Knot Theory

Release 10.2

The Sage Development Team

Dec 06, 2023

CONTENTS

CHAPTER

ONE

KNOTS

AUTHORS:

- Miguel Angel Marco Buzunariz
- Amit Jamadagni

class sage.knots.knot.**Knot**(*data*, *check=True*)

Bases: *Link*, Element

A knot.

A knot is defined as embedding of the circle $\mathbb{S}^1$ in the 3-dimensional sphere $\mathbb{S}^3$, considered up to ambient isotopy. They represent the physical idea of a knotted rope, but with the particularity that the rope is closed. That is, the ends of the rope are joined.

See also:

Link

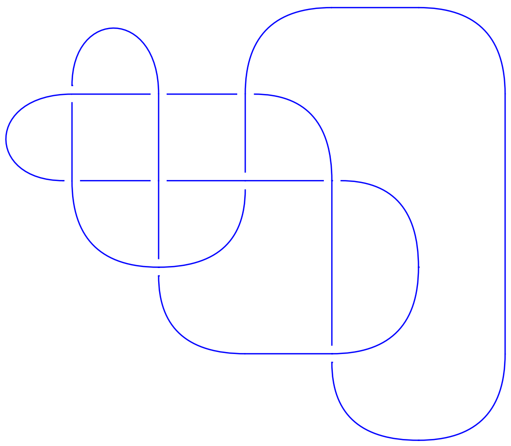
INPUT:

- data – see *Link* for the allowable inputs
- check – optional, default True. If True, make sure that the data define a knot, not a link

EXAMPLES:

We construct the knot 8_{14} and compute some invariants:

```
sage: B = BraidGroup(4)
sage: K = Knot(B([1,1,1,2,-1,2,-3,2,-3]))
```



```
sage: K.alexander_polynomial()
-2*t^-2 + 8*t^-1 - 11 + 8*t - 2*t^2
sage: K.jones_polynomial()
t^7 - 3*t^6 + 4*t^5 - 5*t^4 + 6*t^3 - 5*t^2 + 4*t + 1/t - 2
sage: K.determinant()
31
sage: K.signature()
-2
sage: K.colored_jones_polynomial(2)  # long time
q^-1 - 2 + 4*q - 5*q^2 + 6*q^3 - 5*q^4 + 4*q^5 - 3*q^6 + q^7
```

REFERENCES:

- Wikipedia article Knot_(mathematics)

arf_invariant()

Return the Arf invariant.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: K = Knot(B([-1, 2, 1, 2]))
sage: K.arf_invariant()
0
sage: B = BraidGroup(8)
sage: K = Knot(B([-2, 3, 1, 2, 1, 4]))
sage: K.arf_invariant()
0
sage: K = Knot(B([1, 2, 1, 2]))
sage: K.arf_invariant()
1
```

colored_jones_polynomial(*N*, *variab=None*, *try_inverse=True*)

Return the colored Jones polynomial of the trace closure of the braid.

INPUT:

- `N` – integer; the number of colors
- `variab` – (default: q) the variable in the resulting Laurent polynomial
- `try_inverse` – boolean (default: `True`); if `True`, attempt a faster calculation by using the inverse of the braid

ALGORITHM:

The algorithm used is described in [HL2018] for the corresponding braid representation. We follow their notation, but work in a suitable free algebra over a Laurent polynomial ring in one variable to simplify bookkeeping.

EXAMPLES:

```
sage: W = Knots()
sage: K = W.from_dowker_code([-4,-6,-2])
sage: K.colored_jones_polynomial(2)
-q^-4 + q^-3 + q^-1
sage: K.colored_jones_polynomial(2, 't')
```

(continues on next page)

(continued from previous page)

```
-t^-4 + t^-3 + t^-1
sage: R.<t> = LaurentPolynomialRing(ZZ)
sage: K.colored_jones_polynomial(2, -t)
-t^-4 - t^-3 - t^-1

sage: R.<t> = ZZ[]
sage: K.colored_jones_polynomial(2, t+1)
(t^3 + 3*t^2 + 4*t + 1)/(t^4 + 4*t^3 + 6*t^2 + 4*t + 1)
```

connected_sum(*other*)

Return the oriented connected sum of `self` and `other`.

Note: We give the knots an orientation based upon the braid representation.

INPUT:

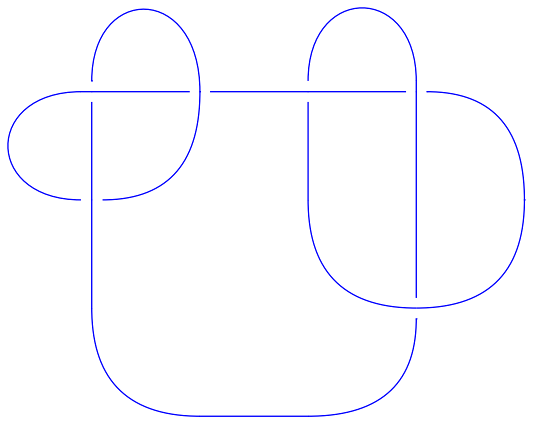
- `other` – a knot

OUTPUT:

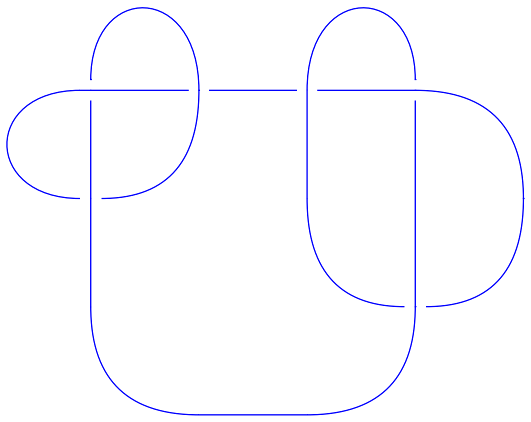
A knot equivalent to the connected sum of `self` and `other`.

EXAMPLES:

```
sage: B = BraidGroup(2)
sage: trefoil = Knot(B([1,1,1]))
sage: K = trefoil.connected_sum(trefoil); K
Knot represented by 6 crossings
sage: K.braid()
s0^3*s1^-1*s0^3*s1
```



```
sage: rev_trefoil = Knot(B([-1,-1,-1]))
sage: K2 = trefoil.connected_sum(rev_trefoil); K2
Knot represented by 6 crossings
sage: K2.braid()
s0^3*s1^-1*s0^-3*s1
```



Observe that both knots have according `dowker_notation` (showing that the constructing from DT-code may not be unique for non prime knots, see `from_dowker_code()`):

```
sage: K.dowker_notation()
[(4, 1), (2, 5), (6, 3), (10, 7), (8, 11), (12, 9)]
sage: K2.dowker_notation()
[(4, 1), (2, 5), (6, 3), (7, 10), (11, 8), (9, 12)]
sage: K.homfly_polynomial() == K2.homfly_polynomial()
False
```

REFERENCES:

- Wikipedia article Connected_sum

dt_code()

Return the DT code of `self`.

ALGORITHM:

The DT code is generated by the following way:

Start moving along the knot, as we encounter the crossings we start numbering them, so every crossing has two numbers assigned to it once we have traced the entire knot. Now we take the even number associated with every crossing.

The following sign convention is to be followed:

Take the even number with a negative sign if it is an overcrossing that we are encountering.

OUTPUT: DT code representation of the knot

EXAMPLES:

```
sage: K = Knot([[1,5,2,4],[5,3,6,2],[3,1,4,6]])
sage: K.dt_code()
[4, 6, 2]
sage: B = BraidGroup(4)
sage: K = Knot(B([1, 2, 1, 2]))
sage: K.dt_code()
[4, -6, 8, -2]
sage: K = Knot([[[1, -2, 3, -4, 5, -1, 2, -3, 4, -5]],
....:           [1, 1, 1, 1, 1]])
```

(continues on next page)

(continued from previous page)

```
sage: K.dt_code()
[6, 8, 10, 2, 4]
```

class sage.knots.knot.**Knots**

Bases: Singleton, Parent

The set for all knots, as a monoid for the connected sum.

Element

alias of *Knot*

an_element()

Return the trefoil knot.

EXAMPLES:

```
sage: Knots().an_element()
Knot represented by 3 crossings
```

from_dowker_code(*code*)

Build a knot from a Dowker-Thistlethwaite code.

The Dowker-Thistlethwaite code of a knot diagram is defined as follows.

Start following the knot diagram at some regular point. Label the crossings by a number (starting from number 1) in the order in which they are met. At the end, every crossing gets numbered twice, once by an even number and once by an odd number. When meeting an over-crossing with even number, use instead the negative of this even number as label.

Then the set of crossings gives a set of pairs (odd, even). Sort this set according to the odd component, and then keep only the even components in the same order. This is the Dowker-Thistlethwaite code.

INPUT:

a list of signed even numbers, the Dowker-Thistlethwaite code of a knot

OUTPUT:

a knot

> **Warning:** In general the Dowker-Thistlethwaite code does not describe a knot uniquely. It is not only insensitive on mirror images, but may also mix up non prime knots. For example `[4, 6, 2, 10, 12, 8]` describes the connected sum of two trefoil knots, as well as the connected sum of a trefoil with its mirror (see the corresponding example in the documentation of `connected_sum()`).

EXAMPLES:

```
sage: W = Knots()
sage: K1 = W.from_dowker_code([8,10,2,12,4,6])
sage: K1.dowker_notation()
[(5, 2), (9, 4), (11, 6), (1, 8), (3, 10), (7, 12)]

sage: W.from_dowker_code([6,10,14,12,16,2,18,4,8])
Knot represented by 9 crossings
```

(continues on next page)

(continued from previous page)

```
sage: W.from_dowker_code([4,8,10,-14,2,-16,-18,-6,-12])
Knot represented by 9 crossings

sage: K3 = W.from_dowker_code([6,-12,2,8,-4,-10]); K3
Knot represented by 6 crossings
sage: K3.dowker_notation()
[(5, 2), (4, 9), (1, 6), (7, 8), (10, 11), (12, 3)]
```

See also:

`dowker_notation()`

REFERENCES:

- Wikipedia article Dowker_notation
- http://katlas.org/wiki/DT_(Dowker-Thistlethwaite)_Codes

from_gauss_code(*gauss*)

Build a knot from a signed Gauss code.

This makes some arbitrary choice of orientation.

INPUT:

- a signed Gauss code

OUTPUT:

- a knot

EXAMPLES:

```
sage: W = Knots()
sage: K1 = W.from_gauss_code([2, -1, 3, -2, 1, -3])
sage: K1.alexander_polynomial()
t^-1 - 1 + t
```

from_table(*n*, *k*)

Return a knot from its index in the Rolfsen table.

INPUT:

- `n` – the crossing number
- `k` – a positive integer

OUTPUT:

the knot $K_{n,k}$ in the Rolfsen table

EXAMPLES:

```
sage: K1 = Knots().from_table(6,3); K1
Knot represented by 6 crossings
sage: K1.alexander_polynomial()
t^-2 - 3*t^-1 + 5 - 3*t + t^2

sage: K2 = Knots().from_table(8,4); K2
Knot represented by 9 crossings
```

(continues on next page)

(continued from previous page)

```
sage: K2.determinant()
19
sage: K2.signature()
2

sage: K3 = Knots().from_table(10,56); K3
Knot represented by 11 crossings
sage: K3.jones_polynomial()
t^10 - 3*t^9 + 6*t^8 - 9*t^7 + 10*t^6 - 11*t^5 + 10*t^4 - 7*t^3
+ 5*t^2 - 2*t + 1

sage: K4 = Knots().from_table(10,100)
sage: K4.genus()
4
```

REFERENCES:

- KnotAtlas, http://katlas.math.toronto.edu/wiki/The_Rolfsen_Knot_Table

one()

Return the unit of the monoid.

This is the trivial knot.

EXAMPLES:

```
sage: Knots().one()
Knot represented by 0 crossings
```

CHAPTER TWO

LINKS

A knot is defined as embedding of the circle $\mathbb{S}^1$ in the 3-dimensional sphere $\mathbb{S}^3$, considered up to ambient isotopy. They represent the physical idea of a knotted rope, but with the particularity that the rope is closed. That is, the ends of the rope are joined.

A link is an embedding of one or more copies of $\mathbb{S}^1$ in $\mathbb{S}^3$, considered up to ambient isotopy. That is, a link represents the idea of one or more tied ropes. Every knot is a link, but not every link is a knot.

Generically, the projection of a link on $\mathbf{R}^2$ is a curve with crossings. The crossings are represented to show which strand goes over the other. This curve is called a planar diagram of the link. If we remove the crossings, the resulting connected components are segments. These segments are called the edges of the diagram.

REFERENCES:

- Wikipedia article Knot_(mathematics)
- [Col2013]
- [KnotAtlas]

See also:

There are also tables of link and knot invariants at web-pages KnotInfo and LinkInfo. These can be used inside Sage after installing the optional package `database_knotinfo` (type `sage -i database_knotinfo` in a command shell, see *`knotinfo`*).

AUTHORS:

- Miguel Angel Marco Buzunariz
- Amit Jamadagni
- Sebastian Oehms (October 2020, add `get_knotinfo()` and `is_isotopic()`)
- Sebastian Oehms (May 2022): add `links_gould_polynomial()`
- Sebastian Oehms (May 2023): change the convention about the `pd_code` from clockwise to anti-clockwise (see github issue #35665).

class `sage.knots.link.`**`Link`**(*data*)

Bases: `SageObject`

A link.

A link is an embedding of one or more copies of $\mathbb{S}^1$ in $\mathbb{S}^3$, considered up to ambient isotopy. That is, a link represents the idea of one or more tied ropes. Every knot is a link, but not every link is a knot.

A link can be created by using one of the conventions mentioned below:

Braid:

- The closure of a braid is a link:

```
sage: B = BraidGroup(8)
sage: L = Link(B([-1, -1, -1, -2, 1, -2, 3, -2, 3]))
sage: L
Link with 1 component represented by 9 crossings
sage: L = Link(B([1, 2, 1, -2, -1]))
sage: L
Link with 2 components represented by 5 crossings
```

Note: The strands of the braid that have no crossings at all are removed.

- Oriented Gauss Code:

  Label the crossings from 1 to n (where n is the number of crossings) and start moving along the link. Trace every component of the link, by starting at a particular point on one component of the link and writing down each of the crossings that you encounter until returning to the starting point. The crossings are written with sign depending on whether we cross them as over or undercrossing. Each component is then represented as a list whose elements are the crossing numbers. A second list of $+1$ and -1's keeps track of the orientation of each crossing:

```
sage: L = Link([[[-1, 2, 3, -4, 5, -6, 7, 8, -2, -5, 6, 1, -8, -3, 4, -7]],
....:           [-1, -1, -1, -1, 1, 1, -1, 1]])
sage: L
Link with 1 component represented by 8 crossings
```

  For links there may be more than one component and the input is as follows:

```
sage: L = Link([[[-1, 2], [-3, 4], [1, 3, -4, -2]], [-1, -1, 1, 1]])
sage: L
Link with 3 components represented by 4 crossings
```

- Planar Diagram (PD) Code:

  The diagram of the link is formed by segments that are adjacent to the crossings. Label each one of this segments with a positive number, and for each crossing, write down the four incident segments. The order of these segments is anti-clockwise, starting with the incoming undercrossing.

  There is no particular distinction between knots and links for this input.

EXAMPLES:

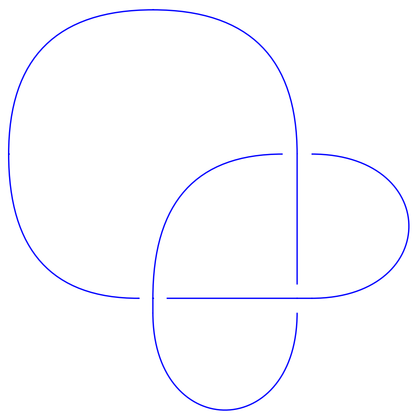
One of the representations of the trefoil knot:

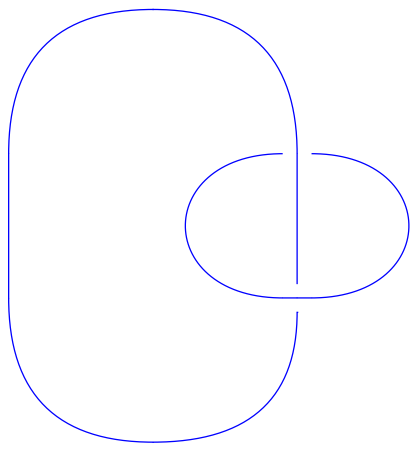
```
sage: L = Link([[1, 5, 2, 4], [5, 3, 6, 2], [3, 1, 4, 6]])
sage: L
Link with 1 component represented by 3 crossings
```

One of the representations of the Hopf link:

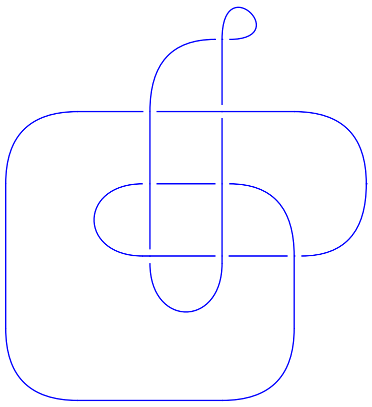
```
sage: L = Link([[1, 4, 2, 3], [4, 1, 3, 2]])
sage: L
Link with 2 components represented by 2 crossings
```

We can construct links from the braid group:

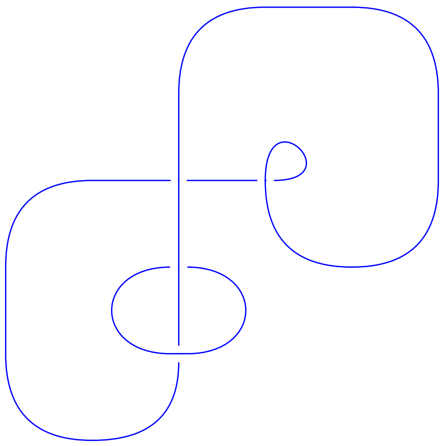




