x = R.an_element(); x
2*R[(0, 0)] + 2*R[(0, 1)] + 3*R[(0, 3)]
sage: x.lift()
2*I[0, 0] + 2*I[0, 1] + 2*I[0, 2] + 3*I[0, 3] + 2*I[1, 1]
 + 2*I[1, 3] + 2*I[2, 2] + 2*I[2, 3] + 2*I[3, 3]
```

to_matrix()

Return `self` as a matrix.

EXAMPLES:

```
sage: P = posets.BooleanLattice(2)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: mu = R.moebius()
sage: mu.to_matrix()
[ 1 -1 -1  1]
[ 0  1  0 -1]
```

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```
[ 0  0  1 -1]
[ 0  0  0  1]
```

delta()

Return the Kronecker delta function in `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.delta()
R[(0, 0)]
```

lift()

Return the lift morphism from `self` to the ambient space.

EXAMPLES:

```
sage: P = posets.BooleanLattice(2)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.lift
Generic morphism:
  From: Reduced incidence algebra of Finite lattice containing 4 elements over␣
↪Rational Field
  To:   Incidence algebra of Finite lattice containing 4 elements over Rational␣
↪Field
sage: R.an_element() - R.one()
R[(0, 0)] + 2*R[(0, 1)] + 3*R[(0, 3)]
sage: R.lift(R.an_element() - R.one())
I[0, 0] + 2*I[0, 1] + 2*I[0, 2] + 3*I[0, 3] + I[1, 1]
 + 2*I[1, 3] + I[2, 2] + 2*I[2, 3] + I[3, 3]
```

moebius()

Return the Möbius function of `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.moebius()
R[(0, 0)] - R[(0, 1)] + R[(0, 3)] - R[(0, 7)] + R[(0, 15)]
```

one_basis()

Return the index of the element 1 in `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.one_basis()
(0, 0)
```

poset()

Return the defining poset of `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.poset()
Finite lattice containing 16 elements
sage: R.poset() == P
True
```

some_elements()

Return a list of elements of `self`.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.some_elements()
[2*R[(0, 0)] + 2*R[(0, 1)] + 3*R[(0, 3)],
 R[(0, 0)] - R[(0, 1)] + R[(0, 3)] - R[(0, 7)] + R[(0, 15)],
 R[(0, 0)] + R[(0, 1)] + R[(0, 3)] + R[(0, 7)] + R[(0, 15)]]
```

zeta()

Return the ζ function in `self`.

The ζ function on a poset P is given by

$$\zeta(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & x \not\leq y. \end{cases}$$

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.zeta()
R[(0, 0)] + R[(0, 1)] + R[(0, 3)] + R[(0, 7)] + R[(0, 15)]
```

5.11 Group algebras

This functionality has been moved to `sage.categories.algebra_functor`.

`sage.algebras.group_algebra.`**GroupAlgebra**(*G*, *R=Integer Ring*)

Return the group algebra of G over R.

INPUT:

- G – a group
- R – (default: $\mathbf{Z}$) a ring

EXAMPLES:

The *group algebra* $A = RG$ is the space of formal linear combinations of elements of G with coefficients in R:

```
sage: G = DihedralGroup(3)
sage: R = QQ
sage: A = GroupAlgebra(G, R); A
Algebra of Dihedral group of order 6 as a permutation group over Rational Field
sage: a = A.an_element(); a
() + (1,2) + 3*(1,2,3) + 2*(1,3,2)
```

This space is endowed with an algebra structure, obtained by extending by bilinearity the multiplication of G to a multiplication on RG:

```
sage: A in Algebras
True
sage: a * a
14*() + 5*(2,3) + 2*(1,2) + 10*(1,2,3) + 13*(1,3,2) + 5*(1,3)
```

`GroupAlgebra()` is just a short hand for a more general construction that covers, e.g., monoid algebras, additive group algebras and so on:

```
sage: G.algebra(QQ)
Algebra of Dihedral group of order 6 as a permutation group over Rational Field

sage: GroupAlgebra(G,QQ) is G.algebra(QQ)
True

sage: M = Monoids().example(); M
An example of a monoid:
the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.algebra(QQ)
Algebra of An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
        over Rational Field
```

See the documentation of `sage.categories.algebra_functor` for details.

class `sage.algebras.group_algebra.`**`GroupAlgebra_class`**(*R*, *basis_keys=None*, *element_class=None*, *category=None*, *prefix=None*, *names=None*, ***kwds*)

Bases: `CombinatorialFreeModule`

5.12 Grossman-Larson Hopf Algebras

AUTHORS:

- Frédéric Chapoton (2017)

class `sage.combinat.grossman_larson_algebras.`**`GrossmanLarsonAlgebra`**(*R*, *names=None*)

Bases: `CombinatorialFreeModule`

The Grossman-Larson Hopf Algebra.

The Grossman-Larson Hopf Algebras are Hopf algebras with a basis indexed by forests of decorated rooted trees. They are the universal enveloping algebras of free pre-Lie algebras, seen as Lie algebras.

The Grossman-Larson Hopf algebra on a given set E has an explicit description using rooted forests. The underlying vector space has a basis indexed by finite rooted forests endowed with a map from their vertices to E (called the "labeling"). In this basis, the product of two (decorated) rooted forests $S * T$ is a sum over all maps from the set of roots of T to the union of a singleton $\{\#\}$ and the set of vertices of S. Given such a map, one defines a new forest as follows. Starting from the disjoint union of all rooted trees of S and T, one adds an edge from every root of T to its image when this image is not the fake vertex labelled #. The coproduct sends a rooted forest T to the sum of all tensors $T_1 \otimes T_2$ obtained by splitting the connected components of T into two subsets and letting T_1 be the forest formed by the first subset and T_2 the forest formed by the second. This yields a connected graded Hopf algebra (the degree of a forest is its number of vertices).

See [Pana2002] (Section 2) and [GroLar1]. (Note that both references use rooted trees rather than rooted forests, so think of each rooted forest grafted onto a new root. Also, the product is reversed, so they are defining the opposite algebra structure.)

> **Warning:** For technical reasons, instead of using forests as labels for the basis, we use rooted trees. Their root vertex should be considered as a fake vertex. This fake root vertex is labelled `'#'` when labels are present.

EXAMPLES:

```
sage: G = algebras.GrossmanLarson(QQ, 'xy')
sage: x, y = G.single_vertex_all()
sage: ascii_art(x*y)
B  + B
 #      #_
 |     / /
 x    x y
 |
 y

sage: ascii_art(x*x*x)
B  + B      + 3*B     + B
 #      #         #_      _#__
 |      |        / /     / / /
 x      x_      x x     x x x
 |     / /        |
 x    x x         x
 |
 x
```

The Grossman-Larson algebra is associative:

```
sage: z = x * y
sage: x * (y * z) == (x * y) * z
True
```

It is not commutative:

```
sage: x * y == y * x
False
```

When `None` is given as input, unlabelled forests are used instead; this corresponds to a 1-element set E:

```
sage: G = algebras.GrossmanLarson(QQ, None)
sage: x = G.single_vertex_all()[0]
sage: ascii_art(x*x)
B  + B
```

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```
o      o_
|     / /
o    o o
|
o
```

Note: Variables names can be `None`, a list of strings, a string or an integer. When `None` is given, unlabelled rooted forests are used. When a single string is given, each letter is taken as a variable. See `sage.combinat.words.alphabet.build_alphabet()`.

Warning: Beware that the underlying combinatorial free module is based either on `RootedTrees` or on `LabelledRootedTrees`, with no restriction on the labellings. This means that all code calling the `basis()` method would not give meaningful results, since `basis()` returns many "chaff" elements that do not belong to the algebra.

REFERENCES:

- [Pana2002]
- [GroLar1]

an_element()

Return an element of `self`.

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(QQ, 'xy')
sage: A.an_element()
B[#[x[]]] + 2*B[#[x[x[]]]] + 2*B[#[x[], x[]]]
```

antipode_on_basis(x)

Return the antipode of a forest.

EXAMPLES:

```
sage: G = algebras.GrossmanLarson(QQ,2)
sage: x, y = G.single_vertex_all()
sage: G.antipode(x)  # indirect doctest
-B[#[0[]]]

sage: G.antipode(y*x)  # indirect doctest
B[#[0[1[]]]] + B[#[0[], 1[]]]
```

change_ring(R)

Return the Grossman-Larson algebra in the same variables over R.

INPUT:

- R – a ring

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(ZZ, 'fgh')
sage: A.change_ring(QQ)
Grossman-Larson Hopf algebra on 3 generators ['f', 'g', 'h']
over Rational Field
```

coproduct_on_basis(*x*)

Return the coproduct of a forest.

EXAMPLES:

```
sage: G = algebras.GrossmanLarson(QQ,2)
sage: x, y = G.single_vertex_all()
sage: ascii_art(G.coproduct(x))  # indirect doctest
1 # B  + B  # 1
     #    #
     |    |
     0    0

sage: Delta_xy = G.coproduct(y*x)
sage: ascii_art(Delta_xy)  # random indirect doctest
1 # B     + 1 # B  + B  # B  + B     # 1 + B  # B  + B  # 1
       #_        #    #    #      #_        #    #    #
      / /        |    |    |     / /        |    |    |
     0 1         1    0    1    0 1         1    0    1
                 |                                    |
                 0                                    0
```

counit_on_basis(*x*)

Return the counit on a basis element.

This is zero unless the forest x is empty.

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(QQ, 'xy')
sage: RT = A.basis().keys()
sage: x = RT([RT([],'x')],'#')
sage: A.counit_on_basis(x)
0
sage: A.counit_on_basis(RT([],'#'))
1
```

degree_on_basis(*t*)

Return the degree of a rooted forest in the Grossman-Larson algebra.

This is the total number of vertices of the forest.

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(QQ, '@')
sage: RT = A.basis().keys()
sage: A.degree_on_basis(RT([RT([])]))
1
```

one_basis()

Return the empty rooted forest.

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(QQ, 'ab')
sage: A.one_basis()
#[]

sage: A = algebras.GrossmanLarson(QQ, None)
sage: A.one_basis()
[]
```

product_on_basis(x, y)

Return the product of two forests x and y.

This is the sum over all possible ways for the components of the forest y to either fall side-by-side with components of x or be grafted on a vertex of x.

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.product_on_basis(x, x)
B[[[[]]]] + B[[[], []]]
```

Check that the product is the correct one:

```
sage: A = algebras.GrossmanLarson(QQ, 'uv')
sage: RT = A.basis().keys()
sage: Tu = RT([RT([],'u')],'#')
sage: Tv = RT([RT([],'v')],'#')
sage: A.product_on_basis(Tu, Tv)
B[#[u[v[]]]] + B[#[u[], v[]]]
```

single_vertex(i)

Return the `i`-th rooted forest with one vertex.

This is the rooted forest with just one vertex, labelled by the `i`-th element of the label list.

See also:

single_vertex_all().

INPUT:

- `i` – a nonnegative integer

EXAMPLES:

```
sage: F = algebras.GrossmanLarson(ZZ, 'xyz')
sage: F.single_vertex(0)
B[#[x[]]]

sage: F.single_vertex(4)
Traceback (most recent call last):
...
IndexError: argument i (= 4) must be between 0 and 2
```

`single_vertex_all()`

Return the rooted forests with one vertex in `self`.

They freely generate the Lie algebra of primitive elements as a pre-Lie algebra.

See also:

single_vertex().

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(ZZ, 'fgh')
sage: A.single_vertex_all()
(B[#[f[]]], B[#[g[]]], B[#[h[]]])

sage: A = algebras.GrossmanLarson(QQ, ['x1','x2'])
sage: A.single_vertex_all()
(B[#[x1[]]], B[#[x2[]]])

sage: A = algebras.GrossmanLarson(ZZ, None)
sage: A.single_vertex_all()
(B[[[]]],)
```

`some_elements()`

Return some elements of the Grossman-Larson Hopf algebra.

EXAMPLES:

```
sage: A = algebras.GrossmanLarson(QQ, None)
sage: A.some_elements()
[B[[[]]], B[[]] + B[[[[]]]] + B[[[], []]],
4*B[[[[]]]] + 4*B[[[], []]]]
```

With several generators:

```
sage: A = algebras.GrossmanLarson(QQ, 'xy')
sage: A.some_elements()
[B[#[x[]]],
 B[#[]] + B[#[x[x[]]]] + B[#[x[], x[]]],
 B[#[x[x[]]]] + 3*B[#[x[y[]]]] + B[#[x[], x[]]] + 3*B[#[x[], y[]]]]
```

`variable_names()`

Return the names of the variables.

This returns the set E (as a family).

EXAMPLES:

```
sage: R = algebras.GrossmanLarson(QQ, 'xy')
sage: R.variable_names()
{'x', 'y'}

sage: R = algebras.GrossmanLarson(QQ, ['a','b'])
sage: R.variable_names()
{'a', 'b'}

sage: R = algebras.GrossmanLarson(QQ, 2)
```

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```
sage: R.variable_names()
{0, 1}

sage: R = algebras.GrossmanLarson(QQ, None)
sage: R.variable_names()
{'o'}
```

5.13 Möbius Algebras

class sage.combinat.posets.moebius_algebra.**BasisAbstract**(*R*, *basis_keys=None*, *element_class=None*, *category=None*, *prefix=None*, *names=None*, ***kwds*)

Bases: `CombinatorialFreeModule`, `BindableClass`

Abstract base class for a basis.

class sage.combinat.posets.moebius_algebra.**MoebiusAlgebra**(*R*, *L*)

Bases: `Parent`, `UniqueRepresentation`

The Möbius algebra of a lattice.

Let L be a lattice. The *Möbius algebra* M_L was originally constructed by Solomon [Solomon67] and has a natural basis $\{E_x \mid x \in L\}$ with multiplication given by $E_x \cdot E_y = E_{x \vee y}$. Moreover this has a basis given by orthogonal idempotents $\{I_x \mid x \in L\}$ (so $I_x I_y = \delta_{xy} I_x$ where δ is the Kronecker delta) related to the natural basis by

$$I_x = \sum_{x \leq y} \mu_L(x, y) E_y,$$

where μ_L is the Möbius function of L.

Note: We use the join $\vee$ for our multiplication, whereas [Greene73] and [Etienne98] define the Möbius algebra using the meet $\wedge$. This is done for compatibility with *`QuantumMoebiusAlgebra`*.

REFERENCES:

class E(*M*, *prefix='E'*)

Bases: *`BasisAbstract`*

The natural basis of a Möbius algebra.

Let E_x and E_y be basis elements of M_L for some lattice L. Multiplication is given by $E_x E_y = E_{x \vee y}$.

one()

Return the element 1 of `self`.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: E = L.moebius_algebra(QQ).E()
sage: E.one()
E[0]
```

product_on_basis(*x*, *y*)

Return the product of basis elements indexed by x and y.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: E = L.moebius_algebra(QQ).E()
sage: E.product_on_basis(5, 14)
E[15]
sage: E.product_on_basis(2, 8)
E[10]
```

class I(*M*, *prefix='I'*)

Bases: BasisAbstract

The (orthogonal) idempotent basis of a Möbius algebra.

Let I_x and I_y be basis elements of M_L for some lattice L. Multiplication is given by $I_x I_y = \delta_{xy} I_x$ where δ_{xy} is the Kronecker delta.

one()

Return the element 1 of self.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: I = L.moebius_algebra(QQ).I()
sage: I.one()
I[0] + I[1] + I[2] + I[3] + I[4] + I[5] + I[6] + I[7] + I[8]
 + I[9] + I[10] + I[11] + I[12] + I[13] + I[14] + I[15]
```

product_on_basis(*x*, *y*)

Return the product of basis elements indexed by x and y.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: I = L.moebius_algebra(QQ).I()
sage: I.product_on_basis(5, 14)
0
sage: I.product_on_basis(2, 2)
I[2]
```

a_realization()

Return a particular realization of self (the B-basis).

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: M = L.moebius_algebra(QQ)
sage: M.a_realization()
Moebius algebra of Finite lattice containing 16 elements
 over Rational Field in the natural basis
```

idempotent

alias of I

lattice()

Return the defining lattice of `self`.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: M = L.moebius_algebra(QQ)
sage: M.lattice()
Finite lattice containing 16 elements
sage: M.lattice() == L
True
```

natural

alias of *E*

class sage.combinat.posets.moebius_algebra.**MoebiusAlgebraBases**(*parent_with_realization*)

Bases: `Category_realization_of_parent`

The category of bases of a Möbius algebra.

INPUT:

- `base` – a Möbius algebra

class **ElementMethods**

Bases: `object`

class **ParentMethods**

Bases: `object`

one()

Return the element 1 of `self`.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: C = L.quantum_moebius_algebra().C()
sage: all(C.one() * b == b for b in C.basis())
True
```

product_on_basis(*x*, *y*)

Return the product of basis elements indexed by `x` and `y`.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: C = L.quantum_moebius_algebra().C()
sage: C.product_on_basis(5, 14)
q^3*C[15]
sage: C.product_on_basis(2, 8)
q^4*C[10]
```

super_categories()

The super categories of `self`.

EXAMPLES:

```
sage: from sage.combinat.posets.moebius_algebra import MoebiusAlgebraBases
sage: M = posets.BooleanLattice(4).moebius_algebra(QQ)
sage: bases = MoebiusAlgebraBases(M)
sage: bases.super_categories()
[Category of finite dimensional commutative algebras with basis over Rational␣
↪Field,
 Category of realizations of Moebius algebra of Finite lattice
    containing 16 elements over Rational Field]
```

class sage.combinat.posets.moebius_algebra.**QuantumMoebiusAlgebra**(*L*, *q=None*)

Bases: Parent, UniqueRepresentation

The quantum Möbius algebra of a lattice.

Let L be a lattice, and we define the *quantum Möbius algebra* $M_L(q)$ as the algebra with basis $\{E_x \mid x \in L\}$ with multiplication given by

$$E_x E_y = \sum_{z \geq a \geq x \vee y} \mu_L(a, z) q^{\operatorname{crk} a} E_z,$$

where μ_L is the Möbius function of L and crk is the corank function (i.e., $\operatorname{crk} a = \operatorname{rank} L - \operatorname{rank} \mathrm{a}$). At $q = 1$, this reduces to the multiplication formula originally given by Solomon.

class **C**(*M*, *prefix='C'*)

Bases: *BasisAbstract*

The characteristic basis of a quantum Möbius algebra.

The characteristic basis $\{C_x \mid x \in L\}$ of M_L for some lattice L is defined by

$$C_x = \sum_{a \geq x} P(F^x; q) E_a,$$

where $F^x = \{y \in L \mid y \geq x\}$ is the principal order filter of x and $P(F^x; q)$ is the characteristic polynomial of the (sub)poset F^x.

class **E**(*M*, *prefix='E'*)

Bases: *BasisAbstract*

The natural basis of a quantum Möbius algebra.

Let E_x and E_y be basis elements of M_L for some lattice L. Multiplication is given by

$$E_x E_y = \sum_{z \geq a \geq x \vee y} \mu_L(a, z) q^{\operatorname{crk} a} E_z,$$

where μ_L is the Möbius function of L and crk is the corank function (i.e., $\operatorname{crk} a = \operatorname{rank} L - \operatorname{rank} \mathrm{a}$).

one()

Return the element 1 of self.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: E = L.quantum_moebius_algebra().E()
sage: all(E.one() * b == b for b in E.basis())
True
```

product_on_basis(*x*, *y*)

Return the product of basis elements indexed by x and y.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: E = L.quantum_moebius_algebra().E()
sage: E.product_on_basis(5, 14)
E[15]
sage: E.product_on_basis(2, 8)
q^2*E[10] + (q-q^2)*E[11] + (q-q^2)*E[14] + (1-2*q+q^2)*E[15]
```

class KL(*M*, *prefix='KL'*)

Bases: *BasisAbstract*

The Kazhdan-Lusztig basis of a quantum Möbius algebra.

The Kazhdan-Lusztig basis $\{B_x \mid x \in L\}$ of M_L for some lattice L is defined by

$$B_x = \sum_{y \geq x} P_{x,y}(q) E_a,$$

where $P_{x,y}(q)$ is the Kazhdan-Lusztig polynomial of L, following the definition given in [EPW14].

EXAMPLES:

We construct some examples of Proposition 4.5 of [EPW14]:

```
sage: M = posets.BooleanLattice(4).quantum_moebius_algebra()
sage: KL = M.KL()
sage: KL[4] * KL[5]
(q^2+q^3)*KL[5] + (q+2*q^2+q^3)*KL[7] + (q+2*q^2+q^3)*KL[13]
 + (1+3*q+3*q^2+q^3)*KL[15]
sage: KL[4] * KL[15]
(1+3*q+3*q^2+q^3)*KL[15]
sage: KL[4] * KL[10]
(q+3*q^2+3*q^3+q^4)*KL[14] + (1+4*q+6*q^2+4*q^3+q^4)*KL[15]
```

a_realization()

Return a particular realization of self (the B-basis).

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: M = L.quantum_moebius_algebra()
sage: M.a_realization()
Quantum Moebius algebra of Finite lattice containing 16 elements
 with q=q over Univariate Laurent Polynomial Ring in q
 over Integer Ring in the natural basis
```

characteristic_basis

alias of *C*

kazhdan_lusztig

alias of *KL*

lattice()

Return the defining lattice of `self`.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: M = L.quantum_moebius_algebra()
sage: M.lattice()
Finite lattice containing 16 elements
sage: M.lattice() == L
True
```

natural

alias of `E`

5.14 Orlik-Terao Algebras

class `sage.algebras.orlik_terao.OrlikTeraoAlgebra`(*R*, *M*, *ordering=None*)

Bases: `CombinatorialFreeModule`

An Orlik-Terao algebra.

Let R be a commutative ring. Let M be a matroid with ground set X with some fixed ordering and representation $A = (a_x)_{x \in X}$ (so a_x is a (column) vector). Let $C(M)$ denote the set of circuits of M. Let P denote the quotient algebra $R[e_x \mid x \in X]/\langle e_x^2 \rangle$, i.e., the polynomial algebra with squares being zero. The *Orlik-Terao ideal* $J(M)$ is the ideal of P generated by

$$\partial e_S := \sum_{i=1}^{t} (-1)^i \chi(S \setminus \{j_i\}) e_{S \setminus \{j_i\}}$$

for all $S = \{j_1 < j_2 < \cdots < j_t\} \in C(M)$, where $\chi(T)$ is defined as follows. If T is linearly dependent, then $\chi(T) = 0$. Otherwise, let $T = \{x_1 < \cdots < x_{|T|}\}$, and for every flat F of M, choose a basis Θ_F. Then define $\chi(T) = \det(b_1, \ldots, b_{|T|})$, where b_i is a_{x_i} expressed in the basis Θ_F.

It is easy to see that $\partial e_S \in J(M)$ not only for circuits S, but also for any dependent set S of M. Moreover, every dependent set S of M satisfies $e_S \in J(M)$.

The *Orlik-Terao algebra* $A(M)$ is the quotient $E/J(M)$. This is a graded finite-dimensional commutative R-algebra. The non-broken circuit (NBC) sets of M (that is, the subsets of X containing `no broken circuit` of M) form a basis of $A(M)$. (Recall that a `broken circuit` of M is defined to be the result of removing the smallest element from a circuit of M.)

In the current implementation, the basis of $A(M)$ is indexed by the NBC sets, which are implemented as frozensets.

INPUT:

- `R` – the base ring
- `M` – the defining matroid
- `ordering` – (optional) an ordering of the ground set

EXAMPLES:

We create the Orlik-Terao algebra of the wheel matroid $W(3)$ and do some basic computations:

```
sage: M = matroids.Wheel(3)
sage: OT = M.orlik_terao_algebra(QQ)
sage: OT.dimension()
24
sage: G = OT.algebra_generators()
sage: sorted(map(sorted, M.broken_circuits()))
[[1, 3], [1, 4, 5], [2, 3, 4], [2, 3, 5], [2, 4], [2, 5], [4, 5]]
sage: G[1] * G[2] * G[3]
OT{0, 1, 2} + OT{0, 2, 3}
sage: G[1] * G[4] * G[5]
-OT{0, 1, 4} - OT{0, 1, 5} - OT{0, 3, 4} - OT{0, 3, 5}
```

We create an example of a linear matroid and do a basic computation:

```
sage: R = ZZ['t'].fraction_field()
sage: t = R.gen()
sage: mat = matrix(R, [[1-3*t/(t+2), t, 5], [-2, 1, 3/(7-t)]])
sage: M = Matroid(mat)
sage: OT = M.orlik_terao_algebra()
sage: G = OT.algebra_generators()
sage: G[1] * G[2]
((2*t^3-12*t^2-12*t-14)/(8*t^2-19*t-70))*OT{0, 1}
 + ((10*t^2-44*t-146)/(-8*t^2+19*t+70))*OT{0, 2}
```

REFERENCES:

- [OT1994]
- [FL2001]
- [CF2005]

algebra_generators()

Return the algebra generators of `self`.

These form a family indexed by the ground set X of M. For each $x \in X$, the x-th element is e_x.

EXAMPLES:

```
sage: M = matroids.Whirl(2)
sage: OT = M.orlik_terao_algebra()
sage: OT.algebra_generators()
Finite family {0: OT{0}, 1: OT{1}, 2: OT{2}, 3: OT{3}}

sage: M = matroids.named_matroids.Fano()
sage: OT = M.orlik_terao_algebra()
sage: OT.algebra_generators()
Finite family {'a': OT{a}, 'b': OT{b}, 'c': OT{c}, 'd': OT{d},
               'e': OT{e}, 'f': OT{f}, 'g': OT{g}}

sage: M = matroids.named_matroids.NonFano()
sage: OT = M.orlik_terao_algebra(GF(3)['t'])
sage: OT.algebra_generators()
Finite family {'a': OT{a}, 'b': OT{b}, 'c': OT{c}, 'd': OT{d},
               'e': OT{e}, 'f': OT{f}, 'g': OT{g}}
```

degree_on_basis(m)

Return the degree of the basis element indexed by m.

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OT = M.orlik_terao_algebra(QQ)
sage: OT.degree_on_basis(frozenset([1]))
1
sage: OT.degree_on_basis(frozenset([0, 2, 3]))
3
```

one_basis()

Return the index of the basis element corresponding to 1 in self.

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OT = M.orlik_terao_algebra(QQ)
sage: OT.one_basis() == frozenset([])
True
```

product_on_basis(a, b)

Return the product in self of the basis elements indexed by a and b.

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OT = M.orlik_terao_algebra(QQ)
sage: OT.product_on_basis(frozenset([2]), frozenset([3,4]))
OT{0, 1, 2} + OT{0, 1, 4} + OT{0, 2, 3} + OT{0, 3, 4}
```

```
sage: G = OT.algebra_generators()
sage: prod(G)
0
sage: G[2] * G[4]
OT{1, 2} + OT{1, 4}
sage: G[3] * G[4] * G[2]
OT{0, 1, 2} + OT{0, 1, 4} + OT{0, 2, 3} + OT{0, 3, 4}
sage: G[2] * G[3] * G[4]
OT{0, 1, 2} + OT{0, 1, 4} + OT{0, 2, 3} + OT{0, 3, 4}
sage: G[3] * G[2] * G[4]
OT{0, 1, 2} + OT{0, 1, 4} + OT{0, 2, 3} + OT{0, 3, 4}
```

subset_image(S)

Return the element e_S of self corresponding to a subset S of the ground set of the defining matroid.

INPUT:

- S – a frozenset which is a subset of the ground set of M

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OT = M.orlik_terao_algebra()
sage: BC = sorted(M.broken_circuits(), key=sorted)
```

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```
sage: for bc in BC: (sorted(bc), OT.subset_image(bc))
([1, 3], OT{0, 1} + OT{0, 3})
([1, 4, 5], -OT{0, 1, 4} - OT{0, 1, 5} - OT{0, 3, 4} - OT{0, 3, 5})
([2, 3, 4], OT{0, 1, 2} + OT{0, 1, 4} + OT{0, 2, 3} + OT{0, 3, 4})
([2, 3, 5], -OT{0, 2, 3} + OT{0, 3, 5})
([2, 4], OT{1, 2} + OT{1, 4})
([2, 5], -OT{0, 2} + OT{0, 5})
([4, 5], -OT{3, 4} - OT{3, 5})

sage: M4 = matroids.CompleteGraphic(4).ternary_matroid()
sage: OT = M4.orlik_terao_algebra()
sage: OT.subset_image(frozenset({2,3,4}))
OT{0, 2, 3} + 2*OT{0, 3, 4}
```

An example of a custom ordering:

```
sage: G = Graph([[3, 4], [4, 1], [1, 2], [2, 3], [3, 5], [5, 6], [6, 3]])
sage: M = Matroid(G).regular_matroid()
sage: s = [(5, 6), (1, 2), (3, 5), (2, 3), (1, 4), (3, 6), (3, 4)]
sage: sorted([sorted(c) for c in M.circuits()])
[[(1, 2), (1, 4), (2, 3), (3, 4)],
 [(3, 5), (3, 6), (5, 6)]]
sage: OT = M.orlik_terao_algebra(QQ, ordering=s)
sage: OT.subset_image(frozenset([]))
OT{}
sage: OT.subset_image(frozenset([(1,2),(3,4),(1,4),(2,3)]))
0
sage: OT.subset_image(frozenset([(2,3),(1,2),(3,4)]))
OT{(1, 2), (2, 3), (3, 4)}
sage: OT.subset_image(frozenset([(1,4),(3,4),(2,3),(3,6),(5,6)]))
-OT{(1, 2), (1, 4), (2, 3), (3, 6), (5, 6)}
 - OT{(1, 2), (1, 4), (3, 4), (3, 6), (5, 6)}
 + OT{(1, 2), (2, 3), (3, 4), (3, 6), (5, 6)}
sage: OT.subset_image(frozenset([(1,4),(3,4),(2,3),(3,6),(3,5)]))
-OT{(1, 2), (1, 4), (2, 3), (3, 5), (5, 6)}
 + OT{(1, 2), (1, 4), (2, 3), (3, 6), (5, 6)}
 - OT{(1, 2), (1, 4), (3, 4), (3, 5), (5, 6)}
 + OT{(1, 2), (1, 4), (3, 4), (3, 6), (5, 6)}
 + OT{(1, 2), (2, 3), (3, 4), (3, 5), (5, 6)}
 - OT{(1, 2), (2, 3), (3, 4), (3, 6), (5, 6)}
```

class `sage.algebras.orlik_terao.`**`OrlikTeraoInvariantAlgebra`**(*R*, *M*, *G*, *action_on_groundset=None*, **args*, ***kwargs*)

Bases: `FiniteDimensionalInvariantModule`

Give the invariant algebra of the Orlik-Terao algebra from the action on $A(M)$ which is induced from the `action_on_groundset`.

INPUT:

- `R` – the ring of coefficients
- `M` – a matroid
- `G` – a semigroup

- `action_on_groundset` – a function defining the action of `G` on the elements of the groundset of `M` default

OUTPUT:

- The invariant algebra of the Orlik-Terao algebra induced by the action of `action_on_groundset`

EXAMPLES:

Lets start with the action of S_3 on the rank-2 braid matroid:

```
sage: A = matrix([[1,1,0],[-1,0,1],[0,-1,-1]])
sage: M = Matroid(A)
sage: M.groundset()
frozenset({0, 1, 2})
sage: G = SymmetricGroup(3)
```

Calling elements `g` of `G` on an element i of $\{1, 2, 3\}$ defines the action we want, but since the groundset is $\{0, 1, 2\}$ we first add 1 and then subtract 1:

```
sage: def on_groundset(g,x):
....:     return g(x+1)-1
```

Now that we have defined an action we can create the invariant, and get its basis:

```
sage: OTG = M.orlik_terao_algebra(QQ, invariant = (G, on_groundset))
sage: OTG.basis()
Finite family {0: B[0], 1: B[1]}
sage: [OTG.lift(b) for b in OTG.basis()]
[OT{}, OT{0} + OT{1} + OT{2}]
```

Since it is invariant, the action of any `g` in `G` is trivial:

```
sage: x = OTG.an_element(); x
2*B[0] + 2*B[1]
sage: g = G.an_element(); g
(2,3)
sage: g*x
2*B[0] + 2*B[1]

sage: x = OTG.random_element()
sage: g = G.random_element()
sage: g*x == x
True
```

The underlying ambient module is the Orlik-Terao algebra, which is accessible via `ambient()`:

```
sage: M.orlik_terao_algebra(QQ) is OTG.ambient()
True
```

For a bigger example, here we will look at the rank-3 braid matroid:

```
sage: A = matrix([[1,1,1,0,0,0],[-1,0,0,1,1,0],
....:             [0,-1,0,-1,0,1],[0,0,-1,0,-1,-1]]); A
[ 1  1  1  0  0  0]
[-1  0  0  1  1  0]
[ 0 -1  0 -1  0  1]
```

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```
[ 0  0 -1  0 -1 -1]
sage: M = Matroid(A); M.groundset()
frozenset({0, 1, 2, 3, 4, 5})
sage: G = SymmetricGroup(6)
sage: OTG = M.orlik_terao_algebra(QQ, invariant = (G, on_groundset))
sage: OTG.ambient()
Orlik-Terao algebra of Linear matroid of rank 3 on 6 elements represented over the␣
↪Rational Field over Rational Field
sage: OTG.basis()
Finite family {0: B[0], 1: B[1]}
sage: [OTG.lift(b) for b in OTG.basis()]
[OT{}, OT{0} + OT{1} + OT{2} + OT{3} + OT{4} + OT{5}]
```

construction()

Return the functorial construction of `self`.

This implementation of the method only returns `None`.

5.15 Orlik-Solomon Algebras

class `sage.algebras.orlik_solomon.`**`OrlikSolomonAlgebra`**(*R*, *M*, *ordering=None*)

Bases: `CombinatorialFreeModule`

An Orlik-Solomon algebra.

Let R be a commutative ring. Let M be a matroid with ground set X. Let $C(M)$ denote the set of circuits of M. Let E denote the exterior algebra over R generated by $\{e_x \mid x \in X\}$. The *Orlik-Solomon ideal* $J(M)$ is the ideal of E generated by

$$\partial e_S := \sum_{i=1}^{t} (-1)^{i-1} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge \widehat{e}_{j_i} \wedge \cdots \wedge e_{j_t}$$

for all $S = \{j_1 < j_2 < \cdots < j_t\} \in C(M)$, where $\widehat{e}_{j_i}$ means that the term e_{j_i} is being omitted. The notation ∂e_S is not a coincidence, as ∂e_S is actually the image of $e_S := e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_t}$ under the unique derivation ∂ of E which sends all e_x to 1.

It is easy to see that $\partial e_S \in J(M)$ not only for circuits S, but also for any dependent set S of M. Moreover, every dependent set S of M satisfies $e_S \in J(M)$.

The *Orlik-Solomon algebra* $A(M)$ is the quotient $E/J(M)$. This is a graded finite-dimensional skew-commutative R-algebra. Fix some ordering on X; then, the NBC sets of M (that is, the subsets of X containing no broken circuit of M) form a basis of $A(M)$. (Here, a *broken circuit* of M is defined to be the result of removing the smallest element from a circuit of M.)

In the current implementation, the basis of $A(M)$ is indexed by the NBC sets, which are implemented as frozensets.

INPUT:

- `R` – the base ring
- `M` – the defining matroid
- `ordering` – (optional) an ordering of the ground set

EXAMPLES:

We create the Orlik-Solomon algebra of the uniform matroid $U(3,4)$ and do some basic computations:

```
sage: M = matroids.Uniform(3, 4)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.dimension()
14
sage: G = OS.algebra_generators()
sage: M.broken_circuits()
frozenset({frozenset({1, 2, 3})})
sage: G[1] * G[2] * G[3]
OS{0, 1, 2} - OS{0, 1, 3} + OS{0, 2, 3}
```

REFERENCES:

- Wikipedia article Arrangement_of_hyperplanes#The_Orlik-Solomon_algebra
- [CE2001]

algebra_generators()

Return the algebra generators of `self`.

These form a family indexed by the ground set X of M. For each $x \in X$, the x-th element is e_x.

EXAMPLES:

```
sage: M = matroids.Uniform(2, 2)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.algebra_generators()
Finite family {0: OS{0}, 1: OS{1}}

sage: M = matroids.Uniform(1, 2)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.algebra_generators()
Finite family {0: OS{0}, 1: OS{0}}

sage: M = matroids.Uniform(1, 3)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.algebra_generators()
Finite family {0: OS{0}, 1: OS{0}, 2: OS{0}}
```

as_cdga()

Return the commutative differential graded algebra corresponding to `self` with the trivial differential.

EXAMPLES:

```
sage: H = hyperplane_arrangements.braid(3)
sage: O = H.orlik_solomon_algebra(QQ)
sage: O.as_cdga()
Commutative Differential Graded Algebra with generators ('e0', 'e1', 'e2')
 in degrees (1, 1, 1) with relations [e0*e1 - e0*e2 + e1*e2] over Rational Field
 with differential:
   e0 --> 0
   e1 --> 0
   e2 --> 0
```

as_gca()

Return the graded commutative algebra corresponding to `self`.

EXAMPLES:

```
sage: H = hyperplane_arrangements.braid(3)
sage: O = H.orlik_solomon_algebra(QQ)
sage: O.as_gca()
Graded Commutative Algebra with generators ('e0', 'e1', 'e2') in degrees (1, 1,␣
↪1)
 with relations [e0*e1 - e0*e2 + e1*e2] over Rational Field
```

```
sage: N = matroids.named_matroids.Fano()
sage: O = N.orlik_solomon_algebra(QQ)
sage: O.as_gca()
Graded Commutative Algebra with generators ('e0', 'e1', 'e2', 'e3', 'e4', 'e5',
↪'e6')
 in degrees (1, 1, 1, 1, 1, 1, 1) with relations
 [e1*e2 - e1*e3 + e2*e3, e0*e1*e3 - e0*e1*e4 + e0*e3*e4 - e1*e3*e4,
  e0*e2 - e0*e4 + e2*e4, e3*e4 - e3*e5 + e4*e5,
  e1*e2*e4 - e1*e2*e5 + e1*e4*e5 - e2*e4*e5,
  e0*e2*e3 - e0*e2*e5 + e0*e3*e5 - e2*e3*e5, e0*e1 - e0*e5 + e1*e5,
  e2*e5 - e2*e6 + e5*e6, e1*e3*e5 - e1*e3*e6 + e1*e5*e6 - e3*e5*e6,
  e0*e4*e5 - e0*e4*e6 + e0*e5*e6 - e4*e5*e6, e1*e4 - e1*e6 + e4*e6,
  e2*e3*e4 - e2*e3*e6 + e2*e4*e6 - e3*e4*e6, e0*e3 - e0*e6 + e3*e6,
  e0*e1*e2 - e0*e1*e6 + e0*e2*e6 - e1*e2*e6] over Rational Field
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.degree_on_basis(frozenset([1]))
1
sage: OS.degree_on_basis(frozenset([0, 2, 3]))
3
```

one_basis()

Return the index of the basis element corresponding to 1 in `self`.

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.one_basis() == frozenset([])
True
```

product_on_basis(*a*, *b*)

Return the product in `self` of the basis elements indexed by `a` and `b`.

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: OS.product_on_basis(frozenset([2]), frozenset([3,4]))
OS{0, 1, 2} - OS{0, 1, 4} + OS{0, 2, 3} + OS{0, 3, 4}
```

```
sage: G = OS.algebra_generators()
sage: prod(G)
0
sage: G[2] * G[4]
-OS{1, 2} + OS{1, 4}
sage: G[3] * G[4] * G[2]
OS{0, 1, 2} - OS{0, 1, 4} + OS{0, 2, 3} + OS{0, 3, 4}
sage: G[2] * G[3] * G[4]
OS{0, 1, 2} - OS{0, 1, 4} + OS{0, 2, 3} + OS{0, 3, 4}
sage: G[3] * G[2] * G[4]
-OS{0, 1, 2} + OS{0, 1, 4} - OS{0, 2, 3} - OS{0, 3, 4}
```

subset_image(S)

Return the element e_S of $A(M)$ (== `self`) corresponding to a subset S of the ground set of M.

INPUT:

- `S` – a frozenset which is a subset of the ground set of M

EXAMPLES:

```
sage: M = matroids.Wheel(3)
sage: OS = M.orlik_solomon_algebra(QQ)
sage: BC = sorted(M.broken_circuits(), key=sorted)
sage: for bc in BC: (sorted(bc), OS.subset_image(bc))
([1, 3], -OS{0, 1} + OS{0, 3})
([1, 4, 5], OS{0, 1, 4} - OS{0, 1, 5} - OS{0, 3, 4} + OS{0, 3, 5})
([2, 3, 4], OS{0, 1, 2} - OS{0, 1, 4} + OS{0, 2, 3} + OS{0, 3, 4})
([2, 3, 5], OS{0, 2, 3} + OS{0, 3, 5})
([2, 4], -OS{1, 2} + OS{1, 4})
([2, 5], -OS{0, 2} + OS{0, 5})
([4, 5], -OS{3, 4} + OS{3, 5})

sage: M4 = matroids.CompleteGraphic(4)
sage: OS = M4.orlik_solomon_algebra(QQ)
sage: OS.subset_image(frozenset({2,3,4}))
OS{0, 2, 3} + OS{0, 3, 4}
```

An example of a custom ordering:

```
sage: G = Graph([[3, 4], [4, 1], [1, 2], [2, 3], [3, 5], [5, 6], [6, 3]])
sage: M = Matroid(G)
sage: s = [(5, 6), (1, 2), (3, 5), (2, 3), (1, 4), (3, 6), (3, 4)]
sage: sorted([sorted(c) for c in M.circuits()])
[[(1, 2), (1, 4), (2, 3), (3, 4)],
 [(3, 5), (3, 6), (5, 6)]]
sage: OS = M.orlik_solomon_algebra(QQ, ordering=s)
sage: OS.subset_image(frozenset([]))
OS{}
```

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```
sage: OS.subset_image(frozenset([(1,2),(3,4),(1,4),(2,3)]))
0
sage: OS.subset_image(frozenset([(2,3),(1,2),(3,4)]))
OS{(1, 2), (2, 3), (3, 4)}
sage: OS.subset_image(frozenset([(1,4),(3,4),(2,3),(3,6),(5,6)]))
-OS{(1, 2), (1, 4), (2, 3), (3, 6), (5, 6)}
 + OS{(1, 2), (1, 4), (3, 4), (3, 6), (5, 6)}
 - OS{(1, 2), (2, 3), (3, 4), (3, 6), (5, 6)}
sage: OS.subset_image(frozenset([(1,4),(3,4),(2,3),(3,6),(3,5)]))
OS{(1, 2), (1, 4), (2, 3), (3, 5), (5, 6)}
 - OS{(1, 2), (1, 4), (2, 3), (3, 6), (5, 6)}
 + OS{(1, 2), (1, 4), (3, 4), (3, 5), (5, 6)}
 + OS{(1, 2), (1, 4), (3, 4), (3, 6), (5, 6)}
 - OS{(1, 2), (2, 3), (3, 4), (3, 5), (5, 6)}
 - OS{(1, 2), (2, 3), (3, 4), (3, 6), (5, 6)}
```

class sage.algebras.orlik_solomon.**OrlikSolomonInvariantAlgebra**(*R*, *M*, *G*, *action_on_groundset=None*, **args*, ***kwargs*)

Bases: `FiniteDimensionalInvariantModule`

The invariant algebra of the Orlik-Solomon algebra from the action on $A(M)$ induced from the `action_on_groundset`.

INPUT:

- `R` – the ring of coefficients
- `M` – a matroid
- `G` – a semigroup
- `action_on_groundset` – (optional) a function defining the action of `G` on the elements of the groundset of `M`; default is `g(x)`

EXAMPLES:

Lets start with the action of S_3 on the rank 2 braid matroid:

```
sage: M = matroids.CompleteGraphic(3)
sage: M.groundset()
frozenset({0, 1, 2})
sage: G = SymmetricGroup(3)
```

Calling elements `g` of `G` on an element i of $\{1, 2, 3\}$ defines the action we want, but since the groundset is $\{0, 1, 2\}$ we first add 1 and then subtract 1:

```
sage: def on_groundset(g, x):
....:     return g(x+1) - 1
```

Now that we have defined an action we can create the invariant, and get its basis:

```
sage: OSG = M.orlik_solomon_algebra(QQ, invariant=(G, on_groundset))
sage: OSG.basis()
Finite family {0: B[0], 1: B[1]}
sage: [OSG.lift(b) for b in OSG.basis()]
[OS{}, OS{0} + OS{1} + OS{2}]
```

Since it is invariant, the action of any g in G is trivial:

```
sage: x = OSG.an_element(); x
2*B[0] + 2*B[1]
sage: g = G.an_element(); g
(2,3)
sage: g * x
2*B[0] + 2*B[1]

sage: x = OSG.random_element()
sage: g = G.random_element()
sage: g * x == x
True
```

The underlying ambient module is the Orlik-Solomon algebra, which is accessible via `ambient()`:

```
sage: M.orlik_solomon_algebra(QQ) is OSG.ambient()
True
```

There is not much structure here, so lets look at a bigger example. Here we will look at the rank 3 braid matroid, and to make things easier, we'll start the indexing at 1 so that the S_6 action on the groundset is simply calling g:

```
sage: M = matroids.CompleteGraphic(4); M.groundset()
frozenset({0, 1, 2, 3, 4, 5})
sage: new_bases = [frozenset(i+1 for i in j) for j in M.bases()]
sage: M = Matroid(bases=new_bases); M.groundset()
frozenset({1, 2, 3, 4, 5, 6})
sage: G = SymmetricGroup(6)
sage: OSG = M.orlik_solomon_algebra(QQ, invariant=G)
sage: OSG.basis()
Finite family {0: B[0], 1: B[1]}
sage: [OSG.lift(b) for b in OSG.basis()]
[OS{}, OS{1} + OS{2} + OS{3} + OS{4} + OS{5} + OS{6}]
sage: (OSG.basis()[1])^2
0
sage: 5 * OSG.basis()[1]
5*B[1]
```

Next, we look at the same matroid but with an $S_3 \times S_3$ action (here realized as a Young subgroup of S_6):

```
sage: H = G.young_subgroup([3, 3])
sage: OSH = M.orlik_solomon_algebra(QQ, invariant=H)
sage: OSH.basis()
Finite family {0: B[0], 1: B[1], 2: B[2]}
sage: [OSH.lift(b) for b in OSH.basis()]
[OS{}, OS{4} + OS{5} + OS{6}, OS{1} + OS{2} + OS{3}]
```

We implement an S_4 action on the vertices:

```
sage: M = matroids.CompleteGraphic(4)
sage: G = SymmetricGroup(4)
sage: edge_map = {i: M.groundset_to_edges([i])[0][:2]
....:             for i in M.groundset()}
sage: inv_map = {v: k for k, v in edge_map.items()}
```

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```
sage: def vert_action(g, x):
....:     a, b = edge_map[x]
....:     return inv_map[tuple(sorted([g(a+1)-1, g(b+1)-1]))]
sage: OSG = M.orlik_solomon_algebra(QQ, invariant=(G, vert_action))
sage: B = OSG.basis()
sage: [OSG.lift(b) for b in B]
[OS{}, OS{0} + OS{1} + OS{2} + OS{3} + OS{4} + OS{5}]
```

We use this to describe the Young subgroup $S_2 \times S_2$ action:

```
sage: H = G.young_subgroup([2,2])
sage: OSH = M.orlik_solomon_algebra(QQ, invariant=(H, vert_action))
sage: B = OSH.basis()
sage: [OSH.lift(b) for b in B]
[OS{}, OS{5}, OS{1} + OS{2} + OS{3} + OS{4}, OS{0},
 -1/2*OS{1, 2} + OS{1, 5} - 1/2*OS{3, 4} + OS{3, 5},
 OS{0, 5}, OS{0, 1} + OS{0, 2} + OS{0, 3} + OS{0, 4},
 -1/2*OS{0, 1, 2} + OS{0, 1, 5} - 1/2*OS{0, 3, 4} + OS{0, 3, 5}]
```

We demonstrate the algebra structure:

```
sage: matrix([[b*bp for b in B] for bp in B])
[   B[0]    B[1]    B[2]    B[3]    B[4]    B[5]    B[6]    B[7]]
[   B[1]       0  2*B[4]    B[5]       0       0  2*B[7]       0]
[   B[2] -2*B[4]       0    B[6]       0 -2*B[7]       0       0]
[   B[3]   -B[5]   -B[6]       0    B[7]       0       0       0]
[   B[4]       0       0    B[7]       0       0       0       0]
[   B[5]       0 -2*B[7]       0       0       0       0       0]
[   B[6]  2*B[7]       0       0       0       0       0       0]
[   B[7]       0       0       0       0       0       0       0]
```

Note: The algebra structure only exists when the action on the groundset yields an equivariant matroid, in the sense that $g \cdot I \in \mathcal{I}$ for every $g \in G$ and for every $I \in \mathcal{I}$.

construction()

Return the functorial construction of `self`.

This implementation of the method only returns `None`.

5.16 Partition/Diagram Algebras

class sage.combinat.partition_algebra.**PartitionAlgebraElement_ak**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebraElement_bk**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebraElement_generic**

Bases: `IndexedFreeModuleElement`

class sage.combinat.partition_algebra.**PartitionAlgebraElement_pk**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebraElement_prk**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebraElement_rk**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebraElement_sk**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebraElement_tk**

Bases: *PartitionAlgebraElement_generic*

class sage.combinat.partition_algebra.**PartitionAlgebra_ak**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_ak(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_bk**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_bk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_generic**(*R*, *cclass*, *n*, *k*, *name=None*, *prefix=None*)

Bases: CombinatorialFreeModule

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: s = PartitionAlgebra_sk(QQ, 3, 1)
sage: TestSuite(s).run()
sage: s == loads(dumps(s))
True
```

one_basis()

Return the basis index for the unit of the algebra.

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: s = PartitionAlgebra_sk(ZZ, 3, 1)
sage: len(s.one().support())    # indirect doctest
1
```

product_on_basis(*left*, *right*)

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: s = PartitionAlgebra_sk(QQ, 3, 1)
sage: t12 = s(Set([Set([1,-2]),Set([2,-1]),Set([3,-3])]))
sage: t12^2 == s(1) #indirect doctest
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_pk**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_pk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_prk**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_prk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_rk**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_rk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_sk**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_sk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.**PartitionAlgebra_tk**(*R*, *k*, *n*, *name=None*)

Bases: *PartitionAlgebra_generic*

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_tk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

sage.combinat.partition_algebra.**SetPartitionsAk**(k)

Return the combinatorial class of set partitions of type A_k.

EXAMPLES:

```
sage: A3 = SetPartitionsAk(3); A3
Set partitions of {1, ..., 3, -1, ..., -3}

sage: A3.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: A3.last() #random
{{-1}, {-2}, {3}, {1}, {-3}, {2}}
sage: A3.random_element()  #random
{{1, 3, -3, -1}, {2, -2}}

sage: A3.cardinality()
203

sage: A2p5 = SetPartitionsAk(2.5); A2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block
sage: A2p5.cardinality()
52

sage: A2p5.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: A2p5.last() #random
{{-1}, {-2}, {2}, {3, -3}, {1}}
sage: A2p5.random_element() #random
{{-1}, {-2}, {3, -3}, {1, 2}}
```

class sage.combinat.partition_algebra.**SetPartitionsAk_k**(k)

Bases: SetPartitions_set

Element

alias of *SetPartitionsXkElement*

class sage.combinat.partition_algebra.**SetPartitionsAkhalf_k**(k)

Bases: SetPartitions_set

Element

alias of *SetPartitionsXkElement*

sage.combinat.partition_algebra.**SetPartitionsBk**(k)

Return the combinatorial class of set partitions of type B_k.

These are the set partitions where every block has size 2.

EXAMPLES:

```
sage: B3 = SetPartitionsBk(3); B3
Set partitions of {1, ..., 3, -1, ..., -3} with block size 2

sage: B3.first() #random
{{2, -2}, {1, -3}, {3, -1}}
sage: B3.last() #random
{{1, 2}, {3, -2}, {-3, -1}}
sage: B3.random_element() #random
{{2, -1}, {1, -3}, {3, -2}}

sage: B3.cardinality()
15

sage: B2p5 = SetPartitionsBk(2.5); B2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block and with␣
↪block size 2

sage: B2p5.first() #random
{{2, -1}, {3, -3}, {1, -2}}
sage: B2p5.last() #random
{{1, 2}, {3, -3}, {-1, -2}}
sage: B2p5.random_element() #random
{{2, -2}, {3, -3}, {1, -1}}

sage: B2p5.cardinality()
3
```

class sage.combinat.partition_algebra.**SetPartitionsBk_k**(k)

Bases: *SetPartitionsAk_k*

cardinality()

Return the number of set partitions in B_k where k is an integer.

This is given by (2k)!! = (2k-1)*(2k-3)*…*5*3*1.

EXAMPLES:

```
sage: SetPartitionsBk(3).cardinality()
15
sage: SetPartitionsBk(2).cardinality()
3
sage: SetPartitionsBk(1).cardinality()
1
sage: SetPartitionsBk(4).cardinality()
105
sage: SetPartitionsBk(5).cardinality()
945
```

class sage.combinat.partition_algebra.**SetPartitionsBkhalf_k**(k)

Bases: *SetPartitionsAkhalf_k*

cardinality()

sage.combinat.partition_algebra.**SetPartitionsIk**(k)

Return the combinatorial class of set partitions of type I_k.

These are set partitions with a propagating number of less than k. Note that the identity set partition $\{\{1, -1\}, \ldots, \{k, -k\}\}$ is not in I_k.

EXAMPLES:

```
sage: I3 = SetPartitionsIk(3); I3
Set partitions of {1, ..., 3, -1, ..., -3} with propagating number < 3
sage: I3.cardinality()
197

sage: I3.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: I3.last() #random
{{-1}, {-2}, {3}, {1}, {-3}, {2}}
sage: I3.random_element() #random
{{-1}, {-3, -2}, {2, 3}, {1}}

sage: I2p5 = SetPartitionsIk(2.5); I2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block and␣
↪propagating number < 3
sage: I2p5.cardinality()
50

sage: I2p5.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: I2p5.last() #random
{{-1}, {-2}, {2}, {3, -3}, {1}}
sage: I2p5.random_element() #random
{{-1}, {-2}, {1, 3, -3}, {2}}
```

class sage.combinat.partition_algebra.**SetPartitionsIk_k**(k)

Bases: *SetPartitionsAk_k*

cardinality()

class sage.combinat.partition_algebra.**SetPartitionsIkhalf_k**(k)

Bases: *SetPartitionsAkhalf_k*

cardinality()

sage.combinat.partition_algebra.**SetPartitionsPRk**(k)

Return the combinatorial class of set partitions of type PR_k.

EXAMPLES:

```
sage: SetPartitionsPRk(3)
Set partitions of {1, ..., 3, -1, ..., -3} with at most 1 positive
 and negative entry in each block and that are planar
```

class sage.combinat.partition_algebra.**SetPartitionsPRk_k**(k)

Bases: *SetPartitionsRk_k*

cardinality()

class sage.combinat.partition_algebra.**SetPartitionsPRkhalf_k**(k)

Bases: *SetPartitionsRkhalf_k*

cardinality()

sage.combinat.partition_algebra.**SetPartitionsPk**(k)

Return the combinatorial class of set partitions of type P_k.

These are the planar set partitions.

EXAMPLES:

```
sage: P3 = SetPartitionsPk(3); P3
Set partitions of {1, ..., 3, -1, ..., -3} that are planar
sage: P3.cardinality()
132

sage: P3.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: P3.last() #random
{{-1}, {-2}, {3}, {1}, {-3}, {2}}
sage: P3.random_element() #random
{{1, 2, -1}, {-3}, {3, -2}}

sage: P2p5 = SetPartitionsPk(2.5); P2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block and that␣
↪are planar
sage: P2p5.cardinality()
42

sage: P2p5.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: P2p5.last() #random
{{-1}, {-2}, {2}, {3, -3}, {1}}
sage: P2p5.random_element() #random
{{1, 2, 3, -3}, {-1, -2}}
```

class sage.combinat.partition_algebra.**SetPartitionsPk_k**(k)

Bases: *SetPartitionsAk_k*

cardinality()

class sage.combinat.partition_algebra.**SetPartitionsPkhalf_k**(k)

Bases: *SetPartitionsAkhalf_k*

cardinality()

sage.combinat.partition_algebra.**SetPartitionsRk**(k)

Return the combinatorial class of set partitions of type R_k.

EXAMPLES:

```
sage: SetPartitionsRk(3)
Set partitions of {1, ..., 3, -1, ..., -3} with at most 1 positive
 and negative entry in each block
```

class sage.combinat.partition_algebra.**SetPartitionsRk_k**(k)

Bases: *SetPartitionsAk_k*

cardinality()

class sage.combinat.partition_algebra.**SetPartitionsRkhalf_k**(k)

Bases: *SetPartitionsAkhalf_k*

cardinality()

sage.combinat.partition_algebra.**SetPartitionsSk**(k)

Return the combinatorial class of set partitions of type S_k.

There is a bijection between these set partitions and the permutations of $1, \ldots, k$.

EXAMPLES:

```
sage: S3 = SetPartitionsSk(3); S3
Set partitions of {1, ..., 3, -1, ..., -3} with propagating number 3
sage: S3.cardinality()
6

sage: S3.list()  #random
[{{2, -2}, {3, -3}, {1, -1}},
 {{1, -1}, {2, -3}, {3, -2}},
 {{2, -1}, {3, -3}, {1, -2}},
 {{1, -2}, {2, -3}, {3, -1}},
 {{1, -3}, {2, -1}, {3, -2}},
 {{1, -3}, {2, -2}, {3, -1}}]
sage: S3.first() #random
{{2, -2}, {3, -3}, {1, -1}}
sage: S3.last() #random
{{1, -3}, {2, -2}, {3, -1}}
sage: S3.random_element() #random
{{1, -3}, {2, -1}, {3, -2}}

sage: S3p5 = SetPartitionsSk(3.5); S3p5
Set partitions of {1, ..., 4, -1, ..., -4} with 4 and -4 in the same block and
↪propagating number 4
sage: S3p5.cardinality()
6

sage: S3p5.list() #random
[{{2, -2}, {3, -3}, {1, -1}, {4, -4}},
 {{2, -3}, {1, -1}, {4, -4}, {3, -2}},
 {{2, -1}, {3, -3}, {1, -2}, {4, -4}},
 {{2, -3}, {1, -2}, {4, -4}, {3, -1}},
 {{1, -3}, {2, -1}, {4, -4}, {3, -2}},
 {{1, -3}, {2, -2}, {4, -4}, {3, -1}}]
sage: S3p5.first() #random
{{2, -2}, {3, -3}, {1, -1}, {4, -4}}
sage: S3p5.last() #random
{{1, -3}, {2, -2}, {4, -4}, {3, -1}}
sage: S3p5.random_element() #random
{{1, -3}, {2, -2}, {4, -4}, {3, -1}}
```

class sage.combinat.partition_algebra.**SetPartitionsSk_k**(k)

Bases: *SetPartitionsAk_k*

cardinality()

Return k!.

class sage.combinat.partition_algebra.**SetPartitionsSkhalf_k**(*k*)

Bases: *SetPartitionsAkhalf_k*

cardinality()

sage.combinat.partition_algebra.**SetPartitionsTk**(*k*)

Return the combinatorial class of set partitions of type T_k.

These are planar set partitions where every block is of size 2.

EXAMPLES:

```
sage: T3 = SetPartitionsTk(3); T3
Set partitions of {1, ..., 3, -1, ..., -3} with block size 2 and that are planar
sage: T3.cardinality()
5

sage: T3.first() #random
{{1, -3}, {2, 3}, {-1, -2}}
sage: T3.last() #random
{{1, 2}, {3, -1}, {-3, -2}}
sage: T3.random_element() #random
{{1, -3}, {2, 3}, {-1, -2}}

sage: T2p5 = SetPartitionsTk(2.5); T2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block and with␣
→block size 2 and that are planar
sage: T2p5.cardinality()
2

sage: T2p5.first() #random
{{2, -2}, {3, -3}, {1, -1}}
sage: T2p5.last() #random
{{1, 2}, {3, -3}, {-1, -2}}
```

class sage.combinat.partition_algebra.**SetPartitionsTk_k**(*k*)

Bases: *SetPartitionsBk_k*

cardinality()

class sage.combinat.partition_algebra.**SetPartitionsTkhalf_k**(*k*)

Bases: *SetPartitionsBkhalf_k*

cardinality()

class sage.combinat.partition_algebra.**SetPartitionsXkElement**(*parent*, *s*, *check=True*)

Bases: SetPartition

An element for the classes of SetPartitionXk where X is some letter.

check()

Check to make sure this is a set partition.

EXAMPLES:

```
sage: A2p5 = SetPartitionsAk(2.5)
sage: x = A2p5.first(); x
{{-3, -2, -1, 1, 2, 3}}
sage: x.check()
sage: y = A2p5.next(x); y
{{-3, 3}, {-2, -1, 1, 2}}
sage: y.check()
```

`sage.combinat.partition_algebra.identity(k)`

Return the identity set partition 1, -1, . . . , k, -k

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: pa.identity(2)
{{2, -2}, {1, -1}}
```

`sage.combinat.partition_algebra.is_planar(sp)`

Return `True` if the diagram corresponding to the set partition is planar; otherwise, it returns `False`.

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: pa.is_planar( pa.to_set_partition([[1,-2],[2,-1]]))
False
sage: pa.is_planar( pa.to_set_partition([[1,-1],[2,-2]]))
True
```

`sage.combinat.partition_algebra.pair_to_graph(sp1, sp2)`

Return a graph consisting of the disjoint union of the graphs of set partitions `sp1` and `sp2` along with edges joining the bottom row (negative numbers) of `sp1` to the top row (positive numbers) of `sp2`.

The vertices of the graph `sp1` appear in the result as pairs `(k, 1)`, whereas the vertices of the graph `sp2` appear as pairs `(k, 2)`.

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: sp1 = pa.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = pa.to_set_partition([[1,-2],[2,-1]])
sage: g = pa.pair_to_graph( sp1, sp2 ); g
Graph on 8 vertices
```

```
sage: g.vertices(sort=False) #random
[(1, 2), (-1, 1), (-2, 2), (-1, 2), (-2, 1), (2, 1), (2, 2), (1, 1)]
sage: g.edges(sort=False) #random
[((1, 2), (-1, 1), None),
 ((1, 2), (-2, 2), None),
 ((-1, 1), (2, 1), None),
 ((-1, 2), (2, 2), None),
 ((-2, 1), (1, 1), None),
 ((-2, 1), (2, 2), None)]
```

Another example which used to be wrong until github issue #15958:

```
sage: sp3 = pa.to_set_partition([[1, -1], [2], [-2]])
sage: sp4 = pa.to_set_partition([[1], [-1], [2], [-2]])
sage: g = pa.pair_to_graph( sp3, sp4 ); g
Graph on 8 vertices

sage: g.vertices(sort=True)
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]
sage: g.edges(sort=True)
[((-2, 1), (2, 2), None), ((-1, 1), (1, 1), None),
 ((-1, 1), (1, 2), None)]
```

sage.combinat.partition_algebra.**propagating_number**(*sp*)

Return the propagating number of the set partition sp.

The propagating number is the number of blocks with both a positive and negative number.

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: sp1 = pa.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = pa.to_set_partition([[1,2],[-2,-1]])
sage: pa.propagating_number(sp1)
2
sage: pa.propagating_number(sp2)
0
```

sage.combinat.partition_algebra.**set_partition_composition**(*sp1*, *sp2*)

Return a tuple consisting of the composition of the set partitions sp1 and sp2 and the number of components removed from the middle rows of the graph.

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: sp1 = pa.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = pa.to_set_partition([[1,-2],[2,-1]])
sage: pa.set_partition_composition(sp1, sp2) == (pa.identity(2), 0)
True
```

sage.combinat.partition_algebra.**to_graph**(*sp*)

Return a graph representing the set partition sp.

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: g = pa.to_graph( pa.to_set_partition([[1,-2],[2,-1]])); g
Graph on 4 vertices

sage: g.vertices(sort=False) #random
[1, 2, -2, -1]
sage: g.edges(sort=False) #random
[(1, -2, None), (2, -1, None)]
```

sage.combinat.partition_algebra.**to_set_partition**(*l*, *k=None*)

Convert a list of a list of numbers to a set partitions.

Each list of numbers in the outer list specifies the numbers contained in one of the blocks in the set partition.

If k is specified, then the set partition will be a set partition of 1, …, k, -1, …, -k. Otherwise, k will default to the minimum number needed to contain all of the specified numbers.

EXAMPLES:

```
sage: import sage.combinat.partition_algebra as pa
sage: pa.to_set_partition([[1,-1],[2,-2]]) == pa.identity(2)
True
```

5.17 Quantum Clifford Algebras

AUTHORS:

- Travis Scrimshaw (2021-05): initial version

class sage.algebras.quantum_clifford.**QuantumCliffordAlgebra**(*n*, *k*, *q*, *F*, *psi*, *indices*)

Bases: `CombinatorialFreeModule`

The quantum Clifford algebra.

The *quantum Clifford algebra*, or q-Clifford algebra, of rank n and twist k is the unital associative algebra $\mathrm{Cl}_q(n,k)$ over a field F with generators $\psi_a, \psi_a^*, \omega_a$ for $a = 1, \ldots, n$ that satisfy the following relations:

$$\begin{aligned} \omega_a \omega_b &= \omega_b \omega_a, & \omega_a^{4k} &= (1 + q^{-2k}) \omega_a^{2k} - q^{-2k}, \\ \omega_a \psi_b &= q^{\delta_{ab}} \psi_b \omega_a, & \omega_a \psi_b^* &= \psi_b^* \omega_a, \\ \psi_a \psi_b + \psi_b \psi_a &= 0, & \psi_a^* \psi_b^* + \psi_b^* \psi_a^* &= 0, \\ \psi_a \psi_a^* + q^k \psi_a^* \psi_a &= \omega_a^{-k}, & \psi_a^* \psi_a + q^{-k} \psi_a^* \psi_a &= \omega_a^k, \\ \psi_a \psi_b^* + \psi_b^* \psi_a &= 0 & \text{if } a &\neq b. \end{aligned}$$

When $k = 2$, we recover the original definition given by Hayashi in [Hayashi1990]. The $k = 1$ version was used in [Kwon2014].

INPUT:

- `n` – positive integer; the rank
- `k` – positive integer (default: 1); the twist
- `q` – (optional) the parameter q
- `F` – (default: $\mathbf{Q}(q)$) the base field that contains `q`

EXAMPLES:

We construct the rank 3 and twist 1 q-Clifford algebra:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl
Quantum Clifford algebra of rank 3 and twist 1 with q=q over
 Fraction Field of Univariate Polynomial Ring in q over Integer Ring
sage: q = Cl.q()
```

Some sample computations:

```
sage: p0, p1, p2, d0, d1, d2, w0, w1, w2 = Cl.gens()
sage: p0 * p1
```

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```
psi0*psi1
sage: p1 * p0
-psi0*psi1
sage: p0 * w0 * p1 * d0 * w2
(1/(q^3-q))*psi1*w2 + (-q/(q^2-1))*psi1*w0^2*w2
sage: w0^4
-1/q^2 + ((q^2+1)/q^2)*w0^2
```

We construct the homomorphism from $U_q(\mathfrak{sl}_3)$ to $\mathrm{Cl}(3,1)$ given in (3.17) of [Hayashi1990]:

```
sage: e1 = p0*d1; e2 = p1*d2
sage: f1 = p1*d0; f2 = p2*d1
sage: k1 = w0*~w1; k2 = w1*~w2
sage: k1i = w1*~w0; k2i = w2*~w1
sage: (e1, e2, f1, f2, k1, k2, k1i, k2i)
(psi0*psid1, psi1*psid2,
 -psid0*psi1, -psid1*psi2,
 (q^2+1)*w0*w1 - q^2*w0*w1^3, (q^2+1)*w1*w2 - q^2*w1*w2^3,
 (q^2+1)*w0*w1 - q^2*w0^3*w1, (q^2+1)*w1*w2 - q^2*w1^3*w2)
```

We check that k_i and k_i^{-1} are inverses:

```
sage: k1 * k1i
1
sage: k2 * k2i
1
```

The relations between e_i, f_i, and k_i:

```
sage: k1 * f1 == q^-2 * f1 * k1
True
sage: k2 * f1 == q^1 * f1 * k2
True
sage: k2 * e1 == q^-1 * e1 * k2
True
sage: k1 * e1 == q^2 * e1 * k1
True
sage: e1 * f1 - f1 * e1 == (k1 - k1i)/(q-q^-1)
True
sage: e2 * f1 - f1 * e2
0
```

The q-Serre relations:

```
sage: e1 * e1 * e2 - (q^1 + q^-1) * e1 * e2 * e1 + e2 * e1 * e1
0
sage: f1 * f1 * f2 - (q^1 + q^-1) * f1 * f2 * f1 + f2 * f1 * f1
0
```

This also can be constructed at the special point when $q^{2k} = 1$, but the basis used is different:

```
sage: Cl = algebras.QuantumClifford(1, 1, -1)
sage: Cl.inject_variables()
```

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```
Defining psi0, psid0, w0
sage: psi0 * psid0
psi0*psid0
sage: psid0 * psi0
-w0 + psi0*psid0
sage: w0^2
1
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl.algebra_generators()
Finite family {'psi0': psi0, 'psi1': psi1, 'psi2': psi2,
               'psid0': psid0, 'psid1': psid1, 'psid2': psid2,
               'w0': w0, 'w1': w1, 'w2': w2}
```

dimension()

Return the dimension of `self`.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl.dimension()
512

sage: Cl = algebras.QuantumClifford(4, 2)  # long time
sage: Cl.dimension()  # long time
65536
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl.gens()
(psi0, psi1, psi2, psid0, psid1, psid2, w0, w1, w2)
```

one_basis()

Return the index of the basis element of 1.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl.one_basis()
((0, 0, 0), (0, 0, 0))
```

q()

Return the q of `self`.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl.q()
q

sage: Cl = algebras.QuantumClifford(3, q=QQ(-5))
sage: Cl.q()
-5
```

rank()

Return the rank k of self.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3, 2)
sage: Cl.rank()
3
```

twist()

Return the twist k of self.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3, 2)
sage: Cl.twist()
2
```

class sage.algebras.quantum_clifford.**QuantumCliffordAlgebraGeneric**(n, k, q, F)

Bases: *QuantumCliffordAlgebra*

The quantum Clifford algebra when $q^{2k} \neq 1$.

The *quantum Clifford algebra*, or q-Clifford algebra, of rank n and twist k is the unital associative algebra $\mathrm{Cl}_q(n, k)$ over a field F with generators $\psi_a, \psi_a^*, \omega_a$ for $a = 1, \ldots, n$ that satisfy the following relations:

$$\begin{aligned}
\omega_a \omega_b &= \omega_b \omega_a, & \omega_a^{4k} &= (1 + q^{-2k}) \omega_a^{2k} - q^{-2k}, \\
\omega_a \psi_b &= q^{\delta_{ab}} \psi_b \omega_a, & \omega_a \psi_b^* &= \psi_b^* \omega_a, \\
\psi_a \psi_b + \psi_b \psi_a &= 0, & \psi_a^* \psi_b^* + \psi_b^* \psi_a^* &= 0, \\
\psi_a \psi_a^* &= \frac{q^k \omega_a^{3k} - q^{-k} \omega_a^k}{q^k - q^{-k}}, & \psi_a^* \psi_a &= \frac{q^{2k} (\omega_a - \omega_a^{3k})}{q^k - q^{-k}}, \\
\psi_a \psi_b^* + \psi_b^* \psi_a &= 0 & & \text{if } a \neq b,
\end{aligned}$$

where $q \in F$ such that $q^{2k} \neq 1$.

When $k = 2$, we recover the original definition given by Hayashi in [Hayashi1990]. The $k = 1$ version was used in [Kwon2014].

class Element

Bases: IndexedFreeModuleElement

inverse()

Return the inverse if self is a basis element.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(2)
sage: Cl.inject_variables()
Defining psi0, psi1, psid0, psid1, w0, w1
sage: w0^-1
(q^2+1)*w0 - q^2*w0^3
sage: w0^-1 * w0
1
sage: w0^-2
(q^2+1) - q^2*w0^2
sage: w0^-2 * w0^2
1
sage: w0^-2 * w0 == w0^-1
True
sage: w = w0 * w1
sage: w^-1
(q^4+2*q^2+1)*w0*w1 + (-q^4-q^2)*w0*w1^3
 + (-q^4-q^2)*w0^3*w1 + q^4*w0^3*w1^3
sage: w^-1 * w
1
sage: w * w^-1
1

sage: (2*w0)^-1
((q^2+1)/2)*w0 - q^2/2*w0^3

sage: (w0 + w1)^-1
Traceback (most recent call last):
...
ValueError: cannot invert self (= w1 + w0)
sage: (psi0 * w0)^-1
Traceback (most recent call last):
...
ValueError: cannot invert self (= psi0*w0)

sage: Cl = algebras.QuantumClifford(1, 2)
sage: Cl.inject_variables()
Defining psi0, psid0, w0
sage: (psi0 + psid0).inverse()
psid0*w0^2 + q^2*psi0*w0^2

sage: Cl = algebras.QuantumClifford(2, 2)
sage: Cl.inject_variables()
Defining psi0, psi1, psid0, psid1, w0, w1
sage: w0^-1
(q^4+1)*w0^3 - q^4*w0^7
sage: w0 * w0^-1
1
```

product_on_basis(*m1*, *m2*)

Return the product of the basis elements indexed by `m1` and `m2`.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3)
sage: Cl.inject_variables()
Defining psi0, psi1, psi2, psid0, psid1, psid2, w0, w1, w2
sage: psi0^2  # indirect doctest
0
sage: psid0^2
0
sage: w0 * psi0
q*psi0*w0
sage: w0 * psid0
1/q*psid0*w0
sage: w2 * w0
w0*w2
sage: w0^4
-1/q^2 + ((q^2+1)/q^2)*w0^2
```

class sage.algebras.quantum_clifford.**QuantumCliffordAlgebraRootUnity**(*n*, *k*, *q*, *F*)

Bases: `QuantumCliffordAlgebra`

The quantum Clifford algebra when $q^{2k} = 1$.

The *quantum Clifford algebra*, or q-Clifford algebra, of rank n and twist k is the unital associative algebra $\mathrm{Cl}_q(n, k)$ over a field F with generators $\psi_a, \psi_a^*, \omega_a$ for $a = 1, \dots, n$ that satisfy the following relations:

$$\begin{aligned} \omega_a \omega_b &= \omega_b \omega_a, & \omega_a^{2k} &= 1, \\ \omega_a \psi_b &= q^{\delta_{ab}} \psi_b \omega_a, & \omega_a \psi_b^* &= \psi_b^* \omega_a, \\ \psi_a \psi_b + \psi_b \psi_a &= 0, & \psi_a^* \psi_b^* + \psi_b^* \psi_a^* &= 0, \\ \psi_a \psi_a^* + q^k \psi_a^* \psi_a &= \omega_a^k & \psi_a \psi_b^* + \psi_b^* \psi_a &= 0 \quad (a \neq b), \end{aligned}$$

where $q \in F$ such that $q^{2k} = 1$. This has further relations of

$$\begin{aligned} \psi_a^* \psi_a \psi_a^* &= \psi_a^* \omega_a^k, \\ \psi_a \psi_a^* \psi_a &= q^k \psi_a \omega_a^k, \\ (\psi_a \psi_a^*)^2 &= \psi_a \psi_a^* \omega_a^k. \end{aligned}$$

class Element

Bases: `IndexedFreeModuleElement`

inverse()

Return the inverse if `self` is a basis element.

EXAMPLES:

```
sage: Cl = algebras.QuantumClifford(3, 3, -1)
sage: Cl.inject_variables()
Defining psi0, psi1, psi2, psid0, psid1, psid2, w0, w1, w2
sage: w0^-1
w0^5
sage: w0^-1 * w0
1
sage: w0^-2
w0^4
```

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```
sage: w0^-2 * w0^2
1
sage: w0^-2 * w0 == w0^-1
True
sage: w = w0 * w1^3
sage: w^-1
w0^5*w1^3
sage: w^-1 * w
1
sage: w * w^-1
1

sage: (2*w0)^-1
1/2*w0^5

sage: Cl = algebras.QuantumClifford(3, 1, -1)
sage: Cl.inject_variables()
Defining psi0, psi1, psi2, psid0, psid1, psid2, w0, w1, w2

sage: (w0 + w1)^-1
Traceback (most recent call last):
...
ValueError: cannot invert self (= w1 + w0)
sage: (psi0 * w0)^-1
Traceback (most recent call last):
...
ValueError: cannot invert self (= psi0*w0)

sage: z = CyclotomicField(6).gen()
sage: Cl = algebras.QuantumClifford(1, 3, z)
sage: Cl.inject_variables()
Defining psi0, psid0, w0
sage: (psi0 + psid0).inverse()
psid0*w0^3 - psi0*w0^3

sage: Cl = algebras.QuantumClifford(2, 2, -1)
sage: Cl.inject_variables()
Defining psi0, psi1, psid0, psid1, w0, w1
sage: w0^-1
w0^3
sage: w0 * w0^-1
1
```

product_on_basis(*m1*, *m2*)

Return the product of the basis elements indexed by `m1` and `m2`.

EXAMPLES:

```
sage: z = CyclotomicField(3).gen()
sage: Cl = algebras.QuantumClifford(3, 3, z)
sage: Cl.inject_variables()
Defining psi0, psi1, psi2, psid0, psid1, psid2, w0, w1, w2
```

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```
sage: psi0^2  # indirect doctest
0
sage: psid0^2
0
sage: w0 * psi0
-(-zeta3)*psi0*w0
sage: w0 * psid0
-(zeta3+1)*psid0*w0
sage: psi0 * psid0
psi0*psid0
sage: psid0 * psi0
w0^3 - psi0*psid0
sage: w2 * w0
w0*w2
sage: w0^6
1
sage: psi0 * psi1
psi0*psi1
sage: psi1 * psi0
-psi0*psi1
sage: psi1 * (psi0 * psi2)
-psi0*psi1*psi2

sage: z = CyclotomicField(6).gen()
sage: Cl = algebras.QuantumClifford(3, 3, z)
sage: Cl.inject_variables()
Defining psi0, psi1, psi2, psid0, psid1, psid2, w0, w1, w2

sage: psid1 * (psi1 * psid1)
psid1*w1^3
sage: (psi1* psid1) * (psi1 * psid1)
psi1*psid1*w1^3
sage: (psi1 * psid1) * psi1
-psi1*w1^3
```

5.18 Quantum Groups Using GAP's QuaGroup Package

AUTHORS:

- Travis Scrimshaw (03-2017): initial version

The documentation for GAP's QuaGroup package, originally authored by Willem Adriaan de Graaf, can be found at https://www.gap-system.org/Packages/quagroup.html.

class sage.algebras.quantum_groups.quantum_group_gap.**CrystalGraphVertex**(*V*, *s*)

Bases: `SageObject`

Helper class used as the vertices of a crystal graph.

class sage.algebras.quantum_groups.quantum_group_gap.**HighestWeightModule**(*Q*, *weight*)

Bases: `QuantumGroupModule`

A highest weight module of a quantum group.

Element

alias of *QuaGroupRepresentationElement*

an_element()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: V.highest_weight_vector()
1*v0
```

highest_weight_vector()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: V.highest_weight_vector()
1*v0
```

tensor(**V*, ***options*)

Return the tensor product of `self` with `V`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: Vp = Q.highest_weight_module([1,0])
sage: Vp.tensor(V)
Highest weight module of weight Lambda[1] of Quantum Group of type ['A', 2]␣
↪with q=q
 # Highest weight module of weight Lambda[1] + Lambda[2] of Quantum Group of␣
↪type ['A', 2] with q=q
```

class sage.algebras.quantum_groups.quantum_group_gap.**HighestWeightSubmodule**(*ambient*, *gen*, *weight*)

Bases: *QuantumGroupModule*

Initialize `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: S = T.highest_weight_decomposition()[0]
sage: TestSuite(S).run()
```

Element

alias of *QuaGroupRepresentationElement*

ambient()

Return the ambient module of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: S = T.highest_weight_decomposition()[0]
sage: S.ambient() is T
True
```

an_element()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: S = T.highest_weight_decomposition()[1]
sage: u = S.highest_weight_vector(); u
(1)*e.1
sage: u.lift()
-q^-1*(1*v0<x>F[a1]*v0) + 1*(F[a1]*v0<x>1*v0)
```

crystal_graph(*use_ambient=True*)

Return the crystal graph of `self`.

INPUT:

- `use_ambient` – boolean (default: `True`); if `True`, the vertices are given in terms of the ambient module

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: S = T.highest_weight_decomposition()[1]
sage: G = S.crystal_graph()
sage: sorted(G.vertices(sort=False), key=str)
[<-q^-1*(1*v0<x>F[a1+a2]*v0) + 1*(F[a1+a2]*v0<x>1*v0)>,
 <-q^-1*(1*v0<x>F[a1]*v0) + 1*(F[a1]*v0<x>1*v0)>,
 <-q^-1*(F[a1]*v0<x>F[a1+a2]*v0) + 1*(F[a1+a2]*v0<x>F[a1]*v0)>]
sage: sorted(S.crystal_graph(False).vertices(sort=False), key=str)
[<(1)*e.1>, <(1)*e.2>, <(1)*e.3>]
```

highest_weight_vector()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: S = T.highest_weight_decomposition()[1]
sage: u = S.highest_weight_vector(); u
(1)*e.1
sage: u.lift()
-q^-1*(1*v0<x>F[a1]*v0) + 1*(F[a1]*v0<x>1*v0)
```

lift()

The lift morphism from `self` to the ambient space.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: S = T.highest_weight_decomposition()[0]
sage: S.lift
Generic morphism:
  From: Highest weight submodule with weight 2*Lambda[1] generated by 1*(1*v0<x>
↪1*v0)
  To:   Highest weight module ... # Highest weight module ...
sage: x = sum(S.basis())
sage: x.lift()
1*(1*v0<x>1*v0) + 1*(1*v0<x>F[a1]*v0) + 1*(1*v0<x>F[a1+a2]*v0)
 + q^-1*(F[a1]*v0<x>1*v0) + 1*(F[a1]*v0<x>F[a1]*v0)
 + 1*(F[a1]*v0<x>F[a1+a2]*v0) + q^-1*(F[a1+a2]*v0<x>1*v0)
 + q^-1*(F[a1+a2]*v0<x>F[a1]*v0) + 1*(F[a1+a2]*v0<x>F[a1+a2]*v0)
```

retract(*elt*)

The retract map from the ambient space to `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: all(S.retract(S.lift(x)) == x
....:     for S in T.highest_weight_decomposition()
....:     for x in S.basis())
True
```

class sage.algebras.quantum_groups.quantum_group_gap.**LowerHalfQuantumGroup**(*Q*)

Bases: `Parent`, `UniqueRepresentation`

The lower half of the quantum group.

class Element(*parent*, *libgap_elt*)

Bases: *QuaGroupModuleElement*

An element of the lower half of the quantum group.

bar()

Return the bar involution on `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: F1, F2 = Q.F_simple()
sage: B = Q.lower_half()
sage: x = B(Q.an_element()); x
1 + (q)*F[a1]
sage: x.bar()
1 + (q^-1)*F[a1]
```

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```
sage: (F1*x).bar() == F1 * x.bar()
True
sage: (F2*x).bar() == F2 * x.bar()
True

sage: Q = QuantumGroup(['G',2])
sage: F1, F2 = Q.F_simple()
sage: q = Q.q()
sage: B = Q.lower_half()
sage: x = B(q^-2*F1*F2^2*F1)
sage: x
(q + q^-5)*F[a1]*F[a1+a2]*F[a2]
 + (q^8 + q^6 + q^2 + 1)*F[a1]^(2)*F[a2]^(2)
sage: x.bar()
(q^5 + q^-1)*F[a1]*F[a1+a2]*F[a2]
 + (q^12 + q^10 + q^6 + q^4)*F[a1]^(2)*F[a2]^(2)
```

braid_group_action(*braid*)

Return the action of the braid group element `braid` projected into `self`.

INPUT:

- `braid` – a reduced word of a braid group element

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: L = Q.lower_half()
sage: v = L.highest_weight_vector().f_tilde([1,2,2,1]); v
F[a1]*F[a1+a2]*F[a2]
sage: v.braid_group_action([1])
(-q^3-q)*F[a2]^(2)
sage: v.braid_group_action([]) == v
True
```

monomial_coefficients(*copy=True*)

Return the dictionary of `self` whose keys are the basis indices and the values are coefficients.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: x = B.retract(Q.an_element()); x
1 + (q)*F[a1]
sage: sorted(x.monomial_coefficients().items(), key=str)
[((0, 0, 0), 1), ((1, 0, 0), q)]
```

tau()

Return the action of the τ anti-automorphism on `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: F1, F2 = Q.F_simple()
sage: B = Q.lower_half()
```

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```
sage: x = B(Q.an_element()); x
1 + (q)*F[a1]
sage: x.tau()
1 + (q)*F[a1]
sage: (F1*x).tau() == x.tau() * F1.tau()
True
sage: (F2*x).tau() == x.tau() * F2.tau()
True

sage: Q = QuantumGroup(['G',2])
sage: F1, F2 = Q.F_simple()
sage: q = Q.q()
sage: B = Q.lower_half()
sage: x = B(q^-2*F1*F2^2*F1)
sage: x
(q + q^-5)*F[a1]*F[a1+a2]*F[a2]
 + (q^8 + q^6 + q^2 + 1)*F[a1]^(2)*F[a2]^(2)
sage: x.tau()
(q + q^-5)*F[a1]*F[a1+a2]*F[a2]
 + (q^8 + q^6 + q^2 + 1)*F[a1]^(2)*F[a2]^(2)
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.algebra_generators()
Finite family {1: F[a1], 2: F[a2]}
```

ambient()

Return the ambient quantum group of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.ambient() is Q
True
```

an_element()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.highest_weight_vector()
1
```

basis()

Return the basis of `self`.

This returns the PBW basis of `self`, which is given by monomials in $\{F_\alpha\}$, where α runs over all positive roots.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: basis = B.basis(); basis
Lazy family (monomial(i))_{i in The Cartesian product of
 (Non negative integers, Non negative integers, Non negative integers)}
sage: basis[1,2,1]
F[a1]*F[a1+a2]^(2)*F[a2]
sage: basis[1,2,4]
F[a1]*F[a1+a2]^(2)*F[a2]^(4)
sage: basis[1,0,4]
F[a1]*F[a2]^(4)
```

canonical_basis_elements()

Construct the monomial elements of `self` indexed by `k`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: C = B.canonical_basis_elements(); C
Lazy family (Canonical basis(i))_{i in The Cartesian product of
 (Non negative integers, Non negative integers)}
sage: C[2,1]
[F[a1]^(2)*F[a2], F[a1]*F[a1+a2] + (q^2)*F[a1]^(2)*F[a2]]
sage: C[1,2]
[F[a1]*F[a2]^(2), (q^2)*F[a1]*F[a2]^(2) + F[a1+a2]*F[a2]]
```

gens()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.algebra_generators()
Finite family {1: F[a1], 2: F[a2]}
```

highest_weight_vector()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.highest_weight_vector()
1
```

lift(*elt*)

Lift `elt` to the ambient quantum group of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: x = B.lift(B.an_element()); x
1
sage: x.parent() is Q
True
```

one()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.highest_weight_vector()
1
```

retract(*elt*)

Retract `elt` from the ambient quantum group to `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: x = Q.an_element(); x
1 + (q)*F[a1] + E[a1] + (q^2-1-q^-2 + q^-4)*[ K1 ; 2 ]
 + K1 + (-q^-1 + q^-3)*K1[ K1 ; 1 ]
sage: B.retract(x)
1 + (q)*F[a1]
```

zero()

Return the zero element of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: B = Q.lower_half()
sage: B.zero()
0
```

class sage.algebras.quantum_groups.quantum_group_gap.**QuaGroupModuleElement**(*parent*, *libgap_elt*)

Bases: `Element`

Base class for elements created using QuaGroup.

e_tilde(*i*)

Return the action of the Kashiwara operator $\widetilde{e}_i$ on `self`.

INPUT:

- `i` – an element of the index set or a list to perform a string of operators

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: x = Q.one().f_tilde([1,2,1,1,2,2])
sage: x.e_tilde([2,2,1,2])
F[a1]^(2)
```

f_tilde(*i*)

Return the action of the Kashiwara operator $\widetilde{f}_i$ on `self`.

INPUT:

- `i` – an element of the index set or a list to perform a string of operators

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: Q.one().f_tilde(1)
F[a1]
sage: Q.one().f_tilde(2)
F[a2]
sage: Q.one().f_tilde([1,2,1,1,2])
F[a1]*F[a1+a2]^(2)
```

gap()

Return the gap representation of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['B',3])
sage: x = Q.an_element()
sage: x.gap()
1+(q)*F1+E1+(q^4-1-q^-4+q^-8)*[ K1 ; 2 ]+K1+(-q^-2+q^-6)*K1[ K1 ; 1 ]
```

class sage.algebras.quantum_groups.quantum_group_gap.**QuaGroupRepresentationElement**(*parent*, *libgap_elt*)

Bases: *QuaGroupModuleElement*

Element of a quantum group representation.

monomial_coefficients(*copy=True*)

Return the dictionary of `self` whose keys are the basis indices and the values are coefficients.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: v = V.highest_weight_vector()
sage: F1, F2 = Q.F_simple()
sage: q = Q.q()
sage: x = v + F1*v + q*F2*F1*v; x
1*v0 + F[a1]*v0 + (q^2)*F[a1]*F[a2]*v0 + (q)*F[a1+a2]*v0
sage: sorted(x.monomial_coefficients().items(), key=str)
[(0, 1), (1, 1), (3, q^2), (4, q)]
```

class sage.algebras.quantum_groups.quantum_group_gap.**QuantumGroup**(*cartan_type*, *q*)

Bases: `UniqueRepresentation`, `Parent`

A Drinfel'd-Jimbo quantum group (implemented using the optional GAP package `QuaGroup`).

EXAMPLES:

We check the quantum Serre relations. We first we import the q-binomial using the q-int for quantum groups:

```
sage: from sage.algebras.quantum_groups.q_numbers import q_binomial
```

We verify the Serre relations for type A_2:

```
sage: Q = algebras.QuantumGroup(['A',2])
sage: F1,F12,F2 = Q.F()
sage: q = Q.q()
sage: F1^2*F2 - q_binomial(2,1,q) * F1*F2*F1 + F2*F1^2
0
```

We verify the Serre relations for type B_2:

```
sage: Q = algebras.QuantumGroup(['B',2])
sage: F1, F12, F122, F2 = Q.F()
sage: F1^2*F2 - q_binomial(2,1,q^2) * F1*F2*F1 + F2*F1^2
0
sage: (F2^3*F1 - q_binomial(3,1,q) * F2^2*F1*F2
....:  + q_binomial(3,2,q) * F2*F1*F2^2 - F1*F2^3)
0
```

REFERENCES:

- Wikipedia article Quantum_group

E()

Return the family of generators $\{E_\alpha\}_{\alpha\in\Phi}$, where Φ is the root system of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: list(Q.E())
[E[a1], E[a1+a2], E[a1+2*a2], E[a2]]
```

E_simple()

Return the family of generators $\{E_i := E_{\alpha_i}\}_{i\in I}$.

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: Q.E_simple()
Finite family {1: E[a1], 2: E[a2]}
```

class Element(*parent*, *libgap_elt*)

Bases: *QuaGroupModuleElement*

bar()

Return the bar involution on `self`.

The bar involution is defined by

$$\overline{E_i} = E_i, \qquad \overline{F_i} = F_i, \qquad \overline{K_i} = K_i^{-1}.$$

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: [gen.bar() for gen in Q.gens()]
[F[a1],
 (q-q^-1)*F[a1]*F[a2] + F[a1+a2],
 F[a2],
 (-q + q^-1)*[ K1 ; 1 ] + K1, K1,
 (-q + q^-1)*[ K2 ; 1 ] + K2, K2,
 E[a1],
 (-q^2 + 1)*E[a1]*E[a2] + (q^2)*E[a1+a2],
 E[a2]]
```

braid_group_action(*braid*)

Return the action of the braid group element braid.

The braid group operator $T_i \colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is defined by

$$\begin{aligned} T_i(E_i) &= -F_i K_i, \\ T_i(E_j) &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} \text{ if } i \neq j, \\ T_i(K_j) &= K_j K_i^{a_{ij}}, \\ T_i(F_i) &= -K_i^{-1} E_i, \\ T_i(F_j) &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} \text{ if } i \neq j, \end{aligned}$$

where $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ is the (i,j)-entry of the Cartan matrix associated to $\mathfrak{g}$.

INPUT:

- braid – a reduced word of a braid group element

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: F1 = Q.F_simple()[1]
sage: F1.braid_group_action([1])
(q-q^-1)*[ K1 ; 1 ]*E[a1] + (-1)*K1*E[a1]
sage: F1.braid_group_action([1,2])
F[a2]
sage: F1.braid_group_action([2,1])
(-q^3 + 3*q-3*q^-1 + q^-3)*[ K1 ; 1 ]*[ K2 ; 1 ]*E[a1]*E[a2]
 + (q^3-2*q + q^-1)*[ K1 ; 1 ]*[ K2 ; 1 ]*E[a1+a2]
 + (q^2-2 + q^-2)*[ K1 ; 1 ]*K2*E[a1]*E[a2]
 + (-q^2 + 1)*[ K1 ; 1 ]*K2*E[a1+a2]
 + (q^2-2 + q^-2)*K1*[ K2 ; 1 ]*E[a1]*E[a2]
 + (-q^2 + 1)*K1*[ K2 ; 1 ]*E[a1+a2]
 + (-q + q^-1)*K1*K2*E[a1]*E[a2] + (q)*K1*K2*E[a1+a2]
sage: F1.braid_group_action([1,2,1]) == F1.braid_group_action([2,1,2])
True
sage: F1.braid_group_action([]) == F1
True
```

omega()

Return the action of the ω automorphism on self.

The ω automorphism is defined by

$$\omega(E_i) = F_i, \qquad \omega(F_i) = E_i, \qquad \omega(K_i) = K_i^{-1}.$$

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: [gen.omega() for gen in Q.gens()]
[E[a1],
 (-q)*E[a1+a2],
 E[a2],
 (-q + q^-1)*[ K1 ; 1 ] + K1,
 K1,
 (-q + q^-1)*[ K2 ; 1 ] + K2,
 K2,
 F[a1],
 (-q^-1)*F[a1+a2],
 F[a2]]
```

tau()

Return the action of the τ anti-automorphism on `self`.

The τ anti-automorphism is defined by

$$\tau(E_i) = E_i, \qquad \tau(F_i) = F_i, \qquad \tau(K_i) = K_i^{-1}.$$

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: [gen.tau() for gen in Q.gens()]
[F[a1],
 (-q^2 + 1)*F[a1]*F[a2] + (-q)*F[a1+a2],
 F[a2],
 (-q + q^-1)*[ K1 ; 1 ] + K1,
 K1,
 (-q + q^-1)*[ K2 ; 1 ] + K2,
 K2,
 E[a1],
 (q-q^-1)*E[a1]*E[a2] + (-q)*E[a1+a2],
 E[a2]]
```

F()

Return the family of generators $\{F_\alpha\}_{\alpha\in\Phi}$, where Φ is the root system of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['G',2])
sage: list(Q.F())
[F[a1], F[3*a1+a2], F[2*a1+a2], F[3*a1+2*a2], F[a1+a2], F[a2]]
```

F_simple()

Return the family of generators $\{F_i := F_{\alpha_i}\}_{i\in I}$.

EXAMPLES:

```
sage: Q = QuantumGroup(['G',2])
sage: Q.F_simple()
Finite family {1: F[a1], 2: F[a2]}
```

K()

Return the family of generators $\{K_i\}_{i \in I}$.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',3])
sage: Q.K()
Finite family {1: K1, 2: K2, 3: K3}
sage: Q.K_inverse()
Finite family {1: (-q + q^-1)*[ K1 ; 1 ] + K1,
               2: (-q + q^-1)*[ K2 ; 1 ] + K2,
               3: (-q + q^-1)*[ K3 ; 1 ] + K3}
```

K_inverse()

Return the family of generators $\{K_i^{-1}\}_{i \in I}$.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',3])
sage: Q.K_inverse()
Finite family {1: (-q + q^-1)*[ K1 ; 1 ] + K1,
               2: (-q + q^-1)*[ K2 ; 1 ] + K2,
               3: (-q + q^-1)*[ K3 ; 1 ] + K3}
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: list(Q.algebra_generators())
[F[a1], F[a2],
 K1, K2,
 (-q + q^-1)*[ K1 ; 1 ] + K1, (-q + q^-1)*[ K2 ; 1 ] + K2,
 E[a1], E[a2]]
```

antipode(*elt*)

Return the antipode of `elt`.

The antipode $S\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the anti-automorphism defined by

$$S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i, \qquad S(K_i) = K_i^{-1}.$$

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: [Q.antipode(f) for f in Q.F()]
[(-1)*F[a1]*K1,
 (-q^6 + q^2)*F[a1]*F[a2]*K1*K2 + (-q^4)*F[a1+a2]*K1*K2,
 (-q^8 + q^6 + q^4-q^2)*F[a1]*F[a2]^(2)*K1
  + (-q^9 + 2*q^7-2*q^3 + q)*F[a1]*F[a2]^(2)*K1*K2[ K2 ; 1 ]
```

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```
 + (-q^5 + q^3)*F[a1+a2]*F[a2]*K1
 + (-q^6 + 2*q^4-q^2)*F[a1+a2]*F[a2]*K1*K2[ K2 ; 1 ]
 + (-q^4)*F[a1+2*a2]*K1 + (-q^5 + q^3)*F[a1+2*a2]*K1*K2[ K2 ; 1 ],
(-1)*F[a2]*K2]
```

cartan_type()

Return the Cartan type of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.cartan_type()
['A', 2]
```

coproduct(*elt*, *n=1*)

Return the coproduct of `elt` (iterated `n` times).

The comultiplication $\Delta: U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is defined by

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, \\ \Delta(K_i) &= K_i \otimes K_i. \end{aligned}$$

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: [Q.coproduct(e) for e in Q.E()]
[1*(E[a1]<x>1) + 1*(K1<x>E[a1]),
 1*(E[a1+a2]<x>1) + 1*(K1*K2<x>E[a1+a2]) + q^2-q^-2*(K2*E[a1]<x>E[a2]),
 q^4-q^2-1 + q^-2*(E[a1]<x>E[a2]^(2)) + 1*(E[a1+2*a2]<x>1)
  + 1*(K1<x>E[a1+2*a2]) + q-q^-1*(K1*K2[ K2 ; 1 ]<x>E[a1+2*a2])
  + q-q^-1*(K2*E[a1+a2]<x>E[a2]) + q^5-2*q^3
  + 2*q^-1-q^-3*(K2[ K2 ; 1 ]*E[a1]<x>E[a2]^(2)),
 1*(E[a2]<x>1) + 1*(K2<x>E[a2])]
sage: [Q.coproduct(f, 2) for f in Q.F_simple()]
[1*(1<x>1<x>F[a1]) + -q^2 + q^-2*(1<x>F[a1]<x>[ K1 ; 1 ])
  + 1*(1<x>F[a1]<x>K1) + q^4-2 + q^-4*(F[a1]<x>[ K1 ; 1 ]<x>[ K1 ; 1 ])
  + -q^2 + q^-2*(F[a1]<x>[ K1 ; 1 ]<x>K1) + -q^2
  + q^-2*(F[a1]<x>K1<x>[ K1 ; 1 ]) + 1*(F[a1]<x>K1<x>K1),
 1*(1<x>1<x>F[a2]) + -q + q^-1*(1<x>F[a2]<x>[ K2 ; 1 ])
  + 1*(1<x>F[a2]<x>K2) + q^2-2 + q^-2*(F[a2]<x>[ K2 ; 1 ]<x>[ K2 ; 1 ])
  + -q + q^-1*(F[a2]<x>[ K2 ; 1 ]<x>K2) + -q
  + q^-1*(F[a2]<x>K2<x>[ K2 ; 1 ]) + 1*(F[a2]<x>K2<x>K2)]
```

counit(*elt*)

Return the counit of `elt`.

The counit $\varepsilon: U_q(\mathfrak{g}) \to \mathbf{Q}(q)$ is defined by

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1.$$

EXAMPLES:

```
sage: Q = QuantumGroup(['B',2])
sage: x = Q.an_element()^2
sage: Q.counit(x)
4
sage: Q.counit(Q.one())
1
sage: Q.counit(Q.zero())
0
```

gap()

Return the gap representation of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.gap()
QuantumUEA( <root system of type A2>, Qpar = q )
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.gens()
(F[a1], F[a1+a2], F[a2],
 K1, (-q + q^-1)*[ K1 ; 1 ] + K1,
 K2, (-q + q^-1)*[ K2 ; 1 ] + K2,
 E[a1], E[a1+a2], E[a2])
```

highest_weight_module(*weight*)

Return the highest weight module of weight `weight` of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.highest_weight_module([1,3])
Highest weight module of weight Lambda[1] + 3*Lambda[2] of
 Quantum Group of type ['A', 2] with q=q
```

lower_half()

Return the lower half of the quantum group `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.lower_half()
Lower Half of Quantum Group of type ['A', 2] with q=q
```

one()

Return the multiplicative identity of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.one()
1
```

q()

Return the parameter q.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',3])
sage: Q.q()
q
sage: zeta3 = CyclotomicField(3).gen()
sage: Q = QuantumGroup(['B',2], q=zeta3)
sage: Q.q()
zeta3
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',1])
sage: Q.some_elements()
[1 + (q)*F[a1] + E[a1] + (q^2-1-q^-2 + q^-4)*[ K1 ; 2 ]
  + K1 + (-q^-1 + q^-3)*K1[ K1 ; 1 ],
 K1, F[a1], E[a1]]
```

zero()

Return the multiplicative identity of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: Q.zero()
0
```

class `sage.algebras.quantum_groups.quantum_group_gap.`**QuantumGroupHomset**(*X*, *Y*, *category=None*, *check=True*, *base=None*)

Bases: `HomsetWithBase`

The homset whose domain is a quantum group.

class `sage.algebras.quantum_groups.quantum_group_gap.`**QuantumGroupModule**(*Q*, *category*)

Bases: `Parent`, `UniqueRepresentation`

Abstract base class for quantum group representations.

R_matrix()

Return the R-matrix of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',1])
sage: V = Q.highest_weight_module([1])
sage: V.R_matrix()
[       1        0        0        0]
[       0        q -q^2 + 1        0]
[       0        0        q        0]
[       0        0        0        1]
```

basis()

Return a basis of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: V.basis()
Family (1*v0, F[a1]*v0, F[a2]*v0, F[a1]*F[a2]*v0, F[a1+a2]*v0,
        F[a1]*F[a1+a2]*v0, F[a1+a2]*F[a2]*v0, F[a1+a2]^(2)*v0)
```

crystal_basis()

Return the crystal basis of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: V.crystal_basis()
Family (1*v0, F[a1]*v0, F[a2]*v0, F[a1]*F[a2]*v0,
        (q)*F[a1]*F[a2]*v0 + F[a1+a2]*v0, F[a1+a2]*F[a2]*v0,
        (-q^-2)*F[a1]*F[a1+a2]*v0, (-q^-1)*F[a1+a2]^(2)*v0)
```

crystal_graph()

Return the crystal graph of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: G = V.crystal_graph(); G
Digraph on 8 vertices

sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: G.is_isomorphic(B.digraph(), edge_labels=True)
True
```

gap()

Return the gap representation of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: V.gap()
<8-dimensional left-module over QuantumUEA( <root system of type A2>,
 Qpar = q )>
```

zero()

Return the zero element of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: V.zero()
0*v0
```

class sage.algebras.quantum_groups.quantum_group_gap.**QuantumGroupMorphism**(*parent*, *im_gens*, *check=True*)

Bases: `Morphism`

A morphism whose domain is a quantum group.

im_gens()

Return the image of the generators under `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',1])
sage: F, K, Ki, E = Q.gens()
sage: phi = Q.hom([E, Ki, K, F])
sage: phi.im_gens()
(E[a1], (-q + q^-1)*[ K1 ; 1 ] + K1, K1, F[a1])
```

class sage.algebras.quantum_groups.quantum_group_gap.**TensorProductOfHighestWeightModules**(**modules*, ***options*)

Bases: *QuantumGroupModule*

Initialize `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,1])
sage: T = tensor([V,V])
sage: TestSuite(T).run()
```

Element

alias of *QuaGroupRepresentationElement*

highest_weight_decomposition()

Return the highest weight decomposition of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: T.highest_weight_decomposition()
[Highest weight submodule with weight 2*Lambda[1] generated by 1*(1*v0<x>1*v0),
 Highest weight submodule with weight Lambda[2] generated by -q^-1*(1*v0<x>
↪F[a1]*v0) + 1*(F[a1]*v0<x>1*v0)]
```

highest_weight_vectors()

Return the highest weight vectors of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: T.highest_weight_vectors()
[1*(1*v0<x>1*v0), -q^-1*(1*v0<x>F[a1]*v0) + 1*(F[a1]*v0<x>1*v0)]
```

some_elements()

Return the highest weight vectors of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: T.highest_weight_vectors()
[1*(1*v0<x>1*v0), -q^-1*(1*v0<x>F[a1]*v0) + 1*(F[a1]*v0<x>1*v0)]
```

tensor_factors()

Return the factors of `self`.

EXAMPLES:

```
sage: Q = QuantumGroup(['A',2])
sage: V = Q.highest_weight_module([1,0])
sage: T = tensor([V,V])
sage: T.tensor_factors()
(Highest weight module of weight Lambda[1] of Quantum Group of type ['A', 2]␣
↪with q=q,
 Highest weight module of weight Lambda[1] of Quantum Group of type ['A', 2]␣
↪with q=q)
```

sage.algebras.quantum_groups.quantum_group_gap.**projection_lower_half**(*Q*)

Return the projection onto the lower half of the quantum group.

EXAMPLES:

```
sage: from sage.algebras.quantum_groups.quantum_group_gap import projection_lower_
↪half
sage: Q = QuantumGroup(['G',2])
sage: phi = projection_lower_half(Q); phi
Quantum group homomorphism endomorphism of Quantum Group of type ['G', 2] with q=q
  Defn: F[a1] |--> F[a1]
        F[a2] |--> F[a2]
        K1 |--> 0
        K2 |--> 0
        (-q + q^-1)*[ K1 ; 1 ] + K1 |--> 0
        (-q^3 + q^-3)*[ K2 ; 1 ] + K2 |--> 0
        E[a1] |--> 0
        E[a2] |--> 0
sage: all(phi(f) == f for f in Q.F())
```

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```
True
sage: all(phi(e) == Q.zero() for e in Q.E())
True
sage: all(phi(K) == Q.zero() for K in Q.K())
True
```

5.19 Quantum Matrix Coordinate Algebras

AUTHORS:

- Travis Scrimshaw (01-2016): initial version

class sage.algebras.quantum_matrix_coordinate_algebra.**QuantumGL**(*n*, *q*, *bar*, *R*)

Bases: *QuantumMatrixCoordinateAlgebra_abstract*

Quantum coordinate algebra of $GL(n)$.

The quantum coordinate algebra of $GL(n)$, or quantum $GL(n)$ for short and denoted by $\mathcal{O}_q(GL(n))$, is the quantum coordinate algebra of $M_R(n, n)$ with the addition of the additional central group-like element c which satisfies $cd = dc = 1$, where d is the quantum determinant.

Quantum $GL(n)$ is a Hopf algebra where $\varepsilon(c) = 1$ and the antipode S is given by the (quantum) matrix inverse. That is to say, we have $S(c) = c^-1 = d$ and

$$S(x_{ij}) = c * (-q)^{i-j} * \tilde{t}_{ji},$$

where we have the quantum minor

$$\tilde{t}_{ij} = \sum_{\sigma} (-q)^{\ell(\sigma)} x_{1,\sigma(1)} \cdots x_{i-1,\sigma(i-1)} x_{i+1,\sigma(i+1)} \cdots x_{n,\sigma(n)}$$

with the sum over permutations $\sigma \colon \{1, \ldots, i-1, i+1, \ldots n\} \to \{1, \ldots, j-1, j+1, \ldots, n\}$.

See also:

QuantumMatrixCoordinateAlgebra

INPUT:

- n – the integer n
- R – (optional) the ring R if q is not specified (the default is $\mathbf{Z}$); otherwise the ring containing q
- q – (optional) the variable q; the default is $q \in R[q, q^{-1}]$
- bar – (optional) the involution on the base ring; the default is $q \mapsto q^{-1}$

EXAMPLES:

We construct $\mathcal{O}_q(GL(3))$ and the variables:

```
sage: O = algebras.QuantumGL(3)
sage: O.inject_variables()
Defining x11, x12, x13, x21, x22, x23, x31, x32, x33, c
```

We do some basic computations:

```
sage: x33 * x12
x[1,2]*x[3,3] + (q^-1-q)*x[1,3]*x[3,2]
sage: x23 * x12 * x11
(q^-1)*x[1,1]*x[1,2]*x[2,3] + (q^-2-1)*x[1,1]*x[1,3]*x[2,2]
 + (q^-3-q^-1)*x[1,2]*x[1,3]*x[2,1]
sage: c * O.quantum_determinant()
1
```

We verify the quantum determinant is in the center and is group-like:

```
sage: qdet = O.quantum_determinant()
sage: all(qdet * g == g * qdet for g in O.algebra_generators())
True
sage: qdet.coproduct() == tensor([qdet, qdet])
True
```

We check that the inverse of the quantum determinant is also in the center and group-like:

```
sage: all(c * g == g * c for g in O.algebra_generators())
True
sage: c.coproduct() == tensor([c, c])
True
```

Moreover, the antipode interchanges the quantum determinant and its inverse:

```
sage: c.antipode() == qdet
True
sage: qdet.antipode() == c
True
```

REFERENCES:

- [DD1991]
- [Kar1993]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: O = algebras.QuantumGL(2)
sage: O.algebra_generators()
Finite family {(1, 1): x[1,1], (1, 2): x[1,2], (2, 1): x[2,1], (2, 2): x[2,2],
↪'c': c}
```

antipode_on_basis(*x*)

Return the antipode of the basis element indexed by `x`.

EXAMPLES:

```
sage: O = algebras.QuantumGL(3)
sage: x = O.indices().monoid_generators()
sage: O.antipode_on_basis(x[1,2])
-(q^-1)*c*x[1,2]*x[3,3] + c*x[1,3]*x[3,2]
sage: O.antipode_on_basis(x[2,2])
```

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```
c*x[1,1]*x[3,3] - q*c*x[1,3]*x[3,1]
sage: O.antipode_on_basis(x['c']) == O.quantum_determinant()
True
```

coproduct_on_basis(x)

Return the coproduct on the basis element indexed by x.

EXAMPLES:

```
sage: O = algebras.QuantumGL(3)
sage: x = O.indices().monoid_generators()
sage: O.coproduct_on_basis(x[1,2])
x[1,1] # x[1,2] + x[1,2] # x[2,2] + x[1,3] # x[3,2]
sage: O.coproduct_on_basis(x[2,2])
x[2,1] # x[1,2] + x[2,2] # x[2,2] + x[2,3] # x[3,2]
sage: O.coproduct_on_basis(x['c'])
c # c
```

product_on_basis(a, b)

Return the product of basis elements indexed by a and b.

EXAMPLES:

```
sage: O = algebras.QuantumGL(2)
sage: I = O.indices().monoid_generators()
sage: O.product_on_basis(I[1,1], I[2,2])
x[1,1]*x[2,2]
sage: O.product_on_basis(I[2,2], I[1,1])
x[1,1]*x[2,2] + (q^-1-q)*x[1,2]*x[2,1]
```

class sage.algebras.quantum_matrix_coordinate_algebra.**QuantumMatrixCoordinateAlgebra**(m, n, q, bar, R)

Bases: *QuantumMatrixCoordinateAlgebra_abstract*

A quantum matrix coordinate algebra.

Let R be a commutative ring. The quantum matrix coordinate algebra of $M(m, n)$ is the associative algebra over $R[q, q^{-1}]$ generated by x_{ij}, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and subject to the following relations:

$$\begin{array}{ll} x_{it}x_{ij} = q^{-1}x_{ij}x_{it} & \text{if } j < t, \\ x_{sj}x_{ij} = q^{-1}x_{ij}x_{sj} & \text{if } i < s, \\ x_{st}x_{ij} = x_{ij}x_{st} & \text{if } i < s, j > t, \\ x_{st}x_{ij} = x_{ij}x_{st} + (q^{-1} - q)x_{it}x_{sj} & \text{if } i < s, j < t. \end{array}$$

The quantum matrix coordinate algebra is denoted by $\mathcal{O}_q(M(m, n))$. For $m = n$, it is also a bialgebra given by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \varepsilon(x_{ij}) = \delta_{ij}.$$

Moreover, there is a central group-like element called the *quantum determinant* that is defined by

$$\det_q = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)}.$$

The quantum matrix coordinate algebra also has natural inclusions when restricting to submatrices. That is, let $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$. Then the subalgebra generated by $\{x_{ij} \mid i \in I, j \in J\}$ is naturally isomorphic to $\mathcal{O}_q(M(|I|, |J|))$.

Note: The q considered here is q^2 in some references, e.g., [ZZ2005].

INPUT:

- m – the integer m
- n – the integer n
- R – (optional) the ring R if q is not specified (the default is $\mathbf{Z}$); otherwise the ring containing q
- q – (optional) the variable q; the default is $q \in R[q, q^{-1}]$
- bar – (optional) the involution on the base ring; the default is $q \mapsto q^{-1}$

EXAMPLES:

We construct $\mathcal{O}_q(M(2,3))$ and the variables:

```
sage: O = algebras.QuantumMatrixCoordinate(2,3)
sage: O.inject_variables()
Defining x11, x12, x13, x21, x22, x23
```

We do some basic computations:

```
sage: x21 * x11
(q^-1)*x[1,1]*x[2,1]
sage: x23 * x12 * x11
(q^-1)*x[1,1]*x[1,2]*x[2,3] + (q^-2-1)*x[1,1]*x[1,3]*x[2,2]
 + (q^-3-q^-1)*x[1,2]*x[1,3]*x[2,1]
```

We construct the maximal quantum minors:

```
sage: q = O.q()
sage: qm12 = x11*x22 - q*x12*x21
sage: qm13 = x11*x23 - q*x13*x21
sage: qm23 = x12*x23 - q*x13*x22
```

However, unlike for the quantum determinant, they are not central:

```
sage: all(qm12 * g == g * qm12 for g in O.algebra_generators())
False
sage: all(qm13 * g == g * qm13 for g in O.algebra_generators())
False
sage: all(qm23 * g == g * qm23 for g in O.algebra_generators())
False
```

REFERENCES:

- [FRT1990]
- [ZZ2005]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(2)
sage: O.algebra_generators()
Finite family {(1, 1): x[1,1], (1, 2): x[1,2], (2, 1): x[2,1], (2, 2): x[2,2]}
```

coproduct_on_basis(x)

Return the coproduct on the basis element indexed by `x`.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: x24 = O.algebra_generators()[2,4]
sage: O.coproduct_on_basis(x24.leading_support())
x[2,1] # x[1,4] + x[2,2] # x[2,4] + x[2,3] # x[3,4] + x[2,4] # x[4,4]
```

m()

Return the value m.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4, 6)
sage: O.m()
4
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: O.m()
4
```

class sage.algebras.quantum_matrix_coordinate_algebra.**QuantumMatrixCoordinateAlgebra_abstract**(*gp_indices*, *n*, *q*, *bar*, *R*, *category*, *indices_key=N*

Bases: `CombinatorialFreeModule`

Abstract base class for quantum coordinate algebras of a set of matrices.

class Element

Bases: `IndexedFreeModuleElement`

An element of a quantum matrix coordinate algebra.

bar()

Return the image of `self` under the bar involution.

The bar involution is the $\mathbf{Q}$-algebra anti-automorphism defined by $x_{ij} \mapsto x_{ji}$ and $q \mapsto q^{-1}$.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: x = O.an_element()
sage: x.bar()
1 + 2*x[1,1] + (q^-16)*x[1,1]^2*x[1,2]^2*x[1,3]^3 + 3*x[1,2]
sage: x = O.an_element() * O.algebra_generators()[2,4]; x
x[1,1]^2*x[1,2]^2*x[1,3]^3*x[2,4] + 2*x[1,1]*x[2,4]
 + 3*x[1,2]*x[2,4] + x[2,4]
sage: xb = x.bar(); xb
(q^-16)*x[1,1]^2*x[1,2]^2*x[1,3]^3*x[2,4]
 + (q^-21-q^-15)*x[1,1]^2*x[1,2]^2*x[1,3]^2*x[1,4]*x[2,3]
 + (q^-22-q^-18)*x[1,1]^2*x[1,2]*x[1,3]^3*x[1,4]*x[2,2]
 + (q^-24-q^-20)*x[1,1]*x[1,2]^2*x[1,3]^3*x[1,4]*x[2,1]
 + 2*x[1,1]*x[2,4] + 3*x[1,2]*x[2,4]
 + (2*q^-1-2*q)*x[1,4]*x[2,1]
 + (3*q^-1-3*q)*x[1,4]*x[2,2] + x[2,4]
sage: xb.bar() == x
True
```

counit_on_basis(x)

Return the counit on the basis element indexed by `x`.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: G = O.algebra_generators()
sage: I = [1,2,3,4]
sage: matrix([[G[i,j].counit() for i in I] for j in I])  # indirect doctest
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

gens()

Return the generators of `self` as a tuple.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(3)
sage: O.gens()
(x[1,1], x[1,2], x[1,3],
 x[2,1], x[2,2], x[2,3],
 x[3,1], x[3,2], x[3,3])
```

n()

Return the value n.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: O.n()
4
sage: O = algebras.QuantumMatrixCoordinate(4, 6)
sage: O.n()
6
```

one_basis()

Return the basis element indexing 1.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: O.one_basis()
1
sage: O.one()
1
```

product_on_basis(a, b)

Return the product of basis elements indexed by a and b.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: x = O.algebra_generators()
sage: b = x[1,4] * x[2,1] * x[3,4]  # indirect doctest
sage: b * (b * b) == (b * b) * b
True
sage: p = prod(list(O.algebra_generators())[:10])
sage: p * (p * p) == (p * p) * p  # long time
True
sage: x = O.an_element()
sage: y = x^2 + x[4,4] * x[3,3] * x[1,2]
sage: z = x[2,2] * x[1,4] * x[3,4] * x[1,1]
sage: x * (y * z) == (x * y) * z
True
```

q()

Return the variable q.

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(4)
sage: O.q()
q
sage: O.q().parent()
Univariate Laurent Polynomial Ring in q over Integer Ring
sage: O.q().parent() is O.base_ring()
True
```

quantum_determinant()

Return the quantum determinant of `self`.

The quantum determinant is defined by

$$\det_q = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)}.$$

EXAMPLES:

```
sage: O = algebras.QuantumMatrixCoordinate(2)
sage: O.quantum_determinant()
x[1,1]*x[2,2] - q*x[1,2]*x[2,1]
```

We verify that the quantum determinant is central:

```
sage: for n in range(2,5):
....:     O = algebras.QuantumMatrixCoordinate(n)
....:     qdet = O.quantum_determinant()
....:     assert all(g * qdet == qdet * g for g in O.algebra_generators())
```

We also verify that it is group-like:

```
sage: for n in range(2,4):
....:     O = algebras.QuantumMatrixCoordinate(n)
....:     qdet = O.quantum_determinant()
....:     assert qdet.coproduct() == tensor([qdet, qdet])
```

5.20 Quaternion Algebras

AUTHORS:

- Jon Bobber (2009): rewrite
- William Stein (2009): rewrite
- Julian Rueth (2014-03-02): use UniqueFactory for caching
- Peter Bruin (2021): do not require the base ring to be a field

This code is partly based on Sage code by David Kohel from 2005.

class sage.algebras.quatalg.quaternion_algebra.**QuaternionAlgebraFactory**

Bases: `UniqueFactory`

Construct a quaternion algebra.

INPUT:

There are three input formats:

- `QuaternionAlgebra(a, b)`, where a and b can be coerced to units in a common field K of characteristic different from 2.
- `QuaternionAlgebra(K, a, b)`, where K is a ring in which 2 is a unit and a and b are units of K.
- `QuaternionAlgebra(D)`, where $D \geq 1$ is a squarefree integer. This constructs a quaternion algebra of discriminant D over $K = \mathbf{Q}$. Suitable nonzero rational numbers a, b as above are deduced from D.

OUTPUT:

The quaternion algebra $(a, b)_K$ over K generated by i, j subject to $i^2 = a$, $j^2 = b$, and $ji = -ij$.

EXAMPLES:

`QuaternionAlgebra(a, b)` – return the quaternion algebra $(a, b)_K$, where the base ring K is a suitably chosen field containing a and b:

```
sage: QuaternionAlgebra(-2,-3)
Quaternion Algebra (-2, -3) with base ring Rational Field
sage: QuaternionAlgebra(GF(5)(2), GF(5)(3))
Quaternion Algebra (2, 3) with base ring Finite Field of size 5
sage: QuaternionAlgebra(2, GF(5)(3))
```

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```
Quaternion Algebra (2, 3) with base ring Finite Field of size 5
sage: QuaternionAlgebra(QQ[sqrt(2)](-1), -5)
Quaternion Algebra (-1, -5) with base ring Number Field in sqrt2 with defining␣
↪polynomial x^2 - 2 with sqrt2 = 1.414213562373095?
sage: QuaternionAlgebra(sqrt(-1), sqrt(-3))
Quaternion Algebra (I, sqrt(-3)) with base ring Symbolic Ring
sage: QuaternionAlgebra(1r,1)
Quaternion Algebra (1, 1) with base ring Rational Field
sage: A.<t> = ZZ[]
sage: QuaternionAlgebra(-1, t)
Quaternion Algebra (-1, t) with base ring Fraction Field of Univariate Polynomial␣
↪Ring in t over Integer Ring
```

Python ints and floats may be passed to the `QuaternionAlgebra(a, b)` constructor, as may all pairs of nonzero elements of a domain not of characteristic 2.

