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# Differential Geometry of Curves and Surfaces

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## DIFFERENTIAL GEOMETRY OF PARAMETRIZED SURFACES

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- Joris Vankerschaver (2010-10-25): implementation, doctests

```
class sage.geometry.riemannian_manifolds.parametrized_surface3d.ParametrizedSurface3D(equation,
                                                                                       vari-
                                                                                       ables,
                                                                                       name=None)
```

Bases: `SageObject`

Class representing a parametrized two-dimensional surface in Euclidian three-space. Provides methods for calculating the main geometrical objects related to such a surface, such as the first and the second fundamental form, the total (Gaussian) and the mean curvature, the geodesic curves, parallel transport, etc.

INPUT:

- `surface_equation` – a 3-tuple of functions specifying a parametric representation of the surface.
- `variables` – a 2-tuple of intrinsic coordinates  $(u, v)$  on the surface, with  $u$  and  $v$  symbolic variables, or a 2-tuple of triples  $(u, u_{min}, u_{max}), (v, v_{min}, v_{max})$  when the parameter range for the coordinates is known.
- `name` – name of the surface (optional).

---

**Note:** Throughout the documentation, we use the Einstein summation convention: whenever an index appears twice, once as a subscript, and once as a superscript, summation over that index is implied. For instance,  $g_{ij}g^{jk}$  stands for  $\sum_j g_{ij}g^{jk}$ .

---

EXAMPLES:

We give several examples of standard surfaces in differential geometry. First, let's construct an elliptic paraboloid by explicitly specifying its parametric equation:

```
sage: u, v = var('u,v', domain='real')
sage: eparaboloid = ParametrizedSurface3D((u, v, u^2 + v^2), (u, v), 'elliptic_
↳paraboloid'); eparaboloid
Parametrized surface ('elliptic paraboloid') with equation (u, v, u^2 + v^2)
```

When the ranges for the intrinsic coordinates are known, they can be specified explicitly. This is mainly useful for plotting. Here we construct half of an ellipsoid:

```

sage: u1, u2 = var('u1, u2', domain='real')
sage: coords = ((u1, -pi/2, pi/2), (u2, 0, pi))
sage: ellipsoid_eq = (cos(u1)*cos(u2), 2*sin(u1)*cos(u2), 3*sin(u2))
sage: ellipsoid = ParametrizedSurface3D(ellipsoid_eq, coords, 'ellipsoid');
↳ ellipsoid
Parametrized surface ('ellipsoid') with equation (cos(u1)*cos(u2),
↳ 2*cos(u2)*sin(u1), 3*sin(u2))
sage: ellipsoid.plot() # optional - sage.plot
Graphics3d Object

```

Standard surfaces can be constructed using the surfaces generator:

```

sage: klein = surfaces.Klein(); klein
Parametrized surface ('Klein bottle') with equation (-(sin(1/2*u)*sin(2*v) - cos(1/
↳ 2*u)*sin(v) - 1)*cos(u), -(sin(1/2*u)*sin(2*v) - cos(1/2*u)*sin(v) - 1)*sin(u),
↳ cos(1/2*u)*sin(2*v) + sin(1/2*u)*sin(v))

```

Latex representation of the surfaces:

```

sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D((cos(u)*cos(v), sin(u)*cos(v), sin(v)), (u, v),
↳ 'sphere')
sage: print(latex(sphere))
\left(\cos\left(u\right)\cos\left(v\right),\cos\left(v\right)\sin\left(u\right),
↳ \sin\left(v\right)\right)
sage: print(sphere._latex_())
\left(\cos\left(u\right)\cos\left(v\right),\cos\left(v\right)\sin\left(u\right),
↳ \sin\left(v\right)\right)
sage: print(sphere)
Parametrized surface ('sphere') with equation (cos(u)*cos(v), cos(v)*sin(u), sin(v))

```

To plot a parametric surface, use the `plot()` member function:

```

sage: enneper = surfaces.Enneper(); enneper
Parametrized surface ('Enneper's surface') with equation (-1/9*(u^2 - 3*v^2 - 3)*u,
↳ -1/9*(3*u^2 - v^2 + 3)*v, 1/3*u^2 - 1/3*v^2)
sage: enneper.plot(aspect_ratio='automatic') # optional - sage.plot
Graphics3d Object

```

We construct an ellipsoid whose axes are given by symbolic variables  $a$ ,  $b$  and  $c$ , and find the natural frame of tangent vectors, expressed in intrinsic coordinates. Note that the result is a dictionary of vector fields:

```

sage: a, b, c = var('a, b, c', domain='real')
sage: u1, u2 = var('u1, u2', domain='real')
sage: ellipsoid_eq = (a*cos(u1)*cos(u2), b*sin(u1)*cos(u2), c*sin(u2))
sage: ellipsoid = ParametrizedSurface3D(ellipsoid_eq, (u1, u2), 'Symbolic ellipsoid
↳ '); ellipsoid
Parametrized surface ('Symbolic ellipsoid') with equation (a*cos(u1)*cos(u2),
↳ b*cos(u2)*sin(u1), c*sin(u2))

sage: ellipsoid.natural_frame()
{1: (-a*cos(u2)*sin(u1), b*cos(u1)*cos(u2), 0), 2: (-a*cos(u1)*sin(u2), -
↳ b*sin(u1)*sin(u2), c*cos(u2))}

```

We find the normal vector field to the surface. The normal vector field is the vector product of the vectors of the natural frame, and is given by:

```
sage: ellipsoid.normal_vector()
(b*c*cos(u1)*cos(u2)^2, a*c*cos(u2)^2*sin(u1), a*b*cos(u2)*sin(u2))
```

By default, the normal vector field is not normalized. To obtain the unit normal vector field of the elliptic paraboloid, we put:

```
sage: u, v = var('u,v', domain='real')
sage: eparaboloid = ParametrizedSurface3D([u,v,u^2+v^2],[u,v], 'elliptic paraboloid')
sage: eparaboloid.normal_vector(normalized=True)
(-2*u/sqrt(4*u^2 + 4*v^2 + 1), -2*v/sqrt(4*u^2 + 4*v^2 + 1), 1/sqrt(4*u^2 + 4*v^2 + 1))
```

Now let us compute the coefficients of the first fundamental form of the torus:

```
sage: u, v = var('u, v', domain='real')
sage: a, b = var('a, b', domain='real')
sage: torus = ParametrizedSurface3D((a + b*cos(u))*cos(v), (a + b*cos(u))*sin(v), b*sin(u), [u,v], 'torus')
sage: torus.first_fundamental_form_coefficients()
{(1, 1): b^2, (1, 2): 0, (2, 1): 0, (2, 2): b^2*cos(u)^2 + 2*a*b*cos(u) + a^2}
```

The first fundamental form can be used to compute the length of a curve on the surface. For example, let us find the length of the curve  $u^1 = t, u^2 = t, t \in [0, 2\pi]$ , on the ellipsoid with axes  $a = 1, b = 1.5$  and  $c = 1$ . So we take the curve:

```
sage: t = var('t', domain='real')
sage: u1 = t
sage: u2 = t
```