```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, -1, -1, -2, 1, -2, 3, -2]))
sage: L
Link with 2 components represented by 8 crossings
```



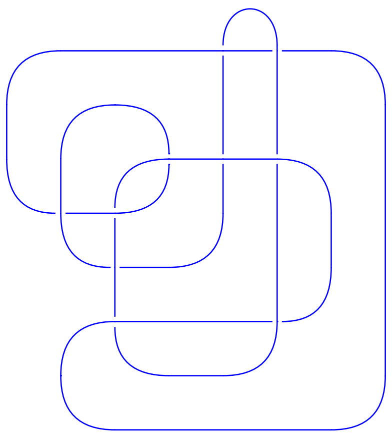
```
sage: L = Link(B([1, 2, 1, 3]))
sage: L
Link with 2 components represented by 4 crossings
```



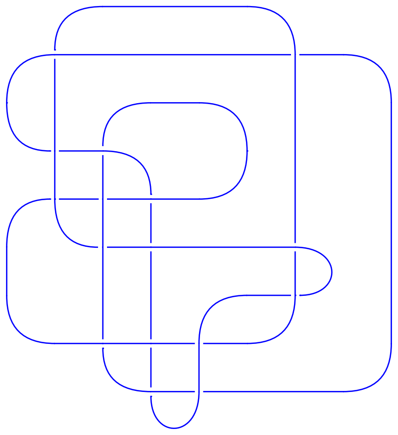
We construct the “monster” unknot using a planar code, and then construct the oriented Gauss code and braid representation:

```
sage: L = Link([[3,4,2,1], [8,7,1,9], [5,3,7,6], [4,5,6,18],
....:           [17,18,8,19], [9,14,11,10], [10,11,13,12],
....:           [12,13,15,19], [20,15,14,16], [16,2,17,20]])
sage: L.oriented_gauss_code()
[[[1, -4, 3, -1, 10, -9, 6, -7, 8, 5, 4, -3, 2, -6, 7, -8, 9, -10, -5, -2]],
 [1, -1, 1, 1, 1, -1, -1, -1, -1, -1]]
sage: L.braid()
s0*s1^-3*s2^-1*s1*s3*s2^2*s1^-1*s0^-1*s2*s1^-1*s3^-1*s2*s1^-1
```

We construct the Ochiai unknot by using an oriented Gauss code:



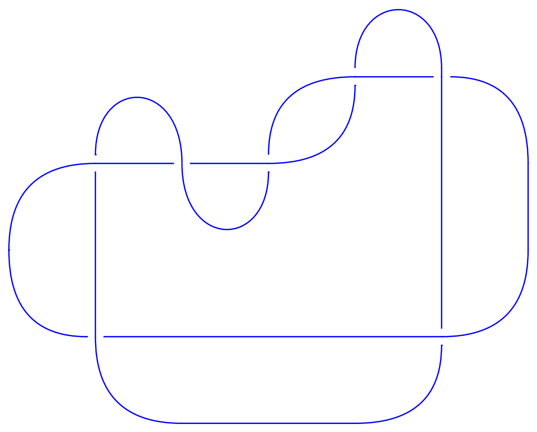
```
sage: L = Link([[[1,-2,-3,-8,-12,13,-14,15,-7,-1,2,-4,10,11,-13,12,
....:             -11,-16,4,3,-5,6,-9,7,-15,14,16,-10,8,9,-6,5]],
....:           [-1,-1,1,1,1,1,-1,1,1,-1,1,-1,-1,-1,-1,-1]])
sage: L.pd_code()
[[10, 1, 11, 2], [2, 11, 3, 12], [3, 21, 4, 20], [12, 20, 13, 19],
 [21, 1, 22, 32], [31, 23, 32, 22], [9, 24, 10, 25], [4, 30, 5, 29],
 [23, 31, 24, 30], [28, 13, 29, 14], [17, 15, 18, 14], [5, 16, 6, 17],
 [15, 6, 16, 7], [7, 26, 8, 27], [25, 8, 26, 9], [18, 27, 19, 28]]
```



We construct the knot 7_1 and compute some invariants:

```
sage: B = BraidGroup(2)
sage: L = Link(B([1]*7))
```

```
sage: L.alexander_polynomial()
t^-3 - t^-2 + t^-1 - 1 + t - t^2 + t^3
sage: L.jones_polynomial()
-t^10 + t^9 - t^8 + t^7 - t^6 + t^5 + t^3
sage: L.determinant()
7
sage: L.signature()
-6
```



The links here have removed components in which no strand is used:

```
sage: B = BraidGroup(8)
sage: b = B([1])
sage: L = Link(b)
sage: b.components_in_closure()
7
sage: L.number_of_components()
1
sage: L.braid().components_in_closure()
1
sage: L.braid().parent()
Braid group on 2 strands
```

Warning: Equality of knots is done by comparing the corresponding braids, which may give false negatives.

Note: The behavior of removing unused strands from an element of a braid group may change without notice in the future. Do not rely on this feature.

Todo: Implement methods to creating new links from previously created links.

alexander_polynomial(*var='t'*)

Return the Alexander polynomial of `self`.

INPUT:

- `var` – (default: `'t'`) the variable in the polynomial

EXAMPLES:

We begin by computing the Alexander polynomial for the figure-eight knot:

```
sage: B = BraidGroup(3)
sage: L = Link(B([1, -2, 1, -2]))
```

(continues on next page)

(continued from previous page)

```
sage: L.alexander_polynomial()
-t^-1 + 3 - t
```

The “monster” unknot:

```
sage: L = Link([[3,1,2,4],[8,9,1,7],[5,6,7,3],[4,18,6,5],
....:           [17,19,8,18],[9,10,11,14],[10,12,13,11],
....:           [12,19,15,13],[20,16,14,15],[16,20,17,2]])
sage: L.alexander_polynomial()
1
```

Some additional examples:

```
sage: B = BraidGroup(2)
sage: L = Link(B([1]))
sage: L.alexander_polynomial()
1
sage: L = Link(B.one())
sage: L.alexander_polynomial()
1
sage: B = BraidGroup(3)
sage: L = Link(B([1, 2, 1, 2]))
sage: L.alexander_polynomial()
t^-1 - 1 + t
```

When the Seifert surface is disconnected, the Alexander polynomial is defined to be 0:

```
sage: B = BraidGroup(4)
sage: L = Link(B([1,3]))
sage: L.alexander_polynomial()
0
```

See also:

conway_polynomial()

arcs(*presentation='pd'*)

Return the arcs of `self`.

Arcs are the connected components of the planar diagram.

INPUT:

- `presentation` – one of the following:
  - `'pd'` - the arcs are returned as lists of parts in the PD code
  - `'gauss_code'` - the arcs are returned as pieces of the Gauss code that start with a negative number, and end with the following negative one; of there exist a closed arc, it is returned as a list of positive numbers only

OUTPUT:

A list of lists representing the arcs based upon `presentation`.

EXAMPLES:

```
sage: K = Knot([[[1,-2,3,-1,2,-3]],[1,1,1]])
sage: K.arcs()
[[1, 2], [3, 4], [5, 6]]
sage: K.arcs(presentation='gauss_code')
[[-3, 1, -2], [-2, 3, -1], [-1, 2, -3]]
```

```
sage: L = Link([[1, 2, 3, 4], [3, 2, 1, 4]])
sage: L.arcs()
[[2, 4], [1], [3]]
sage: L.arcs(presentation='gauss_code')
[[-2, -1], [-1, -2], [2, 1]]
sage: L.gauss_code()
[[-1, -2], [2, 1]]
```

braid()

Return a braid representation of `self`.

OUTPUT: an element in the braid group

Warning: For the unknot with no crossings, this returns the identity of the braid group with 2 strands because this disregards strands with no crossings.

EXAMPLES:

```
sage: L = Link([[2, 4, 1, 3], [4, 2, 3, 1]])
sage: L.braid()
s^2
sage: L = Link([[[-1, 2, -3, 1, -2, 3]], [-1, -1, -1]])
sage: L.braid()
s^-3
sage: L = Link([[1,7,2,8], [8,5,9,4], [3,10,4,9], [10,6,7,1], [5,2,6,3]])
sage: L.braid()
(s0*s1^-1)^2*s1^-1
```

coloring_maps(*n=None*, *finitely_presented=False*)

Return the `n`-coloring maps of `self`. These are group homomorphisms from the fundamental group of `self` to the `n`-th dihedral group.

INPUT:

- `n` – the number of colors to consider (if ommitted the value of the determinant of `self` will be taken). Note that there are no coloring maps if n is coprime to the determinant of `self`
- `finitely_presented` (default `False`) whether to choose the dihedral groups as finitely presented groups. If not set to `True` they are represented as permutation groups.

OUTPUT:

a list of group homomporhisms from the fundamental group of `self` to the `n`-th dihedral group (represented according to the key argument `finitely_presented`).

EXAMPLES:

```
sage: L5a1_1 = Link([[8, 2, 9, 1], [10, 7, 5, 8], [4, 10, 1, 9],
....:                [2, 5, 3, 6], [6, 3, 7, 4]])
sage: L5a1_1.determinant()
8
sage: L5a1_1.coloring_maps(2)
[Group morphism:
   From: Finitely presented group < x0, x1, x2, x3, x4 | x4*x1*x0^-1*x1^-1,␣
↪x0*x4^-1*x3^-1*x4, x2*x0*x1^-1*x0^-1, x1*x3^-1*x2^-1*x3, x3*x2^-1*x4^-1*x2 >
   To:   Dihedral group of order 4 as a permutation group,
 Group morphism:
   From: Finitely presented group < x0, x1, x2, x3, x4 | x4*x1*x0^-1*x1^-1,␣
↪x0*x4^-1*x3^-1*x4, x2*x0*x1^-1*x0^-1, x1*x3^-1*x2^-1*x3, x3*x2^-1*x4^-1*x2 >
   To:   Dihedral group of order 4 as a permutation group]
sage: col_maps = L5a1_1.coloring_maps(4); len(col_maps)
12
sage: col_maps = L5a1_1.coloring_maps(5); len(col_maps)
0
sage: col_maps = L5a1_1.coloring_maps(12); len(col_maps)
36
sage: col_maps = L5a1_1.coloring_maps(); len(col_maps)
56
```

applying the map:

```
sage: cm1 = col_maps[0]
sage: gs = L5a1_1.fundamental_group().gens()
sage: d = cm1(gs[0]); d
(1,8)(2,7)(3,6)(4,5)
sage: d.parent()
Dihedral group of order 16 as a permutation group
```

using the finitely presented dihedral group:

```
sage: col_maps = L5a1_1.coloring_maps(2, finitely_presented=True)
sage: d = col_maps[0](gs[1]); d
b*a
sage: d.parent()
Finitely presented group < a, b | a^2, b^2, (a*b)^2 >
```

REFERENCES:

- Wikipedia article Fox_n-coloring
- Chapter 3 of [Liv1993]

See also:

is_colorable() and *colorings()*

colorings(*n=None*)

Return the n-colorings of `self`.

INPUT:

- `n` – the number of colors to consider (if ommitted the value of the determinant of `self` will be taken). Note that there are no colorings if n is coprime to the determinant of `self`

OUTPUT:

a list with the colorings. Each coloring is represented as a dictionary that maps a tuple of the edges forming each arc (as in the PD code) to the index of the corresponding color.

EXAMPLES:

```
sage: K = Link([[[1, -2, 3, -1, 2, -3]], [1, 1, 1]])
sage: K.colorings(3)
[{(1, 2): 0, (3, 4): 1, (5, 6): 2},
 {(1, 2): 0, (3, 4): 2, (5, 6): 1},
 {(1, 2): 1, (3, 4): 0, (5, 6): 2},
 {(1, 2): 1, (3, 4): 2, (5, 6): 0},
 {(1, 2): 2, (3, 4): 0, (5, 6): 1},
 {(1, 2): 2, (3, 4): 1, (5, 6): 0}]
sage: K.pd_code()
[[4, 2, 5, 1], [2, 6, 3, 5], [6, 4, 1, 3]]
sage: K.arcs('pd')
[[1, 2], [3, 4], [5, 6]]
```

Note that n is not the number of different colors to be used. It can be looked upon the size of the color palette:

```
sage: K = Knots().from_table(9, 15)
sage: cols = K.colorings(13); len(cols)
156
sage: max(cols[0].values())
12
sage: max(cols[13].values())
9
```

REFERENCES:

- Wikipedia article Fox_n-coloring
- Chapter 3 of [Liv1993]

See also:

is_colorable() and *coloring_maps()*

conway_polynomial()

Return the Conway polynomial of self.

This is closely related to the Alexander polynomial.

See Wikipedia article Alexander_polynomial for the definition.

EXAMPLES:

```
sage: B = BraidGroup(3)
sage: L = Link(B([1, -2, 1, -2]))
sage: L.conway_polynomial()
-t^2 + 1
sage: Link([[1, 5, 2, 4], [3, 9, 4, 8], [5, 1, 6, 10],
....:       [7, 3, 8, 2], [9, 7, 10, 6]])
Link with 1 component represented by 5 crossings
sage: _.conway_polynomial()
```

(continues on next page)

(continued from previous page)

```
2*t^2 + 1
sage: B = BraidGroup(4)
sage: L = Link(B([1,3]))
sage: L.conway_polynomial()
0
```

See also:

alexander_polynomial()

determinant()

Return the determinant of `self`.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, 2, 1, 2]))
sage: L.determinant()
1
sage: B = BraidGroup(8)
sage: L = Link(B([2, 4, 2, 3, 1, 2]))
sage: L.determinant()
3
sage: L = Link(B([1]*16 + [2,1,2,1,2,2,2,2,2,2,2,1,2,1,2,-1,2,-2]))
sage: L.determinant()
65
sage: B = BraidGroup(3)
sage: Link(B([1, 2, 1, 1, 2])).determinant()
4
```

REFERENCES:

- Definition 6.6.3 in [Cro2004]

dowker_notation()

Return the Dowker notation of `self`.

Similar to the PD code we number the components, so every crossing is represented by four numbers. We focus on the incoming entities of the under and the overcrossing. It is the pair of incoming undercrossing and the incoming overcrossing. This information at every crossing gives the Dowker notation.

OUTPUT:

A list containing the pair of incoming under cross and the incoming over cross.

EXAMPLES:

```
sage: L = Link([[[-1, 2, -3, 4, 5, 1, -2, 6, 7, 3, -4, -7, -6,-5]], [-1, -1, -1,
↪ -1, 1, -1, 1]])
sage: L.dowker_notation()
[(1, 6), (7, 2), (3, 10), (11, 4), (14, 5), (13, 8), (12, 9)]
sage: B = BraidGroup(4)
sage: L = Link(B([1, 2, 1, 2]))
sage: L.dowker_notation()
[(2, 1), (3, 5), (6, 4), (7, 9)]
sage: L = Link([[1, 4, 2, 3], [4, 1, 3, 2]])
```

(continues on next page)

(continued from previous page)

```
sage: L.dowker_notation()
[(1, 3), (4, 2)]
```

fundamental_group(*presentation='wirtinger'*)

Return the fundamental group of the complement of `self`.

INPUT:

- `presentation` – string; one of the following:
  - `'wirtinger'` - (default) the Wirtinger presentation (see Wikipedia article Link_group)
  - `'braid'` - the presentation is given by the braid action on the free group (see chapter 2 of [Bir1975])

OUTPUT:

- a finitely presented group

EXAMPLES:

```
sage: L = Link([[1, 4, 3, 2], [3, 4, 1, 2]])
sage: L.fundamental_group()
Finitely presented group < x0, x1, x2 | x1*x0^-1*x2^-1*x0, x2*x0*x1^-1*x0^-1 >
sage: L.fundamental_group('braid')
Finitely presented group < x0, x1 | 1, 1 >
```

We can see, for instance, that the two presentations of the group of the figure eight knot correspond to isomorphic groups:

```
sage: K8 = Knot([[[1, -2, 4, -3, 2, -1, 3, -4]], [1, 1, -1, -1]])
sage: GA = K8.fundamental_group()
sage: GA
Finitely presented group < x0, x1, x2, x3 |
 x2*x0*x3^-1*x0^-1, x0*x2*x1^-1*x2^-1,
 x1*x3^-1*x2^-1*x3, x3*x1^-1*x0^-1*x1 >
sage: GB = K8.fundamental_group(presentation='braid')
sage: GB
Finitely presented group < x0, x1, x2 | x1*x2^-1*x1^-1*x0*x1*x2*x1*x2^-1*x1^-
↪1*x0^-1*x1*x2*x1^-1*x0^-1, x1*x2^-1*x1^-1*x0*x1*x2*x1^-1*x2^-1*x1^-1*x0^-
↪1*x1*x2*x1^-1*x0*x1*x2*x1*x2^-1*x1^-1*x0^-1*x1*x2*x1^-2, x1*x2^-1*x1^-
↪1*x0*x1*x2*x1^-1*x2^-1 >
sage: GA.simplified()
Finitely presented group < x0, x1 |
 x1^-1*x0*x1*x0^-1*x1*x0*x1^-1*x0^-1*x1*x0^-1 >
sage: GB.simplified()
Finitely presented group < x0, x2 |
 x2^-1*x0*x2^-1*x0^-1*x2*x0*x2^-1*x0*x2*x0^-1 >
```

gauss_code()

Return the Gauss code of `self`.