The following tests address the issues raised in github issue #10601:

```
sage: QuaternionAlgebra(1r,1)
Quaternion Algebra (1, 1) with base ring Rational Field
sage: QuaternionAlgebra(1,1.0r)
Quaternion Algebra (1.00000000000000, 1.00000000000000) with base ring Real Field␣
↪with 53 bits of precision
sage: QuaternionAlgebra(0,0)
Traceback (most recent call last):
...
ValueError: defining elements of quaternion algebra (0, 0) are not invertible in␣
↪Rational Field
sage: QuaternionAlgebra(GF(2)(1),1)
Traceback (most recent call last):
...
ValueError: 2 is not invertible in Finite Field of size 2
sage: a = PermutationGroupElement([1,2,3])
sage: QuaternionAlgebra(a, a)
Traceback (most recent call last):
...
ValueError: a and b must be elements of a ring with characteristic not 2
```

`QuaternionAlgebra(K, a, b)` – return the quaternion algebra defined by (a, b) over the ring K:

```
sage: QuaternionAlgebra(QQ, -7, -21)
Quaternion Algebra (-7, -21) with base ring Rational Field
sage: QuaternionAlgebra(QQ[sqrt(2)], -2,-3)
Quaternion Algebra (-2, -3) with base ring Number Field in sqrt2 with defining␣
↪polynomial x^2 - 2 with sqrt2 = 1.414213562373095?
```

`QuaternionAlgebra(D)` – D is a squarefree integer; return a rational quaternion algebra of discriminant D:

```
sage: QuaternionAlgebra(1)
Quaternion Algebra (-1, 1) with base ring Rational Field
sage: QuaternionAlgebra(2)
Quaternion Algebra (-1, -1) with base ring Rational Field
sage: QuaternionAlgebra(7)
```

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```
Quaternion Algebra (-1, -7) with base ring Rational Field
sage: QuaternionAlgebra(2*3*5*7)
Quaternion Algebra (-22, 210) with base ring Rational Field
```

If the coefficients a and b in the definition of the quaternion algebra are not integral, then a slower generic type is used for arithmetic:

```
sage: type(QuaternionAlgebra(-1,-3).0)
<... 'sage.algebras.quatalg.quaternion_algebra_element.QuaternionAlgebraElement_
↪rational_field'>
sage: type(QuaternionAlgebra(-1,-3/2).0)
<... 'sage.algebras.quatalg.quaternion_algebra_element.QuaternionAlgebraElement_
↪generic'>
```

Make sure caching is sane:

```
sage: A = QuaternionAlgebra(2,3); A
Quaternion Algebra (2, 3) with base ring Rational Field
sage: B = QuaternionAlgebra(GF(5)(2),GF(5)(3)); B
Quaternion Algebra (2, 3) with base ring Finite Field of size 5
sage: A is QuaternionAlgebra(2,3)
True
sage: B is QuaternionAlgebra(GF(5)(2),GF(5)(3))
True
sage: Q = QuaternionAlgebra(2); Q
Quaternion Algebra (-1, -1) with base ring Rational Field
sage: Q is QuaternionAlgebra(QQ,-1,-1)
True
sage: Q is QuaternionAlgebra(-1,-1)
True
sage: Q.<ii,jj,kk> = QuaternionAlgebra(15); Q.variable_names()
('ii', 'jj', 'kk')
sage: QuaternionAlgebra(15).variable_names()
('i', 'j', 'k')
```

create_key(*arg0*, *arg1=None*, *arg2=None*, *names='i,j,k'*)

Create a key that uniquely determines a quaternion algebra.

create_object(*version*, *key*, ***extra_args*)

Create the object from the key (extra arguments are ignored). This is only called if the object was not found in the cache.

class sage.algebras.quatalg.quaternion_algebra.**QuaternionAlgebra_ab**(*base_ring*, *a*, *b*, *names='i,j,k'*)

Bases: *QuaternionAlgebra_abstract*

A quaternion algebra of the form $(a, b)_K$.

See `QuaternionAlgebra` for many more examples.

INPUT:

- `base_ring` – a commutative ring K in which 2 is invertible
- `a, b` – units of K

- `names` – string (optional, default 'i,j,k') names of the generators

OUTPUT:

The quaternion algebra (a, b) over K generated by i and j subject to $i^2 = a$, $j^2 = b$, and $ji = -ij$.

EXAMPLES:

```
sage: QuaternionAlgebra(QQ, -7, -21)  # indirect doctest
Quaternion Algebra (-7, -21) with base ring Rational Field
```

discriminant()

Given a quaternion algebra A defined over a number field, return the discriminant of A, i.e. the product of the ramified primes of A.

EXAMPLES:

```
sage: QuaternionAlgebra(210,-22).discriminant()
210
sage: QuaternionAlgebra(19).discriminant()
19

sage: x = polygen(ZZ, 'x')
sage: F.<a> = NumberField(x^2 - x - 1)
sage: B.<i,j,k> = QuaternionAlgebra(F, 2*a, F(-1))
sage: B.discriminant()
Fractional ideal (2)

sage: QuaternionAlgebra(QQ[sqrt(2)],3,19).discriminant()
Fractional ideal (1)
```

gen(*i=0*)

Return the i^{th} generator of `self`.

INPUT:

- `i` - integer (optional, default 0)

EXAMPLES:

```
sage: Q.<ii,jj,kk> = QuaternionAlgebra(QQ,-1,-2); Q
Quaternion Algebra (-1, -2) with base ring Rational Field
sage: Q.gen(0)
ii
sage: Q.gen(1)
jj
sage: Q.gen(2)
kk
sage: Q.gens()
[ii, jj, kk]
```

ideal(*gens*, *left_order=None*, *right_order=None*, *check=True*, ***kwds*)

Return the quaternion ideal with given gens over $\mathbf{Z}$.

Neither a left or right order structure need be specified.

INPUT:

- `gens` – a list of elements of this quaternion order

- `check` – bool (default: `True`)
- `left_order` – a quaternion order or `None`
- `right_order` – a quaternion order or `None`

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1)
sage: R.ideal([2*a for a in R.basis()])
Fractional ideal (2, 2*i, 2*j, 2*k)
```

inner_product_matrix()

Return the inner product matrix associated to `self`, i.e. the Gram matrix of the reduced norm as a quadratic form on `self`. The standard basis 1, i, j, k is orthogonal, so this matrix is just the diagonal matrix with diagonal entries 1, a, b, ab.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-5,-19)
sage: Q.inner_product_matrix()
[  2   0   0   0]
[  0  10   0   0]
[  0   0  38   0]
[  0   0   0 190]

sage: R.<a,b> = QQ[]; Q.<i,j,k> = QuaternionAlgebra(Frac(R),a,b)
sage: Q.inner_product_matrix()
[    2     0     0     0]
[    0  -2*a     0     0]
[    0     0  -2*b     0]
[    0     0     0 2*a*b]
```

invariants()

Return the structural invariants a, b of this quaternion algebra: `self` is generated by i, j subject to $i^2 = a$, $j^2 = b$ and $ji = -ij$.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(15)
sage: Q.invariants()
(-3, 5)
sage: i^2
-3
sage: j^2
5
```

maximal_order(*take_shortcuts=True*)

Return a maximal order in this quaternion algebra.

The algorithm used is from [Voi2012].

INPUT:

- `take_shortcuts` – (default: `True`) if the discriminant is prime and the invariants of the algebra are of a nice form, use Proposition 5.2 of [Piz1980].

OUTPUT:

A maximal order in this quaternion algebra.

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order()
Order of Quaternion Algebra (-1, -7) with base ring Rational Field with basis␣
↪(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)

sage: QuaternionAlgebra(-1,-1).maximal_order().basis()
(1/2 + 1/2*i + 1/2*j + 1/2*k, i, j, k)

sage: QuaternionAlgebra(-1,-11).maximal_order().basis()
(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)

sage: QuaternionAlgebra(-1,-3).maximal_order().basis()
(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)

sage: QuaternionAlgebra(-3,-1).maximal_order().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)

sage: QuaternionAlgebra(-2,-5).maximal_order().basis()
(1/2 + 1/2*j + 1/2*k, 1/4*i + 1/2*j + 1/4*k, j, k)

sage: QuaternionAlgebra(-5,-2).maximal_order().basis()
(1/2 + 1/2*i - 1/2*k, 1/2*i + 1/4*j - 1/4*k, i, -k)

sage: QuaternionAlgebra(-17,-3).maximal_order().basis()
(1/2 + 1/2*j, 1/2*i + 1/2*k, -1/3*j - 1/3*k, k)

sage: QuaternionAlgebra(-3,-17).maximal_order().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, -1/3*i + 1/3*k, -k)

sage: QuaternionAlgebra(-17*9,-3).maximal_order().basis()
(1, 1/3*i, 1/6*i + 1/2*j, 1/2 + 1/3*j + 1/18*k)

sage: QuaternionAlgebra(-2, -389).maximal_order().basis()
(1/2 + 1/2*j + 1/2*k, 1/4*i + 1/2*j + 1/4*k, j, k)
```

If you want bases containing 1, switch off `take_shortcuts`:

```
sage: QuaternionAlgebra(-3,-89).maximal_order(take_shortcuts=False)
Order of Quaternion Algebra (-3, -89) with base ring Rational Field with basis␣
↪(1, 1/2 + 1/2*i, j, 1/2 + 1/6*i + 1/2*j + 1/6*k)

sage: QuaternionAlgebra(1,1).maximal_order(take_shortcuts=False)     # Matrix␣
↪ring
Order of Quaternion Algebra (1, 1) with base ring Rational Field with basis (1,␣
↪1/2 + 1/2*i, j, 1/2*j + 1/2*k)

sage: QuaternionAlgebra(-22,210).maximal_order(take_shortcuts=False)
Order of Quaternion Algebra (-22, 210) with base ring Rational Field with basis␣
↪(1, i, 1/2*i + 1/2*j, 1/2 + 17/22*i + 1/44*k)

sage: for d in ( m for m in range(1, 750) if is_squarefree(m) ):        # long␣
```

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```
↪time (3s)
....:     A = QuaternionAlgebra(d)
....:     R = A.maximal_order(take_shortcuts=False)
....:     assert A.discriminant() == R.discriminant()
```

We do not support number fields other than the rationals yet:

```
sage: K = QuadraticField(5)
sage: QuaternionAlgebra(K,-1,-1).maximal_order()
Traceback (most recent call last):
...
NotImplementedError: maximal order only implemented for rational quaternion␣
↪algebras
```

modp_splitting_data(p)

Return mod p splitting data for this quaternion algebra at the unramified prime p.

This is 2×2 matrices I, J, K over the finite field $\mathbf{F}_p$ such that if the quaternion algebra has generators i, j, k, then $I^2 = i^2$, $J^2 = j^2$, $IJ = K$ and $IJ = -JI$.

Note: Currently only implemented when p is odd and the base ring is $\mathbf{Q}$.

INPUT:

- p – unramified odd prime

OUTPUT:

- 2-tuple of matrices over finite field

EXAMPLES:

```
sage: Q = QuaternionAlgebra(-15, -19)
sage: Q.modp_splitting_data(7)
(
[0 6]  [6 1]  [6 6]
[1 0], [1 1], [6 1]
)
sage: Q.modp_splitting_data(next_prime(10^5))
(
[    0 99988]  [97311     4]  [99999 59623]
[    1     0], [13334  2692], [97311     4]
)
sage: I,J,K = Q.modp_splitting_data(23)
sage: I
[0 8]
[1 0]
sage: I^2
[8 0]
[0 8]
sage: J
[19  2]
[17  4]
sage: J^2
```

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```
[4 0]
[0 4]
sage: I*J == -J*I
True
sage: I*J == K
True
```

The following is a good test because of the asserts in the code:

```
sage: v = [Q.modp_splitting_data(p) for p in primes(20,1000)]
```

Proper error handling:

```
sage: Q.modp_splitting_data(5)
Traceback (most recent call last):
...
NotImplementedError: algorithm for computing local splittings not implemented␣
↪in general (currently require the first invariant to be coprime to p)

sage: Q.modp_splitting_data(2)
Traceback (most recent call last):
...
NotImplementedError: p must be odd
```

modp_splitting_map(*p*)

Return Python map from the (p-integral) quaternion algebra to the set of 2×2 matrices over $\mathbf{F}_p$.

INPUT:

- p – prime number

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-1, -7)
sage: f = Q.modp_splitting_map(13)
sage: a = 2+i-j+3*k; b = 7+2*i-4*j+k
sage: f(a*b)
[12  3]
[10  5]
sage: f(a)*f(b)
[12  3]
[10  5]
```

quaternion_order(*basis*, *check=True*)

Return the order of this quaternion order with given basis.

INPUT:

- `basis` - list of 4 elements of `self`
- `check` - bool (default: `True`)

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-11,-1)
sage: Q.quaternion_order([1,i,j,k])
```

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```
Order of Quaternion Algebra (-11, -1) with base ring Rational Field with basis␣
↪(1, i, j, k)
```

We test out `check=False`:

```
sage: Q.quaternion_order([1,i,j,k], check=False)
Order of Quaternion Algebra (-11, -1) with base ring Rational Field with basis␣
↪(1, i, j, k)
sage: Q.quaternion_order([i,j,k], check=False)
Order of Quaternion Algebra (-11, -1) with base ring Rational Field with basis␣
↪(i, j, k)
```

ramified_primes()

Return the primes that ramify in this quaternion algebra.

Currently only implemented over the rational numbers.

EXAMPLES:

```
sage: QuaternionAlgebra(QQ, -1, -1).ramified_primes()
[2]
```

class sage.algebras.quatalg.quaternion_algebra.**QuaternionAlgebra_abstract**

Bases: `Algebra`

basis()

Return the fixed basis of `self`, which is 1, i, j, k, where i, j, k are the generators of `self`.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ,-5,-2)
sage: Q.basis()
(1, i, j, k)

sage: Q.<xyz,abc,theta> = QuaternionAlgebra(GF(9,'a'),-5,-2)
sage: Q.basis()
(1, xyz, abc, theta)
```

The basis is cached:

```
sage: Q.basis() is Q.basis()
True
```

free_module()

Return the free module associated to `self` with inner product given by the reduced norm.

EXAMPLES:

```
sage: A.<t> = LaurentPolynomialRing(GF(3))
sage: B = QuaternionAlgebra(A, -1, t)
sage: B.free_module()
Ambient free quadratic module of rank 4 over the principal ideal domain␣
↪Univariate Laurent Polynomial Ring in t over Finite Field of size 3
Inner product matrix:
```

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```
[2 0 0 0]
[0 2 0 0]
[0 0 t 0]
[0 0 0 t]
```

inner_product_matrix()

Return the inner product matrix associated to `self`.

This is the Gram matrix of the reduced norm as a quadratic form on `self`. The standard basis 1, i, j, k is orthogonal, so this matrix is just the diagonal matrix with diagonal entries 2, $2a$, $2b$, $2ab$.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(-5,-19)
sage: Q.inner_product_matrix()
[  2   0   0   0]
[  0  10   0   0]
[  0   0  38   0]
[  0   0   0 190]
```

is_commutative()

Return `False` always, since all quaternion algebras are noncommutative.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3,-7)
sage: Q.is_commutative()
False
```

is_division_algebra()

Return `True` if the quaternion algebra is a division algebra (i.e. every nonzero element in `self` is invertible), and `False` if the quaternion algebra is isomorphic to the 2x2 matrix algebra.

EXAMPLES:

```
sage: QuaternionAlgebra(QQ,-5,-2).is_division_algebra()
True
sage: QuaternionAlgebra(1).is_division_algebra()
False
sage: QuaternionAlgebra(2,9).is_division_algebra()
False
sage: QuaternionAlgebra(RR(2.),1).is_division_algebra()
Traceback (most recent call last):
...
NotImplementedError: base field must be rational numbers
```

is_exact()

Return `True` if elements of this quaternion algebra are represented exactly, i.e. there is no precision loss when doing arithmetic. A quaternion algebra is exact if and only if its base field is exact.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_exact()
```

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```
True
sage: Q.<i,j,k> = QuaternionAlgebra(Qp(7), -3, -7)
sage: Q.is_exact()
False
```

is_field(*proof=True*)

Return `False` always, since all quaternion algebras are noncommutative and all fields are commutative.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_field()
False
```

is_finite()

Return `True` if the quaternion algebra is finite as a set.

Algorithm: A quaternion algebra is finite if and only if the base field is finite.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_finite()
False
sage: Q.<i,j,k> = QuaternionAlgebra(GF(5), -3, -7)
sage: Q.is_finite()
True
```

is_integral_domain(*proof=True*)

Return `False` always, since all quaternion algebras are noncommutative and integral domains are commutative (in Sage).

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_integral_domain()
False
```

is_matrix_ring()

Return `True` if the quaternion algebra is isomorphic to the 2x2 matrix ring, and `False` if `self` is a division algebra (i.e. every nonzero element in `self` is invertible).

EXAMPLES:

```
sage: QuaternionAlgebra(QQ,-5,-2).is_matrix_ring()
False
sage: QuaternionAlgebra(1).is_matrix_ring()
True
sage: QuaternionAlgebra(2,9).is_matrix_ring()
True
sage: QuaternionAlgebra(RR(2.),1).is_matrix_ring()
Traceback (most recent call last):
...
NotImplementedError: base field must be rational numbers
```

is_noetherian()

Return `True` always, since any quaternion algebra is a noetherian ring (because it is a finitely generated module over a field).

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.is_noetherian()
True
```

ngens()

Return the number of generators of the quaternion algebra as a K-vector space, not including 1.

This value is always 3: the algebra is spanned by the standard basis 1, i, j, k.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ,-5,-2)
sage: Q.ngens()
3
sage: Q.gens()
[i, j, k]
```

order()

Return the number of elements of the quaternion algebra, or `+Infinity` if the algebra is not finite.

EXAMPLES:

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -3, -7)
sage: Q.order()
+Infinity
sage: Q.<i,j,k> = QuaternionAlgebra(GF(5), -3, -7)
sage: Q.order()
625
```

random_element(**args*, ***kwds*)

Return a random element of this quaternion algebra.

The `args` and `kwds` are passed to the `random_element` method of the base ring.

EXAMPLES:

```
sage: g = QuaternionAlgebra(QQ[sqrt(2)], -3, 7).random_element()
sage: g.parent() is QuaternionAlgebra(QQ[sqrt(2)], -3, 7)
True
sage: g = QuaternionAlgebra(-3, 19).random_element()
sage: g.parent() is QuaternionAlgebra(-3, 19)
True
sage: g = QuaternionAlgebra(GF(17)(2), 3).random_element()
sage: g.parent() is QuaternionAlgebra(GF(17)(2), 3)
True
```

Specify the numerator and denominator bounds:

```
sage: g = QuaternionAlgebra(-3,19).random_element(10^6, 10^6)
sage: for h in g:
```

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```
....:     assert h.numerator() in range(-10^6, 10^6 + 1)
....:     assert h.denominator() in range(10^6 + 1)

sage: g = QuaternionAlgebra(-3,19).random_element(5, 4)
sage: for h in g:
....:     assert h.numerator() in range(-5, 5 + 1)
....:     assert h.denominator() in range(4 + 1)
```

vector_space()

Alias for *free_module()*.

EXAMPLES:

```
sage: QuaternionAlgebra(-3,19).vector_space()
Ambient quadratic space of dimension 4 over Rational Field
Inner product matrix:
[   2    0    0    0]
[   0    6    0    0]
[   0    0  -38    0]
[   0    0    0 -114]
```

class sage.algebras.quatalg.quaternion_algebra.**QuaternionFractionalIdeal**(*ring*, *gens*, *coerce=True*)

Bases: Ideal_fractional

class sage.algebras.quatalg.quaternion_algebra.**QuaternionFractionalIdeal_rational**(*Q*, *basis*, *left_order=None*, *right_order=None*, *check=True*)

Bases: *QuaternionFractionalIdeal*

A fractional ideal in a rational quaternion algebra.

INPUT:

- `left_order` – a quaternion order or `None`
- `right_order` – a quaternion order or `None`
- `basis` – tuple of length 4 of elements in of ambient quaternion algebra whose $\mathbf{Z}$-span is an ideal
- `check` – bool (default: `True`); if `False`, do no type checking.

basis()

Return a basis for this fractional ideal.

OUTPUT: tuple

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().unit_ideal().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
```

basis_matrix()

Return basis matrix M in Hermite normal form for self as a matrix with rational entries.

If Q is the ambient quaternion algebra, then the $\mathbf{Z}$-span of the rows of M viewed as linear combinations of Q.basis() = $[1, i, j, k]$ is the fractional ideal self. Also, `M * M.denominator()` is an integer matrix in Hermite normal form.

OUTPUT: matrix over $\mathbf{Q}$

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().unit_ideal().basis_matrix()
[1/2 1/2   0   0]
[  0   1   0   0]
[  0   0 1/2 1/2]
[  0   0   0   1]
```

conjugate()

Return the ideal with generators the conjugates of the generators for self.

OUTPUT: a quaternionic fractional ideal

EXAMPLES:

```
sage: I = BrandtModule(3,5).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
sage: I.conjugate()
Fractional ideal (2 + 2*j + 28*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
```

cyclic_right_subideals(*p*, *alpha=None*)

Let I = `self`. This function returns the right subideals J of I such that I/J is an $\mathbf{F}_p$-vector space of dimension 2.

INPUT:

- `p` – prime number (see below)
- `alpha` – (default: `None`) element of quaternion algebra, which can be used to parameterize the order of the ideals J. More precisely the J's are the right annihilators of $(1, 0)\alpha^i$ for $i = 0, 1, 2, ..., p$

OUTPUT:

- list of right ideals

Note: Currently, p must satisfy a bunch of conditions, or a `NotImplementedError` is raised. In particular, p must be odd and unramified in the quaternion algebra, must be coprime to the index of the right order in the maximal order, and also coprime to the normal of self. (The Brandt modules code has a more general algorithm in some cases.)

EXAMPLES:

```
sage: B = BrandtModule(2,37); I = B.right_ideals()[0]
sage: I.cyclic_right_subideals(3)
[Fractional ideal (2 + 2*i + 10*j + 90*k, 4*i + 4*j + 152*k, 12*j + 132*k,␣
↪444*k), Fractional ideal (2 + 2*i + 2*j + 150*k, 4*i + 8*j + 196*k, 12*j +␣
↪132*k, 444*k), Fractional ideal (2 + 2*i + 6*j + 194*k, 4*i + 8*j + 344*k,␣
↪12*j + 132*k, 444*k), Fractional ideal (2 + 2*i + 6*j + 46*k, 4*i + 4*j + 4*k,
↪ 12*j + 132*k, 444*k)]

sage: B = BrandtModule(5,389); I = B.right_ideals()[0]
```

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```
sage: C = I.cyclic_right_subideals(3); C
[Fractional ideal (2 + 10*j + 546*k, i + 6*j + 133*k, 12*j + 3456*k, 4668*k),␣
↪Fractional ideal (2 + 2*j + 2910*k, i + 6*j + 3245*k, 12*j + 3456*k, 4668*k),␣
↪Fractional ideal (2 + i + 2295*k, 3*i + 2*j + 3571*k, 4*j + 2708*k, 4668*k),␣
↪Fractional ideal (2 + 2*i + 2*j + 4388*k, 3*i + 2*j + 2015*k, 4*j + 4264*k,␣
↪4668*k)]
sage: [(I.free_module()/J.free_module()).invariants() for J in C]
[(3, 3), (3, 3), (3, 3), (3, 3)]
sage: I.scale(3).cyclic_right_subideals(3)
[Fractional ideal (6 + 30*j + 1638*k, 3*i + 18*j + 399*k, 36*j + 10368*k,␣
↪14004*k), Fractional ideal (6 + 6*j + 8730*k, 3*i + 18*j + 9735*k, 36*j +␣
↪10368*k, 14004*k), Fractional ideal (6 + 3*i + 6885*k, 9*i + 6*j + 10713*k,␣
↪12*j + 8124*k, 14004*k), Fractional ideal (6 + 6*i + 6*j + 13164*k, 9*i + 6*j␣
↪+ 6045*k, 12*j + 12792*k, 14004*k)]
sage: C = I.scale(1/9).cyclic_right_subideals(3); C
[Fractional ideal (2/9 + 10/9*j + 182/3*k, 1/9*i + 2/3*j + 133/9*k, 4/3*j +␣
↪384*k, 1556/3*k), Fractional ideal (2/9 + 2/9*j + 970/3*k, 1/9*i + 2/3*j +␣
↪3245/9*k, 4/3*j + 384*k, 1556/3*k), Fractional ideal (2/9 + 1/9*i + 255*k, 1/
↪3*i + 2/9*j + 3571/9*k, 4/9*j + 2708/9*k, 1556/3*k), Fractional ideal (2/9 +␣
↪2/9*i + 2/9*j + 4388/9*k, 1/3*i + 2/9*j + 2015/9*k, 4/9*j + 4264/9*k, 1556/
↪3*k)]
sage: [(I.scale(1/9).free_module()/J.free_module()).invariants() for J in C]
[(3, 3), (3, 3), (3, 3), (3, 3)]

sage: Q.<i,j,k> = QuaternionAlgebra(-2,-5)
sage: I = Q.ideal([Q(1),i,j,k])
sage: I.cyclic_right_subideals(3)
[Fractional ideal (1 + 2*j, i + k, 3*j, 3*k), Fractional ideal (1 + j, i + 2*k,␣
↪3*j, 3*k), Fractional ideal (1 + 2*i, 3*i, j + 2*k, 3*k), Fractional ideal (1␣
↪+ i, 3*i, j + k, 3*k)]
```

The general algorithm is not yet implemented here:

```
sage: I.cyclic_right_subideals(3)[0].cyclic_right_subideals(3)
Traceback (most recent call last):
...
NotImplementedError: general algorithm not implemented (The given basis vectors␣
↪must be linearly independent.)
```

free_module()

Return the underlying free $\mathbf{Z}$-module corresponding to this ideal.

OUTPUT:

Free $\mathbf{Z}$-module of rank 4 embedded in an ambient $\mathbf{Q}^4$.

EXAMPLES:

```
sage: X = BrandtModule(3,5).right_ideals()
sage: X[0]
Fractional ideal (2 + 2*j + 8*k, 2*i + 18*k, 4*j + 16*k, 20*k)
sage: X[0].free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
```

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```
[ 2  0  2  8]
[ 0  2  0 18]
[ 0  0  4 16]
[ 0  0  0 20]
sage: X[0].scale(1/7).free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[ 2/7    0  2/7  8/7]
[   0  2/7    0 18/7]
[   0    0  4/7 16/7]
[   0    0    0 20/7]

sage: QuaternionAlgebra(-11,-1).maximal_order().unit_ideal().basis_matrix()
[1/2 1/2   0   0]
[  0   1   0   0]
[  0   0 1/2 1/2]
[  0   0   0   1]
```

The free module method is also useful since it allows for checking if one ideal is contained in another, computing quotients I/J, etc.:

```
sage: X = BrandtModule(3,17).right_ideals()
sage: I = X[0].intersection(X[2]); I
Fractional ideal (2 + 2*j + 164*k, 2*i + 4*j + 46*k, 16*j + 224*k, 272*k)
sage: I.free_module().is_submodule(X[3].free_module())
False
sage: I.free_module().is_submodule(X[1].free_module())
True
sage: X[0].free_module() / I.free_module()
Finitely generated module V/W over Integer Ring with invariants (4, 4)
```

This shows that the issue at github issue #6760 is fixed:

```
sage: R.<i,j,k> = QuaternionAlgebra(-1, -13)
sage: I = R.ideal([2+i, 3*i, 5*j, j+k]); I
Fractional ideal (2 + i, 3*i, j + k, 5*k)
sage: I.free_module()
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[2 1 0 0]
[0 3 0 0]
[0 0 1 1]
[0 0 0 5]
```

gram_matrix()

Return the Gram matrix of this fractional ideal.

OUTPUT: 4×4 matrix over $\mathbf{Q}$.

EXAMPLES:

```
sage: I = BrandtModule(3,5).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
```

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```
sage: I.gram_matrix()
[  640  1920  2112  1920]
[ 1920 14080 13440 16320]
[ 2112 13440 13056 15360]
[ 1920 16320 15360 19200]
```

intersection(J)

Return the intersection of the ideals self and J.

EXAMPLES:

```
sage: X = BrandtModule(3,5).right_ideals()
sage: I = X[0].intersection(X[1]); I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
```

is_equivalent(J, B=10)

Return `True` if `self` and `J` are equivalent as right ideals.

INPUT:

- `J` – a fractional quaternion ideal with same order as `self`
- `B` – a bound to compute and compare theta series before doing the full equivalence test

OUTPUT: bool

EXAMPLES:

```
sage: R = BrandtModule(3,5).right_ideals(); len(R)
2
sage: R[0].is_equivalent(R[1])
False
sage: R[0].is_equivalent(R[0])
True
sage: OO = R[0].left_order()
sage: S = OO.right_ideal([3*a for a in R[0].basis()])
sage: R[0].is_equivalent(S)
True
```

left_order()

Return the left order associated to this fractional ideal.

OUTPUT: an order in a quaternion algebra

EXAMPLES:

```
sage: B = BrandtModule(11)
sage: R = B.maximal_order()
sage: I = R.unit_ideal()
sage: I.left_order()
Order of Quaternion Algebra (-1, -11) with base ring Rational Field with basis
↪(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)
```

We do a consistency check:

```
sage: B = BrandtModule(11,19); R = B.right_ideals()
sage: [r.left_order().discriminant() for r in R]
[209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209, 209,
↪ 209, 209]
```

multiply_by_conjugate(J)

Return product of self and the conjugate Jbar of J.

INPUT:

- J – a quaternion ideal.

OUTPUT: a quaternionic fractional ideal.

EXAMPLES:

```
sage: R = BrandtModule(3,5).right_ideals()
sage: R[0].multiply_by_conjugate(R[1])
Fractional ideal (8 + 8*j + 112*k, 8*i + 16*j + 136*k, 32*j + 128*k, 160*k)
sage: R[0]*R[1].conjugate()
Fractional ideal (8 + 8*j + 112*k, 8*i + 16*j + 136*k, 32*j + 128*k, 160*k)
```

norm()

Return the reduced norm of this fractional ideal.

OUTPUT: rational number

EXAMPLES:

```
sage: M = BrandtModule(37)
sage: C = M.right_ideals()
sage: [I.norm() for I in C]
[16, 32, 32]

sage: (a,b) = M.quaternion_algebra().invariants()                              ␣
↪        # optional - magma
sage: magma.eval('A<i,j,k> := QuaternionAlgebra<Rationals() | %s, %s>' % (a,b))␣
↪        # optional - magma
''
sage: magma.eval('O := QuaternionOrder(%s)' % str(list(C[0].right_order().
↪basis())))     # optional - magma
''
sage: [ magma('rideal<O | %s>' % str(list(I.basis()))).Norm() for I in C]      ␣
↪        # optional - magma
[16, 32, 32]

sage: A.<i,j,k> = QuaternionAlgebra(-1,-1)
sage: R = A.ideal([i,j,k,1/2 + 1/2*i + 1/2*j + 1/2*k])      # this is actually␣
↪an order, so has reduced norm 1
sage: R.norm()
1
sage: [ J.norm() for J in R.cyclic_right_subideals(3) ]      # enumerate maximal␣
↪right R-ideals of reduced norm 3, verify their norms
[3, 3, 3, 3]
```

quadratic_form()

Return the normalized quadratic form associated to this quaternion ideal.

OUTPUT: quadratic form

EXAMPLES:

```
sage: I = BrandtModule(11).right_ideals()[1]
sage: Q = I.quadratic_form(); Q
Quadratic form in 4 variables over Rational Field with coefficients:
[ 18 22 33 22 ]
[ * 7 22 11 ]
[ * * 22 0 ]
[ * * * 22 ]
sage: Q.theta_series(10)
1 + 12*q^2 + 12*q^3 + 12*q^4 + 12*q^5 + 24*q^6 + 24*q^7 + 36*q^8 + 36*q^9 + O(q^
↪10)
sage: I.theta_series(10)
1 + 12*q^2 + 12*q^3 + 12*q^4 + 12*q^5 + 24*q^6 + 24*q^7 + 36*q^8 + 36*q^9 + O(q^
↪10)
```

quaternion_algebra()

Return the ambient quaternion algebra that contains this fractional ideal.

OUTPUT: a quaternion algebra

EXAMPLES:

```
sage: I = BrandtModule(3,5).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 34*k, 8*j + 32*k, 40*k)
sage: I.quaternion_algebra()
Quaternion Algebra (-1, -3) with base ring Rational Field
```

quaternion_order()

Return the order for which this ideal is a left or right fractional ideal.

If this ideal has both a left and right ideal structure, then the left order is returned. If it has neither structure, then an error is raised.

OUTPUT: QuaternionOrder

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.unit_ideal().quaternion_order() is R
doctest:...: DeprecationWarning: quaternion_order() is deprecated, please use␣
↪left_order() or right_order()
See https://github.com/sagemath/sage/issues/31583 for details.
True
```

right_order()

Return the right order associated to this fractional ideal.

OUTPUT: an order in a quaternion algebra

EXAMPLES:

```
sage: I = BrandtModule(389).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 2*k, i + 2*j + k, 8*j, 8*k)
sage: I.right_order()
Order of Quaternion Algebra (-2, -389) with base ring Rational Field with basis␣
↪(1/2 + 1/2*j + 1/2*k, 1/4*i + 1/2*j + 1/4*k, j, k)
sage: I.left_order()
Order of Quaternion Algebra (-2, -389) with base ring Rational Field with basis␣
↪(1/2 + 1/2*j + 3/2*k, 1/8*i + 1/4*j + 9/8*k, j + k, 2*k)
```

The following is a big consistency check. We take reps for all the right ideal classes of a certain order, take the corresponding left orders, then take ideals in the left orders and from those compute the right order again:

```
sage: B = BrandtModule(11,19); R = B.right_ideals()
sage: O = [r.left_order() for r in R]
sage: J = [O[i].left_ideal(R[i].basis()) for i in range(len(R))]
sage: len(set(J))
18
sage: len(set([I.right_order() for I in J]))
1
sage: J[0].right_order() == B.order_of_level_N()
True
```

ring()

Return ring that this is a fractional ideal for.

The *ring()* method will be removed from this class in the future. Calling *ring()* will then return the ambient quaternion algebra. This is consistent with the behaviour for number fields.

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.unit_ideal().ring() is R
doctest:...: DeprecationWarning: ring() will return the quaternion algebra in␣
↪the future, please use left_order() or right_order()
See https://github.com/sagemath/sage/issues/31583 for details.
True
```

scale(*alpha*, *left=False*)

Scale the fractional ideal `self` by multiplying the basis by `alpha`.

INPUT:

- α – element of quaternion algebra
- `left` – bool (default: False); if true multiply α on the left, otherwise multiply α on the right

OUTPUT:

- a new fractional ideal

EXAMPLES:

```
sage: B = BrandtModule(5,37); I = B.right_ideals()[0]; i,j,k = B.quaternion_
↪algebra().gens(); I
Fractional ideal (2 + 2*j + 106*k, i + 2*j + 105*k, 4*j + 64*k, 148*k)
sage: I.scale(i)
```

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```
Fractional ideal (2*i + 212*j - 2*k, -2 + 210*j - 2*k, 128*j - 4*k, 296*j)
sage: I.scale(i, left=True)
Fractional ideal (2*i - 212*j + 2*k, -2 - 210*j + 2*k, -128*j + 4*k, -296*j)
sage: I.scale(i, left=False)
Fractional ideal (2*i + 212*j - 2*k, -2 + 210*j - 2*k, 128*j - 4*k, 296*j)
sage: i * I.gens()[0]
2*i - 212*j + 2*k
sage: I.gens()[0] * i
2*i + 212*j - 2*k
```

theta_series(*B*, *var='q'*)

Return normalized theta series of `self`, as a power series over $\mathbf{Z}$ in the variable `var`, which is 'q' by default.

The normalized theta series is by definition

$$\theta_I(q) = \sum_{x \in I} q^{\frac{N(x)}{N(I)}}.$$

INPUT:

- `B` – positive integer
- `var` – string (default: 'q')

OUTPUT: power series

EXAMPLES:

```
sage: I = BrandtModule(11).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 4*k, 2*i + 4*j + 2*k, 8*j, 8*k)
sage: I.norm()
32
sage: I.theta_series(5)
1 + 12*q^2 + 12*q^3 + 12*q^4 + O(q^5)
sage: I.theta_series(5,'T')
1 + 12*T^2 + 12*T^3 + 12*T^4 + O(T^5)
sage: I.theta_series(3)
1 + 12*q^2 + O(q^3)
```

theta_series_vector(*B*)

Return theta series coefficients of `self`, as a vector of `B` integers.

INPUT:

- `B` – positive integer

OUTPUT:

Vector over $\mathbf{Z}$ with `B` entries.

EXAMPLES:

```
sage: I = BrandtModule(37).right_ideals()[1]; I
Fractional ideal (2 + 6*j + 2*k, i + 2*j + k, 8*j, 8*k)
sage: I.theta_series_vector(5)
(1, 0, 2, 2, 6)
sage: I.theta_series_vector(10)
```

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```
(1, 0, 2, 2, 6, 4, 8, 6, 10, 10)
sage: I.theta_series_vector(5)
(1, 0, 2, 2, 6)
```

class sage.algebras.quatalg.quaternion_algebra.**QuaternionOrder**(*A*, *basis*, *check=True*)

Bases: Parent

An order in a quaternion algebra.

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order()
Order of Quaternion Algebra (-1, -7) with base ring Rational Field with basis (1/2␣
↪+ 1/2*j, 1/2*i + 1/2*k, j, k)
sage: type(QuaternionAlgebra(-1,-7).maximal_order())
<class 'sage.algebras.quatalg.quaternion_algebra.QuaternionOrder_with_category'>
```

basis()

Return fix choice of basis for this quaternion order.

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
```

discriminant()

Return the discriminant of this order.

This is defined as $\sqrt{det(Tr(e_i\bar{e}_j))}$, where $\{e_i\}$ is the basis of the order.

OUTPUT: rational number

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().discriminant()
11
sage: S = BrandtModule(11,5).order_of_level_N()
sage: S.discriminant()
55
sage: type(S.discriminant())
<... 'sage.rings.rational.Rational'>
```

free_module()

Return the free $\mathbf{Z}$-module that corresponds to this order inside the vector space corresponding to the ambient quaternion algebra.

OUTPUT:

A free $\mathbf{Z}$-module of rank 4.

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.basis()
(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
sage: R.free_module()
```

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```
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[1/2 1/2   0   0]
[  0   1   0   0]
[  0   0 1/2 1/2]
[  0   0   0   1]
```

gen(*n*)

Return the n-th generator.

INPUT:

- n - an integer between 0 and 3, inclusive.

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order(); R
Order of Quaternion Algebra (-11, -1) with base ring Rational Field with basis␣
↪(1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
sage: R.gen(0)
1/2 + 1/2*i
sage: R.gen(1)
1/2*j - 1/2*k
sage: R.gen(2)
i
sage: R.gen(3)
-k
```

gens()

Return generators for self.

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order().gens()
(1/2 + 1/2*j, 1/2*i + 1/2*k, j, k)
```

intersection(*other*)

Return the intersection of this order with other.

INPUT:

- other - a quaternion order in the same ambient quaternion algebra

OUTPUT: a quaternion order

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.intersection(R)
Order of Quaternion Algebra (-11, -1) with base ring Rational Field with basis␣
↪(1/2 + 1/2*i, i, 1/2*j + 1/2*k, k)
```

We intersect various orders in the quaternion algebra ramified at 11:

```
sage: B = BrandtModule(11,3)
sage: R = B.maximal_order(); S = B.order_of_level_N()
```

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```
sage: R.intersection(S)
Order of Quaternion Algebra (-1, -11) with base ring Rational Field with basis␣
↪(1/2 + 1/2*j, 1/2*i + 5/2*k, j, 3*k)
sage: R.intersection(S) == S
True
sage: B = BrandtModule(11,5)
sage: T = B.order_of_level_N()
sage: S.intersection(T)
Order of Quaternion Algebra (-1, -11) with base ring Rational Field with basis␣
↪(1/2 + 1/2*j, 1/2*i + 23/2*k, j, 15*k)
```

left_ideal(*gens*, *check=True*)

Return the ideal with given gens over $\mathbf{Z}$.

INPUT:

- gens – a list of elements of this quaternion order
- check – bool (default: True)

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.left_ideal([2*a for a in R.basis()])
Fractional ideal (1 + i, 2*i, j + k, 2*k)
```

ngens()

Return the number of generators (which is 4).

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order().ngens()
4
```

one()

Return the multiplicative unit of this quaternion order.

EXAMPLES:

```
sage: QuaternionAlgebra(-1,-7).maximal_order().one()
1
```

quadratic_form()

Return the normalized quadratic form associated to this quaternion order.

OUTPUT: quadratic form

EXAMPLES:

```
sage: R = BrandtModule(11,13).order_of_level_N()
sage: Q = R.quadratic_form(); Q
Quadratic form in 4 variables over Rational Field with coefficients:
[ 14 253 55 286 ]
[ * 1455 506 3289 ]
[ * * 55 572 ]
[ * * * 1859 ]
```

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```
sage: Q.theta_series(10)
1 + 2*q + 2*q^4 + 4*q^6 + 4*q^8 + 2*q^9 + O(q^10)
```

quaternion_algebra()

Return ambient quaternion algebra that contains this quaternion order.

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().quaternion_algebra()
Quaternion Algebra (-11, -1) with base ring Rational Field
```

random_element(**args*, ***kwds*)

Return a random element of this order.

The args and kwds are passed to the random_element method of the integer ring, and we return an element of the form

$$ae_1 + be_2 + ce_3 + de_4$$

where e_1, …, e_4 are the basis of this order and a, b, c, d are random integers.

EXAMPLES:

```
sage: QuaternionAlgebra(-11,-1).maximal_order().random_element()   # random
-4 - 4*i + j - k
sage: QuaternionAlgebra(-11,-1).maximal_order().random_element(-10,10)   # random
-9/2 - 7/2*i - 7/2*j - 3/2*k
```

right_ideal(*gens*, *check=True*)

Return the ideal with given gens over $\mathbf{Z}$.

INPUT:

- `gens` – a list of elements of this quaternion order
- `check` – bool (default: `True`)

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: R.right_ideal([2*a for a in R.basis()])
Fractional ideal (1 + i, 2*i, j + k, 2*k)
```

ternary_quadratic_form(*include_basis=False*)

Return the ternary quadratic form associated to this order.

INPUT:

- `include_basis` – bool (default: False), if True also return a basis for the dimension 3 subspace G

OUTPUT:

- QuadraticForm
- optional basis for dimension 3 subspace

This function computes the positive definition quadratic form obtained by letting G be the trace zero subspace of $\mathbf{Z}$ + 2* `self`, which has rank 3, and restricting the pairing `QuaternionAlgebraElement_abstract.pair()`:

```
(x,y) = (x.conjugate()*y).reduced_trace()
```

to G.

APPLICATIONS: Ternary quadratic forms associated to an order in a rational quaternion algebra are useful in computing with Gross points, in decided whether quaternion orders have embeddings from orders in quadratic imaginary fields, and in computing elements of the Kohnen plus subspace of modular forms of weight 3/2.

EXAMPLES:

```
sage: R = BrandtModule(11,13).order_of_level_N()
sage: Q = R.ternary_quadratic_form(); Q
Quadratic form in 3 variables over Rational Field with coefficients:
[ 5820 1012 13156 ]
[ * 55 1144 ]
[ * * 7436 ]
sage: factor(Q.disc())
2^4 * 11^2 * 13^2
```

The following theta series is a modular form of weight 3/2 and level 4*11*13:

```
sage: Q.theta_series(100)
1 + 2*q^23 + 2*q^55 + 2*q^56 + 2*q^75 + 4*q^92 + O(q^100)
```

unit_ideal()

Return the unit ideal in this quaternion order.

EXAMPLES:

```
sage: R = QuaternionAlgebra(-11,-1).maximal_order()
sage: I = R.unit_ideal(); I
Fractional ideal (1/2 + 1/2*i, 1/2*j - 1/2*k, i, -k)
```

sage.algebras.quatalg.quaternion_algebra.**basis_for_quaternion_lattice**(*gens*, *reverse=None*)

Return a basis for the $\mathbf{Z}$-lattice in a quaternion algebra spanned by the given gens.

INPUT:

- gens – list of elements of a single quaternion algebra
- reverse – when computing the HNF do it on the basis $(k, j, i, 1)$ instead of $(1, i, j, k)$; this ensures that if gens are the generators for an order, the first returned basis vector is 1

EXAMPLES:

```
sage: from sage.algebras.quatalg.quaternion_algebra import basis_for_quaternion_
↪lattice
sage: A.<i,j,k> = QuaternionAlgebra(-1,-7)
sage: basis_for_quaternion_lattice([i+j, i-j, 2*k, A(1/3)])
doctest:warning ... DeprecationWarning: ...
[1/3, i + j, 2*j, 2*k]

sage: basis_for_quaternion_lattice([A(1),i,j,k])
[1, i, j, k]
```

sage.algebras.quatalg.quaternion_algebra.**intersection_of_row_modules_over_ZZ**(v)

Intersect the $\mathbf{Z}$-modules with basis matrices the full rank 4×4 $\mathbf{Q}$-matrices in the list v.

The returned intersection is represented by a 4×4 matrix over $\mathbf{Q}$. This can also be done using modules and intersection, but that would take over twice as long because of overhead, hence this function.

EXAMPLES:

```
sage: a = matrix(QQ,4,[-2, 0, 0, 0, 0, -1, -1, 1, 2, -1/2, 0, 0, 1, 1, -1, 0])
sage: b = matrix(QQ,4,[0, -1/2, 0, -1/2, 2, 1/2, -1, -1/2, 1, 2, 1, -2, 0, -1/2, -2,
↪ 0])
sage: c = matrix(QQ,4,[0, 1, 0, -1/2, 0, 0, 2, 2, 0, -1/2, 1/2, -1, 1, -1, -1/2, 0])
sage: v = [a,b,c]
sage: from sage.algebras.quatalg.quaternion_algebra import intersection_of_row_
↪modules_over_ZZ
sage: M = intersection_of_row_modules_over_ZZ(v); M
[   2    0   -1   -1]
[  -4    1    1   -3]
[  -3 19/2   -1   -4]
[   2   -3   -8    4]
sage: M2 = a.row_module(ZZ).intersection(b.row_module(ZZ)).intersection(c.row_
↪module(ZZ))
sage: M.row_module(ZZ) == M2
True
```

sage.algebras.quatalg.quaternion_algebra.**is_QuaternionAlgebra**(A)

Return `True` if `A` is of the QuaternionAlgebra data type.

EXAMPLES:

```
sage: sage.algebras.quatalg.quaternion_algebra.is_
↪QuaternionAlgebra(QuaternionAlgebra(QQ,-1,-1))
True
sage: sage.algebras.quatalg.quaternion_algebra.is_QuaternionAlgebra(ZZ)
False
```

sage.algebras.quatalg.quaternion_algebra.**maxord_solve_aux_eq**(a, b, p)

Given `a` and `b` and an even prime ideal `p` find (y,z,w) with y a unit mod p^{2e} such that

$$1 - ay^2 - bz^2 + abw^2 \equiv 0 mod p^{2e},$$

where e is the ramification index of p.

Currently only $p = 2$ is implemented by hardcoding solutions.

INPUT:

- `a` – integer with $v_p(a) = 0$
- `b` – integer with $v_p(b) \in \{0, 1\}$
- `p` – even prime ideal (actually only `p=ZZ(2)` is implemented)

OUTPUT:

- A tuple (y, z, w)

EXAMPLES:

```
sage: from sage.algebras.quatalg.quaternion_algebra import maxord_solve_aux_eq
sage: for a in [1,3]:
....:     for b in [1,2,3]:
....:         (y,z,w) = maxord_solve_aux_eq(a, b, 2)
....:         assert mod(y, 4) == 1 or mod(y, 4) == 3
....:         assert mod(1 - a*y^2 - b*z^2 + a*b*w^2, 4) == 0
```

sage.algebras.quatalg.quaternion_algebra.**normalize_basis_at_p**(*e*, *p*, *B=<method 'pair' of 'sage.algebras.quatalg.quaternion_algebra_element.Q objects>*)

Compute a (at p) normalized basis from the given basis e of a $\mathbf{Z}$-module.

The returned basis is (at p) a $\mathbf{Z}_p$ basis for the same module, and has the property that with respect to it the quadratic form induced by the bilinear form B is represented as a orthogonal sum of atomic forms multiplied by p-powers.

If $p \neq 2$ this means that the form is diagonal with respect to this basis.

If $p = 2$ there may be additional 2-dimensional subspaces on which the form is represented as $2^e(ax^2+bxy+cx^2)$ with $0 = v_2(b) = v_2(a) \leq v_2(c)$.

INPUT:

- e – list; basis of a $\mathbf{Z}$ module. WARNING: will be modified!
- p – prime for at which the basis should be normalized
- B – (default: QuaternionAlgebraElement_abstract.pair()) a bilinear form with respect to which to normalize

OUTPUT:

- A list containing two-element tuples: The first element of each tuple is a basis element, the second the valuation of the orthogonal summand to which it belongs. The list is sorted by ascending valuation.

EXAMPLES:

```
sage: from sage.algebras.quatalg.quaternion_algebra import normalize_basis_at_p
sage: A.<i,j,k> = QuaternionAlgebra(-1, -1)
sage: e = [A(1), i, j, k]
sage: normalize_basis_at_p(e, 2)
[(1, 0), (i, 0), (j, 0), (k, 0)]

sage: A.<i,j,k> = QuaternionAlgebra(210)
sage: e = [A(1), i, j, k]
sage: normalize_basis_at_p(e, 2)
[(1, 0), (i, 1), (j, 1), (k, 2)]

sage: A.<i,j,k> = QuaternionAlgebra(286)
sage: e = [A(1), k, 1/2*j + 1/2*k, 1/2 + 1/2*i + 1/2*k]
sage: normalize_basis_at_p(e, 5)
[(1, 0), (1/2*j + 1/2*k, 0), (-5/6*j + 1/6*k, 1), (1/2*i, 1)]

sage: A.<i,j,k> = QuaternionAlgebra(-1,-7)
sage: e = [A(1), k, j, 1/2 + 1/2*i + 1/2*j + 1/2*k]
sage: normalize_basis_at_p(e, 2)
[(1, 0), (1/2 + 1/2*i + 1/2*j + 1/2*k, 0), (-34/105*i - 463/735*j + 71/105*k, 1), (-
↪34/105*i - 463/735*j + 71/105*k, 1)]
```

sage.algebras.quatalg.quaternion_algebra.**unpickle_QuaternionAlgebra_v0**(**key*)

The 0th version of pickling for quaternion algebras.

EXAMPLES:

```
sage: Q = QuaternionAlgebra(-5,-19)
sage: t = (QQ, -5, -19, ('i', 'j', 'k'))
sage: sage.algebras.quatalg.quaternion_algebra.unpickle_QuaternionAlgebra_v0(*t)
Quaternion Algebra (-5, -19) with base ring Rational Field
sage: loads(dumps(Q)) == Q
True
sage: loads(dumps(Q)) is Q
True
```

5.21 Rational Cherednik Algebras

class sage.algebras.rational_cherednik_algebra.**RationalCherednikAlgebra**(*ct*, *c*, *t*, *base_ring*, *prefix*)

Bases: CombinatorialFreeModule

A rational Cherednik algebra.

Let k be a field. Let W be a complex reflection group acting on a vector space $\mathfrak{h}$ (over k). Let $\mathfrak{h}^*$ denote the corresponding dual vector space. Let $\cdot$ denote the natural action of w on $\mathfrak{h}$ and $\mathfrak{h}^*$. Let $\mathcal{S}$ denote the set of reflections of W and α_s and $\alpha_s^\vee$ are the associated root and coroot of s. Let $c = (c_s)_{s \in W}$ such that $c_s = c_{tst^{-1}}$ for all $t \in W$.

The *rational Cherednik algebra* is the k-algebra $H_{c,t}(W) = T(\mathfrak{h} \oplus \mathfrak{h}^*) \otimes kW$ with parameters $c, t \in k$ that is subject to the relations:

$$\begin{aligned} w\alpha &= (w \cdot \alpha)w, \\ \alpha^\vee w &= w(w^{-1} \cdot \alpha^\vee), \\ \alpha\alpha^\vee &= \alpha^\vee\alpha + t\langle\alpha^\vee, \alpha\rangle + \sum_{s \in \mathcal{S}} c_s \frac{\langle\alpha^\vee, \alpha_s\rangle\langle\alpha_s^\vee, \alpha\rangle}{\langle\alpha^\vee, \alpha\rangle} s, \end{aligned}$$

where $w \in W$ and $\alpha \in \mathfrak{h}$ and $\alpha^\vee \in \mathfrak{h}^*$.

INPUT:

- `ct` – a finite Cartan type
- `c` – the parameters c_s given as an element or a tuple, where the first entry is the one for the long roots and (for non-simply-laced types) the second is for the short roots
- `t` – the parameter t
- `base_ring` – (optional) the base ring
- `prefix` – (default: `('a', 's', 'ac')`) the prefixes

Todo: Implement a version for complex reflection groups.

REFERENCES:

- [GGOR2003]

- [EM2001]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: list(R.algebra_generators())
[a1, a2, s1, s2, ac1, ac2]
```

an_element()

Return an element of `self`.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: R.an_element()
3*ac1 + 2*s1 + a1
```

deformed_euler()

Return the element eu_k.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: R.deformed_euler()
2*I + 2/3*a1*ac1 + 1/3*a1*ac2 + 1/3*a2*ac1 + 2/3*a2*ac2
 + s1 + s2 + s1*s2*s1
```

degree_on_basis(*m*)

Return the degree on the monomial indexed by `m`.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: [R.degree_on_basis(g.leading_support())
....:  for g in R.algebra_generators()]
[1, 1, 0, 0, -1, -1]
```

one_basis()

Return the index of the element 1.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: R.one_basis()
(1, 1, 1)
```

product_on_basis(*left*, *right*)

Return `left` multiplied by `right` in `self`.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: a2 = R.algebra_generators()['a2']
```

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```
sage: ac1 = R.algebra_generators()['ac1']
sage: a2 * ac1  # indirect doctest
a2*ac1
sage: ac1 * a2
-I + a2*ac1 - s1 - s2 + 1/2*s1*s2*s1
sage: x = R.an_element()
sage: [y * x for y in R.some_elements()]
[0,
 3*ac1 + 2*s1 + a1,
 9*ac1^2 + 10*I + 6*a1*ac1 + 6*s1 + 3/2*s2 + 3/2*s1*s2*s1 + a1^2,
 3*a1*ac1 + 2*a1*s1 + a1^2,
 3*a2*ac1 + 2*a2*s1 + a1*a2,
 3*s1*ac1 + 2*I - a1*s1,
 3*s2*ac1 + 2*s2*s1 + a1*s2 + a2*s2,
 3*ac1^2 - 2*s1*ac1 + 2*I + a1*ac1 + 2*s1 + 1/2*s2 + 1/2*s1*s2*s1,
 3*ac1*ac2 + 2*s1*ac1 + 2*s1*ac2 - I + a1*ac2 - s1 - s2 + 1/2*s1*s2*s1]
sage: [x * y for y in R.some_elements()]
[0,
 3*ac1 + 2*s1 + a1,
 9*ac1^2 + 10*I + 6*a1*ac1 + 6*s1 + 3/2*s2 + 3/2*s1*s2*s1 + a1^2,
 6*I + 3*a1*ac1 + 6*s1 + 3/2*s2 + 3/2*s1*s2*s1 - 2*a1*s1 + a1^2,
 -3*I + 3*a2*ac1 - 3*s1 - 3*s2 + 3/2*s1*s2*s1 + 2*a1*s1 + 2*a2*s1 + a1*a2,
 -3*s1*ac1 + 2*I + a1*s1,
 3*s2*ac1 + 3*s2*ac2 + 2*s1*s2 + a1*s2,
 3*ac1^2 + 2*s1*ac1 + a1*ac1,
 3*ac1*ac2 + 2*s1*ac2 + a1*ac2]
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: R.some_elements()
[0, I, 3*ac1 + 2*s1 + a1, a1, a2, s1, s2, ac1, ac2]
```

trivial_idempotent()

Return the trivial idempotent of `self`.

Let $e = |W|^{-1} \sum_{w \in W} w$ is the trivial idempotent. Thus $e^2 = e$ and $eW = We$. The trivial idempotent is used in the construction of the spherical Cherednik algebra from the rational Cherednik algebra by $U_{c,t}(W) = eH_{c,t}(W)e$.

EXAMPLES:

```
sage: R = algebras.RationalCherednik(['A',2], 1, 1, QQ)
sage: R.trivial_idempotent()
1/6*I + 1/6*s1 + 1/6*s2 + 1/6*s2*s1 + 1/6*s1*s2 + 1/6*s1*s2*s1
```

5.22 Schur algebras for GL_n

This file implements:

- Schur algebras for GL_n over an arbitrary field.
- The canonical action of the Schur algebra on a tensor power of the standard representation.
- Using the above to calculate the characters of irreducible GL_n modules.

AUTHORS:

- Eric Webster (2010-07-01): implement Schur algebra
- Hugh Thomas (2011-05-08): implement action of Schur algebra and characters of irreducible modules

sage.algebras.schur_algebra.**GL_irreducible_character**(*n*, *mu*, *KK*)

Return the character of the irreducible module indexed by mu of $GL(n)$ over the field KK.

INPUT:

- n – a positive integer
- mu – a partition of at most n parts
- KK – a field

OUTPUT:

a symmetric function which should be interpreted in n variables to be meaningful as a character

EXAMPLES:

Over $\mathbf{Q}$, the irreducible character for μ is the Schur function associated to μ, plus garbage terms (Schur functions associated to partitions with more than n parts):

```
sage: from sage.algebras.schur_algebra import GL_irreducible_character
sage: sbasis = SymmetricFunctions(QQ).s()
sage: z = GL_irreducible_character(2, [2], QQ)
sage: sbasis(z)
s[2]

sage: z = GL_irreducible_character(4, [3, 2], QQ)
sage: sbasis(z)
-5*s[1, 1, 1, 1, 1] + s[3, 2]
```

Over a Galois field, the irreducible character for μ will in general be smaller.

In characteristic p, for a one-part partition (r), where $r = a_0 + pa_1 + p^2a_2 + \ldots$, the result is (see [Gr2007], after 5.5d) the product of $h[a_0], h[a_1](pbasis[p]), h[a_2](pbasis[p^2]), \ldots,$ which is consistent with the following

```
sage: from sage.algebras.schur_algebra import GL_irreducible_character
sage: GL_irreducible_character(2, [7], GF(3))
m[4, 3] + m[6, 1] + m[7]
```

class sage.algebras.schur_algebra.**SchurAlgebra**(*R*, *n*, *r*)

Bases: CombinatorialFreeModule

A Schur algebra.

Let R be a commutative ring, n be a positive integer, and r be a non-negative integer. Define $A_R(n,r)$ to be the set of homogeneous polynomials of degree r in n^2 variables x_{ij}. Therefore we can write $R[x_{ij}] = \bigoplus_{r\geq 0} A_R(n,r)$, and $R[x_{ij}]$ is known to be a bialgebra with coproduct given by $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$ and counit $\varepsilon(x_{ij}) = \delta_{ij}$. Therefore $A_R(n,r)$ is a subcoalgebra of $R[x_{ij}]$. The *Schur algebra* $S_R(n,r)$ is the linear dual to $A_R(n,r)$, that is $S_R(n,r) := \hom(A_R(n,r), R)$, and $S_R(n,r)$ obtains its algebra structure naturally by dualizing the comultiplication of $A_R(n,r)$.

Let $V = R^n$. One of the most important properties of the Schur algebra $S_R(n,r)$ is that it is isomorphic to the endomorphisms of $V^{\otimes r}$ which commute with the natural action of S_r.

EXAMPLES:

```
sage: S = SchurAlgebra(ZZ, 2, 2); S
Schur algebra (2, 2) over Integer Ring
```

REFERENCES:

- [Gr2007]
- Wikipedia article Schur_algebra

dimension()

Return the dimension of `self`.

The dimension of the Schur algebra $S_R(n,r)$ is

$$\dim S_R(n,r) = \binom{n^2+r-1}{r}.$$

EXAMPLES:

```
sage: S = SchurAlgebra(QQ, 4, 2)
sage: S.dimension()
136
sage: S = SchurAlgebra(QQ, 2, 4)
sage: S.dimension()
35
```

one()

Return the element 1 of `self`.

EXAMPLES:

```
sage: S = SchurAlgebra(ZZ, 2, 2)
sage: e = S.one(); e
S((1, 1), (1, 1)) + S((1, 2), (1, 2)) + S((2, 2), (2, 2))

sage: x = S.an_element()
sage: x * e == x
True
sage: all(e * x == x for x in S.basis())
True

sage: S = SchurAlgebra(ZZ, 4, 4)
sage: e = S.one()
sage: x = S.an_element()
sage: x * e == x
True
```

product_on_basis(*e_ij*, *e_kl*)

Return the product of basis elements.

EXAMPLES:

```
sage: S = SchurAlgebra(QQ, 2, 3)
sage: B = S.basis()
```

If we multiply two basis elements x and y, such that $x[1]$ and $y[0]$ are not permutations of each other, the result is zero:

```
sage: S.product_on_basis(((1, 1, 1), (1, 1, 2)), ((1, 2, 2), (1, 1, 2)))
0
```

If we multiply a basis element x by a basis element which consists of the same tuple repeated twice (on either side), the result is either zero (if the previous case applies) or x:

```
sage: ww = B[((1, 2, 2), (1, 2, 2))]
sage: x = B[((1, 2, 2), (1, 1, 2))]
sage: ww * x
S((1, 2, 2), (1, 1, 2))
```

An arbitrary product, on the other hand, may have multiplicities:

```
sage: x = B[((1, 1, 1), (1, 1, 2))]
sage: y = B[((1, 1, 2), (1, 2, 2))]
sage: x * y
2*S((1, 1, 1), (1, 2, 2))
```

class sage.algebras.schur_algebra.**SchurTensorModule**(*R*, *n*, *r*)

Bases: CombinatorialFreeModule_Tensor

The space $V^{\otimes r}$ where $V = R^n$ equipped with a left action of the Schur algebra $S_R(n, r)$ and a right action of the symmetric group S_r.

Let R be a commutative ring and $V = R^n$. We consider the module $V^{\otimes r}$ equipped with a natural right action of the symmetric group S_r given by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

The Schur algebra $S_R(n, r)$ is naturally isomorphic to the endomorphisms of $V^{\otimes r}$ which commutes with the S_r action. We get the natural left action of $S_R(n, r)$ by this isomorphism.

EXAMPLES:

```
sage: T = SchurTensorModule(QQ, 2, 3); T
The 3-fold tensor product of a free module of dimension 2
 over Rational Field
sage: A = SchurAlgebra(QQ, 2, 3)
sage: P = Permutations(3)
sage: t = T.an_element(); t
2*B[1] # B[1] # B[1] + 2*B[1] # B[1] # B[2] + 3*B[1] # B[2] # B[1]
sage: a = A.an_element(); a
2*S((1, 1, 1), (1, 1, 1)) + 2*S((1, 1, 1), (1, 1, 2))
 + 3*S((1, 1, 1), (1, 2, 2))
sage: p = P.an_element(); p
```

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```
[3, 1, 2]
sage: y = a * t; y
14*B[1] # B[1] # B[1]
sage: y * p
14*B[1] # B[1] # B[1]
sage: z = t * p; z
2*B[1] # B[1] # B[1] + 3*B[1] # B[1] # B[2] + 2*B[2] # B[1] # B[1]
sage: a * z
14*B[1] # B[1] # B[1]
```

We check the commuting action property:

```
sage: all( (bA * bT) * p == bA * (bT * p)
....:      for bT in T.basis() for bA in A.basis() for p in P)
True
```

class Element

Bases: `IndexedFreeModuleElement`

construction()

Return `None`.

There is no functorial construction for `self`.

EXAMPLES:

```
sage: T = SchurTensorModule(QQ, 2, 3)
sage: T.construction()
```

sage.algebras.schur_algebra.**schur_representative_from_index**(*i0*, *i1*)

Simultaneously reorder a pair of tuples to obtain the equivalent element of the distinguished basis of the Schur algebra.

See also:

schur_representative_indices()

INPUT:

- A pair of tuples of length r with elements in $\{1, \ldots, n\}$

OUTPUT:

- The corresponding pair of tuples ordered correctly.

EXAMPLES:

```
sage: from sage.algebras.schur_algebra import schur_representative_from_index
sage: schur_representative_from_index([2,1,2,2], [1,3,0,0])
((1, 2, 2, 2), (3, 0, 0, 1))
```

sage.algebras.schur_algebra.**schur_representative_indices**(*n*, *r*)

Return a set which functions as a basis of $S_K(n, r)$.

More specifically, the basis for $S_K(n, r)$ consists of equivalence classes of pairs of tuples of length `r` on the alphabet $\{1, \ldots, n\}$, where the equivalence relation is simultaneous permutation of the two tuples. We can therefore fix a representative for each equivalence class in which the entries of the first tuple weakly increase, and the entries of the second tuple whose corresponding values in the first tuple are equal, also weakly increase.

EXAMPLES:

```
sage: from sage.algebras.schur_algebra import schur_representative_indices
sage: schur_representative_indices(2, 2)
[((1, 1), (1, 1)), ((1, 1), (1, 2)),
 ((1, 1), (2, 2)), ((1, 2), (1, 1)),
 ((1, 2), (1, 2)), ((1, 2), (2, 1)),
 ((1, 2), (2, 2)), ((2, 2), (1, 1)),
 ((2, 2), (1, 2)), ((2, 2), (2, 2))]
```

5.23 The Steenrod algebra

AUTHORS:

- John H. Palmieri (2008-07-30): version 0.9: Initial implementation.
- John H. Palmieri (2010-06-30): version 1.0: Implemented sub-Hopf algebras and profile functions; direct multiplication of admissible sequences (rather than conversion to the Milnor basis); implemented the Steenrod algebra using CombinatorialFreeModule; improved the test suite.

This module defines the mod p Steenrod algebra $\mathcal{A}_p$, some of its properties, and ways to define elements of it.

From a topological point of view, $\mathcal{A}_p$ is the algebra of stable cohomology operations on mod p cohomology; thus for any topological space X, its mod p cohomology algebra $H^*(X, \mathbf{F}_p)$ is a module over $\mathcal{A}_p$.

From an algebraic point of view, $\mathcal{A}_p$ is an $\mathbf{F}_p$-algebra; when $p = 2$, it is generated by elements Sq^i for $i \geq 0$ (the *Steenrod squares*), and when p is odd, it is generated by elements $\mathcal{P}^i$ for $i \geq 0$ (the *Steenrod reduced pth powers*) along with an element β (the *mod p Bockstein*). The Steenrod algebra is graded: Sq^i is in degree i for each i, β is in degree 1, and $\mathcal{P}^i$ is in degree $2(p-1)i$.

The unit element is Sq^0 when $p = 2$ and $\mathcal{P}^0$ when p is odd. The generating elements also satisfy the *Adem relations*. At the prime 2, these have the form

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c.$$

At odd primes, they are a bit more complicated; see Steenrod and Epstein [SE1962] or `sage.algebras.steenrod.steenrod_algebra_bases` for full details. These relations lead to the existence of the *Serre-Cartan* basis for $\mathcal{A}_p$.

The mod p Steenrod algebra has the structure of a Hopf algebra, and Milnor [Mil1958] has a beautiful description of the dual, leading to a construction of the *Milnor basis* for $\mathcal{A}_p$. In this module, elements in the Steenrod algebra are represented, by default, using the Milnor basis.

Bases for the Steenrod algebra

There are a handful of other bases studied in the literature; the paper by Monks [Mon1998] is a good reference. Here is a quick summary:

- The *Milnor basis*. When $p = 2$, the Milnor basis consists of symbols of the form $\mathrm{Sq}(m_1, m_2, ..., m_t)$, where each m_i is a non-negative integer and if $t > 1$, then the last entry $m_t > 0$. When p is odd, the Milnor basis consists of symbols of the form $Q_{e_1} Q_{e_2} ... \mathcal{P}(m_1, m_2, ..., m_t)$, where $0 \leq e_1 < e_2 < ...$, each m_i is a non-negative integer, and if $t > 1$, then the last entry $m_t > 0$.

  When $p = 2$, it can be convenient to use the notation $\mathcal{P}(-)$ to mean $\mathrm{Sq}(-)$, so that there is consistent notation for all primes.

- The *Serre-Cartan basis*. This basis consists of 'admissible monomials' in the Steenrod operations. Thus at the prime 2, it consists of monomials $\mathrm{Sq}^{m_1}\mathrm{Sq}^{m_2}...\mathrm{Sq}^{m_t}$ with $m_i \geq 2m_{i+1}$ for each i. At odd primes, this basis consists of monomials $\beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\mathcal{P}^{s_2}...\mathcal{P}^{s_k}\beta^{\epsilon_k}$ with each ϵ_i either 0 or 1, $s_i \geq ps_{i+1} + \epsilon_i$, and $s_k \geq 1$.

Most of the rest of the bases are only defined when $p = 2$. The only exceptions are the P^s_t-bases and the commutator bases, which are defined at all primes.

- *Wood's Y basis*. For pairs of non-negative integers (m, k), let $w(m, k) = \mathrm{Sq}^{2^m(2^{k+1}-1)}$. Wood's Y basis consists of monomials $w(m_0, k_0)...w(m_t, k_t)$ with $(m_i, k_i) > (m_{i+1}, k_{i+1})$, in left lex order.
- *Wood's Z basis*. For pairs of non-negative integers (m, k), let $w(m, k) = \mathrm{Sq}^{2^m(2^{k+1}-1)}$. Wood's Z basis consists of monomials $w(m_0, k_0)...w(m_t, k_t)$ with $(m_i + k_i, m_i) > (m_{i+1} + k_{i+1}, m_{i+1})$, in left lex order.
- *Wall's basis*. For any pair of integers (m, k) with $m \geq k \geq 0$, let $Q^m_k = \mathrm{Sq}^{2^k}\mathrm{Sq}^{2^{k+1}}...\mathrm{Sq}^{2^m}$. The elements of Wall's basis are monomials $Q^{m_0}_{k_0}...Q^{m_t}_{k_t}$ with $(m_i, k_i) > (m_{i+1}, k_{i+1})$, ordered left lexicographically.

  (Note that Q^m_k is the reverse of the element X^m_k used in defining Arnon's A basis.)
- *Arnon's A basis*. For any pair of integers (m, k) with $m \geq k \geq 0$, let $X^m_k = \mathrm{Sq}^{2^m}\mathrm{Sq}^{2^{m-1}}...\mathrm{Sq}^{2^k}$. The elements of Arnon's A basis are monomials $X^{m_0}_{k_0}...X^{m_t}_{k_t}$ with $(m_i, k_i) < (m_{i+1}, k_{i+1})$, ordered left lexicographically.

  (Note that X^m_k is the reverse of the element Q^m_k used in defining Wall's basis.)
- *Arnon's C basis*. The elements of Arnon's C basis are monomials of the form $\mathrm{Sq}^{t_1}...\mathrm{Sq}^{t_m}$ where for each i, we have $t_i \leq 2t_{i+1}$ and $2^i | t_{m-i}$.
- P^s_t *bases*. Let $p = 2$. For integers $s \geq 0$ and $t > 0$, the element P^s_t is the Milnor basis element $\mathcal{P}(0, ..., 0, p^s, 0, ...)$, with the nonzero entry in position t. To obtain a P^s_t-basis, for each set $\{P^{s_1}_{t_1}, ..., P^{s_k}_{t_k}\}$ of (distinct) P^s_t's, one chooses an ordering and forms the monomials

  $$(P^{s_1}_{t_1})^{i_1}...(P^{s_k}_{t_k})^{i_k}$$

  for all exponents i_j with $0 < i_j < p$. When $p = 2$, the set of all such monomials then forms a basis, and when p is odd, if one multiplies each such monomial on the left by products of the form $Q_{e_1}Q_{e_2}...$ with $0 \leq e_1 < e_2 < ...$, one obtains a basis.

  Thus one gets a basis by choosing an ordering on each set of P^s_t's. There are infinitely many orderings possible, and we have implemented four of them:

  - 'rlex': right lexicographic ordering
  - 'llex': left lexicographic ordering
  - 'deg': ordered by degree, which is the same as left lexicographic ordering on the pair $(s + t, t)$
  - 'revz': left lexicographic ordering on the pair $(s + t, s)$, which is the reverse of the ordering used (on elements in the same degrees as the P^s_t's) in Wood's Z basis: 'revz' stands for 'reversed Z'. This is the default: 'pst' is the same as 'pst_revz'.
- *Commutator bases*. Let $c_{i,1} = \mathcal{P}(p^i)$, let $c_{i,2} = [c_{i+1,1}, c_{i,1}]$, and inductively define $c_{i,k} = [c_{i+k-1,1}, c_{i,k-1}]$. Thus $c_{i,k}$ is a k-fold iterated commutator of the elements $\mathcal{P}(p^i)$, …, $\mathcal{P}(p^{i+k-1})$. Note that $\dim c_{i,k} = \dim P^i_k$.

  Commutator bases are obtained in much the same way as P^s_t-bases: for each set $\{c_{s_1,t_1}, ..., c_{s_k,t_k}\}$ of (distinct) $c_{s,t}$'s, one chooses an ordering and forms the resulting monomials

  $$c^{i_1}_{s_1,t_1}...c^{i_k}_{s_k,t_k}$$

  for all exponents i_j with $0 < i_j < p$. When p is odd, one also needs to left-multiply by products of the Q_i's. As for P^s_t-bases, every ordering on each set of iterated commutators determines a basis, and the same four orderings have been defined for these bases as for the P^s_t bases: 'rlex', 'llex', 'deg', 'revz'.

Sub-Hopf algebras of the Steenrod algebra

The sub-Hopf algebras of the Steenrod algebra have been classified. Milnor proved that at the prime 2, the dual of the Steenrod algebra A_* is isomorphic to a polynomial algebra

$$A_* \cong \mathbf{F}_2[\xi_1, \xi_2, \xi_3, ...].$$

The Milnor basis is dual to the monomial basis. Furthermore, any sub-Hopf algebra corresponds to a quotient of this of the form

$$A_*/(\xi_1^{2^{e1}}, \xi_2^{2^{e2}}, \xi_3^{2^{e3}}, ...).$$

The list of exponents $(e_1, e_2, ...)$ may be considered a function e from the positive integers to the extended non-negative integers (the non-negative integers and ∞); this is called the *profile function* for the sub-Hopf algebra. The profile function must satisfy the condition

- $e(r) \geq \min(e(r-i) - i, e(i))$ for all $0 < i < r$.

At odd primes, the situation is similar: the dual is isomorphic to the tensor product of a polynomial algebra and an exterior algebra,

$$A_* = \mathbf{F}_p[\xi_1, \xi_2, \xi_3, ...] \otimes \Lambda(\tau_0, \tau_1, ...),$$

and any sub-Hopf algebra corresponds to a quotient of this of the form

$$A_*/(\xi_1^{p^{e1}}, \xi_2^{p^{e2}}, ...; \tau_0^{k_0}, \tau_1^{k_1}, ...).$$

Here the profile function has two pieces, e as at the prime 2, and k, which maps the non-negative integers to the set $\{1, 2\}$. These must satisfy the following conditions:

- $e(r) \geq \min(e(r-i) - i, e(i))$ for all $0 < i < r$.
- if $k(i+j) = 1$, then either $e(i) \leq j$ or $k(j) = 1$ for all $i \geq 1$, $j \geq 0$.

(See Adams-Margolis [AM1974], for example, for these results on profile functions.)

This module allows one to construct the Steenrod algebra or any of its sub-Hopf algebras, at any prime. When defining a sub-Hopf algebra, you must work with the Milnor basis or a P^s_t-basis.

Elements of the Steenrod algebra

Basic arithmetic, $p = 2$. To construct an element of the mod 2 Steenrod algebra, use the function Sq:

```
sage: a = Sq(1,2)
sage: b = Sq(4,1)
sage: z = a + b
sage: z
Sq(1,2) + Sq(4,1)
sage: Sq(4) * Sq(1,2)
Sq(1,1,1) + Sq(2,3) + Sq(5,2)
sage: z**2          # non-negative exponents work as they should
Sq(1,2,1) + Sq(4,1,1)
sage: z**0
1
```

Basic arithmetic, $p > 2$. To construct an element of the mod p Steenrod algebra when p is odd, you should first define a Steenrod algebra, using the SteenrodAlgebra command:

```
sage: A3 = SteenrodAlgebra(3)
```

Having done this, the newly created algebra A3 has methods Q and P which construct elements of A3:

```
sage: c = A3.Q(1,3,6); c
Q_1 Q_3 Q_6
sage: d = A3.P(2,0,1); d
P(2,0,1)
sage: c * d
Q_1 Q_3 Q_6 P(2,0,1)
sage: e = A3.P(3)
sage: d * e
P(5,0,1)
sage: e * d
P(1,1,1) + P(5,0,1)
sage: c * c
0
sage: e ** 3
2 P(1,2)
```

Note that one can construct an element like c above in one step, without first constructing the algebra:

```
sage: c = SteenrodAlgebra(3).Q(1,3,6)
sage: c
Q_1 Q_3 Q_6
```

And of course, you can do similar constructions with the mod 2 Steenrod algebra:

```
sage: A = SteenrodAlgebra(2); A
mod 2 Steenrod algebra, milnor basis
sage: A.Sq(2,3,5)
Sq(2,3,5)
sage: A.P(2,3,5)   # when p=2, P = Sq
Sq(2,3,5)
sage: A.Q(1,4)     # when p=2, this gives a product of Milnor primitives
Sq(0,1,0,0,1)
```

Associated to each element is its prime (the characteristic of the underlying base field) and its basis (the basis for the Steenrod algebra in which it lies):

```
sage: a = SteenrodAlgebra(basis='milnor').Sq(1,2,1)
sage: a.prime()
2
sage: a.basis_name()
'milnor'
sage: a.degree()
14
```

It can be viewed in other bases:

```
sage: a.milnor() # same as a
Sq(1,2,1)
sage: a.change_basis('adem')
Sq^9 Sq^4 Sq^1 + Sq^11 Sq^2 Sq^1 + Sq^13 Sq^1
```

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```
sage: a.change_basis('adem').change_basis('milnor')
Sq(1,2,1)
```

Regardless of the prime, each element has an `excess`, and if the element is homogeneous, a `degree`. The excess of $\text{Sq}(i_1, i_2, i_3, ...)$ is $i_1 + i_2 + i_3 + ...$; when p is odd, the excess of $Q_0^{\epsilon_0} Q_1^{\epsilon_1} ... \mathcal{P}(r_1, r_2, ...)$ is $\sum e_i + 2 \sum r_i$. The excess of a linear combination of Milnor basis elements is the minimum of the excesses of those basis elements.

The degree of $\text{Sq}(i_1, i_2, i_3, ...)$ is $\sum(2^n - 1) i_n$, and when p is odd, the degree of $Q_0^{\epsilon_0} Q_1^{\epsilon_1} ... \mathcal{P}(r_1, r_2, ...)$ is $\sum \epsilon_i (2p^i - 1) + \sum r_j (2p^j - 2)$. The degree of a linear combination of such terms is only defined if the terms all have the same degree.

Here are some simple examples:

```
sage: z = Sq(1,2) + Sq(4,1)
sage: z.degree()
7
sage: (Sq(0,0,1) + Sq(5,3)).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
sage: Sq(7,2,1).excess()
10
sage: z.excess()
3
sage: B = SteenrodAlgebra(3)
sage: x = B.Q(1,4)
sage: y = B.P(1,2,3)
sage: x.degree()
166
sage: x.excess()
2
sage: y.excess()
12
```

Elements have a `weight` in the May filtration, which (when $p = 2$) is related to the `height` function defined by Wall:

```
sage: Sq(2,1,5).may_weight()
9
sage: Sq(2,1,5).wall_height()
[2, 3, 2, 1, 1]
sage: b = Sq(4)*Sq(8) + Sq(8)*Sq(4)
sage: b.may_weight()
2
sage: b.wall_height()
[0, 0, 1, 1]
```

Odd primary May weights:

```
sage: A5 = SteenrodAlgebra(5)
sage: a = A5.Q(1,2,4)
sage: b = A5.P(1,2,1)
sage: a.may_weight()
10
sage: b.may_weight()
```

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```
8
sage: (a * b).may_weight()
18
sage: A5.P(0,0,1).may_weight()
3
```

Since the Steenrod algebra is a Hopf algebra, every element has a coproduct and an antipode:

```
sage: Sq(5).coproduct()
1 # Sq(5) + Sq(1) # Sq(4) + Sq(2) # Sq(3) + Sq(3) # Sq(2) + Sq(4) # Sq(1) + Sq(5) # 1
sage: Sq(5).antipode()
Sq(2,1) + Sq(5)
sage: d = Sq(0,0,1); d
Sq(0,0,1)
sage: d.antipode()
Sq(0,0,1)
sage: Sq(4).antipode()
Sq(1,1) + Sq(4)
sage: (Sq(4) * Sq(2)).antipode()
Sq(6)
sage: SteenrodAlgebra(7).P(3,1).antipode()
P(3,1)
```

Applying the antipode twice returns the original element:

```
sage: y = Sq(8)*Sq(4)
sage: y == (y.antipode()).antipode()
True
```

Internal representation: you can use any element as an iterator (`for x in a: ...`), and the method `monomial_coefficients()` returns a dictionary with keys tuples representing basis elements and with corresponding value representing the coefficient of that term:

```
sage: c = Sq(5).antipode(); c
Sq(2,1) + Sq(5)
sage: for mono, coeff in c: print((coeff, mono))
(1, (5,))
(1, (2, 1))
sage: c.monomial_coefficients() == {(2, 1): 1, (5,): 1}
True
sage: sorted(c.monomials(), key=lambda x: tuple(x.support()))
[Sq(2,1), Sq(5)]
sage: sorted(c.support())
[(2, 1), (5,)]
sage: Adem = SteenrodAlgebra(basis='adem')
sage: elt = Adem.Sq(10) + Adem.Sq(9) * Adem.Sq(1)
sage: sorted(elt.monomials(), key=lambda x: tuple(x.support()))
[Sq^9 Sq^1, Sq^10]

sage: A7 = SteenrodAlgebra(p=7)
sage: a = A7.P(1) * A7.P(1); a
2 P(2)
sage: a.leading_coefficient()
```

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```
2
sage: a.leading_monomial()
P(2)
sage: a.leading_term()
2 P(2)
sage: a.change_basis('adem').monomial_coefficients()
{(0, 2, 0): 2}
```

The tuple in the previous output stands for the element $\beta^0 P^2 \beta^0$, i.e., P^2. Going in the other direction, if you want to specify a basis element by giving the corresponding tuple, you can use the `monomial()` method on the algebra:

```
sage: SteenrodAlgebra(p=7, basis='adem').monomial((0, 2, 0))
P^2
sage: 10 * SteenrodAlgebra(p=7, basis='adem').monomial((0, 2, 0))
3 P^2
```

In the following example, elements in Wood's Z basis are certain products of the elements $w(m,k) = \mathrm{Sq}^{2^m(2^{k+1}-1)}$. Internally, each $w(m,k)$ is represented by the pair (m,k), and products of them are represented by tuples of such pairs.

```
sage: A = SteenrodAlgebra(basis='wood_z')
sage: t = ((2, 0), (0, 0))
sage: A.monomial(t)
Sq^4 Sq^1
```

See the documentation for *SteenrodAlgebra()* for more details and examples.

`sage.algebras.steenrod.steenrod_algebra.`**`AA`**`(`*n=None*, *p=2*`)`

This returns the Steenrod algebra A or its sub-Hopf algebra $A(n)$.

INPUT:

- n - non-negative integer, optional (default `None`)
- p - prime number, optional (default 2)

OUTPUT:

If n is `None`, then return the full Steenrod algebra. Otherwise, return $A(n)$.

When $p = 2$, $A(n)$ is the sub-Hopf algebra generated by the elements Sq^i for $i \leq 2^n$. Its profile function is $(n+1, n, n-1, \ldots)$. When p is odd, $A(n)$ is the sub-Hopf algebra generated by the elements Q_0 and $\mathcal{P}^i$ for $i \leq p^{n-1}$. Its profile function is $e = (n, n-1, n-2, \ldots)$ and $k = (2, 2, \ldots, 2)$ (length $n+1$).

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra import AA as A
sage: A()
mod 2 Steenrod algebra, milnor basis
sage: A(2)
sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [3, 2, 1]
sage: A(2, p=5)
sub-Hopf algebra of mod 5 Steenrod algebra, milnor basis, profile function ([2, 1],␣
↪[2, 2, 2])
```

`sage.algebras.steenrod.steenrod_algebra.`**`Sq`**`(`**nums*`)`

Milnor element Sq(a,b,c,…).

INPUT:

- `a, b, c, ...` - non-negative integers

OUTPUT: element of the Steenrod algebra

This returns the Milnor basis element $\mathrm{Sq}(a, b, c, ...)$.

EXAMPLES:

```
sage: Sq(5)
Sq(5)
sage: Sq(5) + Sq(2,1) + Sq(5)  # addition is mod 2:
Sq(2,1)
sage: (Sq(4,3) + Sq(7,2)).degree()
13
```

Entries must be non-negative integers; otherwise, an error results.

This function is a good way to define elements of the Steenrod algebra.

`sage.algebras.steenrod.steenrod_algebra.SteenrodAlgebra(p=2, basis='milnor', generic='auto', **kwds)`

The mod p Steenrod algebra

INPUT:

- `p` - positive prime integer (optional, default = 2)
- `basis` - string (optional, default = 'milnor')
- `profile` - a profile function in form specified below (optional, default `None`)
- `truncation_type` - 0 or ∞ or 'auto' (optional, default 'auto')
- `precision` - integer or `None` (optional, default `None`)
- `generic` - (optional, default 'auto')

OUTPUT:

mod p Steenrod algebra or one of its sub-Hopf algebras, elements of which are printed using `basis`

See below for information about `basis`, `profile`, etc.

EXAMPLES:

Some properties of the Steenrod algebra are available:

```
sage: A = SteenrodAlgebra(2)
sage: A.order()
+Infinity
sage: A.is_finite()
False
sage: A.is_commutative()
False
sage: A.is_noetherian()
False
sage: A.is_integral_domain()
False
sage: A.is_field()
False
sage: A.is_division_algebra()
False
```

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```
sage: A.category()
Category of supercocommutative super hopf algebras
 with basis over Finite Field of size 2
```

There are methods for constructing elements of the Steenrod algebra:

```
sage: A2 = SteenrodAlgebra(2); A2
mod 2 Steenrod algebra, milnor basis
sage: A2.Sq(1,2,6)
Sq(1,2,6)
sage: A2.Q(3,4)  # product of Milnor primitives Q_3 and Q_4
Sq(0,0,0,1,1)
sage: A2.pst(2,3)  # Margolis pst element
Sq(0,0,4)
sage: A5 = SteenrodAlgebra(5); A5
mod 5 Steenrod algebra, milnor basis
sage: A5.P(1,2,6)
P(1,2,6)
sage: A5.Q(3,4)
Q_3 Q_4
sage: A5.Q(3,4) * A5.P(1,2,6)
Q_3 Q_4 P(1,2,6)
sage: A5.pst(2,3)
P(0,0,25)
```

You can test whether elements are contained in the Steenrod algebra:

```
sage: w = Sq(2) * Sq(4)
sage: w in SteenrodAlgebra(2)
True
sage: w in SteenrodAlgebra(17)
False
```

Different bases for the Steenrod algebra:

There are two standard vector space bases for the mod p Steenrod algebra: the Milnor basis and the Serre-Cartan basis. When $p = 2$, there are also several other, less well-known, bases. See the documentation for this module (type `sage.algebras.steenrod.steenrod_algebra?`) and the function `steenrod_algebra_basis` for full descriptions of each of the implemented bases.

This module implements the following bases at all primes:

- 'milnor': Milnor basis.
- 'serre-cartan' or 'adem' or 'admissible': Serre-Cartan basis.
- 'pst', 'pst_rlex', 'pst_llex', 'pst_deg', 'pst_revz': various P^s_t-bases.
- 'comm', 'comm_rlex', 'comm_llex', 'comm_deg', 'comm_revz', or these with '_long' appended: various commutator bases.

It implements the following bases when $p = 2$:

- 'wood_y': Wood's Y basis.
- 'wood_z': Wood's Z basis.

- 'wall', 'wall_long': Wall's basis.
- 'arnon_a', 'arnon_a_long': Arnon's A basis.
- 'arnon_c': Arnon's C basis.

When defining a Steenrod algebra, you can specify a basis. Then elements of that Steenrod algebra are printed in that basis:

```
sage: adem = SteenrodAlgebra(2, 'adem')
sage: x = adem.Sq(2,1)  # Sq(-) always means a Milnor basis element
sage: x
Sq^4 Sq^1 + Sq^5
sage: y = Sq(0,1)    # unadorned Sq defines elements w.r.t. Milnor basis
sage: y
Sq(0,1)
sage: adem(y)
Sq^2 Sq^1 + Sq^3
sage: adem5 = SteenrodAlgebra(5, 'serre-cartan')
sage: adem5.P(0,2)
P^10 P^2 + 4 P^11 P^1 + P^12
```

If you add or multiply elements defined using different bases, the left-hand factor determines the form of the output:

```
sage: SteenrodAlgebra(basis='adem').Sq(3) + SteenrodAlgebra(basis='pst').Sq(0,1)
Sq^2 Sq^1
sage: SteenrodAlgebra(basis='pst').Sq(3) + SteenrodAlgebra(basis='milnor').Sq(0,1)
P^0_1 P^1_1 + P^0_2
sage: SteenrodAlgebra(basis='milnor').Sq(2) * SteenrodAlgebra(basis='arnonc').Sq(2)
Sq(1,1)
```

You can get a list of basis elements in a given dimension:

```
sage: A3 = SteenrodAlgebra(3, 'milnor')
sage: A3.basis(13)
Family (Q_1 P(2), Q_0 P(3))
```

Algebras defined over different bases are not equal:

```
sage: SteenrodAlgebra(basis='milnor') == SteenrodAlgebra(basis='pst')
False
```

Bases have various synonyms, and in general Sage tries to figure out what basis you meant:

```
sage: SteenrodAlgebra(basis='MiLNOr')
mod 2 Steenrod algebra, milnor basis
sage: SteenrodAlgebra(basis='MiLNOr') == SteenrodAlgebra(basis='milnor')
True
sage: SteenrodAlgebra(basis='adem')
mod 2 Steenrod algebra, serre-cartan basis
sage: SteenrodAlgebra(basis='adem').basis_name()
'serre-cartan'
sage: SteenrodAlgebra(basis='wood---z---').basis_name()
'woodz'
```

As noted above, several of the bases ('arnon_a', 'wall', 'comm') have alternate, sometimes longer, representations. These provide ways of expressing elements of the Steenrod algebra in terms of the Sq^{2^n}.

```
sage: A_long = SteenrodAlgebra(2, 'arnon_a_long')
sage: A_long(Sq(6))
Sq^1 Sq^2 Sq^1 Sq^2 + Sq^2 Sq^4
sage: SteenrodAlgebra(2, 'wall_long')(Sq(6))
Sq^2 Sq^1 Sq^2 Sq^1 + Sq^2 Sq^4
sage: SteenrodAlgebra(2, 'comm_deg_long')(Sq(6))
s_1 s_2 s_12 + s_2 s_4
```

Sub-Hopf algebras of the Steenrod algebra:

These are specified using the argument `profile`, along with, optionally, `truncation_type` and `precision`. The `profile` argument specifies the profile function for this algebra. Any sub-Hopf algebra of the Steenrod algebra is determined by its *profile function*. When $p = 2$, this is a map e from the positive integers to the set of non-negative integers, plus ∞, corresponding to the sub-Hopf algebra dual to this quotient of the dual Steenrod algebra:

$$\mathbf{F}_2[\xi_1, \xi_2, \xi_3, ...]/(\xi_1^{2^{e(1)}}, \xi_2^{2^{e(2)}}, \xi_3^{2^{e(3)}}, ...).$$

The profile function e must satisfy the condition

- $e(r) \geq \min(e(r-i) - i, e(i))$ for all $0 < i < r$.

This is specified via `profile`, and optionally `precision` and `truncation_type`. First, `profile` must have one of the following forms:

- a list or tuple, e.g., `[3,2,1]`, corresponding to the function sending 1 to 3, 2 to 2, 3 to 1, and all other integers to the value of `truncation_type`.
- a function from positive integers to non-negative integers (and ∞), e.g., `lambda n: n+2`.
- `None` or `Infinity` - use this for the profile function for the whole Steenrod algebra.

In the first and third cases, `precision` is ignored. In the second case, this function is converted to a tuple of length one less than `precision`, which has default value 100. The function is truncated at this point, and all remaining values are set to the value of `truncation_type`.

`truncation_type` may be 0, ∞, or 'auto'. If it's 'auto', then it gets converted to 0 in the first case above (when `profile` is a list), and otherwise (when `profile` is a function, `None`, or `Infinity`) it gets converted to ∞.

For example, the sub-Hopf algebra $A(2)$ has profile function `[3,2,1,0,0,0,...]`, so it can be defined by any of the following:

```
sage: A2 = SteenrodAlgebra(profile=[3,2,1])
sage: B2 = SteenrodAlgebra(profile=[3,2,1,0,0]) # trailing 0's ignored
sage: A2 == B2
True
sage: C2 = SteenrodAlgebra(profile=lambda n: max(4-n, 0), truncation_type=0)
sage: A2 == C2
True
```

In the following case, the profile function is specified by a function and `truncation_type` isn't specified, so it defaults to ∞; therefore this gives a different sub-Hopf algebra:

```
sage: D2 = SteenrodAlgebra(profile=lambda n: max(4-n, 0))
sage: A2 == D2
False
sage: D2.is_finite()
False
sage: E2 = SteenrodAlgebra(profile=lambda n: max(4-n, 0), truncation_type=Infinity)
sage: D2 == E2
True
```

The argument `precision` only needs to be specified if the profile function is defined by a function and you want to control when the profile switches from the given function to the truncation type. For example:

```
sage: D3 = SteenrodAlgebra(profile=lambda n: n, precision=3)
sage: D3
sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [1, 2,␣
↪+Infinity, +Infinity, +Infinity, ...]
sage: D4 = SteenrodAlgebra(profile=lambda n: n, precision=4); D4
sub-Hopf algebra of mod 2 Steenrod algebra, milnor basis, profile function [1, 2, 3,
↪ +Infinity, +Infinity, +Infinity, ...]
sage: D3 == D4
False
```

When p is odd, `profile` is a pair of functions e and k, corresponding to the quotient

$$\mathbf{F}_p[\xi_1, \xi_2, \xi_3, ...] \otimes \Lambda(\tau_0, \tau_1, ...)/(\xi_1^{p^{e_1}}, \xi_2^{p^{e_2}}, ...; \tau_0^{k_0}, \tau_1^{k_1}, ...).$$

Together, the functions e and k must satisfy the conditions

- $e(r) \geq \min(e(r-i) - i, e(i))$ for all $0 < i < r$,
- if $k(i + j) = 1$, then either $e(i) \leq j$ or $k(j) = 1$ for all $i \geq 1$, $j \geq 0$.

Therefore `profile` must have one of the following forms:

- a pair of lists or tuples, the second of which takes values in the set $\{1, 2\}$, e.g., `([3,2,1,1], [1,1,2,2,1])`.
- a pair of functions, one from the positive integers to non-negative integers (and ∞), one from the non-negative integers to the set $\{1, 2\}$, e.g., `(lambda n: n+2, lambda n: 1 if n<3 else 2)`.
- `None` or `Infinity` - use this for the profile function for the whole Steenrod algebra.

You can also mix and match the first two, passing a pair with first entry a list and second entry a function, for instance. The values of `precision` and `truncation_type` are determined by the first entry.

More examples:

```
sage: E = SteenrodAlgebra(profile=lambda n: 0 if n<3 else 3, truncation_type=0)
sage: E.is_commutative()
True

sage: A2 = SteenrodAlgebra(profile=[3,2,1]) # the algebra A(2)
sage: Sq(7,3,1) in A2
True
sage: Sq(8) in A2
False
sage: Sq(8) in SteenrodAlgebra().basis(8)
```

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```
True
sage: Sq(8) in A2.basis(8)
False
sage: A2.basis(8)
Family (Sq(1,0,1), Sq(2,2), Sq(5,1))

sage: A5 = SteenrodAlgebra(p=5)
sage: A51 = SteenrodAlgebra(p=5, profile=([1], [2,2]))
sage: A5.Q(0,1) * A5.P(4) in A51
True
sage: A5.Q(2) in A51
False
sage: A5.P(5) in A51
False
```

For sub-Hopf algebras of the Steenrod algebra, only the Milnor basis or the various P^s_t-bases may be used.

```
sage: SteenrodAlgebra(profile=[1,2,1,1], basis='adem')
Traceback (most recent call last):
...
NotImplementedError: for sub-Hopf algebras of the Steenrod algebra, only the Milnor␣
↪basis and the pst bases are implemented
```

The generic Steenrod algebra at the prime 2:

The structure formulas for the Steenrod algebra at odd primes p also make sense when p is set to 2. We refer to the resulting algebra as the "generic Steenrod algebra" for the prime 2. The dual Hopf algebra is given by

$$A_* = \mathbf{F}_2[\xi_1, \xi_2, \xi_3, ...] \otimes \Lambda(\tau_0, \tau_1, ...)$$

The degree of ξ_k is $2^{k+1} - 2$ and the degree of τ_k is $2^{k+1} - 1$.

The generic Steenrod algebra is an associated graded algebra of the usual Steenrod algebra that is occasionally useful. Its cohomology, for example, is the E_2-term of a spectral sequence that computes the E_2-term of the Novikov spectral sequence. It can also be obtained as a specialisation of Voevodsky's "motivic Steenrod algebra": in the notation of [Voe2003], Remark 12.12, it corresponds to setting $\rho = \tau = 0$. The usual Steenrod algebra is given by $\rho = 0$ and $\tau = 1$.

In Sage this algebra is constructed using the 'generic' keyword.

Example:

```
sage: EA = SteenrodAlgebra(p=2,generic=True) ; EA
generic mod 2 Steenrod algebra, milnor basis
sage: EA[8]
Vector space spanned by (Q_0 Q_2, Q_0 Q_1 P(2), P(1,1), P(4)) over Finite Field of␣
↪size 2
```

class sage.algebras.steenrod.steenrod_algebra.**SteenrodAlgebra_generic**(*p=2*, *basis='milnor'*, ***kwds*)

Bases: CombinatorialFreeModule

The mod p Steenrod algebra.

Users should not call this, but use the function *SteenrodAlgebra()* instead. See that function for extensive documentation.

EXAMPLES:

```
sage: sage.algebras.steenrod.steenrod_algebra.SteenrodAlgebra_generic()
mod 2 Steenrod algebra, milnor basis
sage: sage.algebras.steenrod.steenrod_algebra.SteenrodAlgebra_generic(5)
mod 5 Steenrod algebra, milnor basis
sage: sage.algebras.steenrod.steenrod_algebra.SteenrodAlgebra_generic(5, 'adem')
mod 5 Steenrod algebra, serre-cartan basis
```

class Element

Bases: `IndexedFreeModuleElement`

Class for elements of the Steenrod algebra. Since the Steenrod algebra class is based on `CombinatorialFreeModule`, this is based on `IndexedFreeModuleElement`. It has new methods reflecting its role, like *degree()* for computing the degree of an element.

EXAMPLES:

Since this class inherits from `IndexedFreeModuleElement`, elements can be used as iterators, and there are other useful methods:

```
sage: c = Sq(5).antipode(); c
Sq(2,1) + Sq(5)
sage: for mono, coeff in c: print((coeff, mono))
(1, (5,))
(1, (2, 1))
sage: c.monomial_coefficients() == {(2, 1): 1, (5,): 1}
True
sage: sorted(c.monomials(), key=lambda x: tuple(x.support()))
[Sq(2,1), Sq(5)]
sage: sorted(c.support())
[(2, 1), (5,)]
```

See the documentation for this module (type `sage.algebras.steenrod.steenrod_algebra?`) for more information about elements of the Steenrod algebra.

additive_order()

The additive order of any nonzero element of the mod p Steenrod algebra is p.

OUTPUT: 1 (for the zero element) or p (for anything else)

EXAMPLES:

```
sage: z = Sq(4) + Sq(6) + 1
sage: z.additive_order()
2
sage: (Sq(3) + Sq(3)).additive_order()
1
```

basis_name()

The basis name associated to self.

EXAMPLES:

```
sage: a = SteenrodAlgebra().Sq(3,2,1)
sage: a.basis_name()
'milnor'
sage: a.change_basis('adem').basis_name()
'serre-cartan'
sage: a.change_basis('wood____y').basis_name()
'woody'
sage: b = SteenrodAlgebra(p=7).basis(36)[0]
sage: b.basis_name()
'milnor'
sage: a.change_basis('adem').basis_name()
'serre-cartan'
```

change_basis(*basis='milnor'*)

Representation of element with respect to basis.

INPUT:

- basis - string, basis in which to work.

OUTPUT: representation of self in given basis

The choices for basis are:

- 'milnor' for the Milnor basis.
- 'serre-cartan', 'serre_cartan', 'sc', 'adem', 'admissible' for the Serre-Cartan basis.
- 'wood_y' for Wood's Y basis.
- 'wood_z' for Wood's Z basis.
- 'wall' for Wall's basis.
- 'wall_long' for Wall's basis, alternate representation
- 'arnon_a' for Arnon's A basis.
- 'arnon_a_long' for Arnon's A basis, alternate representation.
- 'arnon_c' for Arnon's C basis.
- 'pst', 'pst_rlex', 'pst_llex', 'pst_deg', 'pst_revz' for various P^s_t-bases.
- 'comm', 'comm_rlex', 'comm_llex', 'comm_deg', 'comm_revz' for various commutator bases.
- 'comm_long', 'comm_rlex_long', etc., for commutator bases, alternate representations.

See documentation for this module (by browsing the reference manual or by typing sage.algebras.steenrod.steenrod_algebra?) for descriptions of the different bases.

EXAMPLES:

```
sage: c = Sq(2) * Sq(1)
sage: c.change_basis('milnor')
Sq(0,1) + Sq(3)
sage: c.change_basis('serre-cartan')
Sq^2 Sq^1
sage: d = Sq(0,0,1)
sage: d.change_basis('arnonc')
Sq^2 Sq^5 + Sq^4 Sq^2 Sq^1 + Sq^4 Sq^3 + Sq^7
```

coproduct(*algorithm='milnor'*)

The coproduct of this element.

INPUT:

- algorithm – None or a string, either 'milnor' or 'serre-cartan' (or anything which will be converted to one of these by the function *get_basis_name*). If None, default to 'serre-cartan' if current basis is 'serre-cartan'; otherwise use 'milnor'.

See *SteenrodAlgebra_generic.coproduct_on_basis()* for more information on computing the coproduct.

EXAMPLES:

```
sage: a = Sq(2)
sage: a.coproduct()
1 # Sq(2) + Sq(1) # Sq(1) + Sq(2) # 1
sage: b = Sq(4)
sage: (a*b).coproduct() == (a.coproduct()) * (b.coproduct())
True

sage: c = a.change_basis('adem'); c.coproduct(algorithm='milnor')
1 # Sq^2 + Sq^1 # Sq^1 + Sq^2 # 1
sage: c = a.change_basis('adem'); c.coproduct(algorithm='adem')
1 # Sq^2 + Sq^1 # Sq^1 + Sq^2 # 1

sage: d = a.change_basis('comm_long'); d.coproduct()
1 # s_2 + s_1 # s_1 + s_2 # 1

sage: A7 = SteenrodAlgebra(p=7)
sage: a = A7.Q(1) * A7.P(1); a
Q_1 P(1)
sage: a.coproduct()
1 # Q_1 P(1) + P(1) # Q_1 + Q_1 # P(1) + Q_1 P(1) # 1
sage: a.coproduct(algorithm='adem')
1 # Q_1 P(1) + P(1) # Q_1 + Q_1 # P(1) + Q_1 P(1) # 1
```

Once you have an element of the tensor product, you may want to extract the tensor factors of its summands.

```
sage: b = Sq(2).coproduct()
sage: b
1 # Sq(2) + Sq(1) # Sq(1) + Sq(2) # 1
sage: supp = sorted(b.support()); supp
[((), (2,)), ((1,), (1,)), ((2,), ())]
sage: Sq(*supp[0][0])
1
sage: Sq(*supp[0][1])
Sq(2)
sage: [(Sq(*x), Sq(*y)) for (x,y) in supp]
[(1, Sq(2)), (Sq(1), Sq(1)), (Sq(2), 1)]
```

The `support` of an element does not include the coefficients, so at odd primes it may be better to use `monomial_coefficients`:

```
sage: A3 = SteenrodAlgebra(p=3)
sage: b = (A3.P(1)**2).coproduct()
sage: b
2*1 # P(2) + 2*P(1) # P(1) + 2*P(2) # 1
sage: sorted(b.support())
[(((), ()), ((), (2,))), (((), (1,)), ((), (1,))), (((), (2,)), ((), ()))]
sage: b.monomial_coefficients()
{(((), ()), ((), (2,))): 2,
```

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```
 (((), (1,)), ((), (1,))): 2,
 (((), (2,)), ((), ())): 2}
sage: mc = b.monomial_coefficients()
sage: sorted([(A3.monomial(x), A3.monomial(y), mc[x,y]) for (x,y) in mc])
[(1, P(2), 2), (P(1), P(1), 2), (P(2), 1, 2)]
```

degree()

The degree of self.

The degree of $\mathrm{Sq}(i_1, i_2, i_3, ...)$ is

$$i_1 + 3i_2 + 7i_3 + ... + (2^k - 1)i_k + ...$$

At an odd prime p, the degree of Q_k is $2p^k - 1$ and the degree of $\mathcal{P}(i_1, i_2, ...)$ is

$$\sum_{k \geq 0} 2(p^k - 1)i_k.$$

ALGORITHM: If *is_homogeneous()* returns True, call *SteenrodAlgebra_generic.degree_on_basis()* on the leading summand.

EXAMPLES:

```
sage: Sq(0,0,1).degree()
7
sage: (Sq(0,0,1) + Sq(7)).degree()
7
sage: (Sq(0,0,1) + Sq(2)).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous

sage: A11 = SteenrodAlgebra(p=11)
sage: A11.P(1).degree()
20
sage: A11.P(1,1).degree()
260
sage: A11.Q(2).degree()
241
```

excess()

Excess of element.

OUTPUT: `excess` - non-negative integer

The excess of a Milnor basis element $\mathrm{Sq}(a, b, c, ...)$ is $a + b + c + \cdots$. When p is odd, the excess of $Q_0^{e_0} Q_1^{e_1} \cdots P(r_1, r_2, ...)$ is $\sum e_i + 2 \sum r_i$. The excess of a linear combination of Milnor basis elements is the minimum of the excesses of those basis elements.

See [Kr1971] for the proofs of these assertions.

EXAMPLES:

```
sage: a = Sq(1,2,3)
sage: a.excess()
6
```

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```
sage: (Sq(0,0,1) + Sq(4,1) + Sq(7)).excess()
1
sage: elt = Sq(0,0,1) + Sq(4,1) + Sq(7)
sage: M = sorted(elt.monomials(), key=lambda x: tuple(x.support()))
sage: [m.excess() for m in M]
[1, 5, 7]
sage: [m for m in M]
[Sq(0,0,1), Sq(4,1), Sq(7)]
sage: B = SteenrodAlgebra(7)
sage: a = B.Q(1,2,5)
sage: b = B.P(2,2,3)
sage: a.excess()
3
sage: b.excess()
14
sage: (a + b).excess()
3
sage: (a * b).excess()
17
```

is_decomposable()

Return True if element is decomposable, False otherwise.

That is, if element is in the square of the augmentation ideal, return True; otherwise, return False.

OUTPUT: boolean

EXAMPLES:

```
sage: a = Sq(6)
sage: a.is_decomposable()
True
sage: for i in range(9):
....:     if not Sq(i).is_decomposable():
....:         print(Sq(i))
1
Sq(1)
Sq(2)
Sq(4)
Sq(8)
sage: A3 = SteenrodAlgebra(p=3, basis='adem')
sage: [A3.P(n) for n in range(30) if not A3.P(n).is_decomposable()]
[1, P^1, P^3, P^9, P^27]
```

is_homogeneous()

Return True iff this element is homogeneous.

EXAMPLES:

```
sage: (Sq(0,0,1) + Sq(7)).is_homogeneous()
True
sage: (Sq(0,0,1) + Sq(2)).is_homogeneous()
False
```

is_nilpotent()

True if element is not a unit, False otherwise.

EXAMPLES:

```
sage: z = Sq(4,2) + Sq(7,1) + Sq(3,0,1)
sage: z.is_nilpotent()
True
sage: u = 1 + Sq(3,1)
sage: u == 1 + Sq(3,1)
True
sage: u.is_nilpotent()
False
```

is_unit()

True if element has a nonzero scalar multiple of P(0) as a summand, False otherwise.

EXAMPLES:

```
sage: z = Sq(4,2) + Sq(7,1) + Sq(3,0,1)
sage: z.is_unit()
False
sage: u = Sq(0) + Sq(3,1)
sage: u == 1 + Sq(3,1)
True
sage: u.is_unit()
True
sage: A5 = SteenrodAlgebra(5)
sage: v = A5.P(0)
sage: (v + v + v).is_unit()
True
```

may_weight()

May's 'weight' of element.

OUTPUT: `weight` - non-negative integer

If we let $F_*(A)$ be the May filtration of the Steenrod algebra, the weight of an element x is the integer k so that x is in $F_k(A)$ and not in $F_{k+1}(A)$. According to Theorem 2.6 in May's thesis [May1964], the weight of a Milnor basis element is computed as follows: first, to compute the weight of $P(r_1, r_2, ...)$, write each r_i in base p as $r_i = \sum_j p^j r_{ij}$. Then each nonzero binary digit r_{ij} contributes i to the weight: the weight is $\sum_{i,j} i r_{ij}$. When p is odd, the weight of Q_i is $i + 1$, so the weight of a product $Q_{i_1} Q_{i_2} ...$ equals $(i_1 + 1) + (i_2 + 1) + ...$. Then the weight of $Q_{i_1} Q_{i_2} ... P(r_1, r_2, ...)$ is the sum of $(i_1 + 1) + (i_2 + 1) + ...$ and $\sum_{i,j} i r_{ij}$.

The weight of a sum of Milnor basis elements is the minimum of the weights of the summands.

When $p = 2$, we compute the weight on Milnor basis elements by adding up the terms in their 'height' - see *wall_height()* for documentation. (When p is odd, the height of an element is not defined.)

EXAMPLES:

```
sage: Sq(0).may_weight()
0
sage: a = Sq(4)
sage: a.may_weight()
1
```

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```
sage: b = Sq(4)*Sq(8) + Sq(8)*Sq(4)
sage: b.may_weight()
2
sage: Sq(2,1,5).wall_height()
[2, 3, 2, 1, 1]
sage: Sq(2,1,5).may_weight()
9
sage: A5 = SteenrodAlgebra(5)
sage: a = A5.Q(1,2,4)
sage: b = A5.P(1,2,1)
sage: a.may_weight()
10
sage: b.may_weight()
8
sage: (a * b).may_weight()
18
sage: A5.P(0,0,1).may_weight()
3
```

milnor()

Return this element in the Milnor basis; that is, as an element of the appropriate Steenrod algebra.

This just calls the method *SteenrodAlgebra_generic.milnor()*.

EXAMPLES:

```
sage: Adem = SteenrodAlgebra(basis='adem')
sage: a = Adem.basis(4)[1]; a
Sq^3 Sq^1
sage: a.milnor()
Sq(1,1)
```

prime()

The prime associated to self.

EXAMPLES:

```
sage: a = SteenrodAlgebra().Sq(3,2,1)
sage: a.prime()
2
sage: a.change_basis('adem').prime()
2
sage: b = SteenrodAlgebra(p=7).basis(36)[0]
sage: b.prime()
7
sage: SteenrodAlgebra(p=3, basis='adem').one().prime()
3
```

wall_height()

Wall's 'height' of element.

OUTPUT: list of non-negative integers

The height of an element of the mod 2 Steenrod algebra is a list of non-negative integers, defined as follows: if the element is a monomial in the generators $\text{Sq}(2^i)$, then the i^{th} entry in the list is the number of times $\mathrm{Sq}(2^i)$ appears. For an arbitrary element, write it as a sum of such monomials; then its height is the maximum, ordered right-lexicographically, of the heights of those monomials.

When p is odd, the height of an element is not defined.

According to Theorem 3 in [Wal1960], the height of the Milnor basis element $\mathrm{Sq}(r_1, r_2, ...)$ is obtained as follows: write each r_i in binary as $r_i = \sum_j 2^j r_{ij}$. Then each nonzero binary digit r_{ij} contributes 1 to the k^{th} entry in the height, for $j \leq k \leq i + j - 1$.

EXAMPLES:

```
sage: Sq(0).wall_height()
[]
sage: a = Sq(4)
sage: a.wall_height()
[0, 0, 1]
sage: b = Sq(4)*Sq(8) + Sq(8)*Sq(4)
sage: b.wall_height()
[0, 0, 1, 1]
sage: Sq(0,0,3).wall_height()
[1, 2, 2, 1]
```

P(**nums*)

The element $P(a, b, c, ...)$

INPUT:

- `a, b, c, ...` - non-negative integers

OUTPUT:

element of the Steenrod algebra given by the Milnor single basis element $P(a, b, c, ...)$

Note that at the prime 2, this is the same element as $\mathrm{Sq}(a, b, c, ...)$.

EXAMPLES:

```
sage: A = SteenrodAlgebra(2)
sage: A.P(5)
Sq(5)
sage: B = SteenrodAlgebra(3)
sage: B.P(5,1,1)
P(5,1,1)
sage: B.P(1,1,-12,1)
Traceback (most recent call last):
...
TypeError: entries must be non-negative integers

sage: SteenrodAlgebra(basis='serre-cartan').P(0,1)
Sq^2 Sq^1 + Sq^3
sage: SteenrodAlgebra(generic=True).P(2,0,1)
P(2,0,1)
```

Q(**nums*)

The element $Q_{n0} Q_{n1} ...$, given by specifying the subscripts.

INPUT:

- `n0, n1, ...` - non-negative integers

OUTPUT: The element $Q_{n0}Q_{n1}...$

Note that at the prime 2, Q_n is the element $\text{Sq}(0, 0, ..., 1)$, where the 1 is in the $(n + 1)^{st}$ position.

Compare this to the method `Q_exp()`, which defines a similar element, but by specifying the tuple of exponents.

EXAMPLES:

```
sage: A2 = SteenrodAlgebra(2)
sage: A2.Q(2,3)
Sq(0,0,1,1)
sage: A5 = SteenrodAlgebra(5)
sage: A5.Q(1,4)
Q_1 Q_4
sage: A5.Q(1,4) == A5.Q_exp(0,1,0,0,1)
True
sage: H = SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]])
sage: H.Q(2)
Q_2
sage: H.Q(4)
Traceback (most recent call last):
...
ValueError: Element not in this algebra
```

Q_exp(**nums*)

The element $Q_0^{e_0} Q_1^{e_1} ...$, given by specifying the exponents.

INPUT:

- `e0, e1, ...` - sequence of 0s and 1s

OUTPUT: The element $Q_0^{e_0} Q_1^{e_1} ...$

Note that at the prime 2, Q_n is the element $\text{Sq}(0, 0, ..., 1)$, where the 1 is in the $(n + 1)^{st}$ position.

Compare this to the method `Q()`, which defines a similar element, but by specifying the tuple of subscripts of terms with exponent 1.

EXAMPLES:

```
sage: A2 = SteenrodAlgebra(2)
sage: A5 = SteenrodAlgebra(5)
sage: A2.Q_exp(0,0,1,1,0)
Sq(0,0,1,1)
sage: A5.Q_exp(0,0,1,1,0)
Q_2 Q_3
sage: A5.Q(2,3)
Q_2 Q_3
sage: A5.Q_exp(0,0,1,1,0) == A5.Q(2,3)
True
sage: SteenrodAlgebra(2,generic=True).Q_exp(1,0,1)
Q_0 Q_2
```

algebra_generators()

Family of generators for this algebra.

OUTPUT: family of elements of this algebra

At the prime 2, the Steenrod algebra is generated by the elements Sq^{2^i} for $i \geq 0$. At odd primes, it is generated by the elements Q_0 and $\mathcal{P}^{p^i}$ for $i \geq 0$. So if this algebra is the entire Steenrod algebra, return an infinite family made up of these elements.

For sub-Hopf algebras of the Steenrod algebra, it is not always clear what a minimal generating set is. The sub-Hopf algebra $A(n)$ is minimally generated by the elements Sq^{2^i} for $0 \leq i \leq n$ at the prime 2. At odd primes, $A(n)$ is minimally generated by Q_0 along with $\mathcal{P}^{p^i}$ for $0 \leq i \leq n-1$. So if this algebra is $A(n)$, return the appropriate list of generators.

For other sub-Hopf algebras: return a non-minimal generating set: the family of P_t^s's and Q_n's contained in the algebra.

EXAMPLES:

```
sage: A3 = SteenrodAlgebra(3, 'adem')
sage: A3.gens()
Lazy family (<bound method SteenrodAlgebra_generic.gen of mod 3 Steenrod␣
↪algebra, serre-cartan basis>(i))_{i in Non negative integers}
sage: A3.gens()[0]
beta
sage: A3.gens()[1]
P^1
sage: A3.gens()[2]
P^3
sage: SteenrodAlgebra(profile=[3,2,1]).gens()
Family (Sq(1), Sq(2), Sq(4))
```

In the following case, return a non-minimal generating set. (It is not minimal because $\text{Sq}(0, 0, 1)$ is the commutator of $\text{Sq}(1)$ and $\text{Sq}(0, 2)$.)

```
sage: SteenrodAlgebra(profile=[1,2,1]).gens()
Family (Sq(1), Sq(0,1), Sq(0,2), Sq(0,0,1))
sage: SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]]).gens()
Family (Q_0, P(1), P(5))
sage: SteenrodAlgebra(profile=lambda n: n).gens()
Lazy family (<bound method SteenrodAlgebra_generic.gen of sub-Hopf algebra of␣
↪mod 2 Steenrod algebra, milnor basis, profile function [1, 2, 3, ..., 98, 99,␣
↪+Infinity, +Infinity, +Infinity, ...]>(i))_{i in Non negative integers}
```

You may also use `algebra_generators` instead of `gens`:

```
sage: SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]]).algebra_generators()
Family (Q_0, P(1), P(5))
```

an_element()

An element of this Steenrod algebra.

The element depends on the basis and whether there is a nontrivial profile function. (This is used by the automatic test suite, so having different elements in different bases may help in discovering bugs.)

EXAMPLES:

```
sage: SteenrodAlgebra().an_element()
Sq(2,1)
sage: SteenrodAlgebra(basis='adem').an_element()
Sq^4 Sq^2 Sq^1
```

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```
sage: SteenrodAlgebra(p=5).an_element()
4 Q_1 Q_3 P(2,1)
sage: SteenrodAlgebra(basis='pst').an_element()
P^3_1
sage: SteenrodAlgebra(basis='pst', profile=[3,2,1]).an_element()
P^0_1
```

antipode_on_basis(*t*)

The antipode of a basis element of this algebra

INPUT:

- `t` – tuple, the index of a basis element of self

OUTPUT:

the antipode of the corresponding basis element, as an element of self.

ALGORITHM: according to a result of Milnor's, the antipode of $\mathrm{Sq}(n)$ is the sum of all of the Milnor basis elements in dimension n. So: convert the element to the Serre-Cartan basis, thus writing it as a sum of products of elements $\mathrm{Sq}(n)$, and use Milnor's formula for the antipode of $\mathrm{Sq}(n)$, together with the fact that the antipode is an antihomomorphism: if we call the antipode c, then $c(ab) = c(b)c(a)$.

At odd primes, a similar method is used: the antipode of $P(n)$ is the sum of the Milnor P basis elements in dimension $n * 2(p-1)$, multiplied by $(-1)^n$, and the antipode of $\beta = Q_0$ is $-Q_0$. So convert to the Serre-Cartan basis, as in the $p = 2$ case. Note that in the odd prime case, there is a sign in the antihomomorphism formula: $c(ab) = (-1)^{\deg a \deg b} c(b)c(a)$.

EXAMPLES:

```
sage: A = SteenrodAlgebra()
sage: A.antipode_on_basis((4,))
Sq(1,1) + Sq(4)
sage: A.Sq(4).antipode()
Sq(1,1) + Sq(4)
sage: Adem = SteenrodAlgebra(basis='adem')
sage: Adem.Sq(4).antipode()
Sq^3 Sq^1 + Sq^4
sage: SteenrodAlgebra(basis='pst').Sq(3).antipode()
P^0_1 P^1_1 + P^0_2
sage: a = SteenrodAlgebra(basis='wall_long').Sq(10)
sage: a.antipode()
Sq^1 Sq^2 Sq^4 Sq^1 Sq^2 + Sq^2 Sq^4 Sq^1 Sq^2 Sq^1 + Sq^8 Sq^2
sage: a.antipode().antipode() == a
True

sage: SteenrodAlgebra(p=3).P(6).antipode()
P(2,1) + P(6)
sage: SteenrodAlgebra(p=3).P(6).antipode().antipode()
P(6)
```

basis(*d=None*)

Return basis for `self`, either the whole basis or the basis in degree d.

INPUT:

- d – integer or `None`, optional (default `None`)

OUTPUT:

If d is None, then return a basis of the algebra. Otherwise, return the basis in degree d.