Then find the tangent vector:

```
sage: du1 = diff(u1,t)
sage: du2 = diff(u2,t)
sage: du = vector([du1, du2]); du
(1, 1)
```

Once we specify numerical values for the axes of the ellipsoid, we can determine the numerical value of the length integral:

```
sage: L = sqrt(ellipsoid.first_fundamental_form(du, du).substitute(u1=u1,u2=u2))
sage: numerical_integral(L.substitute(a=2, b=1.5, c=1),0,1)[0] # rel tol 1e-11
2.00127905972
```

We find the area of the sphere of radius  $R$ :

```
sage: R = var('R', domain='real')
sage: u, v = var('u,v', domain='real')
sage: assume(R>0)
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([R*cos(u)*cos(v), R*sin(u)*cos(v), R*sin(v)], [u, v], 'sphere')
```

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```
sage: integral(integral(sphere.area_form(),u,0,2*pi),v,-pi/2,pi/2)
4*pi*R^2
```

We can find an orthonormal frame field  $\{e_1, e_2\}$  of a surface and calculate its structure functions. Let us first determine the orthonormal frame field for the elliptic paraboloid:

```
sage: u, v = var('u,v', domain='real')
sage: eparaboloid = ParametrizedSurface3D([u,v,u^2+v^2],[u,v],'elliptic paraboloid')
sage: eparaboloid.orthonormal_frame()
{1: (1/sqrt(4*u^2 + 1), 0, 2*u/sqrt(4*u^2 + 1)), 2: (-4*u*v/(sqrt(4*u^2 + 4*v^2 + 1)*sqrt(4*u^2 + 1)), sqrt(4*u^2 + 1)/sqrt(4*u^2 + 4*v^2 + 1), 2*v/(sqrt(4*u^2 + 4*v^2 + 1)*sqrt(4*u^2 + 1)))}
```

We can express the orthogonal frame field both in exterior coordinates (i.e. expressed as vector field fields in the ambient space  $\mathbf{R}^3$ , the default) or in intrinsic coordinates (with respect to the natural frame). Here we use intrinsic coordinates:

```
sage: eparaboloid.orthonormal_frame(coordinates='int')
{1: (1/sqrt(4*u^2 + 1), 0), 2: (-4*u*v/(sqrt(4*u^2 + 4*v^2 + 1)*sqrt(4*u^2 + 1)), sqrt(4*u^2 + 1)/sqrt(4*u^2 + 4*v^2 + 1))}
```

Using the orthonormal frame in interior coordinates, we can calculate the structure functions  $c_{ij}^k$  of the surface, defined by  $[e_i, e_j] = c_{ij}^k e_k$ , where  $[e_i, e_j]$  represents the Lie bracket of two frame vector fields  $e_i, e_j$ . For the elliptic paraboloid, we get:

```
sage: EE = eparaboloid.orthonormal_frame(coordinates='int')
sage: E1 = EE[1]; E2 = EE[2]
sage: CC = eparaboloid.frame_structure_functions(E1,E2)
sage: CC[1,2,1].simplify_full()
4*sqrt(4*u^2 + 4*v^2 + 1)*v/((16*u^4 + 4*(4*u^2 + 1)*v^2 + 8*u^2 + 1)*sqrt(4*u^2 + 4*v^2 + 1))
```

We compute the Gaussian and mean curvatures of the sphere:

```
sage: sphere = surfaces.Sphere(); sphere
Parametrized surface ('Sphere') with equation (cos(u)*cos(v), cos(v)*sin(u), sin(v))
sage: K = sphere.gauss_curvature(); K # Not tested -- see trac 12737
1
sage: H = sphere.mean_curvature(); H # Not tested -- see trac 12737
-1
```

We can easily generate a color plot of the Gaussian curvature of a surface. Here we deal with the ellipsoid:

```
sage: u1, u2 = var('u1,u2', domain='real')
sage: u = [u1,u2]
sage: ellipsoid_equation(u1,u2) = [2*cos(u1)*cos(u2), 1.5*cos(u1)*sin(u2), sin(u1)]
sage: ellipsoid = ParametrizedSurface3D(ellipsoid_equation(u1,u2), [u1, u2],
↳ 'ellipsoid')
sage: # set intervals for variables and the number of division points
sage: u1min, u1max = -1.5, 1.5
sage: u2min, u2max = 0, 6.28
sage: u1num, u2num = 10, 20
sage: # make the arguments array
```

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```

sage: from numpy import linspace
sage: u1_array = linspace(u1min, u1max, u1num)
sage: u2_array = linspace(u2min, u2max, u2num)
sage: u_array = [ (uu1,uu2) for uu1 in u1_array for uu2 in u2_array]
sage: # Find the gaussian curvature
sage: K(u1,u2) = ellipsoid.gauss_curvature()
sage: # Make array of K values
sage: K_array = [K(uu[0],uu[1]) for uu in u_array]
sage: # Find minimum and max of the Gauss curvature
sage: K_max = max(K_array)
sage: K_min = min(K_array)
sage: # Make the array of color coefficients
sage: cc_array = [ (ccc - K_min)/(K_max - K_min) for ccc in K_array ]
sage: points_array = [ellipsoid.equation(u_array[counter][0],u_array[counter][1]),
→for counter in range(0,len(u_array)) ]
sage: curvature_ellipsoid_plot = sum( point([xx for xx in points_array[counter]],
→color=hue(cc_array[counter]/2)) for counter in range(0,len(u_array)) )
sage: curvature_ellipsoid_plot.show(aspect_ratio=1)

```

We can find the principal curvatures and principal directions of the elliptic paraboloid:

```

sage: u, v = var('u, v', domain='real')
sage: eparaboloid = ParametrizedSurface3D([u, v, u^2+v^2], [u, v], 'elliptic_
→paraboloid')
sage: pd = eparaboloid.principal_directions(); pd
[(2*sqrt(4*u^2 + 4*v^2 + 1)/(16*u^4 + 16*v^4 + 8*(4*u^2 + 1)*v^2 + 8*u^2 + 1), [(1,
→v/u)], 1), (2/sqrt(4*u^2 + 4*v^2 + 1), [(1, -u/v)], 1)]

```

We extract the principal curvatures:

```

sage: k1 = pd[0][0].simplify_full()
sage: k1
2*sqrt(4*u^2 + 4*v^2 + 1)/(16*u^4 + 16*v^4 + 8*(4*u^2 + 1)*v^2 + 8*u^2 + 1)
sage: k2 = pd[1][0].simplify_full()
sage: k2
2/sqrt(4*u^2 + 4*v^2 + 1)

```

and check them by comparison with the Gaussian and mean curvature expressed in terms of the principal curvatures:

```

sage: K = eparaboloid.gauss_curvature().simplify_full()
sage: K
4/(16*u^4 + 16*v^4 + 8*(4*u^2 + 1)*v^2 + 8*u^2 + 1)
sage: H = eparaboloid.mean_curvature().simplify_full()
sage: H
2*(2*u^2 + 2*v^2 + 1)/(4*u^2 + 4*v^2 + 1)^(3/2)
sage: (K - k1*k2).simplify_full()
0
sage: (2*H - k1 - k2).simplify_full()
0

```

We can find the intrinsic (local coordinates) of the principal directions:

```
sage: pd[0][1]
[(1, v/u)]
sage: pd[1][1]
[(1, -u/v)]
```

The ParametrizedSurface3D class also contains functionality to compute the coefficients of the second fundamental form, the shape operator, the rotation on the surface at a given angle, the connection coefficients. One can also calculate numerically the geodesics and the parallel translation along a curve.