The Gauss code is generated by the following procedure:

a. Number the crossings from 1 to n.

b. Select a point on the knot and start moving along the component.

c. At each crossing, take the number of the crossing, along with sign, which is − if it is an undercrossing and + if it is a overcrossing.

EXAMPLES:

```
sage: L = Link([[1, 4, 2, 3], [4, 1, 3, 2]])
sage: L.gauss_code()
[[-1, 2], [1, -2]]
sage: B = BraidGroup(8)
sage: L = Link(B([1, -2, 1, -2, -2]))
sage: L.gauss_code()
[[-1, 3, -4, 5], [1, -2, 4, -5, 2, -3]]
sage: L = Link([[[-1, 2], [-3, 4], [1, 3, -4, -2]], [-1, -1, 1, 1]])
sage: L.gauss_code()
[[-1, 2], [-3, 4], [1, 3, -4, -2]]
```

genus()

Return the genus of `self`.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, 3, 1, 3]))
sage: L.genus()
0
sage: L = Link(B([1,3]))
sage: L.genus()
0
sage: B = BraidGroup(8)
sage: L = Link(B([-2, 4, 1, 6, 1, 4]))
sage: L.genus()
0
sage: L = Link(B([1, 2, 1, 2]))
sage: L.genus()
1
```

get_knotinfo(*mirror_version=True*, *unique=True*)

Identify this link as an item of the KnotInfo database (if possible).

INPUT:

- `mirror_version` – boolean (default is `True`). If set to `False` the result of the method will be just the instance of *`KnotInfoBase`* (by default the result is a tuple of the instance and a boolean, see explanation of the output below)
- `unique` – boolean (default is `True`). This only affects the case where a unique identification is not possible. If set to `False` you can obtain a matching list (see explanation of the output below)

OUTPUT:

A tuple `(K, m)` where `K` is an instance of *`KnotInfoBase`* and `m` a boolean (for chiral links) telling if `self` corresponds to the mirrored version of `K` or not. The value of `m` is `None` for amphicheiral links and `?` if it cannot be determined uniquely and the keyword option `unique=False` is given.

For proper links, if the orientation mutant cannot be uniquely determined, K will be a series of links gathering all links having the same unoriented name, that is an instance of *`KnotInfoSeries`*.

If `mirror_version` is set to `False` then the result is just `K` (that is: `m` is suppressed).

If it is not possible to determine a unique result a `NotImplementedError` will be raised. To avoid this you can set `unique` to `False`. You will get a list of matching candidates instead.

Note: The identification of proper links may fail to be unique due to the following fact: In opposite to the database for knots, there are pairs of oriented mutants of an unoriented link which are isotopic to each other. For example `L5a1_0` and `L5a1_1` is such a pair.

This is because all combinatorial possible oriented mutants are listed with individual names regardless whether they are pairwise non isotopic or not. In such a case the identification is not unique and therefore a series of the links will be returned which gathers all having the same unoriented name.

To obtain the individual oriented links being isotopic to `self` use the keyword `unique` (see the examples for `L2a1_1` and `L5a1_0` below).

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = Link([[4,1,5,2], [10,4,11,3], [5,17,6,16], [7,13,8,12],
....:           [18,10,19,9], [2,12,3,11], [13,21,14,20], [15,7,16,6],
....:           [22,17,1,18], [8,20,9,19], [21,15,22,14]])
sage: L.get_knotinfo()          # optional - database_knotinfo
(<KnotInfo.K11n_121: '11n_121'>, True)

sage: K = KnotInfo.K10_25       # optional - database_knotinfo
sage: l = K.link()              # optional - database_knotinfo
sage: l.get_knotinfo()          # optional - database_knotinfo
(<KnotInfo.K10_25: '10_25'>, False)
```

Knots with more than 12 and proper links having more than 11 crossings cannot be identified. In addition non prime links or even links whose HOMFLY-PT polynomial is not irreducible cannot be identified:

```
sage: b, = BraidGroup(2).gens()
sage: Link(b**13).get_knotinfo()
Traceback (most recent call last):
...
NotImplementedError: this knot having more than 12 crossings cannot be␣
↪determined

sage: Link([[1, 4, 2, 5], [3, 8, 4, 1], [5, 2, 6, 3], [6, 10, 7, 9], [10, 8, 9,␣
↪7]])
Link with 2 components represented by 5 crossings
sage: _.get_knotinfo()
Traceback (most recent call last):
...
NotImplementedError: this (possibly non prime) link cannot be determined
```

Lets identify the monster unknot:

```
sage: L = Link([[3,1,2,4], [8,9,1,7], [5,6,7,3], [4,18,6,5],
....:           [17,19,8,18], [9,10,11,14], [10,12,13,11],
....:           [12,19,15,13], [20,16,14,15], [16,20,17,2]])
sage: L.get_knotinfo()
(<KnotInfo.K0_1: '0_1'>, None)
```

Usage of option `mirror_version`:

```
sage: L.get_knotinfo(mirror_version=False) == KnotInfo.K0_1
True
```

Usage of option unique:

```
sage: l = K.link(K.items.gauss_notation)  # optional - database_knotinfo
sage: l.get_knotinfo()                    # optional - database_knotinfo
Traceback (most recent call last):
...
NotImplementedError: this link cannot be uniquely determined
use keyword argument `unique` to obtain more details

sage: l.get_knotinfo(unique=False)        # optional - database_knotinfo
[(<KnotInfo.K10_25: '10_25'>, False), (<KnotInfo.K10_56: '10_56'>, False)]

sage: # optional - database_knotinfo
sage: k11  = KnotInfo.K11n_82.link()
sage: k11m = k11.mirror_image()
sage: k11mr = k11m.reverse()
sage: k11mr.get_knotinfo()
Traceback (most recent call last):
...
NotImplementedError: mirror type of this link cannot be uniquely determined
use keyword argument `unique` to obtain more details

sage: k11mr.get_knotinfo(unique=False)    # optional - database_knotinfo
[(<KnotInfo.K11n_82: '11n_82'>, '?')]

sage: t = (1, -2, 1, 1, -2, 1, -2, -2)
sage: l8 = Link(BraidGroup(3)(t))
sage: l8.get_knotinfo()                   # optional - database_knotinfo
Traceback (most recent call last):
...
NotImplementedError: this link cannot be uniquely determined
use keyword argument `unique` to obtain more details

sage: l8.get_knotinfo(unique=False)       # optional - database_knotinfo
[(<KnotInfo.L8a19_0_0: 'L8a19{0,0}'>, None),
 (<KnotInfo.L8a19_1_1: 'L8a19{1,1}'>, None)]

sage: t = (2, -3, -3, -2, 3, 3, -2, 3, 1, -2, -2, 1)
sage: l12 = Link(BraidGroup(5)(t))
sage: l12.get_knotinfo()                  # optional - database_knotinfo
Traceback (most recent call last):
...
NotImplementedError: this link having more than 11 crossings cannot be uniquely␣
↪determined
use keyword argument `unique` to obtain more details

sage: l12.get_knotinfo(unique=False)      # optional - database_knotinfo
[(<KnotInfo.L10n36_0: 'L10n36{0}'>, '?'),
 (<KnotInfo.L10n36_1: 'L10n36{1}'>, None),
 (<KnotInfo.L10n59_0: 'L10n59{0}'>, None),
```

(continues on next page)

(continued from previous page)

```
 (<KnotInfo.L10n59_1: 'L10n59{1}'>, None)]
```

Furthermore, if the result is a complete series of oriented links having the same unoriented name (according to the note above) the option can be used to achieve more detailed information:

```
sage: L2a1 = Link(b**2)
sage: L2a1.get_knotinfo()
(Series of links L2a1, None)
sage: L2a1.get_knotinfo(unique=False)
[(<KnotInfo.L2a1_0: 'L2a1{0}'>, True), (<KnotInfo.L2a1_1: 'L2a1{1}'>, False)]

sage: KnotInfo.L5a1_0.inject()
Defining L5a1_0
sage: l5 = Link(L5a1_0.braid())
sage: l5.get_knotinfo()
(Series of links L5a1, False)
sage: _[0].inject()
Defining L5a1
sage: list(L5a1)
[<KnotInfo.L5a1_0: 'L5a1{0}'>, <KnotInfo.L5a1_1: 'L5a1{1}'>]
sage: l5.get_knotinfo(unique=False)
[(<KnotInfo.L5a1_0: 'L5a1{0}'>, False), (<KnotInfo.L5a1_1: 'L5a1{1}'>, False)]
```

Clarifying the series around the Perko pair (Wikipedia article Perko_pair):

```
sage: for i in range(160, 166):           # optional - database_knotinfo
....:     K = Knots().from_table(10, i)
....:     print('%s_%s' %(10, i), '--->', K.get_knotinfo())
10_160 ---> (<KnotInfo.K10_160: '10_160'>, False)
10_161 ---> (<KnotInfo.K10_161: '10_161'>, True)
10_162 ---> (<KnotInfo.K10_162: '10_162'>, False)
10_163 ---> (<KnotInfo.K10_163: '10_163'>, False)
10_164 ---> (<KnotInfo.K10_164: '10_164'>, False)
10_165 ---> (<KnotInfo.K10_165: '10_165'>, True)
```

Clarifying ther Perko series against SnapPy:

```
sage: import snappy                     # optional - snappy
Plink failed to import tkinter.

sage: from sage.knots.knotinfo import KnotInfoSeries
sage: KnotInfoSeries(10, True, True)    # optional - database_knotinfo
Series of knots K10
sage: _.inject()                        # optional - database_knotinfo
Defining K10
sage: for i in range(160, 166):         # optional - database_knotinfo snappy
....:     K = K10(i)
....:     k = K.link(K.items.name, snappy=True)
....:     print(k, '--->', k.sage_link().get_knotinfo())
<Link 10_160: 1 comp; 10 cross> ---> (<KnotInfo.K10_160: '10_160'>, False)
<Link 10_161: 1 comp; 10 cross> ---> (<KnotInfo.K10_161: '10_161'>, True)
<Link 10_162: 1 comp; 10 cross> ---> (<KnotInfo.K10_161: '10_161'>, False)
<Link 10_163: 1 comp; 10 cross> ---> (<KnotInfo.K10_162: '10_162'>, False)
```

(continues on next page)

(continued from previous page)

```
<Link 10_164: 1 comp; 10 cross> ---> (<KnotInfo.K10_163: '10_163'>, False)
<Link 10_165: 1 comp; 10 cross> ---> (<KnotInfo.K10_164: '10_164'>, False)

sage: snappy.Link('10_166')            # optional - snappy
<Link 10_166: 1 comp; 10 cross>
sage: _.sage_link().get_knotinfo()     # optional - database_knotinfo snappy
(<KnotInfo.K10_165: '10_165'>, True)
```

Another pair of confusion (see the corresponding Warning):

```
sage: # optional - snappy
sage: Ks10_86 = snappy.Link('10_86')
sage: Ks10_83 = snappy.Link('10_83')
sage: Ks10_86.sage_link().get_knotinfo()
(<KnotInfo.K10_83: '10_83'>, True)
sage: Ks10_83.sage_link().get_knotinfo()
(<KnotInfo.K10_86: '10_86'>, False)
```

homfly_polynomial(*var1=None*, *var2=None*, *normalization='lm'*)

Return the HOMFLY polynomial of `self`.

The HOMFLY polynomial $P(K)$ of a link K is a Laurent polynomial in two variables defined using skein relations and for the unknot U, we have $P(U) = 1$.

INPUT:

- `var1` – (default: `'L'`) the first variable. If `normalization` is set to `az` resp. `vz` the default is `a` resp. `v`
- `var2` – (default: `'M'`) the second variable. If `normalization` is set to `az` resp. `vz` the default is `z`
- `normalization` – (default: `lm`) the system of coordinates and can be one of the following:
  - `'lm'` – corresponding to the Skein relation $L \cdot P(K_+) + L^{-1} \cdot P(K_-) + M \cdot P(K_0) = 0$
  - `'az'` – corresponding to the Skein relation $a \cdot P(K_+) - a^{-1} \cdot P(K_-) = z \cdot P(K_0)$
  - `'vz'` – corresponding to the Skein relation $v^{-1} \cdot P(K_+) - v \cdot P(K_-) = z \cdot P(K_0)$

  where $P(K_+)$, $P(K_-)$ and $P(K_0)$ represent the HOMFLY polynomials of three links that vary only in one crossing; that is the positive, negative, or smoothed links respectively

OUTPUT:

A Laurent polynomial over the integers.

Note: Use the `'az'` normalization to agree with the data in [KnotAtlas]

Use the `'vz'` normalization to agree with the data KnotInfo.

EXAMPLES:

We give some examples:

```
sage: g = BraidGroup(2).gen(0)
sage: K = Knot(g^5)
sage: K.homfly_polynomial()
L^-4*M^4 - 4*L^-4*M^2 + 3*L^-4 - L^-6*M^2 + 2*L^-6
```

The Hopf link:

```
sage: L = Link([[1,4,2,3],[4,1,3,2]])
sage: L.homfly_polynomial('x', 'y')
-x^-1*y + x^-1*y^-1 + x^-3*y^-1
```

Another version of the Hopf link where the orientation has been changed. Therefore we substitute $x \mapsto L^{-1}$ and $y \mapsto M$:

```
sage: L = Link([[1,3,2,4], [4,2,3,1]])
sage: L.homfly_polynomial()
L^3*M^-1 - L*M + L*M^-1
sage: L = Link([[1,3,2,4], [4,2,3,1]])
sage: L.homfly_polynomial(normalization='az')
a^3*z^-1 - a*z - a*z^-1
```

The figure-eight knot:

```
sage: L = Link([[2,5,4,1], [5,3,7,6], [6,9,1,4], [9,7,3,2]])
sage: L.homfly_polynomial()
-L^2 + M^2 - 1 - L^-2
sage: L.homfly_polynomial('a', 'z', 'az')
a^2 - z^2 - 1 + a^-2
```

The "monster" unknot:

```
sage: L = Link([[3,1,2,4], [8,9,1,7], [5,6,7,3], [4,18,6,5],
....:           [17,19,8,18], [9,10,11,14], [10,12,13,11],
....:           [12,19,15,13], [20,16,14,15], [16,20,17,2]])
sage: L.homfly_polynomial()
1
```

Comparison with KnotInfo:

```
sage: KI, m = K.get_knotinfo(); KI, m
 (<KnotInfo.K5_1: '5_1'>, False)
sage: K.homfly_polynomial(normalization='vz') == KI.homfly_polynomial()
True
```

The knot 9_6:

```
sage: B = BraidGroup(3)
sage: K = Knot(B([-1,-1,-1,-1,-1,-1,-2,1,-2,-2]))
sage: K.homfly_polynomial()
L^10*M^4 - L^8*M^6 - 3*L^10*M^2 + 4*L^8*M^4 + L^6*M^6 + L^10
 - 3*L^8*M^2 - 5*L^6*M^4 - L^8 + 7*L^6*M^2 - 3*L^6
sage: K.homfly_polynomial('a', 'z', normalization='az')
-a^10*z^4 + a^8*z^6 - 3*a^10*z^2 + 4*a^8*z^4 + a^6*z^6 - a^10
 + 3*a^8*z^2 + 5*a^6*z^4 - a^8 + 7*a^6*z^2 + 3*a^6
```

REFERENCES:

- Wikipedia article HOMFLY_polynomial
- http://mathworld.wolfram.com/HOMFLYPolynomial.html

`is_alternating()`

Return whether the given knot diagram is alternating.

Alternating diagram implies every overcross is followed by an undercross or the vice-versa.

We look at the Gauss code if the sign is alternating, `True` is returned else the knot is not alternating `False` is returned.

Warning: This does not check if a knot admits an alternating diagram or not. Thus, this term is used differently than in some of the literature, such as in Hoste-Thistlethwaite table.