EXAMPLES:

```
sage: A3 = SteenrodAlgebra(3)
sage: A3.basis(13)
Family (Q_1 P(2), Q_0 P(3))
sage: SteenrodAlgebra(2, 'adem').basis(12)
Family (Sq^12, Sq^11 Sq^1, Sq^9 Sq^2 Sq^1, Sq^8 Sq^3 Sq^1, Sq^10 Sq^2, Sq^9 Sq^
↪3, Sq^8 Sq^4)

sage: A = SteenrodAlgebra(profile=[1,2,1])
sage: A.basis(2)
Family ()
sage: A.basis(3)
Family (Sq(0,1),)
sage: SteenrodAlgebra().basis(3)
Family (Sq(0,1), Sq(3))
sage: A_pst = SteenrodAlgebra(profile=[1,2,1], basis='pst')
sage: A_pst.basis(3)
Family (P^0_2,)

sage: A7 = SteenrodAlgebra(p=7)
sage: B = SteenrodAlgebra(p=7, profile=([1,2,1], [1]))
sage: A7.basis(84)
Family (P(7),)
sage: B.basis(84)
Family ()
sage: C = SteenrodAlgebra(p=7, profile=([1], [2,2]))
sage: A7.Q(0,1) in C.basis(14)
True
sage: A7.Q(2) in A7.basis(97)
True
sage: A7.Q(2) in C.basis(97)
False
```

With no arguments, return the basis of the whole algebra. This does not print in a very helpful way, unfortunately:

```
sage: A7.basis()
Lazy family (Term map from basis key family of mod 7 Steenrod algebra, milnor␣
↪basis to mod 7 Steenrod algebra, milnor basis(i))_{i in basis key family of␣
↪mod 7 Steenrod algebra, milnor basis}
sage: for (idx,a) in zip((1,..,9),A7.basis()):
....:     print("{} {}".format(idx, a))
1 1
2 Q_0
3 P(1)
4 Q_1
5 Q_0 P(1)
6 Q_0 Q_1
7 P(2)
8 Q_1 P(1)
```

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```
9 Q_0 P(2)
sage: D = SteenrodAlgebra(p=3, profile=([1], [2,2]))
sage: sorted(D.basis())
[1, P(1), P(2), Q_0, Q_0 P(1), Q_0 P(2), Q_0 Q_1, Q_0 Q_1 P(1), Q_0 Q_1 P(2), Q_
↪1, Q_1 P(1), Q_1 P(2)]
```

basis_name()

The basis name associated to self.

EXAMPLES:

```
sage: SteenrodAlgebra(p=2, profile=[1,1]).basis_name()
'milnor'
sage: SteenrodAlgebra(basis='serre-cartan').basis_name()
'serre-cartan'
sage: SteenrodAlgebra(basis='adem').basis_name()
'serre-cartan'
```

coproduct(*x*, *algorithm='milnor'*)

Return the coproduct of an element `x` of this algebra.

INPUT:

- `x` – element of self
- `algorithm` – `None` or a string, either 'milnor' or 'serre-cartan' (or anything which will be converted to one of these by the function *get_basis_name*. If `None`, default to 'serre-cartan' if current basis is 'serre-cartan'; otherwise use 'milnor'.

This calls *coproduct_on_basis()* on the summands of `x` and extends linearly.

EXAMPLES:

```
sage: SteenrodAlgebra().Sq(3).coproduct()
1 # Sq(3) + Sq(1) # Sq(2) + Sq(2) # Sq(1) + Sq(3) # 1
```

The element $\mathrm{Sq}(0,1)$ is primitive:

```
sage: SteenrodAlgebra(basis='adem').Sq(0,1).coproduct()
1 # Sq^2 Sq^1 + 1 # Sq^3 + Sq^2 Sq^1 # 1 + Sq^3 # 1
sage: SteenrodAlgebra(basis='pst').Sq(0,1).coproduct()
1 # P^0_2 + P^0_2 # 1

sage: SteenrodAlgebra(p=3).P(4).coproduct()
1 # P(4) + P(1) # P(3) + P(2) # P(2) + P(3) # P(1) + P(4) # 1
sage: SteenrodAlgebra(p=3).P(4).coproduct(algorithm='serre-cartan')
1 # P(4) + P(1) # P(3) + P(2) # P(2) + P(3) # P(1) + P(4) # 1
sage: SteenrodAlgebra(p=3, basis='serre-cartan').P(4).coproduct()
1 # P^4 + P^1 # P^3 + P^2 # P^2 + P^3 # P^1 + P^4 # 1
sage: SteenrodAlgebra(p=11, profile=((), (2,1,2))).Q(0,2).coproduct()
1 # Q_0 Q_2 + Q_0 # Q_2 + Q_0 Q_2 # 1 + 10*Q_2 # Q_0
```

coproduct_on_basis(*t*, *algorithm=None*)

The coproduct of a basis element of this algebra

INPUT:

- t – tuple, the index of a basis element of self
- algorithm – None or a string, either 'milnor' or 'serre-cartan' (or anything which will be converted to one of these by the function *get_basis_name*. If None, default to 'milnor' unless current basis is 'serre-cartan', in which case use 'serre-cartan'.

ALGORITHM: The coproduct on a Milnor basis element $P(n_1, n_2, ...)$ is $\sum P(i_1, i_2, ...) \otimes P(j_1, j_2, ...)$, summed over all $i_k + j_k = n_k$ for each k. At odd primes, each element Q_n is primitive: its coproduct is $Q_n \otimes 1 + 1 \otimes Q_n$.

One can deduce a coproduct formula for the Serre-Cartan basis from this: the coproduct on each P^n is $\sum P^i \otimes P^{n-i}$ and at odd primes β is primitive. Since the coproduct is an algebra map, one can then compute the coproduct on any Serre-Cartan basis element.

Which of these methods is used is controlled by whether algorithm is 'milnor' or 'serre-cartan'.

OUTPUT:

the coproduct of the corresponding basis element, as an element of self tensor self.

EXAMPLES:

```
sage: A = SteenrodAlgebra()
sage: A.coproduct_on_basis((3,))
1 # Sq(3) + Sq(1) # Sq(2) + Sq(2) # Sq(1) + Sq(3) # 1
```

counit_on_basis(*t*)

The counit sends all elements of positive degree to zero.

INPUT:

- t – tuple, the index of a basis element of self

EXAMPLES:

```
sage: A2 = SteenrodAlgebra(p=2)
sage: A2.counit_on_basis(())
1
sage: A2.counit_on_basis((0,0,1))
0
sage: parent(A2.counit_on_basis((0,0,1)))
Finite Field of size 2
sage: A3 = SteenrodAlgebra(p=3)
sage: A3.counit_on_basis(((1,2,3), (1,1,1)))
0
sage: A3.counit_on_basis(((), ()))
1
sage: A3.counit(A3.P(10,5))
0
sage: A3.counit(A3.P(0))
1
```

degree_on_basis(*t*)

The degree of the monomial specified by the tuple t.

INPUT:

- t - tuple, representing basis element in the current basis.

OUTPUT: integer, the degree of the corresponding element.

The degree of $\mathrm{Sq}(i_1, i_2, i_3, ...)$ is

$$i_1 + 3i_2 + 7i_3 + ... + (2^k - 1)i_k + ...$$

At an odd prime p, the degree of Q_k is $2p^k - 1$ and the degree of $\mathcal{P}(i_1, i_2, ...)$ is

$$\sum_{k \geq 0} 2(p^k - 1)i_k.$$

ALGORITHM: Each basis element is represented in terms relevant to the particular basis: 'milnor' basis elements (at the prime 2) are given by tuples `(a,b,c,...)` corresponding to the element $\mathrm{Sq}(a, b, c, ...)$, while 'pst' basis elements are given by tuples of pairs `((a, b), (c, d), ...)`, corresponding to the product $P^a_b P^c_d ...$. The other bases have similar descriptions. The degree of each basis element is computed from this data, rather than converting the element to the Milnor basis, for example, and then computing the degree.

EXAMPLES:

```
sage: SteenrodAlgebra().degree_on_basis((0,0,1))
7
sage: Sq(7).degree()
7

sage: A11 = SteenrodAlgebra(p=11)
sage: A11.degree_on_basis(((), (1,1)))
260
sage: A11.degree_on_basis(((2,), ()))
241
```

dimension()

The dimension of this algebra as a vector space over $\mathbf{F}_p$.

If the algebra is infinite, return `+Infinity`. Otherwise, the profile function must be finite. In this case, at the prime 2, its dimension is 2^s, where s is the sum of the entries in the profile function. At odd primes, the dimension is $p^s * 2^t$ where s is the sum of the e component of the profile function and t is the number of 2's in the k component of the profile function.

EXAMPLES:

```
sage: SteenrodAlgebra(p=7).dimension()
+Infinity
sage: SteenrodAlgebra(profile=[3,2,1]).dimension()
64
sage: SteenrodAlgebra(p=3, profile=([1,1], [])).dimension()
9
sage: SteenrodAlgebra(p=5, profile=([1], [2,2])).dimension()
20
```

gen(*i=0*)

The ith generator of this algebra.

INPUT:

- `i` - non-negative integer

OUTPUT: the ith generator of this algebra

For the full Steenrod algebra, the i^{th} generator is $\mathrm{Sq}(2^i)$ at the prime 2; when p is odd, the 0th generator is $\beta = Q(0)$, and for $i > 0$, the i^{th} generator is $P(p^{i-1})$.

For sub-Hopf algebras of the Steenrod algebra, it is not always clear what a minimal generating set is. The sub-Hopf algebra $A(n)$ is minimally generated by the elements Sq^{2^i} for $0 \leq i \leq n$ at the prime 2. At odd primes, $A(n)$ is minimally generated by Q_0 along with $\mathcal{P}^{p^i}$ for $0 \leq i \leq n-1$. So if this algebra is $A(n)$, return the appropriate generator.

For other sub-Hopf algebras: they are generated (but not necessarily minimally) by the P^s_t's (and Q_n's, if p is odd) that they contain. So order the P^s_t's (and Q_n's) in the algebra by degree and return the i-th one.

EXAMPLES:

```
sage: A = SteenrodAlgebra(2)
sage: A.gen(4)
Sq(16)
sage: A.gen(200)
Sq(1606938044258990275541962092341162602522202993782792835301376)
sage: SteenrodAlgebra(2, basis='adem').gen(2)
Sq^4
sage: SteenrodAlgebra(2, basis='pst').gen(2)
P^2_1
sage: B = SteenrodAlgebra(5)
sage: B.gen(0)
Q_0
sage: B.gen(2)
P(5)

sage: SteenrodAlgebra(profile=[2,1]).gen(1)
Sq(2)
sage: SteenrodAlgebra(profile=[1,2,1]).gen(1)
Sq(0,1)
sage: SteenrodAlgebra(profile=[1,2,1]).gen(5)
Traceback (most recent call last):
...
ValueError: This algebra only has 4 generators, so call gen(i) with 0 <= i < 4

sage: D = SteenrodAlgebra(profile=lambda n: n)
sage: [D.gen(n) for n in range(5)]
[Sq(1), Sq(0,1), Sq(0,2), Sq(0,0,1), Sq(0,0,2)]
sage: D3 = SteenrodAlgebra(p=3, profile=(lambda n: n, lambda n: 2))
sage: [D3.gen(n) for n in range(9)]
[Q_0, P(1), Q_1, P(0,1), Q_2, P(0,3), P(0,0,1), Q_3, P(0,0,3)]
sage: D3 = SteenrodAlgebra(p=3, profile=(lambda n: n, lambda n: 1 if n<1 else␣
↪2))
sage: [D3.gen(n) for n in range(9)]
[P(1), Q_1, P(0,1), Q_2, P(0,3), P(0,0,1), Q_3, P(0,0,3), P(0,0,0,1)]
sage: SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]], basis='pst').gen(2)
P^1_1
```

gens()

Family of generators for this algebra.

OUTPUT: family of elements of this algebra

At the prime 2, the Steenrod algebra is generated by the elements Sq^{2^i} for $i \geq 0$. At odd primes, it is generated by the elements Q_0 and $\mathcal{P}^{p^i}$ for $i \geq 0$. So if this algebra is the entire Steenrod algebra, return an infinite family made up of these elements.

For sub-Hopf algebras of the Steenrod algebra, it is not always clear what a minimal generating set is. The sub-Hopf algebra $A(n)$ is minimally generated by the elements Sq^{2^i} for $0 \leq i \leq n$ at the prime 2. At odd primes, $A(n)$ is minimally generated by Q_0 along with $\mathcal{P}^{p^i}$ for $0 \leq i \leq n-1$. So if this algebra is $A(n)$, return the appropriate list of generators.

For other sub-Hopf algebras: return a non-minimal generating set: the family of P^s_t's and Q_n's contained in the algebra.

EXAMPLES:

```
sage: A3 = SteenrodAlgebra(3, 'adem')
sage: A3.gens()
Lazy family (<bound method SteenrodAlgebra_generic.gen of mod 3 Steenrod␣
↪algebra, serre-cartan basis>(i))_{i in Non negative integers}
sage: A3.gens()[0]
beta
sage: A3.gens()[1]
P^1
sage: A3.gens()[2]
P^3
sage: SteenrodAlgebra(profile=[3,2,1]).gens()
Family (Sq(1), Sq(2), Sq(4))
```

In the following case, return a non-minimal generating set. (It is not minimal because $\text{Sq}(0,0,1)$ is the commutator of $\text{Sq}(1)$ and $\text{Sq}(0,2)$.)

```
sage: SteenrodAlgebra(profile=[1,2,1]).gens()
Family (Sq(1), Sq(0,1), Sq(0,2), Sq(0,0,1))
sage: SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]]).gens()
Family (Q_0, P(1), P(5))
sage: SteenrodAlgebra(profile=lambda n: n).gens()
Lazy family (<bound method SteenrodAlgebra_generic.gen of sub-Hopf algebra of␣
↪mod 2 Steenrod algebra, milnor basis, profile function [1, 2, 3, ..., 98, 99,␣
↪+Infinity, +Infinity, +Infinity, ...]>(i))_{i in Non negative integers}
```

You may also use `algebra_generators` instead of `gens`:

```
sage: SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]]).algebra_generators()
Family (Q_0, P(1), P(5))
```

homogeneous_component(n)

Return the nth homogeneous piece of the Steenrod algebra.

INPUT:

- n - integer

OUTPUT:

a vector space spanned by the basis for this algebra in dimension n

EXAMPLES:

```
sage: A = SteenrodAlgebra()
sage: A.homogeneous_component(4)
Vector space spanned by (Sq(1,1), Sq(4)) over Finite Field of size 2
```

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```
sage: SteenrodAlgebra(profile=[2,1,0]).homogeneous_component(4)
Vector space spanned by (Sq(1,1),) over Finite Field of size 2
```

The notation A[n] may also be used:

```
sage: A[5]
Vector space spanned by (Sq(2,1), Sq(5)) over Finite Field of size 2
sage: SteenrodAlgebra(basis='wall')[4]
Vector space spanned by (Q^1_0 Q^0_0, Q^2_2) over Finite Field of size 2
sage: SteenrodAlgebra(p=5)[17]
Vector space spanned by (Q_1 P(1), Q_0 P(2)) over Finite Field of size 5
```

Note that A[n] is just a vector space, not a Hopf algebra, so its elements don't have products, coproducts, or antipodes defined on them. If you want to use operations like this on elements of some A[n], then convert them back to elements of A:

```
sage: sorted(A[5].basis())
[milnor[(2, 1)], milnor[(5,)]]
sage: a = list(A[5].basis())[1]
sage: a  # not in A, doesn't print like an element of A
milnor[(5,)]
sage: A(a) # in A
Sq(5)
sage: A(a) * A(a)
Sq(7,1)
sage: a * A(a) # only need to convert one factor
Sq(7,1)
sage: a.antipode() # not defined
Traceback (most recent call last):
...
AttributeError: 'CombinatorialFreeModule_with_category.element_class' object␣
↪has no attribute 'antipode'
sage: A(a).antipode() # convert to elt of A, then compute antipode
Sq(2,1) + Sq(5)

sage: G = SteenrodAlgebra(p=5, profile=[[2,1], [2,2,2]], basis='pst')
```

is_commutative()

True if `self` is graded commutative, as determined by the profile function. In particular, a sub-Hopf algebra of the mod 2 Steenrod algebra is commutative if and only if there is an integer $n > 0$ so that its profile function e satisfies

- $e(i) = 0$ for $i < n$,
- $e(i) \leq n$ for $i \geq n$.

When p is odd, there must be an integer $n \geq 0$ so that the profile functions e and k satisfy

- $e(i) = 0$ for $i < n$,
- $e(i) \leq n$ for $i \geq n$.
- $k(i) = 1$ for $i < n$.

EXAMPLES:

```
sage: A = SteenrodAlgebra(p=3)
sage: A.is_commutative()
False
sage: SteenrodAlgebra(profile=[2,1]).is_commutative()
False
sage: SteenrodAlgebra(profile=[0,2,2,1]).is_commutative()
True
```

Note that if the profile function is specified by a function, then by default it has infinite truncation type: the profile function is assumed to be infinite after the 100th term.

```
sage: SteenrodAlgebra(profile=lambda n: 1).is_commutative()
False
sage: SteenrodAlgebra(profile=lambda n: 1, truncation_type=0).is_commutative()
True

sage: SteenrodAlgebra(p=5, profile=([0,2,2,1], [])).is_commutative()
True
sage: SteenrodAlgebra(p=5, profile=([0,2,2,1], [1,1,2])).is_commutative()
True
sage: SteenrodAlgebra(p=5, profile=([0,2,1], [1,2,2,2])).is_commutative()
False
```

is_division_algebra()

The only way this algebra can be a division algebra is if it is the ground field $\mathbf{F}_p$.

EXAMPLES:

```
sage: SteenrodAlgebra(11).is_division_algebra()
False
sage: SteenrodAlgebra(profile=lambda n: 0, truncation_type=0).is_division_
↪algebra()
True
```

is_field(*proof=True*)

The only way this algebra can be a field is if it is the ground field $\mathbf{F}_p$.

EXAMPLES:

```
sage: SteenrodAlgebra(11).is_field()
False
sage: SteenrodAlgebra(profile=lambda n: 0, truncation_type=0).is_field()
True
```

is_finite()

True if this algebra is finite-dimensional.

Therefore true if the profile function is finite, and in particular the `truncation_type` must be finite.

EXAMPLES:

```
sage: A = SteenrodAlgebra(p=3)
sage: A.is_finite()
False
sage: SteenrodAlgebra(profile=[3,2,1]).is_finite()
```

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```
True
sage: SteenrodAlgebra(profile=lambda n: n).is_finite()
False
```

is_generic()

The algebra is generic if it is based on the odd-primary relations, i.e. if its dual is a quotient of

$$A_* = \mathbf{F}_p[\xi_1, \xi_2, \xi_3, ...] \otimes \Lambda(\tau_0, \tau_1, ...)$$

Sage also allows this for $p = 2$. Only the usual Steenrod algebra at the prime 2 and its sub algebras are non-generic.

EXAMPLES:

```
sage: SteenrodAlgebra(3).is_generic()
True
sage: SteenrodAlgebra(2).is_generic()
False
sage: SteenrodAlgebra(2,generic=True).is_generic()
True
```

is_integral_domain(*proof=True*)

The only way this algebra can be an integral domain is if it is the ground field $\mathbf{F}_p$.

EXAMPLES:

```
sage: SteenrodAlgebra(11).is_integral_domain()
False
sage: SteenrodAlgebra(profile=lambda n: 0, truncation_type=0).is_integral_
↪domain()
True
```

is_noetherian()

This algebra is noetherian if and only if it is finite.

EXAMPLES:

```
sage: SteenrodAlgebra(3).is_noetherian()
False
sage: SteenrodAlgebra(profile=[1,2,1]).is_noetherian()
True
sage: SteenrodAlgebra(profile=lambda n: n+2).is_noetherian()
False
```

milnor()

Convert an element of this algebra to the Milnor basis

INPUT:

- x – an element of this algebra

OUTPUT: x converted to the Milnor basis

ALGORITHM: use the method `_milnor_on_basis` and linearity.

EXAMPLES:

```
sage: Adem = SteenrodAlgebra(basis='adem')
sage: a = Adem.Sq(2) * Adem.Sq(1)
sage: Adem.milnor(a)
Sq(0,1) + Sq(3)
```

ngens()

Number of generators of self.

OUTPUT: number or Infinity

The Steenrod algebra is infinitely generated. A sub-Hopf algebra may be finitely or infinitely generated; in general, it is not clear what a minimal generating set is, nor the cardinality of that set. So: if the algebra is infinite-dimensional, this returns Infinity. If the algebra is finite-dimensional and is equal to one of the sub-Hopf algebras $A(n)$, then their minimal generating set is known, and this returns the cardinality of that set. Otherwise, any sub-Hopf algebra is (not necessarily minimally) generated by the P_t^s's that it contains (along with the Q_n's it contains, at odd primes), so this returns the number of P_t^s's and Q_n's in the algebra.

EXAMPLES:

```
sage: A = SteenrodAlgebra(3)
sage: A.ngens()
+Infinity
sage: SteenrodAlgebra(profile=lambda n: n).ngens()
+Infinity
sage: SteenrodAlgebra(profile=[3,2,1]).ngens() # A(2)
3
sage: SteenrodAlgebra(profile=[3,2,1], basis='pst').ngens()
3
sage: SteenrodAlgebra(p=3, profile=[[3,2,1], [2,2,2,2]]).ngens() # A(3) at p=3
4
sage: SteenrodAlgebra(profile=[1,2,1,1]).ngens()
5
```

one_basis()

The index of the element 1 in the basis for the Steenrod algebra.

EXAMPLES:

```
sage: SteenrodAlgebra(p=2).one_basis()
()
sage: SteenrodAlgebra(p=7).one_basis()
((), ())
```

order()

The order of this algebra.

This is computed by computing its vector space dimension d and then returning p^d.

EXAMPLES:

```
sage: SteenrodAlgebra(p=7).order()
+Infinity
sage: SteenrodAlgebra(profile=[2,1]).dimension()
8
sage: SteenrodAlgebra(profile=[2,1]).order()
```

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```
256
sage: SteenrodAlgebra(p=3, profile=([1], [])).dimension()
3
sage: SteenrodAlgebra(p=3, profile=([1], [])).order()
27
sage: SteenrodAlgebra(p=5, profile=([], [2, 2])).dimension()
4
sage: SteenrodAlgebra(p=5, profile=([], [2, 2])).order() == 5**4
True
```

prime()

The prime associated to self.

EXAMPLES:

```
sage: SteenrodAlgebra(p=2, profile=[1,1]).prime()
2
sage: SteenrodAlgebra(p=7).prime()
7
```

product_on_basis(*t1*, *t2*)

The product of two basis elements of this algebra

INPUT:

- `t1`, `t2` – tuples, the indices of two basis elements of self

OUTPUT:

the product of the two corresponding basis elements, as an element of self

ALGORITHM: If the two elements are represented in the Milnor basis, use Milnor multiplication as implemented in *sage.algebras.steenrod.steenrod_algebra_mult*. If the two elements are represented in the Serre-Cartan basis, then multiply them using Adem relations (also implemented in *sage.algebras.steenrod.steenrod_algebra_mult*). This provides a good way of checking work – multiply Milnor elements, then convert them to Adem elements and multiply those, and see if the answers correspond.

If the two elements are represented in some other basis, then convert them both to the Milnor basis and multiply.

EXAMPLES:

```
sage: Milnor = SteenrodAlgebra()
sage: Milnor.product_on_basis((2,), (2,))
Sq(1,1)
sage: Adem = SteenrodAlgebra(basis='adem')
sage: Adem.Sq(2) * Adem.Sq(2) # indirect doctest
Sq^3 Sq^1
```

When multiplying elements from different bases, the left-hand factor determines the form of the output:

```
sage: Adem.Sq(2) * Milnor.Sq(2)
Sq^3 Sq^1
sage: Milnor.Sq(2) * Adem.Sq(2)
Sq(1,1)
```

profile(*i*, *component=0*)

Profile function for this algebra.

INPUT:

- i - integer
- `component` - either 0 or 1, optional (default 0)

OUTPUT: integer or ∞

See the documentation for `sage.algebras.steenrod.steenrod_algebra` and `SteenrodAlgebra()` for information on profile functions.

This applies the profile function to the integer i. Thus when $p = 2$, i must be a positive integer. When p is odd, there are two profile functions, e and k (in the notation of the aforementioned documentation), corresponding, respectively to `component=0` and `component=1`. So when p is odd and `component` is 0, i must be positive, while when `component` is 1, i must be non-negative.

EXAMPLES:

```
sage: SteenrodAlgebra().profile(3)
+Infinity
sage: SteenrodAlgebra(profile=[3,2,1]).profile(1)
3
sage: SteenrodAlgebra(profile=[3,2,1]).profile(2)
2
```

When the profile is specified by a list, the default behavior is to return zero values outside the range of the list. This can be overridden if the algebra is created with an infinite `truncation_type`:

```
sage: SteenrodAlgebra(profile=[3,2,1]).profile(9)
0
sage: SteenrodAlgebra(profile=[3,2,1], truncation_type=Infinity).profile(9)
+Infinity

sage: B = SteenrodAlgebra(p=3, profile=(lambda n: n, lambda n: 1))
sage: B.profile(3)
3
sage: B.profile(3, component=1)
1

sage: EA = SteenrodAlgebra(generic=True, profile=(lambda n: n, lambda n: 1))
sage: EA.profile(4)
4
sage: EA.profile(2, component=1)
1
```

pst(*s*, *t*)

The Margolis element P^s_t.

INPUT:

- `s` - non-negative integer
- `t` - positive integer
- `p` - positive prime number

OUTPUT: element of the Steenrod algebra

This returns the Margolis element P^s_t of the mod p Steenrod algebra: the element equal to $P(0, 0, ..., 0, p^s)$, where the p^s is in position t.

EXAMPLES:

```
sage: A2 = SteenrodAlgebra(2)
sage: A2.pst(3,5)
Sq(0,0,0,0,8)
sage: A2.pst(1,2) == Sq(4)*Sq(2) + Sq(2)*Sq(4)
True
sage: SteenrodAlgebra(5).pst(3,5)
P(0,0,0,0,125)
```

top_class()

Highest dimensional basis element. This is only defined if the algebra is finite.

EXAMPLES:

```
sage: SteenrodAlgebra(2,profile=(3,2,1)).top_class()
Sq(7,3,1)
sage: SteenrodAlgebra(3,profile=((2,2,1),(1,2,2,2,2))).top_class()
Q_1 Q_2 Q_3 Q_4 P(8,8,2)
```

class sage.algebras.steenrod.steenrod_algebra.**SteenrodAlgebra_mod_two**(*p=2*, *basis='milnor'*, ***kwds*)

Bases: *SteenrodAlgebra_generic*

The mod 2 Steenrod algebra.

Users should not call this, but use the function *SteenrodAlgebra()* instead. See that function for extensive documentation. (This differs from *SteenrodAlgebra_generic* only in that it has a method *Sq()* for defining elements.)

Sq(**nums*)

Milnor element $\mathrm{Sq}(a, b, c, ...)$.

INPUT:

- `a, b, c, ...` - non-negative integers

OUTPUT: element of the Steenrod algebra

This returns the Milnor basis element $\mathrm{Sq}(a, b, c, ...)$.

EXAMPLES:

```
sage: A = SteenrodAlgebra(2)
sage: A.Sq(5)
Sq(5)
sage: A.Sq(5,0,2)
Sq(5,0,2)
```

Entries must be non-negative integers; otherwise, an error results.

5.24 Steenrod algebra bases

AUTHORS:

- John H. Palmieri (2008-07-30): version 0.9
- John H. Palmieri (2010-06-30): version 1.0
- Simon King (2011-10-25): Fix the use of cached functions

This package defines functions for computing various bases of the Steenrod algebra, and for converting between the Milnor basis and any other basis.

This packages implements a number of different bases, at least at the prime 2. The Milnor and Serre-Cartan bases are the most familiar and most standard ones, and all of the others are defined in terms of one of these. The bases are described in the documentation for the function *steenrod_algebra_basis()*; also see the papers by Monks [Mon1998] and Wood [Woo1998] for more information about them. For commutator bases, see the preprint by Palmieri and Zhang [PZ2008].

- 'milnor': Milnor basis.
- 'serre-cartan' or 'adem' or 'admissible': Serre-Cartan basis.

Most of the rest of the bases are only defined when $p = 2$. The only exceptions are the P^s_t-bases and the commutator bases, which are defined at all primes.

- 'wood_y': Wood's Y basis.
- 'wood_z': Wood's Z basis.
- 'wall', 'wall_long': Wall's basis.
- 'arnon_a', 'arnon_a_long': Arnon's A basis.
- 'arnon_c': Arnon's C basis.
- 'pst', 'pst_rlex', 'pst_llex', 'pst_deg', 'pst_revz': various P^s_t-bases.
- 'comm', 'comm_rlex', 'comm_llex', 'comm_deg', 'comm_revz', or these with '_long' appended: various commutator bases.

The main functions provided here are

- *steenrod_algebra_basis()*. This computes a tuple representing basis elements for the Steenrod algebra in a given degree, at a given prime, with respect to a given basis. It is a cached function.
- *convert_to_milnor_matrix()*. This returns the change-of-basis matrix, in a given degree, from any basis to the Milnor basis. It is a cached function.
- *convert_from_milnor_matrix()*. This returns the inverse of the previous matrix.

INTERNAL DOCUMENTATION:

If you want to implement a new basis for the Steenrod algebra:

In the file `steenrod_algebra.py`:

For the class *SteenrodAlgebra_generic*, add functionality to the methods:

- `_repr_term`
- *degree_on_basis*
- `_milnor_on_basis`
- *an_element*

In the file steenrod_algebra_misc.py:

- add functionality to *get_basis_name*: this should accept as input various synonyms for the basis, and its output should be a canonical name for the basis.
- add a function BASIS_mono_to_string like *milnor_mono_to_string* or one of the other similar functions.

In this file steenrod_algebra_bases.py:

- add appropriate lines to *steenrod_algebra_basis()*.
- add a function to compute the basis in a given dimension (to be called by *steenrod_algebra_basis()*).
- modify *steenrod_basis_error_check()* so it checks the new basis.

If the basis has an intrinsic way of defining a product, implement it in the file steenrod_algebra_mult.py and also in the *product_on_basis* method for *SteenrodAlgebra_generic* in steenrod_algebra.py.

sage.algebras.steenrod.steenrod_algebra_bases.**arnonC_basis**(*bound=1*)

Arnon's C basis in dimension n.

INPUT:

- n - non-negative integer
- bound - positive integer (optional)

OUTPUT: tuple of basis elements in dimension n

The elements of Arnon's C basis are monomials of the form $\mathrm{Sq}^{t_1}\ldots\mathrm{Sq}^{t_m}$ where for each i, we have $t_i \leq 2t_{i+1}$ and $2^i|t_{m-i}$.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import arnonC_basis
sage: arnonC_basis(7)
((7,), (2, 5), (4, 3), (4, 2, 1))
```

If optional argument bound is present, include only those monomials whose first term is at least as large as bound:

```
sage: arnonC_basis(7,3)
((7,), (4, 3), (4, 2, 1))
```

sage.algebras.steenrod.steenrod_algebra_bases.**atomic_basis**(*n*, *basis*, ***kwds*)

Basis for dimension n made of elements in 'atomic' degrees: degrees of the form $2^i(2^j - 1)$.

This works at the prime 2 only.

INPUT:

- n - non-negative integer
- basis - string, the name of the basis
- profile - profile function (optional, default None). Together with truncation_type, specify the profile function to be used; None means the profile function for the entire Steenrod algebra. See *sage.algebras.steenrod.steenrod_algebra* and *SteenrodAlgebra()* for information on profile functions.
- truncation_type - truncation type, either 0 or Infinity (optional, default Infinity if no profile function is specified, 0 otherwise).

OUTPUT: tuple of basis elements in dimension n

The atomic bases include Wood's Y and Z bases, Wall's basis, Arnon's A basis, the P^s_t-bases, and the commutator bases. (All of these bases are constructed similarly, hence their constructions have been consolidated into a single function. Also, see the documentation for 'steenrod_algebra_basis' for descriptions of them.) For P^s_t-bases, you may also specify a profile function and truncation type; profile functions are ignored for the other bases.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import atomic_basis
sage: atomic_basis(6,'woody')
(((1, 0), (0, 1), (0, 0)), ((2, 0), (1, 0)), ((1, 1),))
sage: atomic_basis(8,'woodz')
(((2, 0), (0, 1), (0, 0)), ((0, 2), (0, 0)), ((1, 1), (1, 0)), ((3, 0),))
sage: atomic_basis(6,'woodz') == atomic_basis(6, 'woody')
True
sage: atomic_basis(9,'woodz') == atomic_basis(9, 'woody')
False
```

Wall's basis:

```
sage: atomic_basis(8,'wall')
(((2, 2), (1, 0), (0, 0)), ((2, 0), (0, 0)), ((2, 1), (1, 1)), ((3, 3),))
```

Arnon's A basis:

```
sage: atomic_basis(7,'arnona')
(((0, 0), (1, 1), (2, 2)), ((0, 0), (2, 1)), ((1, 0), (2, 2)), ((2, 0),))
```

P^s_t-bases:

```
sage: atomic_basis(7,'pst_rlex')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((2, 1), (0, 2)), ((0, 3),))
sage: atomic_basis(7,'pst_llex')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((0, 2), (2, 1)), ((0, 3),))
sage: atomic_basis(7,'pst_deg')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((0, 2), (2, 1)), ((0, 3),))
sage: atomic_basis(7,'pst_revz')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((0, 2), (2, 1)), ((0, 3),))
```

Commutator bases:

```
sage: atomic_basis(7,'comm_rlex')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((2, 1), (0, 2)), ((0, 3),))
sage: atomic_basis(7,'comm_llex')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((0, 2), (2, 1)), ((0, 3),))
sage: atomic_basis(7,'comm_deg')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((0, 2), (2, 1)), ((0, 3),))
sage: atomic_basis(7,'comm_revz')
(((0, 1), (1, 1), (2, 1)), ((0, 1), (1, 2)), ((0, 2), (2, 1)), ((0, 3),))
```

sage.algebras.steenrod.steenrod_algebra_bases.**atomic_basis_odd**(*n*, *basis*, *p*, ***kwds*)

P^s_t-bases and commutator basis in dimension n at odd primes.

This function is called `atomic_basis_odd` in analogy with `atomic_basis()`.

INPUT:

- n - non-negative integer
- basis - string, the name of the basis
- p - positive prime number
- profile - profile function (optional, default None). Together with truncation_type, specify the profile function to be used; None means the profile function for the entire Steenrod algebra. See sage.algebras.steenrod.steenrod_algebra and SteenrodAlgebra() for information on profile functions.
- truncation_type - truncation type, either 0 or Infinity (optional, default Infinity if no profile function is specified, 0 otherwise).

OUTPUT: tuple of basis elements in dimension n

The only possible difference in the implementations for P^s_t bases and commutator bases is that the former make sense, and require filtering, if there is a nontrivial profile function. This function is called by steenrod_algebra_basis(), and it will not be called for commutator bases if there is a profile function, so we treat the two bases exactly the same.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import atomic_basis_odd
sage: atomic_basis_odd(8, 'pst_rlex', 3)
(((), (((0, 1), 2),)),)

sage: atomic_basis_odd(18, 'pst_rlex', 3)
(((0, 2), ()), ((0, 1), (((1, 1), 1),)))
sage: atomic_basis_odd(18, 'pst_rlex', 3, profile=((), (2,2,2)))
(((0, 2), ()),)
```

sage.algebras.steenrod.steenrod_algebra_bases.**convert_from_milnor_matrix**(*n*, *basis*, *p=2*, *generic='auto'*)

Change-of-basis matrix, Milnor to 'basis', in dimension n.

INPUT:

- n - non-negative integer, the dimension
- basis - string, the basis to which to convert
- p - positive prime number (optional, default 2)

OUTPUT:

matrix - change-of-basis matrix, a square matrix over GF(p)

Note: This is called internally. It is not intended for casual users, so no error checking is made on the integer n, the basis name, or the prime.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import convert_from_milnor_
↪matrix, convert_to_milnor_matrix
sage: convert_from_milnor_matrix(12,'wall')
[1 0 0 1 0 0 0]
[0 0 1 1 0 0 0]
[0 0 0 1 0 1 1]
[0 0 0 1 0 0 0]
```

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```
[1 0 1 0 1 0 0]
[1 1 1 0 0 0 0]
[1 0 1 0 1 0 1]
sage: convert_from_milnor_matrix(38,'serre_cartan')
72 x 72 dense matrix over Finite Field of size 2 (use the '.str()' method to see␣
↪the entries)
sage: x = convert_to_milnor_matrix(20,'wood_y')
sage: y = convert_from_milnor_matrix(20,'wood_y')
sage: x*y
[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1]
```

The function takes an optional argument, the prime p over which to work:

```
sage: convert_from_milnor_matrix(17,'adem',3)
[2 1 1 2]
[0 2 0 1]
[1 2 0 0]
[0 1 0 0]
```

`sage.algebras.steenrod.steenrod_algebra_bases.`**`convert_to_milnor_matrix`**(*basis*, *p=2*, *generic='auto'*)

Change-of-basis matrix, 'basis' to Milnor, in dimension n, at the prime p.

INPUT:

- `n` - non-negative integer, the dimension
- `basis` - string, the basis from which to convert
- `p` - positive prime number (optional, default 2)

OUTPUT:

`matrix` - change-of-basis matrix, a square matrix over `GF(p)`

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import convert_to_milnor_
↪matrix
```

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```
sage: convert_to_milnor_matrix(5, 'adem') # indirect doctest
[0 1]
[1 1]
sage: convert_to_milnor_matrix(45, 'milnor')
111 x 111 dense matrix over Finite Field of size 2 (use the '.str()' method to see␣
↪the entries)
sage: convert_to_milnor_matrix(12,'wall')
[1 0 0 1 0 0 0]
[1 1 0 0 0 1 0]
[0 1 0 1 0 0 0]
[0 0 0 1 0 0 0]
[1 1 0 0 1 0 0]
[0 0 1 1 1 0 1]
[0 0 0 0 1 0 1]
```

The function takes an optional argument, the prime p over which to work:

```
sage: convert_to_milnor_matrix(17,'adem',3)
[0 0 1 1]
[0 0 0 1]
[1 1 1 1]
[0 1 0 1]
sage: convert_to_milnor_matrix(48,'adem',5)
[0 1]
[1 1]
sage: convert_to_milnor_matrix(36,'adem',3)
[0 0 1]
[0 1 0]
[1 2 0]
```

sage.algebras.steenrod.steenrod_algebra_bases.**milnor_basis**(*n*, *p=2*, ***kwds*)

Milnor basis in dimension n with profile function `profile`.

INPUT:

- `n` - non-negative integer
- `p` - positive prime number (optional, default 2)
- `profile` - profile function (optional, default None). Together with `truncation_type`, specify the profile function to be used; None means the profile function for the entire Steenrod algebra. See `sage.algebras.steenrod.steenrod_algebra` and `SteenrodAlgebra` for information on profile functions.
- `truncation_type` - truncation type, either 0 or Infinity (optional, default Infinity if no profile function is specified, 0 otherwise)

OUTPUT: tuple of mod p Milnor basis elements in dimension n

At the prime 2, the Milnor basis consists of symbols of the form $\mathrm{Sq}(m_1, m_2, ..., m_t)$, where each m_i is a non-negative integer and if $t > 1$, then $m_t \neq 0$. At odd primes, it consists of symbols of the form $Q_{e_1} Q_{e_2} ... P(m_1, m_2, ..., m_t)$, where $0 \leq e_1 < e_2 < ...$, each m_i is a non-negative integer, and if $t > 1$, then $m_t \neq 0$.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import milnor_basis
sage: milnor_basis(7)
((0, 0, 1), (1, 2), (4, 1), (7,))
sage: milnor_basis(7, 2)
((0, 0, 1), (1, 2), (4, 1), (7,))
sage: milnor_basis(4, 2)
((1, 1), (4,))
sage: milnor_basis(4, 2, profile=[2,1])
((1, 1),)
sage: milnor_basis(4, 2, profile=(), truncation_type=0)
()
sage: milnor_basis(4, 2, profile=(), truncation_type=Infinity)
((1, 1), (4,))
sage: milnor_basis(9, 3)
(((1,), (1,)), ((0,), (2,)))
sage: milnor_basis(17, 3)
(((2,), ()), ((1,), (3,)), ((0,), (0, 1)), ((0,), (4,)))
sage: milnor_basis(48, p=5)
(((), (0, 1)), ((), (6,)))
sage: len(milnor_basis(100,3))
13
sage: len(milnor_basis(200,7))
0
sage: len(milnor_basis(240,7))
3
sage: len(milnor_basis(240,7, profile=((),()), truncation_type=Infinity))
3
sage: len(milnor_basis(240,7, profile=((),()), truncation_type=0))
0
```

sage.algebras.steenrod.steenrod_algebra_bases.**restricted_partitions**(*n*, *l*, *no_repeats=False*)

List of 'restricted' partitions of n: partitions with parts taken from list.

INPUT:

- `n` - non-negative integer
- `l` - list of positive integers
- `no_repeats` - boolean (optional, default = False), if True, only return partitions with no repeated parts

OUTPUT: list of lists

One could also use `Partitions(n, parts_in=l)`, but this function may be faster. Also, while `Partitions(n, parts_in=l, max_slope=-1)` should in theory return the partitions of n with parts in `l` with no repetitions, the `max_slope=-1` argument is ignored, so it doesn't work. (At the moment, the `no_repeats=True` case is the only one used in the code.)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import restricted_
→partitions
sage: restricted_partitions(10, [7,5,1])
[[7, 1, 1, 1], [5, 5], [5, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]]
sage: restricted_partitions(10, [6,5,4,3,2,1], no_repeats=True)
[[6, 4], [6, 3, 1], [5, 4, 1], [5, 3, 2], [4, 3, 2, 1]]
```

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```
sage: restricted_partitions(10, [6,4,2])
[[6, 4], [6, 2, 2], [4, 4, 2], [4, 2, 2, 2], [2, 2, 2, 2, 2]]
sage: restricted_partitions(10, [6,4,2], no_repeats=True)
[[6, 4]]
```

'l' may have repeated elements. If 'no_repeats' is False, this has no effect. If 'no_repeats' is True, and if the repeated elements appear consecutively in 'l', then each element may be used only as many times as it appears in 'l':

```
sage: restricted_partitions(10, [6,4,2,2], no_repeats=True)
[[6, 4], [6, 2, 2]]
sage: restricted_partitions(10, [6,4,2,2,2], no_repeats=True)
[[6, 4], [6, 2, 2], [4, 2, 2, 2]]
```

(If the repeated elements don't appear consecutively, the results are likely meaningless, containing several partitions more than once, for example.)

In the following examples, 'no_repeats' is False:

```
sage: restricted_partitions(10, [6,4,2])
[[6, 4], [6, 2, 2], [4, 4, 2], [4, 2, 2, 2], [2, 2, 2, 2, 2]]
sage: restricted_partitions(10, [6,4,2,2,2])
[[6, 4], [6, 2, 2], [4, 4, 2], [4, 2, 2, 2], [2, 2, 2, 2, 2]]
sage: restricted_partitions(10, [6,4,4,4,2,2,2,2,2,2])
[[6, 4], [6, 2, 2], [4, 4, 2], [4, 2, 2, 2], [2, 2, 2, 2, 2]]
```

sage.algebras.steenrod.steenrod_algebra_bases.**serre_cartan_basis**(*n*, *p=2*, *bound=1*, ***kwds*)

Serre-Cartan basis in dimension n.

INPUT:

- n - non-negative integer
- bound - positive integer (optional)
- prime - positive prime number (optional, default 2)

OUTPUT: tuple of mod p Serre-Cartan basis elements in dimension n

The Serre-Cartan basis consists of 'admissible monomials in the Steenrod squares'. Thus at the prime 2, it consists of monomials $\text{Sq}^{m_1}\text{Sq}^{m_2}...\text{Sq}^{m_t}$ with $m_i \geq 2m_{i+1}$ for each i. At odd primes, it consists of monomials $\beta^{e_0}P^{s_1}\beta^{e_1}P^{s_2}...P^{s_k}\beta^{e_k}$ with each e_i either 0 or 1, $s_i \geq ps_{i+1} + e_i$ for all i, and $s_k \geq 1$.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import serre_cartan_basis
sage: serre_cartan_basis(7)
((7,), (6, 1), (4, 2, 1), (5, 2))
sage: serre_cartan_basis(13,3)
((1, 3, 0), (0, 3, 1))
sage: serre_cartan_basis(50,5)
((1, 5, 0, 1, 1), (1, 6, 1))
```

If optional argument bound is present, include only those monomials whose last term is at least bound (when p=2), or those for which $s_k - e_k \geq bound$ (when p is odd).

```
sage: serre_cartan_basis(7, bound=2)
((7,), (5, 2))
sage: serre_cartan_basis(13, 3, bound=3)
((1, 3, 0),)
```

sage.algebras.steenrod.steenrod_algebra_bases.**steenrod_algebra_basis**(*basis='milnor'*, *p=2*, ***kwds*)

Basis for the Steenrod algebra in degree n.

INPUT:

- `n` - non-negative integer
- `basis` - string, which basis to use (optional, default = 'milnor')
- `p` - positive prime number (optional, default = 2)
- `profile` - profile function (optional, default None). This is just passed on to the functions `milnor_basis()` and `pst_basis()`.
- `truncation_type` - truncation type, either 0 or Infinity (optional, default Infinity if no profile function is specified, 0 otherwise). This is just passed on to the function `milnor_basis()`.
- `generic` - boolean (optional, default = None)

OUTPUT:

Tuple of objects representing basis elements for the Steenrod algebra in dimension n.

The choices for the string `basis` are as follows; see the documentation for `sage.algebras.steenrod.steenrod_algebra` for details on each basis:

- 'milnor': Milnor basis.
- 'serre-cartan' or 'adem' or 'admissible': Serre-Cartan basis.
- 'pst', 'pst_rlex', 'pst_llex', 'pst_deg', 'pst_revz': various P^s_t-bases.
- 'comm', 'comm_rlex', 'comm_llex', 'comm_deg', 'comm_revz', or any of these with '_long' appended: various commutator bases.

The rest of these bases are only defined when $p = 2$.

- 'wood_y': Wood's Y basis.
- 'wood_z': Wood's Z basis.
- 'wall' or 'wall_long': Wall's basis.
- 'arnon_a' or 'arnon_a_long': Arnon's A basis.
- 'arnon_c': Arnon's C basis.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import steenrod_algebra_
↪basis
sage: steenrod_algebra_basis(7,'milnor') # indirect doctest
((0, 0, 1), (1, 2), (4, 1), (7,))
sage: steenrod_algebra_basis(5)   # milnor basis is the default
((2, 1), (5,))
```

Bases in negative dimensions are empty:

```
sage: steenrod_algebra_basis(-2, 'wall')
()
```

The third (optional) argument to 'steenrod_algebra_basis' is the prime p:

```
sage: steenrod_algebra_basis(9, 'milnor', p=3)
(((1,), (1,)), ((0,), (2,)))
sage: steenrod_algebra_basis(9, 'milnor', 3)
(((1,), (1,)), ((0,), (2,)))
sage: steenrod_algebra_basis(17, 'milnor', 3)
(((2,), ()), ((1,), (3,)), ((0,), (0, 1)), ((0,), (4,)))
```

Other bases:

```
sage: steenrod_algebra_basis(7,'admissible')
((7,), (6, 1), (4, 2, 1), (5, 2))
sage: steenrod_algebra_basis(13,'admissible',p=3)
((1, 3, 0), (0, 3, 1))
sage: steenrod_algebra_basis(5,'wall')
(((2, 2), (0, 0)), ((1, 1), (1, 0)))
sage: steenrod_algebra_basis(5,'wall_long')
(((2, 2), (0, 0)), ((1, 1), (1, 0)))
sage: steenrod_algebra_basis(5,'pst-rlex')
(((0, 1), (2, 1)), ((1, 1), (0, 2)))
```

sage.algebras.steenrod.steenrod_algebra_bases.**steenrod_basis_error_check**(*dim*, *p*, ***kwds*)

This performs crude error checking.

INPUT:

- `dim` - non-negative integer
- `p` - positive prime number

OUTPUT: None

This checks to see if the different bases have the same length, and if the change-of-basis matrices are invertible. If something goes wrong, an error message is printed.

This function checks at the prime `p` as the dimension goes up from 0 to `dim`.

If you set the Sage verbosity level to a positive integer (using `set_verbose(n)`), then some extra messages will be printed.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_bases import steenrod_basis_
↪error_check
sage: steenrod_basis_error_check(15,2) # long time
sage: steenrod_basis_error_check(15,2,generic=True) # long time
sage: steenrod_basis_error_check(40,3) # long time
sage: steenrod_basis_error_check(80,5) # long time
```

sage.algebras.steenrod.steenrod_algebra_bases.**xi_degrees**(*n*, *p=2*, *reverse=True*)

Decreasing list of degrees of the xi_i's, starting in degree n.

INPUT:

- n - integer

- p - prime number, optional (default 2)
- `reverse` - bool, optional (default True)

OUTPUT: `list` - list of integers

When $p = 2$: decreasing list of the degrees of the ξ_i's with degree at most n.

At odd primes: decreasing list of these degrees, each divided by $2(p - 1)$.

If `reverse` is False, then return an increasing list rather than a decreasing one.

EXAMPLES:

```
sage: sage.algebras.steenrod.steenrod_algebra_bases.xi_degrees(17)
[15, 7, 3, 1]
sage: sage.algebras.steenrod.steenrod_algebra_bases.xi_degrees(17, reverse=False)
[1, 3, 7, 15]
sage: sage.algebras.steenrod.steenrod_algebra_bases.xi_degrees(17,p=3)
[13, 4, 1]
sage: sage.algebras.steenrod.steenrod_algebra_bases.xi_degrees(400,p=17)
[307, 18, 1]
```

5.25 Miscellaneous functions for the Steenrod algebra and its elements

AUTHORS:

- John H. Palmieri (2008-07-30): initial version (as the file steenrod_algebra_element.py)
- John H. Palmieri (2010-06-30): initial version of steenrod_misc.py. Implemented profile functions. Moved most of the methods for elements to the `Element` subclass of *sage.algebras.steenrod.steenrod_algebra.SteenrodAlgebra_generic*.

The main functions here are

- *get_basis_name()*. This function takes a string as input and attempts to interpret it as the name of a basis for the Steenrod algebra; it returns the canonical name attached to that basis. This allows for the use of synonyms when defining bases, while the resulting algebras will be identical.
- *normalize_profile()*. This function returns the canonical (and hashable) description of any profile function. See *sage.algebras.steenrod.steenrod_algebra* and *SteenrodAlgebra* for information on profile functions.
- functions named `*_mono_to_string` where `*` is a basis name (*milnor_mono_to_string()*, etc.). These convert tuples representing basis elements to strings, for _repr_ and _latex_ methods.

sage.algebras.steenrod.steenrod_algebra_misc.**arnonA_long_mono_to_string**(*mono*, *latex=False*, *p=2*)

Alternate string representation of element of Arnon's A basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- `mono` - tuple of pairs of non-negative integers (m,k) with $m >= k$
- `latex` - boolean (optional, default False), if true, output LaTeX string

OUTPUT:

string - concatenation of strings of the form Sq(2^m)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import arnonA_long_mono_to_
↪string
sage: arnonA_long_mono_to_string(((1,2),(3,0)))
'Sq^{8} Sq^{4} Sq^{2} Sq^{1}'
sage: arnonA_long_mono_to_string(((1,2),(3,0)),latex=True)
'\text{Sq}^{8} \text{Sq}^{4} \text{Sq}^{2} \text{Sq}^{1}'
```

The empty tuple represents the unit element:

```
sage: arnonA_long_mono_to_string(())
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**arnonA_mono_to_string**(*mono*, *latex=False*, *p=2*)

String representation of element of Arnon's A basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- mono - tuple of pairs of non-negative integers (m,k) with $m >= k$
- latex - boolean (optional, default False), if true, output LaTeX string

OUTPUT:

string - concatenation of strings of the form X^{m}_{k} for each pair (m,k)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import arnonA_mono_to_string
sage: arnonA_mono_to_string(((1,2),(3,0)))
'X^{1}_{2} X^{3}_{0}'
sage: arnonA_mono_to_string(((1,2),(3,0)),latex=True)
'X^{1}_{2} X^{3}_{0}'
```

The empty tuple represents the unit element:

```
sage: arnonA_mono_to_string(())
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**comm_long_mono_to_string**(*mono*, *p*, *latex=False*, *generic=False*)

Alternate string representation of element of a commutator basis.

Okay in low dimensions, but gets unwieldy as the dimension increases.

INPUT:

- mono - tuple of pairs of integers (s,t) with $s >= 0$, $t > 0$
- latex - boolean (optional, default False), if true, output LaTeX string
- generic - whether to format generically, or for the prime 2 (default)

OUTPUT:

`string` - concatenation of strings of the form `s_{2^s... 2^(s+t-1)}` for each pair (s,t)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import comm_long_mono_to_
↪string
sage: comm_long_mono_to_string(((1,2),(0,3)), 2)
's_{24} s_{124}'
sage: comm_long_mono_to_string(((1,2),(0,3)), 2, latex=True)
's_{24} s_{124}'
sage: comm_long_mono_to_string(((1, 4), (((1,2), 1),((0,3), 2))), 5, generic=True)
'Q_{1} Q_{4} s_{5,25} s_{1,5,25}^2'
sage: comm_long_mono_to_string(((1, 4), (((1,2), 1),((0,3), 2))), 3, latex=True,␣
↪generic=True)
'Q_{1} Q_{4} s_{3,9} s_{1,3,9}^{2}'
```

The empty tuple represents the unit element:

```
sage: comm_long_mono_to_string((), p=2)
'1'
```

`sage.algebras.steenrod.steenrod_algebra_misc.`**`comm_mono_to_string`**(*mono*, *latex=False*, *generic=False*)

String representation of element of a commutator basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- `mono` - tuple of pairs of integers (s,t) with $s >= 0, t > 0$
- `latex` - boolean (optional, default False), if true, output LaTeX string
- `generic` - whether to format generically, or for the prime 2 (default)

OUTPUT:

`string` - concatenation of strings of the form `c_{s,t}` for each pair (s,t)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import comm_mono_to_string
sage: comm_mono_to_string(((1,2),(0,3)), generic=False)
'c_{1,2} c_{0,3}'
sage: comm_mono_to_string(((1,2),(0,3)), latex=True)
'c_{1,2} c_{0,3}'
sage: comm_mono_to_string(((1, 4), (((1,2), 1),((0,3), 2))), generic=True)
'Q_{1} Q_{4} c_{1,2} c_{0,3}^2'
sage: comm_mono_to_string(((1, 4), (((1,2), 1),((0,3), 2))), latex=True,␣
↪generic=True)
'Q_{1} Q_{4} c_{1,2} c_{0,3}^{2}'
```

The empty tuple represents the unit element:

```
sage: comm_mono_to_string(())
'1'
```

`sage.algebras.steenrod.steenrod_algebra_misc.`**`convert_perm`**`(`*m*`)`

Convert tuple m of non-negative integers to a permutation in one-line form.

INPUT:

- `m` - tuple of non-negative integers with no repetitions

OUTPUT:

`list` - conversion of `m` to a permutation of the set 1,2,. . . ,len(m)

If `m=(3,7,4)`, then one can view `m` as representing the permutation of the set $(3, 4, 7)$ sending 3 to 3, 4 to 7, and 7 to 4. This function converts `m` to the list `[1,3,2]`, which represents essentially the same permutation, but of the set $(1, 2, 3)$. This list can then be passed to `Permutation`, and its signature can be computed.

EXAMPLES:

```
sage: sage.algebras.steenrod.steenrod_algebra_misc.convert_perm((3,7,4))
[1, 3, 2]
sage: sage.algebras.steenrod.steenrod_algebra_misc.convert_perm((5,0,6,3))
[3, 1, 4, 2]
```

`sage.algebras.steenrod.steenrod_algebra_misc.`**`get_basis_name`**`(`*basis*, *p*, *generic=None*`)`

Return canonical basis named by string basis at the prime p.

INPUT:

- `basis` - string
- `p` - positive prime number
- `generic` - boolean, optional, default to 'None'

OUTPUT:

- `basis_name` - string

Specify the names of the implemented bases. The input is converted to lower-case, then processed to return the canonical name for the basis.

For the Milnor and Serre-Cartan bases, use the list of synonyms defined by the variables `_steenrod_milnor_basis_names` and `_steenrod_serre_cartan_basis_names`. Their canonical names are 'milnor' and 'serre-cartan', respectively.

For the other bases, use pattern-matching rather than a list of synonyms:

- Search for 'wood' and 'y' or 'wood' and 'z' to get the Wood bases. Canonical names 'woody', 'woodz'.
- Search for 'arnon' and 'c' for the Arnon C basis. Canonical name: 'arnonc'.
- Search for 'arnon' (and no 'c') for the Arnon A basis. Also see if 'long' is present, for the long form of the basis. Canonical names: 'arnona', 'arnona_long'.
- Search for 'wall' for the Wall basis. Also see if 'long' is present. Canonical names: 'wall', 'wall_long'.
- Search for 'pst' for P^s_t bases, then search for the order type: 'rlex', 'llex', 'deg', 'revz'. Canonical names: 'pst_rlex', 'pst_llex', 'pst_deg', 'pst_revz'.
- For commutator types, search for 'comm', an order type, and also check to see if 'long' is present. Canonical names: 'comm_rlex', 'comm_llex', 'comm_deg', 'comm_revz', 'comm_rlex_long', 'comm_llex_long', 'comm_deg_long', 'comm_revz_long'.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import get_basis_name
sage: get_basis_name('adem', 2)
'serre-cartan'
sage: get_basis_name('milnor', 2)
'milnor'
sage: get_basis_name('MiLNoR', 5)
'milnor'
sage: get_basis_name('pst-llex', 2)
'pst_llex'
sage: get_basis_name('wood_abcdedfg_y', 2)
'woody'
sage: get_basis_name('wood', 2)
Traceback (most recent call last):
...
ValueError: wood is not a recognized basis at the prime 2
sage: get_basis_name('arnon--hello--long', 2)
'arnona_long'
sage: get_basis_name('arnona_long', p=5)
Traceback (most recent call last):
...
ValueError: arnona_long is not a recognized basis at the prime 5
sage: get_basis_name('NOT_A_BASIS', 2)
Traceback (most recent call last):
...
ValueError: not_a_basis is not a recognized basis at the prime 2
sage: get_basis_name('woody', 2, generic=True)
Traceback (most recent call last):
...
ValueError: woody is not a recognized basis for the generic Steenrod algebra at the␣
↪prime 2
```

sage.algebras.steenrod.steenrod_algebra_misc.**is_valid_profile**(*profile*, *truncation_type*, *p=2*, *generic=None*)

True if `profile`, together with `truncation_type`, is a valid profile at the prime p.

INPUT:

- `profile` - when $p = 2$, a tuple or list of numbers; when p is odd, a pair of such lists
- `truncation_type` - either 0 or ∞
- p - prime number, optional, default 2
- $generic$ - boolean, optional, default None

OUTPUT: True if the profile function is valid, False otherwise.

See the documentation for *sage.algebras.steenrod.steenrod_algebra* for descriptions of profile functions and how they correspond to sub-Hopf algebras of the Steenrod algebra. Briefly: at the prime 2, a profile function e is valid if it satisfies the condition

- $e(r) \geq \min(e(r-i) - i, e(i))$ for all $0 < i < r$.

At odd primes, a pair of profile functions e and k are valid if they satisfy

- $e(r) \geq \min(e(r-i) - i, e(i))$ for all $0 < i < r$.
- if $k(i+j) = 1$, then either $e(i) \leq j$ or $k(j) = 1$ for all $i \geq 1$, $j \geq 0$.

In this function, profile functions are lists or tuples, and `truncation_type` is appended as the last element of the list e before testing.

EXAMPLES:

$p = 2$:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import is_valid_profile
sage: is_valid_profile([3,2,1], 0)
True
sage: is_valid_profile([3,2,1], Infinity)
True
sage: is_valid_profile([1,2,3], 0)
False
sage: is_valid_profile([6,2,0], Infinity)
False
sage: is_valid_profile([0,3], 0)
False
sage: is_valid_profile([0,0,4], 0)
False
sage: is_valid_profile([0,0,0,4,0], 0)
True
```

Odd primes:

```
sage: is_valid_profile(([0,0,0], [2,1,1,1,2,2]), 0, p=3)
True
sage: is_valid_profile(([1], [2,2]), 0, p=3)
True
sage: is_valid_profile(([1], [2]), 0, p=7)
False
sage: is_valid_profile(([1,2,1], []), 0, p=7)
True
sage: is_valid_profile(([0,0,0], [2,1,1,1,2,2]), 0, p=2, generic=True)
True
```

sage.algebras.steenrod.steenrod_algebra_misc.**milnor_mono_to_string**(*mono*, *latex=False*, *generic=False*)

String representation of element of the Milnor basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- `mono` - if $generic = False$, tuple of non-negative integers (a,b,c,...); if $generic = True$, pair of tuples of non-negative integers ((e0, e1, e2, ...), (r1, r2, ...))
- `latex` - boolean (optional, default False), if true, output LaTeX string
- `generic` - whether to format generically, or for the prime 2 (default)

OUTPUT: `rep` - string

This returns a string like `Sq(a,b,c,...)` when $generic = False$, or a string like `Q_e0 Q_e1 Q_e2 ... P(r1, r2, ...)` when $generic = True$.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import milnor_mono_to_string
sage: milnor_mono_to_string((1,2,3,4))
'Sq(1,2,3,4)'
sage: milnor_mono_to_string((1,2,3,4),latex=True)
'\text{Sq}(1,2,3,4)'
sage: milnor_mono_to_string(((1,0), (2,3,1)), generic=True)
'Q_{1} Q_{0} P(2,3,1)'
sage: milnor_mono_to_string(((1,0), (2,3,1)), latex=True, generic=True)
'Q_{1} Q_{0} \mathcal{P}(2,3,1)'
```

The empty tuple represents the unit element:

```
sage: milnor_mono_to_string(())
'1'
sage: milnor_mono_to_string((), generic=True)
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**normalize_profile**(*profile*, *precision=None*, *truncation_type='auto'*, *p=2*, *generic=None*)

Given a profile function and related data, return it in a standard form, suitable for hashing and caching as data defining a sub-Hopf algebra of the Steenrod algebra.

INPUT:

- `profile` - a profile function in form specified below
- `precision` - integer or `None`, optional, default `None`
- `truncation_type` - 0 or ∞ or 'auto', optional, default 'auto'
- p - prime, optional, default 2
- $generic$ - boolean, optional, default `None`

OUTPUT:

a triple `profile, precision, truncation_type`, in standard form as described below.

The "standard form" is as follows: `profile` should be a tuple of integers (or ∞) with no trailing zeroes when $p = 2$, or a pair of such when p is odd or $generic$ is `True`. `precision` should be a positive integer. `truncation_type` should be 0 or ∞. Furthermore, this must be a valid profile, as determined by the function `is_valid_profile()`. See also the documentation for the module `sage.algebras.steenrod.steenrod_algebra` for information about profile functions.

For the inputs: when $p = 2$, `profile` should be a valid profile function, and it may be entered in any of the following forms:

- a list or tuple, e.g., `[3,2,1,1]`
- a function from positive integers to non-negative integers (and ∞), e.g., `lambda n: n+2`. This corresponds to the list `[3, 4, 5, ...]`.
- `None` or `Infinity` - use this for the profile function for the whole Steenrod algebra. This corresponds to the list `[Infinity, Infinity, Infinity, ...]`

To make this hashable, it gets turned into a tuple. In the first case it is clear how to do this; also in this case, `precision` is set to be one more than the length of this tuple. In the second case, construct a tuple of length one less than `precision` (default value 100). In the last case, the empty tuple is returned and `precision` is set to 1.

Once a sub-Hopf algebra of the Steenrod algebra has been defined using such a profile function, if the code requires any remaining terms (say, terms after the 100th), then they are given by `truncation_type` if that is 0 or ∞. If `truncation_type` is 'auto', then in the case of a tuple, it gets set to 0, while for the other cases it gets set to ∞.

See the examples below.

When p is odd, `profile` is a pair of "functions", so it may have the following forms:

- a pair of lists or tuples, the second of which takes values in the set $\{1, 2\}$, e.g., `([3,2,1,1], [1,1,2,2, 1])`.
- a pair of functions, one (called e) from positive integers to non-negative integers (and ∞), one (called k) from non-negative integers to the set $\{1, 2\}$, e.g., `(lambda n: n+2, lambda n: 1)`. This corresponds to the pair `([3, 4, 5, ...], [1, 1, 1, ...])`.
- `None` or `Infinity` - use this for the profile function for the whole Steenrod algebra. This corresponds to the pair `([Infinity, Infinity, Infinity, ...], [2, 2, 2, ...])`.

You can also mix and match the first two, passing a pair with first entry a list and second entry a function, for instance. The values of `precision` and `truncation_type` are determined by the first entry.

EXAMPLES:

$p = 2$:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import normalize_profile
sage: normalize_profile([1,2,1,0,0])
((1, 2, 1), 0)
```

The full mod 2 Steenrod algebra:

```
sage: normalize_profile(Infinity)
((), +Infinity)
sage: normalize_profile(None)
((), +Infinity)
sage: normalize_profile(lambda n: Infinity)
((), +Infinity)
```

The `precision` argument has no effect when the first argument is a list or tuple:

```
sage: normalize_profile([1,2,1,0,0], precision=12)
((1, 2, 1), 0)
```

If the first argument is a function, then construct a list of length one less than `precision`, by plugging in the numbers 1, 2, . . ., `precision` - 1:

```
sage: normalize_profile(lambda n: 4-n, precision=4)
((3, 2, 1), +Infinity)
sage: normalize_profile(lambda n: 4-n, precision=4, truncation_type=0)
((3, 2, 1), 0)
```

Negative numbers in profile functions are turned into zeroes:

```
sage: normalize_profile(lambda n: 4-n, precision=6)
((3, 2, 1, 0, 0), +Infinity)
```

If it doesn't give a valid profile, an error is raised:

```
sage: normalize_profile(lambda n: 3, precision=4, truncation_type=0)
Traceback (most recent call last):
...
ValueError: Invalid profile
sage: normalize_profile(lambda n: 3, precision=4, truncation_type = Infinity)
((3, 3, 3), +Infinity)
```

When p is odd, the behavior is similar:

```
sage: normalize_profile(([2,1], [2,2,2]), p=13)
(((2, 1), (2, 2, 2)), 0)
```

The full mod p Steenrod algebra:

```
sage: normalize_profile(None, p=7)
(((), ()), +Infinity)
sage: normalize_profile(Infinity, p=11)
(((), ()), +Infinity)
sage: normalize_profile((lambda n: Infinity, lambda n: 2), p=17)
(((), ()), +Infinity)
```

Note that as at the prime 2, the `precision` argument has no effect on a list or tuple in either entry of `profile`. If `truncation_type` is 'auto', then it gets converted to either `0` or `+Infinity` depending on the *first* entry of `profile`:

```
sage: normalize_profile(([2,1], [2,2,2]), precision=84, p=13)
(((2, 1), (2, 2, 2)), 0)
sage: normalize_profile((lambda n: 0, lambda n: 2), precision=4, p=11)
(((0, 0, 0), ()), +Infinity)
sage: normalize_profile((lambda n: 0, (1,1,1,1,1,1,1)), precision=4, p=11)
(((0, 0, 0), (1, 1, 1, 1, 1, 1, 1)), +Infinity)
sage: normalize_profile(((4,3,2,1), lambda n: 2), precision=6, p=11)
(((4, 3, 2, 1), (2, 2, 2, 2, 2)), 0)
sage: normalize_profile(((4,3,2,1), lambda n: 1), precision=3, p=11, truncation_
↪type=Infinity)
(((4, 3, 2, 1), (1, 1)), +Infinity)
```

As at the prime 2, negative numbers in the first component are converted to zeroes. Numbers in the second component must be either 1 and 2, or else an error is raised:

```
sage: normalize_profile((lambda n: -n, lambda n: 1), precision=4, p=11)
(((0, 0, 0), (1, 1, 1)), +Infinity)
sage: normalize_profile([[0,0,0], [1,2,3,2,1]], p=11)
Traceback (most recent call last):
...
ValueError: Invalid profile
```

sage.algebras.steenrod.steenrod_algebra_misc.**pst_mono_to_string**(*mono*, *latex=False*, *generic=False*)

String representation of element of a P^s_t-basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- `mono` - tuple of pairs of integers (s,t) with $s >= 0$, $t > 0$

- `latex` - boolean (optional, default False), if true, output LaTeX string
- `generic` - whether to format generically, or for the prime 2 (default)

OUTPUT:

`string` - concatenation of strings of the form `P^{s}_{t}` for each pair (s,t)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import pst_mono_to_string
sage: pst_mono_to_string(((1,2),(0,3)), generic=False)
'P^{1}_{2} P^{0}_{3}'
sage: pst_mono_to_string(((1,2),(0,3)),latex=True, generic=False)
'P^{1}_{2} P^{0}_{3}'
sage: pst_mono_to_string(((1,4), (((1,2), 1),((0,3), 2))), generic=True)
'Q_{1} Q_{4} P^{1}_{2} (P^{0}_{3})^2'
sage: pst_mono_to_string(((1,4), (((1,2), 1),((0,3), 2))), latex=True, generic=True)
'Q_{1} Q_{4} P^{1}_{2} (P^{0}_{3})^{2}'
```

The empty tuple represents the unit element:

```
sage: pst_mono_to_string(())
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**serre_cartan_mono_to_string**(*mono*, *latex=False*, *generic=False*)

String representation of element of the Serre-Cartan basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- `mono` - tuple of positive integers (a,b,c,…) when $generic = False$, or tuple (e0, n1, e1, n2, …) when $generic = True$, where each ei is 0 or 1, and each ni is positive
- `latex` - boolean (optional, default False), if true, output LaTeX string
- `generic` - whether to format generically, or for the prime 2 (default)

OUTPUT: `rep` - string

This returns a string like `Sq^{a} Sq^{b} Sq^{c} ...` when $generic = False$, or a string like `\beta^{e0} P^{n1} \beta^{e1} P^{n2} ...` when $generic = True$. is odd.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import serre_cartan_mono_to_
↪string
sage: serre_cartan_mono_to_string((1,2,3,4))
'Sq^{1} Sq^{2} Sq^{3} Sq^{4}'
sage: serre_cartan_mono_to_string((1,2,3,4),latex=True)
'\\text{Sq}^{1} \\text{Sq}^{2} \\text{Sq}^{3} \\text{Sq}^{4}'
sage: serre_cartan_mono_to_string((0,5,1,1,0), generic=True)
'P^{5} beta P^{1}'
sage: serre_cartan_mono_to_string((0,5,1,1,0), generic=True, latex=True)
'\\mathcal{P}^{5} \\beta \\mathcal{P}^{1}'
```

The empty tuple represents the unit element 1:

```
sage: serre_cartan_mono_to_string(())
'1'
sage: serre_cartan_mono_to_string((), generic=True)
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**wall_long_mono_to_string**(*mono*, *latex=False*)

Alternate string representation of element of Wall's basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- mono - tuple of pairs of non-negative integers (m,k) with $m >= k$
- latex - boolean (optional, default False), if true, output LaTeX string

OUTPUT:

string - concatenation of strings of the form Sq^(2^m)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import wall_long_mono_to_
→string
sage: wall_long_mono_to_string(((1,2),(3,0)))
'Sq^{1} Sq^{2} Sq^{4} Sq^{8}'
sage: wall_long_mono_to_string(((1,2),(3,0)),latex=True)
'\text{Sq}^{1} \text{Sq}^{2} \text{Sq}^{4} \text{Sq}^{8}'
```

The empty tuple represents the unit element:

```
sage: wall_long_mono_to_string(())
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**wall_mono_to_string**(*mono*, *latex=False*)

String representation of element of Wall's basis.

This is used by the _repr_ and _latex_ methods.

INPUT:

- mono - tuple of pairs of non-negative integers (m,k) with $m >= k$
- latex - boolean (optional, default False), if true, output LaTeX string

OUTPUT:

string - concatenation of strings Q^{m}_{k} for each pair (m,k)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import wall_mono_to_string
sage: wall_mono_to_string(((1,2),(3,0)))
'Q^{1}_{2} Q^{3}_{0}'
sage: wall_mono_to_string(((1,2),(3,0)),latex=True)
'Q^{1}_{2} Q^{3}_{0}'
```

The empty tuple represents the unit element:

```
sage: wall_mono_to_string(())
'1'
```

sage.algebras.steenrod.steenrod_algebra_misc.**wood_mono_to_string**(*mono*, *latex=False*)

String representation of element of Wood's Y and Z bases.

This is used by the _repr_ and _latex_ methods.

INPUT:

- mono - tuple of pairs of non-negative integers (s,t)
- latex - boolean (optional, default False), if true, output LaTeX string

OUTPUT:

string - concatenation of strings of the form Sq^{2^s (2^{t+1}-1)} for each pair (s,t)

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_misc import wood_mono_to_string
sage: wood_mono_to_string(((1,2),(3,0)))
'Sq^{14} Sq^{8}'
sage: wood_mono_to_string(((1,2),(3,0)),latex=True)
'\text{Sq}^{14} \text{Sq}^{8}'
```

The empty tuple represents the unit element:

```
sage: wood_mono_to_string(())
'1'
```

5.26 Multiplication for elements of the Steenrod algebra

AUTHORS:

- John H. Palmieri (2008-07-30: version 0.9) initial version: Milnor multiplication.
- John H. Palmieri (2010-06-30: version 1.0) multiplication of Serre-Cartan basis elements using the Adem relations.
- Simon King (2011-10-25): Fix the use of cached functions.

Milnor multiplication, $p = 2$

See Milnor's paper [Mil1958] for proofs, etc.

To multiply Milnor basis elements $\mathrm{Sq}(r_1, r_2, ...)$ and $\mathrm{Sq}(s_1, s_2, ...)$ at the prime 2, form all possible matrices M with rows and columns indexed starting at 0, with position (0,0) deleted (or ignored), with s_i equal to the sum of column i for each i, and with r_j equal to the 'weighted' sum of row j. The weights are as follows: elements from column i are multiplied by 2^i. For example, to multiply $\mathrm{Sq}(2)$ and $\mathrm{Sq}(1,1)$, form the matrices

$$\begin{Vmatrix} * & 1 & 1 \\ 2 & 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} * & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}$$

(The $*$ is the ignored (0,0)-entry of the matrix.) For each such matrix M, compute a multinomial coefficient, mod 2: for each diagonal $\{m_{ij} : i + j = n\}$, compute $(\sum m_{i,j}!)/(m_{0,n}!m_{1,n-1}!...m_{n,0}!)$. Multiply these together for all n. (To compute this mod 2, view the entries of the matrix as their base 2 expansions; then this coefficient is zero if and only if there is some diagonal containing two numbers which have a summand in common in their base 2 expansion. For example, if 3 and 10 are in the same diagonal, the coefficient is zero, because $3 = 1 + 2$ and $10 = 2 + 8$: they both have a summand of 2.)

Now, for each matrix with multinomial coefficient 1, let t_n be the sum of the nth diagonal in the matrix; then

$$\mathrm{Sq}(r_1, r_2, ...)\mathrm{Sq}(s_1, s_2, ...) = \sum \mathrm{Sq}(t_1, t_2, ...)$$

The function *milnor_multiplication()* takes as input two tuples of non-negative integers, r and s, which represent $\mathrm{Sq}(r) = \mathrm{Sq}(r_1, r_2, ...)$ and $\mathrm{Sq}(s) = \mathrm{Sq}(s_1, s_2, ...)$; it returns as output a dictionary whose keys are tuples $t = (t_1, t_2, ...)$ of non-negative integers, and for each tuple the associated value is the coefficient of $\mathrm{Sq}(t)$ in the product formula. (Since we are working mod 2, this coefficient is 1 – if it is zero, the element is omitted from the dictionary altogether).

Milnor multiplication, odd primes

As for the $p = 2$ case, see Milnor's paper [Mil1958] for proofs.

Fix an odd prime p. There are three steps to multiply Milnor basis elements $Q_{f_1} Q_{f_2} ...\mathcal{P}(q_1, q_2, ...)$ and $Q_{g_1} Q_{g_2} ...\mathcal{P}(s_1, s_2, ...)$: first, use the formula

$$\mathcal{P}(q_1, q_2, ...)Q_k = Q_k\mathcal{P}(q_1, q_2, ...) + Q_{k+1}\mathcal{P}(q_1 - p^k, q_2, ...) + Q_{k+2}\mathcal{P}(q_1, q_2 - p^k, ...) + ...$$

Second, use the fact that the Q_k's form an exterior algebra: $Q_k^2 = 0$ for all k, and if $i \neq j$, then Q_i and Q_j anticommute: $Q_i Q_j = -Q_j Q_i$. After these two steps, the product is a linear combination of terms of the form

$$Q_{e_1} Q_{e_2} ...\mathcal{P}(r_1, r_2, ...)\mathcal{P}(s_1, s_2, ...).$$

Finally, use Milnor matrices to multiply the pairs of $\mathcal{P}(...)$ terms, as at the prime 2: form all possible matrices M with rows and columns indexed starting at 0, with position (0,0) deleted (or ignored), with s_i equal to the sum of column i for each i, and with r_j equal to the weighted sum of row j: elements from column i are multiplied by p^i. For example when $p = 5$, to multiply $\mathcal{P}(5)$ and $\mathcal{P}(1, 1)$, form the matrices

$$\begin{Vmatrix} * & 1 & 1 \\ 5 & 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} * & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}$$

For each such matrix M, compute a multinomial coefficient, mod p: for each diagonal $\{m_{ij} : i + j = n\}$, compute $(\sum m_{i,j}!)/(m_{0,n}!m_{1,n-1}!...m_{n,0}!)$. Multiply these together for all n.

Now, for each matrix with nonzero multinomial coefficient b_M, let t_n be the sum of the n-th diagonal in the matrix; then

$$\mathcal{P}(r_1, r_2, ...)\mathcal{P}(s_1, s_2, ...) = \sum b_M\mathcal{P}(t_1, t_2, ...)$$

For example when $p = 5$, we have

$$\mathcal{P}(5)\mathcal{P}(1, 1) = \mathcal{P}(6, 1) + 2\mathcal{P}(0, 2).$$

The function *milnor_multiplication()* takes as input two pairs of tuples of non-negative integers, (g, q) and (f, s), which represent $Q_{g_1} Q_{g_2} ...\mathcal{P}(q_1, q_2, ...)$ and $Q_{f_1} Q_{f_2} ...\mathcal{P}(s_1, s_2, ...)$. It returns as output a dictionary whose keys are pairs of tuples (e, t) of non-negative integers, and for each tuple the associated value is the coefficient in the product formula.

The Adem relations and admissible sequences

If $p = 2$, then the mod 2 Steenrod algebra is generated by Steenrod squares Sq^a for $a \geq 0$ (equal to the Milnor basis element $\text{Sq}(a)$). The *Adem relations* are as follows: if $a < 2b$,

$$\text{Sq}^a \text{Sq}^b = \sum_{j=0}^{a/2} \binom{b-j-1}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j$$

A monomial $\text{Sq}^{i_1} \text{Sq}^{i_2} ... \text{Sq}^{i_n}$ is called *admissible* if $i_k \geq 2 i_{k+1}$ for all k. One can use the Adem relations to show that the admissible monomials span the Steenrod algebra, as a vector space; with more work, one can show that the admissible monomials are also linearly independent. They form the *Serre-Cartan* basis for the Steenrod algebra. To multiply a collection of admissible monomials, concatenate them and see if the result is admissible. If it is, you're done. If not, find the first pair $\text{Sq}^a \text{Sq}^b$ where it fails to be admissible and apply the Adem relations there. Repeat with the resulting terms. One can prove that this process terminates in a finite number of steps, and therefore gives a procedure for multiplying elements of the Serre-Cartan basis.

At an odd prime p, the Steenrod algebra is generated by the pth power operations $\mathcal{P}^a$ (the same as $\mathcal{P}(a)$ in the Milnor basis) and the Bockstein operation β (= Q_0 in the Milnor basis). The odd primary *Adem relations* are as follows: if $a < pb$,

$$\mathcal{P}^a \mathcal{P}^b = \sum_{j=0}^{a/p} (-1)^{a+j} \binom{(b-j)(p-1)-1}{a-pj} \mathcal{P}^{a+b-j} \mathcal{P}^j$$

Also, if $a \leq pb$,

$$\mathcal{P}^a \beta \mathcal{P}^b = \sum_{j=0}^{a/p} (-1)^{a+j} \binom{(b-j)(p-1)}{a-pj} \beta \mathcal{P}^{a+b-j} \mathcal{P}^j + \sum_{j=0}^{a/p} (-1)^{a+j-1} \binom{(b-j)(p-1)-1}{a-pj-1} \mathcal{P}^{a+b-j} \beta \mathcal{P}^j$$

The *admissible* monomials at an odd prime are products of the form

$$\beta^{\epsilon_0} \mathcal{P}^{s_1} \beta^{\epsilon_1} \mathcal{P}^{s_2} ... \mathcal{P}^{s_n} \beta^{\epsilon_n}$$

where $s_k \geq \epsilon_{k+1} + p s_{k+1}$ for all k. As at the prime 2, these form a basis for the Steenrod algebra.

The main function for this is `make_mono_admissible()`, which converts a product of Steenrod squares or pth power operations and Bocksteins into a dictionary representing a sum of admissible monomials.

`sage.algebras.steenrod.steenrod_algebra_mult.adem(b, c=0, p=2, generic=None)`

The mod p Adem relations

INPUT:

- a, b, c (optional) - nonnegative integers, corresponding to either $P^a P^b$ or (if c present) to $P^a \beta^b P^c$
- p - positive prime number (optional, default 2)
- $generic$ - whether to use the generic Steenrod algebra, (default: depends on prime)

OUTPUT:

a dictionary representing the mod p Adem relations applied to $P^a P^b$ or (if c present) to $P^a \beta^b P^c$.

The mod p Adem relations for the mod p Steenrod algebra are as follows: if $p = 2$, then if $a < 2b$,

$$\text{Sq}^a \text{Sq}^b = \sum_{j=0}^{a/2} \binom{b-j-1}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j$$

If p is odd, then if $a < pb$,

$$P^a P^b = \sum_{j=0}^{a/p} (-1)^{a+j} \binom{(b-j)(p-1)-1}{a-pj} P^{a+b-j} P^j$$

Also for p odd, if $a \leq pb$,

$$P^a \beta P^b = \sum_{j=0}^{a/p} (-1)^{a+j} \binom{(b-j)(p-1)}{a-pj} \beta P^{a+b-j} P^j + \sum_{j=0}^{a/p} (-1)^{a+j-1} \binom{(b-j)(p-1)-1}{a-pj-1} P^{a+b-j} \beta P^j$$

EXAMPLES:

If two arguments (a and b) are given, then computations are done mod 2. If $a \geq 2b$, then the dictionary {(a,b): 1} is returned. Otherwise, the right side of the mod 2 Adem relation for $\mathrm{Sq}^a \mathrm{Sq}^b$ is returned. For example, since $\mathrm{Sq}^2 \mathrm{Sq}^2 = \mathrm{Sq}^3 \mathrm{Sq}^1$, we have:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import adem
sage: adem(2,2) # indirect doctest
{(3, 1): 1}
sage: adem(4,2)
{(4, 2): 1}
sage: adem(4,4) == {(6, 2): 1, (7, 1): 1}
True
```

If p is given and is odd, then with two inputs a and b, the Adem relation for $P^a P^b$ is computed. With three inputs a, b, c, the Adem relation for $P^a \beta^b P^c$ is computed. In either case, the keys in the output are all tuples of odd length, with `(i_1, i_2, ..., i_m)` representing

$$\beta^{i_1} P^{i_2} \beta^{i_3} P^{i_4} ... \beta^{i_m}$$

For instance:

```
sage: adem(3,1, p=3)
{(0, 3, 0, 1, 0): 1}
sage: adem(3,0,1, p=3)
{(0, 3, 0, 1, 0): 1}
sage: adem(1,0,1, p=7)
{(0, 2, 0): 2}
sage: adem(1,1,1, p=5) == {(0, 2, 1): 1, (1, 2, 0): 1}
True
sage: adem(1,1,2, p=5) == {(0, 3, 1): 1, (1, 3, 0): 2}
True
```

`sage.algebras.steenrod.steenrod_algebra_mult.`**`binomial_mod2`**(n, k)

The binomial coefficient $\binom{n}{k}$, computed mod 2.

INPUT:

- n, k - integers

OUTPUT:

n choose k, mod 2

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import binomial_mod2
sage: binomial_mod2(4,2)
0
sage: binomial_mod2(5,4)
1
sage: binomial_mod2(3 * 32768, 32768)
1
sage: binomial_mod2(4 * 32768, 32768)
0
```

sage.algebras.steenrod.steenrod_algebra_mult.**binomial_modp**(*n*, *k*, *p*)

The binomial coefficient $\binom{n}{k}$, computed mod p.

INPUT:

- n, k - integers
- p - prime number

OUTPUT:

n choose k, mod p

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import binomial_modp
sage: binomial_modp(5,2,3)
1
sage: binomial_modp(6,2,11)  # 6 choose 2 = 15
4
```

sage.algebras.steenrod.steenrod_algebra_mult.**make_mono_admissible**(*p=2*, *generic=None*)

Given a tuple mono, view it as a product of Steenrod operations, and return a dictionary giving data equivalent to writing that product as a linear combination of admissible monomials.

When $p = 2$, the sequence (and hence the corresponding monomial) $(i_1, i_2, ...)$ is admissible if $i_j \geq 2i_{j+1}$ for all j.

When p is odd, the sequence $(e_1, i_1, e_2, i_2, ...)$ is admissible if $i_j \geq e_{j+1} + pi_{j+1}$ for all j.

INPUT:

- mono - a tuple of non-negative integers
- p - prime number, optional (default 2)
- *generic* - whether to use the generic Steenrod algebra, (default: depends on prime)

OUTPUT:

Dictionary of terms of the form (tuple: coeff), where 'tuple' is an admissible tuple of non-negative integers and 'coeff' is its coefficient. This corresponds to a linear combination of admissible monomials. When p is odd, each tuple must have an odd length: it should be of the form $(e_1, i_1, e_2, i_2, ..., e_k)$ where each e_j is either 0 or 1 and each i_j is a positive integer: this corresponds to the admissible monomial

$$\beta^{e_1}\mathcal{P}^{i_2}\beta^{e_2}\mathcal{P}^{i_2}\ldots\mathcal{P}^{i_k}\beta^{e_k}$$

ALGORITHM:

Given $(i_1, i_2, i_3, ...)$, apply the Adem relations to the first pair (or triple when p is odd) where the sequence is inadmissible, and then apply this function recursively to each of the resulting tuples $(i_1, ..., i_{j-1}, NEW, i_{j+2}, ...)$, keeping track of the coefficients.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import make_mono_admissible
sage: make_mono_admissible((12,)) # already admissible, indirect doctest
{(12,): 1}
sage: make_mono_admissible((2,1)) # already admissible
{(2, 1): 1}
sage: make_mono_admissible((2,2))
{(3, 1): 1}
sage: make_mono_admissible((2, 2, 2))
{(5, 1): 1}
sage: make_mono_admissible((0, 2, 0, 1, 0), p=7)
{(0, 3, 0): 3}
```

Test the fix from github issue #13796:

```
sage: SteenrodAlgebra(p=2, basis='adem').Q(2) * (Sq(6) * Sq(2)) # indirect doctest
Sq^10 Sq^4 Sq^1 + Sq^10 Sq^5 + Sq^12 Sq^3 + Sq^13 Sq^2
```

sage.algebras.steenrod.steenrod_algebra_mult.**milnor_multiplication**(*r*, *s*)

Product of Milnor basis elements r and s at the prime 2.

INPUT:

- r – tuple of non-negative integers
- s – tuple of non-negative integers

OUTPUT:

Dictionary of terms of the form (tuple: coeff), where 'tuple' is a tuple of non-negative integers and 'coeff' is 1.

This computes Milnor matrices for the product of $\mathrm{Sq}(r)$ and $\mathrm{Sq}(s)$, computes their multinomial coefficients, and for each matrix whose coefficient is 1, add $\mathrm{Sq}(t)$ to the output, where t is the tuple formed by the diagonals sums from the matrix.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import milnor_multiplication
sage: milnor_multiplication((2,), (1,)) == {(0, 1): 1, (3,): 1}
True
sage: sorted(milnor_multiplication((4,), (2,1)).items())
[((0, 3), 1), ((2, 0, 1), 1), ((6, 1), 1)]
sage: sorted(milnor_multiplication((2,4), (0,1)).items())
[((2, 0, 0, 1), 1), ((2, 5), 1)]
```

These examples correspond to the following product computations:

$$\begin{aligned}\mathrm{Sq}(2)\mathrm{Sq}(1) &= \mathrm{Sq}(0,1) + \mathrm{Sq}(3)\\ \mathrm{Sq}(4)\mathrm{Sq}(2,1) &= \mathrm{Sq}(6,1) + \mathrm{Sq}(0,3) + \mathrm{Sq}(2,0,1)\\ \mathrm{Sq}(2,4)\mathrm{Sq}(0,1) &= \mathrm{Sq}(2,5) + \mathrm{Sq}(2,0,0,1)\end{aligned}$$

This uses the same algorithm Monks does in his Maple package: see http://mathweb.scranton.edu/monks/software/Steenrod/steen.html.

sage.algebras.steenrod.steenrod_algebra_mult.**milnor_multiplication_odd**(*m1*, *m2*, *p*)

Product of Milnor basis elements defined by m1 and m2 at the odd prime p.

INPUT:

- m1 - pair of tuples (e,r), where e is an increasing tuple of non-negative integers and r is a tuple of non-negative integers
- m2 - pair of tuples (f,s), same format as m1
- p – odd prime number

OUTPUT:

Dictionary of terms of the form (tuple: coeff), where 'tuple' is a pair of tuples, as for r and s, and 'coeff' is an integer mod p.

This computes the product of the Milnor basis elements $Q_{e_1}Q_{e_2}...P(r_1, r_2, ...)$ and $Q_{f_1}Q_{f_2}...P(s_1, s_2, ...)$.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import milnor_
↪multiplication_odd
sage: sorted(milnor_multiplication_odd(((0,2),(5,)), ((1,),(1,)), 5).items())
[(((0, 1, 2), (0, 1)), 4), (((0, 1, 2), (6,)), 4)]
sage: milnor_multiplication_odd(((0,2,4),()), ((1,3),()), 7)
{((0, 1, 2, 3, 4), ()): 6}
sage: milnor_multiplication_odd(((0,2,4),()), ((1,5),()), 7)
{((0, 1, 2, 4, 5), ()): 1}
sage: sorted(milnor_multiplication_odd(((),(6,)), ((),(2,)), 3).items())
[(((), (0, 2)), 1), (((), (4, 1)), 1), (((), (8,)), 1)]
```

These examples correspond to the following product computations:

$$p = 5 : \quad Q_0Q_2\mathcal{P}(5)Q_1\mathcal{P}(1) = 4Q_0Q_1Q_2\mathcal{P}(0,1) + 4Q_0Q_1Q_2\mathcal{P}(6)$$
$$p = 7 : \quad (Q_0Q_2Q_4)(Q_1Q_3) = 6Q_0Q_1Q_2Q_3Q_4$$
$$p = 7 : \quad (Q_0Q_2Q_4)(Q_1Q_5) = Q_0Q_1Q_2Q_3Q_5$$
$$p = 3 : \quad \mathcal{P}(6)\mathcal{P}(2) = \mathcal{P}(0,2) + \mathcal{P}(4,1) + \mathcal{P}(8)$$

The following used to fail until the trailing zeroes were eliminated in p_mono:

```
sage: A = SteenrodAlgebra(3)
sage: a = A.P(0,3); b = A.P(12); c = A.Q(1,2)
sage: (a+b)*c == a*c + b*c
True
```

Test that the bug reported in github issue #7212 has been fixed:

```
sage: A.P(36,6)*A.P(27,9,81)
2 P(13,21,83) + P(14,24,82) + P(17,20,83) + P(25,18,83) + P(26,21,82) + P(36,15,80,
↪1) + P(49,12,83) + 2 P(50,15,82) + 2 P(53,11,83) + 2 P(63,15,81)
```

Associativity once failed because of a sign error:

```
sage: a,b,c = A.Q_exp(0,1), A.P(3), A.Q_exp(1,1)
sage: (a*b)*c == a*(b*c)
True
```

This uses the same algorithm Monks does in his Maple package to iterate through the possible matrices: see http://mathweb.scranton.edu/monks/software/Steenrod/steen.html.

sage.algebras.steenrod.steenrod_algebra_mult.**multinomial**(*list*)

Multinomial coefficient of list, mod 2.

INPUT:

- list – list of integers

OUTPUT:

None if the multinomial coefficient is 0, or sum of list if it is 1

Given the input $[n_1, n_2, n_3, ...]$, this computes the multinomial coefficient $(n_1 + n_2 + n_3 + ...)!/(n_1!n_2!n_3!...)$, mod 2. The method is roughly this: expand each n_i in binary. If there is a 1 in the same digit for any n_i and n_j with $i \neq j$, then the coefficient is 0; otherwise, it is 1.

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import multinomial
sage: multinomial([1,2,4])
7
sage: multinomial([1,2,5])
sage: multinomial([1,2,12,192,256])
463
```

This function does not compute any factorials, so the following are actually reasonable to do:

```
sage: multinomial([1,65536])
65537
sage: multinomial([4,65535])
sage: multinomial([32768,65536])
98304
```

sage.algebras.steenrod.steenrod_algebra_mult.**multinomial_odd**(*list*, *p*)

Multinomial coefficient of list, mod p.

INPUT:

- list – list of integers
- p – a prime number

OUTPUT:

Associated multinomial coefficient, mod p

Given the input $[n_1, n_2, n_3, ...]$, this computes the multinomial coefficient $(n_1 + n_2 + n_3 + ...)!/(n_1!n_2!n_3!...)$, mod p. The method is this: expand each n_i in base p: $n_i = \sum_j p^j n_{ij}$. Do the same for the sum of the n_i's, which we call m: $m = \sum_j p^j m_j$. Then the multinomial coefficient is congruent, mod p, to the product of the multinomial coefficients $m_j!/(n_{1j}!n_{2j}!...)$.

Furthermore, any multinomial coefficient $m!/(n_1!n_2!...)$ can be computed as a product of binomial coefficients: it equals

$$\binom{n_1}{n_1}\binom{n_1+n_2}{n_2}\binom{n_1+n_2+n_3}{n_3}...$$

This is convenient because Sage's binomial function returns integers, not rational numbers (as would be produced just by dividing factorials).

EXAMPLES:

```
sage: from sage.algebras.steenrod.steenrod_algebra_mult import multinomial_odd
sage: multinomial_odd([1,2,4], 2)
1
sage: multinomial_odd([1,2,4], 7)
0
sage: multinomial_odd([1,2,4], 11)
6
sage: multinomial_odd([1,2,4], 101)
4
sage: multinomial_odd([1,2,4], 107)
105
```

5.27 Weyl Algebras

AUTHORS:

- Travis Scrimshaw (2013-09-06): Initial version

class `sage.algebras.weyl_algebra.`**`DifferentialWeylAlgebra`**(*R*, *names=None*)

Bases: `Algebra`, `UniqueRepresentation`

The differential Weyl algebra of a polynomial ring.

Let R be a commutative ring. The (differential) Weyl algebra W is the algebra generated by $x_1, x_2, \ldots x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}$ subject to the relations: $[x_i, x_j] = 0$, $[\partial_{x_i}, \partial_{x_j}] = 0$, and $\partial_{x_i} x_j = x_j \partial_{x_i} + \delta_{ij}$. Therefore ∂_{x_i} is acting as the partial differential operator on x_i.

The Weyl algebra can also be constructed as an iterated Ore extension of the polynomial ring $R[x_1, x_2, \ldots, x_n]$ by adding x_i at each step. It can also be seen as a quantization of the symmetric algebra $Sym(V)$, where V is a finite dimensional vector space over a field of characteristic zero, by using a modified Groenewold-Moyal product in the symmetric algebra.

The Weyl algebra (even for $n = 1$) over a field of characteristic 0 has many interesting properties.

- It's a non-commutative domain.
- It's a simple ring (but not in positive characteristic) that is not a matrix ring over a division ring.
- It has no finite-dimensional representations.
- It's a quotient of the universal enveloping algebra of the Heisenberg algebra $\mathfrak{h}_n$.

REFERENCES:

- Wikipedia article Weyl_algebra

INPUT:

- `R` – a (polynomial) ring
- `names` – (default: `None`) if `None` and `R` is a polynomial ring, then the variable names correspond to those of `R`; otherwise if `names` is specified, then `R` is the base ring

EXAMPLES:

There are two ways to create a Weyl algebra, the first is from a polynomial ring:

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R); W
Differential Weyl algebra of polynomials in x, y, z over Rational Field
```

We can call `W.inject_variables()` to give the polynomial ring variables, now as elements of `W`, and the differentials:

```
sage: W.inject_variables()
Defining x, y, z, dx, dy, dz
sage: (dx * dy * dz) * (x^2 * y * z + x * z * dy + 1)
x*z*dx*dy^2*dz + z*dy^2*dz + x^2*y*z*dx*dy*dz + dx*dy*dz
 + x*dx*dy^2 + 2*x*y*z*dy*dz + dy^2 + x^2*z*dx*dz + x^2*y*dx*dy
 + 2*x*z*dz + 2*x*y*dy + x^2*dx + 2*x
```

Or directly by specifying a base ring and variable names:

```
sage: W.<a,b> = DifferentialWeylAlgebra(QQ); W
Differential Weyl algebra of polynomials in a, b over Rational Field
```

Todo: Implement the `graded_algebra()` as a polynomial ring once they are considered to be graded rings (algebras).

Element

alias of *DifferentialWeylAlgebraElement*

algebra_generators()

Return the algebra generators of `self`.

See also:

variables(), *differentials()*

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R)
sage: W.algebra_generators()
Finite family {'x': x, 'y': y, 'z': z, 'dx': dx, 'dy': dy, 'dz': dz}
```

basis()

Return a basis of `self`.

EXAMPLES:

```
sage: W.<x,y> = DifferentialWeylAlgebra(QQ)
sage: B = W.basis()
sage: it = iter(B)
sage: [next(it) for i in range(20)]
[1, x, y, dx, dy, x^2, x*y, x*dx, x*dy, y^2, y*dx, y*dy,
 dx^2, dx*dy, dy^2, x^3, x^2*y, x^2*dx, x^2*dy, x*y^2]
sage: dx, dy = W.differentials()
sage: sorted((dx*x).monomials(), key=str)
[1, x*dx]
sage: B[(x*y).support()[0]]
```

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```
x*y
sage: sorted((dx*x).monomial_coefficients().items())
[(((0, 0), (0, 0)), 1), (((1, 0), (1, 0)), 1)]
```

degree_on_basis(*i*)

Return the degree of the basis element indexed by `i`.

EXAMPLES:

```
sage: W.<a,b> = DifferentialWeylAlgebra(QQ)
sage: W.degree_on_basis( ((1, 3, 2), (0, 1, 3)) )
10

sage: W.<x,y,z> = DifferentialWeylAlgebra(QQ)
sage: dx,dy,dz = W.differentials()
sage: elt = y*dy - (3*x - z)*dx
sage: elt.degree()
2
```

diff_action()

Left action of this Weyl algebra on the underlying polynomial ring by differentiation.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: W = R.weyl_algebra()
sage: dx, dy = W.differentials()
sage: W.diff_action
Left action by Differential Weyl algebra of polynomials in x, y
over Rational Field on Multivariate Polynomial Ring in x, y over
Rational Field
sage: W.diff_action(dx^2 + dy + 1, x^3*y^3)
x^3*y^3 + 3*x^3*y^2 + 6*x*y^3
```

differentials()

Return the differentials of `self`.

See also:

algebra_generators(), *variables()*

EXAMPLES:

```
sage: W.<x,y,z> = DifferentialWeylAlgebra(QQ)
sage: W.differentials()
Finite family {'dx': dx, 'dy': dy, 'dz': dz}
```

gen(*i*)

Return the `i`-th generator of `self`.

See also:

algebra_generators()

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R)
sage: [W.gen(i) for i in range(6)]
[x, y, z, dx, dy, dz]
```

ngens()

Return the number of generators of `self`.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R)
sage: W.ngens()
6
```

one()

Return the multiplicative identity element 1.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R)
sage: W.one()
1
```

options = Current options for DifferentialWeylAlgebra - factor_representation: False

polynomial_ring()

Return the associated polynomial ring of `self`.

EXAMPLES:

```
sage: W.<a,b> = DifferentialWeylAlgebra(QQ)
sage: W.polynomial_ring()
Multivariate Polynomial Ring in a, b over Rational Field
```

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R)
sage: W.polynomial_ring() == R
True
```

variables()

Return the variables of `self`.

See also:

algebra_generators(), *differentials()*

EXAMPLES:

```
sage: W.<x,y,z> = DifferentialWeylAlgebra(QQ)
sage: W.variables()
Finite family {'x': x, 'y': y, 'z': z}
```

zero()

Return the additive identity element 0.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: W = DifferentialWeylAlgebra(R)
sage: W.zero()
0
```

class sage.algebras.weyl_algebra.**DifferentialWeylAlgebraAction**(*G*)

Bases: `Action`

Left action of a Weyl algebra on its underlying polynomial ring by differentiation.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: W = R.weyl_algebra()
sage: dx, dy = W.differentials()
sage: W.diff_action
Left action by Differential Weyl algebra of polynomials in x, y
over Rational Field on Multivariate Polynomial Ring in x, y over
Rational Field
```

```
sage: g = dx^2 + x*dy
sage: p = x^5 + x^3 + y^2*x^2 + 1
sage: W.diff_action(g, p)
2*x^3*y + 20*x^3 + 2*y^2 + 6*x
```

The action is a left action:

```
sage: h = dx*x + x*y
sage: W.diff_action(h, W.diff_action(g, p)) == W.diff_action(h*g, p)
True
```

The action endomorphism of a differential operator:

```
sage: dg = W.diff_action(g); dg
Action of dx^2 + x*dy on Multivariate Polynomial Ring in x, y over
Rational Field under Left action by Differential Weyl algebra...
sage: dg(p) == W.diff_action(g, p) == g.diff(p)
True
```

class sage.algebras.weyl_algebra.**DifferentialWeylAlgebraElement**(*parent*, *monomials*)

Bases: `AlgebraElement`

An element in a differential Weyl algebra.

diff(*p*)

Apply this differential operator to a polynomial.

INPUT:

- `p` – polynomial of the underlying polynomial ring

OUTPUT:

The result of the left action of the Weyl algebra on the polynomial ring via differentiation.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: W = R.weyl_algebra()
sage: dx, dy = W.differentials()
sage: dx.diff(x^3)
3*x^2
sage: (dx*dy).diff(W(x^3*y^3))
9*x^2*y^2
sage: (x*dx + dy + 1).diff(x^4*y^4 + 1)
5*x^4*y^4 + 4*x^4*y^3 + 1
```

factor_differentials()

Return a dict representing `self` with the differentials factored out.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: D = DifferentialWeylAlgebra(R)
sage: t, dt = D.gens()
sage: x = dt^3*t^3 + dt^2*t^4
sage: x
t^3*dt^3 + t^4*dt^2 + 9*t^2*dt^2 + 8*t^3*dt + 18*t*dt + 12*t^2 + 6
sage: x.factor_differentials()
{(0,): 12*t^2 + 6, (1,): 8*t^3 + 18*t, (2,): t^4 + 9*t^2, (3,): t^3}
sage: D.zero().factor_differentials()
{}

sage: R.<x,y,z> = QQ[]
sage: D = DifferentialWeylAlgebra(R)
sage: x, y, z, dx, dy, dz = D.gens()
sage: elt = dx^3*x^3 + (y^3-z*x)*dx^3 + dy^3*x^3 + dx*dy*dz*x*y*z
sage: elt
x^3*dy^3 + x*y*z*dx*dy*dz + y^3*dx^3 + x^3*dx^3 - x*z*dx^3 + y*z*dy*dz
 + x*z*dx*dz + x*y*dx*dy + 9*x^2*dx^2 + z*dz + y*dy + 19*x*dx + 7
sage: elt.factor_differentials()
{(0, 0, 0): 7,
 (0, 0, 1): z,
 (0, 1, 0): y,
 (0, 1, 1): y*z,
 (0, 3, 0): x^3,
 (1, 0, 0): 19*x,
 (1, 0, 1): x*z,
 (1, 1, 0): x*y,
 (1, 1, 1): x*y*z,
 (2, 0, 0): 9*x^2,
 (3, 0, 0): x^3 + y^3 - x*z}
```

list()

Return `self` as a list.

This list consists of pairs (m, c), where m is a pair of tuples indexing a basis element of `self`, and c is the coordinate of `self` corresponding to this basis element. (Only nonzero coordinates are shown.)

EXAMPLES:

```
sage: W.<x,y,z> = DifferentialWeylAlgebra(QQ)
sage: dx,dy,dz = W.differentials()
sage: elt = dy - (3*x - z)*dx
sage: elt.list()
[(((0, 0, 0), (0, 1, 0)), 1),
 (((0, 0, 1), (1, 0, 0)), 1),
 (((1, 0, 0), (1, 0, 0)), -3)]
```

monomial_coefficients(*copy=True*)

Return a dictionary which has the basis keys in the support of `self` as keys and their corresponding coefficients as values.

INPUT:

- `copy` – (default: `True`) if `self` is internally represented by a dictionary `d`, then make a copy of `d`; if `False`, then this can cause undesired behavior by mutating `d`

EXAMPLES:

```
sage: W.<x,y,z> = DifferentialWeylAlgebra(QQ)
sage: dx,dy,dz = W.differentials()
sage: elt = (dy - (3*x - z)*dx)
sage: sorted(elt.monomial_coefficients().items())
[(((0, 0, 0), (0, 1, 0)), 1),
 (((0, 0, 1), (1, 0, 0)), 1),
 (((1, 0, 0), (1, 0, 0)), -3)]
```

support()

Return the support of `self`.

EXAMPLES:

```
sage: W.<x,y,z> = DifferentialWeylAlgebra(QQ)
sage: dx,dy,dz = W.differentials()
sage: elt = dy - (3*x - z)*dx + 1
sage: sorted(elt.support())
[((0, 0, 0), (0, 0, 0)),
((0, 0, 0), (0, 1, 0)),
((0, 0, 1), (1, 0, 0)),
((1, 0, 0), (1, 0, 0))]
```

`sage.algebras.weyl_algebra.`**repr_factored**(*w*, *latex_output=False*)

Return a string representation of `w` with the dx_i generators factored on the right.

EXAMPLES:

```
sage: from sage.algebras.weyl_algebra import repr_factored
sage: R.<t> = QQ[]
sage: D = DifferentialWeylAlgebra(R)
sage: t, dt = D.gens()
sage: x = dt^3*t^3 + dt^2*t^4
sage: x
t^3*dt^3 + t^4*dt^2 + 9*t^2*dt^2 + 8*t^3*dt + 18*t*dt + 12*t^2 + 6
sage: print(repr_factored(x))
```

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```
(12*t^2 + 6) + (8*t^3 + 18*t)*dt + (t^4 + 9*t^2)*dt^2 + (t^3)*dt^3
sage: repr_factored(x, True)
(12 t^{2} + 6) + (8 t^{3} + 18 t) \frac{\partial}{\partial t}
 + (t^{4} + 9 t^{2}) \frac{\partial^{2}}{\partial t^{2}}
 + (t^{3}) \frac{\partial^{3}}{\partial t^{3}}
sage: repr_factored(D.zero())
'0'
```

With multiple variables:

```
sage: R.<x,y,z> = QQ[]
sage: D = DifferentialWeylAlgebra(R)
sage: x, y, z, dx, dy, dz = D.gens()
sage: elt = dx^3*x^3 + (y^3-z*x)*dx^3 + dy^3*x^3 + dx*dy*dz*x*y*z
sage: elt
x^3*dy^3 + x*y*z*dx*dy*dz + y^3*dx^3 + x^3*dx^3 - x*z*dx^3 + y*z*dy*dz
 + x*z*dx*dz + x*y*dx*dy + 9*x^2*dx^2 + z*dz + y*dy + 19*x*dx + 7
sage: print(repr_factored(elt))
(7) + (z)*dz + (y)*dy + (y*z)*dy*dz + (x^3)*dy^3 + (19*x)*dx
 + (x*z)*dx*dz + (x*y)*dx*dy + (x*y*z)*dx*dy*dz
 + (9*x^2)*dx^2 + (x^3 + y^3 - x*z)*dx^3
sage: repr_factored(D.zero(), True)
0
```

sage.algebras.weyl_algebra.**repr_from_monomials**(*monomials*, *term_repr*, *use_latex=False*)

Return a string representation of an element of a free module from the dictionary `monomials`.

INPUT:

- `monomials` – a list of pairs `[m, c]` where `m` is the index and `c` is the coefficient
- `term_repr` – a function which returns a string given an index (can be `repr` or `latex`, for example)
- `use_latex` – (default: `False`) if `True` then the output is in latex format

EXAMPLES:

```
sage: from sage.algebras.weyl_algebra import repr_from_monomials
sage: R.<x,y,z> = QQ[]
sage: d = [(z, 4/7), (y, sqrt(2)), (x, -5)]
sage: repr_from_monomials(d, lambda m: repr(m))
'4/7*z + sqrt(2)*y - 5*x'
sage: a = repr_from_monomials(d, lambda m: latex(m), True); a
\frac{4}{7} z + \sqrt{2} y - 5 x
sage: type(a)
<class 'sage.misc.latex.LatexExpr'>
```

The zero element:

```
sage: repr_from_monomials([], lambda m: repr(m))
'0'
sage: a = repr_from_monomials([], lambda m: latex(m), True); a
0
sage: type(a)
<class 'sage.misc.latex.LatexExpr'>
```

A "unity" element:

```
sage: repr_from_monomials([(1, 1)], lambda m: repr(m))
'1'
sage: a = repr_from_monomials([(1, 1)], lambda m: latex(m), True); a
1
sage: type(a)
<class 'sage.misc.latex.LatexExpr'>
```

```
sage: repr_from_monomials([(1, -1)], lambda m: repr(m))
'-1'
sage: a = repr_from_monomials([(1, -1)], lambda m: latex(m), True); a
-1
sage: type(a)
<class 'sage.misc.latex.LatexExpr'>
```

Leading minus signs are dealt with appropriately:

```
sage: d = [(z, -4/7), (y, -sqrt(2)), (x, -5)]
sage: repr_from_monomials(d, lambda m: repr(m))
'-4/7*z - sqrt(2)*y - 5*x'
sage: a = repr_from_monomials(d, lambda m: latex(m), True); a
-\frac{4}{7} z - \sqrt{2} y - 5 x
sage: type(a)
<class 'sage.misc.latex.LatexExpr'>
```

Indirect doctests using a class that uses this function:

```
sage: R.<x,y> = QQ[]
sage: A = CliffordAlgebra(QuadraticForm(R, 3, [x,0,-1,3,-4,5]))
sage: a,b,c = A.gens()
sage: a*b*c
e0*e1*e2
sage: b*c
e1*e2
sage: (a*a + 2)
x + 2
sage: c*(a*a + 2)*b
(-x - 2)*e1*e2 - 4*x - 8
sage: latex(c*(a*a + 2)*b)
\left( -x - 2 \right)  e_{1} e_{2} - 4 x - 8
```

5.28 Yangians

AUTHORS:

- Travis Scrimshaw (2013-10-08): Initial version

class sage.algebras.yangian.**GradedYangianBase**(*A*, *category=None*)

Bases: *AssociatedGradedAlgebra*

Base class for graded algebras associated to a Yangian.

class sage.algebras.yangian.**GradedYangianLoop**(*Y*)

Bases: *GradedYangianBase*

The associated graded algebra corresponding to a Yangian $\operatorname{gr} Y(\mathfrak{gl}_n)$ with the filtration of $\deg t_{ij}^{(r)} = r - 1$.

Using this filtration for the Yangian, the associated graded algebra is isomorphic to $U(\mathfrak{gl}_n[z])$, the universal enveloping algebra of the loop algebra of $\mathfrak{gl}_n$.

INPUT:

- Y – a Yangian with the loop filtration

antipode_on_basis(*m*)

Return the antipode on a basis element indexed by m.

EXAMPLES:

```
sage: grY = Yangian(QQ, 4).graded_algebra()
sage: grY.antipode_on_basis(grY.gen(2,1,1).leading_support())
-tbar(2)[1,1]

sage: x = grY.an_element(); x
tbar(1)[1,1]*tbar(1)[1,2]^2*tbar(1)[1,3]^3*tbar(42)[1,1]
sage: grY.antipode_on_basis(x.leading_support())
-tbar(1)[1,1]*tbar(1)[1,2]^2*tbar(1)[1,3]^3*tbar(42)[1,1]
 - 2*tbar(1)[1,1]*tbar(1)[1,2]*tbar(1)[1,3]^3*tbar(42)[1,2]
 - 3*tbar(1)[1,1]*tbar(1)[1,2]^2*tbar(1)[1,3]^2*tbar(42)[1,3]
 + 5*tbar(1)[1,2]^2*tbar(1)[1,3]^3*tbar(42)[1,1]
 + 10*tbar(1)[1,2]*tbar(1)[1,3]^3*tbar(42)[1,2]
 + 15*tbar(1)[1,2]^2*tbar(1)[1,3]^2*tbar(42)[1,3]

sage: g = grY.indices().gens()
sage: x = grY(g[1,1,1] * g[1,1,2]^2 * g[1,1,3]^3 * g[3,1,1]); x
tbar(1)[1,1]*tbar(1)[1,2]^2*tbar(1)[1,3]^3*tbar(3)[1,1]
sage: grY.antipode_on_basis(x.leading_support())
-tbar(1)[1,1]*tbar(1)[1,2]^2*tbar(1)[1,3]^3*tbar(3)[1,1]
 - 2*tbar(1)[1,1]*tbar(1)[1,2]*tbar(1)[1,3]^3*tbar(3)[1,2]
 - 3*tbar(1)[1,1]*tbar(1)[1,2]^2*tbar(1)[1,3]^2*tbar(3)[1,3]
 + 5*tbar(1)[1,2]^2*tbar(1)[1,3]^3*tbar(3)[1,1]
 + 10*tbar(1)[1,2]*tbar(1)[1,3]^3*tbar(3)[1,2]
 + 15*tbar(1)[1,2]^2*tbar(1)[1,3]^2*tbar(3)[1,3]
```

coproduct_on_basis(*m*)

Return the coproduct on the basis element indexed by m.

EXAMPLES:

```
sage: grY = Yangian(QQ, 4).graded_algebra()
sage: grY.coproduct_on_basis(grY.gen(2,1,1).leading_support())
1 # tbar(2)[1,1] + tbar(2)[1,1] # 1
sage: grY.gen(2,3,1).coproduct()
1 # tbar(2)[3,1] + tbar(2)[3,1] # 1
```

counit_on_basis(*m*)

Return the antipode on the basis element indexed by m.

EXAMPLES:

```
sage: grY = Yangian(QQ, 4).graded_algebra()
sage: grY.counit_on_basis(grY.gen(2,3,1).leading_support())
0
sage: grY.gen(0,0,0).counit()
1
```

class sage.algebras.yangian.**GradedYangianNatural**(*Y*)

Bases: *GradedYangianBase*

The associated graded algebra corresponding to a Yangian $\operatorname{gr} Y(\mathfrak{gl}_n)$ with the natural filtration of $\deg t_{ij}^{(r)} = r$.

INPUT:

- Y – a Yangian with the natural filtration

product_on_basis(*x*, *y*)

Return the product on basis elements given by the indices x and y.

EXAMPLES:

```
sage: grY = Yangian(QQ, 4, filtration='natural').graded_algebra()
sage: x = grY.gen(12, 2, 1) * grY.gen(2, 1, 1) # indirect doctest
sage: x
tbar(2)[1,1]*tbar(12)[2,1]
sage: x == grY.gen(2, 1, 1) * grY.gen(12, 2, 1)
True
```

class sage.algebras.yangian.**Yangian**(*base_ring*, *n*, *variable_name*, *filtration*)

Bases: CombinatorialFreeModule

The Yangian $Y(\mathfrak{gl}_n)$.

Let A be a commutative ring with unity. The *Yangian* $Y(\mathfrak{gl}_n)$, associated with the Lie algebra $\mathfrak{gl}_n$ for $n \geq 1$, is defined to be the unital associative algebra generated by $\{t_{ij}^{(r)} \mid 1 \leq i, j \leq n, r \geq 1\}$ subject to the relations

$$[t_{ij}^{(M+1)}, t_{k\ell}^{(L)}] - [t_{ij}^{(M)}, t_{k\ell}^{(L+1)}] = t_{kj}^{(M)} t_{i\ell}^{(L)} - t_{kj}^{(L)} t_{i\ell}^{(M)},$$

where $L, M \geq 0$ and $t_{ij}^{(0)} = \delta_{ij} \cdot 1$. This system of quadratic relations is equivalent to the system of commutation relations

$$[t_{ij}^{(r)}, t_{k\ell}^{(s)}] = \sum_{p=0}^{\min\{r,s\}-1} \left(t_{kj}^{(p)} t_{i\ell}^{(r+s-1-p)} - t_{kj}^{(r+s-1-p)} t_{i\ell}^{(p)} \right),$$

where $1 \leq i, j, k, \ell \leq n$ and $r, s \geq 1$.

Let u be a formal variable and, for $1 \leq i, j \leq n$, define

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \in Y(\mathfrak{gl}_n)[\![u^{-1}]\!].$$

Thus, we can write the defining relations as

$$(u - v)[t_{ij}(u), t_{k\ell}(v)] = t_{kj}(u) t_{i\ell}(v) - t_{kj}(v) t_{i\ell}(u).$$

These series can be combined into a single matrix:

$$T(u) := \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij} \in Y(\mathfrak{gl}_n)[\![u^{-1}]\!] \otimes \operatorname{End}(\mathbf{C}^n),$$

where E_{ij} is the matrix with a 1 in the (i,j) position and zeros elsewhere.

For $m \geq 2$, define formal variables $u_1, \ldots, u_m$. For any $1 \leq k \leq m$, set

$$T_k(u_k) := \sum_{i,j=1}^{n} t_{ij}(u_k) \otimes (E_{ij})_k \in Y(\mathfrak{gl}_n)[\![u_1^{-1}, \ldots, u_m^{-1}]\!] \otimes \operatorname{End}(\mathbf{C}^n)^{\otimes m},$$

where $(E_{ij})_k = 1^{\otimes(k-1)} \otimes E_{ij} \otimes 1^{\otimes(m-k)}$. If we consider $m = 2$, we can then also write the defining relations as

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v),$$

where $R(u) = 1 - Pu^{-1}$ and P is the permutation operator that swaps the two factors. Moreover, we can write the Hopf algebra structure as

$$\Delta\colon T(u) \mapsto T_{[1]}(u)T_{[2]}(u), \qquad S\colon T(u) \mapsto T^{-1}(u), \qquad \epsilon\colon T(u) \mapsto 1,$$

where $T_{[a]} = \sum_{i,j=1}^{n} (1^{\otimes a-1} \otimes t_{ij}(u) \otimes 1^{2-a}) \otimes (E_{ij})_1$.

We can also impose two filtrations on $Y(\mathfrak{gl}_n)$: the *natural* filtration $\deg t_{ij}^{(r)} = r$ and the *loop* filtration $\deg t_{ij}^{(r)} = r-1$. The natural filtration has a graded homomorphism with $U(\mathfrak{gl}_n)$ by $t_{ij}^{(r)} \mapsto (E^r)_{ij}$ and an associated graded algebra being polynomial algebra. Moreover, this shows a PBW theorem for the Yangian, that for any fixed order, we can write elements as unique linear combinations of ordered monomials using $t_{ij}^{(r)}$. For the loop filtration, the associated graded algebra is isomorphic (as Hopf algebras) to $U(\mathfrak{gl}_n[z])$ given by $\bar{t}_{ij}^{(r)} \mapsto E_{ij}x^{r-1}$, where $\bar{t}_{ij}^{(r)}$ is the image of $t_{ij}^{(r)}$ in the $(r-1)$-th component of $\operatorname{gr} Y(\mathfrak{gl}_n)$.

INPUT:

- `base_ring` – the base ring
- `n` – the size n
- `level` – (optional) the level of the Yangian
- `variable_name` – (default: `'t'`) the name of the variable
- `filtration` – (default: `'loop'`) the filtration and can be one of the following:
  - `'natural'` – the filtration is given by $\deg t_{ij}^{(r)} = r$
  - `'loop'` – the filtration is given by $\deg t_{ij}^{(r)} = r - 1$

Todo: Implement the antipode.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: t = Y.algebra_generators()
sage: t[6,2,1] * t[2,3,2]
-t(1)[2,2]*t(6)[3,1] + t(1)[3,1]*t(6)[2,2]
 + t(2)[3,2]*t(6)[2,1] - t(7)[3,1]
sage: t[6,2,1] * t[3,1,4]
t(1)[1,1]*t(7)[2,4] + t(1)[1,4]*t(6)[2,1] - t(1)[2,1]*t(6)[1,4]
 - t(1)[2,4]*t(7)[1,1] + t(2)[1,1]*t(6)[2,4] - t(2)[2,4]*t(6)[1,1]
 + t(3)[1,4]*t(6)[2,1] + t(6)[2,4] + t(8)[2,4]
```

We check that the natural filtration has a homomorphism to $U(\mathfrak{gl}_n)$ as algebras:

```
sage: Y = Yangian(QQ, 4, filtration='natural')
sage: t = Y.algebra_generators()
sage: gl4 = lie_algebras.gl(QQ, 4)
sage: Ugl4 = gl4.pbw_basis()
sage: E = matrix(Ugl4, 4, 4, Ugl4.gens())
sage: Esq = E^2
sage: t[2,1,3] * t[1,2,1]
t(1)[2,1]*t(2)[1,3] - t(2)[2,3]
sage: Esq[0,2] * E[1,0] == E[1,0] * Esq[0,2] - Esq[1,2]
True

sage: Em = [E^k for k in range(1,5)]
sage: S = list(t.some_elements())[:30:3]
sage: def convert(x):
....:     return sum(c * prod(Em[t[0]-1][t[1]-1,t[2]-1] ** e
....:                         for t,e in m._sorted_items())
....:                for m,c in x)
sage: for x in S:
....:     for y in S:
....:         ret = x * y
....:         rhs = convert(x) * convert(y)
....:         assert rhs == convert(ret)
....:         assert ret.maximal_degree() == rhs.maximal_degree()
```

REFERENCES:

- Wikipedia article Yangian
- [MNO1994]
- [Mol2007]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.algebra_generators()
Lazy family (generator(i))_{i in The Cartesian product of
 (Positive integers, {1, 2, 3, 4}, {1, 2, 3, 4})}
```

coproduct_on_basis(m)

Return the coproduct on the basis element indexed by `m`.