Here we compute a number of geodesics on the sphere emanating from the point  $(1, 0, 0)$ , in various directions. The geodesics intersect again in the antipodal point  $(-1, 0, 0)$ , indicating that these points are conjugate:

```
sage: S = surfaces.Sphere()
sage: g1 = [c[-1] for c in S.geodesics_numerical((0,0),(1,0),(0,2*pi,100))]
sage: g2 = [c[-1] for c in S.geodesics_numerical((0,0),(cos(pi/3),sin(pi/3)),(0,
↪ 2*pi,100))]
sage: g3 = [c[-1] for c in S.geodesics_numerical((0,0),(cos(2*pi/3),sin(2*pi/3)),(0,
↪ 2*pi,100))]
sage: (S.plot(opacity=0.3) + line3d(g1,color='red') + line3d(g2,color='red') +
↪ line3d(g3,color='red')).show() # optional - sage.plot
```

#### area\_form()

Returns the coefficient of the area form on the surface. In terms of the coefficients  $g_{ij}$  (where  $i, j = 1, 2$ ) of the first fundamental form, the coefficient of the area form is given by  $A = \sqrt{g_{11}g_{22} - g_{12}^2}$ .

See also [area\\_form\\_squared\(\)](#).

OUTPUT:

- Coefficient of the area form

EXAMPLES:

```
sage: u, v = var('u,v', domain='real')
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
sage: sphere.area_form()
cos(v)
```

#### area\_form\_squared()

Returns the square of the coefficient of the area form on the surface. In terms of the coefficients  $g_{ij}$  (where  $i, j = 1, 2$ ) of the first fundamental form, this invariant is given by  $A^2 = g_{11}g_{22} - g_{12}^2$ .

See also [area\\_form\(\)](#).

OUTPUT:

- Square of the area form

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
sage: sphere.area_form_squared()
cos(v)^2
```

**connection\_coefficients()**

Computes the connection coefficients or Christoffel symbols  $\Gamma_{ij}^k$  of the surface. If the coefficients of the first fundamental form are given by  $g_{ij}$  (where  $i, j = 1, 2$ ), then  $\Gamma_{ij}^k = \frac{1}{2}g^{kl} \left( \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^l} \right)$ . Here,  $(g^{kl})$  is the inverse of the matrix  $(g_{ij})$ , with  $i, j = 1, 2$ .

OUTPUT:

Dictionary of connection coefficients, where the keys are 3-tuples  $(i, j, k)$  and the values are the corresponding coefficients  $\Gamma_{ij}^k$ .

EXAMPLES:

```
sage: r = var('r')
sage: assume(r > 0)
sage: u, v = var('u,v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([r*cos(u)*cos(v), r*sin(u)*cos(v), r*sin(v)],
↪ [u,v], 'sphere')
sage: sphere.connection_coefficients()
{(1, 1, 1): 0,
 (1, 1, 2): cos(v)*sin(v),
 (1, 2, 1): -sin(v)/cos(v),
 (1, 2, 2): 0,
 (2, 1, 1): -sin(v)/cos(v),
 (2, 1, 2): 0,
 (2, 2, 1): 0,
 (2, 2, 2): 0}
```

**first\_fundamental\_form(vector1, vector2)**

Evaluate the first fundamental form on two vectors expressed with respect to the natural coordinate frame on the surface. In other words, if the vectors are  $v = (v^1, v^2)$  and  $w = (w^1, w^2)$ , calculate  $g_{11}v^1w^1 + g_{12}(v^1w^2 + v^2w^1) + g_{22}v^2w^2$ , with  $g_{ij}$  the coefficients of the first fundamental form.

INPUT:

- vector1, vector2 - vectors on the surface.

OUTPUT:

- First fundamental form evaluated on the input vectors.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: v1, v2, w1, w2 = var('v1, v2, w1, w2', domain='real')
sage: sphere = ParametrizedSurface3D((cos(u)*cos(v), sin(u)*cos(v), sin(v)), (u,
↪ v), 'sphere')
sage: sphere.first_fundamental_form(vector([v1,v2]),vector([w1,w2]))
v1*w1*cos(v)^2 + v2*w2

sage: vv = vector([1,2])
sage: sphere.first_fundamental_form(vv,vv)
cos(v)^2 + 4

sage: sphere.first_fundamental_form([1,1],[2,1])
2*cos(v)^2 + 1
```

**first\_fundamental\_form\_coefficient(index)**

Compute a single component  $g_{ij}$  of the first fundamental form. If the parametric representation of the surface is given by the vector function  $\vec{r}(u^i)$ , where  $u^i, i = 1, 2$  are curvilinear coordinates, then  $g_{ij} = \frac{\partial \vec{r}}{\partial u^i} \cdot \frac{\partial \vec{r}}{\partial u^j}$ .

INPUT:

- index - tuple (i, j) specifying the index of the component  $g_{ij}$ .

OUTPUT:

- Component  $g_{ij}$  of the first fundamental form

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: eparaboloid = ParametrizedSurface3D((u, v, u^2+v^2), (u, v))
sage: eparaboloid.first_fundamental_form_coefficient((1,2))
4*u*v
```

When the index is invalid, an error is raised:

```
sage: u, v = var('u, v', domain='real')
sage: eparaboloid = ParametrizedSurface3D((u, v, u^2+v^2), (u, v))
sage: eparaboloid.first_fundamental_form_coefficient((1,5))
Traceback (most recent call last):
...
ValueError: Index (1, 5) out of bounds.
```

**first\_fundamental\_form\_coefficients()**

Returns the coefficients of the first fundamental form as a dictionary. The keys are tuples  $(i, j)$ , where  $i$  and  $j$  range over 1, 2, while the values are the corresponding coefficients  $g_{ij}$ .

OUTPUT:

- Dictionary of first fundamental form coefficients.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D((cos(u)*cos(v), sin(u)*cos(v), sin(v)), (u,
↪ v), 'sphere')
sage: sphere.first_fundamental_form_coefficients()
{(1, 1): cos(v)^2, (1, 2): 0, (2, 1): 0, (2, 2): 1}
```

**first\_fundamental\_form\_inverse\_coefficient(index)**

Returns a specific component  $g^{ij}$  of the inverse of the fundamental form.

INPUT:

- index - tuple (i, j) specifying the index of the component  $g^{ij}$ .

OUTPUT:

- Component of the inverse of the fundamental form.

EXAMPLES:

```

sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u, v],
↪ 'sphere')
sage: sphere.first_fundamental_form_inverse_coefficient((1, 2))
0
sage: sphere.first_fundamental_form_inverse_coefficient((1, 1))
cos(v)^(-2)

```

### first\_fundamental\_form\_inverse\_coefficients()

Returns the coefficients  $g^{ij}$  of the inverse of the fundamental form, as a dictionary. The inverse coefficients are defined by  $g^{ij}g_{jk} = \delta_k^i$  with  $\delta_k^i$  the Kronecker delta.