Note: Links with more than one component are considered to not be alternating (knots) even when such a diagram exists.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, -1, -1, -1]))
sage: L.is_alternating()
False
sage: L = Link(B([1, -2, -1, 2]))
sage: L.is_alternating()
False
sage: L = Link(B([-1, 3, 1, 3, 2]))
sage: L.is_alternating()
False
sage: L = Link(B([1]*16 + [2,1,2,1,2,2,2,2,2,2,2,1,2,1,2,-1,2,-2]))
sage: L.is_alternating()
False
sage: L = Link(B([-1,2,-1,2]))
sage: L.is_alternating()
True
```

We give the 5_2 knot with an alternating diagram and a non-alternating diagram:

```
sage: K5_2 = Link([[1, 4, 2, 5], [3, 8, 4, 9], [5, 10, 6, 1],
....:              [7, 2, 8, 3], [9, 6, 10, 7]])
sage: K5_2.is_alternating()
True

sage: K5_2b = Link(K5_2.braid())
sage: K5_2b.is_alternating()
False
```

`is_colorable(`*n=None*`)`

Return whether the link is `n`-colorable.

A link is `n`-colorable if its arcs can be painted with `n` colours, labeled from `0` to `n - 1`, in such a way that at any crossing, the average of the indices of the undercrossings equals twice the index of the overcrossing.

INPUT:

- `n` – the number of colors to consider (if ommitted the value of the determinant of `self` will be taken)

EXAMPLES:

We show that the trefoil knot is 3-colorable:

```
sage: K = Link([[[1, -2, 3, -1, 2, -3]], [1, 1, 1]])
sage: K.is_colorable(3)
True
```

But the figure eight knot is not:

```
sage: K8 = Link([[[1, -2, 4, -3, 2, -1, 3, -4]], [1, 1, -1, -1]])
sage: K8.is_colorable(3)
False
```

But it is colorable with respect to the value of its determinant:

```
sage: K8.determinant()
5
sage: K8.is_colorable()
True
```

An examples with non prime determinant:

```
sage: K = Knots().from_table(6, 1)
sage: K.determinant()
9
sage: K.is_colorable()
True
```

REFERENCES:

- Wikipedia article Fox_n-coloring
- Chapter 3 of [Liv1993]

See also:

colorings() and *coloring_maps()*

is_isotopic(*other*)

Check whether `self` is isotopic to `other`.

INPUT:

- `other` – another instance of *Link*

EXAMPLES:

```
sage: l1 = Link([[2, 9, 3, 10], [4, 13, 5, 14], [6, 11, 7, 12],
....:             [8, 1, 9, 2], [10, 7, 11, 8], [12, 5, 13, 6],
....:             [14, 3, 1, 4]])
sage: l2 = Link([[1, 8, 2, 9], [9, 2, 10, 3], [3, 14, 4, 1],
....:             [13, 4, 14, 5], [5, 12, 6, 13], [11, 6, 12, 7],
....:             [7, 10, 8, 11]])
sage: l1.is_isotopic(l2)
True

sage: l3 = l2.mirror_image()
```

(continues on next page)

(continued from previous page)

```
sage: l1.is_isotopic(l3)
False

sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L7a7_0_0             # optional - database_knotinfo
sage: L.series(oriented=True).inject()  # optional - database_knotinfo
Defining L7a7
sage: L == L7a7(0)                      # optional - database_knotinfo
True
sage: l = L.link()                      # optional - database_knotinfo
sage: l.is_isotopic(L7a7(1).link())     # optional - database_knotinfo
Traceback (most recent call last):
...
NotImplementedError: comparison not possible!
sage: l.is_isotopic(L7a7(2).link())     # optional - database_knotinfo
True
sage: l.is_isotopic(L7a7(3).link())     # optional - database_knotinfo
False
```

is_knot()

Return `True` if `self` is a knot.

Every knot is a link but the converse is not true.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([1, 3, 1, -3]))
sage: L.is_knot()
False
sage: B = BraidGroup(8)
sage: L = Link(B([1, 2, 3, 4, 5, 6]))
sage: L.is_knot()
True
```

jones_polynomial(*variab=None*, *skein_normalization=False*, *algorithm='jonesrep'*)

Return the Jones polynomial of `self`.

The normalization is so that the unknot has Jones polynomial 1. If `skein_normalization` is `True`, the variable of the result is replaced by a itself to the power of 4, so that the result agrees with the conventions of [Lic1997] (which in particular differs slightly from the conventions used otherwise in this class), had one used the conventional Kauffman bracket variable notation directly.

If `variab` is `None` return a polynomial in the variable A or t, depending on the value `skein_normalization`. In particular, if `skein_normalization` is `False`, return the result in terms of the variable t, also used in [Lic1997].

ALGORITHM:

The calculation goes through one of two possible algorithms, depending on the value of `algorithm`. Possible values are `'jonesrep'` which uses the Jones representation of a braid representation of `self` to compute the polynomial of the trace closure of the braid, and `statesum` which recursively computes the Kauffman bracket of `self`. Depending on how the link is given, there might be significant time gains in using one over the other. When the trace closure of the braid is `self`, the algorithms give the same result.

INPUT:

- `variab` – variable (default: `None`); the variable in the resulting polynomial; if unspecified, use either a default variable in $\mathbf{Z}[A, A^{-1}]$ or the variable t in the symbolic ring
- `skein_normalization` – boolean (default: `False`); determines the variable of the resulting polynomial
- `algorithm` – string (default: `'jonesrep'`); algorithm to use and can be one of the following:
  - `'jonesrep'` - use the Jones representation of the braid representation
  - `'statesum'` - recursively computes the Kauffman bracket

OUTPUT:

If `skein_normalization` if `False`, this returns an element in the symbolic ring as the Jones polynomial of the link might have fractional powers when the link is not a knot. Otherwise the result is a Laurent polynomial in `variab`.

EXAMPLES:

The unknot:

```
sage: B = BraidGroup(9)
sage: b = B([1, 2, 3, 4, 5, 6, 7, 8])
sage: Link(b).jones_polynomial()
1
```

The "monster" unknot:

```
sage: L = Link([[3,1,2,4],[8,9,1,7],[5,6,7,3],[4,18,6,5],
....:           [17,19,8,18],[9,10,11,14],[10,12,13,11],
....:           [12,19,15,13],[20,16,14,15],[16,20,17,2]])
sage: L.jones_polynomial()
1
```

The Ochiai unknot:

```
sage: L = Link([[[1,-2,-3,-8,-12,13,-14,15,-7,-1,2,-4,10,11,-13,12,
....:             -11,-16,4,3,-5,6,-9,7,-15,14,16,-10,8,9,-6,5]],
....:           [-1,-1,1,1,1,1,-1,1,1,-1,1,-1,-1,-1,-1,-1]])
sage: L.jones_polynomial()   # long time
1
```

Two unlinked unknots:

```
sage: B = BraidGroup(4)
sage: b = B([1, 3])
sage: Link(b).jones_polynomial()
-sqrt(t) - 1/sqrt(t)
```

The Hopf link:

```
sage: B = BraidGroup(2)
sage: b = B([-1,-1])
sage: Link(b).jones_polynomial()
-1/sqrt(t) - 1/t^(5/2)
```

Different representations of the trefoil and one of its mirror:

```
sage: B = BraidGroup(2)
sage: b = B([-1, -1, -1])
sage: Link(b).jones_polynomial(skein_normalization=True)
-A^-16 + A^-12 + A^-4
sage: Link(b).jones_polynomial()
1/t + 1/t^3 - 1/t^4
sage: B = BraidGroup(3)
sage: b = B([-1, -2, -1, -2])
sage: Link(b).jones_polynomial(skein_normalization=True)
-A^-16 + A^-12 + A^-4
sage: R.<x> = LaurentPolynomialRing(GF(2))
sage: Link(b).jones_polynomial(skein_normalization=True, variab=x)
x^-16 + x^-12 + x^-4
sage: B = BraidGroup(3)
sage: b = B([1, 2, 1, 2])
sage: Link(b).jones_polynomial(skein_normalization=True)
A^4 + A^12 - A^16
```

$K11n42$ (the mirror of the "Kinoshita-Terasaka" knot) and $K11n34$ (the mirror of the "Conway" knot) in [KnotAtlas]:

```
sage: B = BraidGroup(4)
sage: K11n42 = Link(B([1, -2, 3, -2, 3, -2, -2, -1, 2, -3, -3, 2, 2]))
sage: K11n34 = Link(B([1, 1, 2, -3, 2, -3, 1, -2, -2, -3, -3]))
sage: bool(K11n42.jones_polynomial() == K11n34.jones_polynomial())
True
```

The two algorithms for computation give the same result when the trace closure of the braid representation is the link itself:

```
sage: L = Link([[[-1, 2, -3, 4, 5, 1, -2, 6, 7, 3, -4, -7, -6, -5]],
....:           [-1, -1, -1, -1, 1, -1, 1]])
sage: jonesrep = L.jones_polynomial(algorithm='jonesrep')
sage: statesum = L.jones_polynomial(algorithm='statesum')
sage: bool(jonesrep == statesum)
True
```

When we have thrown away unknots so that the trace closure of the braid is not necessarily the link itself, this is only true up to a power of the Jones polynomial of the unknot:

```
sage: B = BraidGroup(3)
sage: b = B([1])
sage: L = Link(b)
sage: b.components_in_closure()
2
sage: L.number_of_components()
1
sage: b.jones_polynomial()
-sqrt(t) - 1/sqrt(t)
sage: L.jones_polynomial()
1
sage: L.jones_polynomial(algorithm='statesum')
1
```

khovanov_homology(*ring=Integer Ring*, *height=None*, *degree=None*)

Return the Khovanov homology of the link.

INPUT:

- `ring` – (default: `ZZ`) the coefficient ring
- `height` – the height of the homology to compute, if not specified, all the heights are computed
- `degree` – the degree of the homology to compute, if not specified, all the degrees are computed

OUTPUT:

The Khovanov homology of the Link. It is given as a dictionary whose keys are the different heights. For each height, the homology is given as another dictionary whose keys are the degrees.

EXAMPLES:

```
sage: K = Link([[[1, -2, 3, -1, 2, -3]],[-1, -1, -1]])
sage: K.khovanov_homology()
{-9: {-3: Z},
 -7: {-3: 0, -2: C2},
 -5: {-3: 0, -2: Z, -1: 0, 0: 0},
 -3: {-3: 0, -2: 0, -1: 0, 0: Z},
 -1: {0: Z}}
```

The figure eight knot:

```
sage: L = Link([[1, 6, 2, 7], [5, 2, 6, 3], [3, 1, 4, 8], [7, 5, 8, 4]])
sage: L.khovanov_homology(height=-1)
{-1: {-2: 0, -1: Z, 0: Z, 1: 0, 2: 0}}
```

The Hopf link:

```
sage: B = BraidGroup(2)
sage: b = B([1, 1])
sage: K = Link(b)
sage: K.khovanov_homology(degree = 2)
{2: {2: 0}, 4: {2: Z}, 6: {2: Z}}
```

khovanov_polynomial(*var1='q'*, *var2='t'*, *base_ring=Integer Ring*)

Return the Khovanov polynomial of `self`.

This is the Poincaré polynomial of the Khovanov homology.

INPUT:

- `var1` – (default: `'q'`) the first variable. Its exponents give the (torsion free) rank of the height of Khovanov homology
- `var2` – (default: `'t'`) the second variable. Its exponents give the (torsion free) rank of the degree of Khovanov homology
- `base_ring` – (default: `ZZ`) the ring of the polynomial's coefficients

OUTPUT:

A two variate Laurent Polynomial over the `base_ring`, more precisely an instance of `LaurentPolynomial`.

EXAMPLES:

```
sage: K = Link([[[1, -2, 3, -1, 2, -3]],[-1, -1, -1]])
sage: K.khovanov_polynomial()
q^-1 + q^-3 + q^-5*t^-2 + q^-9*t^-3
sage: K.khovanov_polynomial(base_ring=GF(2))
q^-1 + q^-3 + q^-5*t^-2 + q^-7*t^-2 + q^-9*t^-3
```

The figure eight knot:

```
sage: L = Link([[1, 6, 2, 7], [5, 2, 6, 3], [3, 1, 4, 8], [7, 5, 8, 4]])
sage: L.khovanov_polynomial(var1='p')
p^5*t^2 + p*t + p + p^-1 + p^-1*t^-1 + p^-5*t^-2
sage: L.khovanov_polynomial(var1='p', var2='s', base_ring=GF(4))
p^5*s^2 + p^3*s^2 + p*s + p + p^-1 + p^-1*s^-1 + p^-3*s^-1 + p^-5*s^-2
```

The Hopf link:

```
sage: B = BraidGroup(2)
sage: b = B([1, 1])
sage: K = Link(b)
sage: K.khovanov_polynomial()
q^6*t^2 + q^4*t^2 + q^2 + 1
```

See also:

khovanov_homology()

links_gould_polynomial(*varnames='t0, t1'*)

Return the Links-Gould polynomial of `self`. See [MW2012], section 3 and references given there. See also the docstring of `links_gould_polynomial()`.

INPUT:

- `varnames` – string (default `t0, t1`)

OUTPUT:

A Laurent polynomial in the given variable names.

EXAMPLES:

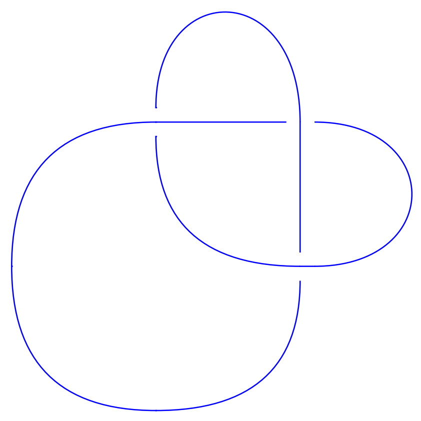
```
sage: Hopf = Link([[1, 3, 2, 4], [4, 2, 3, 1]])
sage: Hopf.links_gould_polynomial()
-1 + t1^-1 + t0^-1 - t0^-1*t1^-1
```

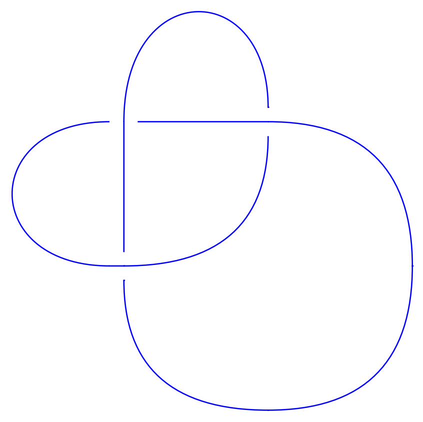
mirror_image()

Return the mirror image of `self`.

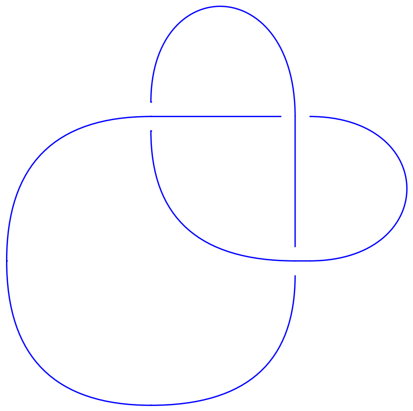
EXAMPLES:

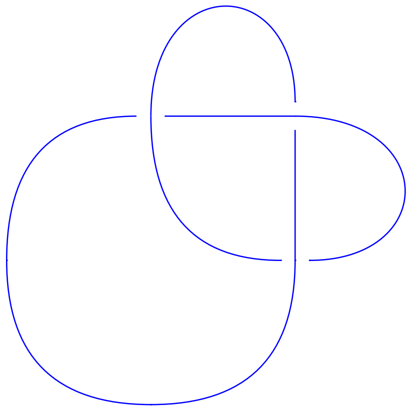
```
sage: g = BraidGroup(2).gen(0)
sage: K = Link(g^3)
sage: K2 = K.mirror_image(); K2
Link with 1 component represented by 3 crossings
sage: K2.braid()
s^-3
```





```
sage: K = Knot([[[1, -2, 3, -1, 2, -3]], [1, 1, 1]])
sage: K2 = K.mirror_image(); K2
Knot represented by 3 crossings
sage: K.pd_code()
[[4, 2, 5, 1], [2, 6, 3, 5], [6, 4, 1, 3]]
sage: K2.pd_code()
[[4, 1, 5, 2], [2, 5, 3, 6], [6, 3, 1, 4]]
```





number_of_components()

Return the number of connected components of `self`.

OUTPUT: number of connected components

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, 3, 1, 3]))
sage: L.number_of_components()
4
sage: B = BraidGroup(8)
sage: L = Link(B([-2, 4, 1, 6, 1, 4]))
sage: L.number_of_components()
5
```

(continues on next page)

(continued from previous page)

```
sage: L = Link(B([1, 2, 1, 2]))
sage: L.number_of_components()
1
sage: L = Link(B.one())
sage: L.number_of_components()
1
```

omega_signature(*omega*)

Compute the ω-signature of `self`.

INPUT:

- ω – a complex number of modulus 1. This is assumed to be coercible to `QQbar`.

This is defined as the signature of the Hermitian matrix

$$(1-\omega)V + (1-\omega^{-1})V^t,$$

where V is the *Seifert matrix*, as explained on page 122 of [Liv1993].

According to [Con2018], this is also known as the Levine-Tristram signature, the equivariant signature or the Tristram-Levine signature.

See also:

signature(), *seifert_matrix()*

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: K = Knot(B([1,1,1,2,-1,2,-3,2,-3]))
sage: omega = QQbar.zeta(3)
sage: K.omega_signature(omega)
-2
```

orientation()

Return the orientation of the crossings of the link diagram of `self`.

EXAMPLES:

```
sage: L = Link([[[1, 2, 5, 4], [3, 7, 6, 5], [4, 6, 9, 8], [7, 11, 10, 9], [8,␣
↪10, 13, 1], [11, 3, 2, 13]]])
sage: L.orientation()
[-1, 1, -1, 1, -1, 1]
sage: L = Link([[1, 6, 2, 7], [7, 2, 8, 3], [3, 10, 4, 11], [11, 4, 12, 5], [14,
↪ 6, 1, 5], [13, 8, 14, 9], [12, 10, 13, 9]])
sage: L.orientation()
[-1, -1, -1, -1, 1, -1, 1]
sage: L = Link([[1, 3, 3, 2], [2, 5, 5, 4], [4, 7, 7, 1]])
sage: L.orientation()
[-1, -1, -1]
```

oriented_gauss_code()

Return the oriented Gauss code of `self`.