The coproduct $\Delta\colon Y(\mathfrak{gl}_n) \longrightarrow Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$ is defined by

$$\Delta(t_{ij}(u)) = \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u).$$

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.gen(2,1,1).coproduct() # indirect doctest
1 # t(2)[1,1] + t(1)[1,1] # t(1)[1,1] + t(1)[1,2] # t(1)[2,1]
 + t(1)[1,3] # t(1)[3,1] + t(1)[1,4] # t(1)[4,1] + t(2)[1,1] # 1
```

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```
sage: Y.gen(2,3,1).coproduct()
1 # t(2)[3,1] + t(1)[3,1] # t(1)[1,1] + t(1)[3,2] # t(1)[2,1]
 + t(1)[3,3] # t(1)[3,1] + t(1)[3,4] # t(1)[4,1] + t(2)[3,1] # 1
sage: Y.gen(2,2,3).coproduct()
1 # t(2)[2,3] + t(1)[2,1] # t(1)[1,3] + t(1)[2,2] # t(1)[2,3]
 + t(1)[2,3] # t(1)[3,3] + t(1)[2,4] # t(1)[4,3] + t(2)[2,3] # 1
```

counit_on_basis(m)

Return the counit on the basis element indexed by `m`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.gen(2,3,1).counit() # indirect doctest
0
sage: Y.gen(0,0,0).counit()
1
```

degree_on_basis(m)

Return the degree of the monomial index by `m`.

The degree of $t_{ij}^{(r)}$ is equal to $r - 1$ if `filtration = 'loop'` and is equal to r if `filtration = 'natural'`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.degree_on_basis(Y.gen(2,1,1).leading_support())
1
sage: x = Y.gen(5,2,3)^4
sage: Y.degree_on_basis(x.leading_support())
16
sage: elt = Y.gen(10,3,1) * Y.gen(2,1,1) * Y.gen(1,2,4); elt
t(1)[1,1]*t(1)[2,4]*t(10)[3,1] - t(1)[2,4]*t(1)[3,1]*t(10)[1,1]
 + t(1)[2,4]*t(2)[1,1]*t(10)[3,1] + t(1)[2,4]*t(10)[3,1]
 + t(1)[2,4]*t(11)[3,1]
sage: for s in sorted(elt.support(), key=str): s, Y.degree_on_basis(s)
(t(1, 1, 1)*t(1, 2, 4)*t(10, 3, 1), 9)
(t(1, 2, 4)*t(1, 3, 1)*t(10, 1, 1), 9)
(t(1, 2, 4)*t(10, 3, 1), 9)
(t(1, 2, 4)*t(11, 3, 1), 10)
(t(1, 2, 4)*t(2, 1, 1)*t(10, 3, 1), 10)

sage: Y = Yangian(QQ, 4, filtration='natural')
sage: Y.degree_on_basis(Y.gen(2,1,1).leading_support())
2
sage: x = Y.gen(5,2,3)^4
sage: Y.degree_on_basis(x.leading_support())
20
sage: elt = Y.gen(10,3,1) * Y.gen(2,1,1) * Y.gen(1,2,4)
sage: for s in sorted(elt.support(), key=str): s, Y.degree_on_basis(s)
(t(1, 1, 1)*t(1, 2, 4)*t(10, 3, 1), 12)
(t(1, 2, 4)*t(1, 3, 1)*t(10, 1, 1), 12)
```

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```
(t(1, 2, 4)*t(10, 3, 1), 11)
(t(1, 2, 4)*t(11, 3, 1), 12)
(t(1, 2, 4)*t(2, 1, 1)*t(10, 3, 1), 13)
```

dimension()

Return the dimension of `self`, which is ∞.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.dimension()
+Infinity
```

gen(*r*, *i=None*, *j=None*)

Return the generator $t_{ij}^{(r)}$ of `self`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.gen(2, 1, 3)
t(2)[1,3]
sage: Y.gen(12, 2, 1)
t(12)[2,1]
sage: Y.gen(0, 1, 1)
1
sage: Y.gen(0, 1, 3)
0
```

graded_algebra()

Return the associated graded algebra of `self`.

EXAMPLES:

```
sage: Yangian(QQ, 4).graded_algebra()
Graded Algebra of Yangian of gl(4) in the loop filtration over Rational Field
sage: Yangian(QQ, 4, filtration='natural').graded_algebra()
Graded Algebra of Yangian of gl(4) in the natural filtration over Rational Field
```

one_basis()

Return the basis index of the element 1.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.one_basis()
1
```

product_on_basis(*x*, *y*)

Return the product of two monomials given by `x` and `y`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.gen(12, 2, 1) * Y.gen(2, 1, 1) # indirect doctest
```

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```
t(1)[1,1]*t(12)[2,1] - t(1)[2,1]*t(12)[1,1]
 + t(2)[1,1]*t(12)[2,1] + t(12)[2,1] + t(13)[2,1]
```

product_on_gens(*a*, *b*)

Return the product on two generators indexed by a and b.

We assume $(r,i,j) \geq (s,k,\ell)$, and we start with the basic relation:

$$[t_{ij}^{(r)}, t_{k\ell}^{(s)}] - [t_{ij}^{(r-1)}, t_{k\ell}^{(s+1)}] = t_{kj}^{(r-1)} t_{i\ell}^{(s)} - t_{kj}^{(s)} t_{i\ell}^{(r-1)}.$$

Solving for the first term and using induction we get:

$$[t_{ij}^{(r)}, t_{k\ell}^{(s)}] = \sum_{a=1}^{s} \left(t_{kj}^{(a-1)} t_{i\ell}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{i\ell}^{(a-1)} \right).$$

Next applying induction on this we get

$$t_{ij}^{(r)} t_{k\ell}^{(s)} = t_{k\ell}^{(s)} t_{ij}^{(r)} + \sum C_{abcd}^{m\ell} t_{ab}^{(m)} t_{cd}^{(\ell)}$$

where $m + \ell < r + s$ and $t_{ab}^{(m)} < t_{cd}^{(\ell)}$.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4)
sage: Y.product_on_gens((2,1,1), (12,2,1))
t(2)[1,1]*t(12)[2,1]
sage: Y.gen(2, 1, 1) * Y.gen(12, 2, 1)
t(2)[1,1]*t(12)[2,1]
sage: Y.product_on_gens((12,2,1), (2,1,1))
t(1)[1,1]*t(12)[2,1] - t(1)[2,1]*t(12)[1,1]
 + t(2)[1,1]*t(12)[2,1] + t(12)[2,1] + t(13)[2,1]
sage: Y.gen(12, 2, 1) * Y.gen(2, 1, 1)
t(1)[1,1]*t(12)[2,1] - t(1)[2,1]*t(12)[1,1]
 + t(2)[1,1]*t(12)[2,1] + t(12)[2,1] + t(13)[2,1]
```

class sage.algebras.yangian.**YangianLevel**(*base_ring*, *n*, *level*, *variable_name*, *filtration*)

Bases: *Yangian*

The Yangian $Y_\ell(\mathfrak{gl}_n)$ of level ℓ.

The Yangian of level ℓ is the quotient of the Yangian $Y(\mathfrak{gl}_n)$ by the two-sided ideal generated by $t_{ij}^{(r)}$ for all $r > p$ and all $i, j \in \{1, \ldots, n\}$.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4, 3)
sage: elt = Y.gen(3,2,1) * Y.gen(1,1,3)
sage: elt * Y.gen(1, 1, 2)
t(1)[1,2]*t(1)[1,3]*t(3)[2,1] + t(1)[1,2]*t(3)[2,3]
 - t(1)[1,3]*t(3)[1,1] + t(1)[1,3]*t(3)[2,2] - t(3)[1,3]
```

defining_polynomial(*i*, *j*, *u=None*)

Return the defining polynomial of i and j.

The defining polynomial is given by:

$$T_{ij}(u) = \delta_{ij} u^\ell + \sum_{k=1}^{\ell} t_{ij}^{(k)} u^{\ell-k}.$$

EXAMPLES:

```
sage: Y = Yangian(QQ, 3, 5)
sage: Y.defining_polynomial(3, 2)
t(1)[3,2]*u^4 + t(2)[3,2]*u^3 + t(3)[3,2]*u^2 + t(4)[3,2]*u + t(5)[3,2]
sage: Y.defining_polynomial(1, 1)
u^5 + t(1)[1,1]*u^4 + t(2)[1,1]*u^3 + t(3)[1,1]*u^2 + t(4)[1,1]*u + t(5)[1,1]
```

gen(*r*, *i=None*, *j=None*)

Return the generator $t_{ij}^{(r)}$ of `self`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 4, 3)
sage: Y.gen(2, 1, 3)
t(2)[1,3]
sage: Y.gen(12, 2, 1)
0
sage: Y.gen(0, 1, 1)
1
sage: Y.gen(0, 1, 3)
0
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 2, 2)
sage: Y.gens()
(t(1)[1,1], t(2)[1,1], t(1)[1,2], t(2)[1,2], t(1)[2,1],
 t(2)[2,1], t(1)[2,2], t(2)[2,2])
```

level()

Return the level of `self`.

EXAMPLES:

```
sage: Y = Yangian(QQ, 3, 5)
sage: Y.level()
5
```

product_on_gens(*a*, *b*)

Return the product on two generators indexed by `a` and `b`.

See also:

Yangian.product_on_gens()

EXAMPLES:

```
sage: Y = Yangian(QQ, 4, 3)
sage: Y.gen(1,2,2) * Y.gen(2,1,3) # indirect doctest
t(1)[2,2]*t(2)[1,3]
sage: Y.gen(1,2,1) * Y.gen(2,1,3) # indirect doctest
t(1)[2,1]*t(2)[1,3]
sage: Y.gen(3,2,1) * Y.gen(1,1,3) # indirect doctest
t(1)[1,3]*t(3)[2,1] + t(3)[2,3]
```

quantum_determinant(*u=None*)

Return the quantum determinant of `self`.

The quantum determinant is defined by:

$$\operatorname{qdet}(u) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{k=1}^{n} T_{\sigma(k),k}(u - k + 1).$$

EXAMPLES:

```
sage: Y = Yangian(QQ, 2, 2)
sage: Y.quantum_determinant()
u^4 + (-2 + t(1)[1,1] + t(1)[2,2])*u^3
 + (1 - t(1)[1,1] + t(1)[1,1]*t(1)[2,2] - t(1)[1,2]*t(1)[2,1]
    - 2*t(1)[2,2] + t(2)[1,1] + t(2)[2,2])*u^2
 + (-t(1)[1,1]*t(1)[2,2] + t(1)[1,1]*t(2)[2,2]
    + t(1)[1,2]*t(1)[2,1] - t(1)[1,2]*t(2)[2,1]
    - t(1)[2,1]*t(2)[1,2] + t(1)[2,2] + t(1)[2,2]*t(2)[1,1]
    - t(2)[1,1] - t(2)[2,2])*u
 - t(1)[1,1]*t(2)[2,2] + t(1)[1,2]*t(2)[2,1] + t(2)[1,1]*t(2)[2,2]
    - t(2)[1,2]*t(2)[2,1] + t(2)[2,2]
```

CHAPTER

SIX

HECKE ALGEBRAS

6.1 Ariki-Koike Algebras

The *Ariki-Koike algebras* were introduced by Ariki and Koike [AK1994] as a natural generalization of the Iwahori-Hecke algebras of types A and B (see `IwahoriHeckeAlgebra`). Soon afterwards, Broué and Malle defined analogues of the Hecke algebras for all complex reflection groups

Fix non-negative integers r an n. The Ariki-Koike algebras are deformations of the group algebra of the complex reflection group $G(r,1,n) = \mathbf{Z}/r\mathbf{Z} \wr \mathfrak{S}_n$. If R is a ring containing a *Hecke parameter* q and *cyclotomic parameters* $u_0,\ldots,u_{r-1}$ then the Ariki-Koike algebra $H_n(q,u_1,\ldots,u_r)$ is the unital associative r-algebra with generators $T_0,T_1,\ldots,T_{n-1}$ an relations:

$$\begin{aligned}
\prod_{i=0}^{r-1}(T_0-u_i) &= 0, \\
T_i^2 &= (q-1)T_i + q && \text{for } 1 \leq i < n, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
T_iT_j &= T_jT_i && \text{if } |i-j| \geq 2, \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && \text{for } 1 \leq i < n.
\end{aligned}$$

AUTHORS:

- Travis Scrimshaw (2016-04): initial version
- Andrew Mathas (2016-07): improved multiplication code

REFERENCES:

- [AK1994]
- [BM1993]
- [MM1998]

class `sage.algebras.hecke_algebras.ariki_koike_algebra.`**`ArikiKoikeAlgebra`**(*r*, *n*, *q*, *u*, *R*)

Bases: `Parent`, `UniqueRepresentation`

The Ariki-Koike algebra $H_{r,n}(q,u)$.

Let R be an unital integral domain. Let $q,u_0,\ldots,u_{r-1} \in R$ such that $q^{-1} \in R$. The *Ariki-Koike algebra* is the unital associative algebra $H_{r,n}(q,u)$ generated by $T_0, \ldots, T_{n-1}$ that satisfies the following relations:

$$\begin{aligned} \prod_{i=0}^{r-1}(T_0 - u_i) &= 0, & \\ T_i^2 &= (q-1)T_i + q & \text{for } 1 \leq i < n, \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0, & \\ T_iT_j &= T_jT_i & \text{if } |i-j| \geq 2, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} & \text{for } 1 \leq i < n. \end{aligned}$$

The parameter q is called the *Hecke parameter* and the parameters $u_0, \ldots, u_{r-1}$ are called the *cyclotomic parameters*. Thus, the Ariki-Koike algebra is a deformation of the group algebra of the complex reflection group $G(r,1,n) = \mathbf{Z}/r\mathbf{Z} \wr \mathfrak{S}_n$.

Next, we define *Jucys-Murphy elements*

$$L_i = q^{-i+1}T_{i-1} \cdots T_1T_0T_1 \cdots T_{i-1}$$

for $1 \leq i \leq n$.

Note: These element differ by a power of q from the corresponding elements in [AK1994]. However, these elements are more commonly used because they lead to nicer representation theoretic formulas.

Ariki and Koike [AK1994] showed that $H_{r,n}(q,u)$ is a free R-module with a basis given by

$$\{L_1^{c_i} \cdots L_n^{c_n} T_w \mid w \in S_n, 0 \leq c_i < r\}.$$

In particular, we have $\dim H_{r,n}(q,u) = r^n n! = |G(r,1,n)|$. Moreover, we have $L_iL_j = L_iL_j$ for all $1 \leq i,j \leq n$.

The Ariki-Koike algebra $H_{r,n}(q,u)$ can be considered as a quotient of the group algebra of the braid group for $G(r,1,n)$ by the ideal generated by $\prod_{i=0}^{r-1}(T_0 - u_i)$ and $(T_i - q)(T_i + 1)$. Furthermore, $H_{r,n}(q,u)$ can be constructed as a quotient of the extended affine Hecke algebra of type $A_{n-1}^{(1)}$ by $\prod_{i=0}^{r-1}(X_1 - u_i)$.

Since the Ariki-Koike algebra is a quotient of the group algebra of the braid group of $G(r,1,n)$, we can recover the group algebra of $G(r,1,n)$ as follows. Consider $u = (1, \zeta_r, \ldots, \zeta_r^{r-1})$, where ζ_r is a primitive r-th root of unity, then we have

$$RG(r,1,n) = H_{r,n}(1,u).$$

INPUT:

- `r` – the maximum power of L_i
- `n` – the rank S_n
- `q` – (optional) an invertible element in a commutative ring; the default is $q \in R[q,q^{-1}]$, where R is the ring containing the variables `u`
- `u` – (optional) the variables $u_1, \ldots, u_r$; the default is the generators of $\mathbf{Z}[u_1, \ldots, u_r]$
- `R` – (optional) a commutative ring containing `q` and `u`; the default is the parent of q and $u_1, \ldots, u_r$

EXAMPLES:

We start by constructing an Ariki-Koike algebra where the values q, u are generic and do some computations:

```
sage: H = algebras.ArikiKoike(3, 4)
```

Next, we do some computations using the LT basis:

```
sage: LT = H.LT()
sage: LT.inject_variables()
Defining L1, L2, L3, L4, T1, T2, T3
sage: T1 * T2 * T1 * T2
q*T[2,1] - (1-q)*T[2,1,2]
sage: T1 * L1 * T2 * L3 * T1 * T2
-(q-q^2)*L2*L3*T[2] + q*L1*L2*T[2,1] - (1-q)*L1*L2*T[2,1,2]
sage: L1^3
u0*u1*u2 + ((-u0*u1-u0*u2-u1*u2))*L1 + ((u0+u1+u2))*L1^2
sage: L3 * L2 * L1
L1*L2*L3
sage: u = LT.u()
sage: q = LT.q()
sage: (q + 2*u[0]) * (T1 * T2) * L3
(-2*u0+(2*u0-1)*q+q^2)*L3*T[1] + (-2*u0+(2*u0-1)*q+q^2)*L2*T[2]
 + (2*u0+q)*L1*T[1,2]
```

We check the defining relations:

```
sage: prod(L1 - val for val in u) == H.zero()
True
sage: L1 * T1 * L1 * T1 == T1 * L1 * T1 * L1
True
sage: T1 * T2 * T1 == T2 * T1 * T2
True
sage: T2 * T3 * T2 == T3 * T2 * T3
True
sage: L2 == q^-1 * T1 * L1 * T1
True
sage: L3 == q^-2 * T2 * T1 * L1 * T1 * T2
True
```

We construct an Ariki-Koike algebra with $u = (1, \zeta_3, \zeta_3^2)$, where ζ_3 is a primitive third root of unity:

```
sage: F = CyclotomicField(3)
sage: zeta3 = F.gen()
sage: R.<q> = LaurentPolynomialRing(F)
sage: H = algebras.ArikiKoike(3, 4, q=q, u=[1, zeta3, zeta3^2], R=R)
sage: H.LT().inject_variables()
Defining L1, L2, L3, L4, T1, T2, T3
sage: L1^3
1
sage: L2^3
1 - (q^-1-1)*T[1] - (q^-1-1)*L1*L2^2*T[1] - (q^-1-1)*L1^2*L2*T[1]
```

Next, we additionally take $q = 1$ to obtain the group algebra of $G(r, 1, n)$:

```
sage: F = CyclotomicField(3)
sage: zeta3 = F.gen()
```

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```
sage: H = algebras.ArikiKoike(3, 4, q=1, u=[1, zeta3, zeta3^2], R=F)
sage: LT = H.LT()
sage: LT.inject_variables()
Defining L1, L2, L3, L4, T1, T2, T3
sage: A = ColoredPermutations(3, 4).algebra(F)
sage: s1, s2, s3, s0 = list(A.algebra_generators())
sage: all(L^3 == LT.one() for L in LT.L())
True
sage: J = [s0, s3*s0*s3, s2*s3*s0*s3*s2, s1*s2*s3*s0*s3*s2*s1]
sage: all(Ji^3 == A.one() for Ji in J)
True
```

class LT(*algebra*)

Bases: `_Basis`

The basis of the Ariki-Koike algebra given by monomials of the form LT, where L is product of Jucys-Murphy elements and T is a product of $\{T_i | 0 < i < n\}$.

This was the basis defined in [AK1994] except using the renormalized Jucys-Murphy elements.

class Element

Bases: `IndexedFreeModuleElement`

L(*i=None*)

Return the generator(s) L_i.

INPUT:

- `i` – (default: `None`) the generator L_i or if `None`, then the list of all generators L_i

EXAMPLES:

```
sage: LT = algebras.ArikiKoike(8, 3).LT()
sage: LT.L(2)
L2
sage: LT.L()
[L1, L2, L3]

sage: LT = algebras.ArikiKoike(1, 3).LT()
sage: LT.L(2)
u + (-u*q^-1+u)*T[1]
sage: LT.L()
[u,
 u + (-u*q^-1+u)*T[1],
 u + (-u*q^-1+u)*T[2] + (-u*q^-2+u*q^-1)*T[2,1,2]]
```

T(*i=None*)

Return the generator(s) T_i of `self`.

INPUT:

- `i` – (default: `None`) the generator T_i or if `None`, then the list of all generators T_i

EXAMPLES:

```
sage: LT = algebras.ArikiKoike(8, 3).LT()
sage: LT.T(1)
T[1]
```

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```
sage: LT.T()
[L1, T[1], T[2]]
sage: LT.T(0)
L1
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: LT = algebras.ArikiKoike(5, 3).LT()
sage: dict(LT.algebra_generators())
{'L1': L1, 'L2': L2, 'L3': L3, 'T1': T[1], 'T2': T[2]}

sage: LT = algebras.ArikiKoike(1, 4).LT()
sage: dict(LT.algebra_generators())
{'T1': T[1], 'T2': T[2], 'T3': T[3]}
```

inverse_T(*i*)

Return the inverse of the generator T_i.

From the quadratic relation, we have

$$T_i^{-1} = q^{-1} T_i + (q^{-1} - 1).$$

EXAMPLES:

```
sage: LT = algebras.ArikiKoike(3, 4).LT()
sage: [LT.inverse_T(i) for i in range(1, 4)]
[(q^-1-1) + (q^-1)*T[1],
 (q^-1-1) + (q^-1)*T[2],
 (q^-1-1) + (q^-1)*T[3]]
```

product_on_basis(*m1*, *m2*)

Return the product of the basis elements indexed by `m1` and `m2`.

EXAMPLES:

```
sage: LT = algebras.ArikiKoike(6, 3).LT()
sage: m = ((1, 0, 2), Permutations(3)([2,1,3]))
sage: LT.product_on_basis(m, m)
q*L1*L2*L3^4

sage: LT = algebras.ArikiKoike(4, 3).LT()
sage: L1,L2,L3,T1,T2 = LT.algebra_generators()
sage: L1 * T1 * L1^2 * T1
q*L1*L2^2 + (1-q)*L1^2*L2*T[1]
sage: L1^2 * T1 * L1^2 * T1
q*L1^2*L2^2 + (1-q)*L1^3*L2*T[1]
sage: L1^3 * T1 * L1^2 * T1
(-u0*u1*u2*u3+u0*u1*u2*u3*q)*L2*T[1]
 + ((u0*u1*u2+u0*u1*u3+u0*u2*u3+u1*u2*u3)+(-u0*u1*u2-u0*u1*u3-u0*u2*u3-
↪u1*u2*u3)*q)*L1*L2*T[1]
 + ((-u0*u1-u0*u2-u1*u2-u0*u3-u1*u3-
```

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```
↪u2*u3)+(u0*u1+u0*u2+u1*u2+u0*u3+u1*u3+u2*u3)*q)*L1^2*L2*T[1]
 + ((u0+u1+u2+u3)+(-u0-u1-u2-u3)*q)*L1^3*L2*T[1] + q*L1^3*L2^2

sage: L1^2 * T1 * L1^3 * T1
(-u0*u1*u2*u3+u0*u1*u2*u3*q)*L2*T[1]
 + ((u0*u1*u2+u0*u1*u3+u0*u2*u3+u1*u2*u3)+(-u0*u1*u2-u0*u1*u3-u0*u2*u3-
↪u1*u2*u3)*q)*L1*L2*T[1]
 + ((-u0*u1-u0*u2-u1*u2-u0*u3-u1*u3-
↪u2*u3)+(u0*u1+u0*u2+u1*u2+u0*u3+u1*u3+u2*u3)*q)*L1^2*L2*T[1]
 + q*L1^2*L2^3
 + ((u0+u1+u2+u3)+(-u0-u1-u2-u3)*q)*L1^3*L2*T[1]
 + (1-q)*L1^3*L2^2*T[1]

sage: L1^2 * T1*T2*T1 * L2 * L3 * T2
(q-2*q^2+q^3)*L1^2*L2*L3 - (1-2*q+2*q^2-q^3)*L1^2*L2*L3*T[2]
 - (q-q^2)*L1^3*L3*T[1] + (1-2*q+q^2)*L1^3*L3*T[1,2]
 + q*L1^3*L2*T[2,1] - (1-q)*L1^3*L2*T[2,1,2]

sage: LT = algebras.ArikiKoike(2, 3).LT()
sage: L3 = LT.L(3)
sage: x = LT.an_element()
sage: (x * L3) * L3 == x * (L3 * L3)
True
```

class T(*algebra*)

Bases: `_Basis`

The basis of the Ariki-Koike algebra given by monomials of the generators $\{T_i | 0 \leq i < n\}$.

We use the choice of reduced expression given by [BM1997]:

$$T_{1,a_1} \cdots T_{n,a_n} T_w,$$

where $T_{i,k} = T_{i-1} \cdots T_2 T_1 T_0^k$ (note that $T_{1,k} = T_0^k$) and w is a reduced expression of an element in $\mathfrak{S}_n$.

L(*i=None*)

Return the Jucys-Murphy element(s) L_i.

The Jucys-Murphy element L_i is defined as

$$L_i = q^{-i+1} T_{i-1} \cdots T_1 T_0 T_1 \cdots T_{i-1} = q^{-1} T_{i-1} L_{i-1} T_{i-1}.$$

INPUT:

- `i` – (default: `None`) the Jucys-Murphy element L_i or if `None`, then the list of all L_i

EXAMPLES:

```
sage: T = algebras.ArikiKoike(8, 3).T()
sage: T.L(2)
(q^-1)*T[1,0,1]
sage: T.L()
[T[0], (q^-1)*T[1,0,1], (q^-2)*T[2,1,0,1,2]]

sage: T0,T1,T2 = T.T()
sage: q = T.q()
```

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```
sage: T.L(1) == T0
True
sage: T.L(2) == q^-1 * T1*T0*T1
True
sage: T.L(3) == q^-2 * T2*T1*T0*T1*T2
True

sage: T = algebras.ArikiKoike(1, 3).T()
sage: T.L(2)
u + (-u*q^-1+u)*T[1]
sage: T.L()
[u,
 u + (-u*q^-1+u)*T[1],
 u + (-u*q^-1+u)*T[2] + (-u*q^-2+u*q^-1)*T[2,1,2]]
```

T(*i=None*)

Return the generator(s) T_i of `self`.

INPUT:

- `i` – (default: `None`) the generator T_i or if `None`, then the list of all generators T_i

EXAMPLES:

```
sage: T = algebras.ArikiKoike(8, 3).T()
sage: T.T(1)
T[1]
sage: T.T()
[T[0], T[1], T[2]]

sage: T = algebras.ArikiKoike(1, 4).T()
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: T = algebras.ArikiKoike(5, 3).T()
sage: dict(T.algebra_generators())
{0: T[0], 1: T[1], 2: T[2]}

sage: T = algebras.ArikiKoike(1, 4).T()
sage: dict(T.algebra_generators())
{1: T[1], 2: T[2], 3: T[3]}
```

product_on_basis(*m1*, *m2*)

Return the product of the basis elements indexed by `m1` and `m2`.

EXAMPLES:

```
sage: T = algebras.ArikiKoike(2, 3).T()
sage: T0, T1, T2 = T.T()
sage: T.product_on_basis(T0.leading_support(), T1.leading_support())
T[0,1]
sage: T1 * T2
```

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```
T[1,2]
sage: T2 * T1
T[2,1]
sage: T2 * (T2 * T1 * T0)
-(1-q)*T[2,1,0] + q*T[1,0]
sage: (T1 * T0 * T1 * T0) * T0
(-u0*u1)*T[1,0,1] + ((u0+u1))*T[0,1,0,1]
sage: (T0 * T1 * T0 * T1) * (T0 * T1)
(-u0*u1*q)*T[1,0] + (u0*u1-u0*u1*q)*T[1,0,1]
 + ((u0+u1)*q)*T[0,1,0] + ((-u0-u1)+(u0+u1)*q)*T[0,1,0,1]
sage: T1 * (T0 * T2 * T1 * T0)
T[1,0,2,1,0]
sage: (T1 * T2) * (T2 * T1 * T0)
-(1-q)*T[2,1,0,2] - (q-q^2)*T[1,0] + q^2*T[0]
sage: (T2*T1*T2) * (T2*T1*T0*T1*T2)
-(q-q^2)*T[2,1,0,1,2] + (1-2*q+q^2)*T[2,1,0,2,1,2]
 - (q-q^2)*T[1,0,2,1,2] + q^2*T[0,2,1,2]
```

We check some relations:

```
sage: T0 * T1 * T0 * T1 == T1 * T0 * T1 * T0
True
sage: T1 * T2 * T1 == T2 * T1 * T2
True
sage: (T1 * T0) * T0 == T1 * (T0 * T0)
True
sage: (T.L(1) * T.L(2)) * T.L(2) - T.L(1) * (T.L(2) * T.L(2))
0
sage: (T.L(2) * T.L(3)) * T.L(3) - T.L(2) * (T.L(3) * T.L(3))
0
```

a_realization()

Return a realization of `self`.

EXAMPLES:

```
sage: H = algebras.ArikiKoike(5, 2)
sage: H.a_realization()
Ariki-Koike algebra of rank 5 and order 2
 with q=q and u=(u0, u1, u2, u3, u4) ... in the LT-basis
```

cyclotomic_parameters()

Return the cyclotomic parameters u of `self`.

EXAMPLES:

```
sage: H = algebras.ArikiKoike(5, 3)
sage: H.cyclotomic_parameters()
(u0, u1, u2, u3, u4)
```

hecke_parameter()

Return the Hecke parameter q of `self`.

EXAMPLES:

```
sage: H = algebras.ArikiKoike(5, 3)
sage: H.hecke_parameter()
q
```

q()

Return the Hecke parameter q of `self`.

EXAMPLES:

```
sage: H = algebras.ArikiKoike(5, 3)
sage: H.hecke_parameter()
q
```

u()

Return the cyclotomic parameters u of `self`.

EXAMPLES:

```
sage: H = algebras.ArikiKoike(5, 3)
sage: H.cyclotomic_parameters()
(u0, u1, u2, u3, u4)
```

6.2 Iwahori-Hecke Algebras

AUTHORS:

- Daniel Bump, Nicolas Thiery (2010): Initial version
- Brant Jones, Travis Scrimshaw, Andrew Mathas (2013): Moved into the category framework and implemented the Kazhdan-Lusztig C and C' bases
- Chase Meadors, Tianyuan Xu (2021): Implemented direct computation of products in the C' basis using du Cloux's Coxeter3 package

class `sage.algebras.iwahori_hecke_algebra.`**`IwahoriHeckeAlgebra`**(*W*, *q1*, *q2*, *base_ring*)

Bases: `Parent`, `UniqueRepresentation`

The Iwahori-Hecke algebra of the Coxeter group `W` with the specified parameters.

INPUT:

- `W` – a Coxeter group or Cartan type
- `q1` – a parameter

OPTIONAL ARGUMENTS:

- `q2` – (default `-1`) another parameter
- `base_ring` – (default `q1.parent()`) a ring containing `q1` and `q2`

The Iwahori-Hecke algebra [Iwa1964] is a deformation of the group algebra of a Weyl group or, more generally, a Coxeter group. These algebras are defined by generators and relations and they depend on a deformation parameter q. Taking $q = 1$, as in the following example, gives a ring isomorphic to the group algebra of the corresponding Coxeter group.

Let (W, S) be a Coxeter system and let R be a commutative ring containing elements q_1 and q_2. Then the *Iwahori-Hecke algebra* $H = H_{q_1,q_2}(W, S)$ of (W, S) with parameters q_1 and q_2 is the unital associative algebra with generators $\{T_s \mid s \in S\}$ and relations:

$$(T_s - q_1)(T_s - q_2) = 0$$
$$T_r T_s T_r \cdots = T_s T_r T_s \cdots ,$$

where the number of terms on either side of the second relations (the braid relations) is the order of rs in the Coxeter group W, for $r, s \in S$.

Iwahori-Hecke algebras are fundamental in many areas of mathematics, ranging from the representation theory of Lie groups and quantum groups, to knot theory and statistical mechanics. For more information see, for example, [KL1979], [HKP2010], [Jon1987] and Wikipedia article Iwahori-Hecke_algebra.

Bases

A reduced expression for an element $w \in W$ is any minimal length word $w = s_1 \cdots s_k$, with $s_i \in S$. If $w = s_1 \cdots s_k$ is a reduced expression for w then Matsumoto's Monoid Lemma implies that $T_w = T_{s_1} \cdots T_{s_k}$ depends on w and not on the choice of reduced expressions. Moreover, $\{T_w \mid w \in W\}$ is a basis for the Iwahori-Hecke algebra H and

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1, \\ (q_1 + q_2)T_w - q_1 q_2 T_{sw}, & \text{if } \ell(sw) = \ell(w) - 1. \end{cases}$$

The T-basis of H is implemented for any choice of parameters `q_1` and `q_2`:

```
sage: R.<u,v> = LaurentPolynomialRing(ZZ,2)
sage: H = IwahoriHeckeAlgebra('A3', u,v)
sage: T = H.T()
sage: T[1]
T[1]
sage: T[1,2,1] + T[2]
T[1,2,1] + T[2]
sage: T[1] * T[1,2,1]
(u+v)*T[1,2,1] + (-u*v)*T[2,1]
sage: T[1]^-1
(-u^-1*v^-1)*T[1] + (v^-1+u^-1)
```

Working over the Laurent polynomial ring $Z[q^{\pm 1/2}]$ Kazhdan and Lusztig proved that there exist two distinguished bases $\{C'_w \mid w \in W\}$ and $\{C_w \mid w \in W\}$ of H which are uniquely determined by the properties that they are invariant under the bar involution on H and have triangular transitions matrices with polynomial entries of a certain form with the T-basis; see [KL1979] for a precise statement.

It turns out that the Kazhdan-Lusztig bases can be defined (by specialization) in H whenever $-q_1 q_2$ is a square in the base ring. The Kazhdan-Lusztig bases are implemented inside H whenever $-q_1 q_2$ has a square root:

```
sage: H = IwahoriHeckeAlgebra('A3', u^2,-v^2)
sage: T=H.T(); Cp= H.Cp(); C=H.C()
sage: T(Cp[1])
(u^-1*v^-1)*T[1] + (u^-1*v)
sage: T(C[1])
(u^-1*v^-1)*T[1] + (-u*v^-1)
sage: Cp(C[1])
Cp[1] + (-u*v^-1-u^-1*v)
sage: elt = Cp[2]*Cp[3]+C[1]; elt
Cp[2,3] + Cp[1] + (-u*v^-1-u^-1*v)
```

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```
sage: c = C(elt); c
C[2,3] + C[1] + (u*v^-1+u^-1*v)*C[3] + (u*v^-1+u^-1*v)*C[2] + (u^2*v^-2+2+u^-2*v^2)
sage: t = T(c); t
(u^-2*v^-2)*T[2,3] + (u^-1*v^-1)*T[1] + (u^-2)*T[3] + (u^-2)*T[2] + (-u*v^-1+u^-2*v^
↪2)
sage: Cp(t)
Cp[2,3] + Cp[1] + (-u*v^-1-u^-1*v)
sage: Cp(c)
Cp[2,3] + Cp[1] + (-u*v^-1-u^-1*v)
```

The conversions to and from the Kazhdan-Lusztig bases are done behind the scenes whenever the Kazhdan-Lusztig bases are well-defined. Once a suitable Iwahori-Hecke algebra is defined they will work without further intervention.

For example, with the "standard parameters", so that $(T_r - q^2)(T_r + 1) = 0$:

```
sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra('A3', q^2)
sage: T=H.T(); Cp=H.Cp(); C=H.C()
sage: C(T[1])
q*C[1] + q^2
sage: elt = Cp(T[1,2,1]); elt
q^3*Cp[1,2,1] - q^2*Cp[2,1] - q^2*Cp[1,2] + q*Cp[1] + q*Cp[2] - 1
sage: C(elt)
q^3*C[1,2,1] + q^4*C[2,1] + q^4*C[1,2] + q^5*C[1] + q^5*C[2] + q^6
```

With the "normalized presentation", so that $(T_r - q)(T_r + q^{-1}) = 0$:

```
sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra('A3', q, -q^-1)
sage: T=H.T(); Cp=H.Cp(); C=H.C()
sage: C(T[1])
C[1] + q
sage: elt = Cp(T[1,2,1]); elt
Cp[1,2,1] - (q^-1)*Cp[2,1] - (q^-1)*Cp[1,2] + (q^-2)*Cp[1] + (q^-2)*Cp[2] - (q^-3)
sage: C(elt)
C[1,2,1] + q*C[2,1] + q*C[1,2] + q^2*C[1] + q^2*C[2] + q^3
```

In the group algebra, so that $(T_r - 1)(T_r + 1) = 0$:

```
sage: H = IwahoriHeckeAlgebra('A3', 1)
sage: T=H.T(); Cp=H.Cp(); C=H.C()
sage: C(T[1])
C[1] + 1
sage: Cp(T[1,2,1])
Cp[1,2,1] - Cp[2,1] - Cp[1,2] + Cp[1] + Cp[2] - 1
sage: C(_)
C[1,2,1] + C[2,1] + C[1,2] + C[1] + C[2] + 1
```

On the other hand, if the Kazhdan-Lusztig bases are not well-defined (when $-q_1 q_2$ is not a square), attempting to use the Kazhdan-Lusztig bases triggers an error:

```
sage: R.<q>=LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra('A3', q)
sage: C=H.C()
Traceback (most recent call last):
...
ValueError: The Kazhdan_Lusztig bases are defined only when -q_1*q_2 is a square
```

We give an example in affine type:

```
sage: R.<v> = LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra(['A',2,1], v^2)
sage: T=H.T(); Cp=H.Cp(); C=H.C()
sage: C(T[1,0,2])
v^3*C[1,0,2] + v^4*C[1,0] + v^4*C[0,2] + v^4*C[1,2]
 + v^5*C[0] + v^5*C[2] + v^5*C[1] + v^6
sage: Cp(T[1,0,2])
v^3*Cp[1,0,2] - v^2*Cp[1,0] - v^2*Cp[0,2] - v^2*Cp[1,2]
 + v*Cp[0] + v*Cp[2] + v*Cp[1] - 1
sage: T(C[1,0,2])
(v^-3)*T[1,0,2] - (v^-1)*T[1,0] - (v^-1)*T[0,2] - (v^-1)*T[1,2]
 + v*T[0] + v*T[2] + v*T[1] - v^3
sage: T(Cp[1,0,2])
(v^-3)*T[1,0,2] + (v^-3)*T[1,0] + (v^-3)*T[0,2] + (v^-3)*T[1,2]
 + (v^-3)*T[0] + (v^-3)*T[2] + (v^-3)*T[1] + (v^-3)
```

EXAMPLES:

We start by creating a Iwahori-Hecke algebra together with the three bases for these algebras that are currently supported:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: T = H.T()
sage: C = H.C()
sage: Cp = H.Cp()
```

It is also possible to define these three bases quickly using the `inject_shorthands()` method.

Next we create our generators for the T-basis and do some basic computations and conversions between the bases:

```
sage: T1,T2,T3 = T.algebra_generators()
sage: T1 == T[1]
True
sage: T1*T2 == T[1,2]
True
sage: T1 + T2
T[1] + T[2]
sage: T1*T1
-(1-v^2)*T[1] + v^2
sage: (T1 + T2)*T3 + T1*T1 - (v + v^-1)*T2
T[3,1] + T[2,3] - (1-v^2)*T[1] - (v^-1+v)*T[2] + v^2
sage: Cp(T1)
v*Cp[1] - 1
```

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```
sage: Cp((v^1 - 1)*T1*T2 - T3)
-(v^2-v^3)*Cp[1,2] + (v-v^2)*Cp[1] - v*Cp[3] + (v-v^2)*Cp[2] + v
sage: C(T1)
v*C[1] + v^2
sage: p = C(T2*T3 - v*T1); p
v^2*C[2,3] - v^2*C[1] + v^3*C[3] + v^3*C[2] - (v^3-v^4)
sage: Cp(p)
v^2*Cp[2,3] - v^2*Cp[1] - v*Cp[3] - v*Cp[2] + (1+v)
sage: Cp(T2*T3 - v*T1)
v^2*Cp[2,3] - v^2*Cp[1] - v*Cp[3] - v*Cp[2] + (1+v)
```

In addition to explicitly creating generators, we have two shortcuts to basis elements. The first is by using elements of the underlying Coxeter group, the other is by using reduced words:

```
sage: s1,s2,s3 = H.coxeter_group().gens()
sage: T[s1*s2*s1*s3] == T[1,2,1,3]
True
sage: T[1,2,1,3] == T1*T2*T1*T3
True
```

Todo: Implement multi-parameter Iwahori-Hecke algebras together with their Kazhdan-Lusztig bases. That is, Iwahori-Hecke algebras with (possibly) different parameters for each conjugacy class of simple reflections in the underlying Coxeter group.

Todo: When given "generic parameters" we should return the generic Iwahori-Hecke algebra with these parameters and allow the user to work inside this algebra rather than doing calculations behind the scenes in a copy of the generic Iwahori-Hecke algebra. The main problem is that it is not clear how to recognise when the parameters are "generic".

---

class A(*IHAlgebra*, *prefix=None*)

Bases: `_Basis`

The A-basis of an Iwahori-Hecke algebra.

The A-basis of the Iwahori-Hecke algebra is the simplest basis that is invariant under the Goldman involution $\#$, up to sign. For w in the underlying Coxeter group define:

$$A_w = T_w + (-1)^{\ell(w)} T_w^{\#} = T_w + (-1)^{\ell(w)} T_{w^{-1}}^{-1}$$

This gives a basis of the Iwahori-Hecke algebra whenever 2 is a unit in the base ring. The A-basis induces a $\mathbf{Z}/2\mathbf{Z}$-grading on the Iwahori-Hecke algebra.

The A-basis is a basis only when 2 is invertible. An error is raised whenever 2 is not a unit in the base ring.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: A=H.A(); T=H.T()
sage: T(A[1])
T[1] + (1/2-1/2*v^2)
```

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```
sage: T(A[1,2])
T[1,2] + (1/2-1/2*v^2)*T[1] + (1/2-1/2*v^2)*T[2] + (1/2-v^2+1/2*v^4)
sage: A[1]*A[2]
A[1,2] - (1/4-1/2*v^2+1/4*v^4)
```

goldman_involution_on_basis(*w*)

Return the effect of applying the Goldman involution to the basis element `self[w]`.

This function is not intended to be called directly. Instead, use `goldman_involution()`.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: A=H.A()
sage: s=H.coxeter_group().simple_reflection(1)
sage: A.goldman_involution_on_basis(s)
-A[1]
sage: A[1,2].goldman_involution()
A[1,2]
```

to_T_basis(*w*)

Return the A-basis element `self[w]` as a linear combination of T-basis elements.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ)
sage: H = IwahoriHeckeAlgebra('A3', v**2); A=H.A(); T=H.T()
sage: s=H.coxeter_group().simple_reflection(1)
sage: A.to_T_basis(s)
T[1] + (1/2-1/2*v^2)
sage: T(A[1,2])
T[1,2] + (1/2-1/2*v^2)*T[1] + (1/2-1/2*v^2)*T[2] + (1/2-v^2+1/2*v^4)
sage: A(T[1,2])
A[1,2] - (1/2-1/2*v^2)*A[1] - (1/2-1/2*v^2)*A[2]
```

class B(*IHAlgebra*, *prefix=None*)

Bases: `_Basis`

The B-basis of an Iwahori-Hecke algebra.

The B-basis is the unique basis of the Iwahori-Hecke algebra that is invariant under the Goldman involution, up to sign, and invariant under the Kazhdan-Lusztig bar involution. In the generic case, the B-basis becomes the group basis of the group algebra of the Coxeter group the B-basis upon setting the Hecke parameters equal to 1. If w is an element of the corresponding Coxeter group then the B-basis element B_w is uniquely determined by the conditions that $B_w^{\#} = (-1)^{\ell(w)} B_w$, where $\#$ is the `Goldman involution` and

$$B_w = T_w + \sum_{v<w} b_{vw}(q) T_v$$

where $b_{vw}(q) \neq 0$ only if $v < w$ in the Bruhat order and $\ell(v) \not\equiv \ell(w) \pmod 2$.

This gives a basis of the Iwahori-Hecke algebra whenever 2 is a unit in the base ring. The B-basis induces a $\mathbf{Z}/2\mathbf{Z}$-grading on the Iwahori-Hecke algebra. The B-basis elements are also invariant under the Kazhdan-Lusztig bar involution and hence are related to the Kazhdan-Lusztig bases.

The B-basis is a basis only when 2 is invertible. An error is raised whenever 2 is not a unit in the base ring.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: A=H.A(); T=H.T(); Cp=H.Cp()
sage: T(A[1])
T[1] + (1/2-1/2*v^2)
sage: T(A[1,2])
T[1,2] + (1/2-1/2*v^2)*T[1] + (1/2-1/2*v^2)*T[2] + (1/2-v^2+1/2*v^4)
sage: A[1]*A[2]
A[1,2] - (1/4-1/2*v^2+1/4*v^4)
sage: Cp(A[1]*A[2])
v^2*Cp[1,2] - (1/2*v+1/2*v^3)*Cp[1] - (1/2*v+1/2*v^3)*Cp[2]
 + (1/4+1/2*v^2+1/4*v^4)
sage: Cp(A[1])
v*Cp[1] - (1/2+1/2*v^2)
sage: Cp(A[1,2])
v^2*Cp[1,2] - (1/2*v+1/2*v^3)*Cp[1]
 - (1/2*v+1/2*v^3)*Cp[2] + (1/2+1/2*v^4)
sage: Cp(A[1,2,1])
v^3*Cp[1,2,1] - (1/2*v^2+1/2*v^4)*Cp[2,1]
 - (1/2*v^2+1/2*v^4)*Cp[1,2] + (1/2*v+1/2*v^5)*Cp[1]
 + (1/2*v+1/2*v^5)*Cp[2] - (1/2+1/2*v^6)
```

goldman_involution_on_basis(w)

Return the Goldman involution to the basis element indexed by `w`.

This function is not intended to be called directly. Instead, use `goldman_involution()`.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: B=H.B()
sage: s=H.coxeter_group().simple_reflection(1)
sage: B.goldman_involution_on_basis(s)
-B[1]
sage: B[1,2].goldman_involution()
B[1,2]
```

to_T_basis(w)

Return the B-basis element `self[w]` as a linear combination of T-basis elements.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ)
sage: H = IwahoriHeckeAlgebra('A3', v**2); B=H.B(); T=H.T()
sage: s=H.coxeter_group().simple_reflection(1)
sage: B.to_T_basis(s)
T[1] + (1/2-1/2*v^2)
sage: T(B[1,2])
T[1,2] + (1/2-1/2*v^2)*T[1] + (1/2-1/2*v^2)*T[2]
sage: B(T[1,2])
B[1,2] - (1/2-1/2*v^2)*B[1] - (1/2-1/2*v^2)*B[2] + (1/2-v^2+1/2*v^4)
```

class C(*IHAlgebra*, *prefix=None*)

Bases: `_KLHeckeBasis`

The Kazhdan-Lusztig C-basis of Iwahori-Hecke algebra.

Assuming the standard quadratic relations of $(T_r - q)(T_r + 1) = 0$, for every element w in the Coxeter group, there is a unique element C_w in the Iwahori-Hecke algebra which is uniquely determined by the two properties:

$$\overline{C_w} = C_w$$
$$C_w = (-1)^{\ell(w)} q^{\ell(w)/2} \sum_{v \leq w} (-q)^{-\ell(v)} \overline{P_{v,w}(q)} T_v$$

where $\leq$ is the Bruhat order on the underlying Coxeter group and $P_{v,w}(q) \in \mathbf{Z}[q, q^{-1}]$ are polynomials in $\mathbf{Z}[q]$ such that $P_{w,w}(q) = 1$ and if $v < w$ then $\deg P_{v,w}(q) \leq \frac{1}{2}(\ell(w) - \ell(v) - 1)$. This is related to the C' Kazhdan-Lusztig basis by $C_i = -\alpha(C_i')$ where α is the $\mathbf{Z}$-linear Hecke involution defined by $q^{1/2} \mapsto q^{-1/2}$ and $\alpha(T_i) = -(q_1 q_2)^{-1/2} T_i$.

More generally, if the quadratic relations are of the form (T_s-q_1)(T_s-q_2)=0` and $\sqrt{-q_1 q_2}$ exists then, for a simple reflection s, the corresponding Kazhdan-Lusztig basis element is:

$$C_s = (-q_1 q_2)^{1/2} (1 - (-q_1 q_2)^{-1/2} T_s).$$

See [KL1979] for more details.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ)
sage: H = IwahoriHeckeAlgebra('A5', v**2)
sage: W = H.coxeter_group()
sage: s1,s2,s3,s4,s5 = W.simple_reflections()
sage: T = H.T()
sage: C = H.C()
sage: T(s1)**2
-(1-v^2)*T[1] + v^2
sage: T(C(s1))
(v^-1)*T[1] - v
sage: T(C(s1)*C(s2)*C(s1))
(v^-3)*T[1,2,1] - (v^-1)*T[2,1] - (v^-1)*T[1,2]
 + (v^-1+v)*T[1] + v*T[2] - (v+v^3)
```

```
sage: R.<v> = LaurentPolynomialRing(QQ)
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: W = H.coxeter_group()
sage: s1,s2,s3 = W.simple_reflections()
sage: C = H.C()
sage: C(s1*s2*s1)
C[1,2,1]
sage: C(s1)**2
-(v^-1+v)*C[1]
sage: C(s1)*C(s2)*C(s1)
C[1,2,1] + C[1]
```

hash_involution_on_basis(*w*)

Return the effect of applying the hash involution to the basis element `self[w]`.

This function is not intended to be called directly. Instead, use `hash_involution()`.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: C=H.C()
sage: s=H.coxeter_group().simple_reflection(1)
sage: C.hash_involution_on_basis(s)
-C[1] - (v^-1+v)
sage: C[s].hash_involution()
-C[1] - (v^-1+v)
```

C_prime

alias of *Cp*

class Cp(*IHAlgebra*, *prefix=None*)

Bases: `_KLHeckeBasis`

The C' Kazhdan-Lusztig basis of Iwahori-Hecke algebra.

Assuming the standard quadratic relations of $(T_r - q)(T_r + 1) = 0$, for every element w in the Coxeter group, there is a unique element C'_w in the Iwahori-Hecke algebra which is uniquely determined by the two properties:

$$\overline{C'_w} = C'_w,$$
$$C'_w = q^{-\ell(w)/2} \sum_{v \leq w} P_{v,w}(q) T_v,$$

where $\leq$ is the Bruhat order on the underlying Coxeter group and $P_{v,w}(q) \in \mathbf{Z}[q, q^{-1}]$ are polynomials in $\mathbf{Z}[q]$ such that $P_{w,w}(q) = 1$ and if $v < w$ then $\deg P_{v,w}(q) \leq \frac{1}{2}(\ell(w) - \ell(v) - 1)$.

More generally, if the quadratic relations are of the form (T_s-q_1)(T_s-q_2)=0` and $\sqrt{-q_1 q_2}$ exists then, for a simple reflection s, the corresponding Kazhdan-Lusztig basis element is:

$$C'_s = (-q_1 q_2)^{-1/2}(T_s + 1).$$

See [KL1979] for more details.

If the optional `coxeter3` package is available and the Iwahori–Hecke algebra was initialized in the "standard" presentation where $\{q_1, q_2\} = \{v^2, 1\}$ as sets or the "normalized" presentation where $\{q_1, q_2\} = \{v, -v^{-1}\}$ as sets, the function :func::$product_on_basis$ in this class computes products in the C'-basis directly in the basis itself, using `coxeter3` to calculate certain μ-coefficients quickly. If the above conditions are not all met, the function computes such products indirectly, by converting elements to the T-basis, computing products there, and converting back. The indirect method can be prohibitively slow for more complex calculations; the direct method is faster.

EXAMPLES:

```
sage: R = LaurentPolynomialRing(QQ, 'v')
sage: v = R.gen(0)
sage: H = IwahoriHeckeAlgebra('A5', v**2)
sage: W = H.coxeter_group()
sage: s1,s2,s3,s4,s5 = W.simple_reflections()
sage: T = H.T()
sage: Cp = H.Cp()
sage: T(s1)**2
```

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```
-(1-v^2)*T[1] + v^2
sage: T(Cp(s1))
(v^-1)*T[1] + (v^-1)
sage: T(Cp(s1)*Cp(s2)*Cp(s1))
(v^-3)*T[1,2,1] + (v^-3)*T[2,1] + (v^-3)*T[1,2]
 + (v^-3+v^-1)*T[1] + (v^-3)*T[2] + (v^-3+v^-1)
```

```
sage: R = LaurentPolynomialRing(QQ, 'v')
sage: v = R.gen(0)
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: W = H.coxeter_group()
sage: s1,s2,s3 = W.simple_reflections()
sage: Cp = H.Cp()
sage: Cp(s1*s2*s1)
Cp[1,2,1]
sage: Cp(s1)**2
(v^-1+v)*Cp[1]
sage: Cp(s1)*Cp(s2)*Cp(s1)
Cp[1,2,1] + Cp[1]
sage: Cp(s1)*Cp(s2)*Cp(s3)*Cp(s1)*Cp(s2)    # long time
Cp[1,2,3,1,2] + Cp[1,2,1] + Cp[3,1,2]
```

In the following product computations, whether `coxeter3` is installed makes a big difference: without `coxeter3` the product in type H_4 takes about 5 seconds to compute and the product in type A_9 seems infeasible, while with `coxeter3` both the computations are instant:

```
sage: H = IwahoriHeckeAlgebra('H4', v**2)   # optional - coxeter3
sage: Cp = H.Cp()                           # optional - coxeter3
sage: Cp[3,4,3]*Cp[3,4,3,4]*Cp[1,2,3,4]     # optional - coxeter3
(v^-2+2+v^2)*Cp[3,4,3,4,1,2,3,4,2]
+ (v^-2+2+v^2)*Cp[3,4,3,4,3,1,2]
+ (v^-3+3*v^-1+3*v+v^3)*Cp[3,4,3,4,3,1]
+ (v^-1+v)*Cp[3,4,1,2,3,4]
+ (v^-1+v)*Cp[3,4,1,2]

sage: H = IwahoriHeckeAlgebra('A9', v**2)   # optional - coxeter3
sage: Cp = H.Cp()                           # optional - coxeter3
sage: Cp[1,2,1,8,9,8]*Cp[1,2,3,7,8,9]       # optional - coxeter3
(v^-2+2+v^2)*Cp[7,8,9,7,8,7,1,2,3,1]
+ (v^-2+2+v^2)*Cp[8,9,8,7,1,2,3,1]
+ (v^-3+3*v^-1+3*v+v^3)*Cp[8,9,8,1,2,3,1]
```

To use `coxeter3` for product computations most efficiently, we recommend creating the Iwahori-Hecke algebra from a Coxeter group implemented with `coxeter3` to avoid unnecessary conversions, as in the following example with the same product computed in the last one:

```
sage: R = LaurentPolynomialRing(QQ, 'v')                       # optional -␣
↪coxeter3
sage: v = R.gen(0)                                             # optional -␣
↪coxeter3
sage: W = CoxeterGroup('A9', implementation='coxeter3')        # optional -␣
↪coxeter3
```

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```
sage: H = IwahoriHeckeAlgebra(W, v**2)                          # optional -
↪coxeter3
sage: Cp = H.Cp()                                               # optional -
↪coxeter3
sage: Cp[1,2,1,8,9,8]*Cp[1,2,3,7,8,9]                           # optional -
↪coxeter3
(v^-2+2+v^2)*Cp[1,2,1,3,7,8,7,9,8,7]
+ (v^-2+2+v^2)*Cp[1,2,1,3,8,9,8,7]
+ (v^-3+3*v^-1+3*v+v^3)*Cp[1,2,1,3,8,9,8]
```

hash_involution_on_basis(*w*)

Return the effect of applying the hash involution to the basis element `self[w]`.

This function is not intended to be called directly. Instead, use `hash_involution()`.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: Cp=H.Cp()
sage: s=H.coxeter_group().simple_reflection(1)
sage: Cp.hash_involution_on_basis(s)
-Cp[1] + (v^-1+v)
sage: Cp[s].hash_involution()
-Cp[1] + (v^-1+v)
```

product_on_basis(*w1*, *w2*)

Return the expansion of $C'_{w_1} \cdot C'_{w_2}$ in the C'-basis.

If `coxeter3` is installed and the Iwahori–Hecke algebra is in the standard or normalized presentation, the product is computed directly using the method described in ALGORITHM. If not, the product is computed indirectly by converting the factors to the T-basis, computing the product there, and converting back.

The following formulas for products of the forms $C'_s \cdot C'_w$ and $C'_w \cdot C'_s$, where s is a generator of the Coxeter group and w an arbitrary element, are key to the direct computation method. The formulas are valid for both the standard and normalized presentation of the Hecke algebra.

$$C'_s \cdot C'_w = \begin{cases} (q+q^{-1})C'_w, & \text{if } \ell(sw) = \ell(w)-1,\\ C'_{sw} + \sum_{v\leq w, sv\leq v} \mu(v,w)C'_v, & \text{if } \ell(sw) = \ell(w)+1. \end{cases}$$

$$C'_w \cdot C'_s = \begin{cases} (q+q^{-1})C'_w, & \text{if } \ell(ws) = \ell(w)-1,\\ C'_{ws} + \sum_{v\leq w, vs\leq v} \mu(v,w)C'_v, & \text{if } \ell(ws) = \ell(w)+1. \end{cases}$$

In the above, $\leq$ is the Bruhat order on the Coxeter group and $\mu(v,w)$ is the "leading coefficient of Kazhdan-Lusztig polynomials"; see [KL1979] and [Lus2013] for more details. The method designates the computation of the μ-coefficients to Sage's interface to Fokko du Cloux's `coxeter3` package, which is why the method requires the creation of the Coxeter group using the `'coxeter3'` implementation.

ALGORITHM:

The direct algorithm for computing $C'_x \cdot C'_y$ runs in two steps as follows.

If $\ell(x) \leq \ell(y)$, we first decompose C'_x into a polynomial in the generators $C'_s (s \in S)$ and then multiply that polynomial with C'_y. If $\ell(x) > \ell(y)$, we decompose C'_y into a polynomial in $C'_s (s \in S)$ and multiply that polynomial with C'_x. The second step (multiplication) is done by repeatedly applying the formulas displayed earlier directly. The first step (decomposition) is done by induction on the Bruhat order as follows: for every element $u \in W$ with length $\ell(u) > 1$, pick a left descent s of u and write $u = sw$ (so $w = su$), then note that

$$C'_u = C'_s \cdot C'_w - \sum_{v \leq u; sv < v} \mu(v, w) C'_v$$

by the earlier formulas, where the element w and all elements v's on the right side are lower than u in the Bruhat order; this allows us to finish the computation by decomposing the lower order terms C'_w and each C'_v. For example, for $u = 121, s = 1, w = 21$ in type A_3 we have $C'_{121} = C'_1 C'_{21} - C'_1$, where the lower order term C'_{21} further decomposes into $C'_2 C'_1$, therefore

$$C'_{121} = C'_1 C'_2 C'_1 - C'_1.$$

We note that the base cases $\ell(x) = 1$ or $\ell(x) = 0$ of the above induction occur when x is itself a Coxeter generator s or the group identity, respectively. The decomposition is trivial in these cases (we have $C'_x = C'_s$ or $C'_x = 1$, the unit of the Hecke algebra).

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(ZZ, 'v')                    # optional -
↪ coxeter3
sage: W = CoxeterGroup('A3', implementation='coxeter3')         # optional -
↪ coxeter3
sage: H = IwahoriHeckeAlgebra(W, v**2); Cp=H.Cp()               # optional -
↪ coxeter3
sage: Cp.product_on_basis(W([1,2,1]), W([3,1]))                 # optional -
↪ coxeter3
(v^-1+v)*Cp[1,2,1,3]
sage: Cp.product_on_basis(W([1,2,1]), W([3,1,2]))               # optional -
↪ coxeter3
(v^-1+v)*Cp[1,2,1,3,2] + (v^-1+v)*Cp[1,2,1]
```

class T(*algebra*, *prefix=None*)

Bases: `_Basis`

The standard basis of Iwahori-Hecke algebra.

For every simple reflection s_i of the Coxeter group, there is a corresponding generator T_i of Iwahori-Hecke algebra. These are subject to the relations:

$$(T_i - q_1)(T_i - q_2) = 0$$

together with the braid relations:

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots ,$$

where the number of terms on each of the two sides is the order of $s_i s_j$ in the Coxeter group.

Weyl group elements form a basis of Iwahori-Hecke algebra H with the property that if w_1 and w_2 are Coxeter group elements such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ then $T_{w_1 w_2} = T_{w_1} T_{w_2}$.

With the default value $q_2 = -1$ and with $q_1 = q$ the generating relation may be written $T_i^2 = (q-1) \cdot T_i + q \cdot 1$ as in [Iwa1964].

EXAMPLES:

```
sage: H = IwahoriHeckeAlgebra("A3", 1)
sage: T = H.T()
sage: T1,T2,T3 = T.algebra_generators()
sage: T1*T2*T3*T1*T2*T1 == T3*T2*T1*T3*T2*T3
True
sage: w0 = T(H.coxeter_group().long_element())
sage: w0
T[1,2,3,1,2,1]
sage: T = H.T(prefix="s")
sage: T.an_element()
s[1,2,3] + 2*s[1] + 3*s[2] + 1
```

class Element

Bases: `IndexedFreeModuleElement`

A class for elements of an Iwahori-Hecke algebra in the T basis.

bar_on_basis(*w*)

Return the bar involution of T_w, which is $T^{-1}_{w^{-1}}$.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ)
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: W = H.coxeter_group()
sage: s1,s2,s3 = W.simple_reflections()
sage: T = H.T()
sage: b = T.bar_on_basis(s1*s2*s3); b
(v^-6)*T[1,2,3] + (v^-6-v^-4)*T[3,1]
 + (v^-6-v^-4)*T[1,2] + (v^-6-v^-4)*T[2,3]
 + (v^-6-2*v^-4+v^-2)*T[1] + (v^-6-2*v^-4+v^-2)*T[3]
 + (v^-6-2*v^-4+v^-2)*T[2] + (v^-6-3*v^-4+3*v^-2-1)
sage: b.bar()
T[1,2,3]
```

goldman_involution_on_basis(*w*)

Return the Goldman involution to the basis element indexed by `w`.

The goldman involution is the algebra involution of the Iwahori-Hecke algebra determined by

$$T_w \mapsto (-q_1 q_2)^{\ell(w)} T^{-1}_{w^{-1}},$$

where w is an element of the corresponding Coxeter group.

This map is defined in [Iwa1964] and it is used to define the alternating subalgebra of the Iwahori-Hecke algebra, which is the fixed-point subalgebra of the Goldman involution.

This function is not intended to be called directly. Instead, use `goldman_involution()`.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: T=H.T()
sage: s=H.coxeter_group().simple_reflection(1)
sage: T.goldman_involution_on_basis(s)
```

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```
-T[1] - (1-v^2)
sage: T[s].goldman_involution()
-T[1] - (1-v^2)
sage: h = T[1]*T[2] + (v^3 - v^-1 + 2)*T[3,1,2,3]
sage: h.goldman_involution()
-(v^-1-2-v^3)*T[1,2,3,2]
 - (v^-1-2-v+2*v^2-v^3+v^5)*T[3,1,2]
 - (v^-1-2-v+2*v^2-v^3+v^5)*T[1,2,3]
 - (v^-1-2-v+2*v^2-v^3+v^5)*T[2,3,2]
 - (v^-1-2-2*v+4*v^2-2*v^4+2*v^5-v^7)*T[3,1]
 - (v^-1-3-2*v+4*v^2-2*v^4+2*v^5-v^7)*T[1,2]
 - (v^-1-2-2*v+4*v^2-2*v^4+2*v^5-v^7)*T[3,2]
 - (v^-1-2-2*v+4*v^2-2*v^4+2*v^5-v^7)*T[2,3]
- (v^-1-3-2*v+5*v^2+v^3-4*v^4+v^5+2*v^6-2*v^7+v^9)*T[1]
 - (v^-1-2-3*v+6*v^2+2*v^3-6*v^4+2*v^5+2*v^6-3*v^7+v^9)*T[3]
 - (v^-1-3-3*v+7*v^2+2*v^3-6*v^4+2*v^5+2*v^6-3*v^7+v^9)*T[2]
 - (v^-1-3-3*v+8*v^2+3*v^3-9*v^4+6*v^6-3*v^7-2*v^8+3*v^9-v^11)
sage: h.goldman_involution().goldman_involution() == h
True
```

hash_involution_on_basis(w)

Return the hash involution on the basis element `self[w]`.

The hash involution α is a $\mathbf{Z}$-algebra involution of the Iwahori-Hecke algebra determined by $q^{1/2} \mapsto q^{-1/2}$, and $T_w \mapsto (-q_1 q_2)^{-\ell(w)} T_w$, for w an element of the corresponding Coxeter group.

This map is defined in [KL1979] and it is used to change between the C and C' bases because $\alpha(C_w) = (-1)^{\ell(w)} C'_w$.

This function is not intended to be called directly. Instead, use `hash_involution()`.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ, 'v')
sage: H = IwahoriHeckeAlgebra('A3', v**2)
sage: T=H.T()
sage: s=H.coxeter_group().simple_reflection(1)
sage: T.hash_involution_on_basis(s)
-(v^-2)*T[1]
sage: T[s].hash_involution()
-(v^-2)*T[1]
sage: h = T[1]*T[2] + (v^3 - v^-1 + 2)*T[3,1,2,3]
sage: h.hash_involution()
(v^-11+2*v^-8-v^-7)*T[1,2,3,2] + (v^-4)*T[1,2]
sage: h.hash_involution().hash_involution() == h
True
```

inverse_generator(i)

Return the inverse of the i-th generator, if it exists.

This method is only available if the Iwahori-Hecke algebra parameters `q1` and `q2` are both invertible. In this case, the algebra generators are also invertible and this method returns the inverse of `self.algebra_generator(i)`.

EXAMPLES:

```
sage: P.<q1, q2>=QQ[]
sage: F = Frac(P)
sage: H = IwahoriHeckeAlgebra("A2", q1, q2=q2, base_ring=F).T()
sage: H.base_ring()
Fraction Field of Multivariate Polynomial Ring in q1, q2 over Rational Field
sage: H.inverse_generator(1)
-1/(q1*q2)*T[1] + ((q1+q2)/(q1*q2))
sage: H = IwahoriHeckeAlgebra("A2", q1, base_ring=F).T()
sage: H.inverse_generator(2)
-(1/(-q1))*T[2] + ((q1-1)/(-q1))
sage: P1.<r1, r2> = LaurentPolynomialRing(QQ)
sage: H1 = IwahoriHeckeAlgebra("B2", r1, q2=r2, base_ring=P1).T()
sage: H1.base_ring()
Multivariate Laurent Polynomial Ring in r1, r2 over Rational Field
sage: H1.inverse_generator(2)
(-r1^-1*r2^-1)*T[2] + (r2^-1+r1^-1)
sage: H2 = IwahoriHeckeAlgebra("C2", r1, base_ring=P1).T()
sage: H2.inverse_generator(2)
(r1^-1)*T[2] + (-1+r1^-1)
```

inverse_generators()

Return the inverses of all the generators, if they exist.

This method is only available if `q1` and `q2` are invertible. In that case, the algebra generators are also invertible.

EXAMPLES:

```
sage: P.<q> = PolynomialRing(QQ)
sage: F = Frac(P)
sage: H = IwahoriHeckeAlgebra("A2", q, base_ring=F).T()
sage: T1,T2 = H.algebra_generators()
sage: U1,U2 = H.inverse_generators()
sage: U1*T1,T1*U1
(1, 1)
sage: P1.<q> = LaurentPolynomialRing(QQ)
sage: H1 = IwahoriHeckeAlgebra("A2", q, base_ring=P1).T(prefix="V")
sage: V1,V2 = H1.algebra_generators()
sage: W1,W2 = H1.inverse_generators()
sage: [W1,W2]
[(q^-1)*V[1] + (q^-1-1), (q^-1)*V[2] + (q^-1-1)]
sage: V1*W1, W2*V2
(1, 1)
```

product_by_generator(*x*, *i*, *side='right'*)

Return $T_i \cdot x$, where T_i is the i-th generator. This is coded individually for use in `x._mul_()`.

EXAMPLES:

```
sage: R.<q> = QQ[]; H = IwahoriHeckeAlgebra("A2", q).T()
sage: T1, T2 = H.algebra_generators()
sage: [H.product_by_generator(x, 1) for x in [T1,T2]]
[(q-1)*T[1] + q, T[2,1]]
sage: [H.product_by_generator(x, 1, side = "left") for x in [T1,T2]]
```

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```
[(q-1)*T[1] + q, T[1,2]]
```

product_by_generator_on_basis(*w*, *i*, *side='right'*)

Return the product $T_w T_i$ (resp. $T_i T_w$) if `side` is `'right'` (resp. `'left'`).

If the quadratic relation is $(T_i - u)(T_i - v) = 0$, then we have

$$T_w T_i = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) = \ell(w) + 1, \\ (u+v)T_{ws_i} - uvT_w & \text{if } \ell(ws_i) = \ell(w) - 1. \end{cases}$$

The left action is similar.

INPUT:

- `w` – an element of the Coxeter group
- `i` – an element of the index set
- `side` – `'right'` (default) or `'left'`

EXAMPLES:

```
sage: R.<q> = QQ[]; H = IwahoriHeckeAlgebra("A2", q)
sage: T = H.T()
sage: s1,s2 = H.coxeter_group().simple_reflections()
sage: [T.product_by_generator_on_basis(w, 1) for w in [s1,s2,s1*s2]]
[(q-1)*T[1] + q, T[2,1], T[1,2,1]]
sage: [T.product_by_generator_on_basis(w, 1, side="left") for w in [s1,s2,
↪s1*s2]]
[(q-1)*T[1] + q, T[1,2], (q-1)*T[1,2] + q*T[2]]
```

product_on_basis(*w1*, *w2*)

Return $T_{w_1} T_{w_2}$, where w_1 and w_2 are words in the Coxeter group.

EXAMPLES:

```
sage: R.<q> = QQ[]; H = IwahoriHeckeAlgebra("A2", q)
sage: T = H.T()
sage: s1,s2 = H.coxeter_group().simple_reflections()
sage: [T.product_on_basis(s1,x) for x in [s1,s2]]
[(q-1)*T[1] + q, T[1,2]]
```

to_C_basis(*w*)

Return T_w as a linear combination of C-basis elements.

EXAMPLES:

```
sage: R = LaurentPolynomialRing(QQ, 'v')
sage: v = R.gen(0)
sage: H = IwahoriHeckeAlgebra('A2', v**2)
sage: s1,s2 = H.coxeter_group().simple_reflections()
sage: T = H.T()
sage: C = H.C()
sage: T.to_C_basis(s1)
v*T[1] + v^2
sage: C(T(s1))
v*C[1] + v^2
sage: C(v^-1*T(s1) - v)
```

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```
C[1]
sage: C(T(s1*s2)+T(s1)+T(s2)+1)
v^2*C[1,2] + (v+v^3)*C[1] + (v+v^3)*C[2] + (1+2*v^2+v^4)
sage: C(T(s1*s2*s1))
v^3*C[1,2,1] + v^4*C[2,1] + v^4*C[1,2] + v^5*C[1] + v^5*C[2] + v^6
```

to_Cp_basis(w)

Return T_w as a linear combination of C'-basis elements.

EXAMPLES:

```
sage: R.<v> = LaurentPolynomialRing(QQ)
sage: H = IwahoriHeckeAlgebra('A2', v**2)
sage: s1,s2 = H.coxeter_group().simple_reflections()
sage: T = H.T()
sage: Cp = H.Cp()
sage: T.to_Cp_basis(s1)
v*Cp[1] - 1
sage: Cp(T(s1))
v*Cp[1] - 1
sage: Cp(T(s1)+1)
v*Cp[1]
sage: Cp(T(s1*s2)+T(s1)+T(s2)+1)
v^2*Cp[1,2]
sage: Cp(T(s1*s2*s1))
v^3*Cp[1,2,1] - v^2*Cp[2,1] - v^2*Cp[1,2] + v*Cp[1] + v*Cp[2] - 1
```

a_realization()

Return a particular realization of `self` (the T-basis).

EXAMPLES:

```
sage: H = IwahoriHeckeAlgebra("B2", 1)
sage: H.a_realization()
Iwahori-Hecke algebra of type B2 in 1,-1 over Integer Ring in the T-basis
```

cartan_type()

Return the Cartan type of `self`.

EXAMPLES:

```
sage: IwahoriHeckeAlgebra("D4", 1).cartan_type()
['D', 4]
```

coxeter_group()

Return the Coxeter group of `self`.

EXAMPLES:

```
sage: IwahoriHeckeAlgebra("B2", 1).coxeter_group()
Finite Coxeter group over Number Field in a with defining polynomial x^2 - 2␣
↪with a = 1.414213562373095? with Coxeter matrix:
[1 4]
[4 1]
```

coxeter_type()

Return the Coxeter type of `self`.

EXAMPLES:

```
sage: IwahoriHeckeAlgebra("D4", 1).coxeter_type()
Coxeter type of ['D', 4]
```

q1()

Return the parameter q_1 of `self`.

EXAMPLES:

```
sage: H = IwahoriHeckeAlgebra("B2", 1)
sage: H.q1()
1
```

q2()

Return the parameter q_2 of `self`.

EXAMPLES:

```
sage: H = IwahoriHeckeAlgebra("B2", 1)
sage: H.q2()
-1
```

standard

alias of *T*

class sage.algebras.iwahori_hecke_algebra.**IwahoriHeckeAlgebra_nonstandard**(*W*)

Bases: *IwahoriHeckeAlgebra*

This is a class which is used behind the scenes by *IwahoriHeckeAlgebra* to compute the Kazhdan-Lusztig bases. It is not meant to be used directly. It implements the slightly idiosyncratic (but convenient) Iwahori-Hecke algebra with two parameters which is defined over the Laurent polynomial ring $\mathbf{Z}[u, u^{-1}, v, v^{-1}]$ in two variables and has quadratic relations:

$$(T_r - u)(T_r + v^2/u) = 0.$$

The point of these relations is that the product of the two parameters is v^2 which is a square in $\mathbf{Z}[u, u^{-1}, v, v^{-1}]$. Consequently, the Kazhdan-Lusztig bases are defined for this algebra.

More generally, if we have a Iwahori-Hecke algebra with two parameters which has quadratic relations of the form:

$$(T_r - q_1)(T_r - q_2) = 0$$

where $-q_1q_2$ is a square then the Kazhdan-Lusztig bases are well-defined for this algebra. Moreover, these bases be computed by specialization from the generic Iwahori-Hecke algebra using the specialization which sends $u \mapsto q_1$ and $v \mapsto \sqrt{-q_1q_2}$, so that $v^2/u \mapsto -q_2$.

For example, if $q_1 = q = Q^2$ and $q_2 = -1$ then $u \mapsto q$ and $v \mapsto \sqrt{q} = Q$; this is the standard presentation of the Iwahori-Hecke algebra with $(T_r - q)(T_r + 1) = 0$. On the other hand, when $q_1 = q$ and $q_2 = -q^{-1}$ then $u \mapsto q$ and $v \mapsto 1$. This is the normalized presentation with $(T_r - v)(T_r + v^{-1}) = 0$.

Warning: This class uses non-standard parameters for the Iwahori-Hecke algebra and are related to the standard parameters by an outer automorphism that is non-trivial on the T-basis.

class C(*IHAlgebra*, *prefix=None*)

Bases: C

The Kazhdan-Lusztig C-basis for the generic Iwahori-Hecke algebra.

to_T_basis(*w*)

Return C_w as a linear combination of T-basis elements.

EXAMPLES:

```
sage: H = sage.algebras.iwahori_hecke_algebra.IwahoriHeckeAlgebra_
↪nonstandard("A3")
sage: s1,s2,s3 = H.coxeter_group().simple_reflections()
sage: T = H.T()
sage: C = H.C()
sage: C.to_T_basis(s1)
(v^-1)*T[1] + (-u*v^-1)
sage: C.to_T_basis(s1*s2)
(v^-2)*T[1,2] + (-u*v^-2)*T[1] + (-u*v^-2)*T[2] + (u^2*v^-2)
sage: C.to_T_basis(s1*s2*s1)
(v^-3)*T[1,2,1] + (-u*v^-3)*T[2,1] + (-u*v^-3)*T[1,2]
 + (u^2*v^-3)*T[1] + (u^2*v^-3)*T[2] + (-u^3*v^-3)
sage: T(C(s1*s2*s1))
(v^-3)*T[1,2,1] + (-u*v^-3)*T[2,1] + (-u*v^-3)*T[1,2]
 + (u^2*v^-3)*T[1] + (u^2*v^-3)*T[2] + (-u^3*v^-3)
sage: T(C(s2*s1*s3*s2))
(v^-4)*T[2,3,1,2] + (-u*v^-4)*T[2,3,1] + (-u*v^-4)*T[1,2,1]
 + (-u*v^-4)*T[3,1,2] + (-u*v^-4)*T[2,3,2] + (u^2*v^-4)*T[2,1]
 + (u^2*v^-4)*T[3,1] + (u^2*v^-4)*T[1,2] + (u^2*v^-4)*T[3,2]
 + (u^2*v^-4)*T[2,3] + (-u^3*v^-4)*T[1] + (-u^3*v^-4)*T[3]
 + (-u^3*v^-4-u*v^-2)*T[2] + (u^4*v^-4+u^2*v^-2)
```

C_prime

alias of Cp

class Cp(*IHAlgebra*, *prefix=None*)

Bases: Cp

The Kazhdan-Lusztig C'-basis for the generic Iwahori-Hecke algebra.

to_T_basis(*w*)

Return C'_w as a linear combination of T-basis elements.

EXAMPLES:

```
sage: H = sage.algebras.iwahori_hecke_algebra.IwahoriHeckeAlgebra_
↪nonstandard("A3")
sage: s1,s2,s3 = H.coxeter_group().simple_reflections()
sage: T = H.T()
sage: Cp = H.Cp()
sage: Cp.to_T_basis(s1)
(v^-1)*T[1] + (u^-1*v)
```

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```
sage: Cp.to_T_basis(s1*s2)
(v^-2)*T[1,2] + (u^-1)*T[1] + (u^-1)*T[2] + (u^-2*v^2)
sage: Cp.to_T_basis(s1*s2*s1)
(v^-3)*T[1,2,1] + (u^-1*v^-1)*T[2,1] + (u^-1*v^-1)*T[1,2]
 + (u^-2*v)*T[1] + (u^-2*v)*T[2] + (u^-3*v^3)
sage: T(Cp(s1*s2*s1))
(v^-3)*T[1,2,1] + (u^-1*v^-1)*T[2,1] + (u^-1*v^-1)*T[1,2]
 + (u^-2*v)*T[1] + (u^-2*v)*T[2] + (u^-3*v^3)
sage: T(Cp(s2*s1*s3*s2))
(v^-4)*T[2,3,1,2] + (u^-1*v^-2)*T[2,3,1] + (u^-1*v^-2)*T[1,2,1]
 + (u^-1*v^-2)*T[3,1,2] + (u^-1*v^-2)*T[2,3,2] + (u^-2)*T[2,1]
 + (u^-2)*T[3,1] + (u^-2)*T[1,2] + (u^-2)*T[3,2]
 + (u^-2)*T[2,3] + (u^-3*v^2)*T[1] + (u^-3*v^2)*T[3]
 + (u^-1+u^-3*v^2)*T[2] + (u^-2*v^2+u^-4*v^4)
```

class T(*algebra*, *prefix=None*)

Bases: *T*

The T-basis for the generic Iwahori-Hecke algebra.

to_C_basis(*w*)

Return T_w as a linear combination of C-basis elements.

To compute this we piggy back off the C'-basis conversion using the observation that the hash involution sends T_w to $(-q_1 q_1)^{\ell(w)} T_w$ and C_w to $(-1)^{\ell(w)} C'_w$. Therefore, if

$$T_w = \sum_v a_{vw} C'_v$$

then

$$T_w = (-q_1 q_2)^{\ell(w)} \Big(\sum_v a_{vw} C'_v \Big)^{\#} = \sum_v (-1)^{\ell(v)} \overline{a_{vw}} C_v$$

Note that we cannot just apply `hash_involution()` here because this involution always returns the answer with respect to the same basis.

EXAMPLES:

```
sage: H = sage.algebras.iwahori_hecke_algebra.IwahoriHeckeAlgebra_
↪nonstandard("A2")
sage: s1,s2 = H.coxeter_group().simple_reflections()
sage: T = H.T()
sage: C = H.C()
sage: T.to_C_basis(s1)
v*T[1] + u
sage: C(T(s1))
v*C[1] + u
sage: C(T( C[1] ))
C[1]
sage: C(T(s1*s2)+T(s1)+T(s2)+1)
v^2*C[1,2] + (u*v+v)*C[1] + (u*v+v)*C[2] + (u^2+2*u+1)
sage: C(T(s1*s2*s1))
v^3*C[1,2,1] + u*v^2*C[2,1] + u*v^2*C[1,2] + u^2*v*C[1] + u^2*v*C[2] + u^3
```

to_Cp_basis(w)

Return T_w as a linear combination of C'-basis elements.

EXAMPLES:

```
sage: H = sage.algebras.iwahori_hecke_algebra.IwahoriHeckeAlgebra_
↪nonstandard("A2")
sage: s1,s2 = H.coxeter_group().simple_reflections()
sage: T = H.T()
sage: Cp = H.Cp()
sage: T.to_Cp_basis(s1)
v*Cp[1] + (-u^-1*v^2)
sage: Cp(T(s1))
v*Cp[1] + (-u^-1*v^2)
sage: Cp(T(s1)+1)
v*Cp[1] + (-u^-1*v^2+1)
sage: Cp(T(s1*s2)+T(s1)+T(s2)+1)
v^2*Cp[1,2] + (-u^-1*v^3+v)*Cp[1] + (-u^-1*v^3+v)*Cp[2]
 + (u^-2*v^4-2*u^-1*v^2+1)
sage: Cp(T(s1*s2*s1))
v^3*Cp[1,2,1] + (-u^-1*v^4)*Cp[2,1] + (-u^-1*v^4)*Cp[1,2]
 + (u^-2*v^5)*Cp[1] + (u^-2*v^5)*Cp[2] + (-u^-3*v^6)
```

sage.algebras.iwahori_hecke_algebra.**index_cmp**(x, y)

Compare two term indices x and y by Bruhat order, then by word length, and then by the generic comparison.

EXAMPLES:

```
sage: from sage.algebras.iwahori_hecke_algebra import index_cmp
sage: W = WeylGroup(['A',2,1])
sage: x = W.from_reduced_word([0,1])
sage: y = W.from_reduced_word([0,2,1])
sage: x.bruhat_le(y)
True
sage: index_cmp(x, y)
1
```

sage.algebras.iwahori_hecke_algebra.**normalized_laurent_polynomial**(R, p)

Return a normalized version of the (Laurent polynomial) p in the ring R.

Various ring operations in sage return an element of the field of fractions of the parent ring even though the element is "known" to belong to the base ring. This function is a hack to recover from this. This occurs somewhat haphazardly with Laurent polynomial rings:

```
sage: R.<q>=LaurentPolynomialRing(ZZ)
sage: [type(c) for c in (q**-1).coefficients()]
[<class 'sage.rings.integer.Integer'>]
```

It also happens in any ring when dividing by units:

```
sage: type ( 3/1 )
<class 'sage.rings.rational.Rational'>
sage: type ( -1/-1 )
<class 'sage.rings.rational.Rational'>
```

This function is a variation on a suggested workaround of Nils Bruin.

EXAMPLES:

```
sage: from sage.algebras.iwahori_hecke_algebra import normalized_laurent_polynomial
sage: type ( normalized_laurent_polynomial(ZZ, 3/1) )
<class 'sage.rings.integer.Integer'>
sage: R.<q>=LaurentPolynomialRing(ZZ)
sage: [type(c) for c in normalized_laurent_polynomial(R, q**-1).coefficients()]
[<class 'sage.rings.integer.Integer'>]
sage: R.<u,v>=LaurentPolynomialRing(ZZ,2)
sage: p=normalized_laurent_polynomial(R, 2*u**-1*v**-1+u*v)
sage: ui=normalized_laurent_polynomial(R, u^-1)
sage: vi=normalized_laurent_polynomial(R, v^-1)
sage: p(ui,vi)
2*u*v + u^-1*v^-1
sage: q= u+v+ui
sage: q(ui,vi)
u + v^-1 + u^-1
```

6.3 Nil-Coxeter Algebra

class sage.algebras.nil_coxeter_algebra.**NilCoxeterAlgebra**(*W*, *base_ring=Rational Field*, *prefix='u'*)

Bases: *T*

Construct the Nil-Coxeter algebra of given type.

This is the algebra with generators u_i for every node i of the corresponding Dynkin diagram. It has the usual braid relations (from the Weyl group) as well as the quadratic relation $u_i^2 = 0$.

INPUT:

- W – a Weyl group

OPTIONAL ARGUMENTS:

- base_ring – a ring (default is the rational numbers)
- prefix – a label for the generators (default "u")

EXAMPLES:

```
sage: U = NilCoxeterAlgebra(WeylGroup(['A',3,1]))
sage: u0, u1, u2, u3 = U.algebra_generators()
sage: u1*u1
0
sage: u2*u1*u2 == u1*u2*u1
True
sage: U.an_element()
u[0,1,2,3] + 2*u[0] + 3*u[1] + 1
```

homogeneous_generator_noncommutative_variables(*r*)

Give the r^{th} homogeneous function inside the Nil-Coxeter algebra. In finite type A this is the sum of all decreasing elements of length r. In affine type A this is the sum of all cyclically decreasing elements of length r. This is only defined in finite type A, B and affine types $A^{(1)}$, $B^{(1)}$, $C^{(1)}$, $D^{(1)}$.

INPUT:

- `r` – a positive integer at most the rank of the Weyl group

EXAMPLES:

```
sage: U = NilCoxeterAlgebra(WeylGroup(['A',3,1]))
sage: U.homogeneous_generator_noncommutative_variables(2)
u[1,0] + u[2,0] + u[0,3] + u[3,2] + u[3,1] + u[2,1]

sage: U = NilCoxeterAlgebra(WeylGroup(['B',4]))
sage: U.homogeneous_generator_noncommutative_variables(2)
u[1,2] + u[2,1] + u[3,1] + u[4,1] + u[2,3] + u[3,2] + u[4,2] + u[3,4] + u[4,3]

sage: U = NilCoxeterAlgebra(WeylGroup(['C',3]))
sage: U.homogeneous_generator_noncommutative_variables(2)
Traceback (most recent call last):
...
AssertionError: Analogue of symmetric functions in noncommutative variables is␣
↪not defined in type ['C', 3]
```

homogeneous_noncommutative_variables(*la*)

Give the homogeneous function indexed by la, viewed inside the Nil-Coxeter algebra. This is only defined in finite type A, B and affine types $A^{(1)}$, $B^{(1)}$, $C^{(1)}$, $D^{(1)}$.

INPUT:

- `la` – a partition with first part bounded by the rank of the Weyl group

EXAMPLES:

```
sage: U = NilCoxeterAlgebra(WeylGroup(['B',2,1]))
sage: U.homogeneous_noncommutative_variables([2,1])
u[1,2,0] + 2*u[2,1,0] + u[0,2,0] + u[0,2,1] + u[1,2,1] + u[2,1,2] + u[2,0,2] +␣
↪u[1,0,2]
```

k_schur_noncommutative_variables(*la*)

In type $A^{(1)}$ this is the k-Schur function in noncommutative variables defined by Thomas Lam [Lam2005].

This function is currently only defined in type $A^{(1)}$.

INPUT:

- `la` – a partition with first part bounded by the rank of the Weyl group

EXAMPLES:

```
sage: A = NilCoxeterAlgebra(WeylGroup(['A',3,1]))
sage: A.k_schur_noncommutative_variables([2,2])
u[0,3,1,0] + u[3,1,2,0] + u[1,2,0,1] + u[3,2,0,3] + u[2,0,3,1] + u[2,3,1,2]
```

6.4 Yokonuma-Hecke Algebras

AUTHORS:

- Travis Scrimshaw (2015-11): initial version

class sage.algebras.yokonuma_hecke_algebra.**YokonumaHeckeAlgebra**(*d*, *n*, *q*, *R*)

Bases: CombinatorialFreeModule

The Yokonuma-Hecke algebra $Y_{d,n}(q)$.

Let R be a commutative ring and q be a unit in R. The *Yokonuma-Hecke algebra* $Y_{d,n}(q)$ is the associative, unital R-algebra generated by $t_1, t_2, \ldots, t_n, g_1, g_2, \ldots, g_{n-1}$ and subject to the relations:

- $g_i g_j = g_j g_i$ for all $|i - j| > 1$,
- $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,
- $t_i t_j = t_j t_i$,
- $t_j g_i = g_i t_{j s_i}$, and
- $t_j^d = 1$,

where s_i is the simple transposition $(i, i+1)$, along with the quadratic relation

$$g_i^2 = 1 + \frac{(q - q^{-1})}{d} \left(\sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s} \right) g_i.$$

Thus the Yokonuma-Hecke algebra can be considered a quotient of the framed braid group $(\mathbf{Z}/d\mathbf{Z}) \wr B_n$, where B_n is the classical braid group on n strands, by the quadratic relations. Moreover, all of the algebra generators are invertible. In particular, we have

$$g_i^{-1} = g_i - (q - q^{-1}) e_i.$$

When we specialize $q = \pm 1$, we obtain the group algebra of the complex reflection group $G(d, 1, n) = (\mathbf{Z}/d\mathbf{Z}) \wr S_n$. Moreover for $d = 1$, the Yokonuma-Hecke algebra is equal to the *Iwahori-Hecke* of type A_{n-1}.

INPUT:

- d – the maximum power of t
- n – the number of generators
- q – (optional) an invertible element in a commutative ring; the default is $q \in \mathbf{Q}[q, q^{-1}]$
- R – (optional) a commutative ring containing q; the default is the parent of q

EXAMPLES:

We construct $Y_{4,3}$ and do some computations:

```
sage: Y = algebras.YokonumaHecke(4, 3)
sage: g1, g2, t1, t2, t3 = Y.algebra_generators()
sage: g1 * g2
g[1,2]
sage: t1 * g1
t1*g[1]
sage: g2 * t2
t3*g[2]
sage: g2 * t3
```

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```
t2*g[2]
sage: (g2 + t1) * (g1 + t2*t3)
g[2,1] + t2*t3*g[2] + t1*g[1] + t1*t2*t3
sage: g1 * g1
1 - (1/4*q^-1-1/4*q)*g[1] - (1/4*q^-1-1/4*q)*t1*t2^3*g[1]
 - (1/4*q^-1-1/4*q)*t1^2*t2^2*g[1] - (1/4*q^-1-1/4*q)*t1^3*t2*g[1]
sage: g2 * g1 * t1
t3*g[2,1]
```

We construct the elements e_i and show that they are idempotents:

```
sage: e1 = Y.e(1); e1
1/4 + 1/4*t1*t2^3 + 1/4*t1^2*t2^2 + 1/4*t1^3*t2
sage: e1 * e1 == e1
True
sage: e2 = Y.e(2); e2
1/4 + 1/4*t2*t3^3 + 1/4*t2^2*t3^2 + 1/4*t2^3*t3
sage: e2 * e2 == e2
True
```

REFERENCES:

- [CL2013]
- [CPdA2014]
- [ERH2015]
- [JPdA15]

class Element

Bases: `IndexedFreeModuleElement`

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(5, 3)
sage: dict(Y.algebra_generators())
{'g1': g[1], 'g2': g[2], 't1': t1, 't2': t2, 't3': t3}
```

e(*i*)

Return the element e_i.

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(4, 3)
sage: Y.e(1)
1/4 + 1/4*t1*t2^3 + 1/4*t1^2*t2^2 + 1/4*t1^3*t2
sage: Y.e(2)
1/4 + 1/4*t2*t3^3 + 1/4*t2^2*t3^2 + 1/4*t2^3*t3
```

g(*i=None*)

Return the generator(s) g_i.

INPUT:

- `i` – (default: `None`) the generator g_i or if `None`, then the list of all generators g_i

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(8, 3)
sage: Y.g(1)
g[1]
sage: Y.g()
[g[1], g[2]]
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(5, 3)
sage: Y.gens()
(g[1], g[2], t1, t2, t3)
```

inverse_g(*i*)

Return the inverse of the generator g_i.

From the quadratic relation, we have

$$g_i^{-1} = g_i - (q - q^{-1})e_i.$$

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(2, 4)
sage: [2*Y.inverse_g(i) for i in range(1, 4)]
[(q^-1+q) + 2*g[1] + (q^-1+q)*t1*t2,
 (q^-1+q) + 2*g[2] + (q^-1+q)*t2*t3,
 (q^-1+q) + 2*g[3] + (q^-1+q)*t3*t4]
```

one_basis()

Return the index of the basis element of 1.

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(5, 3)
sage: Y.one_basis()
((0, 0, 0), [1, 2, 3])
```

product_on_basis(*m1*, *m2*)

Return the product of the basis elements indexed by `m1` and `m2`.

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(4, 3)
sage: m = ((1, 0, 2), Permutations(3)([2,1,3]))
sage: 4 * Y.product_on_basis(m, m)
-(q^-1-q)*t2^2*g[1] + 4*t1*t2 - (q^-1-q)*t1*t2*g[1]
 - (q^-1-q)*t1^2*g[1] - (q^-1-q)*t1^3*t2^3*g[1]
```

Check that we apply the permutation correctly on t_i:

```
sage: Y = algebras.YokonumaHecke(4, 3)
sage: g1, g2, t1, t2, t3 = Y.algebra_generators()
sage: g21 = g2 * g1
sage: g21 * t1
t3*g[2,1]
```

t(*i=None*)

Return the generator(s) t_i.

INPUT:

- i – (default: None) the generator t_i or if None, then the list of all generators t_i

EXAMPLES:

```
sage: Y = algebras.YokonumaHecke(8, 3)
sage: Y.t(2)
t2
sage: Y.t()
[t1, t2, t3]
```

6.5 Cubic Hecke Algebras

6.5.1 Cubic Hecke Algebras

We consider the factors of the group algebra of the Artin braid groups such that the images s_i of the braid generators satisfy a cubic equation:

$$s_i^3 = us_i^2 - vs_i + w.$$

Here u, v, w are elements in an arbitrary integral domain and i is a positive integer less than n, the number of the braid group's strands. By the analogue to the *Iwahori Hecke algebras* (see IwahoriHeckeAlgebra), in which the braid generators satisfy a quadratic relation these algebras have been called *cubic Hecke algebras*. The relations inherited from the braid group are:

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \leq i < n-1 \text{ and } s_i s_j = s_j s_i \text{ for } 1 \leq i < j-1 < n-1.$$

The algebra epimorphism from the braid group algebra over the same base ring is realized inside the element constructor of the present class, for example in the case of the 3 strand cubic Hecke algebra:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: BG3 = CHA3.braid_group()
sage: braid = BG3((1,2,-1,2,2,-1)); braid
c0*c1*c0^-1*c1^2*c0^-1
sage: braid_image = CHA3(braid); braid_image
u*c1*c0^-1*c1 + u*v*c0*c1^-1*c0^-1 + (-u^2)*c0^-1*c1
+ ((u^2*v-v^2)/w)*c0*c1*c0^-1 + ((u^2-v)/w)*c0*c1*c0
+ ((-u^3+u*v)/w)*c0*c1 + (-u*v+w)*c1^-1
```

If the ring elements u, v, w (which will be called the *cubic equation parameters* in the sequel) are taken to be $u = v = 0, w = 1$ the cubic Hecke algebra specializes to the group algebra of the *cubic braid group*, which is the factor group of the Artin braid group under setting the generators order to be three. These groups can be obtained by *CubicHeckeAlgebra.cubic_braid_group()*.

It is well known, that these algebras are free of finite rank as long as the number of braid generators is less than six and infinite dimensional else wise. In the former (non trivial) cases they are also known as *cyclotomic Hecke algebras* corresponding to the complex reflection groups having Shepard-Todd number 4, 25 and 32.

Since the *Broué, Malle, Rouquiere* conjecture has been proved (for references of these cases see [Mar2012]) there exists a finite free basis of the cubic Hecke algebra which is in bijection to the cubic braid group and compatible with the specialization to the cubic braid group algebra as explained above.

For the algebras corresponding to braid groups of less than five strands such a basis has been calculated by Ivan Marin. This one is used here. In the case of 5 strands such a basis is not available, right now. Instead the elements of the cubic braid group class themselves are used as basis elements. This is also the case when the cubic braid group is infinite, even though it is not known if these elements span all of the cubic Hecke algebra.

Accordingly, be aware that the module embedding of the group algebra of the cubicbraid groups is known to be an isomorphism of free modules only in the cases of less than five strands.

EXAMPLES:

Consider the obstruction `b` of the *triple quadratic algebra* from Section 2.6 of [Mar2018]. We verify that the third power of it is a scalar multiple of itself (explicitly `2*w^2` times the *Schur element* of the three dimensional irreducible representation):

```
sage: CHA3 = algebras.CubicHecke(3)
sage: c1, c2 = CHA3.gens()
sage: b = c1^2*c2 - c2*c1^2 - c1*c2^2 + c2^2*c1; b
w*c0^-1*c1 + (-w)*c0*c1^-1 + (-w)*c1*c0^-1 + w*c1^-1*c0
sage: b2 = b*b
sage: b3 = b2*b
sage: BR = CHA3.base_ring()
sage: ER = CHA3.extension_ring()
sage: u, v, w = BR.gens()
sage: f =  BR(b3.coefficients()[0]/w)
sage: try:
....:     sh = CHA3.schur_element(CHA3.irred_repr.W3_111)
....: except NotImplementedError:    # for the case GAP3 / CHEVIE not available
....:     sh = ER(f/(2*w^2))
sage: ER(f/(2*w^2)) == sh
True
sage: b3 == f*b
True
```

Defining the cubic Hecke algebra on 6 strands will need some seconds for initializing. However, you can do calculations inside the infinite algebra as well:

```
sage: CHA6 = algebras.CubicHecke(6)  # optional - database_cubic_hecke
sage: CHA6.inject_variables()        # optional - database_cubic_hecke
Defining c0, c1, c2, c3, c4
sage: s = c0*c1*c2*c3*c4; s          # optional - database_cubic_hecke
c0*c1*c2*c3*c4
sage: s^2                            # optional - database_cubic_hecke
(c0*c1*c2*c3*c4)^2
sage: t = CHA6.an_element() * c4; t  # optional - database_cubic_hecke
(-w)*c0*c1^-1*c4 + v*c0*c2^-1*c4 + u*c2*c1*c4 + ((-v*w+u)/w)*c4
```

REFERENCES:

- [Mar2012]

- [Mar2018]
- [CM2012]

AUTHORS:

- Sebastian Oehms May 2020: initial version

class sage.algebras.hecke_algebras.cubic_hecke_algebra.**CubicHeckeAlgebra**(*names*, *cubic_equation_parameters=None*, *cubic_equation_roots=None*)

Bases: CombinatorialFreeModule

Return the Cubic-Hecke algebra with respect to the Artin braid group on n strands.

This is a quotient of the group algebra of the Artin braid group, such that the images s_i $(1 \leq i < n)$ of the braid generators satisfy a cubic equation (see *cubic_hecke_algebra* for more information, in a session type sage.algebras.hecke_algebras.cubic_hecke_algebra?):

$$s_i^3 = us_i^2 - vs_i + w.$$

The base ring of this algebra can be specified by giving optional keywords described below. If no keywords are given, the base ring will be a *CubicHeckeRingOfDefinition*, which is constructed as the polynomial ring in u, v, w over the integers localized at w. This ring will be called the *ring of definition* or sometimes for short *generic base ring*. However note, that in this context the word *generic* should not remind in a generic point of the corresponding scheme.

In addition to the base ring, another ring containing the roots (a, b and c) of the cubic equation will be needed to handle the split irreducible representations. This ring will be called the *extension ring*. Generically, the extension ring will be a *CubicHeckeExtensionRing*, which is constructed as the Laurent polynomial ring in a, b and c over the integers adjoined with a primitive third root of unity. A special form of this *generic extension ring* is constructed as a *SplittingAlgebra* for the roots of the cubic equation and a primitive third root of unity over the ring of definition. This ring will be called the *default extension ring*.

This class uses a static and a dynamic data library. The first one is defined as instance of CubicHeckeDataBase and contains the complete basis for the algebras with less than 5 strands and various types of representation matrices of the generators. These data have been calculated by Ivan Marin and have been imported from his corresponding web page.

Note that just the data for the cubic Hecke algebras on less than four strands is available in Sage by default. To deal with four strands and more you need to install the optional package database_cubic_hecke by typing

- sage -i database_cubic_hecke (first time installation) or
- sage -f database_cubic_hecke (reinstallation) respective
- sage -i -c database_cubic_hecke (for running all test in concern)
- sage -f -c database_cubic_hecke

This will add a Python wrapper around Ivan Marin's data to the Sage library. For more installation hints see the documentation of this wrapper.

Furthermore, representation matrices can be obtained from the CHEVIE package of GAP3 via the GAP3 interface if GAP3 is installed inside Sage. For more information on how to obtain representation matrices to elements of this class, see the documentation of the element class *CubicHeckeElement* or its method *matrix()*:

> algebras.CubicHecke.Element? or algebras.CubicHecke.Element.matrix?

The second library is created as instance of CubicHeckeFileCache and used while working with the class to achieve a better performance. This file cache contains images of braids and representation matrices of basis elements from former calculations. A refresh of the file cache can be done using the *reset_filecache()*.

INPUT:

- `names` – string containing the names of the generators as images of the braid group generators
- `cubic_equation_parameters` – tuple `(u, v, w)` of three elements in an integral domain used as coefficients in the cubic equation. If this argument is given the base ring will be set to the common parent of `u, v, w`. In addition a conversion map from the generic base ring is supplied. This keyword can also be used to change the variable names of the generic base ring (see example 3 below)
- `cubic_equation_roots` – tuple `(a, b, c)` of three elements in an integral domain which stand for the roots of the cubic equation. If this argument is given the extension ring will be set to the common parent of `a, b, c`. In addition a conversion map from the generic extension ring and the generic base ring is supplied. This keyword can also be used to change the variable names of the generic extension ring (see example 3 below)

EXAMPLES:

Cubic Hecke algebra over the ring of definition:

```
sage: CHA3 = algebras.CubicHecke('s1, s2'); CHA3
Cubic Hecke algebra on 3 strands over Multivariate Polynomial Ring
  in u, v, w
  over Integer Ring localized at (w,)
  with cubic equation: h^3 - u*h^2 + v*h - w = 0
sage: CHA3.gens()
(s1, s2)
sage: GER = CHA3.extension_ring(generic=True); GER
Multivariate Laurent Polynomial Ring in a, b, c
  over Splitting Algebra of x^2 + x + 1
    with roots [e3, -e3 - 1] over Integer Ring
sage: ER = CHA3.extension_ring(); ER
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
  over Splitting Algebra of h^3 - u*h^2 + v*h - w
    with roots [a, b, -b - a + u]
  over Multivariate Polynomial Ring in u, v, w
  over Integer Ring localized at (w,)
```

Element construction:

```
sage: ele = CHA3.an_element(); ele
(-w)*s1*s2^-1 + v*s1 + u*s2 + ((-v*w+u)/w)
sage: ele2 = ele**2; ele2
w^2*(s1^-1*s2)^2 + (-u*w^2)*s1^-1*s2*s1^-1 + (-v*w)*s2*s1^-1*s2
+ (-v*w^2)*s1^-1*s2^-1 + u*w*s1*s2*s1^-1*s2 + (-u*w)*s1^-1*s2*s1
+ (-u*v*w+2*v*w-2*u)*s1*s2^-1 + u*v*w*s2*s1^-1 + u*v*s2*s1 + v^2*w*s1^-1
+ (-u^2*w)*s1*s2*s1^-1 + ((u*v^2*w-2*v^2*w-u*w^2+2*u*v)/w)*s1
+ u*v*s1*s2 + (u^2*w+v^2*w)*s2^-1 + ((u^3*w-2*u*v*w+2*u^2)/w)*s2
+ ((-u^2*v*w^2-v^3*w^2+v^2*w^2-2*u*v*w+u^2)/w^2)
sage: B3 = CHA3.braid_group()
sage: braid = B3((2,-1, 2, 1)); braid
s2*s1^-1*s2*s1
sage: ele3 = CHA3(braid); ele3
s1*s2*s1^-1*s2 + u*s1^-1*s2*s1 + (-v)*s1*s2^-1 + v*s2^-1*s1 + (-u)*s1*s2*s1^-1
sage: ele3t = CHA3((2,-1, 2, 1))
sage: ele3 == ele3t
True
```

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```
sage: CHA4 = algebras.CubicHecke(4)      # optional database_cubic_hecke
sage: ele4 = CHA4(ele3); ele4            # optional database_cubic_hecke
c0*c1*c0^-1*c1 + u*c0^-1*c1*c0 + (-v)*c0*c1^-1 + v*c1^-1*c0 + (-u)*c0*c1*c0^-1
```

Cubic Hecke algebra over the ring of definition using different variable names:

```
sage: algebras.CubicHecke(3, cubic_equation_parameters='u, v, w', cubic_equation_
→roots='p, q, r')
Cubic Hecke algebra on 3 strands over Multivariate Polynomial Ring
  in u, v, w
  over Integer Ring localized at (w,)
    with cubic equation: h^3 - u*h^2 + v*h - w = 0
sage: _.extension_ring()
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
  over Splitting Algebra of h^3 - u*h^2 + v*h - w
    with roots [p, q, -q - p + u]
  over Multivariate Polynomial Ring in u, v, w
  over Integer Ring localized at (w,)
```

Cubic Hecke algebra over a special base ring with respect to a special cubic equation:

```
sage: algebras.CubicHecke('s1, s2', cubic_equation_parameters=(QQ(1),3,1))
Cubic Hecke algebra on 3 strands over Rational Field
  with cubic equation: h^3 - h^2 + 3*h - 1 = 0
sage: CHA3 = _
sage: ER = CHA3.extension_ring(); ER
Number Field in T with defining polynomial T^12 + 4*T^11 + 51*T^10
+ 154*T^9 + 855*T^8 + 1880*T^7 + 5805*T^6 + 8798*T^5 + 15312*T^4
+ 14212*T^3 + 13224*T^2 + 5776*T + 1444
sage: CHA3.cubic_equation_roots()[0]
-4321/1337904*T^11 - 4181/445968*T^10 - 4064/27873*T^9 - 51725/167238*T^8
- 2693189/1337904*T^7 - 1272907/445968*T^6 - 704251/74328*T^5
- 591488/83619*T^4 - 642145/83619*T^3 + 252521/111492*T^2 + 45685/5868*T
+ 55187/17604

sage: F = GF(25,'u')
sage: algebras.CubicHecke('s1, s2', cubic_equation_parameters=(F(1), F.gen(), F(3)))
Cubic Hecke algebra on 3 strands over Finite Field in u of size 5^2
  with cubic equation: h^3 + 4*h^2 + u*h + 2 = 0
sage: CHA3 = _
sage: ER = CHA3.extension_ring(); ER
Finite Field in S of size 5^4
sage: CHA3.cubic_equation_roots()
[2*S^3 + 2*S^2 + 2*S + 1, 2*S^3 + 3*S^2 + 3*S + 2, S^3 + 3]
```

Cubic Hecke algebra over a special extension ring with respect to special roots of the cubic equation:

```
sage: UCF = UniversalCyclotomicField()
sage: e3=UCF.gen(3); e5=UCF.gen(5)
sage: algebras.CubicHecke('s1, s2', cubic_equation_roots=(1, e5, e3))
Cubic Hecke algebra on 3 strands over Universal Cyclotomic Field
  with cubic equation:
```

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```
h^3 + (-E(15) - E(15)^4 - E(15)^7 + E(15)^8)*h^2 + (-E(15)^2 - E(15)^8
- E(15)^11 - E(15)^13 - E(15)^14)*h - E(15)^8 = 0
```

Element

alias of *CubicHeckeElement*

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.algebra_generators()
Finite family {c: c}
```

base_ring(*generic=False*)

Return the base ring of `self`.

INPUT:

- `generic` – boolean (default: `False`); if `True` the ring of definition (here often called the generic base ring) is returned

EXAMMPLES:

```
sage: CHA2 = algebras.CubicHecke(2, cubic_equation_roots=(3, 4, 5))
sage: CHA2.base_ring()
Integer Ring localized at (2, 3, 5)
sage: CHA2.base_ring(generic=True)
Multivariate Polynomial Ring in u, v, w
  over Integer Ring localized at (w,)
```

braid_group()

Return the braid group attached to `self`.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.braid_group()
Braid group on 2 strands
```

braid_group_algebra()

Return the group algebra of braid group attached to `self` over the base ring of `self`.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.braid_group_algebra()
Algebra of Braid group on 2 strands
 over Multivariate Polynomial Ring in u, v, w
 over Integer Ring localized at (w,)
```

characters(*irr=None*, *original=True*)

Return the irreducible characters of `self`.

By default the values are given in the generic extension ring. Setting the keyword `original` to `False` you can obtain the values in the (non generic) extension ring (compare the same keyword for *CubicHeckeElement.matrix()*).

INPUT:

- `irr` – (optional) instance of *AbsIrreducibeRep* selecting the irreducible representation corresponding to the character; if not given a list of all characters is returned
- `original` – (default: `True`) see description above

OUTPUT:

Function or list of Functions from the element class of `self` to the (generic or non generic) extension ring depending on the given keyword arguments.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ch = CHA3.characters()
sage: e = CHA3.an_element()
sage: ch[0](e)
a^2*b + a^2*c + a^2 - b*c + b^-1*c^-1 + a^-1*c^-1 + a^-1*b^-1
sage: _.parent()
Multivariate Laurent Polynomial Ring in a, b, c
  over Splitting Algebra of x^2 + x + 1 with roots [e3, -e3 - 1]
  over Integer Ring
sage: ch_w3_100 = CHA3.characters(irr=CHA3.irred_repr.W3_100)
sage: ch_w3_100(e) == ch[0](e)
True
sage: ch_x = CHA3.characters(original=False)
sage: ch_x[0](e)
(u + v)*a + (-v*w - w^2 + u)/w
sage: _.parent()
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
  over Splitting Algebra of h^3 - u*h^2 + v*h - w
    with roots [a, b, -b - a + u]
  over Multivariate Polynomial Ring in u, v, w
  over Integer Ring localized at (w,)
```

chevie()

Return the `GAP3-CHEVIE` realization of the corresponding cyclotomic Hecke algebra in the finite-dimensional case.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)  # optional gap3
sage: CHA3.chevie()                  # optional gap3
Hecke(G4,[[a,b,c]])
```

cubic_braid_group()

Return the cubic braid group attached to `self`.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.cubic_braid_group()
Cubic Braid group on 2 strands
```

cubic_braid_group_algebra()

Return the group algebra of cubic braid group attached to self over the base ring of self.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.cubic_braid_group_algebra()
Algebra of Cubic Braid group on 2 strands
 over Multivariate Polynomial Ring in u, v, w
 over Integer Ring localized at (w,)
```

cubic_equation(*var='h'*, *as_coefficients=False*, *generic=False*)

Return the cubic equation attached to self.

INPUT:

- var – string (default h) setting the indeterminate of the equation
- as_coefficients – boolean (default: False); if set to True the list of coefficients is returned
- generic – boolean (default: False); if set to True the cubic equation will be given over the generic base ring

OUTPUT:

A polynomial over the base ring (resp. generic base ring if generic is set to True). In case as_coefficients is set to True a list of them is returned.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2, cubic_equation_roots=(E(3), ~E(3), 1))
sage: CHA2.cubic_equation()
h^3 - 1
sage: CHA2.cubic_equation(generic=True)
h^3 - u*h^2 + v*h - w
sage: CHA2.cubic_equation(as_coefficients=True, generic=True)
[-w, v, -u, 1]
sage: CHA2.cubic_equation(as_coefficients=True)
[-1, 0, 0, 1]
```

cubic_equation_parameters(*generic=False*)

Return the coefficients of the underlying cubic equation.

INPUT:

- generic – boolean (default: False); if set to True the coefficients are returned as elements of the generic base ring

OUTPUT:

A tripple consisting of the coefficients.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2, cubic_equation_roots=(3, 4, 5))
sage: CHA2.cubic_equation()
h^3 - 12*h^2 + 47*h - 60
sage: CHA2.cubic_equation_parameters()
[12, 47, 60]
```

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```
sage: CHA2.cubic_equation_parameters(generic=True)
[u, v, w]
```

cubic_equation_roots(*generic=False*)

Return the roots of the underlying cubic equation.

INPUT:

- `generic` – boolean (default: `False`); if set to `True` the roots are returned as elements of the generic extension ring

OUTPUT:

A triple consisting of the roots.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2, cubic_equation_roots=(3, 4, 5))
sage: CHA2.cubic_equation()
h^3 - 12*h^2 + 47*h - 60
sage: CHA2.cubic_equation_roots()
[3, 4, 5]
sage: CHA2.cubic_equation_roots(generic=True)
[a, b, c]
```

cubic_hecke_subalgebra(*nstrands=None*)

Return a *CubicHeckeAlgebra* that realizes a sub-algebra of `self` on the first `n_strands` strands.

INPUT:

- `nstrands` – integer at least 1 and at most *strands()* giving the number of strands for the subgroup; the default is one strand less than `self` has

OUTPUT:

An instance of this class realizing the sub-algebra.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3, cubic_equation_roots=(3, 4, 5))
sage: CHA3.cubic_hecke_subalgebra()
Cubic Hecke algebra on 2 strands
  over Integer Ring localized at (2, 3, 5)
    with cubic equation: h^3 - 12*h^2 + 47*h - 60 = 0
```

cyclotomic_generator(*generic=False*)

Return the third root of unity as element of the extension ring.

The only thing where this is needed is in the nine dimensional irreducible representations of the cubic Hecke algebra on four strands (see the examples of *CubicHeckeElement.matrix()* for instance).

INPUT:

- `generic` – boolean (default: `False`); if `True` the cyclotomic generator is returned as an element extension ring of definition

EXAMMPLES:

```
sage: CHA2 = algebras.CubicHecke(2, cubic_equation_roots=(3, 4, 5))
sage: CHA2.cyclotomic_generator()
E3
sage: CHA2.cyclotomic_generator(generic=True)
e3
```

extension_ring(*generic=False*)

Return the extension ring of `self`.

This is an extension of its base ring containing the roots of the cubic equation.

INPUT:

- `generic` – boolean (default: `False`); if `True` the extension ring of definition (here often called the generic extension ring) is returned

EXAMMPLES:

```
sage: CHA2 = algebras.CubicHecke(2, cubic_equation_roots=(3, 4, 5))
sage: CHA2.extension_ring()
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
over Integer Ring localized at (2, 3, 5)
sage: CHA2.extension_ring(generic=True)
Multivariate Laurent Polynomial Ring in a, b, c
over Splitting Algebra of x^2 + x + 1
  with roots [e3, -e3 - 1] over Integer Ring
```

filecache_section()

Return the `enum` to select a section in the file cache.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: list(CHA2.filecache_section())
[<section.matrix_representations: 'matrix_representations'>,
 <section.braid_images: 'braid_images'>,
 <section.basis_extensions: 'basis_extensions'>,
 <section.markov_trace: 'markov_trace'>]
```

garside_involution(*element*)

Return the image of the given element of `self` under the extension of the Garside involution of braids to `self`.

This method may be invoked by the `revert_garside` method of the element class of `self`, alternatively.

INPUT:

- `element` – instance of the element class of `self`

OUTPUT:

Instance of the element class of `self` representing the image of `element` under the extension of the Garside involution to `self`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element()
```

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```
sage: ele_gar = CHA3.garside_involution(ele); ele_gar
(-w)*c1*c0^-1 + u*c0 + v*c1 + ((-v*w+u)/w)
sage: ele == CHA3.garside_involution(ele_gar)
True
```

gen(*i*)

The i-th generator of the algebra.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.gen(0)
c
```

gens()

Return the generators of self.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.gens()
(c,)
```

get_order()

Return an ordering of the basis of self.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: len(CHA3.get_order())
24
```

irred_repr

alias of *AbsIrreducibeRep*

is_filecache_empty(*section=None*)

Return True if the file cache of the given section is empty. If no section is given the answer is given for the complete file cache.

INPUT:

- section – (default: all sections) an element of enum section that can be selected using *filecache_section()*

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.is_filecache_empty()
False
```

mirror_image()

Return a copy of self with the mirrored cubic equation, that is: the cubic equation has the inverse roots to the roots with respect to self.

This is needed since the mirror involution of the braid group does not factor through self (considered as an algebra over the base ring, just considered as $\mathbf{Z}$-algebra). Therefore, the mirror involution of an element of self belongs to mirror_image.

OUTPUT:

A cubic Hecke algebra over the same base and extension ring, but whose cubic equation is transformed by the mirror involution applied to its coefficients and roots.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: ce = CHA2.cubic_equation(); ce
h^3 - u*h^2 + v*h - w
sage: CHA2m = CHA2.mirror_image()
sage: cem =  CHA2m.cubic_equation(); cem
h^3 + ((-v)/w)*h^2 + u/w*h + (-1)/w
sage: mi = CHA2.base_ring().mirror_involution(); mi
Ring endomorphism of Multivariate Polynomial Ring in u, v, w
                     over Integer Ring localized at (w,)
  Defn: u |--> v/w
        v |--> u/w
        w |--> 1/w
sage: cem == cem.parent()([mi(cf) for cf in ce.coefficients()])
True
```

Note that both cubic Hecke algebras have the same ring of definition and identical generic cubic equation:

```
sage: cemg = CHA2m.cubic_equation(generic=True)
sage: CHA2.cubic_equation(generic=True) == cemg
True
sage: CHA2.cubic_equation() == cemg
True
sage: a, b, c = CHA2.cubic_equation_roots()
sage: CHA2m.cubic_equation_roots(generic=True) == [a, b, c]
True
sage: CHA2m.cubic_equation_roots()
[((-1)/(-w))*a^2 + (u/(-w))*a + (-v)/(-w),
 ((1/(-w))*a)*b + (1/(-w))*a^2 + ((-u)/(-w))*a,
 (((-1)/(-w))*a)*b]
sage: ai, bi, ci = _
sage: ai == ~a, bi == ~b, ci == ~c
(True, True, True)
sage: CHA2.extension_ring(generic=True).mirror_involution()
Ring endomorphism of Multivariate Laurent Polynomial Ring in a, b, c
                     over Splitting Algebra of x^2 + x + 1
                       with roots [e3, -e3 - 1] over Integer Ring
  Defn: a |--> a^-1
        b |--> b^-1
        c |--> c^-1
        with map of base ring
```

The mirror image can not be obtained for specialized cubic Hecke algebras if the specialization does not factor through the mirror involution on the ring if definition:

```
sage: CHA2s = algebras.CubicHecke(2, cubic_equation_roots=(3, 4, 5))
sage: CHA2s
Cubic Hecke algebra on 2 strands
  over Integer Ring localized at (2, 3, 5)
```

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```
    with cubic equation: h^3 - 12*h^2 + 47*h - 60 = 0
```

In the next example it is not clear what the mirror image of 7 should be:

```
sage: CHA2s.mirror_image()
Traceback (most recent call last):
...
RuntimeError: base ring Integer Ring localized at (2, 3, 5)
does not factor through mirror involution
```

mirror_isomorphism(*element*)

Return the image of the given element of `self` under the extension of the mirror involution of braids to `self`. The mirror involution of a braid is given by inverting all generators in the braid word. It does not factor through `self` over the base ring but it factors through `self` considered as a $\mathbf{Z}$-module relative to the mirror automorphism of the generic base ring. Considering `self` as algebra over its base ring this involution defines an isomorphism of `self` onto a different cubic Hecke algebra with a different cubic equation. This is defined over a different base (and extension) ring than `self`. It can be obtained by the method `mirror_image` or as parent of the output of this method.

This method may be invoked by the `CubicHeckeElelemnt.revert_mirror` method of the element class of `self`, alternatively.

INPUT:

- `element` – instance of the element class of `self`

OUTPUT:

Instance of the element class of the mirror image of `self` representing the image of element under the extension of the braid mirror involution to `self`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element()
sage: ele_mirr = CHA3.mirror_isomorphism(ele); ele_mirr
-1/w*c0^-1*c1 + u/w*c0^-1 + v/w*c1^-1 + ((v*w-u)/w)
sage: ele_mirr2 = ele.revert_mirror()  # indirect doctest
sage: ele_mirr == ele_mirr2
True
sage: par_mirr = ele_mirr.parent()
sage: par_mirr == CHA3
False
sage: par_mirr == CHA3.mirror_image()
True
sage: ele == par_mirr.mirror_isomorphism(ele_mirr)
True
```

ngens()

The number of generators of the algebra.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.ngens()
1
```

one_basis()

Return the index of the basis element for the identity element in the cubic braid group.

EXAMPLES:

```
sage: CHA2 = algebras.CubicHecke(2)
sage: CHA2.one_basis()
1
```

orientation_antiinvolution(*element*)

Return the image of the given element of `self` under the extension of the orientation anti involution of braids to `self`. The orientation anti involution of a braid is given by reversing the order of generators in the braid word.

This method may be invoked by the `revert_orientation` method of the element class of `self`, alternatively.

INPUT:

- `element` – instance of the element class of `self`

OUTPUT:

Instance of the element class of `self` representing the image of `element` under the extension of the orientation reversing braid involution to `self`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element()
sage: ele_ori = CHA3.orientation_antiinvolution(ele); ele_ori
(-w)*c1^-1*c0 + v*c0 + u*c1 + ((-v*w+u)/w)
sage: ele == CHA3.orientation_antiinvolution(ele_ori)
True
```

product_on_basis(*g1*, *g2*)

Return product on basis elements indexed by `g1` and `g2`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: g = CHA3.basis().keys().an_element(); g
c0*c1
sage: CHA3.product_on_basis(g, ~g)
1
sage: CHA3.product_on_basis(g, g)
w*c0^-1*c1*c0 + (-v)*c1*c0 + u*c0*c1*c0
```

repr_type

alias of *RepresentationType*

reset_filecache(*section=None*, *commit=True*)

Reset the file cache of the given `section` resp. the complete file cache if no `section` is given.