OUTPUT:

- Dictionary of the inverse coefficients.

EXAMPLES:

```

sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u, v],
↪ 'sphere')
sage: sphere.first_fundamental_form_inverse_coefficients()
{(1, 1): cos(v)^(-2), (1, 2): 0, (2, 1): 0, (2, 2): 1}

```

### frame\_structure\_functions(e1, e2)

Returns the structure functions  $c_{ij}^k$  for a frame field  $e_1, e_2$ , i.e. a pair of vector fields on the surface which are linearly independent at each point. The structure functions are defined using the Lie bracket by  $[e_i, e_j] = c_{ij}^k e_k$ .

INPUT:

- $e_1, e_2$  - vector fields in intrinsic coordinates on the surface, expressed as pairs of functions, or as vectors of length 2.

OUTPUT:

- Dictionary of structure functions, where the key  $(i, j, k)$  refers to the structure function  $c_{i,j}^k$ .

EXAMPLES:

```

sage: u, v = var('u, v', domain='real')
sage: assume(cos(v) > 0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,
↪ v], 'sphere')
sage: sphere.frame_structure_functions([u, v], [-v, u])
{(1, 1, 1): 0,
 (1, 1, 2): 0,
 (1, 2, 1): 0,
 (1, 2, 2): 0,
 (2, 1, 1): 0,
 (2, 1, 2): 0,
 (2, 2, 1): 0,
 (2, 2, 2): 0}

```

We construct the structure functions of the orthonormal frame on the surface:

```

sage: EE_int = sphere.orthonormal_frame(coordinates='int')
sage: CC = sphere.frame_structure_functions(EE_int[1],EE_int[2]); CC
{(1, 1, 1): 0,
 (1, 1, 2): 0,
 (1, 2, 1): sin(v)/cos(v),
 (1, 2, 2): 0,
 (2, 1, 1): -sin(v)/cos(v),
 (2, 1, 2): 0,
 (2, 2, 1): 0,
 (2, 2, 2): 0}
sage: sphere.lie_bracket(EE_int[1],EE_int[2]) - CC[(1,2,1)]*EE_int[1] - CC[(1,2,
↪2)]*EE_int[2]
(0, 0)

```

**gauss\_curvature()**

Finds the gaussian curvature of the surface, given by  $K = \frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}$ , where  $g_{ij}$  and  $h_{ij}$  are the coefficients of the first and second fundamental form, respectively.

OUTPUT:

- Gaussian curvature of the surface.

EXAMPLES:

```

sage: R = var('R')
sage: assume(R>0)
sage: u, v = var('u,v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([R*cos(u)*cos(v),R*sin(u)*cos(v),R*sin(v)],
↪[u,v], 'sphere')
sage: sphere.gauss_curvature()
R^(-2)

```

**geodesics\_numerical(p0, v0, tinterval)**

Numerical integration of the geodesic equations. Explicitly, the geodesic equations are given by  $\frac{d^2 u^i}{dt^2} + \Gamma_{jk}^i \frac{du^j}{dt} \frac{du^k}{dt} = 0$ .

Solving these equations gives the coordinates  $(u^1, u^2)$  of the geodesic on the surface. The coordinates in space can then be found by substituting  $(u^1, u^2)$  into the vector  $\vec{r}(u^1, u^2)$  representing the surface.

ALGORITHM:

The geodesic equations are integrated forward in time using the ode solvers from `sage.calculus.ode`. See the member function `_create_geodesic_ode_system` for more details.

INPUT:

- `p0` - 2-tuple with coordinates of the initial point.
- `v0` - 2-tuple with components of the initial tangent vector to the geodesic.
- `ttinterval` - List `[a, b, M]`, where `(a,b)` is the domain of the geodesic and `M` is the number of subdivision points used when returning the solution.

OUTPUT:

List of lists `[t, [u1(t), u2(t)], [v1(t), v2(t)], [x1(t), x2(t), x3(t)]]`, where

- `t` is a subdivision point;

- $[u_1(t), u_2(t)]$  are the intrinsic coordinates of the geodesic point;
- $[v_1(t), v_2(t)]$  are the intrinsic coordinates of the tangent vector to the geodesic;
- $[x_1(t), x_2(t), x_3(t)]$  are the coordinates of the geodesic point in the three-dimensional space.

EXAMPLES:

```
sage: p, q = var('p,q', domain='real')
sage: assume(cos(q)>0)
sage: sphere = ParametrizedSurface3D([cos(q)*cos(p), sin(q)*cos(p), sin(p)], [p,q],
↳ 'sphere')
sage: geodesic = sphere.geodesics_numerical([0.0,0.0],[1.0,1.0],[0,2*pi,5])
sage: times, points, tangent_vectors, ext_points = zip(*geodesic)

sage: round4 = lambda vec: [N(x, digits=4) for x in vec] # helper function to
↳ round to 4 digits
sage: round4(times)
[0.0000, 1.257, 2.513, 3.770, 5.027, 6.283]
sage: [round4(p) for p in points]
[[0.0000, 0.0000], [0.7644, 1.859], [-0.2876, 3.442], [-0.6137, 5.502], [0.5464,
↳ 6.937], [0.3714, 9.025]]
sage: [round4(p) for p in ext_points]
[[1.000, 0.0000, 0.0000], [-0.2049, 0.6921, 0.6921], [-0.9160, -0.2836, -0.
↳ 2836], [0.5803, -0.5759, -0.5759], [0.6782, 0.5196, 0.5196], [-0.8582, 0.3629,
↳ 0.3629]]
```

**lie\_bracket**( $v, w$ )

Returns the Lie bracket of two vector fields that are tangent to the surface. The vector fields should be given in intrinsic coordinates, i.e. with respect to the natural frame.

INPUT:

- $v$  and  $w$  - vector fields on the surface, expressed as pairs of functions or as vectors of length 2.

OUTPUT:

- The Lie bracket  $[v, w]$ .

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↳ 'sphere')
sage: sphere.lie_bracket([u,v], [-v,u])
(0, 0)

sage: EE_int = sphere.orthonormal_frame(coordinates='int')
sage: sphere.lie_bracket(EE_int[1], EE_int[2])
(sin(v)/cos(v)^2, 0)
```

**mean\_curvature**()

Finds the mean curvature of the surface, given by  $H = \frac{1}{2} \frac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{g_{11}g_{22} - g_{12}^2}$ , where  $g_{ij}$  and  $h_{ij}$  are the components of the first and second fundamental forms, respectively.

OUTPUT:

- Mean curvature of the surface

EXAMPLES:

```
sage: R = var('R')
sage: assume(R>0)
sage: u, v = var('u,v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([R*cos(u)*cos(v), R*sin(u)*cos(v), R*sin(v)],
↪ [u,v], 'sphere')
sage: sphere.mean_curvature()
-1/R
```

### **natural\_frame()**

Returns the natural tangent frame on the parametrized surface. The vectors of this frame are tangent to the coordinate lines on the surface.

OUTPUT:

- The natural frame as a dictionary.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: eparaboloid = ParametrizedSurface3D((u, v, u^2+v^2), (u, v), 'elliptic_
↪ paraboloid')
sage: eparaboloid.natural_frame()
{1: (1, 0, 2*u), 2: (0, 1, 2*v)}
```

### **normal\_vector(normalized=False)**

Returns the normal vector field of the parametrized surface.