The oriented Gauss code has two parts:

a. the Gauss code

b. the orientation of each crossing

The following orientation was taken into consideration for construction of knots:

From the outgoing of the overcrossing if we move in the clockwise direction to reach the outgoing of the undercrossing then we label that crossing as -1.

From the outgoing of the overcrossing if we move in the anticlockwise direction to reach the outgoing of the undercrossing then we label that crossing as $+1$.

One more consideration we take in while constructing the orientation is the order of the orientation is same as the ordering of the crossings in the Gauss code.

Note: Convention: under is denoted by -1, and over by $+1$ in the crossing info.

EXAMPLES:

```
sage: L = Link([[1, 10, 2, 11], [6, 3, 7, 2], [3, 9, 4, 12], [9, 6, 10, 5], [8,␣
↪4, 5, 1], [11, 7, 12, 8]])
sage: L.oriented_gauss_code()
[[[-1, 2, -3, 5], [4, -2, 6, -5], [-4, 1, -6, 3]], [-1, 1, 1, 1, -1, -1]]
sage: L = Link([[1, 3, 2, 4], [6, 2, 3, 1], [7, 5, 8, 4], [5, 7, 6, 8]])
sage: L.oriented_gauss_code()
[[[-1, 2], [-3, 4], [1, 3, -4, -2]], [-1, -1, 1, 1]]
sage: B = BraidGroup(8)
sage: b = B([1, 1, 1, 1, 1])
sage: L = Link(b)
sage: L.oriented_gauss_code()
[[[1, -2, 3, -4, 5, -1, 2, -3, 4, -5]], [1, 1, 1, 1, 1]]
```

pd_code()

Return the planar diagram code of `self`.

The planar diagram is returned in the following format.

We construct the crossing by starting with the entering component of the undercrossing, move in the anti-clockwise direction (see the note below) and then generate the list. If the crossing is given by $[a, b, c, d]$, then we interpret this information as:

1. a is the entering component of the undercrossing;
2. b, d are the components of the overcrossing;
3. c is the leaving component of the undercrossing.

Note: Until version 10.0 the convention to read the `PD` code has been to list the components in clockwise direction. As of version 10.1 the convention has changed, since it was opposite to the usage in most other places.

Thus, if you use `PD` codes from former Sage releases with this version you should check for the correct mirror type.

EXAMPLES:

```
sage: L = Link([[[1, -2, 3, -4, 2, -1, 4, -3]], [1, 1, -1, -1]])
sage: L.pd_code()
```

(continues on next page)

(continued from previous page)

```
[[6, 2, 7, 1], [2, 6, 3, 5], [8, 3, 1, 4], [4, 7, 5, 8]]
sage: B = BraidGroup(2)
sage: b = B([1, 1, 1, 1, 1])
sage: L = Link(b)
sage: L.pd_code()
[[2, 4, 3, 1], [4, 6, 5, 3], [6, 8, 7, 5], [8, 10, 9, 7], [10, 2, 1, 9]]
sage: L = Link([[[2, -1], [1, -2]], [1, 1]])
sage: L.pd_code()
[[2, 4, 1, 3], [4, 2, 3, 1]]
sage: L = Link([[1, 2, 3, 3], [2, 4, 5, 5], [4, 1, 7, 7]])
sage: L.pd_code()
[[1, 2, 3, 3], [2, 4, 5, 5], [4, 1, 7, 7]]
```

plot(*gap=0.1*, *component_gap=0.5*, *solver=None*, *color='blue'*, ***kwargs*)

Plot `self`.

INPUT:

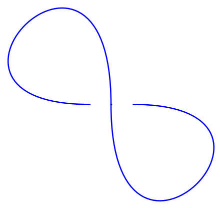
- `gap` – (default: 0.1) the size of the blank gap left for the crossings
- `component_gap` – (default: 0.5) the gap between isolated components
- `solver` – the linear solver to use, see `MixedIntegerLinearProgram`.
- `color` – (default: 'blue') a color or a coloring (as returned by `colorings()`.

The usual keywords for plots can be used here too.

EXAMPLES:

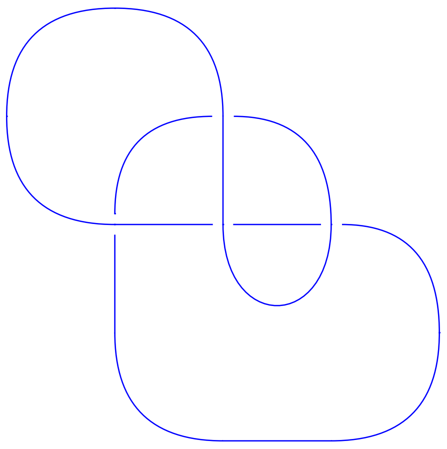
We construct the simplest version of the unknot:

```
sage: L = Link([[2, 1, 1, 2]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```



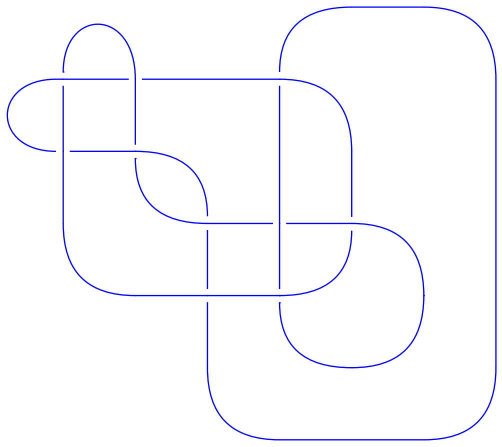
We construct a more interesting example of the unknot:

```
sage: L = Link([[2, 1, 4, 5], [3, 5, 6, 7], [4, 1, 9, 6], [9, 2, 3, 7]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```



The "monster" unknot:

```
sage: L = Link([[3,1,2,4],[8,9,1,7],[5,6,7,3],[4,18,6,5],
....:           [17,19,8,18],[9,10,11,14],[10,12,13,11],
....:           [12,19,15,13],[20,16,14,15],[16,20,17,2]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```



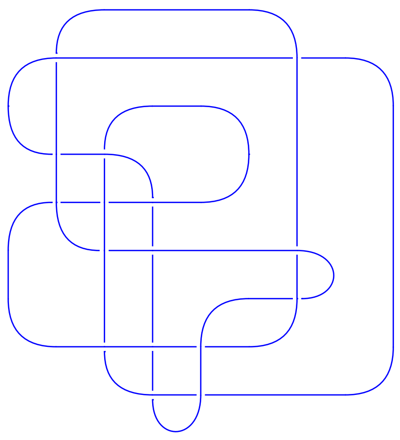
The Ochiai unknot:

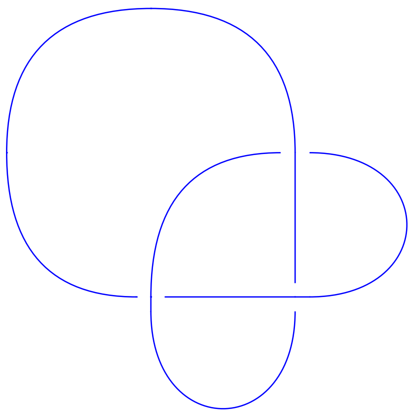
```
sage: L = Link([[[1,-2,-3,-8,-12,13,-14,15,-7,-1,2,-4,10,11,-13,12,
....:             -11,-16,4,3,-5,6,-9,7,-15,14,16,-10,8,9,-6,5]],
....:           [-1,-1,1,1,1,1,-1,1,1,-1,1,-1,-1,-1,-1,-1]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```

One of the representations of the trefoil knot:

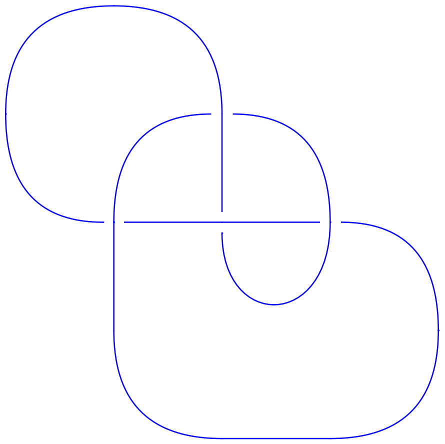
```
sage: L = Link([[1, 5, 2, 4], [5, 3, 6, 2], [3, 1, 4, 6]])
sage: L.plot()
Graphics object consisting of 14 graphics primitives
```

The figure-eight knot:



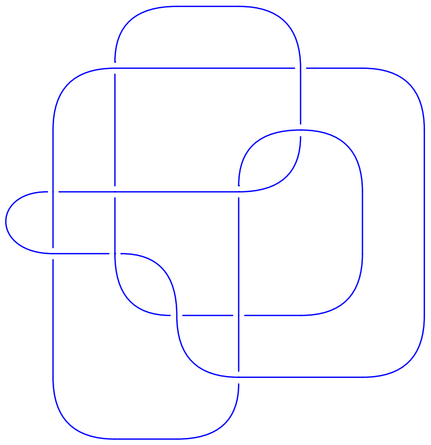


```
sage: L = Link([[2, 1, 4, 5], [5, 6, 7, 3], [6, 4, 1, 9], [9, 2, 3, 7]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```



The knot $K11n121$ in [KnotAtlas]:

```
sage: L = Link([[4,2,5,1], [10,3,11,4], [5,16,6,17], [7,12,8,13],
....:           [18,9,19,10], [2,11,3,12], [13,20,14,21], [15,6,16,7],
....:           [22,18,1,17], [8,19,9,20], [21,14,22,15]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```



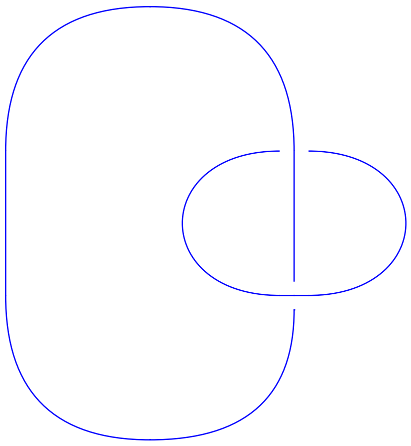
One of the representations of the Hopf link:

```
sage: L = Link([[1, 4, 2, 3], [4, 1, 3, 2]])
sage: L.plot()
Graphics object consisting of ... graphics primitives
```

Plotting links with multiple isolated components:

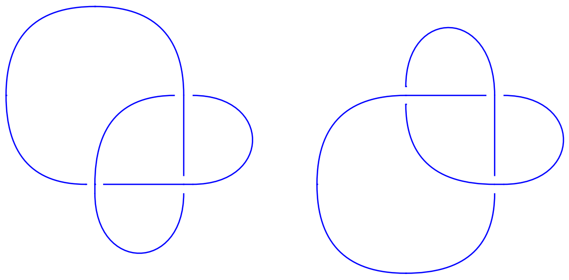
```
sage: L = Link([[[-1, 2, -3, 1, -2, 3], [4, -5, 6, -4, 5, -6]], [1, 1, 1, 1, 1,␣
↪1]])
```

(continues on next page)



(continued from previous page)

```
sage: L.plot()
Graphics object consisting of ... graphics primitives
```



If a coloring is passed, the different arcs are plotted with the corresponding colors (see `colorings()`):

```
sage: B = BraidGroup(4)
sage: b = B([1,2,3,1,2,-1,-3,2,3])
sage: L = Link(b)
sage: L.plot(color=L.colorings()[0])
Graphics object consisting of ... graphics primitives
```

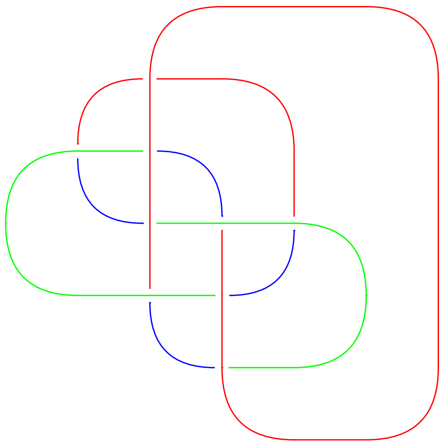
regions()

Return the regions from the link diagram of `self`.

Regions are obtained always turning left at each crossing.

Then the regions are represented as a list with the segments that form its boundary, with a sign depending on the orientation of the segment as part of the boundary.

EXAMPLES:



```
sage: L = Link([[[-1, +2, -3, 4, +5, +1, -2, +6, +7, 3, -4, -7, -6,-5]],[-1, -1,
↪ -1, -1, 1, -1, 1]])
sage: L.regions()
[[14, -5, 12, -9], [13, 9], [11, 5, 1, 7, 3], [10, -3, 8, -13], [6, -1], [4, -
↪11], [2, -7], [-2, -6, -14, -8], [-4, -10, -12]]
sage: L = Link([[[1, -2, 3, -4, 2, -1, 4, -3]],[1, 1, -1, -1]])
sage: L.regions()
[[8, 4], [7, -4, 1], [6, -1, -3], [5, 3, -8], [2, -5, -7], [-2, -6]]
sage: L = Link([[[-1, +2, 3, -4, 5, -6, 7, 8, -2, -5, +6, +1, -8, -3, 4, -7]],[-
↪1, -1, -1, -1, 1, 1, -1, 1]])
sage: L.regions()
[[16, 8, 14, 4], [15, -4], [13, -8, 1], [12, -1, -7], [11, 7, -16, 5], [10, -5,␣
↪-15, -3], [9, 3, -14], [6, -11], [2, -9, -13], [-2, -12, -6, -10]]
sage: B = BraidGroup(2)
sage: L = Link(B([-1, -1, -1]))
sage: L.regions()
[[6, -5], [5, 1, 3], [4, -3], [2, -1], [-2, -6, -4]]
sage: L = Link([[[1, -2, 3, -4], [-1, 5, -3, 2, -5, 4]], [-1, 1, 1, -1, -1]])
sage: L.regions()
[[10, -4, -7], [9, 7, -3], [8, 3], [6, -9, -2], [5, 2, -8, 4], [1, -5], [-1, -
↪10, -6]]
sage: L = Link([[1, 3, 3, 2], [2, 4, 4, 5], [5, 6, 6, 7], [7, 8, 8, 1]])
sage: L.regions()
[[-3], [-4], [-6], [-8], [7, 1, 2, 5], [-1, 8, -7, 6, -5, 4, -2, 3]]
```

Note: The link diagram is assumed to have only one completely isolated component. This is because otherwise some regions would have disconnected boundary.

reverse()

Return the reverse of `self`. This is the link obtained from `self` by reverting the orientation on all components.

EXAMPLES:

```
sage: K3 = Knot([[5, 2, 4, 1], [3, 6, 2, 5], [1, 4, 6, 3]])
sage: K3r = K3.reverse(); K3r.pd_code()
```

(continues on next page)

(continued from previous page)

```
[[4, 1, 5, 2], [2, 5, 3, 6], [6, 3, 1, 4]]
sage: K3 == K3r
True
```

a non reversable knot:

```
sage: K8_17 = Knot([[6, 1, 7, 2], [14, 7, 15, 8], [8, 4, 9, 3],
....:               [2, 14, 3, 13], [12, 6, 13, 5], [4, 10, 5, 9],
....:               [16, 11, 1, 12], [10, 15, 11, 16]])
sage: K8_17r = K8_17.reverse()
sage: b = K8_17.braid(); b
s0^2*s1^-1*(s1^-1*s0)^2*s1^-1
sage: br = K8_17r.braid(); br
s0^-1*s1*s0^-2*s1^2*s0^-1*s1
sage: b.is_conjugated(br)
False
sage: b == br.reverse()
False
sage: b.is_conjugated(br.reverse())
True
sage: K8_17b = Link(b)
sage: K8_17br = K8_17b.reverse()
sage: bbr = K8_17br.braid(); bbr
(s1^-1*s0)^2*s1^-2*s0^2
sage: br == bbr
False
sage: br.is_conjugated(bbr)
True
```

seifert_circles()

Return the Seifert circles from the link diagram of `self`.

Seifert circles are the circles obtained by smoothing all crossings respecting the orientation of the segments.

Each Seifert circle is represented as a list of the segments that form it.

EXAMPLES:

```
sage: L = Link([[[1, -2, 3, -4, 2, -1, 4, -3]], [1, 1, -1, -1]])
sage: L.seifert_circles()
[[1, 7, 5, 3], [2, 6], [4, 8]]
sage: L = Link([[[-1, 2, 3, -4, 5, -6, 7, 8, -2, -5, 6, 1, -8, -3, 4, -7]], [-1,
↪ -1, -1, -1, 1, 1, -1, 1]])
sage: L.seifert_circles()
[[1, 13, 9, 3, 15, 5, 11, 7], [2, 10, 6, 12], [4, 16, 8, 14]]
sage: L = Link([[[-1, 2, -3, 4, 5, 1, -2, 6, 7, 3, -4, -7, -6, -5]], [-1, -1, -
↪1, -1, 1, -1, 1]])
sage: L.seifert_circles()
[[1, 7, 3, 11, 5], [2, 8, 14, 6], [4, 12, 10], [9, 13]]
sage: L = Link([[1, 7, 2, 6], [7, 3, 8, 2], [3, 11, 4, 10], [11, 5, 12, 4], [14,
↪ 5, 1, 6], [13, 9, 14, 8], [12, 9, 13, 10]])
sage: L.seifert_circles()
[[1, 7, 3, 11, 5], [2, 8, 14, 6], [4, 12, 10], [9, 13]]
sage: L = Link([[[-1, 2, -3, 5], [4, -2, 6, -5], [-4, 1, -6, 3]], [-1, 1, 1, 1,␣
```

(continues on next page)

(continued from previous page)

```
↪-1, -1]])
sage: L.seifert_circles()
[[1, 11, 8], [2, 7, 12, 4, 5, 10], [3, 9, 6]]
sage: B = BraidGroup(2)
sage: L = Link(B([1, 1, 1]))
sage: L.seifert_circles()
[[1, 3, 5], [2, 4, 6]]
```

seifert_matrix()

Return the Seifert matrix associated with `self`.