INPUT:

- `section` – (default: all sections) an element of enum `section` that can be selected using *filecache_section()*

- `commit` – boolean (default: `True`); if set to `False` the reset is not written to the filesystem

EXAMPLES:

```
sage: CHA5 = algebras.CubicHecke(5)    # optional - database_cubic_hecke
sage: be = CHA5.filecache_section().basis_extensions # optional - database_
↪cubic_hecke
sage: CHA5.is_filecache_empty(be)      # optional - database_cubic_hecke
False
sage: CHA5.reset_filecache(be)         # optional - database_cubic_hecke
sage: CHA5.is_filecache_empty(be)      # optional - database_cubic_hecke
True
```

schur_element(*item*, *generic=False*)

Return a single Schur element of `self` as elements of the extension ring of `self`.

Note: This method needs `GAP3` installed with package `CHEVIE`.

INPUT:

- `item` – an element of *AbsIrreducibeRep* to give the irreducible representation of `self` to which the Schur element should be returned
- `generic` – boolean (default: `False`); if `True`, the element is returned as element of the generic extension ring

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)                 # optional gap3
sage: CHA3.schur_element(CHA3.irred_repr.W3_111)    # optional gap3
(u^3*w + v^3 - 6*u*v*w + 8*w^2)/w^2
```

schur_elements(*generic=False*)

Return the list of Schur elements of `self` as elements of the extension ring of `self`.

Note: This method needs `GAP3` installed with package `CHEVIE`.

INPUT:

- `generic` – boolean (default: `False`); if `True`, the element is returned as element of the generic extension ring

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)       # optional gap3
sage: sch_eles = CHA3.schur_elements()    # optional gap3
sage: sch_eles[6]                         # optional gap3
(u^3*w + v^3 - 6*u*v*w + 8*w^2)/w^2
```

strands()

Return the number of strands of the braid group whose group algebra image is `self`.

EXAMPLES:

```
sage: CHA4 = algebras.CubicHecke(2)
sage: CHA4.strands()
2
```

class sage.algebras.hecke_algebras.cubic_hecke_algebra.**CubicHeckeElement**

Bases: `IndexedFreeModuleElement`

An element of a *CubicHeckeAlgebra*.

For more information see *CubicHeckeAlgebra*.

EXAMPLES:

```
sage: CHA3s = algebras.CubicHecke('s1, s2'); CHA3s.an_element()
(-w)*s1*s2^-1 + v*s1 + u*s2 + ((-v*w+u)/w)
sage: CHA3.<c1, c2> = algebras.CubicHecke(3)
sage: c1**3*~c2
u*w*c1^-1*c2^-1 + (u^2-v)*c1*c2^-1 + (-u*v+w)*c2^-1
```

Tietze()

Return the Tietze presentation of `self` if `self` belongs to the basis of its parent and `None` otherwise.

OUTPUT:

A tuple representing the pre image braid of `self` if `self` is a monomial from the basis `None` else-wise

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element(); ele
(-w)*c0*c1^-1 + v*c0 + u*c1 + ((-v*w+u)/w)
sage: ele.Tietze() is None
True
sage: [CHA3(sp).Tietze() for sp in ele.support()]
[(), (1,), (1, -2), (2,)]
```

braid_group_algebra_pre_image()

Return a pre image of `self` in the group algebra of the braid group (with respect to the basis given by Ivan Marin).

OUTPUT:

The pre image of `self` as instance of the element class of the group algebra of the BraidGroup

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element(); ele
(-w)*c0*c1^-1 + v*c0 + u*c1 + ((-v*w+u)/w)
sage: b_ele = ele.braid_group_algebra_pre_image(); b_ele
((-v*w+u)/w) + v*c0 + u*c1 + (-w)*c0*c1^-1
sage: ele in CHA3
True
sage: b_ele in CHA3
False
sage: b_ele in CHA3.braid_group_algebra()
True
```

cubic_braid_group_algebra_pre_image()

Return a pre image of `self` in the group algebra of the cubic braid group.

OUTPUT:

The pre image of `self` as instance of the element class of the group algebra of the `CubicBraidGroup`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element(); ele
(-w)*c0*c1^-1 + v*c0 + u*c1 + ((-v*w+u)/w)
sage: cb_ele = ele.cubic_braid_group_algebra_pre_image(); cb_ele
(-w)*c0*c1^-1 + v*c0 + u*c1 + ((-v*w+u)/w)
sage: ele in CHA3
True
sage: cb_ele in CHA3
False
sage: cb_ele in CHA3.cubic_braid_group_algebra()
True
```

formal_markov_trace(*extended=False*, *field_embedding=False*)

Return a formal expression which can be specialized to Markov traces which factor through the cubic Hecke algebra.

This covers Markov traces corresponding to the

- HOMFLY-PT polynomial,
- Kauffman polynomial,
- Links-Gould polynomial.

These expressions are elements of a sub-module of the module of linear forms on `self` the base ring of which is an extension of the generic base ring of `self` by an additional variable `s` representing the writhe factor. All variables of this base ring extension are invertible.

A Markov trace is a family of class functions tr_n on the family of braid groups B_n into some commutative ring R depending on a unit $s \in R$ such that for all $b \in B_n$ the following two conditions are satisfied (see [Kau1991], section 7):

$$\begin{array}{rcl} tr_{n+1}(bg_n) & = & str_n(b), \\ tr_{n+1}(bg_n^{-1}) & = & s^{-1}tr_n(b). \end{array}$$

The unit s is often called the writhe factor and corresponds to the additional variable mentioned above.

Note: Currently it is not known if all linear forms of this sub-module belong to a Markov trace, i.e. can be extended to the full tower of cubic Hecke algebras. Anyway, at least the four basis elements (`U1`, `U2`, `U3` and `K4`) can be reconstructed form the HOMFLY-PT and Kauffman polynomial.

INPUT:

- `extended` – boolean (default: `False`); if set to `True` the base ring of the Markov trace module is constructed as an extension of generic extension ring of `self`; per default it is constructed upon the generic base ring
- `field_embedding` – boolean (default: `False`); if set to `True` the base ring of the module is the smallest field containing the generic extension ring of `self`; ignored if `extended=False`

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: CHA2 = algebras.CubicHecke(2)
sage: K3_1 = KnotInfo.K3_1
sage: b3_1 = CHA2(K3_1.braid())
sage: mt3_1 = b3_1.formal_markov_trace(); mt3_1
((u^2*s^2-v*s^2+u*w)/s)*B[U1] + (-u*v+w)*B[U2]
sage: mt3_1.parent()
Free module generated by {U1, U2}
   over Multivariate Polynomial Ring in u, v, w, s
   over Integer Ring localized at (s, w, v, u)

sage: f = b3_1.formal_markov_trace(extended=True); f
(a^2*b*c*s^-1+a*b^2*c*s^-1+a*b*c^2*s^-1+a^2*s+a*b*s+b^2*s+a*c*s+b*c*s+c^
↪2*s)*B[U1]
  + (-a^2*b-a*b^2-a^2*c+(-2)*a*b*c-b^2*c-a*c^2-b*c^2)*B[U2]
sage: f.parent().base_ring()
Multivariate Laurent Polynomial Ring in a, b, c, s
  over Splitting Algebra of x^2 + x + 1 with roots [e3, -e3 - 1]
  over Integer Ring

sage: f = b3_1.formal_markov_trace(extended=True, field_embedding=True); f
((a^2*b*c+a*b^2*c+a*b*c^2+a^2*s^2+a*b*s^2+b^2*s^2+a*c*s^2+b*c*s^2+c^2*s^2)/
↪s)*B[U1]
  + (-a^2*b-a*b^2-a^2*c-2*a*b*c-b^2*c-a*c^2-b*c^2)*B[U2]
sage: f.parent().base_ring()
Fraction Field of Multivariate Polynomial Ring in a, b, c, s
  over Cyclotomic Field of order 3 and degree 2
```

Obtaining the well known link invariants from it:

```
sage: MT = mt3_1.base_ring()
sage: sup = mt3_1.support()
sage: u, v, w, s = mt3_1.base_ring().gens()
sage: LK3_1 = mt3_1*s**-3 # since the writhe of K3_1 is 3
sage: f = MT.specialize_homfly()
sage: g = sum(f(LK3_1.coefficient(b)) * b.regular_homfly_polynomial() for b in␣
↪sup); g
L^-2*M^2 - 2*L^-2 - L^-4
sage: g == K3_1.link().homfly_polynomial()
True

sage: f = MT.specialize_kauffman()
sage: g = sum(f(LK3_1.coefficient(b)) * b.regular_kauffman_polynomial() for b␣
↪in sup); g
a^-2*z^2 - 2*a^-2 + a^-3*z + a^-4*z^2 - a^-4 + a^-5*z
sage: g == K3_1.kauffman_polynomial()
True

sage: f = MT.specialize_links_gould()
sage: g = sum(f(LK3_1.coefficient(b)) * b.links_gould_polynomial() for b in␣
↪sup); g
-t0^2*t1 - t0*t1^2 + t0^2 + 2*t0*t1 + t1^2 - t0 - t1 + 1
```

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```
sage: g == K3_1.link().links_gould_polynomial()
True
```

matrix(*subdivide=False*, *representation_type=None*, *original=False*)

Return certain types of matrix representations of `self`.

The absolutely irreducible representations of the cubic Hecke algebra are constructed using the `GAP3` interface and the `CHEVIE` package if `GAP3` and `CHEVIE` are installed on the system. Furthermore, the representations given on Ivan Marin's homepage are used:

INPUT:

- `subdivide` – boolean (default: `False`): this boolean is passed to the block_matrix function
- `representation_type` – instance of enum *RepresentationType*; this can be obtained by the attribute *CubicHeckeAlgebra.repr_type* of `self`; the following values are possible:
  - `RegularLeft` – (regular left repr. from the above URL)
  - `RegularRight` – (regular right repr. from the above URL)
  - `SplitIrredChevie` – (split irred. repr. via CHEVIE)
  - `SplitIrredMarin` – (split irred. repr. from the above URL)
  - default: `SplitIrredChevie` taken if GAP3 and CHEVIE are installed on the system, otherwise the default will be `SplitIrredMarin`
- `original` – boolean (default: `False`): if set to true the base ring of the matrix will be the generic base_ring resp. generic extension ring (for the split versions) of the parent of `self`

OUTPUT:

An instance of *CubicHeckeMatrixRep*, which is inherited from `Matrix_generic_dense`. In the case of the irreducible representations the matrix is given as a block matrix. Each single irreducible can be obtained as item indexed by the members of the enum *AbsIrreducibeRep* available via *CubicHeckeAlgebra.irred_repr*. For details type: `CubicHeckeAlgebra.irred_repr?`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: CHA3.inject_variables()
Defining c0, c1
sage: c0m = c0.matrix()
sage: c0m[CHA3.irred_repr.W3_111]
[                       -b - a + u      0     0]
[(-2*a + u)*b - 2*a^2 + 2*u*a - v      b     0]
[                                b      1     a]
```

using the the `representation_type` option:

```
sage: CHA3.<c0, c1> = algebras.CubicHecke(3)      #  optional gap3
sage: chevie = CHA3.repr_type.SplitIrredChevie    #  optional gap3
sage: c0m_ch = c0.matrix(representation_type=chevie) #  optional gap3
sage: c0m_ch[CHA3.irred_repr.W3_011]              #  optional gap3
[         b          0]
[        -b -b - a + u]
sage: c0m[CHA3.irred_repr.W3_011]
```

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```
[            b             0]
[a^2 - u*a + v    -b - a + u]
```

using the the `original` option:

```
sage: c0mo = c0.matrix(original=True)
sage: c0mo_ch = c0.matrix(representation_type=chevie, original=True) # ␣
↪optional gap3
sage: c0mo[CHA3.irred_repr.W3_011]
[  b   0]
[b*c   c]
sage: c0mo_ch[CHA3.irred_repr.W3_011]            #  optional gap3
[ b  0]
[-b  c]
```

specialized matrices:

```
sage: t = (3,7,11)
sage: CHA4 = algebras.CubicHecke(4, cubic_equation_roots=t)  # optional␣
↪database_cubic_hecke
sage: e = CHA4.an_element(); e                         # optional database_cubic_
↪hecke
-231*c0*c1^-1 + 131*c0*c2^-1 + 21*c2*c1 - 1440/11
sage: em = e.matrix()                                  # optional database_cubic_
↪hecke
sage: em.base_ring()                                   # optional database_cubic_
↪hecke
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
   over Integer Ring localized at (3, 7, 11)
sage: em.dimensions()                                  # optional database_cubic_
↪hecke
(108, 108)
sage: em_irr24 = em[23]                                # optional database_cubic_
↪hecke
sage: em_irr24.dimensions()                            # optional database_cubic_
↪hecke
(9, 9)
sage: em_irr24[3,2]                                    # optional database_cubic_
↪hecke
-131*E3 - 393/7
sage: emg = e.matrix(representation_type=chevie)    # optional gap3 database_
↪cubic_hecke
sage: emg_irr24 = emg[23]                              # optional gap3 database_
↪cubic_hecke
sage: emg_irr24[3,2]                                   # optional gap3 database_
↪cubic_hecke
-131*E3 - 393/7
```

max_len()

Return the maximum of the length of Tietze expressions among the support of `self`.

EXAMPLES:

```
sage: CHA3 = algebras.CubicHecke(3)
sage: ele = CHA3.an_element(); ele
(-w)*c0*c1^-1 + v*c0 + u*c1 + ((-v*w+u)/w)
sage: ele.max_len()
2
```

revert_garside()

Return the image of `self` under the Garside involution.

See also:

CubicHeckeAlgebra.garside_involution()

EXAMPLES:

```
sage: roots = (E(3), ~E(3), 1)
sage: CHA3.<c1, c2> = algebras.CubicHecke(3, cubic_equation_roots=roots)
sage: e = CHA3.an_element(); e
-c1*c2^-1
sage: _.revert_garside()
-c2*c1^-1
sage: _.revert_garside()
-c1*c2^-1
```

revert_mirror()

Return the image of `self` under the mirror isomorphism.

See also:

CubicHeckeAlgebra.mirror_isomorphism()

EXAMPLES:

```
sage: CHA3.<c1, c2> = algebras.CubicHecke(3)
sage: e = CHA3.an_element()
sage: e.revert_mirror()
-1/w*c0^-1*c1 + u/w*c0^-1 + v/w*c1^-1 + ((v*w-u)/w)
sage: _.revert_mirror() == e
True
```

revert_orientation()

Return the image of `self` under the anti involution reverting the orientation of braids.

See also:

CubicHeckeAlgebra.orientation_antiinvolution()

EXAMPLES:

```
sage: CHA3.<c1, c2> = algebras.CubicHecke(3)
sage: e = CHA3.an_element()
sage: e.revert_orientation()
(-w)*c2^-1*c1 + v*c1 + u*c2 + ((-v*w+u)/w)
sage: _.revert_orientation() == e
True
```

6.5.2 Cubic Hecke Base Rings

This module contains special classes of polynomial rings (*CubicHeckeRingOfDefinition* and *CubicHeckeExtensionRing*) used in the context of *cubic Hecke algebras*.

AUTHORS:

- Sebastian Oehms May 2020: initial version

class sage.algebras.hecke_algebras.cubic_hecke_base_ring.**CubicHeckeExtensionRing**(*names*, *order='degrevlex'*, *ring_of_definition=None*, *third_unity_root_name='e3'*, *markov_trace_version=False*)

Bases: LaurentPolynomialRing_mpair

The generic splitting algebra for the irreducible representations of the cubic Hecke algebra.

This ring must contain three invertible indeterminates (representing the roots of the cubic equation) together with a third root of unity (needed for the 18-dimensional irreducibles of the cubic Hecke algebra on 4 strands).

Therefore this ring is constructed as a multivariate Laurent polynomial ring in three indeterminates over a polynomial quotient ring over the integers with respect to the minimal polynomial of a third root of unity.

The polynomial quotient ring is constructed as instance of *SplittingAlgebra*.

INPUT:

- `names` – (default: `'u,v,w'`) string containing the names of the indeterminates separated by `,` or a triple of strings each of which are the names of one of the three indeterminates
- `order` – string (default: `'degrevlex'`); the term order; see also LaurentPolynomialRing_mpair
- `ring_of_definition` – (optional) a *CubicHeckeRingOfDefinition* to specify the generic cubic Hecke base ring over which `self` may be realized as splitting ring via the `as_splitting_algebra` method
- `third_unity_root_name` – string (default: `'e3'`); for setting the name of the third root if unity of `self`
- `markov_trace_version` – boolean (default: `False`) if this is set to `True` then `self` contains one invertible indeterminate in addition which is meant to represent the writhe factor of a Markov trace on the cubic Hecke algebra and which default name is `s`

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: chbr.CubicHeckeExtensionRing('a, b, c')
Multivariate Laurent Polynomial Ring in a, b, c
  over Splitting Algebra of x^2 + x + 1
    with roots [e3, -e3 - 1]
  over Integer Ring
sage: _.an_element()
b^2*c^-1 + e3*a
```

as_splitting_algebra()

Return `self` as a *SplittingAlgebra*; that is as an extension ring of the corresponding cubic Hecke algebra base ring (`self._ring_of_definition`, as a *CubicHeckeRingOfDefinition*) splitting its cubic equation into linear factors, such that the roots are images of the generators of `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: GBR = chbr.CubicHeckeRingOfDefinition()
sage: GER = GBR.extension_ring()
sage: ER = GER.as_splitting_algebra(); ER
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
  over Splitting Algebra of h^3 - u*h^2 + v*h - w
    with roots [a, b, -b - a + u]
  over Multivariate Polynomial Ring in u, v, w
  over Integer Ring localized at (w,)
sage: ER(GER.an_element())
a*E3 + ((u/(-w))*a^2 + ((u^2 - v)/w)*a)*b + a - u
sage: ER(GBR.an_element())
(u^2 + v*w)/w

sage: MBR = chbr.CubicHeckeRingOfDefinition(markov_trace_version=True)
sage: MER = MBR.extension_ring()
sage: ES = MER.as_splitting_algebra(); ES
Splitting Algebra of T^2 + T + 1 with roots [E3, -E3 - 1]
  over Splitting Algebra of h^3 - u*h^2 + v*h - w
    with roots [a, b, -b - a + u]
  over Multivariate Polynomial Ring in u, v, w, s
  over Integer Ring localized at (s, w, v, u)
sage: ES(MER.an_element())
(((-1)/(-s))*a)*E3 + ((u/(-w))*a^2 + ((u^2 - v)/w)*a)*b + a - u
sage: ES(MBR.an_element())
(u^2*s + v*w)/(w*s)
```

conjugation()

Return an involution that performs *complex conjugation* with respect to base ring considered as order in the complex field.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('x, y, z')
sage: conj = ER.conjugation()
sage: conj(ER.an_element())
y^2*z^-1 + (-e3 - 1)*x
sage: MER = chbr.CubicHeckeExtensionRing('x, y, z, s', markov_trace_
↪version=True)
sage: conj = MER.conjugation()
sage: conj(MER.an_element())
y^2*z^-1 + (-e3 - 1)*x*s^-1
```

construction()

Return None since this construction is not functorial.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('a, b, c')
sage: ER._test_category()    # indirect doctest
```

create_specialization(*im_cubic_equation_roots*, *im_writhe_parameter=None*, *var='T'*, *third_unity_root_name='E3'*)

Return an appropriate ring containing the elements from the list `im_cubic_equation_roots` defining a conversion map from self mapping the cubic equation roots of `self` to `im_cubic_equation_roots`.

INPUT:

- `im_cubic_equation_roots` – list or tuple of three ring elements such that there exists a ring homomorphism from the corresponding elements of `self` to them

OUTPUT:

A common parent containing the elements of `im_cubic_equation_roots` together with their inverses.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('a, b, c')
sage: t = ER.an_element(); t
b^2*c^-1 + e3*a
sage: Sp1 = ER.create_specialization([E(5), E(7), E(3)]); Sp1
Universal Cyclotomic Field
sage: Sp1(t)
-E(105)^11 - E(105)^16 - E(105)^26 - E(105)^37 - E(105)^41
- E(105)^58 - E(105)^71 - E(105)^79 - E(105)^86 - E(105)^101
sage: MER = chbr.CubicHeckeExtensionRing('a, b, c, s', markov_trace_
↪version=True)
sage: MER.create_specialization([E(5), E(7), E(3)], im_writhe_parameter=E(4))
Universal Cyclotomic Field
sage: a, b, c, s = MER.gens()
sage: Sp1(MER(t)/s)
E(420) + E(420)^29 + E(420)^89 + E(420)^149 + E(420)^169 + E(420)^209
+ E(420)^253 + E(420)^269 + E(420)^337 + E(420)^389

sage: Z3 = CyclotomicField(3); E3=Z3.gen()
sage: Sp2 = ER.create_specialization([E3, E3**2, Z3(1)])
sage: Sp2(t)
-1
sage: MER.create_specialization([E3, E3**2, 1], im_writhe_parameter=2)
Cyclotomic Field of order 3 and degree 2
sage: Sp2(MER(t)*s)
-2

sage: Sp3 = ER.create_specialization([5, 7, 11])
sage: Sp3(t)
5*E3 + 49/11
```

cubic_equation_galois_group()

Return the Galois group of the cubic equation, which is the permutation group on the three generators together with its action on `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('a, b, c')
sage: G = ER.cubic_equation_galois_group()
sage: t = ER.an_element()
```

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```
sage: [(g ,g*t) for g in G]
[((), b^2*c^-1 + e3*a),
((1,3,2), a^2*b^-1 + e3*c),
((1,2,3), e3*b + a^-1*c^2),
((2,3), e3*a + b^-1*c^2),
((1,3), a^-1*b^2 + e3*c),
((1,2), a^2*c^-1 + e3*b)]
```

cyclotomic_generator()

Return the third root of unity as generator of the base ring of `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER  = chbr.CubicHeckeExtensionRing('a, b, c')
sage: ER.cyclotomic_generator()
e3
sage: _**3 == 1
True
```

field_embedding(*characteristic=0*)

Return a field embedding of `self`.

INPUT:

- `characteristic` – integer (default: `0`); the characteristic of the field

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: ER = BR.extension_ring()
sage: ER.field_embedding()
Ring morphism:
From: Multivariate Laurent Polynomial Ring in a, b, c
        over Splitting Algebra of x^2 + x + 1
          with roots [e3, -e3 - 1]
        over Integer Ring
To:   Fraction Field of Multivariate Polynomial Ring in a, b, c
        over Cyclotomic Field of order 3 and degree 2
Defn: a |--> a
      b |--> b
      c |--> c
with map of base ring

sage: ER.field_embedding(characteristic=5)
Ring morphism:
From: Multivariate Laurent Polynomial Ring in a, b, c
        over Splitting Algebra of x^2 + x + 1
          with roots [e3, -e3 - 1]
        over Integer Ring
To:   Fraction Field of Multivariate Polynomial Ring in a, b, c
        over Finite Field in a of size 5^2
Defn: a |--> a
```

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```
      b |--> b
      c |--> c
with map of base ring

sage: MER = ER.markov_trace_version()
sage: MER.field_embedding()
Ring morphism:
From: Multivariate Laurent Polynomial Ring in a, b, c, s
        over Splitting Algebra of x^2 + x + 1
          with roots [e3, -e3 - 1]
        over Integer Ring
To:   Fraction Field of Multivariate Polynomial Ring in a, b, c, s
        over Cyclotomic Field of order 3 and degree 2
Defn: a |--> a
      b |--> b
      c |--> c
      s |--> s
with map of base ring
```

hom(*im_gens*, *codomain=None*, *check=True*, *base_map=None*)

Return a homomorphism of `self`.

INPUT:

- `im_gens` – tuple for the image of the generators of `self`
- `codomain` – (optional) the codomain of the homomorphism

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('a, b, c')
sage: UCF = UniversalCyclotomicField()
sage: map = ER.hom((UCF.gen(3),) + (UCF(3),UCF(4),UCF(5)))
sage: ER.an_element()
b^2*c^-1 + e3*a
sage: map(_)
-1/5*E(3) - 16/5*E(3)^2
```

markov_trace_version()

Return the Markov trace version of `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('a, b, c')
sage: ER.markov_trace_version()
Multivariate Laurent Polynomial Ring in a, b, c, s
  over Splitting Algebra of x^2 + x + 1
    with roots [e3, -e3 - 1] over Integer Ring
```

mirror_involution()

Return the involution of `self` corresponding to the involution of the cubic Hecke algebra (with the same name).

This means that it maps the generators of `self` to their inverses.

Note: The mirror involution of the braid group does not factor through the cubic Hecke algebra over its base ring, but it does if it is considered as **Z**-algebra. The base ring elements are transformed by this automorphism.

OUTPUT:

The involution as automorphism of `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: ER = chbr.CubicHeckeExtensionRing('p, q, r')
sage: ER.mirror_involution()
Ring endomorphism of Multivariate Laurent Polynomial Ring in p, q, r
                     over Splitting Algebra of x^2 + x + 1
                       with roots [e3, -e3 - 1]
                     over Integer Ring
  Defn: p |--> p^-1
        q |--> q^-1
        r |--> r^-1
        with map of base ring
sage: _(ER.an_element())
e3*p^-1 + q^-2*r

sage: MER = chbr.CubicHeckeExtensionRing('p, q, r, s', markov_trace_
↪version=True)
sage: MER.mirror_involution()
Ring endomorphism of Multivariate Laurent Polynomial Ring in p, q, r, s
  over Splitting Algebra of x^2 + x + 1
    with roots [e3, -e3 - 1] over Integer Ring
Defn: p |--> p^-1
q |--> q^-1
r |--> r^-1
s |--> s^-1
with map of base ring
sage: _(MER.an_element())
e3*p^-1*s + q^-2*r
```

class `sage.algebras.hecke_algebras.cubic_hecke_base_ring.`**`CubicHeckeRingOfDefinition`**(*names=('u', 'v', 'w', 's'), order='degrevlex', markov_trace_version=F*

Bases: `Localization`

The *ring of definition* of the cubic Hecke algebra.

It contains one invertible indeterminate (representing the product of the roots of the cubic equation) and two non invertible indeterminates.

Note: We follow a suggestion by Ivan Marin in the name *ring of definition*. We avoid alternative names like *generic* or *universal* base ring as these have some issues. The first option could be misleading in view of the term *generic point* used in algebraic geometry, which would mean the function field in `u, v, w`, here.

The second option is problematic since the base ring itself is not a universal object. Rather, the universal object is the cubic Hecke algebra considered as a **Z**-algebra including `u, v, w` as pairwise commuting indeterminates. From this point of view the base ring appears to be a subalgebra of this universal object generated by `u, v, w`.

INPUT:

- `names` – (default: `'u,v,w'`) string containing the names of the indeterminates separated by `,` or a triple of strings each of which are the names of one of the three indeterminates
- `order` – string (default: `'degrevlex'`); the term order; see also `LaurentPolynomialRing_mpair`
- `ring_of_definition` – (optional) a *CubicHeckeRingOfDefinition* to specify the generic cubic Hecke base ring over which `self` may be realized as splitting ring via the `as_splitting_algebra` method
- `markov_trace_version` – boolean (default: `False`) if this is set to `True` then `self` contains one invertible indeterminate in addition which is meant to represent the writhe factor of a Markov trace on the cubic Hecke algebra and which default name is `s`

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: u, v, w = BR.gens()
sage: ele = 3*u*v-5*w**(-2)
sage: ER = BR.extension_ring()
sage: ER(ele)
3*a^2*b + 3*a*b^2 + 3*a^2*c + 9*a*b*c + 3*b^2*c
+ 3*a*c^2 + 3*b*c^2 + (-5)*a^-2*b^-2*c^-2
sage: phi1 = BR.hom( [4,3,1/1] )
sage: phi1(ele)
31

sage: LL.<t> = LaurentPolynomialRing(ZZ)
sage: phi2=BR.hom( [LL(4),LL(3),t] )
sage: phi2(ele)
-5*t^-2 + 36

sage: BR.create_specialization( [E(5), E(7), E(3)] )
Universal Cyclotomic Field
sage: _(ele)
-3*E(105) - 5*E(105)^2 - 5*E(105)^8 - 5*E(105)^11 - 5*E(105)^17
- 5*E(105)^23 - 5*E(105)^26 - 5*E(105)^29 - 5*E(105)^32 - 5*E(105)^38
- 5*E(105)^41 - 5*E(105)^44 - 5*E(105)^47 - 5*E(105)^53 - 5*E(105)^59
- 5*E(105)^62 - 5*E(105)^68 - 8*E(105)^71 - 5*E(105)^74 - 5*E(105)^83
- 5*E(105)^86 - 5*E(105)^89 - 5*E(105)^92 - 5*E(105)^101 - 5*E(105)^104
```

create_specialization(*im_cubic_equation_parameters*, *im_writhe_parameter=None*)

Return an appropriate Ring containing the elements from the list `im_cubic_equation_parameters` having a conversion map from `self` mapping the cubic equation parameters of `self` to `im_cubic_equation_parameters`.

INPUT:

- `im_cubic_equation_parameters` – list or tuple of three ring elements such that there exists a ring homomorphism from the corresponding elements of `self` to them

OUTPUT:

A common parent containing the elements of `im_cubic_equation_parameters` together with an inverse of the third element.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: t = BR.an_element(); t
(u^2 + v*w)/w
sage: Sp1 = BR.create_specialization([E(5), E(7), E(3)]); Sp1
Universal Cyclotomic Field
sage: Sp1(t)
E(105) + E(105)^8 + E(105)^29 - E(105)^37 + E(105)^43 - E(105)^52
+ E(105)^64 - E(105)^67 + E(105)^71 - E(105)^82 + E(105)^92
- E(105)^97

sage: MBR = chbr.CubicHeckeRingOfDefinition(markov_trace_version=True)
sage: MBR.create_specialization([E(5), E(7), E(3)], im_writhe_parameter=E(4))
Universal Cyclotomic Field
sage: u, v, w, s = MBR.gens()
sage: Sp1(MBR(t)/s)
E(420)^13 - E(420)^53 + E(420)^73 - E(420)^109 - E(420)^137
- E(420)^221 + E(420)^253 - E(420)^277 + E(420)^313 - E(420)^361
+ E(420)^373 - E(420)^389

sage: Z3 = CyclotomicField(3); E3=Z3.gen()
sage: Sp2 = BR.create_specialization([E3, E3**2, 1]); Sp2
Cyclotomic Field of order 3 and degree 2
sage: Sp2(t)
-2*zeta3 - 2
sage: MBR.create_specialization([E3, E3**2, 1], im_writhe_parameter=2)
Cyclotomic Field of order 3 and degree 2
sage: Sp2(MBR(t)/s)
-zeta3 - 1

sage: Sp3 = BR.create_specialization([5, 7, 11]); Sp3
Integer Ring localized at (11,)
sage: Sp3(t)
102/11
```

cubic_equation(*var='h'*, *as_coefficients=False*)

Return the cubic equation over which the cubic Hecke algebra is defined.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: BR.cubic_equation()
h^3 - u*h^2 + v*h - w
sage: BR.cubic_equation(var='t')
t^3 - u*t^2 + v*t - w
sage: BR.cubic_equation(as_coefficients=True)
[-w, v, -u, 1]
```

extension_ring(*names=('a', 'b', 'c', 's')*)

Return the generic extension ring attached to `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: BR.extension_ring()
Multivariate Laurent Polynomial Ring in a, b, c
  over Splitting Algebra of x^2 + x + 1
    with roots [e3, -e3 - 1]
  over Integer Ring
```

markov_trace_version()

Return the extension of the ring of definition needed to treat the formal Markov traces.

This appends an additional variable `s` to measure the writhe of knots and makes `u` and `v` invertible.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: GBR = chbr.CubicHeckeRingOfDefinition()
sage: GBR.markov_trace_version()
Multivariate Polynomial Ring in u, v, w, s
  over Integer Ring localized at (s, w, v, u)
```

mirror_involution()

Return the involution of `self` corresponding to the involution of the cubic Hecke algebra (with the same name).

This means that it maps the last generator of `self` to its inverse and both others to their product with the image of the former.

From the cubic equation for a braid generator β_i:

$$\beta_i^3 - u\beta_i^2 + v\beta_i - w = 0.$$

One deduces the following cubic equation for β_i^{-1}:

$$\beta_i^{-3} - \frac{v}{w}\beta_i^{-2} + \frac{u}{w}\beta_i^{-1} - \frac{1}{w} = 0.$$

Note: The mirror involution of the braid group does not factor through the cubic Hecke algebra over its base ring, but it does if it is considered as $\mathbf{Z}$-algebra. The base ring elements are transformed by this automorphism.

OUTPUT:

The involution as automorphism of `self`.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: BR.mirror_involution()
Ring endomorphism of Multivariate Polynomial Ring in u, v, w
```

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```
                         over Integer Ring localized at (w,)
  Defn: u |--> v/w
        v |--> u/w
        w |--> 1/w
sage: _(BR.an_element())
(v^2 + u)/w

sage: MBR = chbr.CubicHeckeRingOfDefinition(markov_trace_version=True)
sage: MBR.mirror_involution()
Ring endomorphism of Multivariate Polynomial Ring in u, v, w, s
                     over Integer Ring localized at (s, w, v, u)
Defn: u |--> v/w
v |--> u/w
w |--> 1/w
s |--> 1/s
sage: _(MBR.an_element())
(v^2 + u*s)/w
```

specialize_homfly()

Return a map to the two variable Laurent polynomial ring that is the parent of the HOMFLY-PT polynomial.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: CHA2 = algebras.CubicHecke(2)
sage: K5_1 = KnotInfo.K5_1.link()
sage: br = CHA2(K5_1.braid())
sage: mt = br.formal_markov_trace()
sage: MT = mt.base_ring()
sage: f = MT.specialize_homfly(); f
Composite map:
  From: Multivariate Polynomial Ring in u, v, w, s over Integer Ring
        localized at (s, w, v, u)
  To:   Multivariate Laurent Polynomial Ring in L, M over Integer Ring
  Defn:   Ring morphism:
          From: Multivariate Polynomial Ring in u, v, w, s
                over Integer Ring localized at (s, w, v, u)
          To:   Multivariate Polynomial Ring in L, M
                over Integer Ring localized at (M, M - 1, L)
          Defn: u |--> -M + 1
                v |--> -M + 1
                w |--> 1
                s |--> L
        then
          Conversion map:
          From: Multivariate Polynomial Ring in L, M
                over Integer Ring localized at (M, M - 1, L)
          To:   Multivariate Laurent Polynomial Ring in L, M
                over Integer Ring
sage: sup = mt.support()
sage: h1 = sum(f(mt.coefficient(b)) * b.regular_homfly_polynomial() for b in␣
↪sup)
```

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```
sage: L, M = f.codomain().gens()
sage: h2 = K5_1.homfly_polynomial()
sage: h1*L**(-5) == h2  # since the writhe of K5_1 is 5
True
```

specialize_kauffman()

Return a map to the two variable Laurent polynomial ring that is the parent of the Kauffman polynomial.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: CHA2 = algebras.CubicHecke(2)
sage: K5_1 = KnotInfo.K5_1.link()
sage: br = CHA2(K5_1.braid())
sage: mt = br.formal_markov_trace()
sage: MT = mt.base_ring()
sage: f = MT.specialize_kauffman(); f
Composite map:
  From: Multivariate Polynomial Ring in u, v, w, s over Integer Ring
        localized at (s, w, v, u)
  To:   Multivariate Laurent Polynomial Ring in a, z over Integer Ring
  Defn:   Ring morphism:
          From: Multivariate Polynomial Ring in u, v, w, s
                over Integer Ring localized at (s, w, v, u)
          To:   Multivariate Polynomial Ring in a, z
                over Integer Ring localized at (z, a, a + z, a*z + 1)
          Defn: u |--> (a*z + 1)/a
                v |--> (a + z)/a
                w |--> 1/a
                s |--> a
        then
          Conversion map:
          From: Multivariate Polynomial Ring in a, z over Integer Ring
                localized at (z, a, a + z, a*z + 1)
          To:   Multivariate Laurent Polynomial Ring in a, z
                over Integer Ring
sage: sup = mt.support()
sage: k1 = sum(f(mt.coefficient(b)) * b.regular_kauffman_polynomial() for b in␣
↪sup)
sage: a, z = f.codomain().gens()
sage: k2 = KnotInfo.K5_1.kauffman_polynomial()
sage: k1*a**(-5) == k2  # since the writhe of K5_1 is 5
True
```

specialize_links_gould()

Return a map to the two variable Laurent polynomial ring that is the parent of the Links-Gould polynomial.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: CHA2 = algebras.CubicHecke(2)
sage: K5_1 = KnotInfo.K5_1.link()
sage: br = CHA2(K5_1.braid())
```

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```
sage: mt = br.formal_markov_trace()
sage: MT = mt.base_ring()
sage: f = MT.specialize_links_gould(); f
Composite map:
  From: Multivariate Polynomial Ring in u, v, w, s over Integer Ring
        localized at (s, w, v, u)
  To:   Multivariate Laurent Polynomial Ring in t0, t1 over Integer Ring
  Defn:   Ring morphism:
          From: Multivariate Polynomial Ring in u, v, w, s
                over Integer Ring localized at (s, w, v, u)
          To:   Multivariate Polynomial Ring in t0, t1 over Integer Ring
                localized at (t1, t0, t0 + t1 - 1, t0*t1 - t0 - t1)
          Defn: u |--> t0 + t1 - 1
                v |--> t0*t1 - t0 - t1
                w |--> -t0*t1
                s |--> 1
        then
          Conversion map:
          From: Multivariate Polynomial Ring in t0, t1 over Integer Ring
                localized at (t1, t0, t0 + t1 - 1, t0*t1 - t0 - t1)
          To:   Multivariate Laurent Polynomial Ring in t0, t1 over Integer Ring
sage: sup = mt.support()
sage: sum(f(mt.coefficient(b)) * b.links_gould_polynomial() for b in sup)
-t0^4*t1 - t0^3*t1^2 - t0^2*t1^3 - t0*t1^4 + t0^4 + 2*t0^3*t1 + 2*t0^2*t1^2
+ 2*t0*t1^3 + t1^4 - t0^3 - 2*t0^2*t1 - 2*t0*t1^2 - t1^3 + t0^2 + 2*t0*t1
+ t1^2 - t0 - t1 + 1
```

class sage.algebras.hecke_algebras.cubic_hecke_base_ring.**GaloisGroupAction**

Bases: Action

Action on a multivariate polynomial ring by permuting the generators.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: from operator import mul
sage: R.<x, y, z> = ZZ[]
sage: G = SymmetricGroup(3)
sage: p = 5*x*y + 3*z**2
sage: R._unset_coercions_used()
sage: R.register_action(chbr.GaloisGroupAction(G, R, op=mul))
sage: s = G([2,3,1])
sage: s*p
3*x^2 + 5*y*z
```

sage.algebras.hecke_algebras.cubic_hecke_base_ring.**normalize_names_markov**(*names*, *markov_trace_version*)

Return a tuple of strings of variable names of length 3 resp. 4 (if markov_trace_version is True) according to the given input names.

INPUT:

- names – passed to normalize_names()

- `markov_trace_version` – boolean; if set to `True` four names are expected the last of which corresponds to the writhe factor of the Markov trace

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: chbr.normalize_names_markov('a, b, c', False)
('a', 'b', 'c')
sage: chbr.normalize_names_markov(('u', 'v', 'w', 's'), False)
('u', 'v', 'w')
```

`sage.algebras.hecke_algebras.cubic_hecke_base_ring.`**`register_ring_hom`**(*ring_hom*)

Register the given ring homomorphism as conversion map.

EXAMPLES:

```
sage: from sage.algebras.hecke_algebras import cubic_hecke_base_ring as chbr
sage: BR = chbr.CubicHeckeRingOfDefinition()
sage: BR.create_specialization([E(5), E(7), E(3)])  # indirect doctest
Universal Cyclotomic Field
sage: _.convert_map_from(BR)
Ring morphism:
  From: Multivariate Polynomial Ring in u, v, w
          over Integer Ring localized at (w,)
  To:   Universal Cyclotomic Field
  Defn: u |--> E(5)
        v |--> E(7)
        w |--> E(3)
```

6.5.3 Cubic Hecke matrix representations

This module contains the class *CubicHeckeMatrixRep* which is used to treat the matrix representations of the elements of the cubic Hecke algebra (*CubicHeckeAlgebra*) together with its parent class *CubicHeckeMatrixSpace*. Furthermore, it contains enums for their types (*RepresentationType*) and names (*AbsIrreducibeRep*).

AUTHORS:

- Sebastian Oehms May 2020: initial version

class `sage.algebras.hecke_algebras.cubic_hecke_matrix_rep.`**`AbsIrreducibeRep`**(*value*)

Bases: `Enum`

Enum class to select an absolutely irreducible representation for the cubic Hecke algebra (`CHAn`) on n-strands.

The names are build as follows: Take the determinant of one of the generators of the `CHAn`. This is a monomial in the generic extension ring (`GER`) of `CHA`, say `a^ib^jc^k` where `a, b` and `c` are the generators of `GER`. This does not depend on the choice of the generator of `CHA`, since these are conjugated to each other. This monomial might be looked as the weight of the representation. Therefore we use it as a name:

`Wn_ijk`

The only ambiguity among the available irreducible representations occurs for the two nine-dimensional modules, which are conjugated to each other and distinguished by these names:

`W4_333` and `W4_333bar`

Examples of names:

- `W2_100` – one dimensional representation of the cubic Hecke algebra on 2 strands corresponding to the first root of the cubic equation
- `W3_111` – three dimensional irreducible representation of the cubic Hecke algebra on 3 strands
- `W4_242` – eight dimensional irreducible representation of the cubic Hecke algebra on 4 strands having the second root of the cubic equation as weight of dimension 4

Alternative names are taken from [MW2012] and can be shown by *alternative_name()*.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: [irr.name for irr in chmr.AbsIrreducibeRep]
['W2_100', 'W2_001', 'W2_010', 'W3_100', 'W3_001', 'W3_010', 'W3_011', 'W3_110',
 'W3_101', 'W3_111', 'W4_100', 'W4_001', 'W4_010', 'W4_011', 'W4_110', 'W4_101',
 'W4_111', 'W4_120', 'W4_201', 'W4_012', 'W4_102', 'W4_210', 'W4_021', 'W4_213',
 'W4_132', 'W4_321', 'W4_231', 'W4_123', 'W4_312', 'W4_422', 'W4_224', 'W4_242',
 'W4_333', 'W4_333bar', 'W5_100', 'W5_001', 'W5_010', 'W5_013', 'W5_130', 'W5_301',
 'W5_031', 'W5_103', 'W5_310', 'W5_203', 'W5_032', 'W5_320', 'W5_230', 'W5_023',
 'W5_302', 'W5_033', 'W5_330', 'W5_303', 'W5_163', 'W5_631', 'W5_316', 'W5_136',
 'W5_613', 'W5_361', 'W5_366', 'W5_663', 'W5_636', 'W5_933', 'W5_339', 'W5_393']
```

REFERENCES:

- [MW2012]

W2_001 = {'alt_name': 'Sc', 'dim': 1, 'gap_ind': 1, 'intern_ind': 1, 'len_orbit': 3, 'ngens': 1}

W2_010 = {'alt_name': 'Sb', 'dim': 1, 'gap_ind': 2, 'intern_ind': 2, 'len_orbit': 3, 'ngens': 1}

W2_100 = {'alt_name': 'Sa', 'dim': 1, 'gap_ind': 0, 'intern_ind': 0, 'len_orbit': 3, 'ngens': 1}

W3_001 = {'alt_name': 'Sc', 'dim': 1, 'gap_ind': 1, 'intern_ind': 1, 'len_orbit': 3, 'ngens': 2}

W3_010 = {'alt_name': 'Sb', 'dim': 1, 'gap_ind': 2, 'intern_ind': 2, 'len_orbit': 3, 'ngens': 2}

W3_011 = {'alt_name': 'Tbc', 'dim': 2, 'gap_ind': 3, 'intern_ind': 3, 'len_orbit': 3, 'ngens': 2}

W3_100 = {'alt_name': 'Sa', 'dim': 1, 'gap_ind': 0, 'intern_ind': 0, 'len_orbit': 3, 'ngens': 2}

W3_101 = {'alt_name': 'Tac', 'dim': 2, 'gap_ind': 5, 'intern_ind': 5, 'len_orbit': 3, 'ngens': 2}

W3_110 = {'alt_name': 'Tab', 'dim': 2, 'gap_ind': 4, 'intern_ind': 4, 'len_orbit': 3, 'ngens': 2}

W3_111 = {'alt_name': 'V', 'dim': 3, 'gap_ind': 6, 'intern_ind': 6, 'len_orbit': 1, 'ngens': 2}

W4_001 = {'alt_name': 'Sc', 'dim': 1, 'gap_ind': 1, 'intern_ind': 1, 'len_orbit': 3, 'ngens': 3}

W4_010 = {'alt_name': 'Sb', 'dim': 1, 'gap_ind': 2, 'intern_ind': 2, 'len_orbit': 3, 'ngens': 3}

W4_011 = {'alt_name': 'Tbc', 'dim': 2, 'gap_ind': 3, 'intern_ind': 3, 'len_orbit': 3, 'ngens': 3}

W4_012 = {'alt_name': 'Ucb', 'dim': 3, 'gap_ind': 9, 'intern_ind': 9, 'len_orbit': 6, 'ngens': 3}

W4_021 = {'alt_name': 'Ubc', 'dim': 3, 'gap_ind': 12, 'intern_ind': 12, 'len_orbit': 6, 'ngens': 3}

W4_100 = {'alt_name': 'Sa', 'dim': 1, 'gap_ind': 0, 'intern_ind': 0, 'len_orbit': 3, 'ngens': 3}

W4_101 = {'alt_name': 'Tac', 'dim': 2, 'gap_ind': 5, 'intern_ind': 5, 'len_orbit': 3, 'ngens': 3}

W4_102 = {'alt_name': 'Uca', 'dim': 3, 'gap_ind': 10, 'intern_ind': 10, 'len_orbit': 6, 'ngens': 3}

W4_110 = {'alt_name': 'Tab', 'dim': 2, 'gap_ind': 4, 'intern_ind': 4, 'len_orbit': 3, 'ngens': 3}

W4_111 = {'alt_name': 'V', 'dim': 3, 'gap_ind': 6, 'intern_ind': 6, 'len_orbit': 1, 'ngens': 3}

W4_120 = {'alt_name': 'Uba', 'dim': 3, 'gap_ind': 7, 'intern_ind': 7, 'len_orbit': 6, 'ngens': 3}

W4_123 = {'alt_name': 'Vcba', 'dim': 6, 'gap_ind': 17, 'intern_ind': 17, 'len_orbit': 6, 'ngens': 3}

W4_132 = {'alt_name': 'Vbca', 'dim': 6, 'gap_ind': 14, 'intern_ind': 14, 'len_orbit': 6, 'ngens': 3}

W4_201 = {'alt_name': 'Uac', 'dim': 3, 'gap_ind': 8, 'intern_ind': 8, 'len_orbit': 6, 'ngens': 3}

W4_210 = {'alt_name': 'Uab', 'dim': 3, 'gap_ind': 11, 'intern_ind': 11, 'len_orbit': 6, 'ngens': 3}

W4_213 = {'alt_name': 'Vcab', 'dim': 6, 'gap_ind': 13, 'intern_ind': 13, 'len_orbit': 6, 'ngens': 3}

W4_224 = {'alt_name': 'Wc', 'dim': 8, 'gap_ind': 20, 'intern_ind': 20, 'len_orbit': 3, 'ngens': 3}

W4_231 = {'alt_name': 'Vbac', 'dim': 6, 'gap_ind': 16, 'intern_ind': 16, 'len_orbit': 6, 'ngens': 3}

W4_242 = {'alt_name': 'Wb', 'dim': 8, 'gap_ind': 21, 'intern_ind': 21, 'len_orbit': 3, 'ngens': 3}

W4_312 = {'alt_name': 'Vacb', 'dim': 6, 'gap_ind': 18, 'intern_ind': 18, 'len_orbit': 6, 'ngens': 3}

W4_321 = {'alt_name': 'Vabc', 'dim': 6, 'gap_ind': 15, 'intern_ind': 15, 'len_orbit': 6, 'ngens': 3}

W4_333 = {'alt_name': 'X', 'dim': 9, 'gap_ind': 22, 'intern_ind': 22, 'len_orbit': 2, 'ngens': 3}

W4_333bar = {'alt_name': 'Xbar', 'dim': 9, 'gap_ind': 23, 'intern_ind': 23, 'len_orbit': 2, 'ngens': 3}

W4_422 = {'alt_name': 'Wa', 'dim': 8, 'gap_ind': 19, 'intern_ind': 19, 'len_orbit': 3, 'ngens': 3}

W5_001 = {'alt_name': None, 'dim': 1, 'gap_ind': 1, 'intern_ind': 1, 'len_orbit': 3, 'ngens': 4}

W5_010 = {'alt_name': None, 'dim': 1, 'gap_ind': 2, 'intern_ind': 2, 'len_orbit': 3, 'ngens': 4}

W5_013 = {'alt_name': None, 'dim': 4, 'gap_ind': 3, 'intern_ind': 3, 'len_orbit': 6, 'ngens': 4}

W5_023 = {'alt_name': None, 'dim': 5, 'gap_ind': 13, 'intern_ind': 13, 'len_orbit': 6, 'ngens': 4}

W5_031 = {'alt_name': None, 'dim': 4, 'gap_ind': 6, 'intern_ind': 6, 'len_orbit': 6, 'ngens': 4}

W5_032 = {'alt_name': None, 'dim': 5, 'gap_ind': 10, 'intern_ind': 10, 'len_orbit': 6, 'ngens': 4}

W5_033 = {'alt_name': None, 'dim': 6, 'gap_ind': 15, 'intern_ind': 15, 'len_orbit': 3, 'ngens': 4}

W5_100 = {'alt_name': None, 'dim': 1, 'gap_ind': 0, 'intern_ind': 0, 'len_orbit': 3, 'ngens': 4}

W5_103 = {'alt_name': None, 'dim': 4, 'gap_ind': 7, 'intern_ind': 7, 'len_orbit': 6, 'ngens': 4}

W5_130 = {'alt_name': None, 'dim': 4, 'gap_ind': 4, 'intern_ind': 4, 'len_orbit': 6, 'ngens': 4}

W5_136 = {'alt_name': None, 'dim': 10, 'gap_ind': 21, 'intern_ind': 21, 'len_orbit': 6, 'ngens': 4}

W5_163 = {'alt_name': None, 'dim': 10, 'gap_ind': 18, 'intern_ind': 18, 'len_orbit': 6, 'ngens': 4}

W5_203 = {'alt_name': None, 'dim': 5, 'gap_ind': 9, 'intern_ind': 9, 'len_orbit': 6, 'ngens': 4}

W5_230 = {'alt_name': None, 'dim': 5, 'gap_ind': 12, 'intern_ind': 12, 'len_orbit': 6, 'ngens': 4}

W5_301 = {'alt_name': None, 'dim': 4, 'gap_ind': 5, 'intern_ind': 5, 'len_orbit': 6, 'ngens': 4}

W5_302 = {'alt_name': None, 'dim': 5, 'gap_ind': 14, 'intern_ind': 14, 'len_orbit': 6, 'ngens': 4}

W5_303 = {'alt_name': None, 'dim': 6, 'gap_ind': 17, 'intern_ind': 17, 'len_orbit': 3, 'ngens': 4}

W5_310 = {'alt_name': None, 'dim': 4, 'gap_ind': 8, 'intern_ind': 8, 'len_orbit': 6, 'ngens': 4}

W5_316 = {'alt_name': None, 'dim': 10, 'gap_ind': 20, 'intern_ind': 20, 'len_orbit': 6, 'ngens': 4}

W5_320 = {'alt_name': None, 'dim': 5, 'gap_ind': 11, 'intern_ind': 11, 'len_orbit': 6, 'ngens': 4}

W5_330 = {'alt_name': None, 'dim': 6, 'gap_ind': 16, 'intern_ind': 16, 'len_orbit': 3, 'ngens': 4}

W5_339 = {'alt_name': None, 'dim': 15, 'gap_ind': 28, 'intern_ind': 28, 'len_orbit': 3, 'ngens': 4}

W5_361 = {'alt_name': None, 'dim': 10, 'gap_ind': 23, 'intern_ind': 23, 'len_orbit': 6, 'ngens': 4}

W5_366 = {'alt_name': None, 'dim': 15, 'gap_ind': 24, 'intern_ind': 24, 'len_orbit': 3, 'ngens': 4}

W5_393 = {'alt_name': None, 'dim': 15, 'gap_ind': 29, 'intern_ind': 29, 'len_orbit': 3, 'ngens': 4}

W5_613 = {'alt_name': None, 'dim': 10, 'gap_ind': 22, 'intern_ind': 22, 'len_orbit': 6, 'ngens': 4}

W5_631 = {'alt_name': None, 'dim': 10, 'gap_ind': 19, 'intern_ind': 19, 'len_orbit': 6, 'ngens': 4}

W5_636 = {'alt_name': None, 'dim': 15, 'gap_ind': 27, 'intern_ind': 26, 'len_orbit': 3, 'ngens': 4}

W5_663 = {'alt_name': None, 'dim': 15, 'gap_ind': 26, 'intern_ind': 25, 'len_orbit': 3, 'ngens': 4}

W5_933 = {'alt_name': None, 'dim': 15, 'gap_ind': 25, 'intern_ind': 27, 'len_orbit': 3, 'ngens': 4}

alternative_name()

Return the name of the split irreducible representation for cubic Hecke algebras for up to four strands as given in [MW2012].

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.AbsIrreducibeRep.W3_011.alternative_name()
'Tbc'
```

dimension()

Return the dimension of the representation.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.AbsIrreducibeRep.W3_111.dimension()
3
```

gap_index()

Return the array index of this representation for the access to the GAP3 package CHEVIE.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.AbsIrreducibeRep.W3_111.gap_index()
6
```

internal_index()

Return the array index of this representation for the internal access.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.AbsIrreducibeRep.W3_111.internal_index()
6
```

length_orbit()

Return the length of the orbit of this representation under the action of the Galois group of the cubic equation.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.AbsIrreducibeRep.W3_001.length_orbit()
3
sage: chmr.AbsIrreducibeRep.W3_111.length_orbit()
1
```

number_gens()

Return the number of generators of the underlying cubic Hecke algebra.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.AbsIrreducibeRep.W3_001.number_gens()
2
sage: chmr.AbsIrreducibeRep.W4_001.number_gens()
3
```

class sage.algebras.hecke_algebras.cubic_hecke_matrix_rep.**CubicHeckeMatrixRep**

Bases: Matrix_generic_dense

Class to supervise the diagonal block matrix structure arising from cubic Hecke algebra-representations.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: MS = chmr.CubicHeckeMatrixSpace(CHA2)
sage: m1 = MS(c1); m1
[          a           0           0]
[          0           b           0]
[          0           0 -b - a + u]
sage: type(m1)
<class 'sage.algebras.hecke_algebras.cubic_hecke_matrix_rep.CubicHeckeMatrixSpace_
↪with_category.element_class'>
sage: m1.block_diagonal_list()
[[a], [b], [-b - a + u]]

sage: MSo = chmr.CubicHeckeMatrixSpace(CHA2, original=True)
sage: MSo(c1)
[a 0 0]
[0 b 0]
[0 0 c]

sage: reg_left = chmr.RepresentationType.RegularLeft
sage: MSreg = chmr.CubicHeckeMatrixSpace(CHA2, representation_type=reg_left)
sage: MSreg(c1)
[ 0 -v  1]
[ 1  u  0]
[ 0  w  0]
sage: len(_.block_diagonal_list())
1
```

block_diagonal_list()

Return the list of sub-matrix blocks of `self` considered as block diagonal matrix.

OUTPUT:

A list of instances of `Matrix_generic_dense` each of which represents a diagonal block of `self`.

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: c1.matrix().block_diagonal_list()
[[a], [b], [-b - a + u]]
```

reduce_to_irr_block(*irr*)

Return a copy of `self` with zeroes outside the block corresponding to `irr` but the block according to the input identical to that of `self`.

INPUT:

- `irr` – an *AbsIrreducibeRep* specifying an absolute irreducible representation of the cubic Hecke algebra; alternatively, it can be specified by list index (see `internal_index()` respectively `gap_index()`)

OUTPUT:

An instance of `Matrix_generic_dense` with exactly one non zero block according to `irr`.

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: m1 = c1.matrix()
sage: m1.reduce_to_irr_block(0)
[a 0 0]
[0 0 0]
[0 0 0]
sage: m1.reduce_to_irr_block(CHA2.irred_repr.W2_001)
[0 0 0]
[0 b 0]
[0 0 0]
```

class sage.algebras.hecke_algebras.cubic_hecke_matrix_rep.**CubicHeckeMatrixSpace**(*base_ring*, *dimension*, *cubic_hecke_algebra*, *representation_type*, *subdivide*)

Bases: MatrixSpace

The matrix space of cubic Hecke algebra representations.

INPUT:

- cubic_hecke_algebra – (optional) *CubicHeckeAlgebra* must be given if element fails to be an instance of its element class
- representation_type – (default: RepresentationType.SplitIrredChevie) *RepresentationType* specifying the type of the representation
- subdivide – boolean (default: False); whether or not to subdivide the resulting matrices
- original – boolean (default: False) if True, the matrix will have coefficients in the generic base / extension ring

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: c1.matrix()      # indirect doctest
[         a           0           0]
[         0           b           0]
[         0           0 -b - a + u]
sage: c1.matrix(original=True)
[a 0 0]
[0 b 0]
[0 0 c]
sage: c1.matrix(representation_type = CHA2.repr_type.RegularLeft)    # indirect
↪doctest
[ 0 -v  1]
[ 1  u  0]
[ 0  w  0]
```

construction()

Return None since this construction is not functorial.

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: MS = c1.matrix().parent()
sage: MS._test_category()   # indirect doctest
```

one()

Return the one element of `self`.

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: m1   = c1.matrix()
sage: m1rl = c1.matrix(representation_type = CHA2.repr_type.RegularLeft)
sage: o    = m1.parent().one()
sage: orl  = m1rl.parent().one()
sage: matrix(o) == matrix(orl), o.is_one(), orl.is_one()
(True, True, True)
sage: o.block_diagonal_list()
[[1], [1], [1]]
sage: orl.block_diagonal_list()
[
[1 0 0]
[0 1 0]
[0 0 1]
]
```

some_elements()

Return a generator of elements of `self`.

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2, cubic_equation_roots=(2, 3, 5))
sage: M = c1.matrix(); M
[2 0 0]
[0 3 0]
[0 0 5]
sage: MS = M.parent()
sage: MS.some_elements()
(
[ 94/3     0     0]
[    0 187/3     0]
[    0     0 373/3]
)
sage: MS.some_elements() == tuple(MS(x) for x in CHA2.some_elements())
True
```

zero()

Return the zero element of `self`.

EXAMPLES:

```
sage: CHA2.<c1> = algebras.CubicHecke(2)
sage: m1   = c1.matrix()
sage: m1rl = c1.matrix(representation_type = CHA2.repr_type.RegularLeft)
sage: z    = m1.parent().zero()
```

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```
sage: zrl = m1rl.parent().zero()
sage: matrix(z) == matrix(zrl), z.is_zero(), zrl.is_zero()
(True, True, True)
sage: z.block_diagonal_list()
[[0], [0], [0]]
sage: zrl.block_diagonal_list()
[
[0 0 0]
[0 0 0]
[0 0 0]
]
```

class sage.algebras.hecke_algebras.cubic_hecke_matrix_rep.**GenSign**(*value*)

Bases: `Enum`

Enum class to select the braid generators sign.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.GenSign.pos
<GenSign.pos: 1>
sage: chmr.GenSign.neg
<GenSign.neg: -1>
```

neg = -1

pos = 1

class sage.algebras.hecke_algebras.cubic_hecke_matrix_rep.**RepresentationType**(*value*)

Bases: `Enum`

Enum class to select a representation type for the cubic Hecke algebra.

- `RegularLeft` – left regular representations
- `RegularRight` – right regular representations
- `SplitIrredMarin` – split irreducible representations obtained from Ivan Marin's data
- `SplitIrredChevie` – the split irreducible representations obtained from CHEVIE via the `GAP3` interface

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.RepresentationType.RegularLeft.is_regular()
True
```

RegularLeft = {'data': CubicHeckeDataSection.regular_left, 'num_rep': [1, 1, 1, 1], 'regular': True, 'split': False}

RegularRight = {'data': CubicHeckeDataSection.regular_right, 'num_rep': [1, 1, 1, 1], 'regular': True, 'split': False}

SplitIrredChevie = {'data': None, 'num_rep': [1, 3, 7, 24, 30], 'regular': False, 'split': True}

SplitIrredMarin = {'data': CubicHeckeDataSection.split_irred, 'num_rep': [1, 3, 7, 24], 'regular': False, 'split': True}

data_section()

Return the name of the data file. For more information see CubicHeckeDataBase.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: reg_left = chmr.RepresentationType.RegularLeft
sage: reg_left.data_section()
<CubicHeckeDataSection.regular_left: 'regular_left'>
```

is_regular()

Return True if this representation type is regular, False else-wise.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: reg_left = chmr.RepresentationType.RegularLeft
sage: reg_left.is_regular()
True
```

is_split()

Return True if this representation type is absolutely split, False else-wise.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chevie = chmr.RepresentationType.SplitIrredChevie
sage: chevie.is_split()
True
```

number_of_representations(*nstrands*)

Return the number of representations existing to that type.

EXAMPLES:

```
sage: import sage.algebras.hecke_algebras.cubic_hecke_matrix_rep as chmr
sage: chmr.RepresentationType.SplitIrredChevie.number_of_representations(4)
24
sage: chmr.RepresentationType.SplitIrredMarin.number_of_representations(4)
24
```

CHAPTER

SEVEN

GRADED ALGEBRAS

7.1 Finite dimensional graded commutative algebras

AUTHORS:

- Michael Jung (2021): initial version

`class sage.algebras.finite_gca.FiniteGCAlgebra(base, names, degrees, max_degree, category=None, **kwargs)`

Bases: `CombinatorialFreeModule`, `Algebra`

Finite dimensional graded commutative algebras.

A finite dimensional graded commutative algebra A is an integer-graded algebra satisfying the super-algebra relation w.r.t. the degree modulo 2. More precisely, A has a graded ring structure

$$A = \bigoplus_{i=0}^{n} A_i,$$

where $n \in \mathbf{N}$ is the finite maximal degree, and the multiplication satisfies

$$A_i \cdot A_j \subset \begin{cases} A_{i+j} & \text{if } i+j \leq n, \\ 0 & \text{if } i+j > n, \end{cases}$$

as well as the super-algebra relation

$$xy = (-1)^{ij} yx$$

for all homogeneous elements $x \in A_i$ and $y \in A_j$.

Such an algebra is multiplicatively generated by a set of single monomials $\{x_1, \ldots, x_k\}$, where each x_i is given a certain degree $\deg(x_i)$. To that end, this algebra can be given a vector space basis, and the basis vectors are of the form $x_1^{w_1} \cdots x_n^{w_k}$, where $\sum_{i=1}^{k} \deg(x_i) \, w_i \leq n$ and

$$w_i \in \begin{cases} \mathbf{Z}_2 & \text{if } \deg(x_i) \text{ is odd,} \\ \mathbf{N} & \text{if } \deg(x_i) \text{ is even.} \end{cases}$$

Typical examples of finite dimensional graded commutative algebras are cohomology rings over finite dimensional CW-complexes.

INPUT:

- `base` – the base field
- `names` – (optional) names of the generators: a list of strings or a single string with the names separated by commas. If not specified, the generators are named "x0", "x1",…

- degrees – (optional) a tuple or list specifying the degrees of the generators; if omitted, each generator is given degree 1, and if both names and degrees are omitted, an error is raised.
- max_degree – the maximal degree of the graded algebra.
- mul_symbol – (optional) symbol used for multiplication. If omitted, the string "*" is used.
- mul_latex_symbol – (optional) latex symbol used for multiplication. If omitted, the empty string is used.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1,2,2,3), max_degree=6)
sage: A
Graded commutative algebra with generators ('x', 'y', 'z', 't') in degrees (1, 2, 2,
↪ 3) with maximal degree 6
sage: t*x + x*t
0
sage: x^2
0
sage: x*t^2
0
sage: x*y^2+z*t
x*y^2 + z*t
```

The generators can be returned with *algebra_generators()*:

```
sage: F = A.algebra_generators(); F
Family (x, y, z, t)
sage: [g.degree() for g in F]
[1, 2, 2, 3]
```

We can also return the basis:

```
sage: list(A.basis())
[1, x, z, y, t, x*z, x*y, x*t, z^2, y*z, y^2, z*t, y*t, x*z^2, x*y*z, x*y^2]
```

Depending on the context, the multiplication can be given a different symbol:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1,2,6,6), max_degree=10,␣
↪mul_symbol='⌣', mul_latex_symbol=r'\smile')
sage: x*y^2 + x*t
x⌣y^2 + x⌣t
sage: latex(x*y^2 - z*x)
x\smile y^{2} - x\smile z
```

Note: Notice, when the argument max_degree in the global namespace is omitted, an instance of the class *sage.algebras.commutative_dga.GCAlgebra* is created instead:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1,2,6,6))
sage: type(A)
<class 'sage.algebras.commutative_dga.GCAlgebra_with_category'>
```

---

algebra_generators()

Return the generators of self as a sage.sets.family.TrivialFamily.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(4,8,2), max_degree=10)
sage: A.algebra_generators()
Family (x, y, z)
```

degree_on_basis(i)

Return the degree of a homogeneous element with index i.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=(2,4,6), max_degree=7)
sage: a.degree()
2
sage: (2*a*b).degree()
6
sage: (a+b).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

gen(i)

Return the i-th generator of `self`.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(4,8,2), max_degree=10)
sage: A.gen(0)
x
sage: A.gen(1)
y
sage: A.gen(2)
z
```

gens()

Return the generators of `self` as a tuple.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(4,8,2), max_degree=10)
sage: A.gens()
(x, y, z)
```

max_degree()

Return the maximal degree of `self`.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,2,3), max_degree=8)
sage: A.maximal_degree()
8
```

maximal_degree()

Return the maximal degree of `self`.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,2,3), max_degree=8)
sage: A.maximal_degree()
8
```

ngens()

Return the number of generators of `self`.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(4,8,2), max_degree=10)
sage: A.ngens()
3
```

one_basis()

Return the index of the one element of `self`.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(4,8,2), max_degree=10)
sage: ind = A.one_basis(); ind
[0, 0, 0]
sage: A.monomial(ind)
1
sage: A.one()  # indirect doctest
1
```

product_on_basis(*w1*, *w2*)

Return the product of two indices within the algebra.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(4,8,2), max_degree=10)
sage: z*x
x*z
sage: x^3
0
sage: 5*z + 4*z*x
5*z + 4*x*z
```

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,2,3), max_degree=5)
sage: 2*x*y
2*x*y
sage: x^2
0
sage: x*z
x*z
sage: z*x
-x*z
sage: x*y*z
0
```

7.2 Commutative Differential Graded Algebras

An algebra is said to be *graded commutative* if it is endowed with a grading and its multiplication satisfies the Koszul sign convention: $yx = (-1)^{ij} xy$ if x and y are homogeneous of degrees i and j, respectively. Thus the multiplication is anticommutative for odd degree elements, commutative otherwise. *Commutative differential graded algebras* are graded commutative algebras endowed with a graded differential of degree 1. These algebras can be graded over the integers or they can be multi-graded (i.e., graded over a finite rank free abelian group $\mathbf{Z}^n$); if multi-graded, the total degree is used in the Koszul sign convention, and the differential must have total degree 1.

EXAMPLES:

All of these algebras may be constructed with the function `GradedCommutativeAlgebra()`. For most users, that will be the main function of interest. See its documentation for many more examples.

We start by constructing some graded commutative algebras. Generators have degree 1 by default:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ)
sage: x.degree()
1
sage: x^2
0
sage: y*x
-x*y
sage: B.<a,b> = GradedCommutativeAlgebra(QQ, degrees = (2,3))
sage: a.degree()
2
sage: b.degree()
3
```

Once we have defined a graded commutative algebra, it is easy to define a differential on it using the `GCAlgebra.cdg_algebra()` method:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,1,2))
sage: B = A.cdg_algebra({x: x*y, y: -x*y})
sage: B
Commutative Differential Graded Algebra with generators ('x', 'y', 'z') in degrees (1, 1,
↪ 2) over Rational Field with differential:
    x --> x*y
    y --> -x*y
    z --> 0
sage: B.cohomology(3)
Free module generated by {[x*z + y*z]} over Rational Field
sage: B.cohomology(4)
Free module generated by {[z^2]} over Rational Field
```

We can also compute algebra generators for the cohomology in a range of degrees, and in this case we compute up to degree 10:

```
sage: B.cohomology_generators(10)
{1: [x + y], 2: [z]}
```

AUTHORS:

- Miguel Marco, John Palmieri (2014-07): initial version

class sage.algebras.commutative_dga.**CohomologyClass**(*x*, *cdga=None*)

Bases: SageObject, CachedRepresentation

A class for representing cohomology classes.

This just has `_repr_` and `_latex_` methods which put brackets around the object's name.

EXAMPLES:

```
sage: from sage.algebras.commutative_dga import CohomologyClass
sage: CohomologyClass(3)
[3]
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees = (2,2,3,3))
sage: CohomologyClass(x^2+2*y*z, A)
[2*y*z + x^2]
```

representative()

Return the representative of `self`.

EXAMPLES:

```
sage: from sage.algebras.commutative_dga import CohomologyClass
sage: x = CohomologyClass(sin)
sage: x.representative() == sin
True
```

class sage.algebras.commutative_dga.**Differential**(*A*, *im_gens*)

Bases: UniqueRepresentation, Morphism

Differential of a commutative graded algebra.

INPUT:

- `A` – algebra where the differential is defined
- `im_gens` – tuple containing the image of each generator

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 2, 3))
sage: B = A.cdg_algebra({x: x*y, y: -x*y , z: t})
sage: B
Commutative Differential Graded Algebra with generators ('x', 'y', 'z', 't') in␣
↪degrees (1, 1, 2, 3) over Rational Field with differential:
    x --> x*y
    y --> -x*y
    z --> t
    t --> 0
sage: B.differential()(x)
x*y
```

coboundaries(*n*)

The `n`-th coboundary group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the `n`-th homogeneous component has dimension d.

INPUT:

- `n` – degree

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 2))
sage: d = A.differential({z: x*z})
sage: d.coboundaries(2)
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
sage: d.coboundaries(3)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
sage: d.coboundaries(1)
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
```

cocycles(n)

The n-th cocycle group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the n-th homogeneous component has dimension d.

INPUT:

- n – degree

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 2))
sage: d = A.differential({z: x*z})
sage: d.cocycles(2)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
```

cohomology(n)

The n-th cohomology group of self.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

INPUT:

- n – degree

See also:

cohomology_raw()

EXAMPLES:

```
sage: A.<a,b,c,d,e> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 1, 1, 1))
sage: d = A.differential({d: a*b, e: b*c})
sage: d.cohomology(2)
Free module generated by {[a*c], [a*d], [b*d], [c*d - a*e], [b*e], [c*e]} over
↪Rational Field
```

Compare to *cohomology_raw()*:

```
sage: d.cohomology_raw(2)
Vector space quotient V/W of dimension 6 over Rational Field where
V: Vector space of degree 10 and dimension 8 over Rational Field
Basis matrix:
[ 1  0  0  0  0  0  0  0  0  0]
[ 0  1  0  0  0  0  0  0  0  0]
[ 0  0  1  0  0  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0  0  0]
[ 0  0  0  0  1  0  0  0  0  0]
[ 0  0  0  0  0  1 -1  0  0  0]
[ 0  0  0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  0  0  1  0]
W: Vector space of degree 10 and dimension 2 over Rational Field
Basis matrix:
[1 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0]
```

cohomology_raw(*n*)

The n-th cohomology group of self.

This is a vector space over the base ring, and it is returned as the quotient cocycles/coboundaries.

INPUT:

- n – degree

See also:

cohomology()

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(2, 2, 3, 4))
sage: d = A.differential({t: x*z, x: z, y: z})
sage: d.cohomology_raw(4)
Vector space quotient V/W of dimension 2 over Rational Field where
V: Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
[   1    0    0   -2]
[   0    1 -1/2   -1]
W: Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

Compare to *cohomology()*:

```
sage: d.cohomology(4)
Free module generated by {[x^2 - 2*t], [x*y - 1/2*y^2 - t]} over Rational Field
```

differential_matrix(*n*)

The matrix that gives the differential in degree n.

INPUT:

- n – degree

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(GF(5), degrees=(2, 2, 3, 4))
sage: d = A.differential({t: x*z, x: z, y: z})
sage: d.differential_matrix(4)
[2 0]
[1 1]
[0 2]
[1 0]
sage: A.inject_variables()
Defining x, y, z, t
sage: d(t)
x*z
sage: d(y^2)
2*y*z
sage: d(x*y)
x*z + y*z
sage: d(x^2)
2*x*z
```

homology(*n*)

The n-th cohomology group of self.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

INPUT:

- n – degree

See also:

cohomology_raw()

EXAMPLES:

```
sage: A.<a,b,c,d,e> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 1, 1, 1))
sage: d = A.differential({d: a*b, e: b*c})
sage: d.cohomology(2)
Free module generated by {[a*c], [a*d], [b*d], [c*d - a*e], [b*e], [c*e]} over␣
↪Rational Field
```

Compare to *cohomology_raw()*:

```
sage: d.cohomology_raw(2)
Vector space quotient V/W of dimension 6 over Rational Field where
V: Vector space of degree 10 and dimension 8 over Rational Field
Basis matrix:
[ 1  0  0  0  0  0  0  0  0  0]
[ 0  1  0  0  0  0  0  0  0  0]
[ 0  0  1  0  0  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0  0  0]
[ 0  0  0  0  1  0  0  0  0  0]
[ 0  0  0  0  0  1 -1  0  0  0]
[ 0  0  0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  0  0  1  0]
W: Vector space of degree 10 and dimension 2 over Rational Field
Basis matrix:
```

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```
[1 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0]
```

class sage.algebras.commutative_dga.**DifferentialGCAlgebra**(*A*, *differential*)

Bases: *GCAlgebra*

A commutative differential graded algebra.

INPUT:

- A – a graded commutative algebra; that is, an instance of *GCAlgebra*
- differential – a differential

As described in the module-level documentation, these are graded algebras for which oddly graded elements anticommute and evenly graded elements commute, and on which there is a graded differential of degree 1.

These algebras should be graded over the integers; multi-graded algebras should be constructed using *DifferentialGCAlgebra_multigraded* instead.

Note that a natural way to construct these is to use the *GradedCommutativeAlgebra()* function and the *GCAlgebra.cdg_algebra()* method.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(2, 2, 3, 3))
sage: A.cdg_algebra({z: x*y})
Commutative Differential Graded Algebra with generators ('x', 'y', 'z', 't') in␣
↪degrees (2, 2, 3, 3) over Rational Field with differential:
    x --> 0
    y --> 0
    z --> x*y
    t --> 0
```

Alternatively, starting with *GradedCommutativeAlgebra()*:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(2, 2, 3, 3))
sage: A.cdg_algebra(differential={z: x*y})
Commutative Differential Graded Algebra with generators ('x', 'y', 'z', 't') in␣
↪degrees (2, 2, 3, 3) over Rational Field with differential:
    x --> 0
    y --> 0
    z --> x*y
    t --> 0
```

See the function *GradedCommutativeAlgebra()* for more examples.

class **Element**(*A*, *rep*)

Bases: *Element*

cohomology_class()

Return the cohomology class of an homogeneous cycle, as an element of the corresponding cohomology group.

EXAMPLES:

```
sage: A.<e1,e2,e3,e4,e5> = GradedCommutativeAlgebra(QQ)
sage: B = A.cdg_algebra({e5:e1*e2+e3*e4})
sage: B.inject_variables()
Defining e1, e2, e3, e4, e5
sage: a = e1*e3*e5-3*e2*e3*e5
sage: a.cohomology_class()
B[[e1*e3*e5]] - 3*B[[e2*e3*e5]]
```

differential()

The differential on this element.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees = (2, 2, 3, 4))
sage: B = A.cdg_algebra({t: x*z, x: z, y: z})
sage: B.inject_variables()
Defining x, y, z, t
sage: x.differential()
z
sage: (-1/2 * x^2 + t).differential()
0
```

is_coboundary()

Return `True` if `self` is a coboundary and `False` otherwise.

This raises an error if the element is not homogeneous.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=(1,2,2))
sage: B = A.cdg_algebra(differential={b: a*c})
sage: x,y,z = B.gens()
sage: x.is_coboundary()
False
sage: (x*z).is_coboundary()
True
sage: (x*z+x*y).is_coboundary()
False
sage: (x*z+y**2).is_coboundary()
Traceback (most recent call last):
...
ValueError: this element is not homogeneous
```

is_cohomologous_to(*other*)

Return `True` if `self` is cohomologous to `other` and `False` otherwise.

INPUT:

- `other` – another element of this algebra

EXAMPLES:

```
sage: A.<a,b,c,d> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1))
sage: B = A.cdg_algebra(differential={a:b*c-c*d})
sage: w, x, y, z = B.gens()
sage: (x*y).is_cohomologous_to(y*z)
True
```

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```
sage: (x*y).is_cohomologous_to(x*z)
False
sage: (x*y).is_cohomologous_to(x*y)
True
```

Two elements whose difference is not homogeneous are cohomologous if and only if they are both coboundaries:

```
sage: w.is_cohomologous_to(y*z)
False
sage: (x*y-y*z).is_cohomologous_to(x*y*z)
True
sage: (x*y*z).is_cohomologous_to(0) # make sure 0 works
True
```

cdg_algebra(*differential*)

Construct a differential graded commutative algebra from the underlying graded commutative algebra by specifying a differential. This may be used to get a new differential over the same algebra structure.

INPUT:

- `differential` – a dictionary defining a differential or a map defining a valid differential

The keys of the dictionary are generators of the algebra, and the associated values are their targets under the differential. Any generators which are not specified are assumed to have zero differential. Alternatively, the differential can be defined using the `differential()` method; see below for an example.

See also:

`differential()`

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(GF(5), degrees=(2, 3, 2, 4))
sage: B = A.quotient(A.ideal(x^3-z*t))
sage: C = B.cdg_algebra({y:t})
sage: C
Commutative Differential Graded Algebra with generators ('x', 'y', 'z', 't') in␣
↪degrees (2, 3, 2, 4) with relations [x^3 - z*t] over Finite Field of size 5␣
↪with differential:
x --> 0
y --> t
z --> 0
t --> 0
sage: C.cdg_algebra({})
Commutative Differential Graded Algebra with generators ('x', 'y', 'z', 't') in␣
↪degrees (2, 3, 2, 4) with relations [x^3 - z*t] over Finite Field of size 5␣
↪with differential:
x --> 0
y --> 0
z --> 0
t --> 0
```

coboundaries(*n*)

The n-th coboundary group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the n-th homogeneous component has dimension d.

INPUT:

- n – degree

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,1,2))
sage: B = A.cdg_algebra(differential={z: x*z})
sage: B.coboundaries(2)
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
sage: B.coboundaries(3)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
sage: B.basis(3)
[x*z, y*z]
```

cocycles(*n*)

The n-th cocycle group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the n-th homogeneous component has dimension d.

INPUT:

- n – degree

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,1,2))
sage: B = A.cdg_algebra(differential={z: x*z})
sage: B.cocycles(2)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
sage: B.basis(2)
[x*y, z]
```

cohomology(*n*)

The n-th cohomology group of self.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

INPUT:

- n – degree

EXAMPLES:

```
sage: A.<a,b,c,d,e> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1,1))
sage: B = A.cdg_algebra({d: a*b, e: b*c})
sage: B.cohomology(2)
```

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```
Free module generated by {[a*c], [a*d], [b*d], [c*d - a*e], [b*e], [c*e]} over␣
↪Rational Field
```

Compare to *cohomology_raw()*:

```
sage: B.cohomology_raw(2)
Vector space quotient V/W of dimension 6 over Rational Field where
V: Vector space of degree 10 and dimension 8 over Rational Field
Basis matrix:
[ 1  0  0  0  0  0  0  0  0  0]
[ 0  1  0  0  0  0  0  0  0  0]
[ 0  0  1  0  0  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0  0  0]
[ 0  0  0  0  1  0  0  0  0  0]
[ 0  0  0  0  0  1 -1  0  0  0]
[ 0  0  0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  0  0  1  0]
W: Vector space of degree 10 and dimension 2 over Rational Field
Basis matrix:
[1 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0]
```

cohomology_algebra(*max_degree=3*)

Compute a CDGA with trivial differential, that is isomorphic to the cohomology of self up to``max_degree``

INPUT:

- max_degree – integer (default: 3); degree to which the result is required to be isomorphic to self's cohomology.

EXAMPLES:

```
sage: A.<e1, e2, e3, e4, e5, e6, e7> = GradedCommutativeAlgebra(QQ)
sage: d = A.differential({e1:-e1*e6, e2:-e2*e6, e3:-e3*e6, e4:-e5*e6, e5:e4*e6})
sage: B = A.cdg_algebra(d)
sage: M = B.cohomology_algebra()
sage: M
Commutative Differential Graded Algebra with generators ('x0', 'x1', 'x2') in␣
↪degrees (1, 1, 2) over Rational Field with differential:
   x0 --> 0
   x1 --> 0
   x2 --> 0
sage: M.cohomology(1)
Free module generated by {[x0], [x1]} over Rational Field
sage: B.cohomology(1)
Free module generated by {[e6], [e7]} over Rational Field
sage: M.cohomology(2)
Free module generated by {[x0*x1], [x2]} over Rational Field
sage: B.cohomology(2)
Free module generated by {[e4*e5], [e6*e7]} over Rational Field
sage: M.cohomology(3)
Free module generated by {[x0*x2], [x1*x2]} over Rational Field
sage: B.cohomology(3)
Free module generated by {[e4*e5*e6], [e4*e5*e7]} over Rational Field
```

cohomology_generators(*max_degree*)

Return lifts of algebra generators for cohomology in degrees at most `max_degree`.

INPUT:

- `max_degree` – integer

OUTPUT:

A dictionary keyed by degree, where the corresponding value is a list of cohomology generators in that degree. Actually, the elements are lifts of cohomology generators, which means that they lie in this differential graded algebra. It also means that they are only well-defined up to cohomology, not on the nose.

ALGORITHM:

Reduce a basis of the n'th cohomology modulo all the degree n products of the lower degree cohomologies.

EXAMPLES:

```
sage: A.<a,x,y> = GradedCommutativeAlgebra(QQ, degrees=(1,2,2))
sage: B = A.cdg_algebra(differential={y: a*x})
sage: B.cohomology_generators(3)
{1: [a], 2: [x], 3: [a*y]}
```

The previous example has infinitely generated cohomology: ay^n is a cohomology generator for each n:

```
sage: B.cohomology_generators(10)
{1: [a], 2: [x], 3: [a*y], 5: [a*y^2], 7: [a*y^3], 9: [a*y^4]}
```

In contrast, the corresponding algebra in characteristic p has finitely generated cohomology:

```
sage: A3.<a,x,y> = GradedCommutativeAlgebra(GF(3), degrees=(1,2,2))
sage: B3 = A3.cdg_algebra(differential={y: a*x})
sage: B3.cohomology_generators(16)
{1: [a], 2: [x], 3: [a*y], 5: [a*y^2], 6: [y^3]}
```

This method works with both singly graded and multi-graded algebras:

```
sage: Cs.<a,b,c,d> = GradedCommutativeAlgebra(GF(2), degrees=(1,2,2,3))
sage: Ds = Cs.cdg_algebra({a:c, b:d})
sage: Ds.cohomology_generators(10)
{2: [a^2], 4: [b^2]}

sage: Cm.<a,b,c,d> = GradedCommutativeAlgebra(GF(2), degrees=((1,0), (1,1), (0,
↪2), (0,3)))
sage: Dm = Cm.cdg_algebra({a:c, b:d})
sage: Dm.cohomology_generators(10)
{2: [a^2], 4: [b^2]}
```

cohomology_raw(*n*)

The n-th cohomology group of `self`.

This is a vector space over the base ring, and it is returned as the quotient cocycles/coboundaries.

INPUT:

- `n` – degree

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees = (2,2,3,4))
sage: B = A.cdg_algebra({t: x*z, x: z, y: z})
sage: B.cohomology_raw(4)
Vector space quotient V/W of dimension 2 over Rational Field where
V: Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
[   1    0    0   -2]
[   0    1 -1/2   -1]
W: Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

Compare to *cohomology()*:

```
sage: B.cohomology(4)
Free module generated by {[x^2 - 2*t], [x*y - 1/2*y^2 - t]} over Rational Field
```

differential(*x=None*)

The differential of `self`.

This returns a map, and so it may be evaluated on elements of this algebra.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,1,2))
sage: B = A.cdg_algebra({y:x*y, x: y*x})
sage: d = B.differential(); d
Differential of Commutative Differential Graded Algebra with generators ('x', 'y
↪', 'z') in degrees (1, 1, 2) over Rational Field
  Defn: x --> -x*y
        y --> x*y
        z --> 0
sage: d(y)
x*y
```

graded_commutative_algebra()

Return the base graded commutative algebra of `self`.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(2, 2, 3, 3))
sage: D = A.cdg_algebra({z: x*y})
sage: D.graded_commutative_algebra() == A
True
```

homology(*n*)

The n-th cohomology group of `self`.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

INPUT:

- `n` – degree

EXAMPLES:

```
sage: A.<a,b,c,d,e> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1,1,1))
sage: B = A.cdg_algebra({d: a*b, e: b*c})
sage: B.cohomology(2)
Free module generated by {[a*c], [a*d], [b*d], [c*d - a*e], [b*e], [c*e]} over␣
↪Rational Field
```

Compare to *cohomology_raw()*:

```
sage: B.cohomology_raw(2)
Vector space quotient V/W of dimension 6 over Rational Field where
V: Vector space of degree 10 and dimension 8 over Rational Field
Basis matrix:
[ 1  0  0  0  0  0  0  0  0  0]
[ 0  1  0  0  0  0  0  0  0  0]
[ 0  0  1  0  0  0  0  0  0  0]
[ 0  0  0  1  0  0  0  0  0  0]
[ 0  0  0  0  1  0  0  0  0  0]
[ 0  0  0  0  0  1 -1  0  0  0]
[ 0  0  0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  0  0  1  0]
W: Vector space of degree 10 and dimension 2 over Rational Field
Basis matrix:
[1 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0]
```

is_formal(*i*, *max_iterations=3*)

Check if the algebra is `i`-formal. That is, if it is `i`-quasi-isomorphic to its cohomology algebra.

INPUT:

- `i` – integer; the degree up to which the formality is checked
- `max_iterations` – integer (default: 3); the maximum number of iterations used in the computation of the minimal model

Warning: The method is not granted to finish (it can't, since the minimal model could be infinitely generated in some degrees). The parameter `max_iterations` controls how many iterations of the method are attempted at each degree. In case they are not enough, an exception is raised. If you think that the result will be finitely generated, you can try to run it again with a higher value for `max_iterations`.

Moreover, the method uses criteria that are often enough to conclude that the algebra is either formal or non-formal. However, it could happen that the used criteria can not determine the formality. In that case, an error is raised.

EXAMPLES:

```
sage: A.<e1, e2, e3, e4, e5> = GradedCommutativeAlgebra(QQ)
sage: B = A.cdg_algebra({e5 : e1*e2 + e3*e4})
sage: B.is_formal(1)
True
sage: B.is_formal(2)
False
```

ALGORITHM:

Apply the criteria in [Man2019] . Both the i-minimal model of the algebra and its cohomology algebra are computed. If the numerical invariants are different, the algebra is not i-formal.

If the numerical invariants match, the ψ condition is checked.

minimal_model(*i=3*, *max_iterations=3*, *partial_result=False*)

Try to compute a map from a `i`-minimal gcda that is a `i`-quasi-isomorphism to self.

INPUT:

- `i` – integer (default: 3); degree to which the result is required to induce an isomorphism in cohomology, and the domain is required to be minimal.
- `max_iterations` – integer (default: 3); the number of iterations of the method at each degree. If the algorithm does not finish in this many iterations at each degree, an error is raised, or the partial result computed up to that point is returned, deppending on the `partial_result` flag.
- `partial_result` – boolean (default: `False`); wether to return the partial result if the `max_iterations` limit is reached.

OUTPUT:

A morphism from a minimal Sullivan (up to degree `i`) CDGA's to self, that induces an isomorphism in cohomology up to degree `i`, and a monomorphism in degree `i+1`.

EXAMPLES:

```
sage: S.<x, y, z> = GradedCommutativeAlgebra(QQ, degrees = (1, 1, 2))
sage: d = S.differential({x:x*y, y:x*y})
sage: R = S.cdg_algebra(d)
sage: p = R.minimal_model()
sage: T = p.domain()
sage: p
Commutative Differential Graded Algebra morphism:
  From: Commutative Differential Graded Algebra with generators ('x1_0', 'x2_0
↪') in degrees (1, 2) over Rational Field with differential:
   x1_0 --> 0
   x2_0 --> 0
  To:   Commutative Differential Graded Algebra with generators ('x', 'y', 'z')␣
↪in degrees (1, 1, 2) over Rational Field with differential:
   x --> x*y
   y --> x*y
   z --> 0
  Defn: (x1_0, x2_0) --> (x - y, z)
sage: R.cohomology(1)
Free module generated by {[x - y]} over Rational Field
sage: T.cohomology(1)
Free module generated by {[x1_0]} over Rational Field
sage: [p(g.representative()) for g in T.cohomology(1).basis().keys()]
[x - y]
sage: R.cohomology(2)
Free module generated by {[z]} over Rational Field
sage: T.cohomology(2)
Free module generated by {[x2_0]} over Rational Field
sage: [p(g.representative()) for g in T.cohomology(2).basis().keys()]
[z]
```

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```
sage: A.<e1, e2, e3, e4, e5, e6, e7> = GradedCommutativeAlgebra(QQ)
sage: d = A.differential({e1:e1*e7, e2:e2*e7, e3:-e3*e7, e4:-e4*e7})
sage: B = A.cdg_algebra(d)
sage: phi = B.minimal_model(i=3)
sage: M = phi.domain()
sage: M
Commutative Differential Graded Algebra with generators ('x1_0', 'x1_1', 'x1_2',
↪ 'x2_0', 'x2_1', 'x2_2', 'x2_3', 'y3_0', 'y3_1', 'y3_2', 'y3_3', 'y3_4', 'y3_5
↪', 'y3_6', 'y3_7', 'y3_8') in degrees (1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3,␣
↪3, 3, 3) over Rational Field with differential:
   x1_0 --> 0
   x1_1 --> 0
   x1_2 --> 0
   x2_0 --> 0
   x2_1 --> 0
   x2_2 --> 0
   x2_3 --> 0
   y3_0 --> x2_0^2
   y3_1 --> x2_0*x2_1
   y3_2 --> x2_1^2
   y3_3 --> x2_0*x2_2
   y3_4 --> x2_1*x2_2 + x2_0*x2_3
   y3_5 --> x2_2^2
   y3_6 --> x2_1*x2_3
   y3_7 --> x2_2*x2_3
   y3_8 --> x2_3^2

sage: phi
Commutative Differential Graded Algebra morphism:
  From: Commutative Differential Graded Algebra with generators ('x1_0', 'x1_1',
↪ 'x1_2', 'x2_0', 'x2_1', 'x2_2', 'x2_3', 'y3_0', 'y3_1', 'y3_2', 'y3_3', 'y3_4
↪', 'y3_5', 'y3_6', 'y3_7', 'y3_8') in degrees (1, 1, 1, 2, 2, 2, 2, 3, 3, 3,␣
↪3, 3, 3, 3, 3) over Rational Field with differential:
   x1_0 --> 0
   x1_1 --> 0
   x1_2 --> 0
   x2_0 --> 0
   x2_1 --> 0
   x2_2 --> 0
   x2_3 --> 0
   y3_0 --> x2_0^2
   y3_1 --> x2_0*x2_1
   y3_2 --> x2_1^2
   y3_3 --> x2_0*x2_2
   y3_4 --> x2_1*x2_2 + x2_0*x2_3
   y3_5 --> x2_2^2
   y3_6 --> x2_1*x2_3
   y3_7 --> x2_2*x2_3
   y3_8 --> x2_3^2
  To:   Commutative Differential Graded Algebra with generators ('e1', 'e2', 'e3
↪', 'e4', 'e5', 'e6', 'e7') in degrees (1, 1, 1, 1, 1, 1, 1) over Rational␣
```

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```
↪Field with differential:
   e1 --> e1*e7
   e2 --> e2*e7
   e3 --> -e3*e7
   e4 --> -e4*e7
   e5 --> 0
   e6 --> 0
   e7 --> 0
  Defn: (x1_0, x1_1, x1_2, x2_0, x2_1, x2_2, x2_3, y3_0, y3_1, y3_2, y3_3, y3_4,
↪ y3_5, y3_6, y3_7, y3_8) --> (e5, e6, e7, e1*e3, e2*e3, e1*e4, e2*e4, 0, 0, 0,
↪ 0, 0, 0, 0, 0, 0)
sage: [B.cohomology(i).dimension() for i in [1..3]]
[3, 7, 13]
sage: [M.cohomology(i).dimension() for i in [1..3]]
[3, 7, 13]
```

ALGORITHM:

We follow the algorithm described in [Man2019]. It consists in constructing the minimal Sullivan algebra S by iteratively adding generators to it. Start with one closed generator of degree 1 for each element in the basis of the first cohomology of the algebra. Then proceed degree by degree. At each degree d, we keep adding generators of degree $d-1$ whose differential kills the elements in the kernel of the map $H^d(S) \to H^d(self)$. Once this map is made injective, we add the needed closed generators in degree d to make it surjective.

> **Warning:** The method is not granted to finish (it can't, since the minimal model could be infinitely generated in some degrees). The parameter `max_iterations` controls how many iterations of the method are attempted at each degree. In case they are not enough, an exception is raised. If you think that the result will be finitely generated, you can try to run it again with a higher value for `max_iterations`.

See also:

Wikipedia article Rational_homotopy_theory#Sullivan_algebras

REFERENCES:

- [Fel2001]
- [Man2019]

numerical_invariants(*max_degree=3*, *max_iterations=3*)

Return the numerical invariants of the algebra, up to degree `d`. The numerical invariants reflect the number of generators added at each step of the construction of the minimal model.

The numerical invariants are the dimensions of the subsequent Hirsch extensions used at each degree to compute the minimal model.

INPUT:

- `max_degree` – integer (default: 3); the degree up to which the numerical invariants are computed
- `max_iterations` – integer (default: 3); the maximum number of iterations used to compute the minimal model, if it is not already cached

EXAMPLES:

```
sage: A.<e1, e2, e3> = GradedCommutativeAlgebra(QQ)
sage: B = A.cdg_algebra({e3 : e1*e2})
sage: B.minimal_model(4)
Commutative Differential Graded Algebra morphism:
From: Commutative Differential Graded Algebra with generators ('x1_0', 'x1_1',
↪'y1_0') in degrees (1, 1, 1) over Rational Field with differential:
x1_0 --> 0
x1_1 --> 0
y1_0 --> x1_0*x1_1
To:   Commutative Differential Graded Algebra with generators ('e1', 'e2', 'e3
↪') in degrees (1, 1, 1) over Rational Field with differential:
e1 --> 0
e2 --> 0
e3 --> e1*e2
Defn: (x1_0, x1_1, y1_0) --> (e1, e2, e3)
sage: B.numerical_invariants(2)
{1: [2, 1, 0], 2: [0, 0]}
```

ALGORITHM:

The numerical invariants are stored as the minimal model is constructed.

> **Warning:** The method is not granted to finish (it can't, since the minimal model could be infinitely generated in some degrees). The parameter `max_iterations` controls how many iterations of the method are attempted at each degree. In case they are not enough, an exception is raised. If you think that the result will be finitely generated, you can try to run it again with a higher value for `max_iterations`.

REFERENCES:

For a precise definition and properties, see [Man2019] .

quotient(*I*, *check=True*)

Create the quotient of this algebra by a two-sided ideal `I`.

INPUT:

- `I` – a two-sided homogeneous ideal of this algebra
- `check` – (default: `True`) if `True`, check whether `I` is generated by homogeneous elements

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,1,2))
sage: B = A.cdg_algebra({y:x*y, z:x*z})
sage: B.inject_variables()
Defining x, y, z
sage: I = B.ideal([y*z])
sage: C = B.quotient(I)
sage: (y*z).differential()
2*x*y*z
sage: C((y*z).differential())
0
sage: C(y*z)
0
```

It is checked that the differential maps the ideal into itself, to make sure that the quotient inherits a differential structure:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,2,2))
sage: B = A.cdg_algebra({x:y})
sage: B.quotient(B.ideal(y*x))
Traceback (most recent call last):
...
ValueError: the differential does not preserve the ideal
sage: B.quotient(B.ideal(x))
Traceback (most recent call last):
...
ValueError: the differential does not preserve the ideal
```

class sage.algebras.commutative_dga.**DifferentialGCAlgebra_multigraded**(*A*, *differential*)

Bases: *DifferentialGCAlgebra*, *GCAlgebra_multigraded*

A commutative differential multi-graded algebras.

INPUT:

- A – a commutative multi-graded algebra
- differential – a differential

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: B = A.cdg_algebra(differential={a: c})
sage: B.basis((1,0))
[a]
sage: B.basis(1, total=True)
[a, b]
sage: B.cohomology((1, 0))
Free module generated by {} over Rational Field
sage: B.cohomology(1, total=True)
Free module generated by {[b]} over Rational Field
```

class Element(*A*, *rep*)

Bases: *Element*, *Element*

Element class of a commutative differential multi-graded algebra.

coboundaries(*n*, *total=False*)

The n-th coboundary group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the n-th homogeneous component has dimension d.

INPUT:

- n – degree
- total (default False) – if True, return the coboundaries in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: B = A.cdg_algebra(differential={a: c})
sage: B.coboundaries((0,2))
Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]
sage: B.coboundaries(2)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
```

cocycles(*n*, *total=False*)

The n-th cocycle group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the n-th homogeneous component has dimension d.

INPUT:

- n – degree
- total – (default: False) if True, return the cocycles in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: B = A.cdg_algebra(differential={a: c})
sage: B.cocycles((0,1))
Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]
sage: B.cocycles((0,1), total=True)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
```

cohomology(*n*, *total=False*)

The n-th cohomology group of the algebra.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

Compare to *cohomology_raw()*.

INPUT:

- n – degree
- total – (default: False) if True, return the cohomology in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: B = A.cdg_algebra(differential={a: c})
sage: B.cohomology((0,2))
```

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```
Free module generated by {} over Rational Field

sage: B.cohomology(1)
Free module generated by {[b]} over Rational Field
```

cohomology_raw(*n*, *total=False*)

The n-th cohomology group of the algebra.

This is a vector space over the base ring, and it is returned as the quotient cocycles/coboundaries.

Compare to *cohomology()*.

INPUT:

- n – degree
- total – (default: False) if True, return the cohomology in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: B = A.cdg_algebra(differential={a: c})
sage: B.cohomology_raw((0,2))
Vector space quotient V/W of dimension 0 over Rational Field where
V: Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]
W: Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]

sage: B.cohomology_raw(1)
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
W: Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
```

homology(*n*, *total=False*)

The n-th cohomology group of the algebra.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

Compare to *cohomology_raw()*.

INPUT:

- n – degree
- total – (default: False) if True, return the cohomology in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: B = A.cdg_algebra(differential={a: c})
sage: B.cohomology((0,2))
Free module generated by {} over Rational Field

sage: B.cohomology(1)
Free module generated by {[b]} over Rational Field
```

class sage.algebras.commutative_dga.**Differential_multigraded**(*A*, *im_gens*)

Bases: *Differential*

Differential of a commutative multi-graded algebra.

coboundaries(*n*, *total=False*)

The `n`-th coboundary group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the `n`-th homogeneous component has dimension d.

INPUT:

- `n` – degree
- `total` (default `False`) – if `True`, return the coboundaries in total degree `n`

If `n` is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1, 0), (0, 1), (0, 2)))
sage: d = A.differential({a: c})
sage: d.coboundaries((0, 2))
Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]
sage: d.coboundaries(2)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
```

cocycles(*n*, *total=False*)

The `n`-th cocycle group of the algebra.

This is a vector space over the base field F, and it is returned as a subspace of the vector space F^d, where the `n`-th homogeneous component has dimension d.

INPUT:

- `n` – degree
- `total` – (default: `False`) if `True`, return the cocycles in total degree `n`

If `n` is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1, 0), (0, 1), (0, 2)))
sage: d = A.differential({a: c})
sage: d.cocycles((0, 1))
```

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```
Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]
sage: d.cocycles((0, 1), total=True)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
```

cohomology(*n*, *total=False*)

The n-th cohomology group of the algebra.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

INPUT:

- n – degree
- total – (default: False) if True, return the cohomology in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

See also:

cohomology_raw()

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1, 0), (0, 1), (0, 2)))
sage: d = A.differential({a: c})
sage: d.cohomology((0, 2))
Free module generated by {} over Rational Field

sage: d.cohomology(1)
Free module generated by {[b]} over Rational Field
```

cohomology_raw(*n*, *total=False*)

The n-th cohomology group of the algebra.

This is a vector space over the base ring, and it is returned as the quotient cocycles/coboundaries.

INPUT:

- n – degree
- total – (default: False) if True, return the cohomology in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

See also:

cohomology()

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1, 0), (0, 1), (0, 2)))
sage: d = A.differential({a: c})
sage: d.cohomology_raw((0, 2))
Vector space quotient V/W of dimension 0 over Rational Field where
V: Vector space of degree 1 and dimension 1 over Rational Field
```

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```
Basis matrix:
[1]
W: Vector space of degree 1 and dimension 1 over Rational Field
Basis matrix:
[1]

sage: d.cohomology_raw(1)
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
W: Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
```

differential_matrix_multigraded(*n*, *total=False*)

The matrix that gives the differential in degree n.

Todo: Rename this to differential_matrix once inheritance, overriding, and cached methods work together better. See github issue #17201.

INPUT:

- n – degree
- total – (default: False) if True, return the matrix corresponding to total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1, 0), (0, 1), (0, 2)))
sage: d = A.differential({a: c})
sage: d.differential_matrix_multigraded((1, 0))
[1]
sage: d.differential_matrix_multigraded(1, total=True)
[0 1]
[0 0]
sage: d.differential_matrix_multigraded((1, 0), total=True)
[0 1]
[0 0]
sage: d.differential_matrix_multigraded(1)
[0 1]
[0 0]
```

homology(*n*, *total=False*)

The n-th cohomology group of the algebra.

This is a vector space over the base ring, defined as the quotient cocycles/coboundaries. The elements of the quotient are lifted to the vector space of cocycles, and this is described in terms of those lifts.

INPUT:

- n – degree

- total – (default: False) if True, return the cohomology in total degree n

If n is an integer rather than a multi-index, then the total degree is used in that case as well.

See also:

cohomology_raw()

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1, 0), (0, 1), (0, 2)))
sage: d = A.differential({a: c})
sage: d.cohomology((0, 2))
Free module generated by {} over Rational Field

sage: d.cohomology(1)
Free module generated by {[b]} over Rational Field
```

class sage.algebras.commutative_dga.**GCAlgebra**(*base*, *R=None*, *I=None*, *names=None*, *degrees=None*, *category=None*)

Bases: UniqueRepresentation, QuotientRing_nc

A graded commutative algebra.

INPUT:

- base – the base field
- names – (optional) names of the generators: a list of strings or a single string with the names separated by commas. If not specified, the generators are named "x0", "x1", …
- degrees – (optional) a tuple or list specifying the degrees of the generators; if omitted, each generator is given degree 1, and if both names and degrees are omitted, an error is raised.
- R (optional, default None) – the ring over which the algebra is defined: if this is specified, the algebra is defined to be R/I.
- I (optional, default None) – an ideal in R. It is should include, among other relations, the squares of the generators of odd degree

As described in the module-level documentation, these are graded algebras for which oddly graded elements anticommute and evenly graded elements commute.

The arguments R and I are primarily for use by the *quotient()* method.

These algebras should be graded over the integers; multi-graded algebras should be constructed using *GCAlgebra_multigraded* instead.

EXAMPLES:

```
sage: A.<a,b> = GradedCommutativeAlgebra(QQ, degrees = (2, 3))
sage: a.degree()
2
sage: B = A.quotient(A.ideal(a**2*b))
sage: B
Graded Commutative Algebra with generators ('a', 'b') in degrees (2, 3) with␣
↪relations [a^2*b] over Rational Field
sage: A.basis(7)
[a^2*b]
sage: B.basis(7)
[]
```

Note that the function *GradedCommutativeAlgebra()* can also be used to construct these algebras.

class Element(*A*, *rep*)

Bases: QuotientRingElement

An element of a graded commutative algebra.

basis_coefficients(*total=False*)

Return the coefficients of this homogeneous element with respect to the basis in its degree.

For example, if this is the sum of the 0th and 2nd basis elements, return the list [1, 0, 1].

Raise an error if the element is not homogeneous.

INPUT:

- total – boolean (default False); this is only used in the multi-graded case, in which case if True, it returns the coefficients with respect to the basis for the total degree of this element

OUTPUT:

A list of elements of the base field.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1, 2, 2, 3))
sage: A.basis(3)
[x*y, x*z, t]
sage: (t + 3*x*y).basis_coefficients()
[3, 0, 1]
sage: (t + x).basis_coefficients()
Traceback (most recent call last):
...
ValueError: this element is not homogeneous

sage: B.<c,d> = GradedCommutativeAlgebra(QQ, degrees=((2,0), (0,4)))
sage: B.basis(4)
[c^2, d]
sage: (c^2 - 1/2 * d).basis_coefficients(total=True)
[1, -1/2]
sage: (c^2 - 1/2 * d).basis_coefficients()
Traceback (most recent call last):
...
ValueError: this element is not homogeneous
```

degree(*total=False*)

The degree of this element.

If the element is not homogeneous, this returns the maximum of the degrees of its monomials.

INPUT:

- total – ignored, present for compatibility with the multi-graded case

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1, 2, 3, 3))
sage: el = z*t+2*x*y-y^2*z
sage: el.degree()
7
sage: el.monomials()
[y^2*z, z*t, x*y]
```

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```
sage: [i.degree() for i in el.monomials()]
[7, 6, 3]

sage: A(0).degree()
Traceback (most recent call last):
...
ValueError: the zero element does not have a well-defined degree
```

dict()

A dictionary that determines the element.

The keys of this dictionary are the tuples of exponents of each monomial, and the values are the corresponding coefficients.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1, 2, 2, 3))
sage: dic = (x*y - 5*y*z + 7*x*y^2*z^3*t).dict()
sage: sorted(dic.items())
[((0, 1, 1, 0), -5), ((1, 1, 0, 0), 1), ((1, 2, 3, 1), 7)]
```

homogeneous_parts()

Return the homogeneous parts of the element. The result is given as a dictionary indexed by degree.

EXAMPLES:

```
sage: A.<e1,e2,e3,e4,e5> = GradedCommutativeAlgebra(QQ)
sage: a = e1*e3*e5-3*e2*e3*e5 + e1*e2 -2*e3 + e5
sage: a.homogeneous_parts()
{1: -2*e3 + e5, 2: e1*e2, 3: e1*e3*e5 - 3*e2*e3*e5}
```

is_homogeneous(*total=False*)

Return `True` if `self` is homogeneous and `False` otherwise.

INPUT:

- `total` – boolean (default `False`); only used in the multi-graded case, in which case if `True`, check to see if `self` is homogeneous with respect to total degree

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1, 2, 3, 3))
sage: el = z*t + 2*x*y - y^2*z
sage: el.degree()
7
sage: el.monomials()
[y^2*z, z*t, x*y]
sage: [i.degree() for i in el.monomials()]
[7, 6, 3]
sage: el.is_homogeneous()
False
sage: em = y^3 - 5*z*t + 3/2*x*y*t
sage: em.is_homogeneous()
True
sage: em.monomials()
[y^3, x*y*t, z*t]
```

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```
sage: [i.degree() for i in em.monomials()]
[6, 6, 6]
```

The element 0 is homogeneous, even though it doesn't have a well-defined degree:

```
sage: A(0).is_homogeneous()
True
```

A multi-graded example:

```
sage: B.<c,d> = GradedCommutativeAlgebra(QQ, degrees=((2, 0), (0, 4)))
sage: (c^2 - 1/2 * d).is_homogeneous()
False
sage: (c^2 - 1/2 * d).is_homogeneous(total=True)
True
```

basis(*n*)

Return a basis of the n-th homogeneous component of `self`.

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(QQ, degrees=(1, 2, 2, 3))
sage: A.basis(2)
[y, z]
sage: A.basis(3)
[x*y, x*z, t]
sage: A.basis(4)
[y^2, y*z, z^2, x*t]
sage: A.basis(5)
[x*y^2, x*y*z, x*z^2, y*t, z*t]
sage: A.basis(6)
[y^3, y^2*z, y*z^2, z^3, x*y*t, x*z*t]
```

cdg_algebra(*differential*)

Construct a differential graded commutative algebra from `self` by specifying a differential.

INPUT:

- `differential` – a dictionary defining a differential or a map defining a valid differential

The keys of the dictionary are generators of the algebra, and the associated values are their targets under the differential. Any generators which are not specified are assumed to have zero differential. Alternatively, the differential can be defined using the *differential()* method; see below for an example.

See also:

differential()

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 1))
sage: B = A.cdg_algebra({a: b*c, b: a*c})
sage: B
Commutative Differential Graded Algebra with generators ('a', 'b', 'c') in
↪degrees (1, 1, 1) over Rational Field with differential:
   a --> b*c
```

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```
    b --> a*c
    c --> 0
```

Note that `differential` can also be a map:

```
sage: d = A.differential({a: b*c, b: a*c})
sage: d
Differential of Graded Commutative Algebra with generators ('a', 'b', 'c') in␣
↪degrees (1, 1, 1) over Rational Field
  Defn: a --> b*c
        b --> a*c
        c --> 0
sage: A.cdg_algebra(d) is B
True
```

differential(*diff*)

Construct a differential on `self`.

INPUT:

- `diff` – a dictionary defining a differential

The keys of the dictionary are generators of the algebra, and the associated values are their targets under the differential. Any generators which are not specified are assumed to have zero differential.

EXAMPLES:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 2))
sage: A.differential({y:x*y, x: x*y})
Differential of Graded Commutative Algebra with generators ('x', 'y', 'z') in␣
↪degrees (1, 1, 2) over Rational Field
  Defn: x --> x*y
        y --> x*y
        z --> 0
sage: B.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=(1, 2, 2))
sage: d = B.differential({b:a*c, c:a*c})
sage: d(b*c)
a*b*c + a*c^2
```

quotient(*I*, *check=True*)

Create the quotient of this algebra by a two-sided ideal `I`.

INPUT:

- `I` – a two-sided homogeneous ideal of this algebra
- `check` – (default: `True`) if `True`, check whether `I` is generated by homogeneous elements

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(GF(5), degrees=(2, 2, 3, 4))
sage: I = A.ideal([x*t+z^2, x*y - t])
sage: B = A.quotient(I)
sage: B
Graded Commutative Algebra with generators ('x', 'y', 'z', 't') in degrees (2,␣
↪2, 3, 4) with relations [x*t, x*y - t] over Finite Field of size 5
```

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```
sage: B(x*t)
0
sage: B(x*y)
t
sage: A.basis(7)
[x^2*z, x*y*z, y^2*z, z*t]
sage: B.basis(7)
[x^2*z, y^2*z, z*t]
```

class sage.algebras.commutative_dga.**GCAlgebraHomset**(*R*, *S*, *category=None*)

Bases: RingHomset_generic

Set of morphisms between two graded commutative algebras.

Note: Homsets (and thus morphisms) have only been implemented when the base fields are the same for the domain and codomain.

EXAMPLES:

```
sage: A.<x,y> = GradedCommutativeAlgebra(QQ, degrees=(1,2))
sage: H = Hom(A,A)
sage: H([x,y]) == H.identity()
True
sage: H([x,x]) == H.identity()
False

sage: A.<w,x> = GradedCommutativeAlgebra(QQ, degrees=(1,2))
sage: B.<y,z> = GradedCommutativeAlgebra(QQ, degrees=(1,1))
sage: H = Hom(A,B)
sage: H([y,0])
Graded Commutative Algebra morphism:
  From: Graded Commutative Algebra with generators ('w', 'x') in degrees (1, 2) over Rational Field
  To:   Graded Commutative Algebra with generators ('y', 'z') in degrees (1, 1) over Rational Field
  Defn: (w, x) --> (y, 0)
sage: H([y,y*z])
Graded Commutative Algebra morphism:
  From: Graded Commutative Algebra with generators ('w', 'x') in degrees (1, 2) over Rational Field
  To:   Graded Commutative Algebra with generators ('y', 'z') in degrees (1, 1) over Rational Field
  Defn: (w, x) --> (y, y*z)
```

identity()

Construct the identity morphism of this homset.

EXAMPLES:

```
sage: A.<x,y> = GradedCommutativeAlgebra(QQ, degrees=(1,2))
sage: H = Hom(A,A)
sage: H([x,y]) == H.identity()
```

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```
True
sage: H([x,x]) == H.identity()
False
```

zero()

Construct the "zero" morphism of this homset: the map sending each generator to zero.

EXAMPLES:

```
sage: A.<x,y> = GradedCommutativeAlgebra(QQ, degrees=(1,2))
sage: B.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=(1,1,1))
sage: zero = Hom(A,B).zero()
sage: zero(x) == zero(y) == 0
True
```

class sage.algebras.commutative_dga.**GCAlgebraMorphism**(*parent*, *im_gens*, *check=True*)

Bases: RingHomomorphism_im_gens

Create a morphism between two *graded commutative algebras*.

INPUT:

- parent – the parent homset
- im_gens – the images, in the codomain, of the generators of the domain
- check – boolean (default: True); check whether the proposed map is actually an algebra map; if the domain and codomain have differentials, also check that the map respects those.

EXAMPLES:

```
sage: A.<x,y> = GradedCommutativeAlgebra(QQ)
sage: H = Hom(A,A)
sage: f = H([y,x])
sage: f
Graded Commutative Algebra endomorphism of Graded Commutative Algebra with␣
↪generators ('x', 'y') in degrees (1, 1) over Rational Field
  Defn: (x, y) --> (y, x)
sage: f(x*y)
-x*y
```

is_graded(*total=False*)

Return True if this morphism is graded.

That is, return True if $f(x)$ is zero, or if $f(x)$ is homogeneous and has the same degree as x, for each generator x.

INPUT:

- total (optional, default False) – if True, use the total degree to determine whether the morphism is graded (relevant only in the multigraded case)

EXAMPLES:

```
sage: C.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=(1,1,2))
sage: H = Hom(C,C)
sage: H([a, b, a*b + 2*a]).is_graded()
```

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```
False
sage: H([a, b, a*b]).is_graded()
True

sage: A.<w,x> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (1,0)))
sage: B.<y,z> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0,1)))
sage: H = Hom(A,B)
sage: H([y,0]).is_graded()
True
sage: H([z,z]).is_graded()
False
sage: H([z,z]).is_graded(total=True)
True
```

class sage.algebras.commutative_dga.**GCAlgebra_multigraded**(*base*, *degrees*, *names=None*, *R=None*, *I=None*, *category=None*)

Bases: *GCAlgebra*

A multi-graded commutative algebra.

INPUT:

- base – the base field
- degrees – a tuple or list specifying the degrees of the generators
- names – (optional) names of the generators: a list of strings or a single string with the names separated by commas; if not specified, the generators are named x0, x1, …
- R – (optional) the ring over which the algebra is defined
- I – (optional) an ideal in R; it should include, among other relations, the squares of the generators of odd degree

When defining such an algebra, each entry of degrees should be a list, tuple, or element of an additive (free) abelian group. Regardless of how the user specifies the degrees, Sage converts them to group elements.

The arguments R and I are primarily for use by the *GCAlgebra.quotient()* method.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0,1), (1,1)))
sage: A
Graded Commutative Algebra with generators ('a', 'b', 'c') in degrees ((1, 0), (0,␣
↪1), (1, 1)) over Rational Field
sage: a**2
0
sage: c.degree(total=True)
2
sage: c**2
c^2
sage: c.degree()
(1, 1)
```

Although the degree of c was defined using a Python tuple, it is returned as an element of an additive abelian group, and so it can be manipulated via arithmetic operations:

```
sage: type(c.degree())
<class 'sage.groups.additive_abelian.additive_abelian_group.AdditiveAbelianGroup_
↪fixed_gens_with_category.element_class'>
sage: 2 * c.degree()
(2, 2)
sage: (a*b).degree() == a.degree() + b.degree()
True
```

The `basis()` method and the `Element.degree()` method both accept the boolean keyword `total`. If `True`, use the total degree:

```
sage: A.basis(2, total=True)
[a*b, c]
sage: c.degree(total=True)
2
```

class Element(*A*, *rep*)

Bases: `Element`

degree(*total=False*)

Return the degree of this element.

INPUT:

- `total` – if `True`, return the total degree, an integer; otherwise, return the degree as an element of an additive free abelian group

If not requesting the total degree, raise an error if the element is not homogeneous.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(GF(2), degrees=((1,0), (0,1), (1,
↪1)))
sage: (a**2*b).degree()
(2, 1)
sage: (a**2*b).degree(total=True)
3
sage: (a**2*b + c).degree()
Traceback (most recent call last):
...
ValueError: this element is not homogeneous
sage: (a**2*b + c).degree(total=True)
3
sage: A(0).degree()
Traceback (most recent call last):
...
ValueError: the zero element does not have a well-defined degree
```

basis(*n*, *total=False*)

Basis in degree `n`.

- `n` – degree or integer
- `total` (optional, default False) – if True, return the basis in total degree `n`.

If `n` is an integer rather than a multi-index, then the total degree is used in that case as well.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(GF(2), degrees=((1,0), (0,1), (1,1)))
sage: A.basis((1,1))
[a*b, c]
sage: A.basis(2, total=True)
[a^2, a*b, b^2, c]
```

Since 2 is a not a multi-index, we don't need to specify `total=True`:

```
sage: A.basis(2)
[a^2, a*b, b^2, c]
```

If `total==True`, then `n` can still be a tuple, list, etc., and its total degree is used instead:

```
sage: A.basis((1,1), total=True)
[a^2, a*b, b^2, c]
```

cdg_algebra(*differential*)

Construct a differential graded commutative algebra from `self` by specifying a differential.

INPUT:

- `differential` – a dictionary defining a differential or a map defining a valid differential

The keys of the dictionary are generators of the algebra, and the associated values are their targets under the differential. Any generators which are not specified are assumed to have zero differential. Alternatively, the differential can be defined using the *differential()* method; see below for an example.