INPUT:

- `normalized` - default `False` - specifies whether the normal vector should be normalized.

OUTPUT:

- Normal vector field.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: eparaboloid = ParametrizedSurface3D((u, v, u^2 + v^2), (u, v), 'elliptic_
↪ paraboloid')
sage: eparaboloid.normal_vector(normalized=False)
(-2*u, -2*v, 1)
sage: eparaboloid.normal_vector(normalized=True)
(-2*u/sqrt(4*u^2 + 4*v^2 + 1), -2*v/sqrt(4*u^2 + 4*v^2 + 1), 1/sqrt(4*u^2 + 4*v^
↪ 2 + 1))
```

### **orthonormal\_frame(coordinates='ext')**

Returns the orthonormal frame field on the surface, expressed either in exterior coordinates (i.e. expressed as vector fields in the ambient space  $\mathbb{R}^3$ , the default) or interior coordinates (with respect to the natural frame)

INPUT:

- `coordinates` - either `ext` (default) or `int`.

OUTPUT:



- Orthogonal frame field as a dictionary.

ALGORITHM:

We normalize the first vector  $\vec{e}_1$  of the natural frame and then get the second frame vector as  $\vec{e}_2 = [\vec{n}, \vec{e}_1]$ , where  $\vec{n}$  is the unit normal to the surface.

EXAMPLES:

```
sage: u, v = var('u,v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,
↪ v], 'sphere')
sage: frame = sphere.orthonormal_frame(); frame
{1: (-sin(u), cos(u), 0), 2: (-cos(u)*sin(v), -sin(u)*sin(v), cos(v))}
sage: (frame[1]*frame[1]).simplify_full()
1
sage: (frame[1]*frame[2]).simplify_full()
0
sage: frame[1] == sphere.orthonormal_frame_vector(1)
True
```

We compute the orthonormal frame with respect to the natural frame on the surface:

```
sage: frame_int = sphere.orthonormal_frame(coordinates='int'); frame_int
{1: (1/cos(v), 0), 2: (0, 1)}
sage: sphere.first_fundamental_form(frame_int[1], frame_int[1])
1
sage: sphere.first_fundamental_form(frame_int[1], frame_int[2])
0
sage: sphere.first_fundamental_form(frame_int[2], frame_int[2])
1
```

**orthonormal\_frame\_vector**(*index*, *coordinates*='ext')

Returns a specific basis vector field of the orthonormal frame field on the surface, expressed in exterior or interior coordinates. See `orthogonal_frame()` for more details.

INPUT:

- *index* - index of the basis vector;
- *coordinates* - either `ext` (default) or `int`.

OUTPUT:

- Orthonormal frame vector field.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
sage: V1 = sphere.orthonormal_frame_vector(1); V1
(-sin(u), cos(u), 0)
sage: V2 = sphere.orthonormal_frame_vector(2); V2
(-cos(u)*sin(v), -sin(u)*sin(v), cos(v))
sage: (V1*V1).simplify_full()
```

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```

1
sage: (V1*V2).simplify_full()
0

sage: n = sphere.normal_vector(normalized=True)
sage: (V1.cross_product(V2) - n).simplify_full()
(0, 0, 0)

```

**parallel\_translation\_numerical**(curve, t, v0, tinterval)

Numerically solves the equations for parallel translation of a vector along a curve on the surface. Explicitly, the equations for parallel translation are given by  $\frac{du^i}{dt} + u^j \frac{dc^k}{dt} \Gamma_{jk}^i = 0$ , where  $\Gamma_{jk}^i$  are the connection coefficients of the surface, the vector to be transported has components  $u^j$  and the curve along which to transport has components  $c^k$ .

ALGORITHM:

The parallel transport equations are integrated forward in time using the ode solvers from `sage.calculus.ode`. See `_create_pt_ode_system()` for more details.

INPUT:

- `curve` - 2-tuple of functions which determine the curve with respect to the local coordinate system;
- `t` - symbolic variable denoting the curve parameter;
- `v0` - 2-tuple representing the initial vector;
- `tiinterval` - list `[a, b, N]`, where `(a, b)` is the domain of the curve and `N` is the number of subdivision points.

OUTPUT:

The list consisting of lists `[t, [v1(t), v2(t)]]`, where

- `t` is a subdivision point;
- `[v1(t), v2(t)]` is the list of coordinates of the vector parallel translated along the curve.

EXAMPLES:

```

sage: p, q = var('p,q', domain='real')
sage: v = [p,q]
sage: assume(cos(q)>0)
sage: sphere = ParametrizedSurface3D([cos(q)*cos(p), sin(q)*cos(p), sin(p)], v,
↳ 'sphere')
sage: s = var('s')
sage: vector_field = sphere.parallel_translation_numerical([s,s], s, [1.0, 1.0], [0.
↳ 0, pi/4, 5])
sage: times, components = zip(*vector_field)

sage: round4 = lambda vec: [N(x, digits=4) for x in vec] # helper function to
↳ round to 4 digits
sage: round4(times)
[0.0000, 0.1571, 0.3142, 0.4712, 0.6283, 0.7854]
sage: [round4(v) for v in components]
[[1.000, 1.000], [0.9876, 1.025], [0.9499, 1.102], [0.8853, 1.238], [0.7920, 1.
↳ 448], [0.6687, 1.762]]

```

**plot**(*urange=None, vrange=None, \*\*kws*)

Enable easy plotting directly from the surface class.

The optional keywords *urange* and *vrange* specify the range for the surface parameters  $u$  and  $v$ . If either of these parameters is *None*, the method checks whether a parameter range was specified when the surface was created. If not, the default of  $(0, 2\pi)$  is used.

INPUT:

- *urange* - 2-tuple specifying the parameter range for  $u$ .
- *vrange* - 2-tuple specifying the parameter range for  $v$ .

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: eq = (3*u + 3*u*v^2 - u^3, 3*v + 3*u^2*v - v^3, 3*(u^2-v^2))
sage: enneper = ParametrizedSurface3D(eq, (u, v), 'Enneper Surface')
sage: enneper.plot((-5, 5), (-5, 5)) # optional - sage.plot
Graphics3d Object
```

**point**(*coords*)

Returns a point on the surface given its intrinsic coordinates.

INPUT:

- *coords* - 2-tuple specifying the intrinsic coordinates  $(u, v)$  of the point.