ALGORITHM:

This is the algorithm presented in Section 3.3 of [Col2013].

OUTPUT:

The intersection matrix of a (not necessarily minimal) Seifert surface.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, 3, 1, 3]))
sage: L.seifert_matrix()
[ 0  0]
[ 0 -1]
sage: B = BraidGroup(8)
sage: L = Link(B([-1, 3, 1, 5, 1, 7, 1, 6]))
sage: L.seifert_matrix()
[ 0  0  0]
[ 1 -1  0]
[ 0  1 -1]
sage: L = Link(B([-2, 4, 1, 6, 1, 4]))
sage: L.seifert_matrix()
[-1  0]
[ 0 -1]
```

signature()

Return the signature of `self`.

This is defined as the signature of the symmetric matrix

$$V + V^t,$$

where V is the *Seifert matrix*.

See also:

omega_signature(), *seifert_matrix()*

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: L = Link(B([-1, 3, 1, 3]))
sage: L.signature()
-1
sage: B = BraidGroup(8)
```

(continues on next page)

(continued from previous page)

```
sage: L = Link(B([-2, 4, 1, 6, 1, 4]))
sage: L.signature()
-2
sage: L = Link(B([1, 2, 1, 2]))
sage: L.signature()
-2
```

writhe()

Return the writhe of `self`.

EXAMPLES:

```
sage: L = Link([[[1, -2, 3, -4, 2, -1, 4, -3]],[1, 1, -1, -1]])
sage: L.writhe()
0
sage: L = Link([[[-1, 2, -3, 4, 5, 1, -2, 6, 7, 3, -4, -7, -6,-5]],
....:            [-1, -1, -1, -1, 1, -1, 1]])
sage: L.writhe()
-3
sage: L = Link([[[-1, 2, 3, -4, 5, -6, 7, 8, -2, -5, 6, 1, -8, -3, 4, -7]],
....:            [-1, -1, -1, -1, 1, 1, -1, 1]])
sage: L.writhe()
-2
```

CHAPTER

THREE

ACCESS TO THE KNOTINFO DATABASE

This module contains the class *KnotInfoBase* which is derived from Enum and provides knots and links listed in the databases at the web-pages KnotInfo and LinkInfo as its items.

This interface contains a set of about twenty knots and links statically as demonstration cases. The complete database can be installed as an optional Sage package using

- `sage -i database_knotinfo` (does not install if the current version is present)
- `sage -f database_knotinfo` (installs even if the current version is present)

This will install a Python wrapper for the original databases in Sage. This wrapper perfoms an automatic progress of version numbers. For more details and further install instructions please see the correspondig web-page.

To perform all the doctests concerning the usage of the database on the installation add the option `-c`. In this case (for instance `sage -f -c database_knotinfo`) the installation breaks on failing tests.

The installation of the complete database will be necessary in order to have access to all the properties recorded in the databases, as well.

If the entire database is installed as explained above, the import instructions for *KnotInfo* and *KnotInfoSeries*, which can be seen in the opening lines of the examples, are unnecessary.

Be aware that there are a couple of conventions used differently on KnotInfo as in Sage.

For different conventions regarding normalization of the polynomial invariants see the according documentation of *KnotInfoBase.homfly_polynomial()*, *KnotInfoBase.jones_polynomial()* and *KnotInfoBase.alexander_polynomial()*.

Also, note that the braid notation is used according to Sage, even thought in the source where it is taken from, the braid generators are assumed to have a negative crossing which would be opposite to the convention in Sage (see definition 3 of Gittings, T., “Minimum Braids: A Complete Invariant of Knots and Links”).

Furthermore, note that not all columns available in the database are visible on the web pages (see also the related note under *KnotInfoBase.khovanov_polynomial()*). It is planned to remove non-visible columns from the database in the future (see the Python Wrapper for updated information).

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: L.pd_notation()
[[6, 1, 7, 2], [8, 3, 5, 4], [2, 5, 3, 6], [4, 7, 1, 8]]
sage: L.pd_notation(original=True)
'{{6, 1, 7, 2}, {8, 3, 5, 4}, {2, 5, 3, 6}, {4, 7, 1, 8}}'
sage: L.is_knot()
False
```

(continues on next page)

(continued from previous page)

```
sage: L.num_components()
2
```

Items for knots need a leading K for technical reason:

```
sage: K = KnotInfo.K4_1
sage: K.is_knot()
True
```

Injecting the variable name into the namespace:

```
sage: KnotInfo.K5_1.inject()
Defining K5_1
sage: K5_1.dt_notation()
[6, 8, 10, 2, 4]
```

Defining a link from the original name string:

```
sage: KnotInfo('L6a1{1}').inject()
Defining L6a1_1
sage: L6a1_1.is_alternating()
True
```

Obtaining an instance of Braid:

```
sage: L.braid()
s1^-2*s0^-1*s1*s0^-1
sage: type(_)
<class 'sage.groups.braid.BraidGroup_class_with_category.element_class'>
```

Obtaining an instance of *Link*:

```
sage: l = L.link(); l
Link with 2 components represented by 4 crossings
sage: type(l)
<class 'sage.knots.link.Link'>
```

If you have SnapPy installed inside Sage you can obtain an instance of Link, too:

```
sage: L6 = KnotInfo.L6a1_0
sage: l6s = L6.link(snappy=True); l6s      # optional - snappy
Plink failed to import tkinter.
<Link: 2 comp; 6 cross>

sage: type(l6s)                            # optional - snappy
<class 'spherogram.links.invariants.Link'>
sage: l6  = L6.link()
sage: l6 == l6s.sage_link()                # optional - snappy
True
sage: L6.link(L6.items.name, snappy=True)  # optional - snappy
<Link L6a1: 2 comp; 6 cross>
sage: l6sn = _                             # optional - snappy
sage: l6s == l6sn                          # optional - snappy
```

(continues on next page)

(continued from previous page)

```
False
sage: l6m = l6.mirror_image()              # optional - snappy
sage: l6sn.sage_link().is_isotopic(l6m)    # optional - snappy
True
```

But observe that the name conversion to SnapPy does not distinguish orientation types:

```
sage: L6b = KnotInfo.L6a1_1
sage: L6b.link(L6b.items.name, snappy=True)  # optional - snappy
<Link L6a1: 2 comp; 6 cross>
sage: _.PD_code() == l6sn.PD_code()          # optional - snappy
True
```

Obtaining the HOMFLY-PT polynomial:

```
sage: L.homfly_polynomial()
-v^-1*z - v^-3*z - v^-3*z^-1 + v^-5*z^-1
sage: _ == l.homfly_polynomial(normalization='vz')
True
```

Obtaining the original string from the database for an arbitrary property:

```
sage: K[K.items.classical_conway_name]  # optional - database_knotinfo
'4_1'
```

Further methods:

```
sage: K.crossing_number()
4
sage: K.gauss_notation()
[-1, 2, -3, 1, -4, 3, -2, 4]
sage: K.dt_notation()
[4, 6, 8, 2]
sage: K.determinant()
5
sage: K.symmetry_type()
'fully amphicheiral'
sage: _ == K[K.items.symmetry_type]
True
sage: K.is_reversible()
True
sage: K.is_amphicheiral()
True
sage: K.jones_polynomial()
t^2 - t - 1/t + 1/t^2 + 1
sage: K.kauffman_polynomial()
a^2*z^2 + a*z^3 - a^2 - a*z + 2*z^2 + a^-1*z^3 - 1 - a^-1*z + a^-2*z^2 - a^-2
sage: K.alexander_polynomial()
t^2 - 3*t + 1
```

Using the `column_type` of a property:

```
sage: def select_column(i):
....:     return i.column_type() != i.types.OnlyLinks and K[i] == 'Y'
sage: [i.column_name() for i in K.items if select_column(i)]  # optional - database_
↪knotinfo
['Alternating', 'Fibered', 'Quasialternating', 'Adequate']
```

You can launch web-pages attached to the links:

```
sage: # not tested
sage: K.diagram()
True
sage: L.diagram(single=True)
True
sage: L.knot_atlas_webpage()
True
sage: K.knotilus_webpage()
True
```

and the description web-pages of the properties:

```
sage: K.items.positive.description_webpage()  # not tested
True
```

To see all the properties available in this interface you can use "tab-completion". For example type `K.items.` and than hit the `Tab` key. You can select the item you want from the list. If you know some first letters type them first to obtain a reduced selection list.

In a similar way you may select the knots and links. Here you have to type `KnotInfo.` or `KnotInfo.L7` before stroking the `Tab` key. In the latter case the selection list will be reduced to proper links with 7 crossings.

Finally there is a method *`Link.get_knotinfo()`* of class *`Link`* to find an instance in the KnotInfo database:

```
sage: L = Link([[3,1,2,4], [8,9,1,7], [5,6,7,3], [4,18,6,5],
....:           [17,19,8,18], [9,10,11,14], [10,12,13,11],
....:           [12,19,15,13], [20,16,14,15], [16,20,17,2]])
sage: L.get_knotinfo()
(<KnotInfo.K0_1: '0_1'>, None)
```

REFERENCES:

- KnotInfo
- LinkInfo

AUTHORS:

- Sebastian Oehms August 2020: initial version
- Sebastian Oehms June 2022: add `conway_polynomial()` and `khovanov_polynomial()` (github issue #33969)

Thanks to Chuck Livingston and Allison Moore for their support. For further acknowledgments see the correspondig hompages.

class `sage.knots.knotinfo.`**`KnotInfo`**(*value*)

Bases: *`KnotInfoBase`*

An enumeration.

K0_1 = '0_1'

K3_1 = '3_1'

K4_1 = '4_1'

K5_1 = '5_1'

K5_2 = '5_2'

K6_1 = '6_1'

K6_2 = '6_2'

K6_3 = '6_3'

K7_1 = '7_1'

K7_2 = '7_2'

L2a1_0 = 'L2a1{0}'

L2a1_1 = 'L2a1{1}'

L4a1_0 = 'L4a1{0}'

L4a1_1 = 'L4a1{1}'

L5a1_0 = 'L5a1{0}'

L5a1_1 = 'L5a1{1}'

L6a1_0 = 'L6a1{0}'

L6a1_1 = 'L6a1{1}'

L6a2_0 = 'L6a2{0}'

L6a2_1 = 'L6a2{1}'

class sage.knots.knotinfo.**KnotInfoBase**(*value*)

Bases: Enum

Enum class to select the knots and links listed in the databases at the web-pages KnotInfo and LinkInfo.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: [knot.name for knot in KnotInfo if knot.crossing_number() < 5]
['K0_1', 'K3_1', 'K4_1', 'L2a1_0', 'L2a1_1', 'L4a1_0', 'L4a1_1']
```

More examples and information can be seen in the module header *knotinfo* (by typing):

```
sage: import sage.knots.knotinfo    # not tested
sage: sage.knots.knotinfo?          # not tested
```

alexander_polynomial(*var='t'*, *original=False*, *laurent_poly=False*)

Return the Alexander polynomial according to the value of column `alexander_polynomial` for this knot as an instance of `Polynomial`.

It is obtained from the Seifert matrix V of `self` by the following formula (see the KnotInfo description web-page; to launch it see the example below):

$$A(L) = \det(V - tV^t)$$

Here V^t stands for the transpose of V.

INPUT:

- `var` – (default: `'t'`) the variable
- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string
- `laurent_poly` – boolean (default `False`) see the note below

OUTPUT:

A polynomial over the integers, more precisely an instance of `Polynomial`. If `laurent_poly` is set to `True` a Laurent polynomial over the integers, more precisely an instance of `LaurentPolynomial` is returned. If `original` is set to `True` then a string is returned.

Note: As an invariant the Alexander polynomial is only unique up to a unit factor in the Laurent polynomial ring over the integers in the indeterminate t. While the normalization of the exponents in KnotInfo guarantees it to be a proper polynomial, this is not the case for the implementation in Sage. Use the keyword `laurent_poly` to achiev a normalization according to Sage. But, still there may be a difference in sign (see the example below).

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K4_1
sage: Ka = K.alexander_polynomial(); Ka
t^2 - 3*t + 1
```

Comparison with Sage's results:

```
sage: k = K.link()
sage: ka = k.alexander_polynomial(); ka
-t^-1 + 3 - t
sage: K.alexander_polynomial(laurent_poly=True)
t^-1 - 3 + t
sage: _ == -ka
True
```

Launch the KnotInfo description web-page:

```
sage: K.items.alexander_polynomial.description_webpage() # not tested
True
```

braid()

Return the braid notation of self as an instance of `Braid`.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K3_1
sage: K.braid()
s^3
sage: K.braid_notation()
(1, 1, 1)
```

braid_index()

Return the value of column `braid_index` for this link as a Python int.

OUTPUT:

Python int giving the minimum of strands needed to represent `self` as closure of a braid.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: L.braid_index()
3
```

braid_length()

Return the value of column `braid_length` for this link as a Python int.

OUTPUT:

Python int giving the minimum length of a braid word needed to represent `self` as closure of a braid.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K3_1
sage: K.braid_length()
3
```

braid_notation(*original=False*)

Return the value of column `braid_notation` for this link as a Python tuple (Tietze form). For more information type `KnotInfo.K0_1.items.braid_notation.description_webpage()`.

INPUT:

- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

Python tuple representing the braid whose closure is `self` in Tietze form.

..NOTE:

```
There has been a major change to braid representatives for
proper links since version 2021.10.1. The former braid
reresentatives can be obtained by the column
``braid_notation_old`` (see the final example below).
```

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: L.braid_notation()
(-2, -2, -1, 2, -1)
sage: L.braid_notation(original=True)
'{3, {-2, -2, -1, 2, -1}}'
sage: L[L.items.braid_notation_old]
'{4, {1, -2, 3, -2, -1, -2, -3, -2}}'
```

conway_polynomial(*var='t'*, *original=False*)

Return the Conway polynomial according to the value of column `conway_polynomial` for this knot or link as an instance of `Polynomial`.

It is obtained from the Seifert matrix V of `self` by the following formula (see the KnotInfo description web-page; to launch it see the example below):

$$\nabla(L) = \det(t^{\frac{1}{2}} V - t^{\frac{-1}{2}} V^t)$$

Here V^t stands for the transpose of V.

INPUT:

- `var` – (default: `'t'`) the variable
- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

A polynomial over the integers, more precisely an instance of `Polynomial`. If `original` is set to `True` then a string is returned.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K4_1
sage: Kc = K.conway_polynomial(); Kc
-t^2 + 1
sage: L = KnotInfo.L5a1_0
sage: Lc = L.conway_polynomial(); Lc
t^3
```

Comparision to Sage's results:

```
sage: Kc == K.link().conway_polynomial()
True
sage: Lc == L.link().conway_polynomial()
True
```

Launch the KnotInfo description web-page:

```
sage: K.items.conway_polynomial.description_webpage() # not tested
True
```

crossing_number()

Return the minimal number of crossings of `self`.

Note: In contrast to the number of crossings displayed for instances of *Link* this number is the minimum over all possible diagrams of the link. The number of crossings displayed in the representation string of *Link* refers to the special diagram which could be larger.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L4a1_0.crossing_number()
4
sage: KnotInfo.K3_1.crossing_number()
3
sage: Link(KnotInfo.L4a1_0.braid())
Link with 2 components represented by 5 crossings
```

determinant()

Return the determinant of `self`.

From the KnotInfo description page:

> The determinant of a knot is $\det(V + V^t)$, where V is a Seifert matrix for the knot.

To read the complete description type `KnotInfo.K0_1.items.determinant.description_webpage()`.

Note: KnotInfo's value for the unknot `0_1` is zero. This is not compatible whith Sage's result (the value of the Alexander polynomial at -1). Since this method is needed to identify Sage links we take the according value in that case.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L4a1_0.determinant()
4
sage: KnotInfo.K3_1.determinant()
3
sage: KnotInfo.K0_1.determinant()
1
```

diagram(*single=False*, *new=0*, *autoraise=True*)

Launch the diagram of `self` given on the KnotInfo web-page.

INPUT:

- `single` – boolean (default `False`) if set to `True` only one diagram is shown.
- `new` – `int` according to `open()` of `webbrowser` (`0` default, `1` new window, `2` new tab)
- `autoraise` – boolean (default `True`)

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K3_1
sage: K.diagram()             # not tested
True
```

(continues on next page)

(continued from previous page)

```
sage: K.diagram(single=True) # not tested
True
```

`dt_notation`(*original=False*)

Return the value of column `dt_notation` for this link as a Python list of Python lists. For more information type `KnotInfo.K0_1.items.dt_notation.description_webpage()`.

INPUT:

- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

Python list of python lists each entry of the outer list representing a crossing.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: L.dt_notation()
[[6, 8], [2, 4]]
sage: L.dt_notation(original=True)
'[{6, 8}, {2, 4}]'
sage: L = KnotInfo.L4a1_0
sage: K = KnotInfo.K4_1
sage: K.dt_notation()
[4, 6, 8, 2]
```

`gauss_notation`(*original=False*)

Return the value of column `gauss_notation` for this link as a Python list of Python lists. For more information type `KnotInfo.K0_1.items.gauss_notation.description_webpage()`.