See also:

differential()

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: A.cdg_algebra({a: c})
Commutative Differential Graded Algebra with generators ('a', 'b', 'c') in
↪degrees ((1, 0), (0, 1), (0, 2)) over Rational Field with differential:
   a --> c
   b --> 0
   c --> 0
sage: d = A.differential({a: c})
sage: A.cdg_algebra(d)
Commutative Differential Graded Algebra with generators ('a', 'b', 'c') in
↪degrees ((1, 0), (0, 1), (0, 2)) over Rational Field with differential:
   a --> c
   b --> 0
   c --> 0
```

differential(*diff*)

Construct a differential on `self`.

INPUT:

- `diff` – a dictionary defining a differential

The keys of the dictionary are generators of the algebra, and the associated values are their targets under the differential. Any generators which are not specified are assumed to have zero differential.

EXAMPLES:

```
sage: A.<a,b,c> = GradedCommutativeAlgebra(QQ, degrees=((1,0), (0, 1), (0,2)))
sage: A.differential({a: c})
Differential of Graded Commutative Algebra with generators ('a', 'b', 'c') in
↪degrees ((1, 0), (0, 1), (0, 2)) over Rational Field
  Defn: a --> c
        b --> 0
        c --> 0
```

quotient(*I*, *check=True*)

Create the quotient of this algebra by a two-sided ideal `I`.

INPUT:

- `I` – a two-sided homogeneous ideal of this algebra
- `check` – (default: `True`) if `True`, check whether `I` is generated by homogeneous elements

EXAMPLES:

```
sage: A.<x,y,z,t> = GradedCommutativeAlgebra(GF(5), degrees=(2, 2, 3, 4))
sage: I = A.ideal([x*t+z^2, x*y - t])
sage: B = A.quotient(I)
sage: B
Graded Commutative Algebra with generators ('x', 'y', 'z', 't') in degrees (2,
↪2, 3, 4) with relations [x*t, x*y - t] over Finite Field of size 5
sage: B(x*t)
0
sage: B(x*y)
t
sage: A.basis(7)
[x^2*z, x*y*z, y^2*z, z*t]
sage: B.basis(7)
[x^2*z, y^2*z, z*t]
```

`sage.algebras.commutative_dga.`**GradedCommutativeAlgebra**(*ring*, *names=None*, *degrees=None*, *max_degree=None*, ***kwargs*)

A graded commutative algebra.

INPUT:

There are two ways to call this. The first way defines a free graded commutative algebra:

- `ring` – the base field over which to work
- `names` – names of the generators. You may also use Sage's `A.<x,y,...> = ...` syntax to define the names. If no names are specified, the generators are named `x0`, `x1`, …
- `degrees` – degrees of the generators; if this is omitted, the degree of each generator is 1, and if both `names` and `degrees` are omitted, an error is raised
- `max_degree` – the maximal degree of the graded algebra. If omitted, no maximal degree is assumed and an instance of *GCAlgebra* is returned. Otherwise, an instance of `sage.algebras.commutative_graded_algebra.GradedCommutativeAlgebraWithMaxDeg` is created.

Once such an algebra has been defined, one can use its associated methods to take a quotient, impose a differential, etc. See the examples below.

The second way takes a graded commutative algebra and imposes relations:

- `ring` – a graded commutative algebra

- `relations` – a list or tuple of elements of `ring`

EXAMPLES:

Defining a graded commutative algebra:

```
sage: GradedCommutativeAlgebra(QQ, 'x, y, z')
Graded Commutative Algebra with generators ('x', 'y', 'z') in degrees (1, 1, 1)␣
↪over Rational Field
sage: GradedCommutativeAlgebra(QQ, degrees=(2, 3, 4))
Graded Commutative Algebra with generators ('x0', 'x1', 'x2') in degrees (2, 3, 4)␣
↪over Rational Field
```

As usual in Sage, the `A.<...>` notation defines both the algebra and the generator names:

```
sage: A.<x,y,z> = GradedCommutativeAlgebra(QQ, degrees=(1, 1, 2))
sage: x^2
0
sage: y*x # Odd classes anticommute.
-x*y
sage: z*y # z is central since it is in degree 2.
y*z
sage: (x*y*z**3).degree()
8
sage: A.basis(3) # basis of homogeneous degree 3 elements
[x*z, y*z]
```

Defining a quotient:

```
sage: I = A.ideal(x*z)
sage: AQ = A.quotient(I)
sage: AQ
Graded Commutative Algebra with generators ('x', 'y', 'z') in degrees (1, 1, 2)␣
↪with relations [x*z] over Rational Field
sage: AQ.basis(3)
[y*z]
```

Note that `AQ` has no specified differential. This is reflected in its print representation: `AQ` is described as a "graded commutative algebra" – the word "differential" is missing. Also, it has no default `differential`:

```
sage: AQ.differential()
Traceback (most recent call last):
...
TypeError: ...differential() missing 1 required positional argument:
'diff'
```

Now we add a differential to `AQ`:

```
sage: B = AQ.cdg_algebra({z:y*z})
sage: B
Commutative Differential Graded Algebra with generators ('x', 'y', 'z') in degrees␣
↪(1, 1, 2) with relations [x*z] over Rational Field with differential:
    x --> 0
    y --> 0
    z --> y*z
```

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```
sage: B.differential()
Differential of Commutative Differential Graded Algebra with generators ('x', 'y',
↪'z') in degrees (1, 1, 2) with relations [x*z] over Rational Field
  Defn: x --> 0
        y --> 0
        z --> y*z
sage: B.cohomology(1)
Free module generated by {[x], [y]} over Rational Field
sage: B.cohomology(2)
Free module generated by {[x*y]} over Rational Field
```

We compute algebra generators for cohomology in a range of degrees. This cohomology algebra appears to be finitely generated:

```
sage: B.cohomology_generators(15)
{1: [x, y]}
```

We can construct multi-graded rings as well. We work in characteristic 2 for a change, so the algebras here are honestly commutative:

```
sage: C.<a,b,c,d> = GradedCommutativeAlgebra(GF(2), degrees=((1,0), (1,1), (0,2),␣
↪(0,3)))
sage: D = C.cdg_algebra(differential={a:c, b:d})
sage: D
Commutative Differential Graded Algebra with generators ('a', 'b', 'c', 'd') in␣
↪degrees ((1, 0), (1, 1), (0, 2), (0, 3)) over Finite Field of size 2 with␣
↪differential:
    a --> c
    b --> d
    c --> 0
    d --> 0
```

We can examine D using both total degrees and multidegrees. Use tuples, lists, vectors, or elements of additive abelian groups to specify degrees:

```
sage: D.basis(3) # basis in total degree 3
[a^3, a*b, a*c, d]
sage: D.basis((1,2)) # basis in degree (1,2)
[a*c]
sage: D.basis([1,2])
[a*c]
sage: D.basis(vector([1,2]))
[a*c]
sage: G = AdditiveAbelianGroup([0,0]); G
Additive abelian group isomorphic to Z + Z
sage: D.basis(G(vector([1,2])))
[a*c]
```

At this point, a, for example, is an element of C. We can redefine it so that it is instead an element of D in several ways, for instance using gens() method:

```
sage: a, b, c, d = D.gens()
sage: a.differential()
```

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```
c
```

Or the `inject_variables()` method:

```
sage: D.inject_variables()
Defining a, b, c, d
sage: (a*b).differential()
b*c + a*d
sage: (a*b*c**2).degree()
(2, 5)
```

Degrees are returned as elements of additive abelian groups:

```
sage: (a*b*c**2).degree() in G
True

sage: (a*b*c**2).degree(total=True)  # total degree
7
sage: D.cohomology(4)
Free module generated by {[a^4], [b^2]} over Finite Field of size 2
sage: D.cohomology((2,2))
Free module generated by {[b^2]} over Finite Field of size 2
```

Graded algebra with maximal degree:

```
sage: A.<p,e> = GradedCommutativeAlgebra(QQ, degrees=(4,2), max_degree=6)
sage: A
Graded commutative algebra with generators ('p', 'e') in degrees (4, 2)
 with maximal degree 6
sage: p^2
0
```

`sage.algebras.commutative_dga.`**`exterior_algebra_basis`**(*degrees*)

Basis of an exterior algebra in degree `n`, where the generators are in degrees `degrees`.

INPUT:

- `n` - integer
- `degrees` - iterable of integers

Return list of lists, each list representing exponents for the corresponding generators. (So each list consists of 0's and 1's.)

EXAMPLES:

```
sage: from sage.algebras.commutative_dga import exterior_algebra_basis
sage: exterior_algebra_basis(1, (1,3,1))
[[0, 0, 1], [1, 0, 0]]
sage: exterior_algebra_basis(4, (1,3,1))
[[0, 1, 1], [1, 1, 0]]
sage: exterior_algebra_basis(10, (1,5,1,1))
[]
```

`sage.algebras.commutative_dga.`**`sorting_keys`**(*element*)

Auxiliary function to sort the elements of a basis of a Cohomology group.

It is needed to ensure that elements of a cohomology group are represented in a consistent way.

INPUT:

- `element` - A CohomologyClass

OUTPUT:

Its coordinates in the corresponding cohomology_raw quotient vector space

EXAMPLES:

```
sage: from sage.algebras.commutative_dga import sorting_keys
sage: A.<e1,e2,e3,e4,e5> = GradedCommutativeAlgebra(QQ)
sage: B = A.cdg_algebra({e5:e1*e2+e3*e4})
sage: B.inject_variables()
Defining e1, e2, e3, e4, e5
sage: C = B.cohomology(3)
sage: [sorting_keys(el) for el in C.basis().keys()]
[[1, 0, 0, 0, 0],
[0, 1, 0, 0, 0],
[0, 0, 1, 0, 0],
[0, 0, 0, 1, 0],
[0, 0, 0, 0, 1]]
```

`sage.algebras.commutative_dga.`**`total_degree`**`(`*deg*`)`

Total degree of `deg`.

INPUT:

- `deg` - an element of a free abelian group.

In fact, `deg` could be an integer, a Python int, a list, a tuple, a vector, etc. This function returns the sum of the components of `deg`.

EXAMPLES:

```
sage: from sage.algebras.commutative_dga import total_degree
sage: total_degree(12)
12
sage: total_degree(range(5))
10
sage: total_degree(vector(range(5)))
10
sage: G = AdditiveAbelianGroup((0,0))
sage: x = G.gen(0); y = G.gen(1)
sage: 3*x+4*y
(3, 4)
sage: total_degree(3*x+4*y)
7
```

CHAPTER

EIGHT

VARIOUS ASSOCIATIVE ALGEBRAS

8.1 Associated Graded Algebras To Filtered Algebras

AUTHORS:

- Travis Scrimshaw (2014-10-08): Initial version

class sage.algebras.associated_graded.**AssociatedGradedAlgebra**(A, *category=None*)

Bases: CombinatorialFreeModule

The associated graded algebra/module $\operatorname{gr} A$ of a filtered algebra/module with basis A.

Let A be a filtered module over a commutative ring R. Let $(F_i)_{i\in I}$ be the filtration of A, with I being a totally ordered set. Define

$$G_i = F_i / \sum_{j<i} F_j$$

for every $i \in I$, and then

$$\operatorname{gr} A = \bigoplus_{i\in I} G_i.$$

There are canonical projections $p_i : F_i \to G_i$ for every $i \in I$. Moreover $\operatorname{gr} A$ is naturally a graded R-module with G_i being the i-th graded component. This graded R-module is known as the *associated graded module* (or, for short, just *graded module*) of A.

Now, assume that A (endowed with the filtration $(F_i)_{i\in I}$) is not just a filtered R-module, but also a filtered R-algebra. Let $u \in G_i$ and $v \in G_j$, and let $u' \in F_i$ and $v' \in F_j$ be lifts of u and v, respectively (so that $u = p_i(u')$ and $v = p_j(v')$). Then, we define a multiplication $*$ on $\operatorname{gr} A$ (not to be mistaken for the multiplication of the original algebra A) by

$$u * v = p_{i+j}(u'v').$$

The *associated graded algebra* (or, for short, just *graded algebra*) of A is the graded algebra $\operatorname{gr} A$ (endowed with this multiplication).

Now, assume that A is a filtered R-algebra with basis. Let $(b_x)_{x\in X}$ be the basis of A, and consider the partition $X = \bigsqcup_{i\in I} X_i$ of the set X, which is part of the data of a filtered algebra with basis. We know (see FilteredModulesWithBasis) that A (being a filtered R-module with basis) is canonically (when the basis is considered to be part of the data) isomorphic to $\operatorname{gr} A$ as an R-module. Therefore the k-th graded component G_k can be identified with the span of $(b_x)_{x\in X_k}$, or equivalently the k-th homogeneous component of A. Suppose that $u'v' = \sum_{k\leq i+j} m_k$ where $m_k \in G_k$ (which has been identified with the k-th homogeneous component of A). Then $u * v = m_{i+j}$. We also note that the choice of identification of G_k with the k-th homogeneous component of A depends on the given basis.

The basis $(b_x)_{x \in X}$ of A gives rise to a basis of $\operatorname{gr} A$. This latter basis is still indexed by the elements of X, and consists of the images of the b_x under the R-module isomorphism from A to $\operatorname{gr} A$. It makes $\operatorname{gr} A$ into a graded R-algebra with basis.

In this class, the R-module isomorphism from A to $\operatorname{gr} A$ is implemented as `to_graded_conversion()` and also as the default conversion from A to $\operatorname{gr} A$. Its inverse map is implemented as `from_graded_conversion()`. The projection $p_i : F_i \to G_i$ is implemented as `projection() (i)`.

INPUT:

- `A` – a filtered module (or algebra) with basis

OUTPUT:

The associated graded module of A, if A is just a filtered R-module. The associated graded algebra of A, if A is a filtered R-algebra.

EXAMPLES:

Associated graded module of a filtered module:

```
sage: A = Modules(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: grA.category()
Category of graded vector spaces with basis over Rational Field
sage: x = A.basis()[Partition([3,2,1])]
sage: grA(x)
Bbar[[3, 2, 1]]
```

Associated graded algebra of a filtered algebra:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: grA.category()
Category of graded algebras with basis over Rational Field
sage: x,y,z = [grA.algebra_generators()[s] for s in ['x','y','z']]
sage: x
bar(U['x'])
sage: y * x + z
bar(U['x']*U['y']) + bar(U['z'])
sage: A(y) * A(x) + A(z)
U['x']*U['y']
```

We note that the conversion between `A` and `grA` is the canonical `QQ`-module isomorphism stemming from the fact that the underlying `QQ`-modules of `A` and `grA` are isomorphic:

```
sage: grA(A.an_element())
bar(U['x']^2*U['y']^2*U['z']^3) + 2*bar(U['x']) + 3*bar(U['y']) + bar(1)
sage: elt = A.an_element() + A.algebra_generators()['x'] + 2
sage: grelt = grA(elt); grelt
bar(U['x']^2*U['y']^2*U['z']^3) + 3*bar(U['x']) + 3*bar(U['y']) + 3*bar(1)
sage: A(grelt) == elt
True
```

Todo: The algebra `A` must currently be an instance of (a subclass of) `CombinatorialFreeModule`. This should work with any filtered algebra with a basis.

Todo: Implement a version of associated graded algebra for filtered algebras without a distinguished basis.

REFERENCES:

- Wikipedia article Filtered_algebra#Associated_graded_algebra

algebra_generators()

Return the algebra generators of `self`.

This assumes that the algebra generators of A provided by its `algebra_generators` method are homogeneous.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: grA.algebra_generators()
Finite family {'x': bar(U['x']), 'y': bar(U['y']), 'z': bar(U['z'])}
```

degree_on_basis(*x*)

Return the degree of the basis element indexed by `x`.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: all(A.degree_on_basis(x) == grA.degree_on_basis(x)
....:     for g in grA.algebra_generators() for x in g.support())
True
```

gen(**args*, ***kwds*)

Return a generator of `self`.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: grA.gen('x')
bar(U['x'])
```

one_basis()

Return the basis index of the element 1 of $\operatorname{gr} A$.

This assumes that the unity 1 of A belongs to F_0.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: grA.one_basis()
1
```

product_on_basis(*x*, *y*)

Return the product on basis elements given by the indices `x` and `y`.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: grA = A.graded_algebra()
sage: G = grA.algebra_generators()
sage: x,y,z = G['x'], G['y'], G['z']
sage: x * y # indirect doctest
bar(U['x']*U['y'])
sage: y * x
bar(U['x']*U['y'])
sage: z * y * x
bar(U['x']*U['y']*U['z'])
```

8.2 Cellular Basis

Cellular algebras are a class of algebras introduced by Graham and Lehrer [GrLe1996]. The `CellularBasis` class provides a general framework for implementing cellular algebras and their cell modules and simple modules.

Let R be a commutative ring. A R-algebra A is a *cellular algebra* if it has a *cell datum*, which is a tuple (Λ, i, M, C), where Λ is finite poset with order $\geq$, if $\mu \in \Lambda$ then $T(\mu)$ is a finite set and

$$C: \coprod_{\mu\in\Lambda} T(\mu) \times T(\mu) \longrightarrow A; (\mu, s, t) \mapsto c^\mu_{st} \text{ is an injective map}$$

such that the following holds:

- The set $\{c^\mu_{st} \mid \mu \in \Lambda, s, t \in T(\mu)\}$ is a basis of A.
- If $a \in A$ and $\mu \in \Lambda, s, t \in T(\mu)$ then:

  $$ac^\mu_{st} = \sum_{u\in T(\mu)} r_a(s, u)c^\mu_{ut} \pmod{A^{>\mu}},$$

  where $A^{>\mu}$ is spanned by

  $$`\{c^\nu_{ab} | \nu > \mu \text{ and } a, b \in T(\nu)\}`.$$

  Moreover, the scalar $r_a(s, u)$ depends only on a, s and u and, in particular, is independent of t.
- The map $\iota: A \longrightarrow A; c^\mu_{st} \mapsto c^\mu_{ts}$ is an algebra anti-isomorphism.

A *cellular basis* for A is any basis of the form $\{c^\mu_{st} \mid \mu \in \Lambda, s, t \in T(\mu)\}$.

Note that the scalars $r_a(s, u) \in R$ depend only if a, s and u and, in particular, they do not depend on t. It follows from the definition of a cell datum that $A^{>\mu}$ is a two-sided ideal of A. More importantly, if $\mu \in \Lambda$ then the `CellModule` C^μ is the free R-module with basis $\{c^\mu_s \mid \mu \in \Lambda, s \in T(\mu)\}$ and with A-action:

$$ac^\mu_s = \sum_{u\in T(\mu)} r_a(s, u)c^\mu_u,$$

where the scalars $r_a(s, u)$ are those appearing in the definition of the cell datum. It follows from the cellular basis axioms that C^μ comes equipped with a bilinear form $\langle\ ,\ \rangle$ that is determined by:

$$c^\mu_{st}c^\mu_u = \langle c^\mu_s, c^\mu_t\rangle c^\mu_u.$$

The *radical* of C^μ is the A-submodule $\operatorname{rad} C^\mu = \{x \in C^\mu | \langle x, y\rangle = 0\}$. Hence, $D^\mu = C^\mu / \operatorname{rad} C^\mu$ is also an A-module. It is not difficult to show that $\{D^\mu \mid D^\mu \neq 0\}$ is a complete set of pairwise non-isomorphic A-modules.

Hence, a cell datum for A gives an explicit construction of the irreducible A-modules. The module `simple_module()` D^μ is either zero or absolutely irreducible.

EXAMPLES:

We compute a cellular basis and do some basic computations:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C
Cellular basis of Symmetric group algebra of order 3
 over Rational Field
```

See also:

`CellModule`

AUTHOR:

- Travis Scrimshaw (2015-11-5): Initial version

REFERENCES:

- [GrLe1996]
- [KX1998]
- [Mat1999]
- Wikipedia article Cellular_algebra
- http://webusers.imj-prg.fr/~bernhard.keller/ictp2006/lecturenotes/xi.pdf

class sage.algebras.cellular_basis.**CellularBasis**(*A*)

Bases: `CombinatorialFreeModule`

The cellular basis of a cellular algebra, in the sense of Graham and Lehrer [GrLe1996].

INPUT:

- `A` – the cellular algebra

EXAMPLES:

We compute a cellular basis and do some basic computations:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C
Cellular basis of Symmetric group algebra of order 3
 over Rational Field
sage: len(C.basis())
6
sage: len(S.basis())
6
sage: a,b,c,d,e,f = C.basis()
sage: a
C([3], [[1, 2, 3]], [[1, 2, 3]])
sage: c
C([2, 1], [[1, 3], [2]], [[1, 2], [3]])
sage: d
C([2, 1], [[1, 2], [3]], [[1, 3], [2]])
```

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```
sage: a * a
C([3], [[1, 2, 3]], [[1, 2, 3]])
sage: a * c
0
sage: d * c
C([2, 1], [[1, 2], [3]], [[1, 2], [3]])
sage: c * d
C([2, 1], [[1, 3], [2]], [[1, 3], [2]])
sage: S(a)
1/6*[1, 2, 3] + 1/6*[1, 3, 2] + 1/6*[2, 1, 3] + 1/6*[2, 3, 1]
 + 1/6*[3, 1, 2] + 1/6*[3, 2, 1]
sage: S(d)
1/4*[1, 3, 2] - 1/4*[2, 3, 1] + 1/4*[3, 1, 2] - 1/4*[3, 2, 1]
sage: B = list(S.basis())
sage: B[2]
[2, 1, 3]
sage: C(B[2])
-C([1, 1, 1], [[1], [2], [3]], [[1], [2], [3]])
 + C([2, 1], [[1, 2], [3]], [[1, 2], [3]])
 - C([2, 1], [[1, 3], [2]], [[1, 3], [2]])
 + C([3], [[1, 2, 3]], [[1, 2, 3]])
```

cell_module_indices(*la*)

Return the indices of the cell module of `self` indexed by `la` .

This is the finite set $M(\lambda)$.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C.cell_module_indices([2,1])
Standard tableaux of shape [2, 1]
```

cell_poset()

Return the cell poset of `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C.cell_poset()
Finite poset containing 3 elements
```

cellular_basis()

Return the cellular basis of `self`, which is `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C.cellular_basis() is C
True
```

cellular_basis_of()

Return the defining algebra of `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C.cellular_basis_of() is S
True
```

one()

Return the element 1 in `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: C.one()
C([1, 1, 1], [[1], [2], [3]], [[1], [2], [3]])
 + C([2, 1], [[1, 2], [3]], [[1, 2], [3]])
 + C([2, 1], [[1, 3], [2]], [[1, 3], [2]])
 + C([3], [[1, 2, 3]], [[1, 2, 3]])
```

product_on_basis(*x*, *y*)

Return the product of basis indices by `x` and `y`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: C = S.cellular_basis()
sage: la = Partition([2,1])
sage: s = StandardTableau([[1,2],[3]])
sage: t = StandardTableau([[1,3],[2]])
sage: C.product_on_basis((la, s, t), (la, s, t))
0
```

8.3 Q-Systems

AUTHORS:

- Travis Scrimshaw (2013-10-08): Initial version
- Travis Scrimshaw (2017-12-08): Added twisted Q-systems

class sage.algebras.q_system.**QSystem**(*base_ring*, *cartan_type*, *level*, *twisted*)

Bases: `CombinatorialFreeModule`

A Q-system.

Let $\mathfrak{g}$ be a tamely-laced symmetrizable Kac-Moody algebra with index set I and Cartan matrix $(C_{ab})_{a,b\in I}$ over a field k. Follow the presentation given in [HKOTY1999], an unrestricted Q-system is a k-algebra in infinitely many variables $Q^{(a)}_m$, where $a \in I$ and $m \in \mathbf{Z}_{>0}$, that satisfies the relations

$$\left(Q^{(a)}_m\right)^2 = Q^{(a)}_{m+1} Q^{(a)}_{m-1} + \prod_{b\sim a} \prod_{k=0}^{-C_{ab}-1} Q^{(b)}_{\left\lfloor \frac{m C_{ba} - k}{C_{ab}} \right\rfloor},$$

with $Q_0^{(a)} := 1$. Q-systems can be considered as T-systems where we forget the spectral parameter u and for $\mathfrak{g}$ of finite type, have a solution given by the characters of Kirillov-Reshetikhin modules (again without the spectral parameter) for an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ with $\mathfrak{g}$ as its classical subalgebra. See [KNS2011] for more information.

Q-systems have a natural bases given by polynomials of the fundamental representations $Q_1^{(a)}$, for $a \in I$. As such, we consider the Q-system as generated by $\{Q_1^{(a)}\}_{a \in I}$.

There is also a level ℓ restricted Q-system (with unit boundary condition) given by setting $Q_{d_a \ell}^{(a)} = 1$, where d_a are the entries of the symmetrizing matrix for the dual type of $\mathfrak{g}$.

Similarly, for twisted affine types (we omit type $A_{2n}^{(2)}$), we can define the *twisted Q-system* by using the relation:

$$(Q_m^{(a)})^2 = Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{b \neq a} (Q_m^{(b)})^{-C_{ba}}.$$

See [Wil2013] for more information.

EXAMPLES:

We begin by constructing a Q-system and doing some basic computations in type A_4:

```
sage: Q = QSystem(QQ, ['A', 4])
sage: Q.Q(3,1)
Q^(3)[1]
sage: Q.Q(1,2)
Q^(1)[1]^2 - Q^(2)[1]
sage: Q.Q(3,3)
-Q^(1)[1]*Q^(3)[1] + Q^(1)[1]*Q^(4)[1]^2 + Q^(2)[1]^2
 - 2*Q^(2)[1]*Q^(3)[1]*Q^(4)[1] + Q^(3)[1]^3
sage: x = Q.Q(1,1) + Q.Q(2,1); x
Q^(1)[1] + Q^(2)[1]
sage: x * x
Q^(1)[1]^2 + 2*Q^(1)[1]*Q^(2)[1] + Q^(2)[1]^2
```

Next we do some basic computations in type C_4:

```
sage: Q = QSystem(QQ, ['C', 4])
sage: Q.Q(4,1)
Q^(4)[1]
sage: Q.Q(1,2)
Q^(1)[1]^2 - Q^(2)[1]
sage: Q.Q(2,3)
Q^(1)[1]^2*Q^(4)[1] - 2*Q^(1)[1]*Q^(2)[1]*Q^(3)[1]
 + Q^(2)[1]^3 - Q^(2)[1]*Q^(4)[1] + Q^(3)[1]^2
sage: Q.Q(3,3)
Q^(1)[1]*Q^(4)[1]^2 - 2*Q^(2)[1]*Q^(3)[1]*Q^(4)[1] + Q^(3)[1]^3
```

We compare that with the twisted Q-system of type $A_7^{(2)}$:

```
sage: Q = QSystem(QQ, ['A',7,2], twisted=True)
sage: Q.Q(4,1)
Q^(4)[1]
sage: Q.Q(1,2)
Q^(1)[1]^2 - Q^(2)[1]
sage: Q.Q(2,3)
```

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```
Q^(1)[1]^2*Q^(4)[1] - 2*Q^(1)[1]*Q^(2)[1]*Q^(3)[1]
 + Q^(2)[1]^3 - Q^(2)[1]*Q^(4)[1] + Q^(3)[1]^2
sage: Q.Q(3,3)
-Q^(1)[1]*Q^(3)[1]^2 + Q^(1)[1]*Q^(4)[1]^2 + Q^(2)[1]^2*Q^(3)[1]
 - 2*Q^(2)[1]*Q^(3)[1]*Q^(4)[1] + Q^(3)[1]^3
```

REFERENCES:

- [HKOTY1999]
- [KNS2011]

class Element

Bases: IndexedFreeModuleElement

An element of a Q-system.

Q(*a*, *m*)

Return the generator $Q^{(a)}_m$ of `self`.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A', 8])
sage: Q.Q(2, 1)
Q^(2)[1]
sage: Q.Q(6, 2)
-Q^(5)[1]*Q^(7)[1] + Q^(6)[1]^2
sage: Q.Q(7, 3)
-Q^(5)[1]*Q^(7)[1] + Q^(5)[1]*Q^(8)[1]^2 + Q^(6)[1]^2
 - 2*Q^(6)[1]*Q^(7)[1]*Q^(8)[1] + Q^(7)[1]^3
sage: Q.Q(1, 0)
1
```

Twisted Q-system:

```
sage: Q = QSystem(QQ, ['D',4,3], twisted=True)
sage: Q.Q(1,2)
Q^(1)[1]^2 - Q^(2)[1]
sage: Q.Q(2,2)
-Q^(1)[1]^3 + Q^(2)[1]^2
sage: Q.Q(2,3)
3*Q^(1)[1]^4 - 2*Q^(1)[1]^3*Q^(2)[1] - 3*Q^(1)[1]^2*Q^(2)[1]
 + Q^(2)[1]^2 + Q^(2)[1]^3
sage: Q.Q(1,4)
-2*Q^(1)[1]^2 + 2*Q^(1)[1]^3 + Q^(1)[1]^4
 - 3*Q^(1)[1]^2*Q^(2)[1] + Q^(2)[1] + Q^(2)[1]^2
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A',4])
sage: Q.algebra_generators()
Finite family {1: Q^(1)[1], 2: Q^(2)[1], 3: Q^(3)[1], 4: Q^(4)[1]}
```

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```
sage: Q = QSystem(QQ, ['D',4,3], twisted=True)
sage: Q.algebra_generators()
Finite family {1: Q^(1)[1], 2: Q^(2)[1]}
```

cartan_type()

Return the Cartan type of `self`.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A',4])
sage: Q.cartan_type()
['A', 4]

sage: Q = QSystem(QQ, ['D',4,3], twisted=True)
sage: Q.cartan_type()
['G', 2, 1]^* relabelled by {0: 0, 1: 2, 2: 1}
```

dimension()

Return the dimension of `self`, which is ∞.

EXAMPLES:

```
sage: F = QSystem(QQ, ['A',4])
sage: F.dimension()
+Infinity
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A',4])
sage: Q.gens()
(Q^(1)[1], Q^(2)[1], Q^(3)[1], Q^(4)[1])
```

index_set()

Return the index set of `self`.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A',4])
sage: Q.index_set()
(1, 2, 3, 4)

sage: Q = QSystem(QQ, ['D',4,3], twisted=True)
sage: Q.index_set()
(1, 2)
```

level()

Return the restriction level of `self` or `None` if the system is unrestricted.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A',4])
sage: Q.level()

sage: Q = QSystem(QQ, ['A',4], 5)
sage: Q.level()
5
```

one_basis()

Return the basis element indexing 1.

EXAMPLES:

```
sage: Q = QSystem(QQ, ['A',4])
sage: Q.one_basis()
1
sage: Q.one_basis().parent() is Q._indices
True
```

sage.algebras.q_system.**is_tamely_laced**(*ct*)

Check if the Cartan type ct is tamely-laced.

A (symmetrizable) Cartan type with index set I is *tamely-laced* if $A_{ij} < -1$ implies $d_i = -A_{ji} = 1$ for all $i, j \in I$, where $(d_i)_{i \in I}$ is the diagonal matrix symmetrizing the Cartan matrix $(A_{ij})_{i,j \in I}$.

EXAMPLES:

```
sage: from sage.algebras.q_system import is_tamely_laced
sage: all(is_tamely_laced(ct)
....:     for ct in CartanType.samples(crystallographic=True, finite=True))
True
sage: for ct in CartanType.samples(crystallographic=True, affine=True):
....:     if not is_tamely_laced(ct):
....:         print(ct)
['A', 1, 1]
['BC', 1, 2]
['BC', 5, 2]
['BC', 1, 2]^*
['BC', 5, 2]^*
sage: cm = CartanMatrix([[2,-1,0,0],[-3,2,-2,-2],[0,-1,2,-1],[0,-1,-1,2]])
sage: is_tamely_laced(cm)
True
```

8.4 q-Commuting Polynomials

AUTHORS:

- Travis Scrimshaw (2022-08-23): Initial version
- Travis Scrimshaw (2023-02-10): Added Laurent polynomials

class sage.algebras.q_commuting_polynomials.**qCommutingLaurentPolynomials**(*q*, *B*, *names*)

Bases: *qCommutingPolynomials_generic*

The algebra of q-commuting Laurent polynomials.

Let R be a commutative ring, and fix an element $q \in R$. Let $B = (B_{xy})_{x,y \in I}$ be a skew-symmetric bilinear form with index set I. Let $R[I]_{q,B}$ denote the Laurent polynomial ring in the variables I such that we have the q-*commuting* relation for $x, y \in I$:

$$yx = q^{B_{xy}} \cdot xy.$$

This is a graded R-algebra with a natural basis given by monomials written in increasing order with respect to some total order on I.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y> = algebras.qCommutingLaurentPolynomials(q)
```

We verify a case of the q-binomial theorem using inverse variables:

```
sage: f = (x^-1 + y^-1)^10
sage: all(f[b] == q_binomial(10, -b.list()[0]) for b in f.support())
True
```

We now do a computation with a non-standard B matrix:

```
sage: B = matrix([[0,1,2],[-1,0,3],[-2,-3,0]])
sage: B
[ 0  1  2]
[-1  0  3]
[-2 -3  0]
sage: q = ZZ['q'].gen()
sage: R.<x,y,z> = algebras.qCommutingLaurentPolynomials(q, B)
sage: y^-1 * x
1/q*x*y^-1
sage: z^-1 * x
1/q^2*x*z^-1
sage: z^-1 * y^-1
q^3*y^-1*z^-1

sage: f = (x + z^-1)^10
sage: all(f[b] == q_binomial(10, b.list()[0], q^-2) for b in f.support())
True

sage: f = (y^-1 + z^-1)^10
sage: all(f[b] == q_binomial(10, -b.list()[1], q^3) for b in f.support())
True
```

class Element

Bases: `IndexedFreeModuleElement`

one_basis()

Return the basis index of the element 1.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.one_basis()
1
```

product_on_basis(x, y)

Return the product of two monomials given by x and y.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y> = algebras.qCommutingLaurentPolynomials(q)
sage: R.product_on_basis(x.leading_support(), y.leading_support())
x*y
sage: R.product_on_basis(y.leading_support(), x.leading_support())
q*x*y

sage: x * y
x*y
sage: y * x
q*x*y
sage: y^2 * x
q^2*x*y^2
sage: y * x^2
q^2*x^2*y
sage: y^-2 * x
1/q^2*x*y^-2
sage: y * x^-2
1/q^2*x^-2*y
sage: x * y * x
q*x^2*y
sage: x * y * ~x
1/q*y
sage: y^2 * x^2
q^4*x^2*y^2
sage: y^-2 * x^2
1/q^4*x^2*y^-2
sage: y^-2 * x^-2
q^4*x^-2*y^-2
sage: (x + y)^4
x^4 + (q^3+q^2+q+1)*x^3*y + (q^4+q^3+2*q^2+q+1)*x^2*y^2 + (q^3+q^2+q+1)*x*y^3 +
↪y^4
```

With a non-standard B matrix:

```
sage: B = matrix([[0,1,2],[-1,0,3],[-2,-3,0]])
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingLaurentPolynomials(q, B=B)
sage: x * y
x*y
sage: y * x^2
q^2*x^2*y
sage: z^2 * x
q^4*x*z^2
sage: z^2 * x^3
q^12*x^3*z^2
sage: z^2 * y
q^6*y*z^2
sage: z^2 * y^3
```

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```
q^18*y^3*z^2
sage: x * y^-1
x*y^-1
sage: y * x^-2
1/q^2*x^-2*y
sage: z^-2 * x
1/q^4*x*z^-2
sage: z^-2 * x^-3
q^12*x^-3*z^-2
sage: z^2 * y^-1
1/q^6*y^-1*z^2
sage: z^2 * y^-3
1/q^18*y^-3*z^2
```

class sage.algebras.q_commuting_polynomials.**qCommutingPolynomials**(*q*, *B*, *names*)

Bases: *qCommutingPolynomials_generic*

The algebra of q-commuting polynomials.

Let R be a commutative ring, and fix an element $q \in R$. Let $B = (B_{xy})_{x,y \in I}$ be a skew-symmetric bilinear form with index set I. Let $R[I]_{q,B}$ denote the polynomial ring in the variables I such that we have the *q-commuting* relation for $x, y \in I$:

$$yx = q^{B_{xy}} \cdot xy.$$

This is a graded R-algebra with a natural basis given by monomials written in increasing order with respect to some total order on I.

When $B_{xy} = 1$ and $B_{yx} = -1$ for all $x < y$, then we have a q-analog of the classical binomial coefficient theorem:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}.$$

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y> = algebras.qCommutingPolynomials(q)
```

We verify a case of the q-binomial theorem:

```
sage: f = (x + y)^10
sage: all(f[b] == q_binomial(10, b.list()[0]) for b in f.support())
True
```

We now do a computation with a non-standard B matrix:

```
sage: B = matrix([[0,1,2],[-1,0,3],[-2,-3,0]])
sage: B
[ 0  1  2]
[-1  0  3]
[-2 -3  0]
sage: q = ZZ['q'].gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q, B)
```

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```
sage: y * x
q*x*y
sage: z * x
q^2*x*z
sage: z * y
q^3*y*z

sage: f = (x + z)^10
sage: all(f[b] == q_binomial(10, b.list()[0], q^2) for b in f.support())
True

sage: f = (y + z)^10
sage: all(f[b] == q_binomial(10, b.list()[1], q^3) for b in f.support())
True
```

one_basis()

Return the basis index of the element 1.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.one_basis()
1
```

product_on_basis(*x*, *y*)

Return the product of two monomials given by `x` and `y`.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y> = algebras.qCommutingPolynomials(q)
sage: R.product_on_basis(x.leading_support(), y.leading_support())
x*y
sage: R.product_on_basis(y.leading_support(), x.leading_support())
q*x*y

sage: x * y
x*y
sage: y * x
q*x*y
sage: y^2 * x
q^2*x*y^2
sage: y * x^2
q^2*x^2*y
sage: x * y * x
q*x^2*y
sage: y^2 * x^2
q^4*x^2*y^2
sage: (x + y)^2
x^2 + (q+1)*x*y + y^2
sage: (x + y)^3
x^3 + (q^2+q+1)*x^2*y + (q^2+q+1)*x*y^2 + y^3
```

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```
sage: (x + y)^4
x^4 + (q^3+q^2+q+1)*x^3*y + (q^4+q^3+2*q^2+q+1)*x^2*y^2 + (q^3+q^2+q+1)*x*y^3 +
→y^4
```

With a non-standard B matrix:

```
sage: B = matrix([[0,1,2],[-1,0,3],[-2,-3,0]])
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q, B=B)
sage: x * y
x*y
sage: y * x^2
q^2*x^2*y
sage: z^2 * x
q^4*x*z^2
sage: z^2 * x^3
q^12*x^3*z^2
sage: z^2 * y
q^6*y*z^2
sage: z^2 * y^3
q^18*y^3*z^2
```

class sage.algebras.q_commuting_polynomials.**qCommutingPolynomials_generic**(*q*, *B*, *indices*, *names*)

Bases: `CombinatorialFreeModule`

Base class for algebra of q-commuting (Laurent, etc.) polynomials.

Let R be a commutative ring, and fix an element $q \in R$. Let $B = (B_{xy})_{x,y \in I}$ be a skew-symmetric bilinear form with index set I. Let $R[I]_{q,B}$ denote the polynomial ring in the variables I such that we have the *q-commuting* relation for $x, y \in I$:

$$yx = q^{B_{xy}} \cdot xy.$$

This is a graded R-algebra with a natural basis given by monomials written in increasing order with respect to some total order on I.

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.algebra_generators()
Finite family {'x': x, 'y': y, 'z': z}
```

degree_on_basis(*m*)

Return the degree of the monomial index by `m`.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
```

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```
sage: R.degree_on_basis(R.one_basis())
0
sage: f = (x + y)^3 + z^3
sage: f.degree()
3
```

dimension()

Return the dimension of self, which is ∞.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.dimension()
+Infinity
```

gen(i)

Return the i-generator of self.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.gen(0)
x
sage: R.gen(2)
z
```

gens()

Return the generators of self.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.gens()
(x, y, z)
```

q()

Return the parameter q.

EXAMPLES:

```
sage: q = ZZ['q'].fraction_field().gen()
sage: R.<x,y,z> = algebras.qCommutingPolynomials(q)
sage: R.q() == q
True
```

8.5 Splitting Algebras

Splitting algebras have been considered by Dan Laksov, Anders Thorup, Torsten Ekedahl and others (see references below) in order to study intersection theory of Grassmann and other flag schemes. Similarly as *splitting fields* they can be considered as extensions of rings containing all the roots of a given monic polynomial over that ring under the assumption that its Galois group is the symmetric group of order equal to the polynomial's degree.

Thus they can be used as a tool to express elements of a ring generated by n indeterminates in terms of symmetric functions in these indeterminates.

This realization of splitting algebras follows the approach of a recursive quotient ring construction splitting off some linear factor of the polynomial in each recursive step. Accordingly it is inherited from `PolynomialQuotientRing_domain`.

AUTHORS:

- Sebastian Oehms (April 2020): initial version

class `sage.algebras.splitting_algebra.SplittingAlgebra`(*monic_polynomial*, *names='X'*, *iterate=True*, *warning=True*)

Bases: `PolynomialQuotientRing_domain`

For a given monic polynomial $p(t)$ of degree n over a commutative ring R, the splitting algebra is the universal R-algebra in which $p(t)$ has n roots, or, more precisely, over which $p(t)$ factors,

$$p(t) = (t - \xi_1) \cdots (t - \xi_n).$$

This class creates an algebra as extension over the base ring of a given polynomial p such that p splits into linear factors over that extension. It is assumed (and not checked in general) that the Galois group of p is the symmetric Group $S(n)$. The construction is recursive (following [LT2012], 1.3).

INPUT:

- `monic_polynomial` – the monic polynomial which should be split
- `names` – names for the indeterminates to be adjoined to the base ring of `monic_polynomial`
- `warning` – (default: `True`) can be used (by setting to `False`) to suppress a warning which will be thrown whenever it cannot be checked that the Galois group of `monic_polynomial` is maximal

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: Lc.<w> = LaurentPolynomialRing(ZZ)
sage: PabLc.<u,v> = Lc[]; t = polygen(PabLc)
sage: S.<x, y> = SplittingAlgebra(t^3 - u*t^2 + v*t - w)
doctest:...: UserWarning: Assuming x^3 - u*x^2 + v*x - w to have maximal
                          Galois group!

sage: roots = S.splitting_roots(); roots
[x, y, -y - x + u]
sage: all(t^3 -u*t^2 +v*t -w == 0 for t in roots)
True
sage: xi = ~x; xi
(w^-1)*x^2 + ((-w^-1)*u)*x + (w^-1)*v
sage: ~xi == x
True
sage: ~y
```

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```
((-w^-1)*x)*y + (-w^-1)*x^2 + ((w^-1)*u)*x
sage: zi = ((w^-1)*x)*y; ~zi
-y - x + u

sage: cp3 = cyclotomic_polynomial(3).change_ring(GF(5))
sage: CR3.<e3> = SplittingAlgebra(cp3)
sage: CR3.is_field()
True
sage: CR3.cardinality()
25
sage: F.<a> = cp3.splitting_field()
sage: F.cardinality()
25
sage: E3 = cp3.change_ring(F).roots()[0][0]; E3
3*a + 3
sage: f = CR3.hom([E3]); f
Ring morphism:
  From: Splitting Algebra of x^2 + x + 1
        with roots [e3, 4*e3 + 4]
        over Finite Field of size 5
  To:   Finite Field in a of size 5^2
  Defn: e3 |--> 3*a + 3
```

REFERENCES:

- [EL2002]
- [Lak2010]
- [Tho2011]
- [LT2012]

Element

alias of *SplittingAlgebraElement*

defining_polynomial()

Return the defining polynomial of `self`.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: L.<u, v, w > = LaurentPolynomialRing(ZZ)
sage: x = polygen(L)
sage: S = SplittingAlgebra(x^3 - u*x^2 + v*x - w, ('X', 'Y'))
sage: S.defining_polynomial()
x^3 - u*x^2 + v*x - w
```

hom(*im_gens*, *codomain=None*, *check=True*, *base_map=None*)

This version keeps track with the special recursive structure of *SplittingAlgebra*

Type `Ring.hom?` to see the general documentation of this method. Here you see just special examples for the current class.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: L.<u, v, w> = LaurentPolynomialRing(ZZ); x = polygen(L)
sage: S = SplittingAlgebra(x^3 - u*x^2 + v*x - w, ('X', 'Y'))
sage: P.<x, y, z> = PolynomialRing(ZZ)
sage: F = FractionField(P)
sage: im_gens = [F(g) for g in [y, x, x + y + z, x*y+x*z+y*z, x*y*z]]
sage: f = S.hom(im_gens)
sage: f(u), f(v), f(w)
(x + y + z, x*y + x*z + y*z, x*y*z)
sage: roots = S.splitting_roots(); roots
[X, Y, -Y - X + u]
sage: [f(r) for r in roots]
[x, y, z]
```

is_completely_split()

Return True if the defining polynomial of `self` splits into linear factors over `self`.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: L.<u, v, w > = LaurentPolynomialRing(ZZ); x = polygen(L)
sage: S.<a,b> = SplittingAlgebra(x^3 - u*x^2 + v*x - w)
sage: S.is_completely_split()
True
sage: S.base_ring().is_completely_split()
False
```

lifting_map()

Return a section map from `self` to the cover ring. It is implemented according to the same named method of `QuotientRing_nc`.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: x = polygen(ZZ)
sage: S = SplittingAlgebra(x^2+1, ('I',))
sage: lift = S.lifting_map()
sage: lift(5)
5
sage: r1, r2 =S.splitting_roots()
sage: lift(r1)
I
```

scalar_base_ring()

Return the ring of scalars of `self` (considered as an algebra)

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: L.<u, v, w > = LaurentPolynomialRing(ZZ)
sage: x = polygen(L)
sage: S = SplittingAlgebra(x^3 - u*x^2 + v*x - w, ('X', 'Y'))
sage: S.base_ring()
Factorization Algebra of x^3 - u*x^2 + v*x - w with roots [X]
```

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```
 over Multivariate Laurent Polynomial Ring in u, v, w over Integer Ring
sage: S.scalar_base_ring()
Multivariate Laurent Polynomial Ring in u, v, w over Integer Ring
```

splitting_roots()

Return the roots of the split equation.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: x = polygen(ZZ)
sage: S = SplittingAlgebra(x^2+1, ('I',))
sage: S.splitting_roots()
[I, -I]
```

class sage.algebras.splitting_algebra.**SplittingAlgebraElement**(*parent*, *polynomial*, *check=True*)

Bases: PolynomialQuotientRingElement

Element class for *SplittingAlgebra*.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: cp6 = cyclotomic_polynomial(6)
sage: CR6.<e6> = SplittingAlgebra(cp6)
sage: type(e6)
<class 'sage.algebras.splitting_algebra.SplittingAlgebra_with_category.element_class
↪'>

sage: type(CR6(5))
<class 'sage.algebras.splitting_algebra.SplittingAlgebra_with_category.element_class
↪'>
```

dict()

Return the dictionary of `self` according to its lift to the cover.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: CR3.<e3> = SplittingAlgebra(cyclotomic_polynomial(3))
sage: (e3 + 42).dict()
{0: 42, 1: 1}
```

is_unit()

Return `True` if `self` is invertible.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import SplittingAlgebra
sage: CR3.<e3> = SplittingAlgebra(cyclotomic_polynomial(3))
sage: e3.is_unit()
True
```

sage.algebras.splitting_algebra.**solve_with_extension**(*monic_polynomial*, *root_names=None*, *var='x'*, *flatten=False*, *warning=True*)

Return all roots of a monic polynomial in its base ring or in an appropriate extension ring, as far as possible.

INPUT:

- `monic_polynomial` – the monic polynomial whose roots should be created
- `root_names` – names for the indeterminates needed to define the splitting algebra of the `monic_polynomial` (if necessary and possible)
- `var` – (default: `'x'`) for the indeterminate needed to define the splitting field of the `monic_polynomial` (if necessary and possible)
- `flatten` – (default: `True`) if `True` the roots will not be given as a list of pairs `(root, multiplicity)` but as a list of roots repeated according to their multiplicity
- `warning` – (default: `True`) can be used (by setting to `False`) to suppress a warning which will be thrown whenever it cannot be checked that the Galois group of `monic_polynomial` is maximal

OUTPUT:

List of tuples `(root, multiplicity)` respectively list of roots repeated according to their multiplicity if option `flatten` is `True`.

EXAMPLES:

```
sage: from sage.algebras.splitting_algebra import solve_with_extension
sage: t = polygen(ZZ)
sage: p = t^2 -2*t +1
sage: solve_with_extension(p, flatten=True )
[1, 1]
sage: solve_with_extension(p)
[(1, 2)]

sage: cp5 = cyclotomic_polynomial(5, var='T').change_
↪ring(UniversalCyclotomicField())
sage: solve_with_extension(cp5)
[(E(5), 1), (E(5)^4, 1), (E(5)^2, 1), (E(5)^3, 1)]
sage: _[0][0].parent()
Universal Cyclotomic Field
```

CHAPTER

NINE

NON-ASSOCIATIVE ALGEBRAS

9.1 Octonion Algebras

AUTHORS:

- Travis Scrimshaw (2023-05-06): Initial version

class sage.algebras.octonion_algebra.**Octonion**

Bases: *Octonion_generic*

An octonion.

This is an element of the octonion algebra with parameters $a = b = c = -1$, which is a classical octonion number.

norm()

Return the norm of self.

The norm of an octonion x is $\|x\| = \sqrt{xx^*}$, where x^* is the conjugate() of x.

See also:

This is the square root of *quadratic_form()*.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: elt = sum(i * b for i, b in enumerate(O.basis(), start=2))
sage: elt.norm()
2*sqrt(71)
sage: elt = sum(O.basis())
sage: elt.norm()
2*sqrt(2)
```

quadratic_form()

Return the quadratic form of self.

The octonion algebra has a distinguished quadratic form given by $N(x) = xx^*$, where x^* is the conjugate() of x.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: elt = sum(O.basis()); elt
1 + i + j + k + l + li + lj + lk
```

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```
sage: elt.quadratic_form()
8
sage: elt * elt.conjugate()
8
```

class sage.algebras.octonion_algebra.**OctonionAlgebra**(*R*, *a*, *b*, *c*, *names*)

Bases: UniqueRepresentation, Parent

The octonion algebra.

Let R be a commutative ring of characteristic not equal to 2. The *octonion algebra* with parameters a, b, c is a non-associative non-commutative unital 8-dimensional R-algebra that is a deformation of the usual octonions, which are when $a = b = c = -1$. The octonions were originally constructed by Graves and independently discovered by Cayley (due to being published first, these are sometimes called the Cayley numbers) and can also be built from the Cayley-Dickson construction with the quaternions.

We use the multiplication table from [Scha1996]. The octonion algebra $\mathbf{O}_{a,b,c}(R)$ is a composition (Hurwitz) algebra, which means it is also an alternative algebra as it satisfies $x^2y = (xx)y = x(xy)$ and $yx^2 = y(xx) = (yx)x$ for all $x, y \in \mathbf{O}_{a,b,c}$.

EXAMPLES:

We first create the classical octonions and perform some basic computations:

```
sage: O = OctonionAlgebra(QQ)
sage: O
Octonion algebra over Rational Field
sage: i, j, k, l = O.gens()
sage: i * j * k
1
sage: k * j * i
-1
sage: (i * k) * l
-lj
sage: i * (k * l)
lj
sage: elt = sum(O.basis())
sage: elt^2
-6 + 2*i + 2*j + 2*k + 2*l + 2*li + 2*lj + 2*lk
sage: prod(O.basis())
1
sage: (i + l)^2
-2
sage: (1 + l) * (1 + l).conjugate()
2
sage: S = O.some_elements()
sage: B = O.basis()
sage: S.extend(x * (i + j/2 - 5*k/3) for x in O.some_elements())
sage: all((x * x) * y == (x * (x * y)) for x in S for y in S)
True
sage: all(y * (x * x) == (y * x) * x for x in S for y in S)
True
sage: all((x + x.conjugate()) / 2 == x.real_part() for x in S)
True
```

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```
sage: all((x - x.conjugate()) / 2 == x.imag_part() for x in S)
True
sage: all(sum((b*x)*b for b in B) == -6 * x.conjugate() for x in S)
True
```

We construct the (rescaled) E_8 lattice as the integral octonions, which we verify by constructing 240 shortest length elements in the lattice (see also Wikipedia article E8_lattice#Integral_octonions):

```
sage: m = (i + j + k + l) / 2
sage: basis = [i, j, i*j, i^2, m, i * m, j * m, (i * j) * m]
sage: basis
[i,
 j,
 -k,
 -1,
 1/2*i + 1/2*j + 1/2*k + 1/2*l,
 -1/2 + 1/2*j - 1/2*k - 1/2*li,
 -1/2 - 1/2*i + 1/2*k - 1/2*lj,
 1/2 - 1/2*i + 1/2*j + 1/2*lk]
sage: matrix([vector(b) for b in basis]).rank()
8
sage: [b.norm() for b in basis]
[1, 1, 1, 1, 1, 1, 1, 1]
sage: roots = set(basis)
sage: roots.update(-b for b in basis)
sage: new_roots = set(roots)  # make a copy
sage: while new_roots:
....:     prev_roots = new_roots
....:     new_roots = set()
....:     for a in prev_roots:
....:         for b in roots:
....:             c = a + b
....:             if c.quadratic_form() != 1 or c in roots:
....:                 continue
....:             new_roots.update([c, -c])
....:         roots.update(new_roots)
sage: len(roots)
240
```

A classical construction of the Lie algebra of type G_2 is the Lie algebra of all derivations of $\mathbf{O}$ (as the automorphism group is the Lie group of type G_2). We verify that the derivations have the correct dimension:

```
sage: len(O.derivations_basis())
14
```

We can construct the split octonions by taking the parameter $c = 1$:

```
sage: SO = OctonionAlgebra(QQ, c=1)
sage: SO
Octonion algebra over Rational Field with parameters (-1, -1, 1)
sage: i, j, k, l = SO.gens()
sage: i^2 == j^2 == k^2 == -1
True
```

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```
sage: l^2
1
sage: (i + l)^2
0
sage: (1 + l) * (1 + l).conjugate()
0
```

REFERENCES:

- [Scha1996]
- Wikipedia article octonion
- Wikipedia article Split-octonion
- Wikipedia article Hurwitz's_theorem_(composition_algebras)

Element

alias of *Octonion_generic*

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(ZZ)
sage: O.basis()
Family (1, i, j, k, l, li, lj, lk)
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(ZZ)
sage: O.gens()
(i, j, k, l)
```

one_basis()

Return the index for the basis element of 1.

EXAMPLES:

```
sage: O = OctonionAlgebra(ZZ)
sage: O.one_basis()
0
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(ZZ)
sage: O.some_elements()
[2, 1, i, j, k, l, li, lj, lk,
 2 + 3*i + 4*j + 5*k + 6*l + 7*li + 8*lj + 9*lk,
```

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```
 -2*j + 3*k - li - lj,
 8 - 7*i + 2*j + 13*k - 18*l + 45*li - 40*lj + 5*lk]
sage: O = OctonionAlgebra(Zmod(6))
sage: O.some_elements()
[2, 1, i, j, k, l, li, lj, lk,
 2 + 3*i + 4*j + 5*k + li + 2*lj + 3*lk,
 4*j + 3*k + 5*li + 5*lj,
 2 + 5*i + 2*j + k + 3*li + 2*lj + 5*lk]
```

class sage.algebras.octonion_algebra.**Octonion_generic**

Bases: `AlgebraElement`

An octonion with generic parameters.

abs()

Return the absolute value of `self`.

This is equal to the *norm()*.

> **Warning:** If any of the parameters $a, b, c \not> 0$, then this does not define a metric.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ, 1, 3, 7)
sage: elt = sum(i * b for i, b in enumerate(O.basis(), start=2))
sage: elt.abs()
2*sqrt(-61)
sage: elt = sum(O.basis())
sage: elt.abs()
0
```

conjugate()

Return the conjugate of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: elt = sum(O.basis()); elt
1 + i + j + k + l + li + lj + lk
sage: elt.conjugate()
1 - i - j - k - l - li - lj - lk
```

dict(*copy=False*)

Return `self` as a `dict` with keys being indices for the basis and the values being the corresponding nonzero coefficients.

INPUT:

- copy – ignored

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: x = O([2/7, 0, 0, 0, 2/3, 0, -5, 0])
sage: x.monomial_coefficients()
{0: 2/7, 4: 2/3, 6: -5}
```

imag_part()

Return the imginary part of `self`.

OUTPUT:

The imaginary part of `self` as an element in the octonion algebra.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: elt = sum(i * b for i, b in enumerate(O.basis(), start=2)); elt
2 + 3*i + 4*j + 5*k + 6*l + 7*li + 8*lj + 9*lk
sage: elt.imag_part()
3*i + 4*j + 5*k + 6*l + 7*li + 8*lj + 9*lk
```

is_unit()

Return if `self` is a unit or not.

EXAMPLES:

```
sage: O = OctonionAlgebra(ZZ)
sage: x = O([1, 0, 1, 0, 0, 0, 0, 0])
sage: x.quadratic_form()
2
sage: x.is_unit()
False
sage: O([1, 0, -1, 0, 0, 0, 0, 0]).is_unit()
False
sage: x = O([1, 0, 0, 0, 1, 0, 0, 0])
sage: x.quadratic_form()
2
sage: x.is_unit()
False
sage: x = O([0, 0, 0, 0, 1, 0, 0, 0])
sage: x.quadratic_form()
1
sage: x.is_unit()
True

sage: O = OctonionAlgebra(ZZ, -1, 1, 2)
sage: x = O([1, 0, 1, 0, 0, 0, 0, 0])
sage: x.quadratic_form()
0
sage: x.is_unit()
False
sage: O([1, 0, -1, 0, 0, 0, 0, 0]).is_unit()
False
sage: x = O([1, 0, 0, 0, 1, 0, 0, 0])
sage: x.quadratic_form()
-1
```

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```
sage: x.is_unit()
True
sage: x = O([0, 0, 0, 0, 1, 0, 0, 0])
sage: x.quadratic_form()
-2
sage: x.is_unit()
False
```

monomial_coefficients(*copy=False*)

Return self as a dict with keys being indices for the basis and the values being the corresponding nonzero coefficients.

INPUT:

- copy – ignored

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: x = O([2/7, 0, 0, 0, 2/3, 0, -5, 0])
sage: x.monomial_coefficients()
{0: 2/7, 4: 2/3, 6: -5}
```

norm()

Return the norm of self.

The norm of an octonion x is $\|x\| = \sqrt{xx^*}$, where x^* is the *conjugate()* of x.

See also:

This is the square root of *quadratic_form()*.

Warning: If any of the parameters $a, b, c \not> 0$, then this is not an actual norm.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ, 1, 3, 7)
sage: elt = sum(i * b for i, b in enumerate(O.basis(), start=2))
sage: elt.norm()
2*sqrt(-61)
sage: elt = sum(O.basis())
sage: elt.norm()
0
```

quadratic_form()

Return the quadratic form of self.

The octonion algebra has a distinguished quadratic form given by $N(x) = xx^*$, where x^* is the *conjugate()* of x.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ, 1, 3, 7)
sage: elt = sum(O.basis())
```

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```
sage: elt.quadratic_form()
0
sage: elt * elt.conjugate()
0
```

real_part()

Return the real part of self.

OUTPUT:

The real part of self as an element in the base ring.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: elt = sum(i * b for i, b in enumerate(O.basis(), start=2)); elt
2 + 3*i + 4*j + 5*k + 6*l + 7*li + 8*lj + 9*lk
sage: r = elt.real_part(); r
2
sage: r.parent() is QQ
True
```

vector(*new_base_ring=None*)

Return self as a vector in R^8, where R is the base ring of self.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: elt = sum(i * b for i, b in enumerate(O.basis(), start=2))
sage: elt.vector()
(2, 3, 4, 5, 6, 7, 8, 9)
```

9.2 Lie Algebras

9.2.1 Abelian Lie Algebras

AUTHORS:

- Travis Scrimshaw (2016-06-07): Initial version

class sage.algebras.lie_algebras.abelian.**AbelianLieAlgebra**(*R*, *names*, *index_set*, *category*, ***kwds*)

Bases: *LieAlgebraWithStructureCoefficients*

An abelian Lie algebra.

A Lie algebra $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

EXAMPLES:

```
sage: L.<x, y> = LieAlgebra(QQ, abelian=True)
sage: L.bracket(x, y)
0
```

class Element

Bases: *Element*

is_abelian()

Return True since self is an abelian Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 3, 'x', abelian=True)
sage: L.is_abelian()
True
```

is_nilpotent()

Return True since self is an abelian Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 3, 'x', abelian=True)
sage: L.is_abelian()
True
```

is_solvable()

Return True since self is an abelian Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 3, 'x', abelian=True)
sage: L.is_abelian()
True
```

class sage.algebras.lie_algebras.abelian.**InfiniteDimensionalAbelianLieAlgebra**(*R*, *index_set*, *prefix='L'*, ***kwds*)

Bases: *InfinitelyGeneratedLieAlgebra*, IndexedGenerators

An infinite dimensional abelian Lie algebra.

A Lie algebra $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

class Element

Bases: *LieAlgebraElement*

dimension()

Return the dimension of self, which is ∞.

EXAMPLES:

```
sage: L = lie_algebras.abelian(QQ, index_set=ZZ)
sage: L.dimension()
+Infinity
```

is_abelian()

Return True since self is an abelian Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.abelian(QQ, index_set=ZZ)
sage: L.is_abelian()
True
```

is_nilpotent()

Return True since self is an abelian Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.abelian(QQ, index_set=ZZ)
sage: L.is_abelian()
True
```

is_solvable()

Return True since self is an abelian Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.abelian(QQ, index_set=ZZ)
sage: L.is_abelian()
True
```

9.2.2 Affine Lie Algebras

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version

class sage.algebras.lie_algebras.affine_lie_algebra.**AffineLieAlgebra**(*g*, *cartan_type*, *names*, *kac_moody*)

Bases: *FinitelyGeneratedLieAlgebra*

An (untwisted) affine Lie algebra.

Note that the derived subalgebra of the Kac-Moody algebra is the affine Lie algebra.

INPUT:

Can be one of the following:

- a base ring and an affine Cartan type: constructs the affine (Kac-Moody) Lie algebra of the classical Lie algebra in the bracket representation over the base ring
- a classical Lie algebra: constructs the corresponding affine (Kac-Moody) Lie algebra

There is the optional argument kac_moody, which can be set to False to obtain the affine Lie algebra instead of the affine Kac-Moody algebra.

See also:

- *UntwistedAffineLieAlgebra*
- *TwistedAffineLieAlgebra*

REFERENCES:

- [Ka1990]

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['D', 4, 1])
sage: B = g.basis()
sage: al = RootSystem(['D',4]).root_lattice().simple_roots()
sage: B[al[1]+al[2]+al[4],4]
(E[alpha[1] + alpha[2] + alpha[4]])#t^4
sage: B[-al[1]-2*al[2]-al[3]-al[4],2]
(E[-alpha[1] - 2*alpha[2] - alpha[3] - alpha[4]])#t^2
sage: B[al[4],-2]
(E[alpha[4]])#t^-2
sage: B['c']
c
sage: B['d']
d

sage: g = LieAlgebra(QQ, cartan_type=['D', 4, 2], kac_moody=False)
sage: B = g.basis()
sage: it = iter(B)
sage: [next(it) for _ in range(3)]
[c, (E[alpha[1]])#t^0, (E[alpha[2]])#t^0]
sage: B['c']
c
sage: B['d']
0
```

c()

Return the canonical central element c of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A',3,1])
sage: g.c()
c
```

cartan_type()

Return the Cartan type of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['C',3,1])
sage: g.cartan_type()
['C', 3, 1]
```

classical()

Return the classical Lie algebra of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['F',4,1])
sage: g.classical()
Lie algebra of ['F', 4] in the Chevalley basis
```

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```
sage: so5 = lie_algebras.so(QQ, 5, 'matrix')
sage: A = so5.affine()
sage: A.classical() == so5
True
```

d()

Return the canonical derivation d of `self`.

If `self` is the affine Lie algebra, then this returns 0.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A',3,1])
sage: g.d()
d
sage: D = g.derived_subalgebra()
sage: D.d()
0
```

derived_series()

Return the derived series of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B',3,1])
sage: g.derived_series()
[Affine Kac-Moody algebra of ['B', 3] in the Chevalley basis,
 Affine Lie algebra of ['B', 3] in the Chevalley basis]
sage: g.lower_central_series()
[Affine Kac-Moody algebra of ['B', 3] in the Chevalley basis,
 Affine Lie algebra of ['B', 3] in the Chevalley basis]

sage: D = g.derived_subalgebra()
sage: D.derived_series()
[Affine Lie algebra of ['B', 3] in the Chevalley basis]
```

e(*i=None*)

Return the generators e of `self`.

INPUT:

- `i` – (optional) if specified, return just the generator e_i

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B', 3, 1])
sage: list(g.e())
[(E[-alpha[1] - 2*alpha[2] - 2*alpha[3]])#t^1,
 (E[alpha[1]])#t^0, (E[alpha[2]])#t^0, (E[alpha[3]])#t^0]
sage: g.e(2)
(E[alpha[2]])#t^0
```

f(*i=None*)

Return the generators f of `self`.

INPUT:

- i – (optional) if specified, return just the generator f_i

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A', 5, 2])
sage: list(g.f())
[(E[alpha[1] + 2*alpha[2] + alpha[3]])#t^-1,
 (E[-alpha[1]])#t^0, (E[-alpha[2]])#t^0, (E[-alpha[3]])#t^0]
sage: g.f(2)
(E[-alpha[2]])#t^0
```

is_nilpotent()

Return False as self is semisimple.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B',3,1])
sage: g.is_nilpotent()
False
sage: g.is_solvable()
False
```

is_solvable()

Return False as self is semisimple.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B',3,1])
sage: g.is_nilpotent()
False
sage: g.is_solvable()
False
```

lie_algebra_generators()

Return the Lie algebra generators of self.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A',1,1])
sage: list(g.lie_algebra_generators())
[(E[alpha[1]])#t^0,
 (E[-alpha[1]])#t^0,
 (h1)#t^0,
 (E[-alpha[1]])#t^1,
 (E[alpha[1]])#t^-1,
 c,
 d]

sage: L = LieAlgebra(QQ, cartan_type=['A',5,2])
sage: list(L.lie_algebra_generators())
[(E[alpha[1]])#t^0,
 (E[alpha[2]])#t^0,
 (E[alpha[3]])#t^0,
 (E[-alpha[1]])#t^0,
```

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```
 (E[-alpha[2]])#t^0,
 (E[-alpha[3]])#t^0,
 (h1)#t^0,
 (h2)#t^0,
 (h3)#t^0,
 (E[-alpha[1] - 2*alpha[2] - alpha[3]])#t^1,
 (E[alpha[1] + 2*alpha[2] + alpha[3]])#t^-1,
 c,
 d]
```

lower_central_series()

Return the derived series of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B',3,1])
sage: g.derived_series()
[Affine Kac-Moody algebra of ['B', 3] in the Chevalley basis,
 Affine Lie algebra of ['B', 3] in the Chevalley basis]
sage: g.lower_central_series()
[Affine Kac-Moody algebra of ['B', 3] in the Chevalley basis,
 Affine Lie algebra of ['B', 3] in the Chevalley basis]

sage: D = g.derived_subalgebra()
sage: D.derived_series()
[Affine Lie algebra of ['B', 3] in the Chevalley basis]
```

monomial(*m*)

Construct the monomial indexed by `m`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B',4,1])
sage: al = RootSystem(['B',4]).root_lattice().simple_roots()
sage: g.monomial((al[1]+al[2]+al[3],4))
(E[alpha[1] + alpha[2] + alpha[3]])#t^4
sage: g.monomial((-al[1]-al[2]-2*al[3]-2*al[4],2))
(E[-alpha[1] - alpha[2] - 2*alpha[3] - 2*alpha[4]])#t^2
sage: g.monomial((al[4],-2))
(E[alpha[4]])#t^-2
sage: g.monomial('c')
c
sage: g.monomial('d')
d
```

zero()

Return the element 0.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['F',4,1])
sage: g.zero()
0
```

class sage.algebras.lie_algebras.affine_lie_algebra.**TwistedAffineIndices**(*cartan_type*)

Bases: UniqueRepresentation, Set_generic

The indices for the basis of a twisted affine Lie algebra.

INPUT:

- cartan_type – the Cartan type of twisted affine type Lie algebra

EXAMPLES:

```
sage: from sage.algebras.lie_algebras.affine_lie_algebra import TwistedAffineIndices
sage: I = TwistedAffineIndices(['A', 3, 2])
sage: it = iter(I)
sage: [next(it) for _ in range(20)]
[(alpha[1], 0), (alpha[2], 0), (alpha[1] + alpha[2], 0),
 (2*alpha[1] + alpha[2], 0), (-alpha[1], 0), (-alpha[2], 0),
 (-alpha[1] - alpha[2], 0), (-2*alpha[1] - alpha[2], 0),
 (alphacheck[1], 0), (alphacheck[2], 0), (alpha[1], 1),
 (alpha[1] + alpha[2], 1), (-alpha[1], 1), (-alpha[1] - alpha[2], 1),
 (alphacheck[1], 1), (alpha[1], -1), (alpha[1] + alpha[2], -1),
 (-alpha[1], -1), (-alpha[1] - alpha[2], -1), (alphacheck[1], -1)]

sage: I = TwistedAffineIndices(['A', 4, 2])
sage: it = iter(I)
sage: [next(it) for _ in range(20)]
[(alpha[0], 0), (alpha[1], 0), (alpha[0] + alpha[1], 0),
 (2*alpha[0] + alpha[1], 0), (-alpha[0], 0), (-alpha[1], 0),
 (-alpha[0] - alpha[1], 0), (-2*alpha[0] - alpha[1], 0),
 (alphacheck[0], 0), (alphacheck[1], 0), (alpha[0], 1), (alpha[1], 1),
 (alpha[0] + alpha[1], 1), (2*alpha[0] + alpha[1], 1), (-alpha[0], 1),
 (-alpha[1], 1), (-alpha[0] - alpha[1], 1), (-2*alpha[0] - alpha[1], 1),
 (2*alpha[0], 1), (2*alpha[0] + 2*alpha[1], 1)]

sage: I = TwistedAffineIndices(['A', 2, 2])
sage: it = iter(I)
sage: [next(it) for _ in range(10)]
[(alpha[0], 0), (-alpha[0], 0), (alphacheck[0], 0), (alpha[0], 1),
 (-alpha[0], 1), (2*alpha[0], 1), (-2*alpha[0], 1),
 (alphacheck[0], 1), (alpha[0], -1), (-alpha[0], -1)]
```

class sage.algebras.lie_algebras.affine_lie_algebra.**TwistedAffineLieAlgebra**(*R*, *cartan_type*, *kac_moody*)

Bases: *AffineLieAlgebra*

A twisted affine Lie algebra.

A twisted affine Lie algebra is an affine Lie algebra for type $X_N^{(r)}$ with $r > 1$. We realize this inside an untwisted affine Kac–Moody Lie algebra following Chapter 8 of [Ka1990].

Let $\overline{\mathfrak{g}}$ be the classical Lie algebra by taking the index set $I \setminus \{\epsilon\}$, where $\epsilon = 0$ unless $\epsilon = n$ for $X_N^{(r)} = A_{2n}^{(2)}$, for the twisted affine Lie algebra $\widetilde{\mathfrak{g}}$. Let $\mathfrak{g}$ be the basic Lie algebra of type X_N. We realize $\overline{\mathfrak{g}}$ as the fixed-point subalgebra $\mathfrak{g}^{(0)}$ of $\mathfrak{g}$ under the order r diagram automorphism μ. This naturally acts on the ζ_r (a primitive r-th root of unity) eigenspace $\mathfrak{g}^{(1)}$ of μ, which is the highest weight representation corresponding to the small adjoint (where the weight spaces are the short roots of $\overline{\mathfrak{g}}$). The *twisted affine (Kac-Moody) Lie algebra* $\widehat{\mathfrak{g}}$ is constructed as the subalgebra of $X_N^{(1)}$ given by

$$\sum_{i \in \mathbf{Z}} \mathfrak{g}^{(i \mod 2)} \otimes t^i \oplus Rc \oplus Rd,$$

where R is the base ring.

We encode our basis by using the classical Lie algebra except for type $A_{2n}^{(2)}$. For type $A_{2n}^{(2)}$, the fixed-point algebra $\mathfrak{g}^{(0)}$ is of type B_n using the index set $\{0, \ldots, n-1\}$. For $\mathfrak{g}^{(1)}$, we identify the weights in this representation with the roots of type B_n and the double all of its short roots.

class Element

Bases: *UntwistedAffineLieAlgebraElement*

ambient()

Return the ambient untwisted affine Lie algebra of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A', 5, 2])
sage: g.ambient()
Affine Kac-Moody algebra of ['A', 5] in the Chevalley basis
```

derived_subalgebra()

Return the derived subalgebra of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A', 5, 2])
sage: g
Twisted affine Kac-Moody algebra of type ['B', 3, 1]^* over Rational Field
sage: D = g.derived_subalgebra(); D
Twisted affine Lie algebra of type ['B', 3, 1]^* over Rational Field
sage: D.derived_subalgebra() == D
True
```

retract(*x*)

Retract the element `x` from the ambient untwisted affine Lie algebra into `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A', 5, 2])
sage: it = iter(g.basis())
sage: elts = [next(it) for _ in range(20)]
sage: elts
[c,
 d,
 (E[alpha[1]])#t^0,
 (E[alpha[2]])#t^0,
 (E[alpha[3]])#t^0,
 (E[alpha[1] + alpha[2]])#t^0,
 (E[alpha[2] + alpha[3]])#t^0,
 (E[2*alpha[2] + alpha[3]])#t^0,
 (E[alpha[1] + alpha[2] + alpha[3]])#t^0,
 (E[2*alpha[1] + 2*alpha[2] + alpha[3]])#t^0,
 (E[alpha[1] + 2*alpha[2] + alpha[3]])#t^0,
```

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```
 (E[-alpha[1]])#t^0,
 (E[-alpha[2]])#t^0,
 (E[-alpha[3]])#t^0,
 (E[-alpha[1] - alpha[2]])#t^0,
 (E[-alpha[2] - alpha[3]])#t^0,
 (E[-2*alpha[2] - alpha[3]])#t^0,
 (E[-alpha[1] - alpha[2] - alpha[3]])#t^0,
 (E[-2*alpha[1] - 2*alpha[2] - alpha[3]])#t^0,
 (E[-alpha[1] - 2*alpha[2] - alpha[3]])#t^0]
sage: all(g.retract(g.to_ambient(x)) == x for x in elts)
True
```

to_ambient()

Lift the element `x` from the ambient untwisted affine Lie algebra into `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A', 5, 2])
sage: g.to_ambient
Generic morphism:
  From: Twisted affine Kac-Moody algebra of type ['B', 3, 1]^* over Rational␣
↪Field
  To:   Affine Kac-Moody algebra of ['A', 5] in the Chevalley basis
```

class `sage.algebras.lie_algebras.affine_lie_algebra.`**`UntwistedAffineLieAlgebra`**(*g*, *kac_moody*)

Bases: `AffineLieAlgebra`

An untwisted affine Lie algebra.

Let R be a ring. Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ over R, the affine Lie algebra $\widehat{\mathfrak{g}}'$ associated to $\mathfrak{g}$ is defined as

$$\widehat{\mathfrak{g}}' = \bigl(\mathfrak{g} \otimes R[t, t^{-1}]\bigr) \oplus Rc,$$

where c is the canonical central element and $R[t, t^{-1}]$ is the Laurent polynomial ring over R. The Lie bracket is defined as

$$[x \otimes t^m + \lambda c, y \otimes t^n + \mu c] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x|y)c,$$

where $(x|y)$ is the Killing form on $\mathfrak{g}$.

There is a canonical derivation d on $\widehat{\mathfrak{g}}'$ that is defined by

$$d(x \otimes t^m + \lambda c) = a \otimes mt^m,$$

or equivalently by $d = t\frac{d}{dt}$.

The affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is formed by adjoining the derivation d such that

$$\widehat{\mathfrak{g}} = \bigl(\mathfrak{g} \otimes R[t, t^{-1}]\bigr) \oplus Rc \oplus Rd.$$

Specifically, the bracket on $\widehat{\mathfrak{g}}$ is defined as

$$[t^m \otimes x \oplus \lambda c \oplus \mu d, t^n \otimes y \oplus \lambda_1 c \oplus \mu_1 d] = \bigl(t^{m+n}[x, y] + \mu n t^n \otimes y - \mu_1 m t^m \otimes x\bigr) \oplus m\delta_{m,-n}(x|y)c.$$

EXAMPLES:

We begin by constructing an affine Kac-Moody algebra of type $G_2^{(1)}$ from the classical Lie algebra of type G_2:

```
sage: g = LieAlgebra(QQ, cartan_type=['G',2])
sage: A = g.affine()
sage: A
Affine Kac-Moody algebra of ['G', 2] in the Chevalley basis
```

Next, we construct the generators and perform some computations:

```
sage: A.inject_variables()
Defining e1, e2, f1, f2, h1, h2, e0, f0, c, d
sage: e1.bracket(f1)
(h1)#t^0
sage: e0.bracket(f0)
(-h1 - 2*h2)#t^0 + 8*c
sage: e0.bracket(f1)
0
sage: A[d, f0]
(-E[3*alpha[1] + 2*alpha[2]])#t^-1
sage: A([[e0, e2], [[[e1, e2], [e0, [e1, e2]]], e1]])
(-6*E[-3*alpha[1] - alpha[2]])#t^2
sage: f0.bracket(f1)
0
sage: f0.bracket(f2)
(E[3*alpha[1] + alpha[2]])#t^-1
sage: A[h1+3*h2, A[[[f0, f2], f1], [f1,f2]] + f1] - f1
(2*E[alpha[1]])#t^-1
```

We can construct its derived subalgebra, the affine Lie algebra of type $G_2^{(1)}$. In this case, there is no canonical derivation, so the generator d is 0:

```
sage: D = A.derived_subalgebra()
sage: D.d()
0
```

Element

alias of *UntwistedAffineLieAlgebraElement*

derived_subalgebra()

Return the derived subalgebra of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['B',3,1])
sage: g
Affine Kac-Moody algebra of ['B', 3] in the Chevalley basis
sage: D = g.derived_subalgebra(); D
Affine Lie algebra of ['B', 3] in the Chevalley basis
sage: D.derived_subalgebra() == D
True
```

9.2.3 The Baker-Campbell-Hausdorff formula

AUTHORS:

- Eero Hakavuori (2018-09-23): initial version

sage.algebras.lie_algebras.bch.**bch_iterator**(*X=None*, *Y=None*)

A generator function which returns successive terms of the Baker-Campbell-Hausdorff formula.

INPUT:

- X – (optional) an element of a Lie algebra
- Y – (optional) an element of a Lie algebra

The BCH formula is an expression for $\log(\exp(X)\exp(Y))$ as a sum of Lie brackets of X and Y with rational coefficients. In arbitrary Lie algebras, the infinite sum is only guaranteed to converge for X and Y close to zero.

If the elements X and Y are not given, then the iterator will return successive terms of the abstract BCH formula, i.e., the BCH formula for the generators of the free Lie algebra on 2 generators.

If the Lie algebra containing X and Y is not nilpotent, the iterator will output infinitely many elements. If the Lie algebra is nilpotent, the number of elements outputted is equal to the nilpotency step.

EXAMPLES:

The terms of the abstract BCH formula up to fifth order brackets:

```
sage: from sage.algebras.lie_algebras.bch import bch_iterator
sage: bch = bch_iterator()
sage: next(bch)
X + Y
sage: next(bch)
1/2*[X, Y]
sage: next(bch)
1/12*[X, [X, Y]] + 1/12*[[X, Y], Y]
sage: next(bch)
1/24*[X, [[X, Y], Y]]
sage: next(bch)
-1/720*[X, [X, [X, [X, Y]]]] + 1/180*[X, [X, [[X, Y], Y]]]
+ 1/360*[[X, [X, Y]], [X, Y]] + 1/180*[X, [[[X, Y], Y], Y]]
+ 1/120*[[X, Y], [[X, Y], Y]] - 1/720*[[[[X, Y], Y], Y], Y]
```

For nilpotent Lie algebras the BCH formula only has finitely many terms:

```
sage: L = LieAlgebra(QQ, 2, step=3)
sage: L.inject_variables()
Defining X_1, X_2, X_12, X_112, X_122
sage: [Z for Z in bch_iterator(X_1, X_2)]
[X_1 + X_2, 1/2*X_12, 1/12*X_112 + 1/12*X_122]
sage: [Z for Z in bch_iterator(X_1 + X_2, X_12)]
[X_1 + X_2 + X_12, 1/2*X_112 - 1/2*X_122, 0]
```

The elements X and Y don't need to be elements of the same Lie algebra if there is a coercion from one to the other:

```
sage: L = LieAlgebra(QQ, 3, step=2)
sage: L.inject_variables()
```

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```
Defining X_1, X_2, X_3, X_12, X_13, X_23
sage: S = L.subalgebra(X_1, X_2)
sage: bch1 = [Z for Z in bch_iterator(S(X_1), S(X_2))]; bch1
[X_1 + X_2, 1/2*X_12]
sage: bch1[0].parent() == S
True
sage: bch2 = [Z for Z in bch_iterator(S(X_1), X_3)]; bch2
[X_1 + X_3, 1/2*X_13]
sage: bch2[0].parent() == L
True
```

The BCH formula requires a coercion from the rationals:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, 2, step=2)
sage: bch = bch_iterator(X, Y); next(bch)
Traceback (most recent call last):
...
TypeError: the BCH formula is not well defined since Integer Ring has no coercion␣
↪from Rational Field
```

ALGORITHM:

The BCH formula $\log(\exp(X)\exp(Y)) = \sum_k Z_k$ is computed starting from $Z_1 = X + Y$, by the recursion

$$(m+1)Z_{m+1} = \frac{1}{2}[X - Y, Z_m] + \sum_{2\leq 2p\leq m} \frac{B_{2p}}{(2p)!} \sum_{k_1+\cdots+k_{2p}=m} [Z_{k_1}, [\cdots [Z_{k_{2p}}, X + Y]\cdots],$$

where B_{2p} are the Bernoulli numbers, see Lemma 2.15.3. in [Var1984].

Warning: The time needed to compute each successive term increases exponentially. For example on one machine iterating through $Z_{11}, ..., Z_{18}$ for a free Lie algebra, computing each successive term took 4-5 times longer, going from 0.1s for Z_{11} to 21 minutes for Z_{18}.

9.2.4 Classical Lie Algebras

These are the Lie algebras corresponding to types A_n, B_n, C_n, and D_n. We also include support for the exceptional types $E_{6,7,8}$, F_4, and G_2 in the Chevalley basis, and we give the matrix representation given in [HRT2000].

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version
- Sebastian Oehms (2018-03-18): matrix method of the element class of ClassicalMatrixLieAlgebra added
- Travis Scrimshaw (2019-07-09): Implemented compact real form

class sage.algebras.lie_algebras.classical_lie_algebra.**ClassicalMatrixLieAlgebra**(*R*, *ct*, *e*, *f*, *h*, *sparse=True*)

Bases: *MatrixLieAlgebraFromAssociative*

A classical Lie algebra represented using matrices.

This means a classical Lie algebra given as a Lie algebra of matrices, with commutator as Lie bracket.

INPUT:

- R – the base ring
- ct – the finite Cartan type

EXAMPLES:

```
sage: lie_algebras.ClassicalMatrix(QQ, ['A', 4])
Special linear Lie algebra of rank 5 over Rational Field
sage: lie_algebras.ClassicalMatrix(QQ, CartanType(['B',4]))
Special orthogonal Lie algebra of rank 9 over Rational Field
sage: lie_algebras.ClassicalMatrix(QQ, 'C4')
Symplectic Lie algebra of rank 8 over Rational Field
sage: lie_algebras.ClassicalMatrix(QQ, cartan_type=['D',4])
Special orthogonal Lie algebra of rank 8 over Rational Field
```

affine(*kac_moody=True*)

Return the affine (Kac-Moody) Lie algebra of self.

EXAMPLES:

```
sage: so5 = lie_algebras.so(QQ, 5, 'matrix')
sage: so5
Special orthogonal Lie algebra of rank 5 over Rational Field
sage: so5.affine()
Affine Special orthogonal Kac-Moody algebra of rank 5 over Rational Field
sage: so5.affine(False)
Affine Special orthogonal Lie algebra of rank 5 over Rational Field
```

basis()

Return a basis of self.

EXAMPLES:

```
sage: M = LieAlgebra(ZZ, cartan_type=['A',2], representation='matrix')
sage: list(M.basis())
[
[ 1  0  0]  [0 1 0]  [0 0 1]  [0 0 0]  [ 0  0  0]  [0 0 0]  [0 0 0]
[ 0  0  0]  [0 0 0]  [0 0 0]  [1 0 0]  [ 0  1  0]  [0 0 1]  [0 0 0]
[ 0  0 -1], [0 0 0], [0 0 0], [0 0 0], [ 0  0 -1], [0 0 0], [1 0 0],

[0 0 0]
[0 0 0]
[0 1 0]
]
```

Sparse version:

```
sage: e6 = LieAlgebra(QQ, cartan_type=['E',6], representation='matrix')
sage: len(e6.basis())  # long time
78
```

cartan_type()

Return the Cartan type of self.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.cartan_type()
['A', 2]
```

e(*i*)

Return the generator e_i.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.e(2)
[0 0 0]
[0 0 1]
[0 0 0]
```

epsilon(*i*, *h*)

Return the action of the functional $\varepsilon_i : \mathfrak{h} \to R$, where R is the base ring of `self`, on the element `h`.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.epsilon(1, g.h(1))
1
sage: g.epsilon(2, g.h(1))
-1
sage: g.epsilon(3, g.h(1))
0
```

f(*i*)

Return the generator f_i.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.f(2)
[0 0 0]
[0 0 0]
[0 1 0]
```

h(*i*)

Return the generator h_i.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.h(2)
[ 0  0  0]
[ 0  1  0]
[ 0  0 -1]
```

highest_root_basis_elt(*pos=True*)

Return the basis element corresponding to the highest root θ. If `pos` is `True`, then returns e_θ, otherwise it returns f_θ.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.highest_root_basis_elt()
[0 0 1]
[0 0 0]
[0 0 0]
```

index_set()

Return the index_set of `self`.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.index_set()
(1, 2)
```

simple_root(*i*, *h*)

Return the action of the simple root $\alpha_i \colon \mathfrak{h} \to R$, where R is the base ring of `self`, on the element h.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 3, representation='matrix')
sage: g.simple_root(1, g.h(1))
2
sage: g.simple_root(1, g.h(2))
-1
```

class sage.algebras.lie_algebras.classical_lie_algebra.**ExceptionalMatrixLieAlgebra**(*R*, *cartan_type*, *e*, *f*, *h=None*, *sparse=False*)

Bases: `ClassicalMatrixLieAlgebra`

A matrix Lie algebra of exceptional type.

class sage.algebras.lie_algebras.classical_lie_algebra.**LieAlgebraChevalleyBasis**(*R*, *cartan_type*)

Bases: `LieAlgebraWithStructureCoefficients`

A simple finite dimensional Lie algebra in the Chevalley basis.

Let L be a simple (complex) Lie algebra with roots Φ, then the Chevalley basis is given by e_α for all $\alpha \in \Phi$ and $h_{\alpha_i} := h_i$ where α_i is a simple root subject. These generators are subject to the relations:

$$\begin{aligned}
[h_i, h_j] &= 0, \\
[h_i, e_\beta] &= A_{\alpha_i,\beta} e_\beta, \\
[e_\beta, e_{-\beta}] &= \sum_i A_{\beta,\alpha_i} h_i, \\
[e_\beta, e_\gamma] &= \begin{cases} N_{\beta,\gamma} e_{\beta+\gamma} & \beta + \gamma \in \Phi, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where $A_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ and $N_{\alpha,\beta}$ is the maximum such that $\alpha - N_{\alpha,\beta}\beta \in \Phi$.

For computing the signs of the coefficients, see Section 3 of [CMT2003].

See also:

For simply-laced types, an alternative construction using an asymmetry function is given by *LieAlgebraChevalleyBasis_simply_laced*.

affine(*kac_moody=True*)

Return the affine Lie algebra of `self`.

EXAMPLES:

```
sage: sp6 = lie_algebras.sp(QQ, 6)
sage: sp6
Lie algebra of ['C', 3] in the Chevalley basis
sage: sp6.affine()
Affine Kac-Moody algebra of ['C', 3] in the Chevalley basis

sage: L = LieAlgebra(QQ, cartan_type=['A',3], epsilon=[(1,2),(3,2)])
sage: L.affine(False)
Affine Lie algebra of ['A', 3] in the Chevalley basis
sage: L.affine(True)
Affine Kac-Moody algebra of ['A', 3] in the Chevalley basis
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['G', 2])
sage: [L.degree_on_basis(m) for m in L.basis().keys()]
[alpha[2], alpha[1], alpha[1] + alpha[2],
 2*alpha[1] + alpha[2], 3*alpha[1] + alpha[2],
 3*alpha[1] + 2*alpha[2],
 0, 0,
 -alpha[2], -alpha[1], -alpha[1] - alpha[2],
 -2*alpha[1] - alpha[2], -3*alpha[1] - alpha[2],
 -3*alpha[1] - 2*alpha[2]]
```

gens()

Return the generators of `self` in the order of e_i, f_i, and h_i.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2])
sage: L.gens()
(E[alpha[1]], E[alpha[2]], E[-alpha[1]], E[-alpha[2]], h1, h2)
```

highest_root_basis_elt(*pos=True*)

Return the basis element corresponding to the highest root θ.

INPUT:

- `pos` – (default: `True`) if `True`, then return e_θ, otherwise return f_θ

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2])
sage: L.highest_root_basis_elt()
```

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```
E[alpha[1] + alpha[2]]
sage: L.highest_root_basis_elt(False)
E[-alpha[1] - alpha[2]]
```

indices_to_positive_roots_map()

Return the map from indices to positive roots.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2])
sage: L.indices_to_positive_roots_map()
{1: alpha[1], 2: alpha[2], 3: alpha[1] + alpha[2]}
```

killing_form(x, y)

Return the Killing form on `x` and `y`, where `x` and `y` are two elements of `self`.

The Killing form is defined as

$$\langle x \mid y \rangle = \operatorname{tr}\left(\operatorname{ad}_x \circ \operatorname{ad}_y\right).$$

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2])
sage: L.killing_form(L.an_element(), L.an_element())
36
sage: B = L.basis()
sage: matrix([[L.killing_form(a, b) for a in B] for b in B])
[ 0  0  0  0  0  6  0  0]
[ 0  0  0  0  0  0  6  0]
[ 0  0  0  0  0  0  0  6]
[ 0  0  0 12 -6  0  0  0]
[ 0  0  0 -6 12  0  0  0]
[ 6  0  0  0  0  0  0  0]
[ 0  6  0  0  0  0  0  0]
[ 0  0  6  0  0  0  0  0]
```

killing_form_matrix()

Return the matrix of the Killing form of `self`.

The rows and the columns of this matrix are indexed by the elements of the basis of `self` (in the order provided by `basis()`).

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['A', 2])
sage: g.killing_form_matrix()
[ 0  0  0  0  0  6  0  0]
[ 0  0  0  0  0  0  6  0]
[ 0  0  0  0  0  0  0  6]
[ 0  0  0 12 -6  0  0  0]
[ 0  0  0 -6 12  0  0  0]
[ 6  0  0  0  0  0  0  0]
[ 0  6  0  0  0  0  0  0]
[ 0  0  6  0  0  0  0  0]
```

lie_algebra_generators(*str_keys=False*)

Return the Chevalley Lie algebra generators of `self`.

INPUT:

- `str_keys` – (default: `False`) set to `True` to have the indices indexed by strings instead of simple (co)roots

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 1])
sage: L.lie_algebra_generators()
Finite family {alpha[1]: E[alpha[1]], -alpha[1]: E[-alpha[1]], alphacheck[1]:␣
↪h1}
sage: L.lie_algebra_generators(True)
Finite family {'e1': E[alpha[1]], 'f1': E[-alpha[1]], 'h1': h1}
```

class sage.algebras.lie_algebras.classical_lie_algebra.**LieAlgebraChevalleyBasis_simply_laced**(*R*, *cartan_type*, *epsilon*)

Bases: `LieAlgebraChevalleyBasis`

A finite dimensional simply-laced Lie algebra in the Chevalley basis with structure coefficients given by an orientation of the Dynkin diagram.

We follow Chapter 7.7 of [Ka1990], where the structure coefficients are given by an `asymmetry function` defined by $\varepsilon(\alpha_i, \alpha_j) = -1$ if there is an arrow $i \to j$ in the Dynkin quiver (an orientation of the Dynkin diagram). However we twist E_α by $\mathrm{sign}(\alpha)$ so that $F_i = E_{-\alpha_i}$ rather than its negative.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2], epsilon=[(2, 1)])
sage: L.e(1).bracket(L.e(2))
E[alpha[1] + alpha[2]]

sage: L = LieAlgebra(QQ, cartan_type=['A', 2], epsilon=[(1, 2)])
sage: L.e(1).bracket(L.e(2))
-E[alpha[1] + alpha[2]]
```

asymmetry_function()

Return the asymmetry function of `self`.

An *asymmetry function* is a function $\varepsilon : Q \times Q \to \{1, -1\}$ that satisfies the following properties:

1. $\varepsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2}$
2. bimultiplicativity $\varepsilon(alpha + \alpha', \beta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha', \beta)$ and $\varepsilon(alpha, \beta + \beta') = \varepsilon(\alpha, \beta)\varepsilon(\alpha', \beta)$,

where $(\alpha|\beta)$ is the symmetric bilinear form on Q given by the Cartan matrix. Some consequences of these properties are that $\varepsilon(\alpha, 0) = \varepsilon(0, \beta) = 1$ and $varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}$.

OUTPUT:

The asymmetry function as a `dict` consisting of pairs of all of the roots of Q and 0.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A',2], epsilon=[(2,1)])
sage: ep = L.asymmetry_function()
sage: al = L.cartan_type().root_system().root_lattice().simple_roots()
sage: ep[al[1], al[2]]
1
sage: ep[al[2],al[1]]
-1

sage: L = LieAlgebra(QQ, cartan_type=['A',2], epsilon=[(1,2)])
sage: ep = L.asymmetry_function()
sage: al = L.cartan_type().root_system().root_lattice().simple_roots()
sage: ep[al[1], al[2]]
-1
sage: ep[al[2],al[1]]
1
```

class sage.algebras.lie_algebras.classical_lie_algebra.**MatrixCompactRealForm**(*R*, *cartan_type*)

Bases: *FinitelyGeneratedLieAlgebra*

The compact real form of a matrix Lie algebra.

Let L be a classical (i.e., type $ABCD$) Lie algebra over $\mathbf{R}$ given as matrices that is invariant under matrix transpose (i.e., $X^T \in L$ for all $X \in L$). Then we can perform the *Cartan decomposition* of L by $L = K \oplus S$, where K (resp. S) is the set of skew-symmetric (resp. symmetric) matrices in L. Then the Lie algebra $U = K \oplus iS$ is an $\mathbf{R}$-subspace of the complexification of L that is closed under commutators and has skew-hermitian matrices. Hence, the Killing form is negative definitive (i.e., U is a compact Lie algebra), and thus U is the complex real form of the complexification of L.

EXAMPLES:

```
sage: U = LieAlgebra(QQ, cartan_type=['A',1], representation="compact real")
sage: list(U.basis())
[
[ 0  1]  [ i  0]  [0 i]
[-1  0], [ 0 -i], [i 0]
]
sage: U.killing_form_matrix()
[-8  0  0]
[ 0 -8  0]
[ 0  0 -8]
```

Computations are only (currently) possible if this is defined over a field:

```
sage: U = LieAlgebra(ZZ, cartan_type=['A',1], representation="compact real")
sage: list(U.basis())
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

class Element(*parent*, *real*, *imag*)

Bases: Element

An element of a matrix Lie algebra in its compact real form.

monomial_coefficients(*copy=False*)

Return the monomial coefficients of self.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['C',3], representation="compact real")
sage: B = L.basis()
sage: x = L.sum(i*B[i] for i in range(len(B)))
sage: x.monomial_coefficients() == {i: i for i in range(1,len(B))}
True
```

basis()

Compute a basis of self.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['B',2], representation="compact real")
sage: list(L.basis())
[
[ 0  1  0  0  0]  [ 0  0  0  1  0]  [ 0  0  0  0  1]  [ 0  0  0  0  0]
[-1  0  0  0  0]  [ 0  0 -1  0  0]  [ 0  0  0  0  0]  [ 0  0  0  0  1]
[ 0  0  0  1  0]  [ 0  1  0  0  0]  [ 0  0  0  0  1]  [ 0  0  0  0  0]
[ 0  0 -1  0  0]  [-1  0  0  0  0]  [ 0  0  0  0  0]  [ 0  0  0  0  1]
[ 0  0  0  0  0], [ 0  0  0  0  0], [-1  0 -1  0  0], [ 0 -1  0 -1  0],

[ i  0  0  0  0]  [ 0  i  0  0  0]  [ 0  0  0  i  0]  [ 0  0  0  0  i]
[ 0  0  0  0  0]  [ i  0  0  0  0]  [ 0  0 -i  0  0]  [ 0  0  0  0  0]
[ 0  0 -i  0  0]  [ 0  0  0 -i  0]  [ 0 -i  0  0  0]  [ 0  0  0  0 -i]
[ 0  0  0  0  0]  [ 0  0 -i  0  0]  [ i  0  0  0  0]  [ 0  0  0  0  0]
[ 0  0  0  0  0], [ 0  0  0  0  0], [ 0  0  0  0  0], [ i  0 -i  0  0],

[ 0  0  0  0  0]  [ 0  0  0  0  0]
[ 0  i  0  0  0]  [ 0  0  0  0  i]
[ 0  0  0  0  0]  [ 0  0  0  0  0]
[ 0  0  0 -i  0]  [ 0  0  0  0 -i]
[ 0  0  0  0  0], [ 0  i  0 -i  0]
]
```

monomial(*i*)

Return the monomial indexed by i.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A',3], representation="compact real")
sage: L.monomial(0)
[ 0  1  0  0]
[-1  0  0  0]
[ 0  0  0  0]
[ 0  0  0  0]
```

term(*i*, *c=None*)

Return the term indexed by i with coefficient c.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['C',3], representation="compact real")
sage: L.term(4, 7/2)
[   0    0    0    0    0  7/2]
```

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```
[   0    0    0    0    0    0]
[   0    0    0  7/2    0    0]
[   0    0 -7/2    0    0    0]
[   0    0    0    0    0    0]
[-7/2    0    0    0    0    0]
```

zero()

Return the element 0.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['D',4], representation="compact real")
sage: L.zero()
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
```

class sage.algebras.lie_algebras.classical_lie_algebra.**e6**(*R*)

Bases: *ExceptionalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{e}_6$.

The simple Lie algebra $\mathfrak{e}_6$ of type E_6. The matrix representation is given following [HRT2000].

class sage.algebras.lie_algebras.classical_lie_algebra.**e7**(*R*)

Bases: *ExceptionalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{e}_7$.

The simple Lie algebra $\mathfrak{e}_7$ of type E_7. The matrix representation is given following [HRT2000].

class sage.algebras.lie_algebras.classical_lie_algebra.**e8**(*R*)

Bases: *ExceptionalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{e}_8$.

The simple Lie algebra $\mathfrak{e}_8$ of type E_8 built from the adjoint representation in the Chevalley basis.

basis()

Return a basis of `self`.

EXAMPLES:

```
sage: g = LieAlgebra(QQ, cartan_type=['E', 8], representation='matrix')  # long␣
↪time
sage: len(g.basis())  # long time
248
```

class sage.algebras.lie_algebras.classical_lie_algebra.**f4**(*R*)

Bases: *ExceptionalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{f}_4$.

The simple Lie algebra $\mathfrak{f}_f$ of type F_4. The matrix representation is given following [HRT2000] but indexed in the reversed order (i.e., interchange 1 with 4 and 2 with 3).

class sage.algebras.lie_algebras.classical_lie_algebra.**g2**(*R*)

Bases: *ExceptionalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{g}_2$.

The simple Lie algebra $\mathfrak{g}_2$ of type G_2. The matrix representation is given following [HRT2000].

class sage.algebras.lie_algebras.classical_lie_algebra.**gl**(*R*, *n*)

Bases: *MatrixLieAlgebraFromAssociative*

The matrix Lie algebra $\mathfrak{gl}_n$.

The Lie algebra $\mathfrak{gl}_n$ which consists of all $n \times n$ matrices.

INPUT:

- R – the base ring
- n – the size of the matrix

class Element

Bases: *Element*

monomial_coefficients(*copy=True*)

Return the monomial coefficients of self.

EXAMPLES:

```
sage: gl4 = lie_algebras.gl(QQ, 4)
sage: x = gl4.monomial('E_2_1') + 3*gl4.monomial('E_0_3')
sage: x.monomial_coefficients()
{'E_0_3': 3, 'E_2_1': 1}
```

basis()

Return the basis of self.

EXAMPLES:

```
sage: g = lie_algebras.gl(QQ, 2)
sage: tuple(g.basis())
(
[1 0]  [0 1]  [0 0]  [0 0]
[0 0], [0 0], [1 0], [0 1]
)
```

killing_form(*x*, *y*)

Return the Killing form on x and y.

The Killing form on $\mathfrak{gl}_n$ is:

$$\langle x \mid y \rangle = 2n\mathrm{tr}(xy) - 2\mathrm{tr}(x)\mathrm{tr}(y).$$

EXAMPLES:

```
sage: g = lie_algebras.gl(QQ, 4)
sage: x = g.an_element()
sage: y = g.gens()[1]
sage: g.killing_form(x, y)
8
```

monomial(*i*)

Return the basis element indexed by i.

INPUT:

- i – an element of the index set

EXAMPLES:

```
sage: gl4 = lie_algebras.gl(QQ, 4)
sage: gl4.monomial('E_2_1')
[0 0 0 0]
[0 0 0 0]
[0 1 0 0]
[0 0 0 0]
sage: gl4.monomial((2,1))
[0 0 0 0]
[0 0 0 0]
[0 1 0 0]
[0 0 0 0]
```

class sage.algebras.lie_algebras.classical_lie_algebra.**sl**(*R*, *n*)

Bases: *ClassicalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{sl}_n$.

The Lie algebra $\mathfrak{sl}_n$, which consists of all $n \times n$ matrices with trace 0. This is the Lie algebra of type A_{n-1}.

killing_form(*x*, *y*)

Return the Killing form on x and y.

The Killing form on $\mathfrak{sl}_n$ is:

$$\langle x \mid y \rangle = 2n \mathrm{tr}(xy).$$

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 5, representation='matrix')
sage: x = g.an_element()
sage: y = g.lie_algebra_generators()['e1']
sage: g.killing_form(x, y)
10
```

simple_root(*i*, *h*)

Return the action of the simple root $\alpha_i : \mathfrak{h} \to R$, where R is the base ring of self, on the element j.

EXAMPLES:

```
sage: g = lie_algebras.sl(QQ, 5, representation='matrix')
sage: matrix([[g.simple_root(i, g.h(j)) for i in g.index_set()] for j in g.
```

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```
↪index_set()])
[ 2 -1  0  0]
[-1  2 -1  0]
[ 0 -1  2 -1]
[ 0  0 -1  2]
```

class sage.algebras.lie_algebras.classical_lie_algebra.**so**(*R*, *n*)

Bases: *ClassicalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{so}_n$.

The Lie algebra $\mathfrak{so}_n$, which is isomorphic to the Lie algebra of all anti-symmetric $n \times n$ matrices. The implementation here uses a different bilinear form and follows the description in Chapter 8 of [HK2002]. More precisely, this is the set of matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that $A^t = -D$, $B^t = -B$, $C^t = -C$ for n even and

$$\begin{pmatrix} A & B & a \\ C & D & b \\ c & d & 0 \end{pmatrix}$$

such that $A^t = -D$, $B^t = -B$, $C^t = -C$, $a^t = -d$, and $b^t = -c$ for n odd.

This is the Lie algebra of type $B_{(n-1)/2}$ or $D_{n/2}$ if n is odd or even respectively.

killing_form(*x*, *y*)

Return the Killing form on `x` and `y`.

The Killing form on $\mathfrak{so}_n$ is:

$$\langle x \mid y \rangle = (n-2)\mathrm{tr}(xy).$$

EXAMPLES:

```
sage: g = lie_algebras.so(QQ, 8, representation='matrix')
sage: x = g.an_element()
sage: y = g.lie_algebra_generators()['e1']
sage: g.killing_form(x, y)
12
sage: g = lie_algebras.so(QQ, 9, representation='matrix')
sage: x = g.an_element()
sage: y = g.lie_algebra_generators()['e1']
sage: g.killing_form(x, y)
14
```

simple_root(*i*, *h*)

Return the action of the simple root $\alpha_i \colon \mathfrak{h} \to R$, where R is the base ring of `self`, on the element `j`.

EXAMPLES:

The even or type D case:

```
sage: g = lie_algebras.so(QQ, 8, representation='matrix')
sage: matrix([[g.simple_root(i, g.h(j)) for i in g.index_set()] for j in g.
↪index_set()])
[ 2 -1  0  0]
[-1  2 -1 -1]
[ 0 -1  2  0]
[ 0 -1  0  2]
```

The odd or type B case:

```
sage: g = lie_algebras.so(QQ, 9, representation='matrix')
sage: matrix([[g.simple_root(i, g.h(j)) for i in g.index_set()] for j in g.
↪index_set()])
[ 2 -1  0  0]
[-1  2 -1  0]
[ 0 -1  2 -1]
[ 0  0 -2  2]
```

class sage.algebras.lie_algebras.classical_lie_algebra.**sp**(*R*, *n*)

Bases: *ClassicalMatrixLieAlgebra*

The matrix Lie algebra $\mathfrak{sp}_n$.

The Lie algebra $\mathfrak{sp}_{2k}$, which consists of all $2k \times 2k$ matrices X that satisfy the equation:

$$X^T M - MX = 0$$

where

$$M = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

This is the Lie algebra of type C_k.

killing_form(*x*, *y*)

Return the Killing form on x and y.

The Killing form on $\mathfrak{sp}_n$ is:

$$\langle x \mid y \rangle = (2n+2)\mathrm{tr}(xy).$$

EXAMPLES:

```
sage: g = lie_algebras.sp(QQ, 8, representation='matrix')
sage: x = g.an_element()
sage: y = g.lie_algebra_generators()['e1']
sage: g.killing_form(x, y)
36
```

simple_root(*i*, *h*)

Return the action of the simple root $\alpha_i : \mathfrak{h} \to R$, where R is the base ring of self, on the element j.

EXAMPLES:

```
sage: g = lie_algebras.sp(QQ, 8, representation='matrix')
sage: matrix([[g.simple_root(i, g.h(j)) for i in g.index_set()] for j in g.
↪index_set()])
[ 2 -1  0  0]
[-1  2 -1  0]
[ 0 -1  2 -2]
[ 0  0 -1  2]
```

9.2.5 Examples of Lie Algebras

There are the following examples of Lie algebras:

- A rather comprehensive family of 3-dimensional Lie algebras
- The Lie algebra of affine transformations of the line
- All abelian Lie algebras on free modules
- The Lie algebra of upper triangular matrices
- The Lie algebra of strictly upper triangular matrices
- The symplectic derivation Lie algebra
- The rank two Heisenberg Virasoro algebra

See also *sage.algebras.lie_algebras.virasoro.LieAlgebraRegularVectorFields* and *sage.algebras.lie_algebras.virasoro.VirasoroAlgebra* for other examples.

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

`sage.algebras.lie_algebras.examples.`**`Heisenberg`**(*R*, *n*, *representation='structure'*)

Return the rank `n` Heisenberg algebra in the given representation.

INPUT:

- `R` – the base ring
- `n` – the rank (a nonnegative integer or infinity)
- `representation` – (default: "structure") can be one of the following:
  - `"structure"` – using structure coefficients
  - `"matrix"` – using matrices

EXAMPLES:

```
sage: lie_algebras.Heisenberg(QQ, 3)
Heisenberg algebra of rank 3 over Rational Field
```

`sage.algebras.lie_algebras.examples.`**`abelian`**(*R*, *names=None*, *index_set=None*)

Return the abelian Lie algebra generated by `names`.

EXAMPLES:

```
sage: lie_algebras.abelian(QQ, 'x, y, z')
Abelian Lie algebra on 3 generators (x, y, z) over Rational Field
```

sage.algebras.lie_algebras.examples.**affine_transformations_line**(*R*, *names=['X', 'Y']*, *representation='bracket'*)

The Lie algebra of affine transformations of the line.

EXAMPLES:

```
sage: L = lie_algebras.affine_transformations_line(QQ)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): Y}
sage: X, Y = L.lie_algebra_generators()
sage: L[X, Y] == Y
True
sage: TestSuite(L).run()
sage: L = lie_algebras.affine_transformations_line(QQ, representation="matrix")
sage: X, Y = L.lie_algebra_generators()
sage: L[X, Y] == Y
True
sage: TestSuite(L).run()
```

sage.algebras.lie_algebras.examples.**cross_product**(*R*, *names=['X', 'Y', 'Z']*)

The Lie algebra of $\mathbf{R}^3$ defined by the usual cross product $\times$.

EXAMPLES:

```
sage: L = lie_algebras.cross_product(QQ)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): Z, ('X', 'Z'): -Y, ('Y', 'Z'): X}
sage: TestSuite(L).run()
```

sage.algebras.lie_algebras.examples.**pwitt**(*R*, *p*)

Return the p-Witt Lie algebra over R.

INPUT:

- R – the base ring
- p – a positive integer that is 0 in R

EXAMPLES:

```
sage: lie_algebras.pwitt(GF(5), 5)
The 5-Witt Lie algebra over Finite Field of size 5
```

sage.algebras.lie_algebras.examples.**regular_vector_fields**(*R*)

Return the Lie algebra of regular vector fields on $\mathbf{C}^{\times}$.

This is also known as the Witt (Lie) algebra.

See also:

LieAlgebraRegularVectorFields

EXAMPLES:

```
sage: lie_algebras.regular_vector_fields(QQ)
The Lie algebra of regular vector fields over Rational Field
```

`sage.algebras.lie_algebras.examples.sl(R, n, representation='bracket')`

The Lie algebra $\mathfrak{sl}_n$.

The Lie algebra $\mathfrak{sl}_n$ is the type A_{n-1} Lie algebra and is finite dimensional. As a matrix Lie algebra, it is given by the set of all $n \times n$ matrices with trace 0.

INPUT:

- `R` – the base ring
- `n` – the size of the matrix
- `representation` – (default: `'bracket'`) can be one of the following:
  - `'bracket'` - use brackets and the Chevalley basis
  - `'matrix'` - use matrices

EXAMPLES:

We first construct $\mathfrak{sl}_2$ using the Chevalley basis:

```
sage: sl2 = lie_algebras.sl(QQ, 2); sl2
Lie algebra of ['A', 1] in the Chevalley basis
sage: E,F,H = sl2.gens()
sage: E.bracket(F) == H
True
sage: H.bracket(E) == 2*E
True
sage: H.bracket(F) == -2*F
True
```

We now construct $\mathfrak{sl}_2$ as a matrix Lie algebra:

```
sage: sl2 = lie_algebras.sl(QQ, 2, representation='matrix')
sage: E,F,H = sl2.gens()
sage: E.bracket(F) == H
True
sage: H.bracket(E) == 2*E
True
sage: H.bracket(F) == -2*F
True
```

`sage.algebras.lie_algebras.examples.so(R, n, representation='bracket')`

The Lie algebra $\mathfrak{so}_n$.

The Lie algebra $\mathfrak{so}_n$ is the type B_k Lie algebra if $n = 2k - 1$ or the type D_k Lie algebra if $n = 2k$, and in either case is finite dimensional.

A classical description of this as a matrix Lie algebra is the set of all anti-symmetric $n \times n$ matrices. However, the implementation here uses a different bilinear form for the Lie group and follows the description in Chapter 8 of [HK2002]. See `sage.algebras.lie_algebras.classical_lie_algebra.so` for a precise description.

INPUT:

- `R` – the base ring
- `n` – the size of the matrix
- `representation` – (default: `'bracket'`) can be one of the following:
  - `'bracket'` - use brackets and the Chevalley basis

– 'matrix' - use matrices

EXAMPLES:

We first construct $\mathfrak{so}_5$ using the Chevalley basis:

```
sage: so5 = lie_algebras.so(QQ, 5); so5
Lie algebra of ['B', 2] in the Chevalley basis
sage: E1,E2, F1,F2, H1,H2 = so5.gens()
sage: so5([E1, [E1, E2]])
0
sage: X = so5([E2, [E2, E1]]); X
-2*E[alpha[1] + 2*alpha[2]]
sage: H1.bracket(X)
0
sage: H2.bracket(X)
-4*E[alpha[1] + 2*alpha[2]]
sage: so5([H1, [E1, E2]])
-E[alpha[1] + alpha[2]]
sage: so5([H2, [E1, E2]])
0
```

We do the same construction of $\mathfrak{so}_4$ using the Chevalley basis:

```
sage: so4 = lie_algebras.so(QQ, 4); so4
Lie algebra of ['D', 2] in the Chevalley basis
sage: E1,E2, F1,F2, H1,H2 = so4.gens()
sage: H1.bracket(E1)
2*E[alpha[1]]
sage: H2.bracket(E1) == so4.zero()
True
sage: E1.bracket(E2) == so4.zero()
True
```

We now construct $\mathfrak{so}_4$ as a matrix Lie algebra:

```
sage: sl2 = lie_algebras.sl(QQ, 2, representation='matrix')
sage: E1,E2, F1,F2, H1,H2 = so4.gens()
sage: H2.bracket(E1) == so4.zero()
True
sage: E1.bracket(E2) == so4.zero()
True
```

`sage.algebras.lie_algebras.examples.sp`(*R*, *n*, *representation='bracket'*)

The Lie algebra $\mathfrak{sp}_n$.

The Lie algebra $\mathfrak{sp}_n$ where $n = 2k$ is the type C_k Lie algebra and is finite dimensional. As a matrix Lie algebra, it is given by the set of all matrices X that satisfy the equation:

$$X^T M - MX = 0$$

where

$$M = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

This is the Lie algebra of type C_k.

INPUT:

- R – the base ring
- n – the size of the matrix
- representation – (default: 'bracket') can be one of the following:
  - 'bracket' - use brackets and the Chevalley basis
  - 'matrix' - use matrices

EXAMPLES:

We first construct $\mathfrak{sp}_4$ using the Chevalley basis:

```
sage: sp4 = lie_algebras.sp(QQ, 4); sp4
Lie algebra of ['C', 2] in the Chevalley basis
sage: E1,E2, F1,F2, H1,H2 = sp4.gens()
sage: sp4([E2, [E2, E1]])
0
sage: X = sp4([E1, [E1, E2]]); X
2*E[2*alpha[1] + alpha[2]]
sage: H1.bracket(X)
4*E[2*alpha[1] + alpha[2]]
sage: H2.bracket(X)
0
sage: sp4([H1, [E1, E2]])
0
sage: sp4([H2, [E1, E2]])
-E[alpha[1] + alpha[2]]
```

We now construct $\mathfrak{sp}_4$ as a matrix Lie algebra:

```
sage: sp4 = lie_algebras.sp(QQ, 4, representation='matrix'); sp4
Symplectic Lie algebra of rank 4 over Rational Field
sage: E1,E2, F1,F2, H1,H2 = sp4.gens()
sage: H1.bracket(E1)
[ 0  2  0  0]
[ 0  0  0  0]
[ 0  0  0  0]
[ 0  0 -2  0]
sage: sp4([E1, [E1, E2]])
[0 0 2 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

sage.algebras.lie_algebras.examples.**strictly_upper_triangular_matrices**(R, n)

Return the Lie algebra $\mathfrak{n}_k$ of strictly $k \times k$ upper triangular matrices.

Todo: This implementation does not know it is finite-dimensional and does not know its basis.

EXAMPLES:

```
sage: L = lie_algebras.strictly_upper_triangular_matrices(QQ, 4); L
Lie algebra of 4-dimensional strictly upper triangular matrices over Rational Field
sage: TestSuite(L).run()
sage: n0, n1, n2 = L.lie_algebra_generators()
sage: L[n2, n1]
[ 0  0  0  0]
[ 0  0  0 -1]
[ 0  0  0  0]
[ 0  0  0  0]
```

sage.algebras.lie_algebras.examples.**su**(*R*, *n*, *representation='matrix'*)

The Lie algebra $\mathfrak{su}_n$.

The Lie algebra $\mathfrak{su}_n$ is the compact real form of the type A_{n-1} Lie algebra and is finite-dimensional. As a matrix Lie algebra, it is given by the set of all $n \times n$ skew-Hermitian matrices with trace 0.

INPUT:

- `R` – the base ring
- `n` – the size of the matrix
- `representation` – (default: `'matrix'`) can be one of the following:
  - `'bracket'` - use brackets and the Chevalley basis
  - `'matrix'` - use matrices

EXAMPLES:

We construct $\mathfrak{su}_2$, where the default is as a matrix Lie algebra:

```
sage: su2 = lie_algebras.su(QQ, 2)
sage: E,H,F = su2.basis()
sage: E.bracket(F) == 2*H
True
sage: H.bracket(E) == 2*F
True
sage: H.bracket(F) == -2*E
True
```

Since $\mathfrak{su}_n$ is the same as the type A_{n-1} Lie algebra, the bracket is the same as *sl()*:

```
sage: su2 = lie_algebras.su(QQ, 2, representation='bracket')
sage: su2 is lie_algebras.sl(QQ, 2, representation='bracket')
True
```

sage.algebras.lie_algebras.examples.**three_dimensional**(*R*, *a*, *b*, *c*, *d*, *names=['X', 'Y', 'Z']*)

The 3-dimensional Lie algebra over a given commutative ring R with basis $\{X, Y, Z\}$ subject to the relations:

$$[X, Y] = aZ + dY, \quad [Y, Z] = bX, \quad [Z, X] = cY + dZ$$

where $a, b, c, d \in R$.

This is always a well-defined 3-dimensional Lie algebra, as can be easily proven by computation.

EXAMPLES:

```
sage: L = lie_algebras.three_dimensional(QQ, 4, 1, -1, 2)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): 2*Y + 4*Z, ('X', 'Z'): Y - 2*Z, ('Y', 'Z'): X}
sage: TestSuite(L).run()
sage: L = lie_algebras.three_dimensional(QQ, 1, 0, 0, 0)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): Z}
sage: L = lie_algebras.three_dimensional(QQ, 0, 0, -1, -1)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): -Y, ('X', 'Z'): Y + Z}
sage: L = lie_algebras.three_dimensional(QQ, 0, 1, 0, 0)
sage: L.structure_coefficients()
Finite family {('Y', 'Z'): X}
sage: lie_algebras.three_dimensional(QQ, 0, 0, 0, 0)
Abelian Lie algebra on 3 generators (X, Y, Z) over Rational Field
sage: Q.<a,b,c,d> = PolynomialRing(QQ)
sage: L = lie_algebras.three_dimensional(Q, a, b, c, d)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): d*Y + a*Z, ('X', 'Z'): (-c)*Y + (-d)*Z, ('Y', 'Z'): b*X}
sage: TestSuite(L).run()
```

sage.algebras.lie_algebras.examples.**three_dimensional_by_rank**(*R*, *n*, *a=None*, *names=['X', 'Y', 'Z']*)

Return a 3-dimensional Lie algebra of rank n, where $0 \leq n \leq 3$.

Here, the *rank* of a Lie algebra L is defined as the dimension of its derived subalgebra $[L, L]$. (We are assuming that R is a field of characteristic 0; otherwise the Lie algebras constructed by this function are still well-defined but no longer might have the correct ranks.) This is not to be confused with the other standard definition of a rank (namely, as the dimension of a Cartan subalgebra, when L is semisimple).

INPUT:

- R – the base ring
- n – the rank
- a – the deformation parameter (used for $n = 2$); this should be a nonzero element of R in order for the resulting Lie algebra to actually have the right rank(?)
- names – (optional) the generator names

EXAMPLES:

```
sage: lie_algebras.three_dimensional_by_rank(QQ, 0)
Abelian Lie algebra on 3 generators (X, Y, Z) over Rational Field
sage: L = lie_algebras.three_dimensional_by_rank(QQ, 1)
sage: L.structure_coefficients()
Finite family {('Y', 'Z'): X}
sage: L = lie_algebras.three_dimensional_by_rank(QQ, 2, 4)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): Y, ('X', 'Z'): Y + Z}
sage: L = lie_algebras.three_dimensional_by_rank(QQ, 2, 0)
sage: L.structure_coefficients()
Finite family {('X', 'Y'): Y}
sage: lie_algebras.three_dimensional_by_rank(QQ, 3)
sl2 over Rational Field
```

sage.algebras.lie_algebras.examples.**upper_triangular_matrices**(*R*, *n*)

Return the Lie algebra $\mathfrak{b}_k$ of $k \times k$ upper triangular matrices.

Todo: This implementation does not know it is finite-dimensional and does not know its basis.

EXAMPLES:

```
sage: L = lie_algebras.upper_triangular_matrices(QQ, 4); L
Lie algebra of 4-dimensional upper triangular matrices over Rational Field
sage: TestSuite(L).run()
sage: n0, n1, n2, t0, t1, t2, t3 = L.lie_algebra_generators()
sage: L[n2, t2] == -n2
True
```

sage.algebras.lie_algebras.examples.**witt**(*R*)

Return the Lie algebra of regular vector fields on $\mathbf{C}^\times$.

This is also known as the Witt (Lie) algebra.

See also:

LieAlgebraRegularVectorFields

EXAMPLES:

```
sage: lie_algebras.regular_vector_fields(QQ)
The Lie algebra of regular vector fields over Rational Field
```

9.2.6 Free Lie Algebras

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version

REFERENCES:

- [Bou1989]
- [Reu2003]

class sage.algebras.lie_algebras.free_lie_algebra.**FreeLieAlgebra**(*R*, *names*, *index_set*)

Bases: Parent, UniqueRepresentation

The free Lie algebra of a set X.

The free Lie algebra $\mathfrak{g}_X$ of a set X is the Lie algebra with generators $\{g_x\}_{x \in X}$ where there are no other relations beyond the defining relations. This can be constructed as the free magmatic algebra M_X quotiented by the ideal generated by $(xx, xy + yx, x(yz) + y(zx) + z(xy))$.

EXAMPLES:

We first construct the free Lie algebra in the Hall basis:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: H = L.Hall()
sage: x,y,z = H.gens()
sage: h_elt = H([x, [y, z]]) + H([x - H([y, x]), H([x, z])]); h_elt
[x, [x, z]] + [y, [x, z]] - [z, [x, y]] + [[x, y], [x, z]]
```

We can also use the Lyndon basis and go between the two:

```
sage: Lyn = L.Lyndon()
sage: l_elt = Lyn([x, [y, z]]) + Lyn([x - Lyn([y, x]), Lyn([x, z])]); l_elt
[x, [x, z]] + [[x, y], [x, z]] + [x, [y, z]]
sage: Lyn(h_elt) == l_elt
True
sage: H(l_elt) == h_elt
True
```

class Hall(*lie*)

Bases: *FreeLieBasis_abstract*

The free Lie algebra in the Hall basis.

The basis keys are objects of class *LieObject*, each of which is either a *LieGenerator* (in degree 1) or a *GradedLieBracket* (in degree > 1).

graded_basis(k)

Return the basis for the k-th graded piece of `self`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: H = L.Hall()
sage: H.graded_basis(2)
([x, y], [x, z], [y, z])
sage: H.graded_basis(4)
([x, [x, [x, y]]], [x, [x, [x, z]]],
 [y, [x, [x, y]]], [y, [x, [x, z]]],
 [y, [y, [x, y]]], [y, [y, [x, z]]],
 [y, [y, [y, z]]], [z, [x, [x, y]]],
 [z, [x, [x, z]]], [z, [y, [x, y]]],
 [z, [y, [x, z]]], [z, [y, [y, z]]],
 [z, [z, [x, y]]], [z, [z, [x, z]]],
 [z, [z, [y, z]]], [[x, y], [x, z]],
 [[x, y], [y, z]], [[x, z], [y, z]])
```

class Lyndon(*lie*)

Bases: *FreeLieBasis_abstract*

The free Lie algebra in the Lyndon basis.

The basis keys are objects of class *LieObject*, each of which is either a *LieGenerator* (in degree 1) or a *LyndonBracket* (in degree > 1).

graded_basis(k)

Return the basis for the k-th graded piece of `self`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x', 3)
sage: Lyn = L.Lyndon()
sage: Lyn.graded_basis(1)
(x0, x1, x2)
sage: Lyn.graded_basis(2)
([x0, x1], [x0, x2], [x1, x2])
```

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```
sage: Lyn.graded_basis(4)
([x0, [x0, [x0, x1]]],
 [x0, [x0, [x0, x2]]],
 [x0, [[x0, x1], x1]],
 [x0, [x0, [x1, x2]]],
 [x0, [[x0, x2], x1]],
 [x0, [[x0, x2], x2]],
 [[x0, x1], [x0, x2]],
 [[[x0, x1], x1], x1],
 [x0, [x1, [x1, x2]]],
 [[x0, [x1, x2]], x1],
 [x0, [[x1, x2], x2]],
 [[[x0, x2], x1], x1],
 [[x0, x2], [x1, x2]],
 [[[x0, x2], x2], x1],
 [[[x0, x2], x2], x2],
 [x1, [x1, [x1, x2]]],
 [x1, [[x1, x2], x2]],
 [[[x1, x2], x2], x2])
```

pbw_basis(***kwds*)

Return the Poincare-Birkhoff-Witt basis corresponding to `self`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z', 3)
sage: Lyn = L.Lyndon()
sage: Lyn.pbw_basis()
The Poincare-Birkhoff-Witt basis of Free Algebra on 3 generators (x, y, z)␣
↪over Rational Field
```

poincare_birkhoff_witt_basis(***kwds*)

Return the Poincare-Birkhoff-Witt basis corresponding to `self`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z', 3)
sage: Lyn = L.Lyndon()
sage: Lyn.pbw_basis()
The Poincare-Birkhoff-Witt basis of Free Algebra on 3 generators (x, y, z)␣
↪over Rational Field
```

a_realization()

Return a particular realization of `self` (the Lyndon basis).