OUTPUT:

- 3-vector specifying the coordinates in  $\mathbf{R}^3$  of the point.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: torus = ParametrizedSurface3D(((2 + cos(u))*cos(v), (2 + cos(u))*sin(v),
↪ sin(u)), [u, v], 'torus')
sage: torus.point((0, pi/2))
(0, 3, 0)
sage: torus.point((pi/2, pi))
(-2, 0, 1)
sage: torus.point((pi, pi/2))
(0, 1, 0)
```

**principal\_directions**()

Finds the principal curvatures and principal directions of the surface

OUTPUT:

For each principal curvature, returns a list of the form  $(\rho, V, n)$ , where  $\rho$  is the principal curvature,  $V$  is the corresponding principal direction, and  $n$  is the multiplicity.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: R, r = var('R, r', domain='real')
sage: assume(R>r, r>0)
sage: torus = ParametrizedSurface3D([(R+r*cos(v))*cos(u), (R+r*cos(v))*sin(u),
↪ r*sin(v)], [u, v], 'torus')
```

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```
sage: torus.principal_directions()
[(-cos(v)/(r*cos(v) + R), [(1, 0)], 1), (-1/r, [(0, 1)], 1)]
```

```
sage: u, v = var('u, v', domain='real')
sage: V = vector([u*cos(u+v), u*sin(u+v), u+v])
sage: helicoid = ParametrizedSurface3D(V, (u, v))
sage: helicoid.principal_directions()
[(-1/(u^2 + 1), [(1, -(u^2 - sqrt(u^2 + 1) + 1)/(u^2 + 1))], 1),
(1/(u^2 + 1), [(1, -(u^2 + sqrt(u^2 + 1) + 1)/(u^2 + 1))], 1)]
```

### rotation(theta)

Gives the matrix of the rotation operator over a given angle  $\theta$  with respect to the natural frame.

INPUT:

- theta - rotation angle

OUTPUT:

- Rotation matrix with respect to the natural frame.

ALGORITHM:

The operator of rotation over  $\pi/2$  is  $J_j^i = g^{ik}\omega_{jk}$ , where  $\omega$  is the area form. The operator of rotation over an angle  $\theta$  is  $\cos(\theta)I + \sin(\theta)J$ .

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
```

We first compute the matrix of rotation over  $\pi/3$ :

```
sage: rotation = sphere.rotation(pi/3); rotation
[
      1/2  -1/2*sqrt(3)/cos(v)]
[ 1/2*sqrt(3)*cos(v)      1/2]
```

We verify that three successive rotations over  $\pi/3$  yield minus the identity:

```
sage: rotation^3
[-1  0]
[ 0 -1]
```

### second\_fundamental\_form(vector1, vector2)

Evaluates the second fundamental form on two vectors on the surface. If the vectors are given by  $v = (v^1, v^2)$  and  $w = (w^1, w^2)$ , the result of this function is  $h_{11}v^1w^1 + h_{12}(v^1w^2 + v^2w^1) + h_{22}v^2w^2$ .

INPUT:

- vector1, vector2 - 2-tuples representing the input vectors.

OUTPUT:

- Value of the second fundamental form evaluated on the given vectors.

EXAMPLES:

We evaluate the second fundamental form on two symbolic vectors:

```
sage: u, v = var('u, v', domain='real')
sage: v1, v2, w1, w2 = var('v1, v2, w1, w2', domain='real')
sage: assume(cos(v) > 0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u, v],
↪ 'sphere')
sage: sphere.second_fundamental_form(vector([v1, v2]), vector([w1, w2]))
-v1*w1*cos(v)^2 - v2*w2
```

We evaluate the second fundamental form on vectors with numerical components:

```
sage: vect = vector([1, 2])
sage: sphere.second_fundamental_form(vect, vect)
-cos(v)^2 - 4
sage: sphere.second_fundamental_form([1, 1], [2, 1])
-2*cos(v)^2 - 1
```

### `second_fundamental_form_coefficient(index)`

Returns the coefficient  $h_{ij}$  of the second fundamental form corresponding to the index  $(i, j)$ . If the equation of the surface is  $\vec{r}(u^1, u^2)$ , then  $h_{ij} = \vec{r}_{u^i u^j} \cdot \vec{n}$ , where  $\vec{n}$  is the unit normal.

INPUT:

- `index` - a 2-tuple  $(i, j)$

OUTPUT:

- Component  $h_{ij}$  of the second fundamental form.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u, v],
↪ 'sphere')
sage: sphere.second_fundamental_form_coefficient((1, 1))
-cos(v)^2
sage: sphere.second_fundamental_form_coefficient((2, 1))
0
```

### `second_fundamental_form_coefficients()`

Returns the coefficients  $h_{ij}$  of the second fundamental form as a dictionary, where the keys are the indices  $(i, j)$  and the values are the corresponding components  $h_{ij}$ .

When only one component is needed, consider instead the function `second_fundamental_form_coefficient()`.

OUTPUT:

Dictionary of second fundamental form coefficients.

EXAMPLES:

```
sage: u, v = var('u, v', domain='real')
sage: assume(cos(v)>0)
```

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```

sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
sage: sphere.second_fundamental_form_coefficients()
{(1, 1): -cos(v)^2, (1, 2): 0, (2, 1): 0, (2, 2): -1}

```

**second\_order\_natural\_frame()**

Returns the second-order frame of the surface, i.e. computes the second-order derivatives (with respect to the parameters on the surface) of the parametric expression  $\vec{r} = \vec{r}(u^1, u^2)$  of the surface.

OUTPUT:

- Dictionary where the keys are 2-tuples  $(i, j)$  and the values are the corresponding derivatives  $r_{ij}$ .

EXAMPLES:

We compute the second-order natural frame of the sphere:

```

sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
sage: sphere.second_order_natural_frame()
{(1, 1): (-cos(u)*cos(v), -cos(v)*sin(u), 0),
 (1, 2): (sin(u)*sin(v), -cos(u)*sin(v), 0),
 (2, 1): (sin(u)*sin(v), -cos(u)*sin(v), 0),
 (2, 2): (-cos(u)*cos(v), -cos(v)*sin(u), -sin(v))}

```

**second\_order\_natural\_frame\_element(index)**

Returns a vector in the second-order frame of the surface, i.e. computes the second-order derivatives of the parametric expression  $\vec{r}$  of the surface with respect to the parameters listed in the argument.

INPUT:

- index - a 2-tuple  $(i, j)$  specifying the element of the second-order frame.

OUTPUT:

- The second-order derivative  $r_{ij}$ .

EXAMPLES:

```

sage: u, v = var('u, v', domain='real')
sage: sphere = ParametrizedSurface3D([cos(u)*cos(v), sin(u)*cos(v), sin(v)], [u,v],
↪ 'sphere')
sage: sphere.second_order_natural_frame_element((1, 2))
(sin(u)*sin(v), -cos(u)*sin(v), 0)

```

**shape\_operator()**

Returns the shape operator of the surface as a matrix. The shape operator is defined as the derivative of the Gauss map, and is computed here in terms of the first and second fundamental form by means of the Weingarten equations.