INPUT:

- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

Python list of

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: L.gauss_notation()
[[1, -3, 2, -4], [3, -1, 4, -2]]
sage: L.gauss_notation(original=True)
'{{1, -3, 2, -4}, {3, -1, 4, -2}}'
```

`homfly_polynomial`(*var1='v'*, *var2='z'*, *original=False*)

Return the HOMFLY-PT polynomial according to the value of column `homfly_polynomial` for this knot or link (in the latter case the column `homflypt_polynomial` is used) as an instance of the element class according to the output of *`Link.homfly_polynomial()`* of *`Link`*.

The HOMFLY-PT polynomial $P(L)$ of a link L satisfies the following skein relation (see the corresponding KnotInfo description page):

$$P(O) = 1, \;\; v^{-1}P(L_+) - vP(L_-) = zP(L_0)$$

INPUT:

- `var1` – string (default `v`) for the name of the first variable
- `var2` – string (default `z`) for the name of the second variable
- `original` – boolean (default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

A Laurent polynomial over the integers, more precisely an instance of `LaurentPolynomial_mpair`. If `original` is set to `True` then a string is returned.

Note: The skein-relation for the HOMFLY-PT polynomial given on KnotInfo does not match the default used in Sage. For comparison you have to use the keyword argument `normalization='vz'` on the side of Sage.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K3_1 = KnotInfo.K3_1
sage: PK3_1 = K3_1.homfly_polynomial(); PK3_1
-v^4 + v^2*z^2 + 2*v^2
sage: K3_1.homfly_polynomial(original=True)
'(2*v^2-v^4)+(v^2)*z^2'
sage: PK3_1 == K3_1.link().homfly_polynomial(normalization='vz')
True
```

for proper links:

```
sage: L4a1_1 = KnotInfo.L4a1_1
sage: PL4a1_1 = L4a1_1.homfly_polynomial(var1='x', var2='y'); PL4a1_1
-x^5*y + x^3*y^3 - x^5*y^-1 + 3*x^3*y + x^3*y^-1
sage: _ == L4a1_1.link().homfly_polynomial('x', 'y', 'vz')
True
```

check the skein-relation from the KnotInfo description page (applied to one of the positive crossings of the right-handed trefoil):

```
sage: R = PK3_1.parent()
sage: PO = R.one()
sage: L2a1_1 = KnotInfo.L2a1_1
sage: PL2a1_1 = L2a1_1.homfly_polynomial()
sage: v, z = R.gens()
sage: ~v*PK3_1 -v*PO == z*PL2a1_1
True
```

REFERENCES:

- Wikipedia article HOMFLY_polynomial

inject(*verbose=True*)

Inject `self` with its name into the namespace of the Python code from which this function is called.

INPUT:

- `verbose` – boolean (optional, default `True`) to suppress the message printed on the invocation

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.inject()
Defining K5_2
sage: K5_2.is_alternating()
True
```

is_adequate()

Return whether `self` is adequate.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_adequate()         # optional - database_knotinfo
True
```

is_almost_alternating()

Return whether `self` is almost alternating.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_almost_alternating() # optional - database_knotinfo
False
```

is_alternating()

Return whether `self` is alternating.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_alternating()
True
```

is_amphicheiral(*positive=False*)

Return whether `self` is amphicheiral.

INPUT:

- `positive` – boolean (optional, default False) whether to check if `self` is positive or negative amphicheiral (see documentation of *`symmetry_type()`*)

OUTPUT:

Boolean or `None` if this cannot be determined.

`True` if `self` is fully or negative amphicheiral per default. If `positive` is set to `True` than fully and positive amphicheiral links give `True`.

Note: For proper links this property is not provided in the database. Anyway, we support it here in this case, as well, except for a few items where it cannot be determined easily and where `None` is returned as answer.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: Kp = KnotInfo.K12a_427              # optional - database_knotinfo
sage: Kp.is_amphicheiral()                # optional - database_knotinfo
False
sage: Kp.is_amphicheiral(positive=True)   # optional - database_knotinfo
True

sage: Kn = KnotInfo.K10_88                # optional - database_knotinfo
sage: Kn.is_amphicheiral()                # optional - database_knotinfo
True
sage: Kn.is_amphicheiral(positive=True)   # optional - database_knotinfo
False

sage: KnotInfo.L4a1_0.is_amphicheiral()
False
sage: KnotInfo.L10n59_1.is_amphicheiral() # optional - database_knotinfo
True
sage: KnotInfo.L10n36_0.inject()          # optional - database_knotinfo
Defining L10n36_0
sage: L10n36_0.is_amphicheiral() is None  # optional - database_knotinfo
True
```

is_fibered()

Return whether `self` is fibered.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K6_3.is_fibered()
True
```

is_knot()

Return whether `self` is a knot or a proper link.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L7a1_0.is_knot()      # optional - database_knotinfo
False
sage: KnotInfo.K6_3.is_knot()
True
```

is_oriented()

Return whether `self` is oriented.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L6a2_1.is_oriented()
True
```

is_positive()

Return whether `self` is positive.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_positive()
True
```

is_positive_braid()

Return whether `self` is a positive braid.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_positive_braid()         # optional - database_knotinfo
False
```

is_quasi_alternating()

Return whether `self` is quasi alternating.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_quasi_alternating() # optional - database_knotinfo
True
```

is_quasipositive()

Return whether `self` is quasi-positive.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_quasipositive()     # optional - database_knotinfo
True
```

is_recoverable(*unique=True*)

Return if `self` can be recovered from its conversion to Sage links using the `pd_notation` and the `braid_notation` and their mirror images.

The method is indirectly used by the `TestSuite` of the series of `self`.

INPUT:

- `unique` – boolean (optional, default=``True``) if set to `False` it is only checked if `self` is among the recovered items

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L4a1_0.inject()
Defining L4a1_0
```

(continues on next page)

(continued from previous page)

```
sage: L4a1_0.is_recoverable()
True
sage: L4a1_0.is_recoverable(unique=False)
True
sage: KnotInfo.L5a1_0.inject()
Defining L5a1_0
sage: L5a1_0.is_recoverable()
False
sage: L5a1_0.is_recoverable(unique=False)
True
```

is_reversible()

Return whether `self` is reversible.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K6_3.is_reversible()
True
```

is_strongly_quasipositive()

Return whether `self` is strongly quasi-positive.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.is_strongly_quasipositive() # optional - database_knotinfo
True
```

is_unique()

Return whether there is no other isotopic link in the database or not.

OUTPUT:

Boolean or `None` if this cannot be determined.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L4a1_0.is_unique()
True
sage: KnotInfo.L5a1_0.is_unique()
False
sage: L = KnotInfo.L9a43_0_1            # optional - database_knotinfo
sage: L.series(oriented=True).inject()  # optional - database_knotinfo
Defining L9a43
sage: [(L,L.is_unique()) for L in L9a43] # optional - database_knotinfo
[(<KnotInfo.L9a43_0_0: 'L9a43{0,0}'>, True),
 (<KnotInfo.L9a43_1_0: 'L9a43{1,0}'>, False),
 (<KnotInfo.L9a43_0_1: 'L9a43{0,1}'>, None),
 (<KnotInfo.L9a43_1_1: 'L9a43{1,1}'>, False)]
```

property items

Return an Enum class to select a column item of the KnotInfo database.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: it = L.items
sage: [i.name for i in it if i.name.startswith('braid')]
['braid_index', 'braid_length', 'braid_notation', 'braid_notation_old']
sage: L.items.dt_notation.column_name()
'DT Notation'
```

To check if the item is available for proper links or only knots type:

```
sage: it.gauss_notation.column_type()
<KnotInfoColumnTypes.KnotsAndLinks: 'B'>
sage: it.dt_notation.column_type()
<KnotInfoColumnTypes.OnlyKnots: 'K'>
```

To see the description of the item in your web browser type:

```
sage: it.gauss_notation.description_webpage()    # not tested
True
```

jones_polynomial(*variab=None*, *skein_normalization=False*, *puiseux=False*, *original=False*, *use_sqrt=False*)

Return the Jones polynomial according to the value of column `jones_polynomial` for this knot or link as an element of the symbolic ring `SR` or an instance of `LaurentPolynomial` depending on the keyword `skein_normalization`. Using the keyword `puiseux` instead of an element of the symbolic ring an instance of `PuiseuxSeries` can be returned.

The Jones polynomial $V(L)$ of a link L satisfies the following skein relation (see the corresponding KnotInfo description page):

$$V(O) = 1, \ \ t^{-1}V(L_+) - tV(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0)$$

INPUT:

- `variab` – variable (default: `None`) used according to *`Link.jones_polynomial()`*
- `skein_normalization` – boolean (default: `False`) used according to *`Link.jones_polynomial()`*
- `puiseux` – boolean (default `True`) only used in case `skein_normalization=False`. If set to `True` instead of an element of the symbolic ring an instance of `PuiseuxSeries` is returned
- `original` – boolean (default `False`) if set to `True` the original table entry is returned as a string
- `use_sqrt` – boolean (default `False`) see the note below

OUTPUT:

Depends on the keywords (in excluding order):

- `original=True` a string according to the original value from the database
- `skein_normalization=True` a Laurent polynomial over the integers, more precisely an instance of `LaurentPolynomial`
- `puiseux=True` a puiseux series over the integers, more precisely an instance of `PuiseuxSeries`

In all other cases an element of the symbolic ring `SR`.

Note: There is a difference to Sage's conventions concerning the Jones polynomial in the case of proper links. KnotInfo does not display these polynomials in the indeterminate t used in the skein relation. Instead a variable x is used defined by $x^2 = t$. Sage uses t in both cases, knots and proper links. Thus, to obtain the Jones polynomial for a proper link in t you have to set the keyword `use_sqrt` to `True`.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K4_1
sage: Kj = K.jones_polynomial(); Kj
t^2 - t - 1/t + 1/t^2 + 1
sage: Kjs = K.jones_polynomial(skein_normalization=True); Kjs
A^-8 - A^-4 + 1 - A^4 + A^8
sage: Kjp = K.jones_polynomial(puiseux=True); Kjp
t^-2 - t^-1 + 1 - t + t^2
```

for proper links:

```
sage: L = KnotInfo.L2a1_1
sage: Lj = L.jones_polynomial(); Lj
-x^5 - x
sage: Ljt = L.jones_polynomial(use_sqrt=True); Ljt
-t^(5/2) - sqrt(t)
sage: Ljp = L.jones_polynomial(puiseux=True); Ljp
-t^(1/2) - t^(5/2)
sage: Ljs = L.jones_polynomial(skein_normalization=True); Ljs
-A^2 - A^10
sage: Lj.parent()
Symbolic Ring
sage: Ljt.parent()
Symbolic Ring
sage: Ljp.parent()
Puiseux Series Ring in t over Integer Ring
sage: Ljs.parent()
Univariate Laurent Polynomial Ring in A over Integer Ring
```

Comparison with Sage's results:

```
sage: k = K.link()
sage: kj = k.jones_polynomial()
sage: bool(Kj == kj)
True
sage: kjs = k.jones_polynomial(skein_normalization=True)
sage: Kjs == kjs
True
sage: l = L.link()
sage: lj = l.jones_polynomial()
sage: bool(Lj == lj)
False
sage: bool(Ljt == lj)   # see note above
True
sage: ljs = l.jones_polynomial(skein_normalization=True)
sage: Ljs == ljs
True
```

Check the skein-relation from the KnotInfo description page (applied to one of the positive crossings of the right-handed trefoil):

```
sage: K3_1  = KnotInfo.K3_1
sage: K3_1j = K3_1.jones_polynomial()
sage: L2a1_1j = Ljt     # see note above
sage: R = L2a1_1j.parent()
sage: Oj = R(1)
sage: t = R('t')
sage: lhs = expand(~t*K3_1j - t*Oj)
sage: rhs = expand((sqrt(t) - ~sqrt(t))*L2a1_1j)
sage: bool(lhs == rhs)
True
```

The same with the Puiseux series version:

```
sage: K3_1jp = K3_1.jones_polynomial(puiseux=True)
sage: L2a1_1jp = Ljp
sage: R = L2a1_1jp.parent()
sage: Ojp = R(1)
sage: t = R('t')
sage: ~t*K3_1jp - t*Ojp == (t^(1/2)-~t^(1/2))*L2a1_1jp
True
```

The same in the case of skein normalization (using $t = A^4$):

```
sage: K3_1js = K3_1.jones_polynomial(skein_normalization=True)
sage: L2a1_1js = L.jones_polynomial(skein_normalization=True)
sage: Rs = K3_1js.parent()
sage: Ojs = Rs.one()
sage: A, = Rs.gens()
sage: ~A^4*K3_1js - A^4*Ojs == (A^2-~A^2)*L2a1_1js
True
```

REFERENCES:

- Wikipedia article Jones_polynomial

kauffman_polynomial(*var1='a'*, *var2='z'*, *original=False*)

Return the Kauffman polynomial according to the value of column `kauffman_polynomial` for this knot or link as an instance of `LaurentPolynomial_mpair`.

The Kauffman polynomial $F(L)$ respectivlely its corresponding invariant under regular isotopy $\Delta(L) = a^{w(L)}F(L)$ where $w(L)$ is the writhe of the link L satisfies the following skein relation (see the corresponding KnotInfo description page):

$$\Delta(O) = 1, \;\; \Delta(L_+) - \Delta(L_-) = z(\Delta(L_0 + \Delta(L_\infty))$$

Furthermore, removing a curl of sign ϵ leads to a multiplication of $\Delta(L)$ with a^ϵ.

INPUT:

- `var1` – (default: `'a'`) the first variable
- `var2` – (default: `'z'`) the second variable
- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

A Laurent polynomial over the integers, more precisely an instance of `LaurentPolynomial_mpair`. If `original` is set to `False` then a string is returned.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L2a1_1
sage: L.kauffman_polynomial()
a^-1*z - a^-1*z^-1 + a^-2 + a^-3*z - a^-3*z^-1
sage: K = KnotInfo.K4_1
sage: K.kauffman_polynomial()
a^2*z^2 + a*z^3 - a^2 - a*z + 2*z^2 + a^-1*z^3 - 1 - a^-1*z + a^-2*z^2 - a^-2
```

Comparison with Jones polynomial:

```
sage: k    = _
sage: a, z = k.variables()
sage: j    = K.jones_polynomial(skein_normalization=True)
sage: t,   = j.variables()
sage: k.subs(a=-t^3, z=~t+t) == j.subs(t=t^4)
True
```

Check the skein relation:

```
sage: K3_1    = KnotInfo.K3_1
sage: FK3_1   = K3_1.kauffman_polynomial()
sage: FL2a1_1 = L.kauffman_polynomial()
sage: z, a    = FK3_1.variables()
sage: ΔK3_1   = FK3_1   * a**K3_1.link().writhe()
sage: ΔL2a1_1 = FL2a1_1 * a**L.link().writhe()
sage: Δ01p    = a          # unknot with one positive curl
sage: Δ02n    = a**-2      # unknot with two negative curls
sage: ΔK3_1 + Δ01p == z*(ΔL2a1_1 + Δ02n)
True
```

REFERENCES:

- Wikipedia article Kauffman_polynomial

`khovanov_polynomial`(*var1='q'*, *var2='t'*, *base_ring=Integer Ring*, *original=False*, *reduced=False*, *odd=False*, *KhoHo=False*)

Return the Khovanov polynomial according to the value of column `khovanov_polynomial` for this knot or link as an instance of `LaurentPolynomial_mpair`.

INPUT:

- `var1` – (default: `'q'`) the first variable
- `var2` – (default: `'t'`) the second variable
- `base_ring` – (default: `ZZ`) the ring of the polynomial's coefficients
- `original` – boolean (default: `False`); if set to `True` the original table entry is returned as a string
- `reduced` – boolean (default: `False`); if set to `True` the reduced version of the homology is used
- `odd` – boolean (default: `False`); if set to `True` the odd version of the homology is used

- KhoHo – boolean (default: False for knots and True for multi-component links); if set to True the data calculated using KhoHo is used (see the note below)

OUTPUT:

A Laurent polynomial over the integers, more precisely an instance of LaurentPolynomial_mpair. If original is set to True then a string is returned.

Note: The data used for multi-component links were calculated with the program KhoHo. These can still be used for knots by setting the optional argument KhoHo to True, even though they will no longer be visible on the Knot website as of October 30, 2022. Otherwise, for knots data calculated with KnotJob are used. The latter program is more accurate in terms of orientation and reflection as it is based on PD code.

Note that in the future columns that are not visible on the web page may also be removed in the database (see the Python wrapper for updated information). Therefore, the KhoHo option cannot be guaranteed to work after upgrading the database_knotinfo-SPKG.

Furthermore, since the results of KhoHo were computed using the DT notation, the Khovanov polynomial returned by this option belongs to the mirror image of the given knot for a list of 140 exceptions.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K6_3
sage: Kk = K.khovanov_polynomial(); Kk
q^7*t^3 + q^5*t^2 + q^3*t^2 + q^3*t + q*t + 2*q + 2*q^-1 + q^-1*t^-1
+ q^-3*t^-1 + q^-3*t^-2 + q^-5*t^-2 + q^-7*t^-3
sage: Kk2 = K.khovanov_polynomial(var1='p', base_ring=GF(2)); Kk2
p^7*t^3 + p^5*t^3 + p^5*t^2 + p^3*t + p^-1 + p^-1*t^-1 + p^-3*t^-2 + p^-7*t^-3

sage: L = KnotInfo.L5a1_0
sage: Lk = L.khovanov_polynomial(); Lk
q^4*t^2 + t + 2 + 2*q^-2 + q^-2*t^-1 + q^-4*t^-2 + q^-6*t^-2 + q^-8*t^-3
sage: L.khovanov_polynomial(original=True)
 '2 + 2/q^2 + 1/(q^8*t^3) + 1/(q^6*t^2) + 1/(q^4*t^2) + 1/(q^2*t) + t + q^4*t^2'
```