EXAMPLES:

```
sage: L.<x, y> = LieAlgebra(QQ)
sage: L.a_realization()
Free Lie algebra generated by (x, y) over Rational Field in the Lyndon basis
```

gen(*i*)

Return the `i`-th generator of `self` in the Lyndon basis.

EXAMPLES:

```
sage: L.<x, y> = LieAlgebra(QQ)
sage: L.gen(0)
x
sage: L.gen(1)
y
sage: L.gen(0).parent()
Free Lie algebra generated by (x, y) over Rational Field in the Lyndon basis
```

gens()

Return the generators of `self` in the Lyndon basis.

EXAMPLES:

```
sage: L.<x, y> = LieAlgebra(QQ)
sage: L.gens()
(x, y)
sage: L.gens()[0].parent()
Free Lie algebra generated by (x, y) over Rational Field in the Lyndon basis
```

lie_algebra_generators()

Return the Lie algebra generators of `self` in the Lyndon basis.

EXAMPLES:

```
sage: L.<x, y> = LieAlgebra(QQ)
sage: L.lie_algebra_generators()
Finite family {'x': x, 'y': y}
sage: L.lie_algebra_generators()['x'].parent()
Free Lie algebra generated by (x, y) over Rational Field in the Lyndon basis
```

class sage.algebras.lie_algebras.free_lie_algebra.**FreeLieAlgebraBases**(*base*)

Bases: `Category_realization_of_parent`

The category of bases of a free Lie algebra.

super_categories()

The super categories of `self`.

EXAMPLES:

```
sage: from sage.algebras.lie_algebras.free_lie_algebra import FreeLieAlgebraBases
sage: L.<x, y> = LieAlgebra(QQ)
sage: bases = FreeLieAlgebraBases(L)
sage: bases.super_categories()
[Category of lie algebras with basis over Rational Field,
 Category of realizations of Free Lie algebra generated by (x, y) over Rational Field]
```

class sage.algebras.lie_algebras.free_lie_algebra.**FreeLieBasis_abstract**(*lie*, *basis_name*)

Bases: *FinitelyGeneratedLieAlgebra*, `IndexedGenerators`, `BindableClass`

Abstract base class for all bases of a free Lie algebra.

Element

alias of *FreeLieAlgebraElement*

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 3, 'x')
sage: L.Hall().basis()
Disjoint union of Lazy family (graded basis(i))_{i in Positive integers}
```

graded_basis(k)

Return the basis for the `k`-th graded piece of `self`.

EXAMPLES:

```
sage: H = LieAlgebra(QQ, 3, 'x').Hall()
sage: H.graded_basis(2)
([x0, x1], [x0, x2], [x1, x2])
```

graded_dimension(k)

Return the dimension of the `k`-th graded piece of `self`.

The k-th graded part of a free Lie algebra on n generators has dimension

$$\frac{1}{k}\sum_{d|k}\mu(d)n^{k/d},$$

where μ is the Mobius function.

REFERENCES:

[MKO1998]

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x', 3)
sage: H = L.Hall()
sage: [H.graded_dimension(i) for i in range(1, 11)]
[3, 3, 8, 18, 48, 116, 312, 810, 2184, 5880]
sage: H.graded_dimension(0)
0
```

is_abelian()

Return `True` if this is an abelian Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 3, 'x')
sage: L.is_abelian()
False
sage: L = LieAlgebra(QQ, 1, 'x')
sage: L.is_abelian()
True
```

monomial(x)

Return the monomial indexed by `x`.

EXAMPLES:

```
sage: Lyn = LieAlgebra(QQ, 'x,y').Lyndon()
sage: x = Lyn.monomial('x'); x
x
sage: x.parent() is Lyn
True
```

sage.algebras.lie_algebras.free_lie_algebra.**is_lyndon**(*w*)

Modified form of Word(w).is_lyndon() which uses the default order (this will either be the natural integer order or lex order) and assumes the input w behaves like a nonempty list. This function here is designed for speed.

EXAMPLES:

```
sage: from sage.algebras.lie_algebras.free_lie_algebra import is_lyndon
sage: is_lyndon([1])
True
sage: is_lyndon([1,3,1])
False
sage: is_lyndon((2,2,3))
True
sage: all(is_lyndon(x) for x in LyndonWords(3, 5))
True
sage: all(is_lyndon(x) for x in LyndonWords(6, 4))
True
```

9.2.7 Heisenberg Algebras

AUTHORS:

- Travis Scrimshaw (2013-08-13): Initial version

class sage.algebras.lie_algebras.heisenberg.**HeisenbergAlgebra**(*R*, *n*)

Bases: *HeisenbergAlgebra_fd*, *HeisenbergAlgebra_abstract*, *LieAlgebraWithGenerators*

A Heisenberg algebra defined using structure coefficients.

The n-th Heisenberg algebra (where n is a nonnegative integer or infinity) is the Lie algebra with basis $\{p_i\}_{1\le i\le n} \cup \{q_i\}_{1\le i\le n} \cup \{z\}$ with the following relations:

$$[p_i, q_j] = \delta_{ij} z, \quad [p_i, z] = [q_i, z] = [p_i, p_j] = [q_i, q_j] = 0.$$

This Lie algebra is also known as the Heisenberg algebra of rank n.

Note: The relations $[p_i, q_j] = \delta_{ij} z$, $[p_i, z] = 0$, and $[q_i, z] = 0$ are known as canonical commutation relations. See Wikipedia article Canonical_commutation_relations.

Warning: The n in the above definition is called the "rank" of the Heisenberg algebra; it is not, however, a rank in any of the usual meanings that this word has in the theory of Lie algebras.

INPUT:

- R – the base ring
- n – the rank of the Heisenberg algebra

REFERENCES:

- Wikipedia article Heisenberg_algebra

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 2)
```

class sage.algebras.lie_algebras.heisenberg.**HeisenbergAlgebra_abstract**(*I*)

Bases: IndexedGenerators

The common methods for the (non-matrix) Heisenberg algebras.

class Element

Bases: *LieAlgebraElement*

bracket_on_basis(*x*, *y*)

Return the bracket of basis elements indexed by x and y where x < y.

The basis of a Heisenberg algebra is ordered in such a way that the p_i come first, the q_i come next, and the z comes last.

EXAMPLES:

```
sage: H = lie_algebras.Heisenberg(QQ, 3)
sage: p1 = ('p', 1)
sage: q1 = ('q', 1)
sage: H.bracket_on_basis(p1, q1)
z
```

p(*i*)

The generator p_i of the Heisenberg algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.p(2)
p2
```

q(*i*)

The generator q_i of the Heisenberg algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.q(2)
q2
```

step()

Return the nilpotency step of self.

EXAMPLES:

```
sage: h = lie_algebras.Heisenberg(ZZ, 10)
sage: h.step()
2

sage: h = lie_algebras.Heisenberg(ZZ, oo)
```

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```
sage: h.step()
2
```

z()

Return the basis element z of the Heisenberg algebra.

The element z spans the center of the Heisenberg algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.z()
z
```

class sage.algebras.lie_algebras.heisenberg.**HeisenbergAlgebra_fd**(*n*)

Bases: object

Common methods for finite-dimensional Heisenberg algebras.

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: H = lie_algebras.Heisenberg(QQ, 1)
sage: H.basis()
Finite family {'p1': p1, 'q1': q1, 'z': z}
```

gen(*i*)

Return the `i`-th generator of `self`.

EXAMPLES:

```
sage: H = lie_algebras.Heisenberg(QQ, 2)
sage: H.gen(0)
p1
sage: H.gen(3)
q2
```

gens()

Return the Lie algebra generators of `self`.

EXAMPLES:

```
sage: H = lie_algebras.Heisenberg(QQ, 2)
sage: H.gens()
(p1, p2, q1, q2)
sage: H = lie_algebras.Heisenberg(QQ, 0)
sage: H.gens()
(z,)
```

lie_algebra_generators()

Return the Lie algebra generators of `self`.

EXAMPLES:

```
sage: H = lie_algebras.Heisenberg(QQ, 1)
sage: H.lie_algebra_generators()
Finite family {'p1': p1, 'q1': q1}
sage: H = lie_algebras.Heisenberg(QQ, 0)
sage: H.lie_algebra_generators()
Finite family {'z': z}
```

n()

Return the rank of the Heisenberg algebra `self`.

This is the `n` such that `self` is the n-th Heisenberg algebra. The dimension of this Heisenberg algebra is then $2n + 1$.

EXAMPLES:

```
sage: H = lie_algebras.Heisenberg(QQ, 3)
sage: H.n()
3
sage: H = lie_algebras.Heisenberg(QQ, 3, representation="matrix")
sage: H.n()
3
```

class sage.algebras.lie_algebras.heisenberg.**HeisenbergAlgebra_matrix**(*R*, *n*)

Bases: *HeisenbergAlgebra_fd*, *LieAlgebraFromAssociative*

A Heisenberg algebra represented using matrices.

The n-th Heisenberg algebra over R is a Lie algebra which is defined as the Lie algebra of the $(n+2) \times (n+2)$-matrices:

$$\begin{bmatrix} 0 & p^T & k \\ 0 & 0_n & q \\ 0 & 0 & 0 \end{bmatrix}$$

where $p, q \in R^n$ and 0_n in the $n \times n$ zero matrix. It has a basis consisting of

$$p_i = \begin{bmatrix} 0 & e_i^T & 0 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for } 1 \le i \le n,$$
$$q_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0_n & e_i \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for } 1 \le i \le n,$$
$$z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\{e_i\}$ is the standard basis of R^n. In other words, it has the basis $(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, z)$, where $p_i = E_{1,i+1}$, $q_i = E_{i+1,n+2}$ and $z = E_{1,n+2}$ are elementary matrices.

This Lie algebra is isomorphic to the n-th Heisenberg algebra constructed in *HeisenbergAlgebra*; the bases correspond to each other.

INPUT:

- `R` – the base ring
- `n` – the nonnegative integer n

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 1, representation="matrix")
sage: p = L.p(1)
sage: q = L.q(1)
sage: z = L.bracket(p, q); z
[0 0 1]
[0 0 0]
[0 0 0]
sage: z == L.z()
True
sage: L.dimension()
3

sage: L = lie_algebras.Heisenberg(QQ, 2, representation="matrix")
sage: sorted(dict(L.basis()).items())
[(
      [0 1 0 0]
      [0 0 0 0]
      [0 0 0 0]
'p1', [0 0 0 0]
),
 (
      [0 0 1 0]
      [0 0 0 0]
      [0 0 0 0]
'p2', [0 0 0 0]
),
 (
      [0 0 0 0]
      [0 0 0 1]
      [0 0 0 0]
'q1', [0 0 0 0]
),
 (
      [0 0 0 0]
      [0 0 0 0]
      [0 0 0 1]
'q2', [0 0 0 0]
),
 (
     [0 0 0 1]
     [0 0 0 0]
     [0 0 0 0]
'z', [0 0 0 0]
)]

sage: L = lie_algebras.Heisenberg(QQ, 0, representation="matrix")
sage: sorted(dict(L.basis()).items())
[(
     [0 1]
'z', [0 0]
)]
```

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```
sage: L.gens()
(
[0 1]
[0 0]
)
sage: L.lie_algebra_generators()
Finite family {'z': [0 1]
[0 0]}
```

class Element

Bases: *LieAlgebraMatrixWrapper*, *Element*

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- copy – ignored

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 3, representation="matrix")
sage: elt = L(Matrix(QQ, [[0, 1, 3, 0, 3], [0, 0, 0, 0, 0], [0, 0, 0, 0, -
↪3],
....:                     [0, 0, 0, 0, 7], [0, 0, 0, 0, 0]]))
sage: elt
[ 0  1  3  0  3]
[ 0  0  0  0  0]
[ 0  0  0  0 -3]
[ 0  0  0  0  7]
[ 0  0  0  0  0]
sage: sorted(elt.monomial_coefficients().items())
[('p1', 1), ('p2', 3), ('q2', -3), ('q3', 7), ('z', 3)]
```

p(*i*)

Return the generator p_i of the Heisenberg algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 1, representation="matrix")
sage: L.p(1)
[0 1 0]
[0 0 0]
[0 0 0]
```

q(*i*)

Return the generator q_i of the Heisenberg algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 1, representation="matrix")
sage: L.q(1)
[0 0 0]
[0 0 1]
[0 0 0]
```

step()

Return the nilpotency step of `self`.

EXAMPLES:

```
sage: h = lie_algebras.Heisenberg(ZZ, 2, representation="matrix")
sage: h.step()
2
```

z()

Return the basis element z of the Heisenberg algebra.

The element z spans the center of the Heisenberg algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 1, representation="matrix")
sage: L.z()
[0 0 1]
[0 0 0]
[0 0 0]
```

class sage.algebras.lie_algebras.heisenberg.**InfiniteHeisenbergAlgebra**(*R*)

Bases: *HeisenbergAlgebra_abstract*, *LieAlgebraWithGenerators*

The infinite Heisenberg algebra.

This is the Heisenberg algebra on an infinite number of generators. In other words, this is the Heisenberg algebra of rank ∞. See *HeisenbergAlgebra* for more information.

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.basis()
Lazy family (basis map(i))_{i in Disjoint union of Family ({'z'},
 The Cartesian product of (Positive integers, {'p', 'q'}))}
sage: L.basis()['z']
z
sage: L.basis()[(12, 'p')]
p12
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.lie_algebra_generators()
Lazy family (generator map(i))_{i in The Cartesian product of
                                (Positive integers, {'p', 'q'})}
```

9.2.8 Lie Algebras

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version

class sage.algebras.lie_algebras.lie_algebra.**FinitelyGeneratedLieAlgebra**(*R*, *names=None*, *index_set=None*, *category=None*)

Bases: `LieAlgebraWithGenerators`

A finitely generated Lie algebra.

class sage.algebras.lie_algebras.lie_algebra.**InfinitelyGeneratedLieAlgebra**(*R*, *names=None*, *index_set=None*, *category=None*, *prefix='L'*, ***kwds*)

Bases: `LieAlgebraWithGenerators`

An infinitely generated Lie algebra.

class sage.algebras.lie_algebras.lie_algebra.**LieAlgebra**(*R*, *names=None*, *category=None*)

Bases: `Parent`, `UniqueRepresentation`

A Lie algebra L over a base ring R.

A Lie algebra is an R-module L with a bilinear operation called Lie bracket $[\cdot, \cdot] : L \times L \to L$ such that $[x, x] = 0$ and the following relation holds:

$$\bigl[x, [y, z]\bigr] + \bigl[y, [z, x]\bigr] + \bigl[z, [x, y]\bigr] = 0.$$

This relation is known as the *Jacobi identity* (or sometimes the Jacobi relation). We note that from $[x, x] = 0$, we have $[x + y, x + y] = 0$. Next from bilinearity, we see that

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x],$$

thus $[x, y] = -[y, x]$ and the Lie bracket is antisymmetric.

Lie algebras are closely related to Lie groups. Let G be a Lie group and fix some $g \in G$. We can construct the Lie algebra L of G by considering the tangent space at g. We can also (partially) recover G from L by using what is known as the exponential map.

Given any associative algebra A, we can construct a Lie algebra L on the R-module A by defining the Lie bracket to be the commutator $[a, b] = ab - ba$. We call an associative algebra A which contains L in this fashion an *enveloping algebra* of L. The embedding $L \to A$ which sends the Lie bracket to the commutator will be called a Lie embedding. Now if we are given a Lie algebra L, we can construct an enveloping algebra U_L with Lie embedding $h : L \to U_L$ which has the following universal property: for any enveloping algebra A with Lie embedding $f : L \to A$, there exists a unique unital algebra homomorphism $g : U_L \to A$ such that $f = g \circ h$. The algebra U_L is known as the *universal enveloping algebra* of L.

INPUT:

See examples below for various input options.

EXAMPLES:

1. The simplest examples of Lie algebras are *abelian Lie algebras*. These are Lie algebras whose Lie bracket is (identically) zero. We can create them using the `abelian` keyword:

```
sage: L.<x,y,z> = LieAlgebra(QQ, abelian=True); L
Abelian Lie algebra on 3 generators (x, y, z) over Rational Field
```

2. A Lie algebra can be built from any associative algebra by defining the Lie bracket to be the commutator. For example, we can start with the descent algebra:

```
sage: D = DescentAlgebra(QQ, 4).D()
sage: L = LieAlgebra(associative=D); L
Lie algebra of Descent algebra of 4 over Rational Field
 in the standard basis
sage: L(D[2]).bracket(L(D[3]))
D{1, 2} - D{1, 3} + D{2} - D{3}
```

Next we use a free algebra and do some simple computations:

```
sage: R.<a,b,c> = FreeAlgebra(QQ, 3)
sage: L.<x,y,z> = LieAlgebra(associative=R.gens())
sage: x-y+z
a - b + c
sage: L.bracket(x-y, x-z)
a*b - a*c - b*a + b*c + c*a - c*b
sage: L.bracket(x-y, L.bracket(x,y))
a^2*b - 2*a*b*a + a*b^2 + b*a^2 - 2*b*a*b + b^2*a
```

We can also use a subset of the elements as a generating set of the Lie algebra:

```
sage: R.<a,b,c> = FreeAlgebra(QQ, 3)
sage: L.<x,y> = LieAlgebra(associative=[a,b+c])
sage: L.bracket(x, y)
a*b + a*c - b*a - c*a
```

Now for a more complicated example using the group ring of S_3 as our base algebra:

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L.<x,y> = LieAlgebra(associative=S.gens())
sage: L.bracket(x, y)
(2,3) - (1,3)
sage: L.bracket(x, y-x)
(2,3) - (1,3)
sage: L.bracket(L.bracket(x, y), y)
2*(1,2,3) - 2*(1,3,2)
sage: L.bracket(x, L.bracket(x, y))
(2,3) - 2*(1,2) + (1,3)
sage: L.bracket(x, L.bracket(L.bracket(x, y), y))
0
```

Here is an example using matrices:

```
sage: MS = MatrixSpace(QQ,2)
sage: m1 = MS([[0, -1], [1, 0]])
sage: m2 = MS([[-1, 4], [3, 2]])
sage: L.<x,y> = LieAlgebra(associative=[m1, m2])
sage: x
```

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```
[ 0 -1]
[ 1  0]
sage: y
[-1  4]
[ 3  2]
sage: L.bracket(x,y)
[-7 -3]
[-3  7]
sage: L.bracket(y,y)
[0 0]
[0 0]
sage: L.bracket(y,x)
[ 7  3]
[ 3 -7]
sage: L.bracket(x, L.bracket(y,x))
[-6 14]
[14  6]
```

(See `LieAlgebraFromAssociative` for other examples.)

3. We can also creating a Lie algebra by inputting a set of structure coefficients. For example, we can create the Lie algebra of $\mathbf{Q}^3$ under the Lie bracket $\times$ (cross-product):

```
sage: d = {('x','y'): {'z':1}, ('y','z'): {'x':1}, ('z','x'): {'y':1}}
sage: L.<x,y,z> = LieAlgebra(QQ, d)
sage: L
Lie algebra on 3 generators (x, y, z) over Rational Field
```

To compute the Lie bracket of two elements, you cannot use the * operator. Indeed, this automatically lifts up to the universal enveloping algebra and takes the (associative) product there. To get elements in the Lie algebra, you must use `bracket()`:

```
sage: L = LieAlgebra(QQ, {('e','h'): {'e':-2}, ('f','h'): {'f':2},
....:                     ('e','f'): {'h':1}}, names='e,f,h')
sage: e,f,h = L.lie_algebra_generators()
sage: L.bracket(h, e)
2*e
sage: elt = h*e; elt
e*h + 2*e
sage: P = elt.parent(); P
Noncommutative Multivariate Polynomial Ring in e, f, h over Rational Field,
 nc-relations: {...}
sage: R = P.relations()
sage: for rhs in sorted(R, key=str): print("{} = {}".format(rhs, R[rhs]))
f*e = e*f - h
h*e = e*h + 2*e
h*f = f*h - 2*f
```

For convenience, there are two shorthand notations for computing Lie brackets:

```
sage: L([h,e])
2*e
sage: L([h,[e,f]])
```

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```
0
sage: L([[h,e],[e,f]])
-4*e
sage: L[h, e]
2*e
sage: L[h, L[e, f]]
0
```

Warning: Because this is a modified (abused) version of python syntax, it does **NOT** work with addition. For example `L([e + [h, f], h])` and `L[e + [h, f], h]` will both raise errors. Instead you must use `L[e + L[h, f], h]`.

4. We can construct a Lie algebra from a Cartan type by using the `cartan_type` option:

```
sage: L = LieAlgebra(ZZ, cartan_type=['C', 3])
sage: L.inject_variables()
Defining e1, e2, e3, f1, f2, f3, h1, h2, h3
sage: e1.bracket(e2)
-E[alpha[1] + alpha[2]]
sage: L([[e1, e2], e2])
0
sage: L([[e2, e3], e3])
0
sage: L([e2, [e2, e3]])
2*E[2*alpha[2] + alpha[3]]

sage: L = LieAlgebra(ZZ, cartan_type=['E', 6])
sage: L
Lie algebra of ['E', 6] in the Chevalley basis
```

When the Cartan type is finite type and simply-laced, we can also specify an asymmetry function from [Ka1990] using a Dynkin diagram orientation with the `epsilon` option:

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2], epsilon=[(1, 2)])
sage: e1, e2 = L.e()
sage: L[e1, e2]
-E[alpha[1] + alpha[2]]

sage: L = LieAlgebra(QQ, cartan_type=['A', 2], epsilon=[(2, 1)])
sage: e1, e2 = L.e()
sage: L[e1, e2]
E[alpha[1] + alpha[2]]
```

We also have matrix versions of the classical Lie algebras:

```
sage: L = LieAlgebra(ZZ, cartan_type=['A', 2], representation='matrix')
sage: L.gens()
(
[0 1 0]  [0 0 0]  [0 0 0]  [0 0 0]  [ 1  0  0]  [ 0  0  0]
[0 0 0]  [0 0 1]  [1 0 0]  [0 0 0]  [ 0 -1  0]  [ 0  1  0]
[0 0 0], [0 0 0], [0 0 0], [0 1 0], [ 0  0  0], [ 0  0 -1]
```

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```
)
```

There is also the compact real form of matrix Lie algebras implemented (the base ring must currently be a field):

```
sage: L = LieAlgebra(QQ, cartan_type=['A', 2], representation="compact real")
sage: list(L.basis())
[
[ 0  1  0]  [ 0  0  1]  [ 0  0  0]  [ i  0  0]  [0 i 0]  [0 0 i]
[-1  0  0]  [ 0  0  0]  [ 0  0  1]  [ 0  0  0]  [i 0 0]  [0 0 0]
[ 0  0  0], [-1  0  0], [ 0 -1  0], [ 0  0 -i], [0 0 0], [i 0 0],

[ 0  0  0]  [0 0 0]
[ 0  i  0]  [0 0 i]
[ 0  0 -i], [0 i 0]
]
```

5. We construct a free Lie algebra in a few different ways. There are two primary representations, as brackets and as polynomials:

```
sage: L = LieAlgebra(QQ, 'x,y,z'); L
Free Lie algebra generated by (x, y, z) over Rational Field
sage: P.<a,b,c> = LieAlgebra(QQ, representation="polynomial"); P
Lie algebra generated by (a, b, c) in
 Free Algebra on 3 generators (a, b, c) over Rational Field
```

This has the basis given by Hall and the one indexed by Lyndon words. We do some computations and convert between the bases:

```
sage: H = L.Hall()
doctest:warning...:
FutureWarning: The Hall basis has not been fully proven correct, but currently no␣
↪bugs are known
See https://github.com/sagemath/sage/issues/16823 for details.
sage: H
Free Lie algebra generated by (x, y, z) over Rational Field in the Hall basis
sage: Lyn = L.Lyndon()
sage: Lyn
Free Lie algebra generated by (x, y, z) over Rational Field in the Lyndon basis
sage: x,y,z = Lyn.lie_algebra_generators()
sage: a = Lyn([x, [[z, [x, y]], [y, x]]]); a
-[x, [[x, y], [x, [y, z]]]] - [x, [[x, y], [[x, z], y]]]
sage: H(a)
[[x, y], [z, [x, [x, y]]]] - [[x, y], [[x, y], [x, z]]]
 + [[x, [x, y]], [z, [x, y]]]
```

We also have the free Lie algebra given in the polynomial representation, which is the canonical embedding of the free Lie algebra into the free algebra (i.e., the ring of noncommutative polynomials). So the generators of the free Lie algebra are the generators of the free algebra and the Lie bracket is the commutator:

```
sage: P.<a,b,c> = LieAlgebra(QQ, representation="polynomial"); P
Lie algebra generated by (a, b, c) in
 Free Algebra on 3 generators (a, b, c) over Rational Field
```

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```
sage: P.bracket(a, b) + P.bracket(a - c, b + 3*c)
2*a*b + 3*a*c - 2*b*a + b*c - 3*c*a - c*b
```

6. Nilpotent Lie algebras are Lie algebras such that there exists an integer s such that all iterated brackets of length longer than s are zero. They can be constructed from structural coefficients using the `nilpotent` keyword:

```
sage: L.<X,Y,Z> = LieAlgebra(QQ, {('X','Y'): {'Z': 1}}, nilpotent=True)
sage: L
Nilpotent Lie algebra on 3 generators (X, Y, Z) over Rational Field
sage: L.category()
Category of finite dimensional nilpotent lie algebras with basis over Rational Field
```

A second example defining the Engel Lie algebra:

```
sage: sc = {('X','Y'): {'Z': 1}, ('X','Z'): {'W': 1}}
sage: E.<X,Y,Z,W> = LieAlgebra(QQ, sc, nilpotent=True); E
Nilpotent Lie algebra on 4 generators (X, Y, Z, W) over Rational Field
sage: E.step()
3
sage: E[X, Y + Z]
Z + W
sage: E[X, [X, Y + Z]]
W
sage: E[X, [X, [X, Y + Z]]]
0
```

A nilpotent Lie algebra will also be constructed if given a `category` of a nilpotent Lie algebra:

```
sage: C = LieAlgebras(QQ).Nilpotent().FiniteDimensional().WithBasis()
sage: L.<X,Y,Z> = LieAlgebra(QQ, {('X','Y'): {'Z': 1}}, category=C); L
Nilpotent Lie algebra on 3 generators (X, Y, Z) over Rational Field
```

7. Free nilpotent Lie algebras are the truncated versions of the free Lie algebras. That is, the only relations other than anticommutativity and the Jacobi identity among the Lie brackets are that brackets of length higher than the nilpotency step vanish. They can be created by using the `step` keyword:

```
sage: L = LieAlgebra(ZZ, 2, step=3); L
Free Nilpotent Lie algebra on 5 generators (X_1, X_2, X_12, X_112, X_122) over␣
↪Integer Ring
sage: L.step()
3
```

REFERENCES:

- [deG2000] Willem A. de Graaf. *Lie Algebras: Theory and Algorithms*.
- [Ka1990] Victor Kac, *Infinite dimensional Lie algebras*.
- Wikipedia article Lie_algebra

get_order()

Return an ordering of the basis indices.

Todo: Remove this method and in `CombinatorialFreeModule` in favor of a method in the category of (finite dimensional) modules with basis.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, {})
sage: L.get_order()
('x', 'y')
```

monomial(*i*)

Return the monomial indexed by `i`.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.monomial('p1')
p1
```

term(*i*, *c=None*)

Return the term indexed by `i` with coefficient `c`.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, oo)
sage: L.term('p1', 4)
4*p1
```

zero()

Return the element 0.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, representation="polynomial")
sage: L.zero()
0
```

class sage.algebras.lie_algebras.lie_algebra.**LieAlgebraFromAssociative**(*A*, *gens=None*, *names=None*, *index_set=None*, *category=None*)

Bases: *LieAlgebraWithGenerators*

A Lie algebra whose elements are from an associative algebra and whose bracket is the commutator.

Todo: Split this class into 2 classes, the base class for the Lie algebra corresponding to the full associative algebra and a subclass for the Lie subalgebra (of the full algebra) generated by a generating set?

Todo: Return the subalgebra generated by the basis elements of `self` for the universal enveloping algebra.

EXAMPLES:

For the first example, we start with a commutative algebra. Note that the bracket of everything will be 0:

```
sage: R = SymmetricGroupAlgebra(QQ, 2)
sage: L = LieAlgebra(associative=R)
sage: x, y = L.basis()
sage: L.bracket(x, y)
0
```

Next we use a free algebra and do some simple computations:

```
sage: R.<a,b> = FreeAlgebra(QQ, 2)
sage: L = LieAlgebra(associative=R)
sage: x,y = L(a), L(b)
sage: x-y
a - b
sage: L.bracket(x-y, x)
a*b - b*a
sage: L.bracket(x-y, L.bracket(x,y))
a^2*b - 2*a*b*a + a*b^2 + b*a^2 - 2*b*a*b + b^2*a
```

We can also use a subset of the generators as a generating set of the Lie algebra:

```
sage: R.<a,b,c> = FreeAlgebra(QQ, 3)
sage: L.<x,y> = LieAlgebra(associative=[a,b])
```

Now for a more complicated example using the group ring of S_3 as our base algebra:

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L.<x,y> = LieAlgebra(associative=S.gens())
sage: L.bracket(x, y)
(2,3) - (1,3)
sage: L.bracket(x, y-x)
(2,3) - (1,3)
sage: L.bracket(L.bracket(x, y), y)
2*(1,2,3) - 2*(1,3,2)
sage: L.bracket(x, L.bracket(x, y))
(2,3) - 2*(1,2) + (1,3)
sage: L.bracket(x, L.bracket(L.bracket(x, y), y))
0
```

Here is an example using matrices:

```
sage: MS = MatrixSpace(QQ,2)
sage: m1 = MS([[0, -1], [1, 0]])
sage: m2 = MS([[-1, 4], [3, 2]])
sage: L.<x,y> = LieAlgebra(associative=[m1, m2])
sage: x
[ 0 -1]
[ 1  0]
sage: y
[-1  4]
[ 3  2]
sage: L.bracket(x,y)
[-7 -3]
```

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```
[-3  7]
sage: L.bracket(y,y)
[0 0]
[0 0]
sage: L.bracket(y,x)
[ 7  3]
[ 3 -7]
sage: L.bracket(x, L.bracket(y,x))
[-6 14]
[14  6]
```

class Element

Bases: *LieAlgebraElementWrapper*

lift_associative()

Lift `self` to the ambient associative algebra (which might be smaller than the universal enveloping algebra).

EXAMPLES:

```
sage: R = FreeAlgebra(QQ, 3, 'x,y,z')
sage: L.<x,y,z> = LieAlgebra(associative=R.gens())
sage: x.lift_associative()
x
sage: x.lift_associative().parent()
Free Algebra on 3 generators (x, y, z) over Rational Field
```

monomial_coefficients(*copy=True*)

Return the monomial coefficients of `self` (if this notion makes sense for `self.parent()`).

EXAMPLES:

```
sage: R.<x,y,z> = FreeAlgebra(QQ)
sage: L = LieAlgebra(associative=R)
sage: elt = L(x) + 2*L(y) - L(z)
sage: sorted(elt.monomial_coefficients().items())
[(x, 1), (y, 2), (z, -1)]

sage: L = LieAlgebra(associative=[x,y])
sage: elt = L(x) + 2*L(y)
sage: elt.monomial_coefficients()
Traceback (most recent call last):
...
NotImplementedError: the basis is not defined
```

associative_algebra()

Return the associative algebra used to construct `self`.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L = LieAlgebra(associative=S)
```

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```
sage: L.associative_algebra() is S
True
```

is_abelian()

Return True if self is abelian.

EXAMPLES:

```
sage: R = FreeAlgebra(QQ, 2, 'x,y')
sage: L = LieAlgebra(associative=R.gens())
sage: L.is_abelian()
False

sage: R = PolynomialRing(QQ, 'x,y')
sage: L = LieAlgebra(associative=R.gens())
sage: L.is_abelian()
True
```

An example with a Lie algebra from the group algebra:

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L = LieAlgebra(associative=S)
sage: L.is_abelian()
False
```

Now we construct a Lie algebra from commuting elements in the group algebra:

```
sage: G = SymmetricGroup(5)
sage: S = GroupAlgebra(G, QQ)
sage: gens = map(S, [G((1, 2)), G((3, 4))])
sage: L.<x,y> = LieAlgebra(associative=gens)
sage: L.is_abelian()
True
```

lie_algebra_generators()

Return the Lie algebra generators of self.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L = LieAlgebra(associative=S)
sage: L.lie_algebra_generators()
Finite family {(): (), (1,3,2): (1,3,2), (1,2,3): (1,2,3),
               (2,3): (2,3), (1,3): (1,3), (1,2): (1,2)}
```

monomial(*i*)

Return the monomial indexed by i.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(QQ)
sage: L = LieAlgebra(associative=F)
```

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```
sage: L.monomial(x.leading_support())
x
```

term(*i*, *c=None*)

Return the term indexed by `i` with coefficient `c`.

EXAMPLES:

```
sage: F.<x,y> = FreeAlgebra(QQ)
sage: L = LieAlgebra(associative=F)
sage: L.term(x.leading_support(), 4)
4*x
```

zero()

Return the element 0 in `self`.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L = LieAlgebra(associative=S)
sage: L.zero()
0
```

class sage.algebras.lie_algebras.lie_algebra.**LieAlgebraWithGenerators**(*R*, *names=None*, *index_set=None*, *category=None*, *prefix='L'*, ***kwds*)

Bases: `LieAlgebra`

A Lie algebra with distinguished generators.

gen(*i*)

Return the `i`-th generator of `self`.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)
sage: L.gen(0)
x
```

gens()

Return a tuple whose entries are the generators for this object, in some order.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)
sage: L.gens()
(x, y)
```

indices()

Return the indices of `self`.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, representation="polynomial")
sage: L.indices()
{'x', 'y'}
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, representation="polynomial")
sage: L.lie_algebra_generators()
Finite family {'x': x, 'y': y}
```

class sage.algebras.lie_algebras.lie_algebra.**LiftMorphismToAssociative**(*domain*, *codomain*)

Bases: `LiftMorphism`

The natural lifting morphism from a Lie algebra constructed from an associative algebra A to A.

preimage(x)

Return the preimage of `x` under `self`.

EXAMPLES:

```
sage: R = FreeAlgebra(QQ, 3, 'a,b,c')
sage: L = LieAlgebra(associative=R)
sage: x,y,z = R.gens()
sage: f = R.coerce_map_from(L)
sage: p = f.preimage(x*y - z); p
-c + a*b
sage: p.parent() is L
True
```

section()

Return the section map of `self`.

EXAMPLES:

```
sage: R = FreeAlgebra(QQ, 3, 'x,y,z')
sage: L.<x,y,z> = LieAlgebra(associative=R.gens())
sage: f = R.coerce_map_from(L)
sage: f.section()
Generic morphism:
  From: Free Algebra on 3 generators (x, y, z) over Rational Field
  To:   Lie algebra generated by (x, y, z) in Free Algebra on 3 generators (x,␣
↪y, z) over Rational Field
```

class sage.algebras.lie_algebras.lie_algebra.**MatrixLieAlgebraFromAssociative**(*A*, *gens=None*, *names=None*, *index_set=None*, *category=None*)

Bases: `LieAlgebraFromAssociative`

A Lie algebra constructed from a matrix algebra.

This means a Lie algebra consisting of matrices, with commutator as Lie bracket.

class Element

Bases: *LieAlgebraMatrixWrapper*, *Element*

matrix()

Return self as element of the underlying matrix algebra.

OUTPUT:

An instance of the element class of MatrixSpace.

EXAMPLES:

```
sage: sl3m = lie_algebras.sl(ZZ, 3, representation='matrix')
sage: e1,e2, f1, f2, h1, h2 = sl3m.gens()
sage: h1m = h1.matrix(); h1m
[ 1  0  0]
[ 0 -1  0]
[ 0  0  0]
sage: h1m.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
sage: matrix(h2)
[ 0  0  0]
[ 0  1  0]
[ 0  0 -1]
sage: L = lie_algebras.so(QQ['z'], 5, representation='matrix')
sage: matrix(L.an_element())
[ 1  1  0  0  0]
[ 1  1  0  0  2]
[ 0  0 -1 -1  0]
[ 0  0 -1 -1 -1]
[ 0  1  0 -2  0]

sage: gl2 = lie_algebras.gl(QQ, 2)
sage: matrix(gl2.an_element())
[1 1]
[1 1]
```

9.2.9 Lie Algebra Elements

AUTHORS:

- Travis Scrimshaw (2013-05-04): Initial implementation

class sage.algebras.lie_algebras.lie_algebra_element.**FreeLieAlgebraElement**

Bases: *LieAlgebraElement*

An element of a free Lie algebra.

lift()

Lift self to the universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: Lyn = L.Lyndon()
sage: x,y,z = Lyn.gens()
```

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```
sage: a = Lyn([z, [[x, y], x]]); a
[x, [x, [y, z]]] + [x, [[x, z], y]] - [[x, y], [x, z]]
sage: a.lift()
x^2*y*z - 2*x*y*x*z + y*x^2*z - z*x^2*y + 2*z*x*y*x - z*y*x^2
```

list()

Return `self` as a list of pairs `(m, c)` where `m` is a basis key (i.e., a key of one of the basis elements) and `c` is its coefficient.

This list is sorted from highest to lowest degree.

EXAMPLES:

```
sage: L.<x, y> = LieAlgebra(QQ)
sage: elt = x + L.bracket(y, x)
sage: elt.list()
[([x, y], -1), (x, 1)]
```

class sage.algebras.lie_algebras.lie_algebra_element.**GradedLieBracket**

Bases: *LieBracket*

A Lie bracket (*LieBracket*) for a graded Lie algebra.

Unlike the vanilla Lie bracket class, this also stores a degree, and uses it as a first criterion when comparing graded Lie brackets. (Graded Lie brackets still compare greater than Lie generators.)

class sage.algebras.lie_algebras.lie_algebra_element.**LieAlgebraElement**

Bases: `IndexedFreeModuleElement`

A Lie algebra element.

lift()

Lift `self` to the universal enveloping algebra.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'):{'z':1}})
sage: x.lift().parent() == L.universal_enveloping_algebra()
True
```

class sage.algebras.lie_algebras.lie_algebra_element.**LieAlgebraElementWrapper**

Bases: `ElementWrapper`

Wrap an element as a Lie algebra element.

class sage.algebras.lie_algebras.lie_algebra_element.**LieAlgebraMatrixWrapper**

Bases: *LieAlgebraElementWrapper*

Lie algebra element wrapper around a matrix.

class sage.algebras.lie_algebras.lie_algebra_element.**LieBracket**

Bases: *LieObject*

An abstract Lie bracket (formally, just a binary tree).

lift(*UEA_gens_dict*)

Lift `self` to the universal enveloping algebra.

`UEA_gens_dict` should be the dictionary for the generators of the universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: Lyn = L.Lyndon()
sage: x,y,z = Lyn.gens()
sage: a = Lyn([z, [[x, y], x]]); a
[x, [x, [y, z]]] + [x, [[x, z], y]] - [[x, y], [x, z]]
sage: a.lift() # indirect doctest
x^2*y*z - 2*x*y*x*z + y*x^2*z - z*x^2*y + 2*z*x*y*x - z*y*x^2
```

to_word()

Return the word ("flattening") of `self`.

If `self` is a tree of Lie brackets, this word is usually obtained by "forgetting the brackets".

EXAMPLES:

```
sage: from sage.algebras.lie_algebras.lie_algebra_element import LieGenerator,␣
↪LieBracket
sage: x = LieGenerator('x', 0)
sage: y = LieGenerator('y', 1)
sage: b = LieBracket(x, y)
sage: c = LieBracket(b, x)
sage: c.to_word()
('x', 'y', 'x')
```

class sage.algebras.lie_algebras.lie_algebra_element.**LieGenerator**

Bases: *LieObject*

A wrapper around an object so it can ducktype with and do comparison operations with *LieBracket*.

to_word()

Return the word ("flattening") of `self`.

If `self` is a tree of Lie brackets, this word is usually obtained by "forgetting the brackets".

EXAMPLES:

```
sage: from sage.algebras.lie_algebras.lie_algebra_element import LieGenerator
sage: x = LieGenerator('x', 0)
sage: x.to_word()
('x',)
```

class sage.algebras.lie_algebras.lie_algebra_element.**LieObject**

Bases: `SageObject`

Abstract base class for *LieGenerator* and *LieBracket*.

to_word()

Return the word ("flattening") of `self`.

If `self` is a tree of Lie brackets, this word is usually obtained by "forgetting the brackets".

class sage.algebras.lie_algebras.lie_algebra_element.**LieSubalgebraElementWrapper**

Bases: *LieAlgebraElementWrapper*

Wrap an element of the ambient Lie algebra as an element.

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- `copy` – (default: `True`) if `self` is internally represented by a dictionary `d`, then make a copy of `d`; if `False`, then this can cause undesired behavior by mutating `d`

EXAMPLES:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, {('X','Y'): {'Z': 3}})
sage: S = L.subalgebra([X, Y])
sage: S(2*Y + 9*Z).monomial_coefficients()
{1: 2, 2: 3}
sage: S2 = L.subalgebra([Y, Z])
sage: S2(2*Y + 9*Z).monomial_coefficients()
{0: 2, 1: 9}
```

to_vector(*order=None*, *sparse=False*)

Return the vector in `g.module()` corresponding to the element `self` of `g` (where `g` is the parent of `self`).

EXAMPLES:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, {('X','Y'): {'Z': 3}})
sage: S = L.subalgebra([X, Y])
sage: S.basis()
Family (X, Y, 3*Z)
sage: S(2*Y + 9*Z).to_vector()
(0, 2, 9)
sage: S2 = L.subalgebra([Y, Z])
sage: S2.basis()
Family (Y, Z)
sage: S2(2*Y + 9*Z).to_vector()
(0, 2, 9)
```

class sage.algebras.lie_algebras.lie_algebra_element.**LyndonBracket**

Bases: *GradedLieBracket*

A Lie bracket (*LieBracket*) tailored for the Lyndon basis.

The order on these brackets is defined by $l < r$ if $w(l) < w(r)$, where $w(l)$ is the word corresponding to l. (This is also true if one or both of l and r is a *LieGenerator*.)

class sage.algebras.lie_algebras.lie_algebra_element.**StructureCoefficientsElement**

Bases: *LieAlgebraMatrixWrapper*

An element of a Lie algebra given by structure coefficients.

bracket(*right*)

Return the Lie bracket `[self, right]`.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}, ('y','z'): {'x':1}, ('z',
↪'x'): {'y':1}})
sage: x.bracket(y)
z
sage: y.bracket(x)
-z
sage: (x + y - z).bracket(x - y + z)
-2*y - 2*z
```

lift()

Return the lift of `self` to the universal enveloping algebra.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: elt = x - 3/2 * y
sage: l = elt.lift(); l
x - 3/2*y
sage: l.parent()
Noncommutative Multivariate Polynomial Ring in x, y
 over Rational Field, nc-relations: {y*x: x*y - x}
```

monomial_coefficients(*copy=True*)

Return the monomial coefficients of `self` as a dictionary.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}})
sage: a = 2*x - 3/2*y + z
sage: a.monomial_coefficients()
{'x': 2, 'y': -3/2, 'z': 1}
sage: a = 2*x - 3/2*z
sage: a.monomial_coefficients()
{'x': 2, 'z': -3/2}
```

to_vector(*sparse=False*)

Return `self` as a vector.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}})
sage: a = x + 3*y - z/2
sage: a.to_vector()
(1, 3, -1/2)
```

class sage.algebras.lie_algebras.lie_algebra_element.**UntwistedAffineLieAlgebraElement**

Bases: `Element`

An element of an untwisted affine Lie algebra.

bracket(*right*)

Return the Lie bracket `[self, right]`.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['A',1,1])
sage: e1,f1,h1,e0,f0,c,d = list(L.lie_algebra_generators())
sage: e0.bracket(f0)
(-h1)#t^0 + 4*c
sage: e1.bracket(0)
0
sage: e1.bracket(1)
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents:
 'Affine Kac-Moody algebra of ['A', 1] in the Chevalley basis'
 and 'Integer Ring'
```

c_coefficient()

Return the coefficient of c of `self`.

EXAMPLES:

```
sage: L = lie_algebras.Affine(QQ, ['A',1,1])
sage: x = L.an_element() - 3 * L.c()
sage: x.c_coefficient()
-2
```

canonical_derivation()

Return the canonical derivation d applied to `self`.

The canonical derivation d is defined as

$$d(a \otimes t^m + \alpha c) = a \otimes m t^m.$$

Another formulation is by $d = t \frac{d}{dt}$.

EXAMPLES:

```
sage: L = lie_algebras.Affine(QQ, ['E',6,1])
sage: al = RootSystem(['E',6]).root_lattice().simple_roots()
sage: x = L.basis()[al[2]+al[3]+2*al[4]+al[5],5] + 4*L.c() + L.d()
sage: x.canonical_derivation()
(5*E[alpha[2] + alpha[3] + 2*alpha[4] + alpha[5]])#t^5
```

d_coefficient()

Return the coefficient of d of `self`.

EXAMPLES:

```
sage: L = lie_algebras.Affine(QQ, ['A',1,1])
sage: x = L.an_element() + L.d()
sage: x.d_coefficient()
2
```

monomial_coefficients(*copy=True*)

Return the monomial coefficients of `self`.

EXAMPLES:

```
sage: L = lie_algebras.Affine(QQ, ['C',2,1])
sage: x = L.an_element()
sage: sorted(x.monomial_coefficients(), key=str)
[(-2*alpha[1] - alpha[2], 1),
 (-alpha[1], 0),
 (-alpha[2], 0),
 (2*alpha[1] + alpha[2], -1),
 (alpha[1], 0),
 (alpha[2], 0),
 (alphacheck[1], 0),
 (alphacheck[2], 0),
 'c',
 'd']
```

t_dict()

Return the dict, whose keys are powers of t and values are elements of the classical Lie algebra, of self.

EXAMPLES:

```
sage: L = lie_algebras.Affine(QQ, ['A',1,1])
sage: x = L.an_element()
sage: x.t_dict()
{-1: E[alpha[1]],
 0: E[alpha[1]] + h1 + E[-alpha[1]],
 1: E[-alpha[1]]}
```

9.2.10 Homomorphisms of Lie Algebras

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation
- Eero Hakavuori (08-09-2018): Morphisms defined by a generating subset

class sage.algebras.lie_algebras.morphism.**LieAlgebraHomomorphism_im_gens**(*parent*, *im_gens*, *base_map=None*, *check=True*)

Bases: Morphism

A homomorphism of Lie algebras.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras. A linear map $f: \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism (of Lie algebras) if $f([x,y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$. Thus homomorphisms are completely determined by the image of the generators of $\mathfrak{g}$.

INPUT:

- parent – a homset between two Lie algebras
- im_gens – the image of the generators of the domain
- base_map – a homomorphism to apply to the coefficients. It should be a map from the base ring of the domain to the base ring of the codomain. Note that if base_map is nontrivial then the result will not be a morphism in the category of lie algebras over the base ring.
- check – whether to run checks on the validity of the defining data

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: Lyn = L.Lyndon()
sage: H = L.Hall()
doctest:warning...:
FutureWarning: The Hall basis has not been fully proven correct, but currently no␣
↪bugs are known
See https://github.com/sagemath/sage/issues/16823 for details.
sage: phi = Lyn.coerce_map_from(H); phi
Lie algebra morphism:
  From: Free Lie algebra generated by (x, y, z) over Rational Field in the Hall␣
↪basis
  To:   Free Lie algebra generated by (x, y, z) over Rational Field in the Lyndon␣
↪basis
  Defn: x |--> x
        y |--> y
        z |--> z
```

You can provide a base map, creating a semilinear map that (sometimes) preserves the Lie bracket:

```
sage: R.<x> = ZZ[]
sage: K.<i> = NumberField(x^2 + 1)
sage: cc = K.hom([-i])
sage: L.<X,Y,Z,W> = LieAlgebra(K, {('X','Y'): {'Z':1}, ('X','Z'): {'W':1}})
sage: M.<A,B,C,D> = LieAlgebra(K, {('A','B'): {'C':1}, ('A','C'): {'D':1}})
sage: phi = L.morphism({X:A, Y:B, Z:C, W:D}, base_map=cc)
sage: phi(X)
A
sage: phi(i*X)
-i*A
sage: all(phi(x.bracket(y)) == phi(x).bracket(phi(y)) for x,y in cartesian_product_
↪iterator([[X,Y,Z,W],[X,Y,Z,W]]))
True
```

Note that the Lie bracket should still be preserved, even though the map is no longer linear over the base ring:

```
sage: L.<X,Y,Z,W> = LieAlgebra(K, {('X','Y'): {'Z':i}, ('X','Z'): {'W':1}})
sage: M.<A,B,C,D> = LieAlgebra(K, {('A','B'): {'C':-i}, ('A','C'): {'D':1}})
sage: phi = L.morphism({X:A, Y:B, Z:C, W:D}, base_map=cc)
sage: phi(X.bracket(Y))
-i*C
sage: phi(X).bracket(phi(Y))
-i*C
```

base_map()

Return the map on the base ring that is part of the defining data for this morphism. May return None if a coercion is used.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: K.<i> = NumberField(x^2 + 1)
sage: cc = K.hom([-i])
sage: L.<X,Y,Z,W> = LieAlgebra(K, {('X','Y'): {'Z':1}, ('X','Z'): {'W':1}})
sage: M.<A,B> = LieAlgebra(K, abelian=True)
```

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```
sage: phi = L.morphism({X: A, Y: B}, base_map=cc)
sage: phi(X)
A
sage: phi(i*X)
-i*A
sage: phi.base_map()
Ring endomorphism of Number Field in i with defining polynomial x^2 + 1
  Defn: i |--> -i
```

im_gens()

Return the images of the generators of the domain.

OUTPUT:

- `list` – a copy of the list of gens (it is safe to change this)

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: Lyn = L.Lyndon()
sage: H = L.Hall()
sage: f = Lyn.coerce_map_from(H)
sage: f.im_gens()
[x, y, z]
```

class sage.algebras.lie_algebras.morphism.**LieAlgebraHomset**(*X*, *Y*, *category=None*, *base=None*, *check=True*)

Bases: `Homset`

Homset between two Lie algebras.

Todo: This is a very minimal implementation which does not have coercions of the morphisms.

zero()

Return the zero morphism.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z')
sage: Lyn = L.Lyndon()
sage: H = L.Hall()
sage: HS = Hom(Lyn, H)
sage: HS.zero()
Generic morphism:
  From: Free Lie algebra generated by (x, y, z) over Rational Field in the␣
↪Lyndon basis
  To:   Free Lie algebra generated by (x, y, z) over Rational Field in the Hall␣
↪basis
```

class sage.algebras.lie_algebras.morphism.**LieAlgebraMorphism_from_generators**(*on_generators*, *domain=None*, *codomain=None*, *check=True*, *base_map=None*, *category=None*)

Bases: `LieAlgebraHomomorphism_im_gens`

A morphism between two Lie algebras defined by images of a generating set as a Lie algebra.

This is the Lie algebra morphism $\phi: L \to K$ defined on the chosen basis of L to that of K be using the image of some generating set (as a Lie algebra) of L.

INPUT:

- `on_generators` – dictionary `{X: Y}` of the images Y in `codomain` of elements X of `domain`
- `codomain` – a Lie algebra (optional); this is inferred from the values of `on_generators` if not given
- `base_map` – a homomorphism to apply to the coefficients. It should be a map from the base ring of the domain to the base ring of the codomain. Note that if base_map is nontrivial then the result will not be a morphism in the category of lie algebras over the base ring.
- `check` – (default: `True`) boolean; if `False` the values on the Lie brackets implied by `on_generators` will not be checked for contradictory values

EXAMPLES:

A reflection of one horizontal vector in the Heisenberg algebra:

```
sage: L.<X,Y,Z> = LieAlgebra(QQ, {('X','Y'): {'Z':1}})
sage: phi = L.morphism({X:-X, Y:Y}); phi
Lie algebra endomorphism of Lie algebra on 3 generators (X, Y, Z) over Rational␣
↪Field
  Defn: X |--> -X
        Y |--> Y
        Z |--> -Z
```

There is no Lie algebra morphism that reflects one horizontal vector, but not the vertical one:

```
sage: L.morphism({X:-X, Y:Y, Z:Z})
Traceback (most recent call last):
...
ValueError: this does not define a Lie algebra morphism;
 contradictory values for brackets of length 2
```

Checking for mistakes can be disabled, which can produce invalid results:

```
sage: phi = L.morphism({X:-X, Y:Y, Z:Z}, check=False); phi
Lie algebra endomorphism of Lie algebra on 3 generators (X, Y, Z) over Rational␣
↪Field
  Defn: X |--> -X
        Y |--> Y
        Z |--> Z
sage: L[phi(X), phi(Y)] == phi(L[X,Y])
False
```

The set of keys must generate the Lie algebra:

```
sage: L.morphism({X: X})
Traceback (most recent call last):
...
ValueError: [X] is not a generating set of Lie algebra on 3 generators
(X, Y, Z) over Rational Field
```

Over non-fields, generating subsets are more restricted:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, {('X','Y'): {'Z':2}})
sage: L.morphism({X: X, Y: Y})
Traceback (most recent call last):
...
ValueError: [X, Y] is not a generating set of Lie algebra on 3
generators (X, Y, Z) over Integer Ring
```

The generators do not have to correspond to the defined generating set of the domain:

```
sage: L.<X,Y,Z,W> = LieAlgebra(QQ, {('X','Y'): {'Z':1}, ('X','Z'): {'W':1}})
sage: K.<A,B,C> = LieAlgebra(QQ, {('A','B'): {'C':2}})
sage: phi = L.morphism({X+2*Y: A, X-Y: B}); phi
Lie algebra morphism:
  From: Lie algebra on 4 generators (X, Y, Z, W) over Rational Field
  To:   Lie algebra on 3 generators (A, B, C) over Rational Field
  Defn: X |--> 1/3*A + 2/3*B
        Y |--> 1/3*A - 1/3*B
        Z |--> -2/3*C
        W |--> 0
sage: phi(X+2*Y)
A
sage: phi(X)
1/3*A + 2/3*B
sage: phi(W)
0
sage: phi(Z)
-2/3*C
sage: all(K[phi(p), phi(q)] == phi(L[p,q])
....:     for p in L.basis() for q in L.basis())
True
```

A quotient type Lie algebra morphism:

```
sage: K.<A,B> = LieAlgebra(SR, abelian=True)
sage: L.morphism({X: A, Y: B})
Lie algebra morphism:
  From: Lie algebra on 4 generators (X, Y, Z, W) over Rational Field
  To:   Abelian Lie algebra on 2 generators (A, B) over Symbolic Ring
  Defn: X |--> A
        Y |--> B
        Z |--> 0
        W |--> 0
```

9.2.11 Nilpotent Lie algebras

AUTHORS:

- Eero Hakavuori (2018-08-16): initial version

class sage.algebras.lie_algebras.nilpotent_lie_algebra.**FreeNilpotentLieAlgebra**(*R*, *r*, *s*, *names*, *naming*, *category*, ***kwds*)

Bases: *NilpotentLieAlgebra_dense*

Return the free nilpotent Lie algebra of step `s` with `r` generators.

The free nilpotent Lie algebra L of step s with r generators is the quotient of the free Lie algebra on r generators by the $(s+1)$-th term of the lower central series. That is, the only relations in the Lie algebra L are anticommutativity, the Jacobi identity, and the vanishing of all brackets of length more than s.

INPUT:

- `R` – the base ring
- `r` – an integer; the number of generators
- `s` – an integer; the nilpotency step of the algebra
- `names` – (optional) a string or a list of strings used to name the basis elements; if `names` is a string, then names for the basis will be autogenerated as determined by the `naming` parameter
- `naming` – (optional) a string; the naming scheme to use for the basis; valid values are:
  - `'index'` - (default for $r < 10$) the basis elements are `names_w`, where `w` are Lyndon words indexing the basis
  - `'linear'` - (default for $r \geq 10$) the basis is indexed `names_1`, …, `names_n` in the ordering of the Lyndon basis

Note: The `'index'` naming scheme is not supported if $r \geq 10$ since it leads to ambiguous names.

EXAMPLES:

We compute the free step 4 Lie algebra on 2 generators and verify the only non-trivial relation $[[X_1, [X_1, X_2]], X_2] = [X_1, [[X_1, X_2], X_2]]$:

```
sage: L = LieAlgebra(QQ, 2, step=4)
sage: L.basis().list()
[X_1, X_2, X_12, X_112, X_122, X_1112, X_1122, X_1222]
sage: X_1, X_2 = L.basis().list()[:2]
sage: L[[X_1, [X_1, X_2]], X_2]
X_1122
sage: L[[X_1, [X_1, X_2]], X_2] == L[X_1, [[X_1, X_2], X_2]]
True
```

The linear naming scheme on the same Lie algebra:

```
sage: K = LieAlgebra(QQ, 2, step=4, names='Y', naming='linear')
sage: K.basis().list()
[Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8]
```

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```
sage: K.inject_variables()
Defining Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8
sage: Y_2.bracket(Y_3)
-Y_5
sage: Y_5.bracket(Y_1)
-Y_7
sage: Y_3.bracket(Y_4)
0
```

A fully custom naming scheme on the Heisenberg algebra:

```
sage: L = LieAlgebra(ZZ, 2, step=2, names=('X', 'Y', 'Z'))
sage: a, b, c = L.basis()
sage: L.basis().list()
[X, Y, Z]
sage: a.bracket(b)
Z
```

An equivalent way to define custom names for the basis elements and bind them as local variables simultaneously:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, 2, step=2)
sage: L.basis().list()
[X, Y, Z]
sage: X.bracket(Y)
Z
```

A free nilpotent Lie algebra is a stratified nilpotent Lie algebra:

```
sage: L = LieAlgebra(QQ, 3, step=3)
sage: L.category()
Category of finite dimensional stratified lie algebras with basis over Rational␣
↪Field
sage: L in LieAlgebras(QQ).Nilpotent()
True
```

Being graded means that each basis element has a degree:

```
sage: L in LieAlgebras(QQ).Graded()
True
sage: L.homogeneous_component_basis(1).list()
[X_1, X_2, X_3]
sage: L.homogeneous_component_basis(2).list()
[X_12, X_13, X_23]
sage: L.homogeneous_component_basis(3).list()
[X_112, X_113, X_122, X_123, X_132, X_133, X_223, X_233]
```

class sage.algebras.lie_algebras.nilpotent_lie_algebra.**NilpotentLieAlgebra_dense**(*R*, *s_coeff*, *names*, *index_set*, *step=None*, ***kwds*)

Bases: *LieAlgebraWithStructureCoefficients*

A nilpotent Lie algebra L over a base ring.

INPUT:

- `R` – the base ring
- `s_coeff` – a dictionary of structural coefficients
- `names` – (default:`None`) list of strings to use as names of basis elements; if `None`, the names will be inferred from the structural coefficients
- `index_set` – (default:`None`) list of hashable and comparable elements to use for indexing
- `step` – (optional) an integer; the nilpotency step of the Lie algebra if known; otherwise it will be computed when needed
- `category` – (optional) a subcategory of finite dimensional nilpotent Lie algebras with basis

EXAMPLES:

The input to a *NilpotentLieAlgebra_dense* should be of the same form as to a *LieAlgebraWithStructureCoefficients*:

```
sage: L.<X,Y,Z,W> = LieAlgebra(QQ, {('X','Y'): {'Z': 1}}, nilpotent=True)
sage: L
Nilpotent Lie algebra on 4 generators (X, Y, Z, W) over Rational Field
sage: L[X, Y]
Z
sage: L[X, W]
0
```

If the parameter `names` is omitted, then the terms appearing in the structural coefficients are used as names:

```
sage: L = LieAlgebra(QQ, {('X','Y'): {'Z': 1}}, nilpotent=True); L
Nilpotent Lie algebra on 3 generators (X, Y, Z) over Rational Field
```

9.2.12 Onsager Algebra

AUTHORS:

- Travis Scrimshaw (2017-07): Initial version

class `sage.algebras.lie_algebras.onsager.`**`OnsagerAlgebra`**`(R)`

Bases: *LieAlgebraWithGenerators*, `IndexedGenerators`

The Onsager (Lie) algebra.

The Onsager (Lie) algebra $\mathcal{O}$ is a Lie algebra with generators A_0, A_1 that satisfy

$$[A_0, [A_0, [A_0, A_1]]] = -4[A_0, A_1], \qquad [A_1, [A_1, [A_1, A_0]]] = -4[A_1, A_0].$$

Note: We are using a rescaled version of the usual defining generators.

There exist a basis $\{A_m, G_n \mid m \in \mathbf{Z}, n \in \mathbf{Z}_{>0}\}$ for $\mathcal{O}$ with structure coefficients

$$[A_m, A_{m'}] = G_{m-m'}, \qquad [G_n, G_{n'}] = 0, \qquad [G_n, A_m] = 2A_{m-n} - 2A_{m+n},$$

where $m > m'$.

The Onsager algebra is isomorphic to the subalgebra of the affine Lie algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}K \oplus \mathbf{C}d$ that is invariant under the Chevalley involution. In particular, we have

$$A_i \mapsto f \otimes t^i - e \otimes t^{-i}, \qquad G_i \mapsto h \otimes t^{-i} - h \otimes t^i.$$

where e, f, h are the Chevalley generators of $\mathfrak{sl}_2$.

EXAMPLES:

We construct the Onsager algebra and do some basic computations:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: O.inject_variables()
Defining A0, A1
```

We verify the defining relations:

```
sage: O([A0, [A0, [A0, A1]]]) == -4 * O([A0, A1])
True
sage: O([A1, [A1, [A1, A0]]]) == -4 * O([A1, A0])
True
```

We check the embedding into $\widehat{\mathfrak{sl}}_2$:

```
sage: L = LieAlgebra(QQ, cartan_type=['A',1,1])
sage: B = L.basis()
sage: al = RootSystem(['A',1]).root_lattice().simple_root(1)
sage: ac = al.associated_coroot()
sage: def emb_A(i): return B[-al,i] - B[al,-i]
sage: def emb_G(i): return B[ac,i] - B[ac,-i]
sage: a0 = emb_A(0)
sage: a1 = emb_A(1)
sage: L([a0, [a0, [a0, a1]]]) == -4 * L([a0, a1])
True
sage: L([a1, [a1, [a1, a0]]]) == -4 * L([a1, a0])
True

sage: all(emb_G(n).bracket(emb_A(m)) == 2*emb_A(m-n) - 2*emb_A(m+n)
....:     for m in range(-10, 10) for n in range(1,10))
True
sage: all(emb_A(m).bracket(emb_A(mp)) == emb_G(m-mp)
....:     for m in range(-10,10) for mp in range(m-10, m))
True
```

REFERENCES:

- [Onsager1944]
- [DG1982]

Element

alias of *LieAlgebraElement*

alternating_central_extension()

Return the alternating central extension of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: ACE = O.alternating_central_extension()
sage: ACE
Alternating central extension of the Onsager algebra over Rational Field
```

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: O.basis()
Lazy family (Onsager monomial(i))_{i in
 Disjoint union of Family (Integer Ring, Positive integers)}
```

bracket_on_basis(*x*, *y*)

Return the bracket of basis elements indexed by `x` and `y` where `x < y`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: O.bracket_on_basis((1,3), (1,9))  # [G, G]
0
sage: O.bracket_on_basis((0,8), (1,13))  # [A, G]
-2*A[-5] + 2*A[21]
sage: O.bracket_on_basis((0,-9), (0, 7))  # [A, A]
-G[16]
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: O.lie_algebra_generators()
Finite family {'A0': A[0], 'A1': A[1]}
```

quantum_group(*q=None*, *c=None*)

Return the quantum group of `self`.

The corresponding quantum group is the *QuantumOnsagerAlgebra*. The parameter c must be such that $c(1) = 1$

INPUT:

- `q` – (optional) the quantum parameter; the default is $q \in R(q)$, where R is the base ring of `self`
- `c` – (optional) the parameter c; the default is `q`

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: Q
q-Onsager algebra with c=q over Fraction Field of
 Univariate Polynomial Ring in q over Rational Field
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: O.some_elements()
[A[0], A[2], A[-1], G[4], -2*A[-3] + A[2] + 3*G[2]]
```

class sage.algebras.lie_algebras.onsager.**OnsagerAlgebraACE**(*R*)

Bases: *InfinitelyGeneratedLieAlgebra*, IndexedGenerators

The alternating central extension of the Onsager algebra.

The *alternating central extension* of the *Onsager algebra* is the Lie algebra with basis elements $\{\mathcal{A}_k, \mathcal{B}_k\}_{k\in\mathbf{Z}}$ that satisfy the relations

$$\begin{aligned}[][\mathcal{A}_k, \mathcal{A}_m] &= \mathcal{B}_{k-m} - \mathcal{B}_{m-k},\\ [\mathcal{A}_k, \mathcal{B}_m] &= \mathcal{A}_{k+m} - \mathcal{A}_{k-m},\\ [\mathcal{B}_k, \mathcal{B}_m] &= 0.\end{aligned}$$

This has a natural injection from the Onsager algebra by the map ι defined by

$$\iota(A_k) = \mathcal{A}_k, \qquad \iota(B_k) = \mathcal{B}_k - \mathcal{B}_{-k}.$$

Note that the map ι differs slightly from Lemma 4.18 in [Ter2021b] due to our choice of basis of the Onsager algebra.

> **Warning:** We have added an extra basis vector $\mathcal{B}_0$, which would be 0 in the definition given in [Ter2021b].

EXAMPLES:

We begin by constructing the ACE and doing some sample computations:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: ACE = O.alternating_central_extension()
sage: ACE
Alternating central extension of the Onsager algebra over Rational Field

sage: B = ACE.basis()
sage: A1, A2, Am2 = B[0,1], B[0,2], B[0,-2]
sage: B1, B2, Bm2 = B[1,1], B[1,2], B[1,-2]
sage: A1.bracket(Am2)
-B[-3] + B[3]
sage: A1.bracket(A2)
B[-1] - B[1]
sage: A1.bracket(B2)
-A[-1] + A[3]
sage: A1.bracket(Bm2)
A[-1] - A[3]
sage: B2.bracket(B1)
0
sage: Bm2.bracket(B2)
0
```

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```
sage: (A2 + Am2).bracket(B1 + A2 + B2 + Bm2)
-A[-3] + A[-1] - A[1] + A[3] + B[-4] - B[4]
```

The natural inclusion map ι is implemented as a coercion map:

```
sage: iota = ACE.coerce_map_from(O)
sage: b = O.basis()
sage: am1, a2, b4 = b[0,-1], b[0,2], b[1,4]
sage: iota(am1.bracket(a2)) == iota(am1).bracket(iota(a2))
True
sage: iota(am1.bracket(b4)) == iota(am1).bracket(iota(b4))
True
sage: iota(b4.bracket(a2)) == iota(b4).bracket(iota(a2))
True

sage: am1 + B2
A[-1] + B[2]
sage: am1.bracket(B2)
-A[-3] + A[1]
sage: Bm2.bracket(a2)
-A[0] + A[4]
```

We have the projection map ρ from Lemma 4.19 in [Ter2021b]:

$$\rho(\mathcal{A}_k) = A_k, \qquad \rho(\mathcal{B}_k) = \mathrm{sgn}(k) B_{|k|}.$$

The kernel of ρ is the center $\mathcal{Z}$, which has a basis $\{B_k + B_{-k}\}_{k \in \mathbf{Z}}$:

```
sage: rho = ACE.projection()
sage: rho(A1)
A[1]
sage: rho(Am2)
A[-2]
sage: rho(B1)
1/2*G[1]
sage: rho(Bm2)
-1/2*G[2]
sage: all(rho(B[1,k] + B[1,-k]) == 0 for k in range(-6,6))
True
sage: all(B[0,m].bracket(B[1,k] + B[1,-k]) == 0
....:     for k in range(-4,4) for m in range(-4,4))
True
```

Element

alias of *LieAlgebraElement*

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ).alternating_central_extension()
sage: O.basis()
```

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```
Lazy family (Onsager ACE monomial(i))_{i in
 Disjoint union of Family (Integer Ring, Integer Ring)}
```

bracket_on_basis(*x*, *y*)

Return the bracket of basis elements indexed by x and y where x < y.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ).alternating_central_extension()
sage: O.bracket_on_basis((1,3), (1,9))  # [B, B]
0
sage: O.bracket_on_basis((0,8), (1,13))  # [A, B]
-A[-5] + A[21]
sage: O.bracket_on_basis((0,-9), (0, 7))  # [A, A]
B[-16] - B[16]
```

lie_algebra_generators()

Return the generators of self as a Lie algebra.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ).alternating_central_extension()
sage: O.lie_algebra_generators()
Lazy family (Onsager ACE monomial(i))_{i in
 Disjoint union of Family (Integer Ring, Integer Ring)}
```

projection()

Return the projection map ρ from Lemma 4.19 in [Ter2021b] to the Onsager algebra.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: ACE = O.alternating_central_extension()
sage: rho = ACE.projection()
sage: B = ACE.basis()
sage: A1, A2, Am2 = B[0,1], B[0,2], B[0,-2]
sage: B1, B2, Bm2 = B[1,1], B[1,2], B[1,-2]

sage: rho(A1)
A[1]
sage: rho(Am2)
A[-2]
sage: rho(B1)
1/2*G[1]
sage: rho(B2)
1/2*G[2]
sage: rho(Bm2)
-1/2*G[2]

sage: rho(A1.bracket(A2))
-G[1]
sage: rho(A1).bracket(rho(A2))
-G[1]
```

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```
sage: rho(B1.bracket(Am2))
A[-3] - A[-1]
sage: rho(B1).bracket(rho(Am2))
A[-3] - A[-1]
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ).alternating_central_extension()
sage: O.some_elements()
[A[0], A[2], A[-1], B[4], B[-3], -2*A[-3] + A[2] + B[-1] + 3*B[2]]
```

class sage.algebras.lie_algebras.onsager.**QuantumOnsagerAlgebra**(*g*, *q*, *c*)

Bases: `CombinatorialFreeModule`

The quantum Onsager algebra.

The *quantum Onsager algebra*, or q-Onsager algebra, is a quantum group analog of the Onsager algebra. It is the left (or right) coideal subalgebra of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ and is the simplest example of a quantum symmetric pair coideal subalgebra of affine type.

The q-Onsager algebra depends on a parameter c such that $c(1) = 1$. The q-Onsager algebra with parameter c is denoted $U_q(\mathcal{O}_R)_c$, where R is the base ring of the defining Onsager algebra.

EXAMPLES:

We create the q-Onsager algebra and its generators:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: G = Q.algebra_generators()
```

The generators are given as pairs, where $G[0, n]$ is the generator $B_{n\delta+\alpha_1}$ and $G[1, n]$ is the generator $B_{n\delta}$. We use the convention that $n\delta + \alpha_1 \equiv (-n-1)\delta + \alpha_0$.

```
sage: G[0,5]
B[5d+a1]
sage: G[0,-5]
B[4d+a0]
sage: G[1,5]
B[5d]
sage: (G[0,5] + G[0,-3]) * (G[1,2] - G[0,3])
B[2d+a0]*B[2d] - B[2d+a0]*B[3d+a1]
 + ((-q^4+1)/q^2)*B[1d]*B[6d+a1]
 + ((q^4-1)/q^2)*B[1d]*B[4d+a1] + B[2d]*B[5d+a1]
 - B[5d+a1]*B[3d+a1] + ((q^2+1)/q^2)*B[7d+a1]
 + ((q^6+q^4-q^2-1)/q^2)*B[5d+a1] + (-q^4-q^2)*B[3d+a1]
sage: (G[0,5] + G[0,-3] + G[1,4]) * (G[0,2] - G[1,3])
-B[2d+a0]*B[3d] + B[2d+a0]*B[2d+a1]
 + ((q^4-1)/q^4)*B[1d]*B[7d+a1]
 + ((q^8-2*q^4+1)/q^4)*B[1d]*B[5d+a1]
 + (-q^4+1)*B[1d]*B[3d+a1] + ((q^4-1)/q^2)*B[2d]*B[6d+a1]
 + ((-q^4+1)/q^2)*B[2d]*B[4d+a1] - B[3d]*B[4d]
```

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```
- B[3d]*B[5d+a1] + B[4d]*B[2d+a1] + B[5d+a1]*B[2d+a1]
+ ((-q^2-1)/q^4)*B[8d+a1] + ((-q^6-q^4+q^2+1)/q^4)*B[6d+a1]
+ (-q^6-q^4+q^2+1)*B[4d+a1] + (q^6+q^4)*B[2d+a1]
```

We check the q-Dolan-Grady relations:

```
sage: def q_dolan_grady(a, b, q):
....:     x = q*a*b - ~q*b*a
....:     y = ~q*a*x - q*x*a
....:     return a*y - y*a
sage: A0, A1 = G[0,-1], G[0,0]
sage: q = Q.q()
sage: q_dolan_grady(A1, A0, q) == (q^4 + 2*q^2 + 1) * (A0*A1 - A1*A0)
True
sage: q_dolan_grady(A0, A1, q) == (q^4 + 2*q^2 + 1) * (A1*A0 - A0*A1)
True
```

REFERENCES:

- [BK2017]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: Q.algebra_generators()
Lazy family (generator map(i))_{i in Disjoint union of
 Family (Integer Ring, Positive integers)}
```

c()

Return the parameter c of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group(c=-3)
sage: Q.c()
-3
```

degree_on_basis(m)

Return the degree of the basis element indexed by `m`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: G = Q.algebra_generators()
sage: B0 = G[0,0]
sage: B1 = G[0,-1]
sage: Q.degree_on_basis(B0.leading_support())
1
```

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```
sage: Q.degree_on_basis((B1^10 * B0^10).leading_support())
20
sage: ((B0 * B1)^3).maximal_degree()
6
```

gens()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: Q.algebra_generators()
Lazy family (generator map(i))_{i in Disjoint union of
 Family (Integer Ring, Positive integers)}
```

lie_algebra()

Return the underlying Lie algebra of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: Q.lie_algebra()
Onsager algebra over Rational Field
sage: Q.lie_algebra() is O
True
```

one_basis()

Return the basis element indexing 1.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: ob = Q.one_basis(); ob
1
sage: ob.parent()
Free abelian monoid indexed by
 Disjoint union of Family (Integer Ring, Positive integers)
```

product_on_basis(*lhs*, *rhs*)

Return the product of the two basis elements `lhs` and `rhs`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: I = Q._indices.gens()
sage: Q.product_on_basis(I[1,21]^2, I[1,31]^3)
B[21d]^2*B[31d]^3
sage: Q.product_on_basis(I[1,31]^3, I[1,21]^2)
B[21d]^2*B[31d]^3
sage: Q.product_on_basis(I[0,8], I[0,6])
```

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```
B[8d+a1]*B[6d+a1]
sage: Q.product_on_basis(I[0,-8], I[0,6])
B[7d+a0]*B[6d+a1]
sage: Q.product_on_basis(I[0,-6], I[0,-8])
B[5d+a0]*B[7d+a0]
sage: Q.product_on_basis(I[0,-6], I[1,2])
B[5d+a0]*B[2d]
sage: Q.product_on_basis(I[1,6], I[0,2])
B[6d]*B[2d+a1]

sage: Q.product_on_basis(I[0,1], I[0,2])
1/q^2*B[2d+a1]*B[1d+a1] - B[1d]
sage: Q.product_on_basis(I[0,-3], I[0,-1])
1/q^2*B[a0]*B[2d+a0] + ((-q^2+1)/q^2)*B[1d+a0]^2 - B[2d]
sage: Q.product_on_basis(I[0,2], I[0,-1])
q^2*B[a0]*B[2d+a1] + ((q^4-1)/q^2)*B[1d+a1]*B[a1]
 + (-q^2+1)*B[1d] + q^2*B[3d]
sage: Q.product_on_basis(I[0,2], I[1,1])
B[1d]*B[2d+a1] + (q^2+1)*B[3d+a1] + (-q^2-1)*B[1d+a1]
sage: Q.product_on_basis(I[0,1], I[1,2])
((-q^4+1)/q^2)*B[1d]*B[2d+a1] + ((q^4-1)/q^2)*B[1d]*B[a1]
 + B[2d]*B[1d+a1] + (-q^4-q^2)*B[a0]
 + ((q^2+1)/q^2)*B[3d+a1] + ((q^6+q^4-q^2-1)/q^2)*B[1d+a1]
sage: Q.product_on_basis(I[1,2], I[0,-1])
B[a0]*B[2d] + ((-q^4+1)/q^2)*B[1d+a0]*B[1d]
 + ((q^4-1)/q^2)*B[1d]*B[a1] + ((q^2+1)/q^2)*B[2d+a0]
 + ((-q^2-1)/q^2)*B[1d+a1]
sage: Q.product_on_basis(I[1,2], I[0,-4])
((q^4-1)/q^2)*B[2d+a0]*B[1d] + B[3d+a0]*B[2d]
 + ((-q^4+1)/q^2)*B[4d+a0]*B[1d] + (-q^4-q^2)*B[1d+a0]
 + ((q^6+q^4-q^2-1)/q^2)*B[3d+a0] + ((q^2+1)/q^2)*B[5d+a0]
```

q()

Return the parameter q of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: Q.q()
q
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: O = lie_algebras.OnsagerAlgebra(QQ)
sage: Q = O.quantum_group()
sage: Q.some_elements()
[B[a1], B[3d+a1], B[a0], B[1d], B[4d]]
```

9.2.13 The Poincare-Birkhoff-Witt Basis For A Universal Enveloping Algebra

AUTHORS:

- Travis Scrimshaw (2013-11-03): Initial version

class sage.algebras.lie_algebras.poincare_birkhoff_witt.**PoincareBirkhoffWittBasis**(*g*, *basis_key*, *prefix*, ***kwds*)

Bases: CombinatorialFreeModule

The Poincare-Birkhoff-Witt (PBW) basis of the universal enveloping algebra of a Lie algebra.

Consider a Lie algebra $\mathfrak{g}$ with ordered basis $(b_1, \dots, b_n)$. Then the universal enveloping algebra $U(\mathfrak{g})$ is generated by $b_1, \dots, b_n$ and subject to the relations

$$[b_i, b_j] = \sum_{k=1}^{n} c_{ij}^k b_k$$

where c_{ij}^k are the structure coefficients of $\mathfrak{g}$. The Poincare-Birkhoff-Witt (PBW) basis is given by the monomials $b_1^{e_1} b_2^{e_2} \cdots b_n^{e_n}$. Specifically, we can rewrite $b_j b_i = b_i b_j + [b_j, b_i]$ where $j > i$, and we can repeat this to sort any monomial into

$$b_{i_1} \cdots b_{i_k} = b_1^{e_1} \cdots b_n^{e_n} + LOT$$

where LOT are lower order terms. Thus the PBW basis is a filtered basis for $U(\mathfrak{g})$.

EXAMPLES:

We construct the PBW basis of $\mathfrak{sl}_2$:

```
sage: L = lie_algebras.three_dimensional_by_rank(QQ, 3, names=['E','F','H'])
sage: PBW = L.pbw_basis()
```

We then do some computations; in particular, we check that $[E, F] = H$:

```
sage: E,F,H = PBW.algebra_generators()
sage: E*F
PBW['E']*PBW['F']
sage: F*E
PBW['E']*PBW['F'] - PBW['H']
sage: E*F - F*E
PBW['H']
```

Next we construct another instance of the PBW basis, but sorted in the reverse order:

```
sage: def neg_key(x):
....:     return -L.basis().keys().index(x)
sage: PBW2 = L.pbw_basis(prefix='PBW2', basis_key=neg_key)
```

We then check the multiplication is preserved:

```
sage: PBW2(E) * PBW2(F)
PBW2['F']*PBW2['E'] + PBW2['H']
sage: PBW2(E*F)
PBW2['F']*PBW2['E'] + PBW2['H']
```

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```
sage: F * E + H
PBW['E']*PBW['F']
```

We now construct the PBW basis for Lie algebra of regular vector fields on $\mathbf{C}^\times$:

```
sage: L = lie_algebras.regular_vector_fields(QQ)
sage: PBW = L.pbw_basis()
sage: G = PBW.algebra_generators()
sage: G[2] * G[3]
PBW[2]*PBW[3]
sage: G[3] * G[2]
PBW[2]*PBW[3] + PBW[5]
sage: G[-2] * G[3] * G[2]
PBW[-2]*PBW[2]*PBW[3] + PBW[-2]*PBW[5]
```

class Element

Bases: `IndexedFreeModuleElement`

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)
sage: PBW = L.pbw_basis()
sage: PBW.algebra_generators()
Finite family {alpha[1]: PBW[alpha[1]], alphacheck[1]: PBW[alphacheck[1]], -
↪alpha[1]: PBW[-alpha[1]]}
```

casimir_element(*order=2*)

Return the Casimir element of `self`.

See also:

`casimir_element()`

INPUT:

- `order` – (default: 2) the order of the Casimir element

EXAMPLES:

```
sage: L = LieAlgebra(QQ, cartan_type=['G', 2])
sage: U = L.pbw_basis()
sage: C = U.casimir_element(); C
1/4*PBW[alpha[2]]*PBW[-alpha[2]] + 1/12*PBW[alpha[1]]*PBW[-alpha[1]]
 + 1/12*PBW[alpha[1] + alpha[2]]*PBW[-alpha[1] - alpha[2]] + 1/
↪12*PBW[2*alpha[1] + alpha[2]]*PBW[-2*alpha[1] - alpha[2]]
 + 1/4*PBW[3*alpha[1] + alpha[2]]*PBW[-3*alpha[1] - alpha[2]]
 + 1/4*PBW[3*alpha[1] + 2*alpha[2]]*PBW[-3*alpha[1] - 2*alpha[2]]
 + 1/12*PBW[alphacheck[1]]^2 + 1/4*PBW[alphacheck[1]]*PBW[alphacheck[2]]
 + 1/4*PBW[alphacheck[2]]^2 - 5/12*PBW[alphacheck[1]] - 3/4*PBW[alphacheck[2]]
sage: all(g * C == C * g for g in U.algebra_generators())
True
```

degree_on_basis(*m*)

Return the degree of the basis element indexed by m.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)
sage: PBW = L.pbw_basis()
sage: E,H,F = PBW.algebra_generators()
sage: PBW.degree_on_basis(E.leading_support())
1
sage: m = ((H*F)^10).trailing_support(key=PBW._monomial_key)  # long time
sage: PBW.degree_on_basis(m)  # long time
20
sage: ((H*F*E)^4).maximal_degree()  # long time
12
```

gens()

Return the algebra generators of self.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)
sage: PBW = L.pbw_basis()
sage: PBW.algebra_generators()
Finite family {alpha[1]: PBW[alpha[1]], alphacheck[1]: PBW[alphacheck[1]], -
↪alpha[1]: PBW[-alpha[1]]}
```

lie_algebra()

Return the underlying Lie algebra of self.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)
sage: PBW = L.pbw_basis()
sage: PBW.lie_algebra() is L
True
```

one_basis()

Return the basis element indexing 1.

EXAMPLES:

```
sage: L = lie_algebras.three_dimensional_by_rank(QQ, 3, names=['E','F','H'])
sage: PBW = L.pbw_basis()
sage: ob = PBW.one_basis(); ob
1
sage: ob.parent()
Free abelian monoid indexed by {'E', 'F', 'H'}
```

product_on_basis(*lhs, rhs*)

Return the product of the two basis elements lhs and rhs.

EXAMPLES:

```
sage: L = lie_algebras.three_dimensional_by_rank(QQ, 3, names=['E','F','H'])
sage: PBW = L.pbw_basis()
sage: I = PBW.indices()
sage: PBW.product_on_basis(I.gen('E'), I.gen('F'))
PBW['E']*PBW['F']
sage: PBW.product_on_basis(I.gen('E'), I.gen('H'))
PBW['E']*PBW['H']
sage: PBW.product_on_basis(I.gen('H'), I.gen('E'))
PBW['E']*PBW['H'] + 2*PBW['E']
sage: PBW.product_on_basis(I.gen('F'), I.gen('E'))
PBW['E']*PBW['F'] - PBW['H']
sage: PBW.product_on_basis(I.gen('F'), I.gen('H'))
PBW['F']*PBW['H']
sage: PBW.product_on_basis(I.gen('H'), I.gen('F'))
PBW['F']*PBW['H'] - 2*PBW['F']
sage: PBW.product_on_basis(I.gen('H')**2, I.gen('F')**2)
PBW['F']^2*PBW['H']^2 - 8*PBW['F']^2*PBW['H'] + 16*PBW['F']^2

sage: E,F,H = PBW.algebra_generators()
sage: E*F - F*E
PBW['H']
sage: H * F * E
PBW['E']*PBW['F']*PBW['H'] - PBW['H']^2
sage: E * F * H * E
PBW['E']^2*PBW['F']*PBW['H'] + 2*PBW['E']^2*PBW['F']
 - PBW['E']*PBW['H']^2 - 2*PBW['E']*PBW['H']
```

9.2.14 Quotients of Lie algebras

AUTHORS:

- Eero Hakavuori (2018-09-02): initial version

class sage.algebras.lie_algebras.quotient.**LieQuotient_finite_dimensional_with_basis**(*I*, *L*, *names*, *index_set*, *category=None*)

Bases: *LieAlgebraWithStructureCoefficients*

A quotient Lie algebra.

INPUT:

- I – an ideal or a list of generators of the ideal
- ambient – (optional) the Lie algebra to be quotiented; will be deduced from I if not given
- names – (optional) a string or a list of strings; names for the basis elements of the quotient. If names is a string, the basis will be named names_1,...,``names_n``.

EXAMPLES:

The Engel Lie algebra as a quotient of the free nilpotent Lie algebra of step 3 with 2 generators:

```
sage: L = LieAlgebra(QQ, 2, step=3)
sage: L.inject_variables()
Defining X_1, X_2, X_12, X_112, X_122
sage: I = L.ideal(X_122)
sage: E = L.quotient(I); E
Lie algebra quotient L/I of dimension 4 over Rational Field where
L: Free Nilpotent Lie algebra on 5 generators (X_1, X_2, X_12, X_112, X_122) over
↪Rational Field
I: Ideal (X_122)
sage: E.category()
Join of Category of finite dimensional nilpotent lie algebras with basis
over Rational Field and Category of subquotients of sets
sage: E.basis().list()
[X_1, X_2, X_12, X_112]
sage: E.inject_variables()
Defining X_1, X_2, X_12, X_112
sage: X_1.bracket(X_2)
X_12
sage: X_1.bracket(X_12)
X_112
sage: X_2.bracket(X_12)
0
```

Shorthand for taking a quotient without creating an ideal first:

```
sage: E2 = L.quotient(X_122); E2
Lie algebra quotient L/I of dimension 4 over Rational Field where
L: Free Nilpotent Lie algebra on 5 generators (X_1, X_2, X_12, X_112, X_122) over
↪Rational Field
I: Ideal (X_122)
sage: E is E2
True
```

Custom names for the basis can be given:

```
sage: E.<X,Y,Z,W> = L.quotient(X_122)
sage: E.basis().list()
[X, Y, Z, W]
sage: X.bracket(Z)
W
sage: Y.bracket(Z)
0
```

The elements can be relabeled linearly by passing a string to the `names` parameter:

```
sage: E = L.quotient(X_122, names='Y')
sage: E.basis().list()
[Y_1, Y_2, Y_3, Y_4]
sage: E.inject_variables()
Defining Y_1, Y_2, Y_3, Y_4
sage: Y_1.bracket(Y_3)
Y_4
sage: Y_2.bracket(Y_3)
0
```

Conversion from the ambient Lie algebra uses the quotient projection:

```
sage: L = LieAlgebra(QQ, 2, step=3)
sage: L.inject_variables()
Defining X_1, X_2, X_12, X_112, X_122
sage: E = L.quotient(X_122, names='Y')
sage: E(X_1), E(X_2), E(X_12), E(X_112), E(X_122)
(Y_1, Y_2, Y_3, Y_4, 0)
```

A non-stratifiable Lie algebra as a quotient of the free nilpotent Lie algebra of step 4 on 2 generators by the relation $[X_2, [X_1, X_2]] = [X_1, [X_1, [X_1, X_2]]]$:

```
sage: L = LieAlgebra(QQ, 2, step=4)
sage: X_1, X_2 = L.homogeneous_component_basis(1)
sage: rel = L[X_2, [X_1, X_2]] - L[X_1, [X_1, [X_1, X_2]]]
sage: Q = L.quotient(rel, names='Y')
sage: Q.dimension()
5
sage: Q.inject_variables()
Defining Y_1, Y_2, Y_3, Y_4, Y_5
sage: lcs = Q.lower_central_series()
sage: [I.basis().list() for I in lcs]
[[Y_1, Y_2, Y_3, Y_4, Y_5], [Y_3, Y_4, Y_5], [Y_4, Y_5], [Y_5], []]
sage: Y_2.bracket(Y_3)
-Y_5
```

Quotients when the base ring is not a field are not implemented:

```
sage: L = lie_algebras.Heisenberg(ZZ, 1)
sage: L.quotient(L.an_element())
Traceback (most recent call last):
...
NotImplementedError: quotients over non-fields not implemented
```

ambient()

Return the ambient Lie algebra of `self`.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, 2, step=2)
sage: Q = L.quotient(z)
sage: Q.ambient() == L
True
```

defining_ideal()

Return the ideal generating this quotient Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.Heisenberg(QQ, 1)
sage: p,q,z = L.basis()
sage: Q = L.quotient(p)
sage: Q.defining_ideal()
Ideal (p1) of Heisenberg algebra of rank 1 over Rational Field
```

from_vector(*v*, *order=None*, *coerce=False*)

Return the element of `self` corresponding to the vector `v`.

INPUT:

- `v` – a vector in `self.module()` or `self.ambient().module()`

EXAMPLES:

An element from a vector of the intrinsic module:

```
sage: L.<X,Y,Z> = LieAlgebra(QQ, 3, abelian=True)
sage: Q = L.quotient(X + Y + Z)
sage: Q.dimension()
2
sage: el = Q.from_vector([1, 2]); el
X + 2*Y
sage: el.parent() == Q
True
```

An element from a vector of the ambient module

sage: el = Q.from_vector([1, 2, 3]); el -2*X - Y sage: el.parent() == Q True

lift(*X*)

Return some preimage of `X` under the quotient projection into `self`.

INPUT:

- `X` – an element of `self`

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, 2, step=2)
sage: Q = L.quotient(x + y)
sage: Q(y)
-x
sage: el = Q.lift(Q(y)); el
-x
sage: el.parent()
Free Nilpotent Lie algebra on 3 generators (x, y, z) over Rational Field
```

retract(*X*)

Map `X` under the quotient projection to `self`.

INPUT:

- `X` – an element of the ambient Lie algebra

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 3, step=2)
sage: L.inject_variables()
Defining X_1, X_2, X_3, X_12, X_13, X_23
sage: Q = L.quotient(X_1 + X_2 + X_3)
sage: Q.retract(X_1), Q.retract(X_2), Q.retract(X_3)
(X_1, X_2, -X_1 - X_2)
sage: all(Q.retract(Q.lift(X)) == X for X in Q.basis())
True
```

9.2.15 Rank Two Heisenberg-Virasoro Algebras

AUTHORS:

- Travis Scrimshaw (2018-08): Initial version

class sage.algebras.lie_algebras.rank_two_heisenberg_virasoro.**RankTwoHeisenbergVirasoro**(*R*)

Bases: *InfinitelyGeneratedLieAlgebra*, IndexedGenerators

The rank 2 Heisenberg-Virasoro algebra.

The *rank 2 Heisenberg-Virasoro* (Lie) algebra is the Lie algebra L spaned by the elements

$$\{t^{\alpha}, E(\alpha) \mid \alpha \in \mathbf{Z}^2 \setminus \{(0,0)\}\} \cup \{K_1, K_2, K_3, K_4\},$$

which satisfy the relations

$$\begin{aligned}[][t^{\alpha}, t^{\beta}] &= [K_i, L] = 0,\\ [t^{\alpha}, E(\beta)] &= \det \begin{pmatrix} \beta \\ \alpha \end{pmatrix} t^{\alpha+\beta} + \delta_{\alpha,-\beta}(\alpha_1 K_1 + \alpha_2 K_2),\\ [E(\alpha), E(\beta)] &= \det \begin{pmatrix} \beta \\ \alpha \end{pmatrix} E(\alpha+\beta) + \delta_{\alpha,-\beta}(\alpha_1 K_3 + \alpha_2 K_4),\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$ and δ_{xy} is the Kronecker delta.

EXAMPLES:

```
sage: L = lie_algebras.RankTwoHeisenbergVirasoro(QQ)
sage: K1,K2,K3,K4 = L.K()
sage: E2m1 = L.E(2,-1)
sage: Em21 = L.E(-2,1)
sage: t2m1 = L.t(2,-1)
sage: t53 = L.t(5,3)

sage: Em21.bracket(t2m1)
-2*K1 + K2
sage: t53.bracket(E2m1)
11*t(7, 2)
sage: E2m1.bracket(Em21)
2*K3 - K4
sage: E2m1.bracket(t2m1)
0

sage: all(x.bracket(y) == 0 for x in [K1,K2,K3,K4] for y in [E2m1, Em21, t2m1])
True
```

REFERENCES:

- [LT2018]

E(*a*, *b*)

Return the basis element $E(a, b)$ of self.

EXAMPLES:

```
sage: L = lie_algebras.RankTwoHeisenbergVirasoro(QQ)
sage: L.E(1,-2)
E(1, -2)
```

class Element

Bases: *LieAlgebraElement*

K(*i=None*)

Return the basis element K_i of `self`.

EXAMPLES:

```
sage: L = lie_algebras.RankTwoHeisenbergVirasoro(QQ)
sage: L.K(1)
K1
sage: list(L.K())
[K1, K2, K3, K4]
```

bracket_on_basis(*i, j*)

Return the bracket of basis elements indexed by `i` and `j`, where `i < j`.

EXAMPLES:

```
sage: L = lie_algebras.RankTwoHeisenbergVirasoro(QQ)
sage: v = L._v
sage: L.bracket_on_basis(('K',2), ('t', v(3,-1)))
0
sage: L.bracket_on_basis(('K', 4), ('E', v(3,-1)))
0
sage: L.bracket_on_basis(('t', v(3,-1)), ('t', v(4,3)))
0
sage: L.bracket_on_basis(('t', v(3,-1)), ('E', v(4,3)))
-13*t(7, 2)
sage: L.bracket_on_basis(('t', v(2,2)),  ('E', v(1,1)))
0
sage: L.bracket_on_basis(('t', v(3,-1)), ('E', v(-3,1)))
3*K1 - K2
sage: L.bracket_on_basis(('E', v(3,-1)), ('E', v(4,3)))
-13*E(7, 2)
sage: L.bracket_on_basis(('E', v(2,2)),  ('E', v(1,1)))
0
sage: L.bracket_on_basis(('E', v(3,-1)), ('E', v(-3,1)))
3*K3 - K4
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: L = lie_algebras.RankTwoHeisenbergVirasoro(QQ)
sage: L.some_elements()
[E(1, 1), E(-2, -2), E(0, 1),
 t(1, 1), t(4, -1), t(2, 3),
 K2, K4,
 K3 - 1/2*t(-1, 3) + E(1, -3) + E(2, 2)]
```

t(*a, b*)

Return the basis element $t^{(a,b)}$ of `self`.

EXAMPLES:

```
sage: L = lie_algebras.RankTwoHeisenbergVirasoro(QQ)
sage: L.t(1,-2)
t(1, -2)
```

9.2.16 Lie Algebras Given By Structure Coefficients

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version

class sage.algebras.lie_algebras.structure_coefficients.**LieAlgebraWithStructureCoefficients**(*R*, *s_coeff*, *names*, *index_set*, *category=None*, *prefix=None*, *bracket=None*, *latex_bracket=None*, *string_quotes=...*, ***kwds*)

Bases: *FinitelyGeneratedLieAlgebra*, IndexedGenerators

A Lie algebra with a set of specified structure coefficients.

The structure coefficients are specified as a dictionary d whose keys are pairs of basis indices, and whose values are dictionaries which in turn are indexed by basis indices. The value of d at a pair (u, v) of basis indices is the dictionary whose w-th entry (for w a basis index) is the coefficient of b_w in the Lie bracket $[b_u, b_v]$ (where b_x means the basis element with index x).

INPUT:

- R – a ring, to be used as the base ring
- s_coeff – a dictionary, indexed by pairs of basis indices (see below), and whose values are dictionaries which are indexed by (single) basis indices and whose values are elements of R
- names – list or tuple of strings
- index_set – (default: names) list or tuple of hashable and comparable elements

OUTPUT:

A Lie algebra over R which (as an R-module) is free with a basis indexed by the elements of index_set. The i-th basis element is displayed using the name names[i]. If we let b_i denote this i-th basis element, then the Lie bracket is given by the requirement that the b_k-coefficient of $[b_i, b_j]$ is s_coeff[(i, j)][k] if s_coeff[(i, j)] exists, otherwise -s_coeff[(j, i)][k] if s_coeff[(j, i)] exists, otherwise 0.

EXAMPLES:

We create the Lie algebra of $\mathbf{Q}^3$ under the Lie bracket defined by $\times$ (cross-product):

```
sage: L = LieAlgebra(QQ, 'x,y,z', {('x','y'): {'z':1}, ('y','z'): {'x':1}, ('z','x
↪'): {'y':1}})
sage: (x,y,z) = L.gens()
sage: L.bracket(x, y)
z
sage: L.bracket(y, x)
-z
```

class Element

Bases: *StructureCoefficientsElement*

change_ring(*R*)

Return a Lie algebra with identical structure coefficients over R.

INPUT:

- R – a ring

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(ZZ, {('x','y'): {'z':1}})
sage: L.structure_coefficients()
Finite family {('x', 'y'): z}
sage: LQQ = L.change_ring(QQ)
sage: LQQ.structure_coefficients()
Finite family {('x', 'y'): z}
sage: LSR = LQQ.change_ring(SR)
sage: LSR.structure_coefficients()
Finite family {('x', 'y'): z}
```

dimension()

Return the dimension of self.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y', {('x','y'):{'x':1}})
sage: L.dimension()
2
```

from_vector(*v*, *order=None*, *coerce=True*)

Return an element of self from the vector v.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}})
sage: L.from_vector([1, 2, -2])
x + 2*y - 2*z
```

module(*sparse=True*)

Return self as a free module.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'):{'z':1}})
sage: L.module()
Sparse vector space of dimension 3 over Rational Field
```

monomial(*k*)

Return the monomial indexed by k.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}})
sage: L.monomial('x')
x
```

some_elements()

Return some elements of self.

EXAMPLES:

```
sage: L = lie_algebras.three_dimensional(QQ, 4, 1, -1, 2)
sage: L.some_elements()
[X, Y, Z, X + Y + Z]
```

structure_coefficients(*include_zeros=False*)

Return the dictionary of structure coefficients of self.

EXAMPLES:

```
sage: L = LieAlgebra(QQ, 'x,y,z', {('x','y'): {'x':1}})
sage: L.structure_coefficients()
Finite family {('x', 'y'): x}
sage: S = L.structure_coefficients(True); S
Finite family {('x', 'y'): x, ('x', 'z'): 0, ('y', 'z'): 0}
sage: S['x','z'].parent() is L
True
```

term(*k*, *c=None*)

Return the term indexed by i with coefficient c.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}})
sage: L.term('x', 4)
4*x
```

zero()

Return the element 0 in self.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z':1}})
sage: L.zero()
0
```

9.2.17 Subalgebras and ideals of Lie algebras

AUTHORS:

- Eero Hakavuori (2018-08-29): initial version

class sage.algebras.lie_algebras.subalgebra.**LieSubalgebra_finite_dimensional_with_basis**(*ambient*, *gens*, *ideal*, *order=None*, *category=None*)

Bases: `Parent`, `UniqueRepresentation`

A Lie subalgebra of a finite dimensional Lie algebra with basis.

INPUT:

- `ambient` – the Lie algebra containing the subalgebra
- `gens` – a list of generators of the subalgebra
- `ideal` – (default: `False`) a boolean; if `True`, then `gens` is interpreted as the generating set of an ideal instead of a subalgebra
- `order` – (optional) the key used to sort the indices of `ambient`
- `category` – (optional) a subcategory of subobjects of finite dimensional Lie algebras with basis

EXAMPLES:

Subalgebras and ideals are defined by giving a list of generators:

```
sage: L = lie_algebras.Heisenberg(QQ, 1)
sage: X, Y, Z = L.basis()
sage: S =  L.subalgebra([X, Z]); S
Subalgebra generated by (p1, z) of Heisenberg algebra of rank 1 over Rational Field
sage: I =  L.ideal([X, Z]); I
Ideal (p1, z) of Heisenberg algebra of rank 1 over Rational Field
```

An ideal is in general larger than the subalgebra with the same generators:

```
sage: S = L.subalgebra(Y)
sage: S.basis()
Family (q1,)
sage: I = L.ideal(Y)
sage: I.basis()
Family (q1, z)
```

The zero dimensional subalgebra can be created by giving 0 as a generator or with an empty list of generators:

```
sage: L.<X,Y,Z> = LieAlgebra(QQ, {('X','Y'): {'Z': 1}})
sage: S1 = L.subalgebra(0)
sage: S2 = L.subalgebra([])
sage: S1 is S2
True
```

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```
sage: S1.basis()
Family ()
```

Elements of the ambient Lie algebra can be reduced modulo an ideal or subalgebra:

```
sage: L.<X,Y,Z> = LieAlgebra(SR, {('X','Y'): {'Z': 1}})
sage: I = L.ideal(Y)
sage: I.reduce(X + 2*Y + 3*Z)
X
sage: S = L.subalgebra(Y)
sage: S.reduce(X + 2*Y + 3*Z)
X + 3*Z
```

The reduction gives elements in a fixed complementary subspace. When the base ring is a field, the complementary subspace is spanned by those basis elements which are not leading supports of the basis:

```
sage: I =  L.ideal(X + Y)
sage: I.basis()
Family (X + Y, Z)
sage: el = var('x')*X + var('y')*Y + var('z')*Z; el
x*X + y*Y + z*Z
sage: I.reduce(el)
(x-y)*X
```

Giving a different `order` may change the reduction of elements:

```
sage: I =  L.ideal(X + Y, order=lambda s: ['Z','Y','X'].index(s))
sage: I.basis()
Family (Z, X + Y)
sage: I.reduce(el)
(-x+y)*Y
```

A subalgebra of a subalgebra is a subalgebra of the original:

```
sage: sc = {('X','Y'): {'Z': 1}, ('X','Z'): {'W': 1}}
sage: L.<X,Y,Z,W> = LieAlgebra(QQ, sc)
sage: S1 = L.subalgebra([Y, Z, W]); S1
Subalgebra generated by (Y, Z, W) of Lie algebra on 4 generators (X, Y, Z, W) over
↪Rational Field
sage: S2 = S1.subalgebra(S1.gens()[1:]); S2
Subalgebra generated by (Z, W) of Lie algebra on 4 generators (X, Y, Z, W) over
↪Rational Field
sage: S3 = S2.subalgebra(S2.gens()[1:]); S3
Subalgebra generated by (W) of Lie algebra on 4 generators (X, Y, Z, W) over
↪Rational Field
```

An ideal of an ideal is not necessarily an ideal of the original:

```
sage: I = L.ideal(Y); I
Ideal (Y) of Lie algebra on 4 generators (X, Y, Z, W) over Rational Field
sage: J = I.ideal(Z); J
Ideal (Z) of Ideal (Y) of Lie algebra on 4 generators (X, Y, Z, W) over Rational
↪Field
```

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```
sage: J.basis()
Family (Z,)
sage: J.is_ideal(L)
False
sage: K = L.ideal(J.basis().list())
sage: K.basis()
Family (Z, W)
```

class Element

Bases: *LieSubalgebraElementWrapper*

adjoint_matrix(*sparse=False*)

Return the matrix of the adjoint action of `self`.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ, 2)
sage: m = MS([[0, -1], [1, 0]])
sage: L = LieAlgebra(associative=MS)
sage: S = L.subalgebra([m])
sage: x = S.basis()[0]
sage: x.parent() is S
True
sage: x.adjoint_matrix()
[0]

sage: m1 = MS([[0, 1], [0, 0]])
sage: m2 = MS([[0, 0], [1, 0]])
sage: S = L.subalgebra([m1, m2])
sage: e,f = S.lie_algebra_generators()
sage: ascii_art([b.value.value for b in S.basis()])
[ [0 1]  [0 0]  [-1  0] ]
[ [0 0], [1 0], [ 0  1] ]
sage: E = e.adjoint_matrix(); E
[ 0  0  2]
[ 0  0  0]
[ 0 -1  0]
sage: F = f.adjoint_matrix(); F
[ 0  0  0]
[ 0  0 -2]
[ 1  0  0]
sage: h = e.bracket(f)
sage: E * F - F * E == h.adjoint_matrix()
True
```

ambient()

Return the ambient Lie algebra of `self`.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)
sage: S = L.subalgebra(x)
sage: S.ambient() is L
True
```

basis()

Return a basis of `self`.

EXAMPLES:

A basis of a subalgebra:

```
sage: sc = {('a','b'): {'c': 1}, ('a','c'): {'d': 1}}
sage: L.<a,b,c,d> = LieAlgebra(QQ, sc)
sage: L.subalgebra([a + b, c + d]).basis()
Family (a + b, c, d)
```

A basis of an ideal:

```
sage: sc = {('x','y'): {'z': 1}, ('x','z'): {'w': 1}}
sage: L.<x,y,z,w> = LieAlgebra(QQ, sc)
sage: L.ideal([x + y + z + w]).basis()
Family (x + y, z, w)
```

This also works for Lie algebras whose natural basis elements are not comparable (but have a well-defined basis ordering):

```
sage: sl3 = LieAlgebra(QQ, cartan_type=['A',2])
sage: D = sl3.derived_subalgebra()
sage: len(D.basis())
8
sage: e = list(sl3.e())
sage: sl3.ideal(e).dimension()
8
sage: sl3.subalgebra(e).dimension()
3
```

basis_matrix()

Return the basis matrix of `self` as a submodule of the ambient Lie algebra.

EXAMPLES:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, {('X','Y'): {'Z': 3}})
sage: S1 = L.subalgebra([4*X + Y, Y])
sage: S1.basis_matrix()
[ 4  0  0]
[ 0  1  0]
[ 0  0 12]
sage: K.<X,Y,Z> = LieAlgebra(QQ, {('X','Y'): {'Z': 3}})
sage: S2 = K.subalgebra([4*X + Y, Y])
sage: S2.basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

from_vector(*v*, *order=None*, *coerce=False*)

Return the element of `self` corresponding to the vector `v`

INPUT:

- `v` – a vector in `self.module()` or `self.ambient().module()`

EXAMPLES:

An element from a vector of the intrinsic module:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, abelian=True)
sage: L.dimension()
3
sage: S = L.subalgebra([X, Y])
sage: S.dimension()
2
sage: el = S.from_vector([1, 2]); el
X + 2*Y
sage: el.parent() == S
True
```

An element from a vector of the ambient module

sage: el = S.from_vector([1, 2, 0]); el X + 2*Y sage: el.parent() == S True

gens()

Return the generating set of `self`.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z': 1}})
sage: S = L.subalgebra(x)
sage: S.gens()
(x,)
```

indices()

Return the set of indices for the basis of `self`.

EXAMPLES:

```
sage: L.<x,y,z> = LieAlgebra(QQ, abelian=True)
sage: S = L.subalgebra([x, y])
sage: S.indices()
{0, 1}
sage: [S.basis()[k] for k in S.indices()]
[x, y]
```

is_ideal(*A*)

Return if `self` is an ideal of `A`.

EXAMPLES:

Some subalgebras are ideals:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z': 1}})
sage: S1 = L.subalgebra([x])
sage: S1.is_ideal(L)
False
sage: S2 = L.subalgebra([x, y])
sage: S2.is_ideal(L)
True
sage: S3 = L.subalgebra([y, z])
```

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```
sage: S3.is_ideal(L)
True
```

All ideals are ideals:

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x': 1}})
sage: I = L.ideal(x)
sage: I.is_ideal(L)
True
sage: I.is_ideal(I)
True
```

leading_monomials()

Return the set of leading monomials of the basis of `self`.

EXAMPLES:

A basis of an ideal and the corresponding leading monomials:

```
sage: sc = {('a','b'): {'c': 2}, ('a','c'): {'d': 4}}
sage: L.<a,b,c,d> = LieAlgebra(ZZ, sc)
sage: I = L.ideal(a + b)
sage: I.basis()
Family (a + b, 2*c, 4*d)
sage: I.leading_monomials()
Family (b, c, d)
```

A different ordering can give different leading monomials:

```
sage: key = lambda s: ['d','c','b','a'].index(s)
sage: I = L.ideal(a + b, order=key)
sage: I.basis()
Family (4*d, 2*c, a + b)
sage: I.leading_monomials()
Family (d, c, a)
```

lie_algebra_generators()

Return the generating set of `self` as a Lie algebra.

EXAMPLES:

The Lie algebra generators of a subalgebra are the original generators:

```
sage: L.<x,y,z> = LieAlgebra(QQ, {('x','y'): {'z': 1}})
sage: S = L.subalgebra(x)
sage: S.lie_algebra_generators()
(x,)
```

The Lie algebra generators of an ideal is usually a larger set:

```
sage: I = L.ideal(x)
sage: I.lie_algebra_generators()
Family (x, z)
```

lift(*X*)

Coerce an element X of self into the ambient Lie algebra.

INPUT:

- X – an element of self

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)
sage: S = L.subalgebra(x)
sage: sx = S(x); sx
x
sage: sx.parent()
Subalgebra generated by (x) of Abelian Lie algebra on 2 generators (x, y) over␣
↪Rational Field
sage: a = S.lift(sx); a
x
sage: a.parent()
Abelian Lie algebra on 2 generators (x, y) over Rational Field
```

module(*sparse=False*)

Return the submodule of the ambient Lie algebra corresponding to self.

EXAMPLES:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, {('X','Y'): {'Z': 3}})
sage: S = L.subalgebra([X, Y])
sage: S.module()
Free module of degree 3 and rank 3 over Integer Ring
User basis matrix:
[1 0 0]
[0 1 0]
[0 0 3]
```

reduce(*X*)

Reduce an element of the ambient Lie algebra modulo the ideal self.

INPUT:

- X – an element of the ambient Lie algebra

OUTPUT:

An element Y of the ambient Lie algebra that is contained in a fixed complementary submodule V to self such that $X = Y$ mod self.

When the base ring of self is a field, the complementary submodule V is spanned by the elements of the basis that are not the leading supports of the basis of self.

EXAMPLES:

An example reduction in a 6 dimensional Lie algebra:

```
sage: sc = {('a','b'): {'d': 1}, ('a','c'): {'e': 1},
....:       ('b','c'): {'f': 1}}
sage: L.<a,b,c,d,e,f> = LieAlgebra(QQ, sc)
sage: I =  L.ideal(c)
```

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```
sage: I.reduce(a + b + c + d + e + f)
a + b + d
```

The reduction of an element is zero if and only if the element belongs to the subalgebra:

```
sage: I.reduce(c + e)
0
sage: c + e in I
True
```

Over non-fields, the complementary submodule may not be spanned by a subset of the basis of the ambient Lie algebra:

```
sage: L.<X,Y,Z> = LieAlgebra(ZZ, {('X','Y'): {'Z': 3}})
sage: I = L.ideal(Y)
sage: I.basis()
Family (Y, 3*Z)
sage: I.reduce(3*Z)
0
sage: I.reduce(Y + 14*Z)
2*Z
```

retract(*X*)

Retract X to self.

INPUT:

- X – an element of the ambient Lie algebra

EXAMPLES:

Retraction to a subalgebra of a free nilpotent Lie algebra:

```
sage: L = LieAlgebra(QQ, 3, step=2)
sage: L.inject_variables()
Defining X_1, X_2, X_3, X_12, X_13, X_23
sage: S = L.subalgebra([X_1, X_2])
sage: el = S.retract(2*X_1 + 3*X_2 + 5*X_12); el
2*X_1 + 3*X_2 + 5*X_12
sage: el.parent()
Subalgebra generated by (X_1, X_2) of Free Nilpotent Lie algebra on
6 generators (X_1, X_2, X_3, X_12, X_13, X_23) over Rational Field
```

Retraction raises an error if the element is not contained in the subalgebra:

```
sage: S.retract(X_3)
Traceback (most recent call last):
...
ValueError: the element X_3 is not in Subalgebra generated
by (X_1, X_2) of Free Nilpotent Lie algebra on 6 generators
(X_1, X_2, X_3, X_12, X_13, X_23) over Rational Field
```

zero()

Return the element 0.

EXAMPLES:

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)
sage: S = L.subalgebra(x)
sage: S.zero()
0
sage: S.zero() == S(L.zero())
True
```

9.2.18 Symplectic Derivation Lie Algebras

AUTHORS:

- Travis Scrimshaw (2020-10): Initial version

class sage.algebras.lie_algebras.symplectic_derivation.**SymplecticDerivationLieAlgebra**(*R*, *g*)

Bases: *InfinitelyGeneratedLieAlgebra*, IndexedGenerators

The symplectic derivation Lie algebra.

Fix a $g \geq 4$ and let R be a commutative ring. Let $H = R^{2g}$ be equipped with a symplectic form μ with the basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ such that

$$\mu(a_i, a_j) = \mu(b_i, b_j) = 0, \qquad \mu(a_i, b_j) = -\mu(b_j, a_i) = \delta_{ij},$$

for all i, j. The *symplectic derivation Lie algebra* is the Lie algebra

$$\mathfrak{c}_g := \bigoplus_{w \geq 0} S^{w+2} H$$

with the Lie bracket on basis elements

$$[x_1 \cdots x_{m+2}, y_1 \cdots y_{n+2}] = \sum_{i,j} \mu(x_i, y_j) x_1 \cdots \widehat{x}_i \cdots x_{m+2} \cdot y_1 \cdots \widehat{y}_j \cdots y_{n+2},$$

where $\widehat{z}$ denotes that factor is missing. When $R = \mathbf{Q}$, this corresponds to the classical Poisson bracket on $C^\infty(\mathbf{R}^{2g})$ restricted to polynomials with coefficients in $\mathbf{Q}$.

EXAMPLES:

```
sage: L = lie_algebras.SymplecticDerivation(QQ, 5)
sage: elts = L.some_elements()
sage: list(elts)
[a1*a2, b1*b3, a1*a1*a2, b3*b4,
 a1*a4*b3, a1*a2 - 1/2*a1*a2*a2*a5 + a1*a1*a2*b1*b4]
sage: [[elts[i].bracket(elts[j]) for i in range(len(elts))]
....:  for j in range(len(elts))]
[[0, -a2*b3, 0, 0, 0, -a1*a1*a2*a2*b4],
 [a2*b3, 0, 2*a1*a2*b3, 0, a4*b3*b3, a2*b3 - 1/2*a2*a2*a5*b3 + 2*a1*a2*b1*b3*b4],
 [0, -2*a1*a2*b3, 0, 0, 0, -2*a1*a1*a1*a2*a2*b4],
 [0, 0, 0, 0, a1*b3*b3, 0],
 [0, -a4*b3*b3, 0, -a1*b3*b3, 0, -a1*a1*a1*a2*b1*b3 - a1*a1*a2*a4*b3*b4],
 [a1*a1*a2*a2*b4, -a2*b3 + 1/2*a2*a2*a5*b3 - 2*a1*a2*b1*b3*b4, 2*a1*a1*a1*a2*a2*b4,
  0, a1*a1*a1*a2*b1*b3 + a1*a1*a2*a4*b3*b4, 0]]
sage: x = L.monomial(Partition([8,8,6,6,4,2,2,1,1,1])); x
a1*a1*a1*a2*a2*a4*b1*b1*b3*b3
```

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```
sage: [L[x, elt] for elt in elts]
[-2*a1*a1*a1*a2*a2*a2*a4*b1*b3*b3,
 3*a1*a1*a2*a2*a4*b1*b1*b3*b3*b3,
 -4*a1*a1*a1*a1*a2*a2*a2*a4*b1*b3*b3,
 a1*a1*a1*a2*a2*b1*b1*b3*b3*b3,
 -2*a1*a1*a1*a2*a2*a4*a4*b1*b3*b3*b3,
 -2*a1*a1*a1*a2*a2*a2*a4*b1*b3*b3 + a1*a1*a1*a2*a2*a2*a2*a4*a5*b1*b3*b3
  + a1*a1*a1*a1*a1*a2*a2*a2*b1*b1*b1*b3*b3 - a1*a1*a1*a1*a2*a2*a2*a4*b1*b1*b3*b3*b4]
```

REFERENCES:

- [Harako2020]

class Element

Bases: *LieAlgebraElement*

bracket_on_basis(*x*, *y*)

Return the bracket of basis elements indexed by `x` and `y`, where `i < j`.

EXAMPLES:

```
sage: L = lie_algebras.SymplecticDerivation(QQ, 5)
sage: L.bracket_on_basis([5,2,1], [5,1,1])
0
sage: L.bracket_on_basis([6,1], [3,1,1])
-2*a1*a1*a3
sage: L.bracket_on_basis([9,2,1], [4,1,1])
-a1*a1*a1*a2
sage: L.bracket_on_basis([5,5,2], [6,1,1])
0
sage: L.bracket_on_basis([5,5,5], [10,3])
3*a3*a5*a5
sage: L.bracket_on_basis([10,10,10], [5,3])
-3*a3*b5*b5
```

degree_on_basis(*x*)

Return the degree of the basis element indexed by `x`.

EXAMPLES:

```
sage: L = lie_algebras.SymplecticDerivation(QQ, 5)
sage: L.degree_on_basis([5,2,1])
1
sage: L.degree_on_basis([1,1])
0
sage: elt = L.monomial(Partition([5,5,2,1])) + 3*L.monomial(Partition([3,3,2,
↪1]))
sage: elt.degree()
2
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: L = lie_algebras.SymplecticDerivation(QQ, 5)
sage: L.some_elements()
[a1*a2, b1*b3, a1*a1*a2, b3*b4, a1*a4*b3,
 a1*a2 - 1/2*a1*a2*a2*a5 + a1*a1*a2*b1*b4]
```

9.2.19 Verma Modules

AUTHORS:

- Travis Scrimshaw (2017-06-30): Initial version

Todo: Implement a `sage.categories.pushout.ConstructionFunctor` and return as the `construction()`.

class `sage.algebras.lie_algebras.verma_module.`**`VermaModule`**(*g*, *weight*, *basis_key=None*, *prefix='f'*, ***kwds*)

Bases: `CombinatorialFreeModule`

A Verma module.

Let λ be a weight and $\mathfrak{g}$ be a Kac–Moody Lie algebra with a fixed Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}^+$. The *Verma module* M_λ is a $U(\mathfrak{g})$-module given by

$$M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} F_\lambda,$$

where F_λ is the $U(\mathfrak{b})$ module such that $h \in U(\mathfrak{h})$ acts as multiplication by $\langle \lambda, h \rangle$ and $U\mathfrak{g}^+)F_\lambda = 0$.

INPUT:

- `g` – a Lie algebra
- `weight` – a weight

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(2*La[1] + 3*La[2])
sage: pbw = M.pbw_basis()
sage: E1,E2,F1,F2,H1,H2 = [pbw(g) for g in L.gens()]
sage: v = M.highest_weight_vector()
sage: x = F2^3 * F1 * v
sage: x
f[-alpha[2]]^3*f[-alpha[1]]*v[2*Lambda[1] + 3*Lambda[2]]
sage: F1 * x
f[-alpha[2]]^3*f[-alpha[1]]^2*v[2*Lambda[1] + 3*Lambda[2]]
 + 3*f[-alpha[2]]^2*f[-alpha[1]]*f[-alpha[1] - alpha[2]]*v[2*Lambda[1] +
→3*Lambda[2]]
sage: E1 * x
2*f[-alpha[2]]^3*v[2*Lambda[1] + 3*Lambda[2]]
sage: H1 * x
3*f[-alpha[2]]^3*f[-alpha[1]]*v[2*Lambda[1] + 3*Lambda[2]]
sage: H2 * x
-2*f[-alpha[2]]^3*f[-alpha[1]]*v[2*Lambda[1] + 3*Lambda[2]]
```

REFERENCES:

- Wikipedia article Verma_module

class Element

Bases: `IndexedFreeModuleElement`

degree_on_basis(*m*)

Return the degree (or weight) of the basis element indexed by `m`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(2*La[1] + 3*La[2])
sage: v = M.highest_weight_vector()
sage: M.degree_on_basis(v.leading_support())
2*Lambda[1] + 3*Lambda[2]

sage: pbw = M.pbw_basis()
sage: G = list(pbw.gens())
sage: f1, f2 = L.f()
sage: x = pbw(f1.bracket(f2)) * pbw(f1) * v
sage: x.degree()
-Lambda[1] + 3*Lambda[2]
```

gens()

Return the generators of `self` as a $U(\mathfrak{g})$-module.

EXAMPLES:

```
sage: L = lie_algebras.sp(QQ, 6)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] - 3*La[2])
sage: M.gens()
(v[Lambda[1] - 3*Lambda[2]],)
```

highest_weight()

Return the highest weight of `self`.

EXAMPLES:

```
sage: L = lie_algebras.so(QQ, 7)
sage: La = L.cartan_type().root_system().weight_space().fundamental_weights()
sage: M = L.verma_module(4*La[1] - 3/2*La[2])
sage: M.highest_weight()
4*Lambda[1] - 3/2*Lambda[2]
```

highest_weight_vector()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: L = lie_algebras.sp(QQ, 6)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] - 3*La[2])
sage: M.highest_weight_vector()
v[Lambda[1] - 3*Lambda[2]]
```

homogeneous_component_basis(*d*)

Return a basis for the d-th homogeneous component of `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: P = L.cartan_type().root_system().weight_lattice()
sage: La = P.fundamental_weights()
sage: al = P.simple_roots()
sage: mu = 2*La[1] + 3*La[2]
sage: M = L.verma_module(mu)
sage: M.homogeneous_component_basis(mu - al[2])
[f[-alpha[2]]*v[2*Lambda[1] + 3*Lambda[2]]]
sage: M.homogeneous_component_basis(mu - 3*al[2])
[f[-alpha[2]]^3*v[2*Lambda[1] + 3*Lambda[2]]]
sage: M.homogeneous_component_basis(mu - 3*al[2] - 2*al[1])
[f[-alpha[2]]*f[-alpha[1] - alpha[2]]^2*v[2*Lambda[1] + 3*Lambda[2]],
 f[-alpha[2]]^2*f[-alpha[1]]*f[-alpha[1] - alpha[2]]*v[2*Lambda[1] +
↪3*Lambda[2]],
 f[-alpha[2]]^3*f[-alpha[1]]^2*v[2*Lambda[1] + 3*Lambda[2]]]
sage: M.homogeneous_component_basis(mu - La[1])
Family ()
```

is_singular()

Return if `self` is a singular Verma module.