OUTPUT:

- Matrix of the shape operator

EXAMPLES:

```

sage: R = var('R')
sage: assume(R>0)
sage: u, v = var('u,v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([R*cos(u)*cos(v), R*sin(u)*cos(v), R*sin(v)],
↪ [u,v], 'sphere')
sage: S = sphere.shape_operator(); S
[-1/R      0]
[      0 -1/R]

```

The eigenvalues of the shape operator are the principal curvatures of the surface:

```

sage: u, v = var('u,v', domain='real')
sage: paraboloid = ParametrizedSurface3D([u, v, u^2+v^2], [u, v], 'paraboloid')
sage: S = paraboloid.shape_operator(); S
[2*(4*v^2 + 1)/(4*u^2 + 4*v^2 + 1)^(3/2)      -8*u*v/(4*u^2 + 4*v^2 + 1)^(3/2)
↪ 2)]
[      -8*u*v/(4*u^2 + 4*v^2 + 1)^(3/2)  2*(4*u^2 + 1)/(4*u^2 + 4*v^2 + 1)^(3/2)
↪ 2)]
sage: S.eigenvalues()
[2*sqrt(4*u^2 + 4*v^2 + 1)/(16*u^4 + 16*v^4 + 8*(4*u^2 + 1)*v^2 + 8*u^2 + 1), 2/
↪ sqrt(4*u^2 + 4*v^2 + 1)]

```

### shape\_operator\_coefficients()

Returns the components of the shape operator of the surface as a dictionary. See `shape_operator` for more information.

OUTPUT:

- Dictionary where the keys are two-tuples (i, j), with values the corresponding component of the shape operator.

EXAMPLES:

```

sage: R = var('R')
sage: u, v = var('u,v', domain='real')
sage: assume(cos(v)>0)
sage: sphere = ParametrizedSurface3D([R*cos(u)*cos(v), R*sin(u)*cos(v), R*sin(v)],
↪ [u,v], 'sphere')
sage: sphere.shape_operator_coefficients()
{(1, 1): -1/R, (1, 2): 0, (2, 1): 0, (2, 2): -1/R}

```

### tangent\_vector(coords, components)

Returns the components of a tangent vector given the intrinsic coordinates of the base point and the components of the vector in the intrinsic frame.

INPUT:

- `coords` - 2-tuple specifying the intrinsic coordinates (u, v) of the point.
- `components` - 2-tuple specifying the components of the tangent vector in the intrinsic coordinate frame.

OUTPUT:

- 3-vector specifying the components in  $\mathbf{R}^3$  of the vector.

EXAMPLES:

We compute two tangent vectors to Enneper's surface along the coordinate lines and check that their cross product gives the normal vector:

```
sage: u, v = var('u,v', domain='real')
sage: eq = (3*u + 3*u*v^2 - u^3, 3*v + 3*u^2*v - v^3, 3*(u^2-v^2))
sage: e = ParametrizedSurface3D(eq, (u, v), 'Enneper Surface')

sage: w1 = e.tangent_vector((1, 2), (1, 0)); w1
(12, 12, 6)
sage: w2 = e.tangent_vector((1, 2), (0, 1)); w2
(12, -6, -12)
sage: w1.cross_product(w2)
(-108, 216, -216)

sage: n = e.normal_vector().subs({u: 1, v: 2}); n
(-108, 216, -216)
sage: n == w1.cross_product(w2)
True
```



## COMMON PARAMETRIZED SURFACES IN 3D.

AUTHORS:

- Joris Vankerschaver (2012-06-16)

**class** sage.geometry.riemannian\_manifolds.surface3d\_generators.SurfaceGenerators

Bases: object

A class consisting of generators for several common parametrized surfaces in 3D.

**static** Catenoid( $c=1$ ,  $name='Catenoid'$ )

Return a catenoid surface, with parametric representation

$$x(u, v) = c \cosh(v/c) \cos(u);$$

$$y(u, v) = c \cosh(v/c) \sin(u);$$

$$z(u, v) = v.$$

INPUT:

- $c$  – surface parameter.
- $name$  – string. Name of the surface.

For more information, see [Wikipedia article Catenoid](#).

EXAMPLES:

```
sage: cat = surfaces.Catenoid(); cat
Parametrized surface ('Catenoid') with equation (cos(u)*cosh(v), cosh(v)*sin(u),
↪ v)
sage: cat.plot() # optional - sage.plot
Graphics3d Object
```

**static** Crosscap( $r=1$ ,  $name='Crosscap'$ )

Return a crosscap surface, with parametrization

$$x(u, v) = r(1 + \cos(v)) \cos(u);$$

$$y(u, v) = r(1 + \cos(v)) \sin(u);$$

$$z(u, v) = -r \tanh(u - \pi) \sin(v).$$

INPUT:

- $r$  – surface parameter.
- $name$  – string. Name of the surface.

For more information, see [Wikipedia article Cross-cap](#).

EXAMPLES:

```
sage: crosscap = surfaces.Crosscap(); crosscap
Parametrized surface ('Crosscap') with equation ((cos(v) + 1)*cos(u), (cos(v) +
↪1)*sin(u), -sin(v)*tanh(-pi + u))
sage: crosscap.plot() # optional - sage.plot
Graphics3d Object
```

**static Dini**(*a=1, b=1, name="Dini's surface"*)

Return Dini's surface, with parametrization

$$\begin{aligned}x(u, v) &= a \cos(u) \sin(v); \\y(u, v) &= a \sin(u) \sin(v); \\z(u, v) &= u + \log(\tan(v/2)) + \cos(v).\end{aligned}$$

INPUT:

- *a*, *b* – surface parameters.
- *name* – string. Name of the surface.

For more information, see [Wikipedia article Dini's surface](#).

EXAMPLES:

```
sage: dini = surfaces.Dini(a=3, b=4); dini
Parametrized surface ('Dini's surface') with equation (3*cos(u)*sin(v),
↪3*sin(u)*sin(v), 4*u + 3*cos(v) + 3*log(tan(1/2*v)))
sage: dini.plot() # optional - sage.plot
Graphics3d Object
```

**static Ellipsoid**(*center=(0, 0, 0), axes=(1, 1, 1), name='Ellipsoid'*)

Return an ellipsoid centered at *center* whose semi-principal axes have lengths given by the components of *axes*. The parametrization of the ellipsoid is given by

$$\begin{aligned}x(u, v) &= x_0 + a \cos(u) \cos(v); \\y(u, v) &= y_0 + b \sin(u) \cos(v); \\z(u, v) &= z_0 + c \sin(v).\end{aligned}$$

INPUT:

- *center* – 3-tuple. Coordinates of the center of the ellipsoid.
- *axes* – 3-tuple. Lengths of the semi-principal axes.
- *name* – string. Name of the ellipsoid.

For more information, see [Wikipedia article Ellipsoid](#).

EXAMPLES:

```
sage: ell = surfaces.Ellipsoid(axes=(1, 2, 3)); ell
Parametrized surface ('Ellipsoid') with equation (cos(u)*cos(v),
↪2*cos(v)*sin(u), 3*sin(v))
sage: ell.plot() # optional - sage.plot
Graphics3d Object
```

**static Enneper**(name="Enneper's surface")

Return Enneper's surface, with parametrization

$$\begin{aligned}x(u, v) &= u(1 - u^2/3 + v^2)/3; \\y(u, v) &= -v(1 - v^2/3 + u^2)/3; \\z(u, v) &= (u^2 - v^2)/3.\end{aligned}$$

INPUT:

- name – string. Name of the surface.