Obtaining the reduced homology (for knots only):

```
sage: Kkr = K.khovanov_polynomial(reduced=True); Kkr
q^6*t^3 + 2*q^4*t^2 + 2*q^2*t + 3 + 2*q^-2*t^-1 + 2*q^-4*t^-2 + q^-6*t^-3
sage: K.khovanov_polynomial(base_ring=QQ, reduced=True) == Kkr
True
sage: Kkr2 = K.khovanov_polynomial(base_ring=GF(2), reduced=True); Kkr2
q^6*t^3 + 1 + q^-6*t^-3
sage: KnotInfo.K8_19.inject()                              # optional database_
↪knotinfo
Defining K8_19
sage: K8kr = K8_19.khovanov_polynomial(reduced=True); K8kr  # optional database_
↪knotinfo
q^16*t^5 + q^12*t^4 + q^12*t^3 + q^10*t^2 + q^6
```

Obtaining the odd Khovanov homology (for knots only):

```
sage: K.khovanov_polynomial(odd=True) == Kkr
True
```

(continues on next page)

(continued from previous page)

```
sage: K.khovanov_polynomial(base_ring=QQ, odd=True) == Kkr
True
sage: K.khovanov_polynomial(base_ring=GF(2), odd=True) == Kkr2
True
sage: K8ko = K8_19.khovanov_polynomial(odd=True); K8ko     # optional database_
↪knotinfo
q^16*t^5 + q^10*t^2 + q^6
sage: K8kr == K8ko                                         # optional database_
↪knotinfo
False
```

Comparision to Sage's results:

```
sage: Kk == K.link().khovanov_polynomial()
True
sage: Kk2 == K.link().khovanov_polynomial(var1='p', base_ring=GF(2))
True
sage: Lk == L.link().khovanov_polynomial()
True
```

REFERENCES:

- Wikipedia article Khovanov_homology
- Wikipedia article Reduced_homology
- [ORS2013]

knot_atlas_webpage(*new=0*, *autoraise=True*)

Launch the Knot Atlas web-page for `self`.

INPUT:

- `new` – `int` according to `open()` of `webbrowser` (`0` default, 1 new window, 2 new tab)
- `autoraise` – boolean (default `True`)

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K3_1
sage: K.knot_atlas_webpage()        # not tested
True
```

knotilus_webpage(*new=0*, *autoraise=True*)

Launch the Knotilus web-page for `self`.

INPUT:

- `new` – `int` according to `open()` of `webbrowser` (`0` default, 1 new window, 2 new tab)
- `autoraise` – boolean (default `True`)

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K3_1
sage: K.knotilus_webpage(new=1)    # not tested
True
```

link(*use_item=<Columns.pd_notation: ['PD Notation', <KnotInfoColumnTypes.OnlyKnots: 'K'>]>, snappy=False*)

Return `self` as an instance of *Link* or optional `spherogram.links.invariants.Link` (SnapPy).

INPUT:

- `use_item` – (optional, default `self.items.pd_notation`) instance of `KnotInfoColumns` to choose the column that should be used to construct the link. Allowed values are:
  - `self.items.pd_notation`
  - `self.items.braid_notation`
  - `self.items.name` (only for `snappy=True`)
  - `self.items.dt_notation` (only for knots and `snappy=False`)
  - `self.items.gauss_notation` (only for knots and `snappy=False`)
- `snappy` boolean (default `False`) if set to `True` the target of the conversion is the `pip` installable package SnapPy (explicitely, `spherogram.links.invariants.Link`). If SnapPy is not installed an `ImportError` is raised. To install SnapPy use `sage -pip install snappy`.

Note: We use the PD-notation to construct `self` as default. This ensures that the number of crossings displayed in the representation string of the link coincides with the crossing number as a topological invariant.

Furthermore, note that the mirror version may depend on the used KnotInfo-notation. For instance, regarding to the knot `5_1` the Gauss- and the DT-notation refer to the mirror image (see example below).

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K = KnotInfo.K3_1
sage: K.link()
Knot represented by 3 crossings
sage: _.braid()
s^3
sage: _ == K.braid()
True
```

using `dt_notation`:

```
sage: K.link(use_item=K.items.dt_notation)
Knot represented by 3 crossings
sage: _.braid()
s^3

sage: L = KnotInfo.L4a1_0
sage: L.link()
Link with 2 components represented by 4 crossings

sage: L.link(use_item=L.items.dt_notation)
Traceback (most recent call last):
...
ValueError: Link construction using Columns.dt_notation not possible
```

using `snappy`:

```
sage: L2  = KnotInfo.L2a1_1
sage: l2  = L2.link()
sage: l2s = L2.link(snappy=True).sage_link()  # optional -  snappy
sage: l2 == l2s                               # optional -  snappy
True
```

but observe:

```
sage: K7   = KnotInfo.K7_2
sage: k7s  = K7.link(snappy=True); k7s        # optional - snappy
<Link: 1 comp; 7 cross>
sage: k7sn = K7.link(K7.items.name, snappy=True); k7sn     # optional - snappy
<Link 7_2: 1 comp; 7 cross>
sage: k7s.sage_link().is_isotopic(k7sn)       # optional - snappy
False
sage: k7snm = k7sn.sage_link().mirror_image() # optional - snappy
sage: k7s.sage_link().is_isotopic(k7snm)      # optional - snappy
True
```

using `braid_notation`:

```
sage: L2.link(use_item=L.items.braid_notation) == l2
True
```

observe:

```
sage: L.link(use_item=L.items.braid_notation)
Link with 2 components represented by 5 crossings

sage: K6_1 = KnotInfo.K6_1
sage: K6_1.link().braid() == K6_1.braid()
False
```

also observe:

```
sage: K4_1 = KnotInfo.K4_1
sage: K4_1.link().pd_code()
[[4, 2, 5, 1], [8, 6, 1, 5], [6, 3, 7, 4], [2, 7, 3, 8]]
sage: K4_1.pd_notation()
[[4, 2, 5, 1], [8, 6, 1, 5], [6, 3, 7, 4], [2, 7, 3, 8]]

sage: K5_1 = KnotInfo.K5_1
sage: K5_1.link().braid()
s^5
sage: K5_1.link(K5_1.items.dt_notation).braid()
s^-5
sage: K5_1.link(K5_1.items.gauss_notation).braid()
s^-5
```

name_unoriented()

Return the part of the name of `self` which is independent on the orientation.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L10a122_1_0.name_unoriented()  # optional - database_knotinfo
'L10a122'
```

num_components()

Return the number of components of `self`.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L6a1_0.num_components()
2
```

pd_notation(*original=False*)

Return the value of column `pd_notation` for this link as a Python list of Python lists. For more information type `KnotInfo.K0_1.items.pd_notation.description_webpage()`.

INPUT:

- `original` – boolean (optional, default `False`) if set to `True` the original table entry is returned as a string

OUTPUT:

Python list of python lists each entry of the outer list representing a crossing.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: L = KnotInfo.L4a1_0
sage: L.pd_notation()
[[6, 1, 7, 2], [8, 3, 5, 4], [2, 5, 3, 6], [4, 7, 1, 8]]
sage: L.pd_notation(original=True)
'{{6, 1, 7, 2}, {8, 3, 5, 4}, {2, 5, 3, 6}, {4, 7, 1, 8}}'
sage: K = KnotInfo.K4_1
sage: K.pd_notation()
[[4, 2, 5, 1], [8, 6, 1, 5], [6, 3, 7, 4], [2, 7, 3, 8]]
```

series(*oriented=False*)

Return the series of links `self` belongs to.

INPUT:

- `oriented` – boolean (default False) it only affects proper links. By default the items of the series will be again series of links collecting all orientation mutants of an unoriented name. To obtain the series of the individual links this keyword has to be set to `True`.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: K5 = KnotInfo.K5_2.series()
sage: K5(1)
<KnotInfo.K5_1: '5_1'>
sage: KnotInfo.L4a1_1.series().inject()
Defining L4a
sage: L4a(1)
Series of links L4a1
```

(continues on next page)

(continued from previous page)

```
sage: KnotInfo.L4a1_1.series(oriented=True).inject()
Defining L4a1
sage: L4a(1) == L4a1
True
sage: L4a1(1)
<KnotInfo.L4a1_1: 'L4a1{1}'>
```

`signature()`

Return the signature of `self`.

From the KnotInfo description page:

> The signature of a knot, $\sigma(K)$, is equal to $\sigma(V + V^t)$, the signature of $V + V^t$ where V is a Seifert matrix for the knot and V^t is its transpose.

To read the complete description type `KnotInfo.K0_1.items.signatur.description_webpage()`.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.signature()       # optional - databsase_knotinfo
1
```

`symmetry_type()`

Return the symmetry type of `self`.

From the KnotInfo description page:

> If a knot is viewed as the oriented diffeomorphism class of an oriented pair, $K = (S_3, S_1)$, with S_i diffeomorphic to S^i, there are four oriented knots associated to any particular knot K. In addition to K itself, there is the reverse, $K^r = (S_3, -S_1)$, the concordance inverse, $-K = (-S_3, -S_1)$, and the mirror image, $K^m = (-S_3, S_1)$. A knot is called reversible if $K = K^r$, negative amphicheiral if $K = -K$, and positive amphicheiral if $K = K^m$.
>
> A knot possessing any two of these types of symmetry has all three. Thus, in the table, a knot is called reversible if that is the only type of symmetry it has, and likewise for negative amphicheiral. If it has none of these types of symmetry it is called chiral, and if it has all three it is called fully amphicheiral.
>
> For prime knots with fewer than 12 crossings, all amphicheiral knots are negative amphicheiral.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K6_1.series().inject()
Defining K6
sage: [(K.name, K.symmetry_type()) for K in K6]
[('K6_1', 'reversible'),
 ('K6_2', 'reversible'),
 ('K6_3', 'fully amphicheiral')]
```

`three_genus()`

Return the three genus of `self`.

From the KnotInfo description page:

> The three-genus of a knot is defined to be the minimal genus of a Seifert surface for a knot.

To read the complete description type `KnotInfo.K0_1.items.three_genus.description_webpage()`.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.K5_2.three_genus()      # optional - databsase_knotinfo
1
```

Note that this differs from the corresponding result in Sage since the latter is obtained for a Seifert surface that does not have the minimal genus:

```
sage: KnotInfo.K5_2.link().genus()
3
```

class `sage.knots.knotinfo.`**`KnotInfoSeries`**(*crossing_number*, *is_knot*, *is_alternating*, *name_unoriented=None*)

Bases: `UniqueRepresentation`, `SageObject`

This class can be used to access knots and links via their index according to the series they belong to.

INPUT:

- `crossing_number` – integer giving the crossing numer of this series of links
- `is_knot` – boolean whether this series is a series of knots or proper links
- `is_alternating` – boolean whether this series is restriced to alternating links or not This is not relevant for knots with less than 11 crossings
- `name_unoriented` – string restricting the series to all links with that `name_unoriented`

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfoSeries
sage: K6 = KnotInfoSeries(6, True, True); K6
Series of knots K6
sage: K6(3)
<KnotInfo.K6_3: '6_3'>
sage: list(K6)
[<KnotInfo.K6_1: '6_1'>, <KnotInfo.K6_2: '6_2'>, <KnotInfo.K6_3: '6_3'>]
sage: L6a = KnotInfoSeries(6, False, True); L6a
Series of links L6a
sage: L6a(2)
Series of links L6a2
sage: _.inject()
Defining L6a2
sage: list(L6a2)
[<KnotInfo.L6a2_0: 'L6a2{0}'>, <KnotInfo.L6a2_1: 'L6a2{1}'>]
sage: L6a2(0).series() == L6a
True
sage: L6a2(0) == L6a2('0')
True
```

`inject`(*verbose=True*)

Inject `self` with its name into the namespace of the Python code from which this function is called.

INPUT:

- `verbose` – boolean (optional, default `True`) to suppress the message printed on the invocation

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfoSeries
sage: KnotInfoSeries(6, True, True).inject()
Defining K6
sage: K6(2)
<KnotInfo.K6_2: '6_2'>
```

is_recoverable(*unique=True*, *max_samples=8*)

Return if all items of `self` can be recovered from its conversion to Sage links using the `pd_notation` and the `braid_notation` and their mirror images.

The method is indirectly used by the `TestSuite`.

INPUT:

- `unique` – boolean (optional, default=``True``) see *`KnotInfoBase.is_recoverable()`*
- `max_samples` – non negative integer or `infinity` (optional, default `8`) limits the number of items to check (random sample). If set to `infinity` then no limit is set.

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo
sage: KnotInfo.L4a1_0.series().inject()
Defining L4a
sage: L4a.is_recoverable()
True
sage: L4a.is_recoverable(unique=False)
True
sage: KnotInfo.L5a1_0.series().inject()
Defining L5a
sage: L5a.is_recoverable()
False
sage: L5a.is_recoverable(unique=False)
True
```

list(*oriented=False*, *comp=None*, *det=None*, *homfly=None*)

Return this series as a Python list.

INPUT:

- `oriented` – boolean (optional, default `False`) it only affects series of proper links. By default the list items of a series of proper links are again series of links collecting all orientation types of an unoriented name. To obtain the list of the individual links this keyword has to be set to `True`
- `comp` (optional, default `None`) if given an integer for this keyword the list is restriced to links having the according number of components. This keyword implies `oriented=True`
- `det` (optional, default `None`) if given an integer for this keyword the list is restriced to links having the according value for its determinant. This keyword implies `oriented=True`
- `homfly` (optional, default `None`) if given a HOMFLY-PT polynomial having `normalization='vz'` for this keyword the list is restriced to links having the according value for its HOMFLY-PT polynomial. This keyword implies `oriented=True`

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfoSeries
sage: K6 = KnotInfoSeries(6, True, True); K6
Series of knots K6
sage: K6.list()
[<KnotInfo.K6_1: '6_1'>, <KnotInfo.K6_2: '6_2'>, <KnotInfo.K6_3: '6_3'>]
sage: KnotInfoSeries(2, False, True).inject()
Defining L2a
sage: L2a.list()
[Series of links L2a1]
sage: L2a.list(oriented=True)
[<KnotInfo.L2a1_0: 'L2a1{0}'>, <KnotInfo.L2a1_1: 'L2a1{1}'>]
```

lower_list(*oriented=False*, *comp=None*, *det=None*, *homfly=None*)

Return this series together with all series with smaller crossing number as a Python list.

INPUT:

- `oriented` – boolean (optional, default `False`) see the description for *list()*
- `comp` (optional, default `None`) see the description for *list()*
- `det` (optional, default `None`) see the description for *list()*
- `homfly` (optional, default `None`) see the description for *list()*

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfoSeries
sage: KnotInfoSeries(5, True, True).lower_list()
[<KnotInfo.K0_1: '0_1'>,
 <KnotInfo.K3_1: '3_1'>,
 <KnotInfo.K4_1: '4_1'>,
 <KnotInfo.K5_1: '5_1'>,
 <KnotInfo.K5_2: '5_2'>]
sage: KnotInfoSeries(4, False, True).lower_list()
[Series of links L2a1, Series of links L4a1]
sage: KnotInfoSeries(4, False, True).lower_list(oriented=True)
[<KnotInfo.L2a1_0: 'L2a1{0}'>,
 <KnotInfo.L2a1_1: 'L2a1{1}'>,
 <KnotInfo.L4a1_0: 'L4a1{0}'>,
 <KnotInfo.L4a1_1: 'L4a1{1}'>]
```

sage.knots.knotinfo.**eval_knotinfo**(*string*, *locals={}*, *to_tuple=True*)

Preparse a string from the KnotInfo database and evaluate it by `sage_eval`.

INPUT:

- `string` – string that gives a value of some database entry
- `locals` – dictionary of locals passed to `sage_eval`

EXAMPLES:

```
sage: from sage.knots.knotinfo import KnotInfo, eval_knotinfo
sage: L = KnotInfo.L4a1_0
sage: L.braid_notation(original=True)
'{3, {-2, -2, -1, 2, -1}}'
```

(continues on next page)

(continued from previous page)

```
sage: eval_knotinfo(_)
(3, (-2, -2, -1, 2, -1))
```

`sage.knots.knotinfo.`**`knotinfo_bool`**`(`*string*`)`

Preparse a string from the KnotInfo database representing a boolean.

INPUT:

- `string` – string that gives a value of some database entry

EXAMPLES:

```
sage: from sage.knots.knotinfo import knotinfo_bool
sage: knotinfo_bool('Y')
True
```

CHAPTER FOUR

INDICES AND TABLES

- Index
- Module Index
- Search Page

PYTHON MODULE INDEX

INDEX