A Verma module M_λ is *singular* if there does not exist a dominant weight $\tilde{\lambda}$ that is in the dot orbit of λ. We call a Verma module *regular* otherwise.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: M.is_singular()
False
sage: M = L.verma_module(La[1] - La[2])
sage: M.is_singular()
True
sage: M = L.verma_module(2*La[1] - 10*La[2])
sage: M.is_singular()
False
sage: M = L.verma_module(-2*La[1] - 2*La[2])
sage: M.is_singular()
False
sage: M = L.verma_module(-4*La[1] - La[2])
sage: M.is_singular()
True
```

lie_algebra()

Return the underlying Lie algebra of `self`.

EXAMPLES:

```
sage: L = lie_algebras.so(QQ, 9)
sage: La = L.cartan_type().root_system().weight_space().fundamental_weights()
sage: M = L.verma_module(La[3] - 1/2*La[1])
sage: M.lie_algebra()
Lie algebra of ['B', 4] in the Chevalley basis
```

pbw_basis()

Return the PBW basis of the underlying Lie algebra used to define `self`.

EXAMPLES:

```
sage: L = lie_algebras.so(QQ, 8)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[2] - 2*La[3])
sage: M.pbw_basis()
Universal enveloping algebra of Lie algebra of ['D', 4] in the Chevalley basis
 in the Poincare-Birkhoff-Witt basis
```

poincare_birkhoff_witt_basis()

Return the PBW basis of the underlying Lie algebra used to define `self`.

EXAMPLES:

```
sage: L = lie_algebras.so(QQ, 8)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[2] - 2*La[3])
sage: M.pbw_basis()
Universal enveloping algebra of Lie algebra of ['D', 4] in the Chevalley basis
 in the Poincare-Birkhoff-Witt basis
```

class sage.algebras.lie_algebras.verma_module.**VermaModuleHomset**(*X*, *Y*, *category=None*, *base=None*, *check=True*)

Bases: `Homset`

The set of morphisms from one Verma module to another considered as $U(\mathfrak{g})$-representations.

Let $M_{w \cdot \lambda}$ and $M_{w' \cdot \lambda'}$ be Verma modules, $\cdot$ is the dot action, and $\lambda + \rho$, $\lambda' + \rho$ are dominant weights. Then we have

$$\dim \hom(M_{w \cdot \lambda}, M_{w' \cdot \lambda'}) = 1$$

if and only if $\lambda = \lambda'$ and $w' \leq w$ in Bruhat order. Otherwise the homset is 0 dimensional.

Element

alias of *VermaModuleMorphism*

basis()

Return a basis of `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: Mp = L.verma_module(M.highest_weight().dot_action([2]))
sage: H = Hom(Mp, M)
```

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```
sage: list(H.basis()) == [H.natural_map()]
True

sage: Mp = L.verma_module(La[1] + 2*La[2])
sage: H = Hom(Mp, M)
sage: H.basis()
Family ()
```

dimension()

Return the dimension of `self` (as a vector space over the base ring).

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: Mp = L.verma_module(M.highest_weight().dot_action([2]))
sage: H = Hom(Mp, M)
sage: H.dimension()
1

sage: Mp = L.verma_module(La[1] + 2*La[2])
sage: H = Hom(Mp, M)
sage: H.dimension()
0
```

natural_map()

Return the "natural map" of `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: Mp = L.verma_module(M.highest_weight().dot_action([2]))
sage: H = Hom(Mp, M)
sage: H.natural_map()
Verma module morphism:
  From: Verma module with highest weight 3*Lambda[1] - 3*Lambda[2]
        of Lie algebra of ['A', 2] in the Chevalley basis
  To:   Verma module with highest weight Lambda[1] + Lambda[2]
        of Lie algebra of ['A', 2] in the Chevalley basis
  Defn: v[3*Lambda[1] - 3*Lambda[2]] |-->
        f[-alpha[2]]^2*v[Lambda[1] + Lambda[2]]

sage: Mp = L.verma_module(La[1] + 2*La[2])
sage: H = Hom(Mp, M)
sage: H.natural_map()
Verma module morphism:
  From: Verma module with highest weight Lambda[1] + 2*Lambda[2]
        of Lie algebra of ['A', 2] in the Chevalley basis
  To:   Verma module with highest weight Lambda[1] + Lambda[2]
        of Lie algebra of ['A', 2] in the Chevalley basis
```

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```
Defn: v[Lambda[1] + 2*Lambda[2]] |--> 0
```

singular_vector()

Return the singular vector in the codomain corresponding to the domain's highest weight element or None if no such element exists.

ALGORITHM:

We essentially follow the algorithm laid out in [deG2005]. We use the $\mathfrak{sl}_2$ relation on $M_{s_i \cdot \lambda} \to M_\lambda$, where $\langle \lambda + \delta, \alpha_i^\vee \rangle = m > 0$, i.e., the weight λ is i-dominant with respect to the dot action. From here, we construct the singular vector $f_i^m v_\lambda$. We iterate this until we reach μ.

EXAMPLES:

```
sage: L = lie_algebras.sp(QQ, 6)
sage: La = L.cartan_type().root_system().weight_space().fundamental_weights()
sage: la = La[1] - La[3]
sage: mu = la.dot_action([1,2])
sage: M = L.verma_module(la)
sage: Mp = L.verma_module(mu)
sage: H = Hom(Mp, M)
sage: H.singular_vector()
f[-alpha[2]]*f[-alpha[1]]^3*v[Lambda[1] - Lambda[3]]
 + 3*f[-alpha[1]]^2*f[-alpha[1] - alpha[2]]*v[Lambda[1] - Lambda[3]]
```

```
sage: L = LieAlgebra(QQ, cartan_type=['F',4])
sage: La = L.cartan_type().root_system().weight_space().fundamental_weights()
sage: la = La[1] + La[2] - La[3]
sage: mu = la.dot_action([1,2,3,2])
sage: M = L.verma_module(la)
sage: Mp = L.verma_module(mu)
sage: H = Hom(Mp, M)
sage: v = H.singular_vector()
sage: pbw = M.pbw_basis()
sage: E = [pbw(e) for e in L.e()]
sage: all(e * v == M.zero() for e in E)
True
```

When $w \cdot \lambda \notin \lambda + Q^-$, there does not exist a singular vector:

```
sage: L = lie_algebras.sl(QQ, 4)
sage: La = L.cartan_type().root_system().weight_space().fundamental_weights()
sage: la = 3/7*La[1] - 1/2*La[3]
sage: mu = la.dot_action([1,2])
sage: M = L.verma_module(la)
sage: Mp = L.verma_module(mu)
sage: H = Hom(Mp, M)
sage: H.singular_vector() is None
True
```

zero()

Return the zero morphism of self.

EXAMPLES:

```
sage: L = lie_algebras.sp(QQ, 6)
sage: La = L.cartan_type().root_system().weight_space().fundamental_weights()
sage: M = L.verma_module(La[1] + 2/3*La[2])
sage: Mp = L.verma_module(La[2] - La[3])
sage: H = Hom(Mp, M)
sage: H.zero()
Verma module morphism:
  From: Verma module with highest weight Lambda[2] - Lambda[3]
        of Lie algebra of ['C', 3] in the Chevalley basis
  To:   Verma module with highest weight Lambda[1] + 2/3*Lambda[2]
        of Lie algebra of ['C', 3] in the Chevalley basis
  Defn: v[Lambda[2] - Lambda[3]] |--> 0
```

class sage.algebras.lie_algebras.verma_module.**VermaModuleMorphism**(*parent*, *scalar*)

Bases: `Morphism`

A morphism of Verma modules.

is_injective()

Return if `self` is injective or not.

A Verma module morphism $\phi : M \to M'$ is injective if and only if $\dim \hom(M, M') = 1$ and $\phi \neq 0$.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: Mp = L.verma_module(M.highest_weight().dot_action([1,2]))
sage: Mpp = L.verma_module(M.highest_weight().dot_action([1,2]) + La[1])
sage: phi = Hom(Mp, M).natural_map()
sage: phi.is_injective()
True
sage: (0 * phi).is_injective()
False
sage: psi = Hom(Mpp, Mp).natural_map()
sage: psi.is_injective()
False
```

is_surjective()

Return if `self` is surjective or not.

A Verma module morphism is surjective if and only if the domain is equal to the codomain and it is not the zero morphism.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 3)
sage: La = L.cartan_type().root_system().weight_lattice().fundamental_weights()
sage: M = L.verma_module(La[1] + La[2])
sage: Mp = L.verma_module(M.highest_weight().dot_action([1,2]))
sage: phi = Hom(M, M).natural_map()
sage: phi.is_surjective()
True
sage: (0 * phi).is_surjective()
```

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```
False
sage: psi = Hom(Mp, M).natural_map()
sage: psi.is_surjective()
False
```

9.2.20 Virasoro Algebra and Related Lie Algebras

AUTHORS:

- Travis Scrimshaw (2013-05-03): Initial version

class sage.algebras.lie_algebras.virasoro.**ChargelessRepresentation**(*V*, *a*, *b*)

Bases: CombinatorialFreeModule

A chargeless representation of the Virasoro algebra.

Let L be the Virasoro algebra over the field F of characteristic 0. For $\alpha, \beta \in R$, we denote $V_{a,b}$ as the (a, b)-*chargeless representation* of L, which is the F-span of $\{v_k \mid k \in \mathbf{Z}\}$ with L action

$$\begin{aligned} d_n \cdot v_k &= (an + b - k)v_{n+k}, \\ c \cdot v_k &= 0, \end{aligned}$$

This comes from the action of $d_n = -t^{n+1}\frac{d}{dt}$ on $F[t, t^{-1}]$ (recall that L is the central extension of the *algebra of derivations* of $F[t, t^{-1}]$), where

$$V_{a,b} = F[t, t^{-1}]t^{a-b}(dt)^{-a}$$

and $v_k = t^{a-b+k}(dz)^{-a}$.

The chargeless representations are either irreducible or contains exactly two simple subquotients, one of which is the trivial representation and the other is $F[t, t^{-1}]/F$. The non-trivial simple subquotients are called the *intermediate series*.

The module $V_{a,b}$ is irreducible if and only if $a \neq 0, -1$ or $b \notin \mathbf{Z}$. When $a = 0$ and $b \in \mathbf{Z}$, then there exists a subrepresentation isomorphic to the trivial representation. If $a = -1$ and $b \in \mathbf{Z}$, then there exists a subrepresentation V such that $V_{a,b}/V$ is isomorphic to $K\frac{dt}{t}$ and V is irreducible.

In characteristic p, the non-trivial simple subquotient is isomorphic to $F[t, t^{-1}]/F[t^p, t^{-p}]$. For $p \neq 2, 3$, then the action is given as above.

EXAMPLES:

We first construct the irreducible $V_{1/2,3/4}$ and do some basic computations:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.chargeless_representation(1/2, 3/4)
sage: d = L.basis()
sage: v = M.basis()
sage: d[3] * v[2]
1/4*v[5]
sage: d[3] * v[-1]
13/4*v[2]
sage: (d[3] - d[-2]) * (v[-1] + 1/2*v[0] - v[4])
-3/4*v[-3] + 1/8*v[-2] - v[2] + 9/8*v[3] + 7/4*v[7]
```

We construct the reducible $V_{0,2}$ and the trivial subrepresentation given by the span of v_2. We verify this for $\{d_i \mid -10 \leq i < 10\}$:

```
sage: M = L.chargeless_representation(0, 2)
sage: v = M.basis()
sage: all(d[i] * v[2] == M.zero() for i in range(-10, 10))
True
```

REFERENCES:

- [Mat1992]
- [IK2010]

class Element

Bases: IndexedFreeModuleElement

degree_on_basis(i)

Return the degree of the basis element indexed by i, which is i.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.chargeless_representation(1/2, 3/4)
sage: M.degree_on_basis(-3)
-3
```

parameters()

Return the parameters (a, b) of self.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.chargeless_representation(1/2, 3/4)
sage: M.parameters()
(1/2, 3/4)
```

virasoro_algebra()

Return the Virasoro algebra self is a representation of.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.chargeless_representation(1/2, 3/4)
sage: M.virasoro_algebra() is L
True
```

class sage.algebras.lie_algebras.virasoro.**LieAlgebraRegularVectorFields**(R)

Bases: *InfinitelyGeneratedLieAlgebra*, IndexedGenerators

The Lie algebra of regular vector fields on $\mathbf{C}^{\times}$.

This is the Lie algebra with basis $\{d_i\}_{i \in \mathbf{Z}}$ and subject to the relations

$$[d_i, d_j] = (i - j) d_{i+j}.$$

This is also known as the Witt (Lie) algebra.

Note: This differs from some conventions (e.g., [Ka1990]), where we have $d'_i \mapsto -d_i$.

REFERENCES:

- Wikipedia article Witt_algebra

See also:

WittLieAlgebra_charp

class Element

Bases: *LieAlgebraElement*

bracket_on_basis(*i*, *j*)

Return the bracket of basis elements indexed by `x` and `y` where `x < y`.

(This particular implementation actually does not require `x < y`.)

EXAMPLES:

```
sage: L = lie_algebras.regular_vector_fields(QQ)
sage: L.bracket_on_basis(2, -2)
4*d[0]
sage: L.bracket_on_basis(2, 4)
-2*d[6]
sage: L.bracket_on_basis(4, 4)
0
```

degree_on_basis(*i*)

Return the degree of the basis element indexed by `i`, which is `i`.

EXAMPLES:

```
sage: L = lie_algebras.regular_vector_fields(QQ)
sage: L.degree_on_basis(2)
2
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.regular_vector_fields(QQ)
sage: L.lie_algebra_generators()
Lazy family (generator map(i))_{i in Integer Ring}
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: L = lie_algebras.regular_vector_fields(QQ)
sage: L.some_elements()
[d[0], d[2], d[-2], d[-1] + d[0] - 3*d[1]]
```

class sage.algebras.lie_algebras.virasoro.**VermaModule**(*V*, *c*, *h*)

Bases: `CombinatorialFreeModule`

A Verma module of the Virasoro algebra.

The Virasoro algebra admits a triangular decomposition

$$V_- \oplus R d_0 \oplus R\hat{c} \oplus V_+,$$

where V_- (resp. V_+) is the span of $\{d_i \mid i < 0\}$ (resp. $\{d_i \mid i > 0\}$). We can construct the *Verma module* $M_{c,h}$ as the induced representation of the $Rd_0 \oplus R\hat{c} \oplus V_+$ representation $R_{c,H} = Rv$, where

$$V_+ v = 0, \qquad \hat{c}v = cv, \qquad d_0 v = hv.$$

Therefore, we have a basis of $M_{c,h}$

$$\{L_{i_1} \cdots L_{i_k} v \mid i_1 \leq \cdots \leq i_k < 0\}.$$

Moreover, the Verma modules are the free objects in the category of highest weight representations of V and are indecomposable. The Verma module $M_{c,h}$ is irreducible for generic values of c and h and when it is reducible, the quotient by the maximal submodule is the unique irreducible highest weight representation $V_{c,h}$.

EXAMPLES:

We construct a Verma module and do some basic computations:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.verma_module(3, 0)
sage: d = L.basis()
sage: v = M.highest_weight_vector()
sage: d[3] * v
0
sage: d[-3] * v
d[-3]*v
sage: d[-1] * (d[-3] * v)
2*d[-4]*v + d[-3]*d[-1]*v
sage: d[2] * (d[-1] * (d[-3] * v))
12*d[-2]*v + 5*d[-1]*d[-1]*v
```

We verify that $d_{-1}v$ is a singular vector for $\{d_i \mid 1 \leq i < 20\}$:

```
sage: w = M.basis()[-1]; w
d[-1]*v
sage: all(d[i] * w == M.zero() for i in range(1,20))
True
```

We also verify a singular vector for $V_{-2,1}$:

```
sage: M = L.verma_module(-2, 1)
sage: B = M.basis()
sage: w = B[-1,-1] - 2 * B[-2]
sage: d = L.basis()
sage: all(d[i] * w == M.zero() for i in range(1,20))
True
```

REFERENCES:

- Wikipedia article Virasoro_algebra#Representation_theory

class Element

Bases: `IndexedFreeModuleElement`

central_charge()

Return the central charge of `self`.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.verma_module(3, 0)
sage: M.central_charge()
3
```

conformal_weight()

Return the conformal weight of `self`.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.verma_module(3, 0)
sage: M.conformal_weight()
3
```

degree_on_basis(*d*)

Return the degree of the basis element indexed by `d`, which is the sum of the entries of `d`.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.verma_module(-2/7, 3)
sage: M.degree_on_basis((-3,-3,-1))
-7
```

highest_weight_vector()

Return the highest weight vector of `self`.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.verma_module(-2/7, 3)
sage: M.highest_weight_vector()
v
```

virasoro_algebra()

Return the Virasoro algebra `self` is a representation of.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: M = L.verma_module(1/2, 3/4)
sage: M.virasoro_algebra() is L
True
```

class sage.algebras.lie_algebras.virasoro.**VirasoroAlgebra**(*R*)

Bases: *InfinitelyGeneratedLieAlgebra*, IndexedGenerators

The Virasoro algebra.

This is the Lie algebra with basis $\{d_i\}_{i\in\mathbf{Z}} \cup \{c\}$ and subject to the relations

$$[d_i, d_j] = (i-j)d_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j}c$$

and

$$[d_i, c] = 0.$$

(Here, it is assumed that the base ring R has 2 invertible.)

This is the universal central extension $\widetilde{\mathfrak{d}}$ of the Lie algebra $\mathfrak{d}$ of *regular vector fields* on $\mathbf{C}^\times$.

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
```

REFERENCES:

- Wikipedia article Virasoro_algebra

class Element

Bases: *LieAlgebraElement*

basis()

Return a basis of self.

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
sage: B = d.basis(); B
Lazy family (basis map(i))_{i in Disjoint union of
                            Family ({'c'}, Integer Ring)}
sage: B['c']
c
sage: B[3]
d[3]
sage: B[-15]
d[-15]
```

bracket_on_basis(*i*, *j*)

Return the bracket of basis elements indexed by x and y where x < y.

(This particular implementation actually does not require x < y.)

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
sage: d.bracket_on_basis('c', 2)
0
sage: d.bracket_on_basis(2, -2)
4*d[0] + 1/2*c
```

c()

The central element c in self.

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
sage: d.c()
c
```

chargeless_representation(a, b)

Return the chargeless representation of `self` with parameters `a` and `b`.

See also:

ChargelessRepresentation

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: L.chargeless_representation(3, 2)
Chargeless representation (3, 2) of
 The Virasoro algebra over Rational Field
```

d(i)

Return the element d_i in `self`.

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: L.d(2)
d[2]
```

degree_on_basis(i)

Return the degree of the basis element indexed by `i`, which is `i` and 0 for `'c'`.

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
sage: d.degree_on_basis(2)
2
sage: d.c().degree()
0
sage: (d.c() + d.basis()[0]).is_homogeneous()
True
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
sage: d.lie_algebra_generators()
Lazy family (generator map(i))_{i in Integer Ring}
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: d = lie_algebras.VirasoroAlgebra(QQ)
sage: d.some_elements()
[d[0], d[2], d[-2], c, d[-1] + d[0] - 1/2*d[1] + c]
```

verma_module(c, h)

Return the Verma module with central charge `c` and conformal (or highest) weight `h`.

See also:

VermaModule

EXAMPLES:

```
sage: L = lie_algebras.VirasoroAlgebra(QQ)
sage: L.verma_module(3, 2)
Verma module with charge 3 and conformal weight 2 of
 The Virasoro algebra over Rational Field
```

class sage.algebras.lie_algebras.virasoro.**WittLieAlgebra_charp**(*R*, *p*)

Bases: *FinitelyGeneratedLieAlgebra*, IndexedGenerators

The p-Witt Lie algebra over a ring R in which $p \cdot 1_R = 0$.

Let R be a ring and p be a positive integer such that $p \cdot 1_R = 0$. The p-Witt Lie algebra over R is the Lie algebra with basis $\{d_0, d_1, \ldots, d_{p-1}\}$ and subject to the relations

$$[d_i, d_j] = (i - j) d_{i+j},$$

where the $i + j$ on the right hand side is identified with its remainder modulo p.

See also:

LieAlgebraRegularVectorFields

class Element

Bases: *LieAlgebraElement*

bracket_on_basis(*i*, *j*)

Return the bracket of basis elements indexed by x and y where x < y.

(This particular implementation actually does not require x < y.)

EXAMPLES:

```
sage: L = lie_algebras.pwitt(Zmod(5), 5)
sage: L.bracket_on_basis(2, 3)
4*d[0]
sage: L.bracket_on_basis(3, 2)
d[0]
sage: L.bracket_on_basis(2, 2)
0
sage: L.bracket_on_basis(1, 3)
3*d[4]
```

degree_on_basis(*i*)

Return the degree of the basis element indexed by i, which is i mod p.

EXAMPLES:

```
sage: L = lie_algebras.pwitt(Zmod(5), 5)
sage: L.degree_on_basis(7)
2
sage: L.degree_on_basis(2).parent()
Ring of integers modulo 5
```

lie_algebra_generators()

Return the generators of `self` as a Lie algebra.

EXAMPLES:

```
sage: L = lie_algebras.pwitt(Zmod(5), 5)
sage: L.lie_algebra_generators()
Finite family {0: d[0], 1: d[1], 2: d[2], 3: d[3], 4: d[4]}
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: L = lie_algebras.pwitt(Zmod(5), 5)
sage: L.some_elements()
[d[0], d[2], d[3], d[0] + 2*d[1] + d[4]]
```

9.3 Lie Conformal Algebras

9.3.1 The main classes to work with Lie conformal algebras

Lie Conformal Algebra

Let R be a commutative ring, a *super Lie conformal algebra* [Kac1997] over R (also known as a *vertex Lie algebra*) is an $R[T]$ super module L together with a $\mathbb{Z}/2\mathbb{Z}$-graded R-bilinear operation (called the λ-bracket) $L \otimes L \to L[\lambda]$ (polynomials in λ with coefficients in L), $a \otimes b \mapsto [a_\lambda b]$ satisfying

1. Sesquilinearity:

$$[Ta_\lambda b] = -\lambda[a_\lambda b], \qquad [a_\lambda Tb] = (\lambda + T)[a_\lambda b].$$

2. Skew-Symmetry:

$$[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-T}a],$$

   where $p(a)$ is 0 if a is *even* and 1 if a is *odd*. The bracket in the RHS is computed as follows. First we evaluate $[b_\mu a]$ with the formal parameter μ to the *left*, then replace each appearance of the formal variable μ by $-\lambda - T$. Finally apply T to the coefficients in L.

3. Jacobi identity:

$$[a_\lambda[b_\mu c]] = [[a_{\lambda+\mu}b]_\mu c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]],$$

   which is understood as an equality in $L[\lambda, \mu]$.

   T is usually called the *translation operation* or the *derivative*. For an element $a \in L$ we will say that Ta is the *derivative of* a. We define the *n-th products* $a_{(n)}b$ for $a, b \in L$ by

$$[a_\lambda b] = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)}b.$$

   A Lie conformal algebra is called *H-Graded* [DSK2006] if there exists a decomposition $L = \oplus L_n$ such that the λ-bracket becomes graded of degree -1, that is:

$$a_{(n)}b \in L_{p+q-n-1} \qquad a \in L_p,\ b \in L_q,\ n \geq 0.$$

In particular this implies that the action of T increases degree by 1.

Note: In the literature arbitrary gradings are allowed. In this implementation we only support non-negative rational gradings.

EXAMPLES:

1. The **Virasoro** Lie conformal algebra Vir over a ring R where 12 is invertible has two generators L, C as an $R[T]$-module. It is the direct sum of a free module of rank 1 generated by L, and a free rank one R module generated by C satisfying $TC = 0$. C is central (the λ-bracket of C with any other vector vanishes). The remaining λ-bracket is given by

$$[L_\lambda L] = TL + 2\lambda L + \frac{\lambda^3}{12}C.$$

2. The **affine** or current Lie conformal algebra $L(\mathfrak{g})$ associated to a finite dimensional Lie algebra $\mathfrak{g}$ with non-degenerate, invariant R-bilinear form $(,)$ is given as a central extension of the free $R[T]$ module generated by $\mathfrak{g}$ by a central element K. The λ-bracket of generators is given by

$$[a_\lambda b] = [a, b] + \lambda(a, b)K, \qquad a, b \in \mathfrak{g}$$

3. The **Weyl** Lie conformal algebra, or $\beta - \gamma$ system is given as the central extension of a free $R[T]$ module with two generators β and γ, by a central element K. The only non-trivial brackets among generators are

$$[\beta_\lambda \gamma] = -[\gamma_\lambda \beta] = K$$

4. The **Neveu-Schwarz** super Lie conformal algebra is a super Lie conformal algebra which is an extension of the Virasoro Lie conformal algebra. It consists of a Virasoro generator L as in example 1 above and an *odd* generator G. The remaining brackets are given by:

$$[L_\lambda G] = \left(T + \frac{3}{2}\lambda\right) G \qquad [G_\lambda G] = 2L + \frac{\lambda^2}{3}C$$

See also:

sage.algebras.lie_conformal_algebras.examples

The base class for all Lie conformal algebras is *LieConformalAlgebra*. All subclasses are called through its method `__classcall_private__`. This class provides no functionality besides calling the appropriate constructor.

We provide some convenience classes to define named Lie conformal algebras. See *sage.algebras.lie_conformal_algebras.examples*.

EXAMPLES:

- We construct the Virasoro Lie conformal algebra, its universal enveloping vertex algebra and lift some elements:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)
sage: Vir.inject_variables()
Defining L, C
sage: L.bracket(L)
{0: TL, 1: 2*L, 3: 1/2*C}
```

- We construct the Current algebra for $\mathfrak{sl}_2$:

```
sage: R = lie_conformal_algebras.Affine(QQ, 'A1', names = ('e', 'h', 'f'))
sage: R.gens()
(e, h, f, K)
sage: R.inject_variables()
Defining e, h, f, K
sage: e.bracket(f.T())
{0: Th, 1: h, 2: 2*K}
sage: e.T(3)
6*T^(3)e
```

- We construct the $\beta - \gamma$ system by directly giving the λ-brackets of the generators:

```
sage: betagamma_dict = {('b','a'):{0:{('K',0):1}}}
sage: V = LieConformalAlgebra(QQ, betagamma_dict, names=('a','b'), weights=(1,0),
↪central_elements=('K',))
sage: V.category()
Category of H-graded finitely generated Lie conformal algebras with basis over
↪Rational Field
sage: V.inject_variables()
Defining a, b, K
sage: a.bracket(b)
{0: -K}
```

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.lie_conformal_algebra.**LieConformalAlgebra**

Bases: UniqueRepresentation, Parent

Lie Conformal Algebras base class and factory.

INPUT:

- R – a commutative ring (default: None); the base ring of this Lie conformal algebra. Behaviour is undefined if it is not a field of characteristic zero.
- arg0 – a dictionary (default: None); a dictionary containing the λ brackets of the generators of this Lie conformal algebra. The keys of this dictionary are pairs of either names or indices of the generators and the values are themselves dictionaries. For a pair of generators 'a' and 'b', the value of arg0[('a', 'b')] is a dictionary whose keys are positive integer numbers and the corresponding value for the key j is a dictionary itself representing the j-th product $a_{(j)}b$. Thus, for a positive integer number j, the value of arg0[('a','b')][j] is a dictionary whose entries are pairs ('c',n) where 'c' is the name of a generator and n is a positive number. The value for this key is the coefficient of $\frac{T^n}{n!}c$ in $a_{(j)}b$. For example the arg0 for the *Virasoro* Lie conformal algebra is:

```
{('L','L'):{0:{('L',1):1}, 1:{('L',0):2}, 3:{('C',0):1/2}}}
```

  Do not include central elements as keys in this dictionary. Also, if the key ('a','b') is present, there is no need to include ('b','a') as it is defined by skew-symmetry. Any missing pair (besides the ones defined by skew-symmetry) is assumed to have vanishing λ-bracket.

- names – tuple of str (default: None); the list of names for generators of this Lie conformal algebra. Do not include central elements in this list.
- central_elements – tuple of str (default: None); A list of names for central elements of this Lie conformal algebra.

- `index_set` – enumerated set (default: `None`); an indexing set for the generators of this Lie conformal algebra. Do not include central elements in this list.
- `weights` – tuple of non-negative rational numbers (default: `None`); a list of degrees for this Lie conformal algebra. The returned Lie conformal algebra is H-Graded. This tuple needs to have the same cardinality as `index_set` or `names`. Central elements are assumed to have weight 0.
- `parity` – tuple of 0 or 1 (default: tuple of 0); if this is a super Lie conformal algebra, this tuple specifies the parity of each of the non-central generators of this Lie conformal algebra. Central elements are assumed to be even. Notice that if this tuple is present, the category of this Lie conformal algebra is set to be a subcategory of `LieConformalAlgebras(R).Super()`, even if all generators are even.
- `category` The category that this Lie conformal algebra belongs to.

In addition we accept the following keywords:

- `graded` – a boolean (default: `False`); if `True`, the returned algebra is H-Graded. If `weights` is not specified, all non-central generators are assigned degree 1. This keyword is ignored if `weights` is specified
- `super` – a boolean (default: `False`); if `True`, the returned algebra is a super Lie conformal algebra even if all generators are even. If `parity` is not specified, all generators are assigned even parity. This keyword is ignored if `parity` is specified.

Note: Any remaining keyword is currently passed to `CombinatorialFreeModule`.

EXAMPLES:

We construct the $\beta - \gamma$ system or *Weyl* Lie conformal algebra:

```
sage: betagamma_dict = {('b','a'):{0:{('K',0):1}}}
sage: V = LieConformalAlgebra(QQbar, betagamma_dict, names=('a','b'), weights=(1,0),
↪ central_elements=('K',))
sage: V.category()
Category of H-graded finitely generated Lie conformal algebras with basis over␣
↪Algebraic Field
sage: V.inject_variables()
Defining a, b, K
sage: a.bracket(b)
{0: -K}
```

We construct the current algebra for $\mathfrak{sl}_2$:

```
sage: sl2dict = {('e','f'):{0:{('h',0):1}, 1:{('K',0):1}}, ('e','h'):{0:{('e',0):-2}
↪}, ('f','h'):{0:{('f',0):2}}, ('h', 'h'):{1:{('K', 0):2}}}
sage: V = LieConformalAlgebra(QQ, sl2dict, names=('e', 'h', 'f'), central_elements=(
↪'K',), graded=True)
sage: V.inject_variables()
Defining e, h, f, K
sage: e.bracket(f)
{0: h, 1: K}
sage: h.bracket(e)
{0: 2*e}
sage: e.bracket(f.T())
{0: Th, 1: h, 2: 2*K}
sage: V.category()
Category of H-graded finitely generated Lie conformal algebras with basis over␣
```

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```
↪Rational Field
sage: e.degree()
1
```

Todo: This class checks that the provided dictionary is consistent with skew-symmetry. It does not check that it is consistent with the Jacobi identity.

See also:

sage.algebras.lie_conformal_algebras.graded_lie_conformal_algebra

Examples of Lie Conformal Algebras

We implement the following examples of Lie conformal algebras:

- *Abelian Lie conformal algebra*
- *Affine Lie conformal algebra*
- *Bosonic Ghosts*
- *Fermionic Ghosts*
- *Free Bosons*
- *Free Fermions*
- *N=2 super Lie Conformal algebra*
- *Neveu-Schwarz super Lie conformal algebra*
- *Virasoro Lie conformal algebra*
- *Weyl Lie conformal algebra*

AUTHORS:

- Reimundo Heluani (2020-06-15): Initial implementation.

Lie Conformal Algebra Element

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.lie_conformal_algebra_element.**LCAStructureCoefficientsElement**

Bases: *LCAWithGeneratorsElement*

An element of a Lie conformal algebra given by structure coefficients.

class sage.algebras.lie_conformal_algebras.lie_conformal_algebra_element.**LCAWithGeneratorsElement**

Bases: IndexedFreeModuleElement

The element class of a Lie conformal algebra with a preferred set of generators.

T(*n=1*)

The n-th derivative of this element.

INPUT:

- n – a non-negative integer (default:1); how many times to apply T to this element.

We use the *divided powers* notation $T^{(j)} = \frac{T^j}{j!}$.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)
sage: Vir.inject_variables()
Defining L, C
sage: L.T()
TL
sage: L.T(3)
6*T^(3)L
sage: C.T()
0

sage: R = lie_conformal_algebras.NeveuSchwarz(QQbar); R.inject_variables()
Defining L, G, C
sage: (L + 2*G.T() + 4*C).T(2)
2*T^(2)L + 12*T^(3)G
```

is_monomial()

Whether this element is a monomial.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ); L = Vir.0
sage: (L + L.T()).is_monomial()
False
sage: L.T().is_monomial()
True
```

See also:

The Category of Lie Conformal Algebras

9.3.2 Implemented examples of Lie Conformal Algebras

Abelian Lie Conformal Algebra

For a commutative ring R and a free R-module M. The *Abelian Lie conformal algebra* generated by M is the free $R[T]$ module generated by M with vanishing λ-brackets.

AUTHORS:

- Reimundo Heluani (2020-06-15): Initial implementation.

class sage.algebras.lie_conformal_algebras.abelian_lie_conformal_algebra.**AbelianLieConformalAlgebra**(*R*, *ngens*, *weights*, *parity*, *names*, *index_*

Bases: *GradedLieConformalAlgebra*

The Abelian Lie conformal algebra.

INPUT:

- R – a commutative ring; the base ring of this Lie conformal algebra
- ngens – a positive integer (default: 1); the number of generators of this Lie conformal algebra
- weights – a list of positive rational numbers (default: 1 for each generator); the weights of the generators. The resulting Lie conformal algebra is H-graded.
- parity – None or a list of 0 or 1 (default: None); The parity of the generators. If not None the resulting Lie Conformal algebra is a Super Lie conformal algebra
- names – a tuple of str or None (default: None); the list of names of the generators of this algebra.
- index_set – an enumerated set or None (default: None); A set indexing the generators of this Lie conformal algebra.

OUTPUT:

The Abelian Lie conformal algebra with generators $a_i, i = 1, ..., n$ and vanishing λ-brackets, where n is ngens.

EXAMPLES:

```
sage: R = lie_conformal_algebras.Abelian(QQ,2); R
The Abelian Lie conformal algebra with generators (a0, a1) over Rational Field
sage: R.inject_variables()
Defining a0, a1
sage: a0.bracket(a1.T(2))
{}
```

Todo: implement its own class to speed up arithmetics in this case.

Affine Lie Conformal Algebra

The affine Kac-Moody Lie conformal algebra associated to the finite dimensional simple Lie algebra $\mathfrak{g}$. For a commutative ring R, it is the $R[T]$-module freely generated by $\mathfrak{g}$ plus a central element K satisfying $TK = 0$. The non-vanishing λ-brackets are given by

$$[a_\lambda b] = [a, b] + \lambda(a, b)K,$$

where $a, b \in \mathfrak{g}$ and (a, b) is the normalized form of $\mathfrak{g}$ so that its longest root has square-norm 2.

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.affine_lie_conformal_algebra.**AffineLieConformalAlgebra**(*R*, *ct*, *names=*, *pre-fix=No*, *bracke*

Bases: *GradedLieConformalAlgebra*

The current or affine Kac-Moody Lie conformal algebra.

INPUT:

- R – a commutative Ring; the base ring for this Lie conformal algebra.
- ct – a str or a CartanType; the Cartan Type for the corresponding finite dimensional Lie algebra. It must correspond to a simple finite dimensional Lie algebra.
- names – a list of str or None (default: None) ; alternative names for the generators. If None the generators are labeled by the corresponding root and coroot vectors.
- prefix – a str; parameter passed to IndexedGenerators
- bracket – a str; parameter passed to IndexedGenerators.

EXAMPLES:

```
sage: R = lie_conformal_algebras.Affine(QQ, 'A1')
sage: R
The affine Lie conformal algebra of type ['A', 1] over Rational Field
sage: R.an_element()
B[alpha[1]] + B[alphacheck[1]] + B[-alpha[1]] + B['K']

sage: R = lie_conformal_algebras.Affine(QQ, 'A1', names = ('e', 'h','f'))
sage: R.inject_variables()
Defining e, h, f, K
sage: Family(e.bracket(f.T(3)))
Finite family {0: 6*T^(3)h, 1: 6*T^(2)h, 2: 6*Th, 3: 6*h, 4: 24*K}

sage: V = lie_conformal_algebras.Affine(QQ, CartanType(["A",2,1]))
Traceback (most recent call last):
...
ValueError: only affine algebras of simple finite dimensionalLie algebras are␣
↪implemented
```

OUTPUT:

The Affine Lie conformal algebra associated with the finite dimensional simple Lie algebra of Cartan type ct.

cartan_type()

The Cartan type of this Lie conformal algbera.

EXAMPLES:

```
sage: R = lie_conformal_algebras.Affine(QQ, 'B3')
sage: R
The affine Lie conformal algebra of type ['B', 3] over Rational Field
sage: R.cartan_type()
['B', 3]
```

Bosonic Ghosts Lie Conformal Algebra

The *Bosonic-ghosts* or $\beta - \gamma$-system Lie conformal algebra with $2n$ generators is the H-graded Lie conformal algebra generated by $\beta_i, \gamma_i, i = 1, \ldots, n$ and a central element K, with non-vanishing λ-brackets:

$$[{\beta_i}_\lambda \gamma_j] = \delta_{ij} K.$$

The generators β_i have degree 1 while the generators γ_i have degree 0.

AUTHORS:

- Reimundo Heluani (2020-06-15): Initial implementation.

class sage.algebras.lie_conformal_algebras.bosonic_ghosts_lie_conformal_algebra.**BosonicGhostsLieConforma**

Bases: *GradedLieConformalAlgebra*

The Bosonic ghosts or $\beta - \gamma$-system Lie conformal algebra.

INPUT:

- R – a commutative ring.
- ngens – an even positive Integer (default: 2); the number of non-central generators of this Lie conformal algebra.
- names – a list of str; alternative names for the generators
- index_set – an enumerated set; An indexing set for the generators.

OUTPUT:

The Bosonic Ghosts Lie conformal algebra with generators $\beta_i, \gamma_i, i = 1, \ldots, n$ and K, where $2n$ is ngens.

EXAMPLES:

```
sage: R = lie_conformal_algebras.BosonicGhosts(QQ); R
The Bosonic ghosts Lie conformal algebra with generators (beta, gamma, K) over␣
↪Rational Field
sage: R.inject_variables(); beta.bracket(gamma)
Defining beta, gamma, K
{0: K}
sage: beta.degree()
1
sage: gamma.degree()
0

sage: R = lie_conformal_algebras.BosonicGhosts(QQbar, ngens = 4, names = 'abcd'); R
The Bosonic ghosts Lie conformal algebra with generators (a, b, c, d, K) over␣
↪Algebraic Field
sage: R.structure_coefficients()
Finite family {('a', 'c'): ((0, K),),  ('b', 'd'): ((0, K),),  ('c', 'a'): ((0, -K),
↪),  ('d', 'b'): ((0, -K),)}
```

Fermionic Ghosts Super Lie Conformal Algebra

The *Fermionic-ghosts* or b–c system super Lie conformal algebra with $2n$ generators is the H-graded super Lie conformal algebra generated by odd vectors $b_i, c_i, i = 1, \ldots, n$ and a central element K, with non-vanishing λ-brackets:

$$[{b_i}_\lambda c_j] = \delta_{ij} K.$$

The generators b_i have degree 1 while the generators c_i have degree 0.

AUTHORS:

- Reimundo Heluani (2020-06-03): Initial implementation.

class sage.algebras.lie_conformal_algebras.fermionic_ghosts_lie_conformal_algebra.**FermionicGhostsLieCon**

Bases: *GradedLieConformalAlgebra*

The Fermionic ghosts or bc-system super Lie conformal algebra.

INPUT:

- `R` – a commutative ring; the base ring of this Lie conformal algebra
- `ngens` – an even positive Integer (default: `2`); The number of non-central generators of this Lie conformal algebra.
- `names` – a tuple of `str`; alternative names for the generators
- `index_set` – an enumerated set; alternative indexing set for the generators.

OUTPUT:

The Fermionic Ghosts super Lie conformal algebra with generators $b_i, c_i, i = 1, \ldots, n$ and K where $2n$ is `ngens`.

EXAMPLES:

```
sage: R = lie_conformal_algebras.FermionicGhosts(QQ); R
The Fermionic ghosts Lie conformal algebra with generators (b, c, K) over Rational␣
↪Field
sage: R.inject_variables()
Defining b, c, K
sage: b.bracket(c) == c.bracket(b)
True
sage: b.degree()
1
sage: c.degree()
0
sage: R.category()
Category of H-graded super finitely generated Lie conformal algebras with basis␣
↪over Rational Field

sage: R = lie_conformal_algebras.FermionicGhosts(QQbar, ngens=4, names = 'abcd');R
The Fermionic ghosts Lie conformal algebra with generators (a, b, c, d, K) over␣
↪Algebraic Field
sage: R.structure_coefficients()
Finite family {('a', 'c'): ((0, K),),  ('b', 'd'): ((0, K),),  ('c', 'a'): ((0, K),
↪),  ('d', 'b'): ((0, K),)}
```

Free Bosons Lie Conformal Algebra

Given an R-module M with a symmetric, bilinear pairing $(\cdot, \cdot) : M \otimes_R M \to R$. The *Free Bosons* Lie conformal algebra associated to this datum is the free $R[T]$-module generated by M plus a central vector K satisfying $TK = 0$. The remaining λ-brackets are given by:

$$[v_\lambda w] = \lambda(v, w)K,$$

where $v, w \in M$.

This is an H-graded Lie conformal algebra where every generator $v \in M$ has degree 1.

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.free_bosons_lie_conformal_algebra.**FreeBosonsLieConformalAlge**

Bases: *GradedLieConformalAlgebra*

The Free Bosons Lie conformal algebra.

INPUT:

- R – a commutative ring.
- ngens – a positive Integer (default 1); the number of non-central generators of this Lie conformal algebra.
- gram_matrix: a symmetric square matrix with coefficients in R (default: identity_matrix(ngens)); the Gram matrix of the inner product
- names – a tuple of str; alternative names for the generators
- index_set – an enumerated set; alternative indexing set for the generators.

OUTPUT:

The Free Bosons Lie conformal algebra with generators
α_i, $i = 1, ..., n$ and λ-brackets

$$[\alpha_{i\lambda}\alpha_j] = \lambda M_{ij} K,$$

where n is the number of generators ngens and M is the gram_matrix. This Lie conformal algebra is H-graded where every generator has conformal weight 1.

EXAMPLES:

```
sage: R = lie_conformal_algebras.FreeBosons(AA); R
The free Bosons Lie conformal algebra with generators (alpha, K) over Algebraic␣
↪Real Field
sage: R.inject_variables()
Defining alpha, K
sage: alpha.bracket(alpha)
{1: K}
sage: M = identity_matrix(QQ,2); R = lie_conformal_algebras.FreeBosons(QQ,gram_
↪matrix=M, names='alpha,beta'); R
```

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```
The free Bosons Lie conformal algebra with generators (alpha, beta, K) over␣
↪Rational Field
sage: R.inject_variables(); alpha.bracket(beta)
Defining alpha, beta, K
{}
sage: alpha.bracket(alpha)
{1: K}
sage: R = lie_conformal_algebras.FreeBosons(QQbar, ngens=3); R
The free Bosons Lie conformal algebra with generators (alpha0, alpha1, alpha2, K)␣
↪over Algebraic Field
```

gram_matrix()

The Gram matrix that specifies the λ-brackets of the generators.

EXAMPLES:

```
sage: R = lie_conformal_algebras.FreeBosons(QQ,ngens=2);
sage: R.gram_matrix()
[1 0]
[0 1]
```

Free Fermions Super Lie Conformal Algebra.

Given an R-module M with a skew-symmetric, bilinear pairing $\langle\cdot,\cdot\rangle : M \otimes_R M \to R$. The *Free Fermions* super Lie conformal algebra associated to this datum is the free $R[T]$-super module generated by ΠM (a purely odd copy of M) plus a central vector K satisfying $TK = 0$. The remaining λ-brackets are given by:

$$[v_\lambda w] = \langle v, w\rangle K,$$

where $v, w \in M$.

This is an H-graded Lie conformal algebra where every generator $v \in M$ has degree $1/2$.

AUTHORS:

- Reimundo Heluani (2020-06-03): Initial implementation.

class sage.algebras.lie_conformal_algebras.free_fermions_lie_conformal_algebra.**FreeFermionsLieConformal**

Bases: *GradedLieConformalAlgebra*

The Free Fermions Super Lie conformal algebra.

INPUT:

- R: a commutative ring.
- ngens: a positive Integer (default 1); the number of non-central generators of this Lie conformal algebra.
- gram_matrix: a symmetric square matrix with coefficients in R (default: identity_matrix(ngens)); the Gram matrix of the inner product

OUTPUT:

The Free Fermions Lie conformal algebra with generators
ψ_i, $i = 1, ..., n$ and λ-brackets

$$[{\psi_i}_\lambda \psi_j] = M_{ij} K,$$

where n is the number of generators `ngens` and M is the `gram_matrix`. This super Lie conformal algebra is H-graded where every generator has degree $1/2$.

EXAMPLES:

```
sage: R = lie_conformal_algebras.FreeFermions(QQbar); R
The free Fermions super Lie conformal algebra with generators (psi, K) over␣
↪Algebraic Field
sage: R.inject_variables()
Defining psi, K
sage: psi.bracket(psi)
{0: K}

sage: R = lie_conformal_algebras.FreeFermions(QQbar,gram_matrix=Matrix([[0,1],[1,
↪0]])); R
The free Fermions super Lie conformal algebra with generators (psi_0, psi_1, K)␣
↪over Algebraic Field
sage: R.inject_variables()
Defining psi_0, psi_1, K
sage: psi_0.bracket(psi_1)
{0: K}
sage: psi_0.degree()
1/2
sage: R.category()
Category of H-graded super finitely generated Lie conformal algebras with basis␣
↪over Algebraic Field
```

gram_matrix()

The Gram matrix that specifies the λ-brackets of the generators.

EXAMPLES:

```
sage: R = lie_conformal_algebras.FreeFermions(QQ,ngens=2);
sage: R.gram_matrix()
[1 0]
[0 1]
```

N=2 Super Lie Conformal Algebra

The $N = 2$ super Lie conformal algebra is an extension of the Virasoro Lie conformal algebra (with generators L, C) by an even generator J which is primary of conformal weight 1 and two odd generators G_1, G_2 which are primary of conformal weight $3/2$. The remaining λ-brackets are given by:

$$\begin{aligned}[J_\lambda J] &= \frac{\lambda}{3} C,\\ [J_\lambda G_1] &= G_1,\\ [J_\lambda G_2] &= -G_2,\\ [{G_1}_\lambda G_1] &= [{G_2}_\lambda G_2] = 0,\\ [{G_1}_\lambda G_2] &= L + \frac{1}{2} TJ + \lambda J + \frac{\lambda^2}{6} C.\end{aligned}$$

AUTHORS:

- Reimundo Heluani (2020-06-03): Initial implementation.

class sage.algebras.lie_conformal_algebras.n2_lie_conformal_algebra.**N2LieConformalAlgebra**(*R*)

Bases: *GradedLieConformalAlgebra*

The N=2 super Lie conformal algebra.

INPUT:

- R – a commutative ring; the base ring of this super Lie conformal algebra.

EXAMPLES:

```
sage: x = polygen(ZZ, 'x')
sage: F.<x> = NumberField(x^2 - 2)
sage: R = lie_conformal_algebras.N2(F); R
The N=2 super Lie conformal algebra over Number Field in x with defining polynomial␣
↪x^2 - 2
sage: R.inject_variables()
Defining L, J, G1, G2, C
sage: G1.bracket(G2)
{0: L + 1/2*TJ, 1: J, 2: 1/3*C}
sage: G2.bracket(G1)
{0: L - 1/2*TJ, 1: -J, 2: 1/3*C}
sage: G1.degree()
3/2
sage: J.degree()
1
```

The topological twist is a Virasoro vector with central charge 0:

```
sage: L2 = L - 1/2*J.T()
sage: L2.bracket(L2) == {0: L2.T(), 1: 2*L2}
True
```

The sum of the fermions is a generator of the Neveu-Schwarz Lie conformal algebra:

```
sage: G = (G1 + G2)
sage: G.bracket(G)
{0: 2*L, 2: 2/3*C}
```

Neveu-Schwarz Super Lie Conformal Algebra

The $N = 1$ or *Neveu-Schwarz* super Lie conformal algebra is a super extension of the Virasoro Lie conformal algebra with generators L and C by an odd primary generator G of conformal weight $3/2$. The remaining λ-bracket is given by:

$$[G_\lambda G] = 2L + \frac{\lambda^2}{3}C.$$

AUTHORS:

- Reimundo Heluani (2020-06-03): Initial implementation.

class sage.algebras.lie_conformal_algebras.neveu_schwarz_lie_conformal_algebra.**NeveuSchwarzLieConformal**

Bases: *GradedLieConformalAlgebra*

The Neveu-Schwarz super Lie conformal algebra.

INPUT:

- R – a commutative Ring; the base ring of this Lie conformal algebra.

EXAMPLES:

```
sage: R = lie_conformal_algebras.NeveuSchwarz(AA); R
The Neveu-Schwarz super Lie conformal algebra over Algebraic Real Field
sage: R.structure_coefficients()
Finite family {('G', 'G'): ((0, 2*L), (2, 2/3*C)),  ('G', 'L'): ((0, 1/2*TG), (1, 3/
↪2*G)),  ('L', 'G'): ((0, TG), (1, 3/2*G)),  ('L', 'L'): ((0, TL), (1, 2*L), (3, 1/
↪2*C))}
sage: R.inject_variables()
Defining L, G, C
sage: G.nproduct(G,0)
2*L
sage: G.degree()
3/2
```

Virasoro Lie Conformal Algebra

The Virasoro Lie conformal algebra is generated by L and a central element C. The λ-brackets are given by:

$$[L_\lambda L] = TL + 2\lambda L + \frac{\lambda^3}{12}C.$$

It is an H-graded Lie conformal algebra with L of degree 2.

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.virasoro_lie_conformal_algebra.**VirasoroLieConformalAlgebra**(*R*

Bases: *GradedLieConformalAlgebra*

The Virasoro Lie Conformal algebra over R.

INPUT:

- R – a commutative ring; behaviour is undefined if R is not a Field of characteristic zero.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)
sage: Vir.category()
Category of H-graded finitely generated Lie conformal algebras with basis over␣
↪Rational Field
sage: Vir.inject_variables()
Defining L, C
sage: L.bracket(L)
{0: TL, 1: 2*L, 3: 1/2*C}
```

Weyl Lie Conformal Algebra

Given a commutative ring R, a free R-module M and a non-degenerate, skew-symmetric, bilinear pairing $\langle\cdot,\cdot\rangle : M \otimes_R M \to R$. The *Weyl* Lie conformal algebra associated to this datum is the free $R[T]$-module generated by M plus a central vector K. The non-vanishing λ-brackets are given by:

$$[v_\lambda w] = \langle v, w\rangle K.$$

This is not an H-graded Lie conformal algebra. The choice of a Lagrangian decomposition $M = L \oplus L^*$ determines an H-graded structure. For this H-graded Lie conformal algebra see the *Bosonic Ghosts Lie conformal algebra*

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.weyl_lie_conformal_algebra.**WeylLieConformalAlgebra**(*R*, *ngens=None*, *gram_matri*, *names=Non*, *in-dex_set=No*

Bases: *LieConformalAlgebraWithStructureCoefficients*

The Weyl Lie conformal algebra.

INPUT:

- `R` – a commutative ring; the base ring of this Lie conformal algebra.
- `ngens`: an even positive Integer (default 2); The number of non-central generators of this Lie conformal algebra.
- `gram_matrix`: a matrix (default: `None`); A non-singular skew-symmetric square matrix with coefficients in R.
- `names` – a list or tuple of `str`; alternative names for the generators
- `index_set` – an enumerated set; alternative indexing set for the generators

OUTPUT:

The Weyl Lie conformal algebra with generators
$\alpha_i, i = 1, ..., ngens$ and λ-brackets

$$[{\alpha_i}_\lambda \alpha_j] = M_{ij} K,$$

where M is the `gram_matrix` above.

Note: The returned Lie conformal algebra is not H-graded. For a related H-graded Lie conformal algebra see *BosonicGhostsLieConformalAlgebra*.

EXAMPLES:

```
sage: lie_conformal_algebras.Weyl(QQ)
The Weyl Lie conformal algebra with generators (alpha0, alpha1, K) over Rational␣
↪Field
sage: R = lie_conformal_algebras.Weyl(QQbar, gram_matrix=Matrix(QQ,[[0,1],[-1,0]]),␣
↪names = ('a','b'))
```

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```
sage: R.inject_variables()
Defining a, b, K
sage: a.bracket(b)
{0: K}
sage: b.bracket(a)
{0: -K}

sage: R = lie_conformal_algebras.Weyl(QQbar, ngens=4)
sage: R.gram_matrix()
[ 0  0| 1  0]
[ 0  0| 0  1]
[-----+-----]
[-1  0| 0  0]
[ 0 -1| 0  0]
sage: R.inject_variables()
Defining alpha0, alpha1, alpha2, alpha3, K
sage: alpha0.bracket(alpha2)
{0: K}

sage: R = lie_conformal_algebras.Weyl(QQ); R.category()
Category of finitely generated Lie conformal algebras with basis over Rational Field
sage: R in LieConformalAlgebras(QQ).Graded()
False
sage: R.inject_variables()
Defining alpha0, alpha1, K
sage: alpha0.degree()
Traceback (most recent call last):
...
AttributeError: 'WeylLieConformalAlgebra_with_category.element_class' object has no␣
↪attribute 'degree'
```

gram_matrix()

The Gram matrix that specifies the λ-brackets of the generators.

EXAMPLES:

```
sage: R = lie_conformal_algebras.Weyl(QQbar, ngens=4)
sage: R.gram_matrix()
[ 0  0| 1  0]
[ 0  0| 0  1]
[-----+-----]
[-1  0| 0  0]
[ 0 -1| 0  0]
```

9.3.3 See also

Finitely and Freely Generated Lie Conformal Algebras.

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.finitely_freely_generated_lca.**FinitelyFreelyGeneratedLCA**(*R*, *index_set*, *central_elements*, *category*, *element_class*, *prefix*, *names*, *latex_names*, ***kws*)

Bases: *FreelyGeneratedLieConformalAlgebra*

Abstract base class for finitely generated Lie conformal algebras.

This class provides minimal functionality, simply sets the number of generators.

central_elements()

The central elements of this Lie conformal algebra.

EXAMPLES:

```
sage: R = lie_conformal_algebras.NeveuSchwarz(QQ); R.central_elements()
(C,)
```

gens()

The generators for this Lie conformal algebra.

OUTPUT:

This method returns a tuple with the (finite) generators of this Lie conformal algebra.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ);
sage: Vir.gens()
(L, C)
```

See also:

lie_conformal_algebra_generators

ngens()

The number of generators of this Lie conformal algebra.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ); Vir.ngens()
2
sage: V = lie_conformal_algebras.Affine(QQ, 'A1'); V.ngens()
4
```

Freely Generated Lie Conformal Algebras

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation

class sage.algebras.lie_conformal_algebras.freely_generated_lie_conformal_algebra.**FreelyGeneratedLieCon**

Bases: *LieConformalAlgebraWithBasis*

Base class for a central extension of a freely generated Lie conformal algebra.

This class provides minimal functionality, it sets up the family of Lie conformal algebra generators.

Note: We now only accept direct sums of free modules plus some central generators C_i such that $TC_i = 0$.

central_elements()

The central generators of this Lie conformal algebra.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)
sage: Vir.central_elements()
(C,)
sage: V = lie_conformal_algebras.Affine(QQ, 'A1')
sage: V.central_elements()
(B['K'],)
```

lie_conformal_algebra_generators()

The generators of this Lie conformal algebra.

OUTPUT: a (possibly infinite) family of generators (as an $R[T]$-module) of this Lie conformal algebra.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(QQ)
sage: Vir.lie_conformal_algebra_generators()
(L, C)
sage: V = lie_conformal_algebras.Affine(QQ,'A1')
sage: V.lie_conformal_algebra_generators()
(B[alpha[1]], B[alphacheck[1]], B[-alpha[1]], B['K'])
```

Graded Lie Conformal Algebras

A (super) Lie conformal algebra V is called H-graded if there exists a decomposition $V = \oplus_n V_n$ such that the λ-bracket is graded of degree -1, that is for homogeneous elements $a \in V_p$, $b \in V_q$ with λ-brackets:

$$[a_\lambda b] = \sum \frac{\lambda^n}{n!} c_n,$$

we have $c_n \in V_{p+q-n-1}$. This situation arises typically when V has a vector $L \in V$ that generates the Virasoro Lie conformal algebra. Such that for every $a \in V$ we have

$$[L_\lambda a] = Ta + \lambda \Delta_a a + O(\lambda^2).$$

In this situation V is graded by the eigenvalues Δ_a of $L_{(1)}$, the (1)-th product with L. When the higher order terms $O(\lambda^2)$ vanish we say that a is a *primary vector* of *conformal weight* or degree Δ_a.

Note: Although arbitrary gradings are allowed, many of the constructions we implement in these classes work only for positive rational gradings.

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.graded_lie_conformal_algebra.**GradedLieConformalAlgebra**(*R*, *s_coeff*, *index_se*, *central_el*, *category=I*, *prefix=No*, *names=*, *latex_na*, *parity=No*, *weight*, ***kwds*

Bases: `LieConformalAlgebraWithStructureCoefficients`

An H-Graded Lie conformal algebra.

INPUT:

- R – a commutative ring (default: None); the base ring of this Lie conformal algebra. Behaviour is undefined if it is not a field of characteristic zero
- s_coeff – a dictionary (default: None); as in the input of *LieConformalAlgebra*
- names – tuple of str (default: None); as in the input of *LieConformalAlgebra*
- central_elements – tuple of str (default: None); as in the input of *LieConformalAlgebra*
- index_set – enumerated set (default: None); as in the input of *LieConformalAlgebra*
- weights – tuple of non-negative rational numbers (default: tuple of 1); a list of degrees for this Lie conformal algebra. This tuple needs to have the same cardinality as index_set or names. Central elements are assumed to have weight 0.
- category The category that this Lie conformal algebra belongs to.
- parity – tuple of 0 or 1 (Default: tuple of 0); a tuple specifying the parity of each non-central generator.

EXAMPLES:

```
sage: bosondict = {('a','a'):{1:{('K',0):1}}}
sage: R = LieConformalAlgebra(QQ,bosondict,names=('a',),central_elements=('K',),␣
↪weights=(1,))
sage: R.inject_variables()
Defining a, K
sage: a.T(3).degree()
4
sage: K.degree()
0
sage: R.category()
Category of H-graded finitely generated Lie conformal algebras with basis over␣
↪Rational Field
```

Lie Conformal Algebras With Basis

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation

class sage.algebras.lie_conformal_algebras.lie_conformal_algebra_with_basis.**LieConformalAlgebraWithBasis**

Bases: CombinatorialFreeModule

Abstract base class for a Lie conformal algebra with a preferred basis.

This class provides no functionality, it simply passes the arguments to CombinatorialFreeModule.

EXAMPLES:

```
sage: R = lie_conformal_algebras.Virasoro(QQbar);R
The Virasoro Lie conformal algebra over Algebraic Field
```

Lie Conformal Algebras With Structure Coefficients

AUTHORS:

- Reimundo Heluani (2019-08-09): Initial implementation.

class sage.algebras.lie_conformal_algebras.lie_conformal_algebra_with_structure_coefs.**LieConformalAlgeb**

Bases: *FinitelyFreelyGeneratedLCA*

A Lie conformal algebra with a set of specified structure coefficients.

INPUT:

- R – a ring (Default: None); The base ring of this Lie conformal algebra. Behaviour is undefined if it is not a field of characteristic zero.
- s_coeff – Dictionary (Default: None); a dictionary containing the λ brackets of the generators of this Lie conformal algebra. The family encodes a dictionary whose keys are pairs of either names or indices of the generators and the values are themselves dictionaries. For a pair of generators a and b, the value of s_coeff[('a','b')] is a dictionary whose keys are positive integer numbers and the corresponding value for the key j is a dictionary itself representing the j-th product $a_{(j)}b$. Thus, for a positive integer number j, the value of s_coeff[('a','b')][j] is a dictionary whose entries are pairs ('c',n) where 'c' is the name of a generator and n is a positive number. The value for this key is the coefficient of $\frac{T^n}{n!}c$ in $a_{(j)}b$. For example the s_coeff for the *Virasoro* Lie conformal algebra is:

  ```
  {('L','L'):{0:{('L',1):1}, 1:{('L',0):2}, 3:{('C',0):1/2}}}
  ```

  Do not include central elements in this dictionary. Also, if the key ('a','b') is present, there is no need to include ('b','a') as it is defined by skew-symmetry. Any missing pair (besides the ones defined by skew-symmetry) is assumed to have vanishing λ-bracket.
- names – tuple of str (Default: None); The list of names for generators of this Lie conformal algebra. Do not include central elements in this list.

- `central_elements` – tuple of `str` (Default: `None`); A list of names for central elements of this Lie conformal algebra.
- `index_set` – enumerated set (Default: `None`); an indexing set for the generators of this Lie conformal algebra. Do not include central elements in this list.
- **`parity` – tuple of 0 or 1 (Default: tuple of 0);**
  a tuple specifying the parity of each non-central generator.

EXAMPLES:

- We construct the $\beta - \gamma$ system by directly giving the λ-brackets of the generators:

```
sage: betagamma_dict = {('b','a'):{0:{('K',0):1}}}
sage: V = LieConformalAlgebra(QQ, betagamma_dict, names=('a','b'), weights=(1,
↪0), central_elements=('K',))
sage: V.category()
Category of H-graded finitely generated Lie conformal algebras with basis over␣
↪Rational Field
sage: V.inject_variables()
Defining a, b, K
sage: a.bracket(b)
{0: -K}
```

- We construct the centerless Virasoro Lie conformal algebra:

```
sage: virdict =  {('L','L'):{0:{('L',1):1}, 1:{('L',0): 2}}}
sage: R = LieConformalAlgebra(QQbar, virdict, names='L')
sage: R.inject_variables()
Defining L
sage: L.bracket(L)
{0: TL, 1: 2*L}
```

- The construction checks that skew-symmetry is violated:

```
sage: wrongdict =  {('L','L'):{0:{('L',1):2}, 1:{('L',0): 2}}}
sage: LieConformalAlgebra(QQbar, wrongdict, names='L')
Traceback (most recent call last):
...
ValueError: two distinct values given for one and the same bracket. Skew-
↪symmetry is not satisfied?
```

`structure_coefficients()`

The structure coefficients of this Lie conformal algebra.

EXAMPLES:

```
sage: Vir = lie_conformal_algebras.Virasoro(AA)
sage: Vir.structure_coefficients()
Finite family {('L', 'L'): ((0, TL), (1, 2*L), (3, 1/2*C))}

sage: lie_conformal_algebras.NeveuSchwarz(QQ).structure_coefficients()
Finite family {('G', 'G'): ((0, 2*L), (2, 2/3*C)),  ('G', 'L'): ((0, 1/2*TG),␣
↪(1, 3/2*G)),  ('L', 'G'): ((0, TG), (1, 3/2*G)),  ('L', 'L'): ((0, TL), (1,␣
↪2*L), (3, 1/2*C))}
```

9.4 Jordan Algebras

AUTHORS:

- Travis Scrimshaw (2014-04-02): initial version
- Travis Scrimshaw (2023-05-09): added the 27 dimensional exceptional Jordan algebra

class sage.algebras.jordan_algebra.**ExceptionalJordanAlgebra**(*O*)

Bases: *JordanAlgebra*

The exceptional 27 dimensional Jordan algebra as self-adjoint 3×3 matrix over an octonion algebra.

Let $\mathbf{O}$ be the *OctonionAlgebra* over a commutative ring R of characteristic not equal to 2. The *exceptional Jordan algebra* $\mathfrak{h}_3(\mathbf{O})$ is a 27 dimensional free R-module spanned by the matrices

$$\begin{bmatrix} \alpha & x & y \\ x^* & \beta & z \\ y^* & z^* & \gamma \end{bmatrix}$$

for $\alpha, \beta, \gamma \in R$ and $x, y, z \in \mathbf{O}$, with multiplication given by the usual symmetrizer operation $X \circ Y = \frac{1}{2}(XY + YX)$.

These are also known as *Albert algebras* due to the work of Abraham Adrian Albert on these algebras over $\mathbf{R}$.

EXAMPLES:

We construct an exceptional Jordan algebra over $\mathbf{Q}$ and perform some basic computations:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: gens = J.gens()
sage: gens[1]
[0 0 0]
[0 1 0]
[0 0 0]
sage: gens[3]
[0 1 0]
[1 0 0]
[0 0 0]
sage: gens[1] * gens[3]
[  0 1/2   0]
[1/2   0   0]
[  0   0   0]
```

The Lie algebra of derivations of the exceptional Jordan algebra is isomorphic to the simple Lie algebra of type F_4. We verify that we the derivation module has the correct dimension:

```
sage: len(J.derivations_basis())  # long time
52
sage: LieAlgebra(QQ, cartan_type='F4').dimension()
52
```

REFERENCES:

- Wikipedia article Albert_algebra
- Wikipedia article Jordan_algebra#Examples
- Wikipedia article Hurwitz's_theorem_(composition_algebras)#Applications_to_Jordan_algebras

- https://math.ucr.edu/home/baez/octonions/octonions.pdf

class Element(*parent*, *data*)

Bases: AlgebraElement

An element of an exceptional Jordan algebra.

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- copy – ignored

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: elt = sum(~QQ(ind) * b for ind, b in enumerate(J.basis()[::6],␣
↪start=1)); elt
[              1          1/2*k  1/3*i + 1/4*lk]
[         -1/2*k              0          1/5*li]
[-1/3*i - 1/4*lk         -1/5*li               0]
sage: elt.monomial_coefficients()
{0: 1, 6: 1/2, 12: 1/3, 18: 1/4, 24: 1/5}
```

algebra_generators()

Return a basis of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: B = J.basis()
sage: B[::6]
([1 0 0]
 [0 0 0]
 [0 0 0],
 [ 0  k  0]
 [-k  0  0]
 [ 0  0  0],
 [ 0  0  i]
 [ 0  0  0]
 [-i  0  0],
 [  0   0  lk]
 [  0   0   0]
 [-lk   0   0],
 [  0   0   0]
 [  0   0  li]
 [  0 -li   0])
sage: len(B)
27
```

basis()

Return a basis of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: B = J.basis()
sage: B[::6]
([1 0 0]
 [0 0 0]
 [0 0 0],
 [ 0  k  0]
 [-k  0  0]
 [ 0  0  0],
 [ 0  0  i]
 [ 0  0  0]
 [-i  0  0],
 [  0   0  lk]
 [  0   0   0]
 [-lk   0   0],
 [  0   0   0]
 [  0   0  li]
 [  0 -li   0])
sage: len(B)
27
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: G = J.gens()
sage: G[0]
[1 0 0]
[0 0 0]
[0 0 0]
sage: G[5]
[ 0  j  0]
[-j  0  0]
[ 0  0  0]
sage: G[22]
[ 0  0  0]
[ 0  0  k]
[ 0 -k  0]
```

one()

Return multiplicative identity.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: J.one()
[1 0 0]
[0 1 0]
[0 0 1]
```

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```
sage: all(J.one() * b == b for b in J.basis())
True
```

some_elements()

Return some elements of `self`.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: J.some_elements()
[[6/5   0   0]
 [  0 6/5   0]
 [  0   0 6/5],
 [1 0 0]
 [0 1 0]
 [0 0 1],
 [0 0 0]
 [0 0 0]
 [0 0 0],
 [0 0 0]
 [0 1 0]
 [0 0 0],
 [ 0  j  0]
 [-j  0  0]
 [ 0  0  0],
 [  0   0  lj]
 [  0   0   0]
 [-lj   0   0],
 [      0       0       0]
 [      0       1  1/2*lj]
 [      0 -1/2*lj       0],
 [        1         0  j + 2*li]
 [        0         1         0]
 [-j - 2*li         0         1],
 [      1  j + lk       l]
 [-j - lk       0  i + lj]
 [     -l -i - lj       0],
 [      1  3/2*l    2*k]
 [-3/2*l       0  5/2*j]
 [  -2*k -5/2*j      0]]

sage: O = OctonionAlgebra(GF(3))
sage: J = JordanAlgebra(O)
sage: J.some_elements()
[[-1  0  0]
 [ 0 -1  0]
 [ 0  0 -1],
 [1 0 0]
 [0 1 0]
 [0 0 1],
 [0 0 0]
```

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```
[0 0 0]
[0 0 0],
[0 0 0]
[0 1 0]
[0 0 0],
[ 0  j  0]
[-j  0  0]
[ 0  0  0],
[  0   0  lj]
[  0   0   0]
[-lj   0   0],
[  0   0   0]
[  0   1 -lj]
[  0  lj   0],
[      1       0  j - li]
[      0       1       0]
[-j + li       0       1],
[      1  j + lk       1]
[-j - lk       0  i + lj]
[     -l -i - lj       0],
[ 1  0 -k]
[ 0  0  j]
[ k -j  0]]
```

zero()

Return the additive identity.

EXAMPLES:

```
sage: O = OctonionAlgebra(QQ)
sage: J = JordanAlgebra(O)
sage: J.zero()
[0 0 0]
[0 0 0]
[0 0 0]
```

class sage.algebras.jordan_algebra.**JordanAlgebra**

Bases: `UniqueRepresentation`, `Parent`

A Jordan algebra.

A *Jordan algebra* is a magmatic algebra (over a commutative ring R) whose multiplication satisfies the following axioms:

- $xy = yx$, and
- $(xy)(xx) = x(y(xx))$ (the Jordan identity).

See [Ja1971], [Ch2012], and [McC1978], for example.

These axioms imply that a Jordan algebra is power-associative and the following generalization of Jordan's identity holds [Al1947]: $(x^m y)x^n = x^m(yx^n)$ for all $m, n \in \mathbf{Z}_{>0}$.

Let A be an associative algebra over a ring R in which 2 is invertible. We construct a Jordan algebra A^+ with ground set A by defining the multiplication as

$$x \circ y = \frac{xy + yx}{2}.$$

Often the multiplication is written as $x \circ y$ to avoid confusion with the product in the associative algebra A. We note that if A is commutative then this reduces to the usual multiplication in A.

Jordan algebras constructed in this fashion, or their subalgebras, are called *special*. All other Jordan algebras are called *exceptional*.

Jordan algebras can also be constructed from a module M over R with a symmetric bilinear form $(\cdot, \cdot) : M \times M \to R$. We begin with the module $M^* = R \oplus M$ and define multiplication in M^* by

$$(\alpha + x) \circ (\beta + y) = \underbrace{\alpha\beta + (x, y)}_{\in R} + \underbrace{\beta x + \alpha y}_{\in M},$$

where $\alpha, \beta \in R$ and $x, y \in M$.

INPUT:

Can be either an associative algebra A or a symmetric bilinear form given as a matrix (possibly followed by, or preceded by, a base ring argument).

EXAMPLES:

We let the base algebra A be the free algebra on 3 generators:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F); J
Jordan algebra of Free Algebra on 3 generators (x, y, z) over Rational Field
sage: a,b,c = map(J, F.gens())
sage: a*b
1/2*x*y + 1/2*y*x
sage: b*a
1/2*x*y + 1/2*y*x
```

Jordan algebras are typically non-associative:

```
sage: (a*b)*c
1/4*x*y*z + 1/4*y*x*z + 1/4*z*x*y + 1/4*z*y*x
sage: a*(b*c)
1/4*x*y*z + 1/4*x*z*y + 1/4*y*z*x + 1/4*z*y*x
```

We check the Jordan identity:

```
sage: (a*b)*(a*a) == a*(b*(a*a))
True
sage: x = a + c
sage: y = b - 2*a
sage: (x*y)*(x*x) == x*(y*(x*x))
True
```

Next we construct a Jordan algebra from a symmetric bilinear form:

```
sage: m = matrix([[-2,3],[3,4]])
sage: J.<a,b,c> = JordanAlgebra(m); J
Jordan algebra over Integer Ring given by the symmetric bilinear form:
[-2  3]
[ 3  4]
sage: a
1 + (0, 0)
```

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```
sage: b
0 + (1, 0)
sage: x = 3*a - 2*b + c; x
3 + (-2, 1)
```

We again show that Jordan algebras are usually non-associative:

```
sage: (x*b)*b
-6 + (7, 0)
sage: x*(b*b)
-6 + (4, -2)
```

We verify the Jordan identity:

```
sage: y = -a + 4*b - c
sage: (x*y)*(x*x) == x*(y*(x*x))
True
```

The base ring, while normally inferred from the matrix, can also be explicitly specified:

```
sage: J.<a,b,c> = JordanAlgebra(m, QQ); J
Jordan algebra over Rational Field given by the symmetric bilinear form:
[-2  3]
[ 3  4]
sage: J.<a,b,c> = JordanAlgebra(QQ, m); J # either order work
Jordan algebra over Rational Field given by the symmetric bilinear form:
[-2  3]
[ 3  4]
```

REFERENCES:

- Wikipedia article Jordan_algebra
- [Ja1971]
- [Ch2012]
- [McC1978]
- [Al1947]

class sage.algebras.jordan_algebra.**JordanAlgebraSymmetricBilinear**(*R*, *form*, *names=None*)

Bases: *JordanAlgebra*

A Jordan algebra given by a symmetric bilinear form m.

class Element(*parent*, *s*, *v*)

Bases: AlgebraElement

An element of a Jordan algebra defined by a symmetric bilinear form.

bar()

Return the result of the bar involution of `self`.

The bar involution $\bar{\cdot}$ is the R-linear endomorphism of M^* defined by $\bar{1} = 1$ and $\bar{x} = -x$ for $x \in M$.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: x = 4*a - b + 3*c
sage: x.bar()
4 + (1, -3)
```

We check that it is an algebra morphism:

```
sage: y = 2*a + 2*b - c
sage: x.bar() * y.bar() == (x*y).bar()
True
```

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- `copy` – ignored

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: elt = a + 2*b - c
sage: elt.monomial_coefficients()
{0: 1, 1: 2, 2: -1}
```

norm()

Return the norm of `self`.

The norm of an element $\alpha + x \in M^*$ is given by $n(\alpha + x) = \alpha^2 - (x, x)$.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: x = 4*a - b + 3*c; x
4 + (-1, 3)
sage: x.norm()
13
```

trace()

Return the trace of `self`.

The trace of an element $\alpha + x \in M^*$ is given by $t(\alpha + x) = 2\alpha$.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: x = 4*a - b + 3*c
sage: x.trace()
8
```

algebra_generators()

Return a basis of `self`.

The basis returned begins with the unity of R and continues with the standard basis of M.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J = JordanAlgebra(m)
sage: J.basis()
Family (1 + (0, 0), 0 + (1, 0), 0 + (0, 1))
```

basis()

Return a basis of `self`.

The basis returned begins with the unity of R and continues with the standard basis of M.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J = JordanAlgebra(m)
sage: J.basis()
Family (1 + (0, 0), 0 + (1, 0), 0 + (0, 1))
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J = JordanAlgebra(m)
sage: J.gens()
(1 + (0, 0), 0 + (1, 0), 0 + (0, 1))
```

one()

Return the element 1 if it exists.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J = JordanAlgebra(m)
sage: J.one()
1 + (0, 0)
```

zero()

Return the element 0.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J = JordanAlgebra(m)
sage: J.zero()
0 + (0, 0)
```

class sage.algebras.jordan_algebra.**SpecialJordanAlgebra**(*A*, *names=None*)

Bases: `JordanAlgebra`

A (special) Jordan algebra A^+ from an associative algebra A.

class Element(*parent*, *x*)

Bases: AlgebraElement

An element of a special Jordan algebra.

monomial_coefficients(*copy=True*)

Return a dictionary whose keys are indices of basis elements in the support of `self` and whose values are the corresponding coefficients.

INPUT:

- `copy` – (default: `True`) if `self` is internally represented by a dictionary `d`, then make a copy of `d`; if `False`, then this can cause undesired behavior by mutating `d`

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F)
sage: a,b,c = map(J, F.gens())
sage: elt = a + 2*b - c
sage: elt.monomial_coefficients()
{x: 1, y: 2, z: -1}
```

algebra_generators()

Return the basis of `self`.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F)
sage: J.basis()
Lazy family (Term map(i))_{i in Free monoid on 3 generators (x, y, z)}
```

basis()

Return the basis of `self`.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F)
sage: J.basis()
Lazy family (Term map(i))_{i in Free monoid on 3 generators (x, y, z)}
```

gens()

Return the generators of `self`.

EXAMPLES:

```
sage: cat = Algebras(QQ).WithBasis().FiniteDimensional()
sage: C = CombinatorialFreeModule(QQ, ['x','y','z'], category=cat)
sage: J = JordanAlgebra(C)
sage: J.gens()
(B['x'], B['y'], B['z'])

sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F)
sage: J.gens()
Traceback (most recent call last):
```

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```
...
NotImplementedError: infinite set
```

one()

Return the element 1 if it exists.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F)
sage: J.one()
1
```

zero()

Return the element 0.

EXAMPLES:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: J = JordanAlgebra(F)
sage: J.zero()
0
```

9.5 Free Dendriform Algebras

AUTHORS:

Frédéric Chapoton (2017)

class sage.combinat.free_dendriform_algebra.**DendriformFunctor**(*vars*)

Bases: ConstructionFunctor

A constructor for dendriform algebras.

EXAMPLES:

```
sage: P = algebras.FreeDendriform(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F = P.construction()[0]; F
Dendriform[x,y]

sage: A = GF(5)['a,b']
sage: a, b = A.gens()
sage: F(A)
Free Dendriform algebra on 2 generators ['x', 'y']
 over Multivariate Polynomial Ring in a, b over Finite Field of size 5

sage: f = A.hom([a+b,a-b],A)
sage: F(f)
Generic endomorphism of Free Dendriform algebra on 2 generators ['x', 'y']
 over Multivariate Polynomial Ring in a, b over Finite Field of size 5
```

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```
sage: F(f)(a * F(A)(x))
(a+b)*B[x[., .]]
```

merge(*other*)

Merge `self` with another construction functor, or return `None`.

EXAMPLES:

```
sage: F = sage.combinat.free_dendriform_algebra.DendriformFunctor(['x','y'])
sage: G = sage.combinat.free_dendriform_algebra.DendriformFunctor(['t'])
sage: F.merge(G)
Dendriform[x,y,t]
sage: F.merge(F)
Dendriform[x,y]
```

Now some actual use cases:

```
sage: R = algebras.FreeDendriform(ZZ, 'x,y,z')
sage: x,y,z = R.gens()
sage: 1/2 * x
1/2*B[x[., .]]
sage: parent(1/2 * x)
Free Dendriform algebra on 3 generators ['x', 'y', 'z'] over Rational Field

sage: S = algebras.FreeDendriform(QQ, 'zt')
sage: z,t = S.gens()
sage: x + t
B[t[., .]] + B[x[., .]]
sage: parent(x + t)
Free Dendriform algebra on 4 generators ['z', 't', 'x', 'y'] over Rational Field
```

rank = 9

class sage.combinat.free_dendriform_algebra.**FreeDendriformAlgebra**(*R*, *names=None*)

Bases: `CombinatorialFreeModule`

The free dendriform algebra.

Dendriform algebras are associative algebras, where the associative product $*$ is decomposed as a sum of two binary operations

$$x * y = x \succ y + x \prec y$$

that satisfy the axioms:

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \prec z = x \prec (y * z).$$

$$(x * y) \succ z = x \succ (y \succ z).$$

The free Dendriform algebra on a given set E has an explicit description using (planar) binary trees, just as the free associative algebra can be described using words. The underlying vector space has a basis indexed by finite binary trees endowed with a map from their vertices to E. In this basis, the associative product of two (decorated) binary trees $S * T$ is the sum over all possible ways of identifying (glueing) the rightmost path in S and the leftmost path in T.

The decomposition of the associative product as the sum of two binary operations $\succ$ and $\prec$ is made by separating the terms according to the origin of the root vertex. For $x \succ y$, one keeps the terms where the root vertex comes from y, whereas for $x \prec y$ one keeps the terms where the root vertex comes from x.

The free dendriform algebra can also be considered as the free algebra over the Dendriform operad.

Note: The usual binary operator $*$ is used for the associative product.

EXAMPLES:

```
sage: F = algebras.FreeDendriform(ZZ, 'xyz')
sage: x,y,z = F.gens()
sage: (x * y) * z
B[x[., y[., z[., .]]]] + B[x[., z[y[., .], .]]] + B[y[x[., .], z[., .]]] + B[z[x[.,␣
↪y[., .]], .]] + B[z[y[x[., .], .], .]]
```

The free dendriform algebra is associative:

```
sage: x * (y * z) == (x * y) * z
True
```

The associative product decomposes in two parts:

```
sage: x * y == F.prec(x, y) + F.succ(x, y)
True
```

The axioms hold:

```
sage: F.prec(F.succ(x, y), z) == F.succ(x, F.prec(y, z))
True
sage: F.prec(F.prec(x, y), z) == F.prec(x, y * z)
True
sage: F.succ(x * y, z) == F.succ(x, F.succ(y, z))
True
```

When there is only one generator, unlabelled trees are used instead:

```
sage: F1 = algebras.FreeDendriform(QQ)
sage: w = F1.gen(0); w
B[[., .]]
sage: w * w * w
B[[., [., [., .]]]] + B[[., [[., .], .]]] + B[[[., .], [., .]]] + B[[[., [., .]], .
↪]] + B[[[[., .], .], .]]
```

The set E can be infinite:

```
sage: F = algebras.FreeDendriform(QQ, ZZ)
sage: w = F.gen(1); w
B[1[., .]]
sage: x = F.gen(2); x
B[-1[., .]]
sage: w*x
B[-1[1[., .], .]] + B[1[., -1[., .]]]
```

REFERENCES:

- [LR1998]

algebra_generators()

Return the generators of this algebra.

These are the binary trees with just one vertex.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(ZZ, 'fgh'); A
Free Dendriform algebra on 3 generators ['f', 'g', 'h']
 over Integer Ring
sage: list(A.algebra_generators())
[B[f[., .]], B[g[., .]], B[h[., .]]]

sage: A = algebras.FreeDendriform(QQ, ['x1','x2'])
sage: list(A.algebra_generators())
[B[x1[., .]], B[x2[., .]]]
```

an_element()

Return an element of `self`.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ, 'xy')
sage: A.an_element()
B[x[., .]] + 2*B[x[., x[., .]]] + 2*B[x[x[., .], .]]
```

change_ring(R)

Return the free dendriform algebra in the same variables over R.

INPUT:

- R – a ring

EXAMPLES:

```
sage: A = algebras.FreeDendriform(ZZ, 'fgh')
sage: A.change_ring(QQ)
Free Dendriform algebra on 3 generators ['f', 'g', 'h'] over
Rational Field
```

construction()

Return a pair `(F, R)`, where `F` is a *DendriformFunctor* and R is a ring, such that `F(R)` returns `self`.

EXAMPLES:

```
sage: P = algebras.FreeDendriform(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F, R = P.construction()
sage: F
Dendriform[x,y]
sage: R
Integer Ring
sage: F(ZZ) is P
True
sage: F(QQ)
Free Dendriform algebra on 2 generators ['x', 'y'] over Rational Field
```

coproduct_on_basis(*x*)

Return the coproduct of a binary tree.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: x = A.gen(0)
sage: ascii_art(A.coproduct(A.one()))  # indirect doctest
1 # 1

sage: ascii_art(A.coproduct(x))  # indirect doctest
1 # B  + B  # 1
     o    o

sage: A = algebras.FreeDendriform(QQ, 'xyz')
sage: x, y, z = A.gens()
sage: w = A.under(z,A.over(x,y))
sage: A.coproduct(z)
B[.] # B[z[., .]] + B[z[., .]] # B[.]
sage: A.coproduct(w)
B[.] # B[x[z[., .], y[., .]]] + B[x[., .]] # B[z[., y[., .]]] +
B[x[., .]] # B[y[z[., .], .]] + B[x[., y[., .]]] # B[z[., .]] +
B[x[z[., .], .]] # B[y[., .]] + B[x[z[., .], y[., .]]] # B[.]
```

degree_on_basis(*t*)

Return the degree of a binary tree in the free Dendriform algebra.

This is the number of vertices.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ,'@')
sage: RT = A.basis().keys()
sage: u = RT([], '@')
sage: A.degree_on_basis(u.over(u))
2
```

gen(*i*)

Return the `i`-th generator of the algebra.

INPUT:

- `i` – an integer

EXAMPLES:

```
sage: F = algebras.FreeDendriform(ZZ, 'xyz')
sage: F.gen(0)
B[x[., .]]

sage: F.gen(4)
Traceback (most recent call last):
...
IndexError: argument i (= 4) must be between 0 and 2
```

gens()

Return the generators of `self` (as an algebra).

EXAMPLES:

```
sage: A = algebras.FreeDendriform(ZZ, 'fgh')
sage: A.gens()
(B[f[., .]], B[g[., .]], B[h[., .]])
```

one_basis()

Return the index of the unit.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ, '@')
sage: A.one_basis()
.
sage: A = algebras.FreeDendriform(QQ, 'xy')
sage: A.one_basis()
.
```

over()

Return the over product.

The over product x/y is the binary tree obtained by grafting the root of y at the rightmost leaf of x.

The usual symbol for this operation is $/$.

See also:

product(), *succ()*, *prec()*, *under()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.over(x, x)
B[[., [., .]]]
```

prec()

Return the $\prec$ dendriform product.

This is the sum over all possible ways to identify the rightmost path in x and the leftmost path in y, with the additional condition that the root vertex of the result comes from x.

The usual symbol for this operation is $\prec$.

See also:

product(), *succ()*, *over()*, *under()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.prec(x, x)
B[[., [., .]]]
```

prec_product_on_basis(x, y)

Return the $\prec$ dendriform product of two trees.

This is the sum over all possible ways of identifying the rightmost path in x and the leftmost path in y, with the additional condition that the root vertex of the result comes from x.

The usual symbol for this operation is $\prec$.

See also:

- *product_on_basis()*, *succ_product_on_basis()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.prec_product_on_basis(x, x)
B[[., [., .]]]
```

product_on_basis(x, y)

Return the $*$ associative dendriform product of two trees.

This is the sum over all possible ways of identifying the rightmost path in x and the leftmost path in y. Every term corresponds to a shuffle of the vertices on the rightmost path in x and the vertices on the leftmost path in y.

See also:

- *succ_product_on_basis()*, *prec_product_on_basis()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.product_on_basis(x, x)
B[[., [., .]]] + B[[[., .], .]]
```

some_elements()

Return some elements of the free dendriform algebra.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: A.some_elements()
[B[.],
 B[[., .]],
 B[[., [., .]]] + B[[[., .], .]],
 B[.] + B[[., [., .]]] + B[[[., .], .]]]
```

With several generators:

```
sage: A = algebras.FreeDendriform(QQ, 'xy')
sage: A.some_elements()
[B[.],
```

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```
B[x[., .]],
B[x[., x[., .]]] + B[x[x[., .], .]],
B[.] + B[x[., x[., .]]] + B[x[x[., .], .]]]
```

succ()

Return the $\succ$ dendriform product.

This is the sum over all possible ways of identifying the rightmost path in x and the leftmost path in y, with the additional condition that the root vertex of the result comes from y.

The usual symbol for this operation is $\succ$.

See also:

product(), *prec()*, *over()*, *under()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.succ(x, x)
B[[[., .], .]]
```

succ_product_on_basis(x, y)

Return the $\succ$ dendriform product of two trees.

This is the sum over all possible ways to identify the rightmost path in x and the leftmost path in y, with the additional condition that the root vertex of the result comes from y.

The usual symbol for this operation is $\succ$.

See also:

- *product_on_basis()*, *prec_product_on_basis()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.succ_product_on_basis(x, x)
B[[[., .], .]]
```

under()

Return the under product.

The over product $x\backslash y$ is the binary tree obtained by grafting the root of x at the leftmost leaf of y.

The usual symbol for this operation is $\backslash$.

See also:

product(), *succ()*, *prec()*, *over()*

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.under(x, x)
B[[[., .], .]]
```

variable_names()

Return the names of the variables.

EXAMPLES:

```
sage: R = algebras.FreeDendriform(QQ, 'xy')
sage: R.variable_names()
{'x', 'y'}
```

9.6 Free Pre-Lie Algebras

AUTHORS:

- Florent Hivert, Frédéric Chapoton (2011)

class sage.combinat.free_prelie_algebra.**FreePreLieAlgebra**(*R*, *names=None*)

Bases: `CombinatorialFreeModule`

The free pre-Lie algebra.

Pre-Lie algebras are non-associative algebras, where the product $*$ satisfies

$$(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y).$$

We use here the convention where the associator

$$(x, y, z) := (x * y) * z - x * (y * z)$$

is symmetric in its two rightmost arguments. This is sometimes called a right pre-Lie algebra.

They have appeared in numerical analysis and deformation theory.

The free Pre-Lie algebra on a given set E has an explicit description using rooted trees, just as the free associative algebra can be described using words. The underlying vector space has a basis indexed by finite rooted trees endowed with a map from their vertices to E. In this basis, the product of two (decorated) rooted trees $S * T$ is the sum over vertices of S of the rooted tree obtained by adding one edge from the root of T to the given vertex of S. The root of these trees is taken to be the root of S. The free pre-Lie algebra can also be considered as the free algebra over the PreLie operad.

> **Warning:** The usual binary operator `*` can be used for the pre-Lie product. Beware that it but must be parenthesized properly, as the pre-Lie product is not associative. By default, a multiple product will be taken with left parentheses.

EXAMPLES:

```
sage: F = algebras.FreePreLie(ZZ, 'xyz')
sage: x,y,z = F.gens()
sage: (x * y) * z
B[x[y[z[]]]] + B[x[y[], z[]]]
sage: (x * y) * z - x * (y * z) == (x * z) * y - x * (z * y)
True
```

The free pre-Lie algebra is non-associative:

```
sage: x * (y * z) == (x * y) * z
False
```

The default product is with left parentheses:

```
sage: x * y * z == (x * y) * z
True
sage: x * y * z * x == ((x * y) * z) * x
True
```

The NAP product as defined in [Liv2006] is also implemented on the same vector space:

```
sage: N = F.nap_product
sage: N(x*y,z*z)
B[x[y[], z[z[]]]]
```

When None is given as input, unlabelled trees are used instead:

```
sage: F1 = algebras.FreePreLie(QQ, None)
sage: w = F1.gen(0); w
B[[]]
sage: w * w * w * w
B[[[[[]]]]] + B[[[[], []]]] + 3*B[[[], [[]]]] + B[[[], [], []]]
```

However, it is equally possible to use labelled trees instead:

```
sage: F1 = algebras.FreePreLie(QQ, 'q')
sage: w = F1.gen(0); w
B[q[]]
sage: w * w * w * w
B[q[q[q[q[]]]]] + B[q[q[q[], q[]]]] + 3*B[q[q[], q[q[]]]] + B[q[q[], q[], q[]]]
```

The set E can be infinite:

```
sage: F = algebras.FreePreLie(QQ, ZZ)
sage: w = F.gen(1); w
B[1[]]
sage: x = F.gen(2); x
B[-1[]]
sage: y = F.gen(3); y
B[2[]]
sage: w*x
B[1[-1[]]]
sage: (w*x)*y
B[1[-1[2[]]]] + B[1[-1[], 2[]]]
```

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```
sage: w*(x*y)
B[1[-1[2[]]]]
```

Elements of a free pre-Lie algebra can be lifted to the universal enveloping algebra of the associated Lie algebra. The universal enveloping algebra is the Grossman-Larson Hopf algebra:

```
sage: F = algebras.FreePreLie(QQ,'abc')
sage: a,b,c = F.gens()
sage: (a*b+b*c).lift()
B[#[a[b[]]]] + B[#[b[c[]]]]
```

Note: Variables names can be `None`, a list of strings, a string or an integer. When `None` is given, unlabelled rooted trees are used. When a single string is given, each letter is taken as a variable. See `sage.combinat.words.alphabet.build_alphabet()`.

Warning: Beware that the underlying combinatorial free module is based either on `RootedTrees` or on `LabelledRootedTrees`, with no restriction on the labellings. This means that all code calling the `basis()` method would not give meaningful results, since `basis()` returns many "chaff" elements that do not belong to the algebra.

REFERENCES:

- [ChLi]
- [Liv2006]

class Element

Bases: `IndexedFreeModuleElement`

lift()

Lift element to the Grossman-Larson algebra.

EXAMPLES:

```
sage: F = algebras.FreePreLie(QQ,'abc')
sage: elt = F.an_element().lift(); elt
B[#[a[a[a[a[]]]]]] + B[#[a[a[], a[]]]]
sage: parent(elt)
Grossman-Larson Hopf algebra on 3 generators ['a', 'b', 'c']
over Rational Field
```

algebra_generators()

Return the generators of this algebra.

These are the rooted trees with just one vertex.

EXAMPLES:

```
sage: A = algebras.FreePreLie(ZZ, 'fgh'); A
Free PreLie algebra on 3 generators ['f', 'g', 'h']
 over Integer Ring
sage: list(A.algebra_generators())
```

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```
[B[f[]], B[g[]], B[h[]]]

sage: A = algebras.FreePreLie(QQ, ['x1','x2'])
sage: list(A.algebra_generators())
[B[x1[]], B[x2[]]]
```

an_element()

Return an element of `self`.

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, 'xy')
sage: A.an_element()
B[x[x[x[x[]]]]] + B[x[x[], x[x[]]]]
```

bracket_on_basis(x, y)

Return the Lie bracket of two trees.

This is the commutator $[x, y] = x * y - y * x$ of the pre-Lie product.

See also:

pre_Lie_product_on_basis()

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: y = RT([x])
sage: A.bracket_on_basis(x, y)
-B[[[[], [[]]]]] + B[[[], [[[]]]]] - B[[[[]], [[]]]]
```

change_ring(R)

Return the free pre-Lie algebra in the same variables over R.

INPUT:

- R – a ring

EXAMPLES:

```
sage: A = algebras.FreePreLie(ZZ, 'fgh')
sage: A.change_ring(QQ)
Free PreLie algebra on 3 generators ['f', 'g', 'h'] over
Rational Field
```

construction()

Return a pair `(F, R)`, where `F` is a *PreLieFunctor* and R is a ring, such that `F(R)` returns `self`.

EXAMPLES:

```
sage: P = algebras.FreePreLie(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F, R = P.construction()
sage: F
```

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```
PreLie[x,y]
sage: R
Integer Ring
sage: F(ZZ) is P
True
sage: F(QQ)
Free PreLie algebra on 2 generators ['x', 'y'] over Rational Field
```

degree_on_basis(*t*)

Return the degree of a rooted tree in the free Pre-Lie algebra.

This is the number of vertices.

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: A.degree_on_basis(RT([RT([])]))
2
```

gen(*i*)

Return the `i`-th generator of the algebra.

INPUT:

- `i` – an integer

EXAMPLES:

```
sage: F = algebras.FreePreLie(ZZ, 'xyz')
sage: F.gen(0)
B[x[]]

sage: F.gen(4)
Traceback (most recent call last):
...
IndexError: argument i (= 4) must be between 0 and 2
```

gens()

Return the generators of `self` (as an algebra).

EXAMPLES:

```
sage: A = algebras.FreePreLie(ZZ, 'fgh')
sage: A.gens()
(B[f[]], B[g[]], B[h[]])
```

nap_product()

Return the NAP product.

See also:

nap_product_on_basis()

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = A(RT([RT([])]))
sage: A.nap_product(x, x)
B[[[], [[]]]]
```

nap_product_on_basis(*x*, *y*)

Return the NAP product of two trees.

This is the grafting of the root of y over the root of x. The root of the resulting tree is the root of x.

See also:

nap_product()

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.nap_product_on_basis(x, x)
B[[[], [[]]]]
```

pre_Lie_product()

Return the pre-Lie product.

See also:

pre_Lie_product_on_basis()

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = A(RT([RT([])]))
sage: A.pre_Lie_product(x, x)
B[[[[[]]]]] + B[[[], [[]]]]
```

pre_Lie_product_on_basis(*x*, *y*)

Return the pre-Lie product of two trees.

This is the sum over all graftings of the root of y over a vertex of x. The root of the resulting trees is the root of x.

See also:

pre_Lie_product()

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.product_on_basis(x, x)
B[[[[[]]]]] + B[[[], [[]]]]
```

product_on_basis(*x*, *y*)

Return the pre-Lie product of two trees.

This is the sum over all graftings of the root of y over a vertex of x. The root of the resulting trees is the root of x.

See also:

pre_Lie_product()

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.product_on_basis(x, x)
B[[[[[]]]]] + B[[[], [[]]]]
```

some_elements()

Return some elements of the free pre-Lie algebra.

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: A.some_elements()
[B[[]], B[[[]]], B[[[[[]]]]] + B[[[], [[]]]], B[[[[]]]] + B[[[], []]], B[[[]]]]
```

With several generators:

```
sage: A = algebras.FreePreLie(QQ, 'xy')
sage: A.some_elements()
[B[x[]],
 B[x[x[]]],
 B[x[x[x[x[]]]]] + B[x[x[], x[x[]]]],
 B[x[x[x[]]]] + B[x[x[], x[]]],
 B[x[x[y[]]]] + B[x[x[], y[]]]]
```

variable_names()

Return the names of the variables.

EXAMPLES:

```
sage: R = algebras.FreePreLie(QQ, 'xy')
sage: R.variable_names()
{'x', 'y'}

sage: R = algebras.FreePreLie(QQ, None)
sage: R.variable_names()
{'o'}
```

class sage.combinat.free_prelie_algebra.**PreLieFunctor**(*vars*)

Bases: ConstructionFunctor

A constructor for pre-Lie algebras.

EXAMPLES:

```
sage: P = algebras.FreePreLie(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F = P.construction()[0]; F
```

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```
PreLie[x,y]

sage: A = GF(5)['a,b']
sage: a, b = A.gens()
sage: F(A)
Free PreLie algebra on 2 generators ['x', 'y'] over Multivariate Polynomial Ring in␣
↪a, b over Finite Field of size 5

sage: f = A.hom([a+b,a-b],A)
sage: F(f)
Generic endomorphism of Free PreLie algebra on 2 generators ['x', 'y']
over Multivariate Polynomial Ring in a, b over Finite Field of size 5

sage: F(f)(a * F(A)(x))
(a+b)*B[x[]]
```

merge(*other*)

Merge `self` with another construction functor, or return None.

EXAMPLES:

```
sage: F = sage.combinat.free_prelie_algebra.PreLieFunctor(['x','y'])
sage: G = sage.combinat.free_prelie_algebra.PreLieFunctor(['t'])
sage: F.merge(G)
PreLie[x,y,t]
sage: F.merge(F)
PreLie[x,y]
```

Now some actual use cases:

```
sage: R = algebras.FreePreLie(ZZ, 'xyz')
sage: x,y,z = R.gens()
sage: 1/2 * x
1/2*B[x[]]
sage: parent(1/2 * x)
Free PreLie algebra on 3 generators ['x', 'y', 'z'] over Rational Field

sage: S = algebras.FreePreLie(QQ, 'zt')
sage: z,t = S.gens()
sage: x + t
B[t[]] + B[x[]]
sage: parent(x + t)
Free PreLie algebra on 4 generators ['z', 't', 'x', 'y'] over Rational Field
```

rank = 9

9.7 Shuffle algebras

AUTHORS:

- Frédéric Chapoton (2013-03): Initial version
- Matthieu Deneufchatel (2013-07): Implemented dual PBW basis

class sage.algebras.shuffle_algebra.**DualPBWBasis**(*R*, *names*)

Bases: CombinatorialFreeModule

The basis dual to the Poincaré-Birkhoff-Witt basis of the free algebra.

We recursively define the dual PBW basis as the basis of the shuffle algebra given by

$$S_w = \begin{cases} w & |w| = 1, \\ xS_u & w = xu \text{ and } w \in \mathrm{Lyn}(X), \\ \dfrac{S_{\ell_{i_1}}^{*\alpha_1} * \cdots * S_{\ell_{i_k}}^{*\alpha_k}}{\alpha_1! \cdots \alpha_k!} & w = \ell_{i_1}^{\alpha_1} \cdots \ell_{i_k}^{\alpha_k} \text{ with } \ell_1 > \cdots > \ell_k \in \mathrm{Lyn}(X). \end{cases}$$

where $S * T$ denotes the shuffle product of S and T and $\mathrm{Lyn}(X)$ is the set of Lyndon words in the alphabet X.

The definition may be found in Theorem 5.3 of [Reu1993].

INPUT:

- R – ring
- `names` – names of the generators (string or an alphabet)

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S
The dual Poincare-Birkhoff-Witt basis of Shuffle Algebra on 2 generators ['a', 'b']␣
↪over Rational Field
sage: S.one()
S[]
sage: S.one_basis()
word:
sage: T = ShuffleAlgebra(QQ, 'abcd').dual_pbw_basis(); T
The dual Poincare-Birkhoff-Witt basis of Shuffle Algebra on 4 generators ['a', 'b',
↪'c', 'd'] over Rational Field
sage: T.algebra_generators()
(S[a], S[b], S[c], S[d])
```

class Element

Bases: IndexedFreeModuleElement

An element in the dual PBW basis.

expand()

Expand `self` in words of the shuffle algebra.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: f = S('ab') + S('bab')
sage: f.expand()
B[ab] + 2*B[abb] + B[bab]
```

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S.algebra_generators()
(S[a], S[b])
```

antipode(*elt*)

Return the antipode of the element `elt`.

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ, 'ab')
sage: S = A.dual_pbw_basis()
sage: w = S('abaab').antipode(); w
S[abaab] - 2*S[ababa] - S[baaba]
 + 3*S[babaa] - 6*S[bbaaa]
sage: w.antipode()
S[abaab]
```

coproduct(*elt*)

Return the coproduct of the element `elt`.

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ, 'ab')
sage: S = A.dual_pbw_basis()
sage: S('ab').coproduct()
S[] # S[ab] + S[a] # S[b]
 + S[ab] # S[]
sage: S('ba').coproduct()
S[] # S[ba] + S[a] # S[b]
 + S[b] # S[a] + S[ba] # S[]
```

counit(*S*)

Return the counit of `S`.

EXAMPLES:

```
sage: F = ShuffleAlgebra(QQ,'ab').dual_pbw_basis()
sage: (3*F.gen(0)+5*F.gen(1)**2).counit()
0
sage: (4*F.one()).counit()
4
```

degree_on_basis(*w*)

Return the degree of the element `w`.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: [S.degree_on_basis(x.leading_support()) for x in S.some_elements() if x !
↪= 0]
[0, 1, 1, 2]
```

expansion()

Return the morphism corresponding to the expansion into words of the shuffle algebra.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: f = S('ab') + S('aba')
sage: S.expansion(f)
2*B[aab] + B[ab] + B[aba]
```

expansion_on_basis(w)

Return the expansion of S_w in words of the shuffle algebra.

INPUT:

- `w` – a word

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S.expansion_on_basis(Word())
B[]
sage: S.expansion_on_basis(Word()).parent()
Shuffle Algebra on 2 generators ['a', 'b'] over Rational Field
sage: S.expansion_on_basis(Word('abba'))
2*B[aabb] + B[abab] + B[abba]
sage: S.expansion_on_basis(Word())
B[]
sage: S.expansion_on_basis(Word('abab'))
2*B[aabb] + B[abab]
```

gen(i)

Return the `i`-th generator of `self`.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S.gen(0)
S[a]
sage: S.gen(1)
S[b]
```

gens()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S.algebra_generators()
(S[a], S[b])
```

one_basis()

Return the indexing element of the basis element 1.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S.one_basis()
word:
```

product(*u*, *v*)

Return the product of two elements u and v.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: a,b = S.gens()
sage: S.product(a, b)
S[ba]
sage: S.product(b, a)
S[ba]
sage: S.product(b^2*a, a*b*a)
36*S[bbbaaa]
```

shuffle_algebra()

Return the associated shuffle algebra of `self`.

EXAMPLES:

```
sage: S = ShuffleAlgebra(QQ, 'ab').dual_pbw_basis()
sage: S.shuffle_algebra()
Shuffle Algebra on 2 generators ['a', 'b'] over Rational Field
```

some_elements()

Return some typical elements.

EXAMPLES:

```
sage: F = ShuffleAlgebra(QQ,'xyz').dual_pbw_basis()
sage: F.some_elements()
[0, S[], S[x], S[y], S[z], S[zx]]
```

class sage.algebras.shuffle_algebra.**ShuffleAlgebra**(*R*, *names*, *prefix*)

Bases: `CombinatorialFreeModule`

The shuffle algebra on some generators over a base ring.

Shuffle algebras are commutative and associative algebras, with a basis indexed by words. The product of two words $w_1 \cdot w_2$ is given by the sum over the shuffle product of w_1 and w_2.

See also:

For more on shuffle products, see `shuffle_product` and `shuffle()`.

REFERENCES:

- Wikipedia article Shuffle algebra

INPUT:

- `R` – ring
- `names` – generator names (string or an alphabet)

EXAMPLES:

```
sage: F = ShuffleAlgebra(QQ, 'xyz'); F
Shuffle Algebra on 3 generators ['x', 'y', 'z'] over Rational Field

sage: mul(F.gens())
B[xyz] + B[xzy] + B[yxz] + B[yzx] + B[zxy] + B[zyx]

sage: mul([ F.gen(i) for i in range(2) ]) + mul([ F.gen(i+1) for i in range(2) ])
B[xy] + B[yx] + B[yz] + B[zy]

sage: S = ShuffleAlgebra(ZZ, 'abcabc'); S
Shuffle Algebra on 3 generators ['a', 'b', 'c'] over Integer Ring
sage: S.base_ring()
Integer Ring

sage: G = ShuffleAlgebra(S, 'mn'); G
Shuffle Algebra on 2 generators ['m', 'n'] over Shuffle Algebra on 3 generators ['a
↪', 'b', 'c'] over Integer Ring
sage: G.base_ring()
Shuffle Algebra on 3 generators ['a', 'b', 'c'] over Integer Ring
```

Shuffle algebras commute with their base ring:

```
sage: K = ShuffleAlgebra(QQ,'ab')
sage: a,b = K.gens()
sage: K.is_commutative()
True
sage: L = ShuffleAlgebra(K,'cd')
sage: c,d = L.gens()
sage: L.is_commutative()
True
sage: s = a*b^2 * c^3; s
(12*B[abb]+12*B[bab]+12*B[bba])*B[ccc]
sage: parent(s)
Shuffle Algebra on 2 generators ['c', 'd'] over Shuffle Algebra on 2 generators ['a
↪', 'b'] over Rational Field
sage: c^3 * a * b^2
(12*B[abb]+12*B[bab]+12*B[bba])*B[ccc]
```

Shuffle algebras are commutative:

```
sage: c^3 * b * a * b == c * a * c * b^2 * c
True
```

We can also manipulate elements in the basis and coerce elements from our base field:

```
sage: F = ShuffleAlgebra(QQ, 'abc')
sage: B = F.basis()
sage: B[Word('bb')] * B[Word('ca')]
B[bbca] + B[bcab] + B[bcba] + B[cabb]
 + B[cbab] + B[cbba]
sage: 1 - B[Word('bb')] * B[Word('ca')] / 2
B[] - 1/2*B[bbca] - 1/2*B[bcab] - 1/2*B[bcba]
 - 1/2*B[cabb] - 1/2*B[cbab] - 1/2*B[cbba]
```

algebra_generators()

Return the generators of this algebra.

EXAMPLES:

```
sage: A = ShuffleAlgebra(ZZ,'fgh'); A
Shuffle Algebra on 3 generators ['f', 'g', 'h'] over Integer Ring
sage: A.algebra_generators()
Family (B[f], B[g], B[h])

sage: A = ShuffleAlgebra(QQ, ['x1','x2'])
sage: A.algebra_generators()
Family (B[x1], B[x2])
```

antipode_on_basis(*w*)

Return the antipode on the basis element w.

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ,'abc')
sage: W = A.basis().keys()
sage: A.antipode_on_basis(W("acb"))
-B[bca]
```

coproduct_on_basis(*w*)

Return the coproduct of the element of the basis indexed by the word w.

The coproduct is given by deconcatenation.

INPUT:

- w – a word

EXAMPLES:

```
sage: F = ShuffleAlgebra(QQ,'ab')
sage: F.coproduct_on_basis(Word('a'))
B[] # B[a] + B[a] # B[]
sage: F.coproduct_on_basis(Word('aba'))
B[] # B[aba] + B[a] # B[ba]
 + B[ab] # B[a] + B[aba] # B[]
sage: F.coproduct_on_basis(Word())
B[] # B[]
```

counit(*S*)

Return the counit of S.

EXAMPLES:

```
sage: F = ShuffleAlgebra(QQ,'ab')
sage: S = F.an_element(); S
B[] + 2*B[a] + 3*B[b] + B[bab]
sage: F.counit(S)
1
```

degree_on_basis(*w*)

Return the degree of the element w.

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ, 'ab')
sage: [A.degree_on_basis(x.leading_support()) for x in A.some_elements() if x !
↪= 0]
[0, 1, 1, 2]
```

dual_pbw_basis()

Return the dual PBW of self.

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ, 'ab')
sage: A.dual_pbw_basis()
The dual Poincare-Birkhoff-Witt basis of Shuffle Algebra on 2 generators ['a',
↪'b'] over Rational Field
```

gen(*i*)

Return the i-th generator of the algebra.

INPUT:

- i – an integer

EXAMPLES:

```
sage: F = ShuffleAlgebra(ZZ,'xyz')
sage: F.gen(0)
B[x]

sage: F.gen(4)
Traceback (most recent call last):
...
IndexError: argument i (= 4) must be between 0 and 2
```

gens()

Return the generators of this algebra.

EXAMPLES:

```
sage: A = ShuffleAlgebra(ZZ,'fgh'); A
Shuffle Algebra on 3 generators ['f', 'g', 'h'] over Integer Ring
sage: A.algebra_generators()
Family (B[f], B[g], B[h])

sage: A = ShuffleAlgebra(QQ, ['x1','x2'])
sage: A.algebra_generators()
Family (B[x1], B[x2])
```

one_basis()

Return the empty word, which index of 1 of this algebra, as per AlgebrasWithBasis.ParentMethods.one_basis().

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ,'a')
sage: A.one_basis()
word:
sage: A.one()
B[]
```

product_on_basis(*w1*, *w2*)

Return the product of basis elements `w1` and `w2`, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

INPUT:

- `w1, w2` – Basis elements

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ,'abc')
sage: W = A.basis().keys()
sage: A.product_on_basis(W("acb"), W("cba"))
B[acbacb] + B[acbcab] + 2*B[acbcba]
 + 2*B[accbab] + 4*B[accbba] + B[cabacb]
 + B[cabcab] + B[cabcba] + B[cacbab]
 + 2*B[cacbba] + 2*B[cbaacb] + B[cbacab]
 + B[cbacba]

sage: (a,b,c) = A.algebra_generators()
sage: a * (1-b)^2 * c
2*B[abbc] - 2*B[abc] + 2*B[abcb] + B[ac]
 - 2*B[acb] + 2*B[acbb] + 2*B[babc]
 - 2*B[bac] + 2*B[bacb] + 2*B[bbac]
 + 2*B[bbca] - 2*B[bca] + 2*B[bcab]
 + 2*B[bcba] + B[ca] - 2*B[cab] + 2*B[cabb]
 - 2*B[cba] + 2*B[cbab] + 2*B[cbba]
```

some_elements()

Return some typical elements.

EXAMPLES:

```
sage: F = ShuffleAlgebra(ZZ,'xyz')
sage: F.some_elements()
[0, B[], B[x], B[y], B[z], B[xz] + B[zx]]
```

to_dual_pbw_element(*w*)

Return the element w of `self` expressed in the dual PBW basis.

INPUT:

- `w` – an element of the shuffle algebra

EXAMPLES:

```
sage: A = ShuffleAlgebra(QQ, 'ab')
sage: f = 2 * A(Word()) + A(Word('ab')); f
2*B[] + B[ab]
sage: A.to_dual_pbw_element(f)
```

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```
2*S[] + S[ab]
sage: A.to_dual_pbw_element(A.one())
S[]
sage: S = A.dual_pbw_basis()
sage: elt = S.expansion_on_basis(Word('abba')); elt
2*B[aabb] + B[abab] + B[abba]
sage: A.to_dual_pbw_element(elt)
S[abba]
sage: A.to_dual_pbw_element(2*A(Word('aabb')) + A(Word('abab')))
S[abab]
sage: S.expansion(S('abab'))
2*B[aabb] + B[abab]
```

variable_names()

Return the names of the variables.

EXAMPLES:

```
sage: R = ShuffleAlgebra(QQ,'xy')
sage: R.variable_names()
{'x', 'y'}
```

9.8 Free Zinbiel Algebras

AUTHORS:

- Travis Scrimshaw (2015-09): initial version

class sage.algebras.free_zinbiel_algebra.**FreeZinbielAlgebra**(*R*, *n*, *names*, *prefix*, *side*)

Bases: `CombinatorialFreeModule`

The free Zinbiel algebra on n generators.

Let R be a ring. A *Zinbiel algebra* is a non-associative algebra with multiplication $\circ$ that satisfies

$$(a \circ b) \circ c = a \circ (b \circ c) + a \circ (c \circ b).$$

Zinbiel algebras were first introduced by Loday (see [Lod1995] and [LV2012]) as the Koszul dual to Leibniz algebras (hence the name coined by Lemaire).

By default, the convention above is used. The opposite product, which satisfy the opposite axiom, can be used instead by setting the `side` parameter to `'>'` instead of the default value `'<'`.

Zinbiel algebras are divided power algebras, in that for

$$x^{\circ n} = \bigl(x \circ (x \circ \cdots \circ (x \circ x) \cdots)\bigr)$$

we have

$$x^{\circ m} \circ x^{\circ n} = \binom{n+m-1}{m} x^{n+m}$$

and

$$\underbrace{\bigl((x \circ \cdots \circ x \circ (x \circ x) \cdots)\bigr)}_{n+1 \text{ times}} = n! x^n.$$

Note: This implies that Zinbiel algebras are not power associative.

To every Zinbiel algebra, we can construct a corresponding commutative associative algebra by using the symmetrized product:

$$a * b = a \circ b + b \circ a.$$

The free Zinbiel algebra on n generators is isomorphic as R-modules to the reduced tensor algebra $\bar{T}(R^n)$ with the product

$$(x_0 x_1 \cdots x_p) \circ (x_{p+1} x_{p+2} \cdots x_{p+q}) = \sum_{\sigma \in S_{p,q}} x_0 (x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(p+q)},$$

where $S_{p,q}$ is the set of (p, q)-shuffles.

The free Zinbiel algebra is free as a divided power algebra. Moreover, the corresponding commutative algebra is isomorphic to the (non-unital) shuffle algebra.

INPUT:

- `R` – a ring
- `n` – (optional) the number of generators
- `names` – the generator names

Warning: Currently the basis is indexed by all finite words over the variables, including the empty word. This is a slight abuse as it is supposed to be indexed by all non-empty words.

EXAMPLES:

We create the free Zinbiel algebra and check the defining relation:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ)
sage: (x*y)*z
Z[xyz] + Z[xzy]
sage: x*(y*z) + x*(z*y)
Z[xyz] + Z[xzy]
```

We see that the Zinbiel algebra is not associative, not even power associative:

```
sage: x*(y*z)
Z[xyz]
sage: x*(x*x)
Z[xxx]
sage: (x*x)*x
2*Z[xxx]
```

We verify that it is a divided power algebra:

```
sage: (x*(x*x)) * (x*(x*(x*x)))
15*Z[xxxxxxx]
sage: binomial(3+4-1,4)
15
```

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```
sage: (x*(x*(x*x))) * (x*(x*x))
20*Z[xxxxxxx]
sage: binomial(3+4-1,3)
20
sage: ((x*x)*x)*x
6*Z[xxxx]
sage: (((x*x)*x)*x)*x
24*Z[xxxxx]
```

A few tests with the opposite convention for the product:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ, side='>')
sage: (x*y)*z
Z[xyz]
sage: x*(y*z)
Z[xyz] + Z[yxz]
```

REFERENCES:

- Wikipedia article Zinbiel_algebra
- [Lod1995]
- [LV2012]

algebra_generators()

Return the algebra generators of `self`.

EXAMPLES:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ)
sage: list(Z.algebra_generators())
[Z[x], Z[y], Z[z]]
```

change_ring(*R*)

Return the free Zinbiel algebra in the same variables over R.

INPUT:

- `R` – a ring

The same side convention is used for the product.

EXAMPLES:

```
sage: A = algebras.FreeZinbiel(ZZ, 'f,g,h')
sage: A.change_ring(QQ)
Free Zinbiel algebra on generators (Z[f], Z[g], Z[h])
over Rational Field
```

construction()

Return a pair `(F, R)`, where `F` is a *ZinbielFunctor* and `R` is a ring, such that `F(R)` returns `self`.

EXAMPLES:

```
sage: P = algebras.FreeZinbiel(ZZ, 'x,y')
sage: x,y = P.gens()
```

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```
sage: F, R = P.construction()
sage: F
Zinbiel[x,y]
sage: R
Integer Ring
sage: F(ZZ) is P
True
sage: F(QQ)
Free Zinbiel algebra on generators (Z[x], Z[y]) over Rational Field
```

coproduct_on_basis(*w*)

Return the coproduct of the element of the basis indexed by the word w.

The coproduct is given by deconcatenation.

INPUT:

- w – a word

EXAMPLES:

```
sage: F = algebras.FreeZinbiel(QQ,['a','b'])
sage: F.coproduct_on_basis(Word('a'))
Z[] # Z[a] + Z[a] # Z[]
sage: F.coproduct_on_basis(Word('aba'))
Z[] # Z[aba] + Z[a] # Z[ba] + Z[ab] # Z[a] + Z[aba] # Z[]
sage: F.coproduct_on_basis(Word())
Z[] # Z[]
```

counit(*S*)

Return the counit of S.

EXAMPLES:

```
sage: F = algebras.FreeZinbiel(QQ,['a','b'])
sage: S = F.an_element(); S
Z[] + 2*Z[a] + 3*Z[b] + Z[bab]
sage: F.counit(S)
1
```

degree_on_basis(*t*)

Return the degree of a word in the free Zinbiel algebra.

This is the length.

EXAMPLES:

```
sage: A = algebras.FreeZinbiel(QQ, 'x,y')
sage: W = A.basis().keys()
sage: A.degree_on_basis(W('xy'))
2
```

gens()

Return the generators of self.

EXAMPLES:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ)
sage: Z.gens()
(Z[x], Z[y], Z[z])
```

product_on_basis_left(*x*, *y*)

Return the product < of the basis elements indexed by x and y.

This is one half of the shuffle product, where the first letter comes from the first letter of the first argument.

INPUT:

- x, y – two words

EXAMPLES:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ)
sage: (x*y)*z  # indirect doctest
Z[xyz] + Z[xzy]
```

product_on_basis_right(*x*, *y*)

Return the product > of the basis elements indexed by x and y.

This is one half of the shuffle product, where the last letter comes from the last letter of the second argument.

INPUT:

- x, y – two words

EXAMPLES:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ, side='>')
sage: (x*y)*z  # indirect doctest
Z[xyz]
```

side()

Return the choice of side for the product.

This is either '<' or '>'.

EXAMPLES:

```
sage: Z.<x,y,z> = algebras.FreeZinbiel(QQ)
sage: Z.side()
'<'
```

class sage.algebras.free_zinbiel_algebra.**ZinbielFunctor**(*variables*, *side*)

Bases: ConstructionFunctor

A constructor for free Zinbiel algebras.

EXAMPLES:

```
sage: P = algebras.FreeZinbiel(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F = P.construction()[0]; F
Zinbiel[x,y]

sage: A = GF(5)['a,b']
```

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```
sage: a, b = A.gens()
sage: F(A)
Free Zinbiel algebra on generators (Z[x], Z[y])
over Multivariate Polynomial Ring in a, b over Finite Field of size 5

sage: f = A.hom([a+b,a-b],A)
sage: F(f)
Generic endomorphism of Free Zinbiel algebra on generators (Z[x], Z[y])
over Multivariate Polynomial Ring in a, b over Finite Field of size 5

sage: F(f)(a * F(A)(x))
(a+b)*Z[x]
```

merge(*other*)

Merge `self` with another construction functor, or return `None`.

EXAMPLES:

```
sage: functor = sage.algebras.free_zinbiel_algebra.ZinbielFunctor
sage: F = functor(['x','y'], '<')
sage: G = functor(['t'], '<')
sage: F.merge(G)
Zinbiel[x,y,t]
sage: F.merge(F)
Zinbiel[x,y]
```

With an infinite generating set:

```
sage: H = functor(ZZ, '<')
sage: H.merge(H) is H
True
sage: H.merge(F) is None
True
sage: F.merge(H) is None
True
```

Now some actual use cases:

```
sage: R = algebras.FreeZinbiel(ZZ, 'x,y,z')
sage: x,y,z = R.gens()
sage: 1/2 * x
1/2*Z[x]
sage: parent(1/2 * x)
Free Zinbiel algebra on generators (Z[x], Z[y], Z[z])
over Rational Field

sage: S = algebras.FreeZinbiel(QQ, 'z,t')
sage: z,t = S.gens()
sage: x * t
Z[xt]
sage: parent(x * t)
Free Zinbiel algebra on generators (Z[z], Z[t], Z[x], Z[y])
over Rational Field
```

```
rank = 9
```

CHAPTER

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