For more information, see [Wikipedia article Enneper\\_surface](#).

EXAMPLES:

```
sage: enn = surfaces.Enneper(); enn
Parametrized surface ('Enneper's surface') with equation (-1/9*(u^2 - 3*v^2 - 3)*u,
↪ -1/9*(3*u^2 - v^2 + 3)*v, 1/3*u^2 - 1/3*v^2)
sage: enn.plot() # optional - sage.plot
Graphics3d Object
```

**static Helicoid**(h=1, name='Helicoid')

Return a helicoid surface, with parametrization

$$\begin{aligned}x(\rho, \theta) &= \rho \cos(\theta); \\y(\rho, \theta) &= \rho \sin(\theta); \\z(\rho, \theta) &= h\theta/(2\pi).\end{aligned}$$

INPUT:

- h – distance along the z-axis between two successive turns of the helicoid.
- name – string. Name of the surface.

For more information, see [Wikipedia article Helicoid](#).

EXAMPLES:

```
sage: helicoid = surfaces.Helicoid(h=2); helicoid
Parametrized surface ('Helicoid') with equation (rho*cos(theta), rho*sin(theta),
↪ theta/pi)
sage: helicoid.plot() # optional - sage.plot
Graphics3d Object
```

**static Klein**(r=1, name='Klein bottle')

Return the Klein bottle, in the figure-8 parametrization given by

$$\begin{aligned}x(u, v) &= (r + \cos(u/2) \cos(v) - \sin(u/2) \sin(2v)) \cos(u); \\y(u, v) &= (r + \cos(u/2) \cos(v) - \sin(u/2) \sin(2v)) \sin(u); \\z(u, v) &= \sin(u/2) \cos(v) + \cos(u/2) \sin(2v).\end{aligned}$$

INPUT:

- r – radius of the “figure-8” circle.
- name – string. Name of the surface.

For more information, see [Wikipedia article Klein\\_bottle](#).

EXAMPLES:

```

sage: klein = surfaces.Klein(); klein
Parametrized surface ('Klein bottle') with equation  $(-\sin(1/2*u)*\sin(2*v) - \cos(1/2*u)*\sin(v) - 1)*\cos(u), -(\sin(1/2*u)*\sin(2*v) - \cos(1/2*u)*\sin(v) - 1)*\sin(u), \cos(1/2*u)*\sin(2*v) + \sin(1/2*u)*\sin(v))$ 
sage: klein.plot() # optional - sage.plot
Graphics3d Object

```

**static MonkeySaddle**(name='Monkey saddle')

Return a monkey saddle surface, with equation

$$z = x^3 - 3xy^2.$$

INPUT:

- name – string. Name of the surface.

For more information, see [Wikipedia article Monkey\\_saddle](#).

EXAMPLES:

```

sage: saddle = surfaces.MonkeySaddle(); saddle
Parametrized surface ('Monkey saddle') with equation  $(u, v, u^3 - 3*u*v^2)$ 
sage: saddle.plot() # optional - sage.plot
Graphics3d Object

```

**static Paraboloid**(a=1, b=1, c=1, elliptic=True, name=None)

Return a paraboloid with equation

$$\frac{z}{c} = \pm \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

When the plus sign is selected, the paraboloid is elliptic. Otherwise the surface is a hyperbolic paraboloid.

INPUT:

- a, b, c – Surface parameters.
- elliptic (default: True) – whether to create an elliptic or hyperbolic paraboloid.
- name – string. Name of the surface.

For more information, see [Wikipedia article Paraboloid](#).

EXAMPLES:

```

sage: epar = surfaces.Paraboloid(1, 3, 2); epar
Parametrized surface ('Elliptic paraboloid') with equation  $(u, v, 2*u^2 + 2/9*v^2)$ 
sage: epar.plot() # optional - sage.plot
Graphics3d Object

sage: hpar = surfaces.Paraboloid(2, 3, 1, elliptic=False); hpar
Parametrized surface ('Hyperbolic paraboloid') with equation  $(u, v, -1/4*u^2 + 1/9*v^2)$ 
sage: hpar.plot() # optional - sage.plot
Graphics3d Object

```

**static Sphere**(center=(0, 0, 0), R=1, name='Sphere')

Return a sphere of radius R centered at center.

INPUT:

- center – 3-tuple, center of the sphere.
- R – Radius of the sphere.
- name – string. Name of the surface.

For more information, see [Wikipedia article Sphere](#).

EXAMPLES:

```
sage: sphere = surfaces.Sphere(center=(0, 1, -1), R=2); sphere
Parametrized surface ('Sphere') with equation (2*cos(u)*cos(v), 2*cos(v)*sin(u),
↪ + 1, 2*sin(v) - 1)
sage: sphere.plot() # optional - sage.plot
Graphics3d Object
```

Note that the radius of the sphere can be negative. The surface thus obtained is equal to the sphere (or part thereof) with positive radius, whose coordinate functions have been multiplied by -1. Compare for instant the first octant of the unit sphere with positive radius:

```
sage: octant1 = surfaces.Sphere(R=1); octant1
Parametrized surface ('Sphere') with equation (cos(u)*cos(v), cos(v)*sin(u),
↪ sin(v))
sage: octant1.plot((0, pi/2), (0, pi/2)) # optional - sage.plot
Graphics3d Object
```

with the first octant of the unit sphere with negative radius:

```
sage: octant2 = surfaces.Sphere(R=-1); octant2
Parametrized surface ('Sphere') with equation (-cos(u)*cos(v), -cos(v)*sin(u), -
↪ sin(v))
sage: octant2.plot((0, pi/2), (0, pi/2)) # optional - sage.plot
Graphics3d Object
```

**static Torus**(r=2, R=3, name='Torus')

Return a torus obtained by revolving a circle of radius r around a coplanar axis R units away from the center of the circle. The parametrization used is

$$\begin{aligned}x(u, v) &= (R + r \cos(v)) \cos(u); \\y(u, v) &= (R + r \cos(v)) \sin(u); \\z(u, v) &= r \sin(v).\end{aligned}$$

INPUT:

- r, R – Minor and major radius of the torus.
- name – string. Name of the surface.

For more information, see [Wikipedia article Torus](#).

EXAMPLES:

```
sage: torus = surfaces.Torus(); torus
Parametrized surface ('Torus') with equation ((2*cos(v) + 3)*cos(u), (2*cos(v)
↪+ 3)*sin(u), 2*sin(v))
sage: torus.plot() # optional - sage.plot
Graphics3d Object
```

**static WhitneyUmbrella**(*name*="Whitney's umbrella")

Return Whitney's umbrella, with parametric representation

$$x(u, v) = uv, \quad y(u, v) = u, \quad z(u, v) = v^2.$$

INPUT:

- *name* – string. Name of the surface.

For more information, see [Wikipedia article Whitney\\_umbrella](#).

EXAMPLES:

```
sage: whitney = surfaces.WhitneyUmbrella(); whitney
Parametrized surface ('Whitney's umbrella') with equation (u*v, u, v^2)
sage: whitney.plot() # optional - sage.plot
Graphics3d Object
```

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