Matrices and Spaces of Matrices

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The Sage Development Team

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CONTENTS

Sage provides native support for working with matrices over any commutative or noncommutative ring. The parent object for a matrix is a matrix space `MatrixSpace(R, n, m)` of all $n \times m$ matrices over a ring R.

To create a matrix, either use the `matrix(...)` function or create a matrix space using the `MatrixSpace` command and coerce an object into it.

Matrices also act on row vectors, which you create using the `vector(...)` command or by making a `VectorSpace` and coercing lists into it. The natural action of matrices on row vectors is from the right. Sage currently does not have a column vector class (on which matrices would act from the left), but this is planned.

In addition to native Sage matrices, Sage also includes the following additional ways to compute with matrices:

- Several math software systems included with Sage have their own native matrix support, which can be used from Sage. E.g., PARI, GAP, Maxima, and Singular all have a notion of matrices.
- The GSL C-library is included with Sage, and can be used via Cython.
- The `scipy` module provides support for *sparse* numerical linear algebra, among many other things.
- The `numpy` module, which you load by typing `import numpy` is included standard with Sage. It contains a very sophisticated and well developed array class, plus optimized support for *numerical linear algebra*. Sage's matrices over RDF and CDF (native floating-point real and complex numbers) use numpy.

Finally, this module contains some data-structures for matrix-like objects like operation tables (e.g. the multiplication table of a group).

CHAPTER

ONE

MATRIX SPACES

You can create any space $\mathrm{Mat}_{n\times m}(R)$ of either dense or sparse matrices with given number of rows and columns over any commutative or noncommutative ring.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,6,6,sparse=True); MS
Full MatrixSpace of 6 by 6 sparse matrices over Rational Field
sage: MS.base_ring()
Rational Field
sage: MS = MatrixSpace(ZZ,3,5,sparse=False); MS
Full MatrixSpace of 3 by 5 dense matrices over Integer Ring
```

class sage.matrix.matrix_space.**MatrixSpace**(*base_ring*, *nrows*, *ncols*, *sparse*, *implementation*)

Bases: UniqueRepresentation, Parent

The space of matrices of given size and base ring

EXAMPLES:

Some examples of square 2 by 2 rational matrices:

```
sage: MS = MatrixSpace(QQ, 2)
sage: MS.dimension()
4
sage: MS.dims()
(2, 2)
sage: B = MS.basis()
sage: list(B)
[
[1 0]  [0 1]  [0 0]  [0 0]
[0 0], [0 0], [1 0], [0 1]
]
sage: B[0,0]
[1 0]
[0 0]
sage: B[0,1]
[0 1]
[0 0]
sage: B[1,0]
[0 0]
[1 0]
sage: B[1,1]
```

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```
[0 0]
[0 1]
sage: A = MS.matrix([1,2,3,4])
sage: A
[1 2]
[3 4]
```

The above matrix `A` can be multiplied by a 2 by 3 integer matrix:

```
sage: MS2 = MatrixSpace(ZZ, 2, 3)
sage: B = MS2.matrix([1,2,3,4,5,6])
sage: A * B
[ 9 12 15]
[19 26 33]
```

Check categories:

```
sage: MatrixSpace(ZZ,10,5)
Full MatrixSpace of 10 by 5 dense matrices over Integer Ring
sage: MatrixSpace(ZZ,10,5).category()
Category of infinite enumerated finite dimensional modules with basis over
 (euclidean domains and infinite enumerated sets and metric spaces)
sage: MatrixSpace(ZZ,10,10).category()
Category of infinite enumerated finite dimensional algebras with basis over
 (euclidean domains and infinite enumerated sets and metric spaces)
sage: MatrixSpace(QQ,10).category()
Category of infinite finite dimensional algebras with basis over
 (number fields and quotient fields and metric spaces)
```

base_extend(*R*)

Return base extension of this matrix space to R.

INPUT:

- R - ring

OUTPUT: a matrix space

EXAMPLES:

```
sage: Mat(ZZ,3,5).base_extend(QQ)
Full MatrixSpace of 3 by 5 dense matrices over Rational Field
sage: Mat(QQ,3,5).base_extend(GF(7))
Traceback (most recent call last):
...
TypeError: no base extension defined
```

basis()

Return a basis for this matrix space.

Warning: This will of course compute every generator of this matrix space. So for large dimensions, this could take a long time, waste a massive amount of memory (for dense matrices), and is likely not very useful. Don't use this on large matrix spaces.

EXAMPLES:

```
sage: list(Mat(ZZ,2,2).basis())
[
[1 0]  [0 1]  [0 0]  [0 0]
[0 0], [0 0], [1 0], [0 1]
]
```

cached_method(*f*, *name=None*, *key=None*, *do_pickle=None*)

A decorator for cached methods.

EXAMPLES:

In the following examples, one can see how a cached method works in application. Below, we demonstrate what is done behind the scenes:

```
sage: class C:
....:     @cached_method
....:     def __hash__(self):
....:         print("compute hash")
....:         return int(5)
....:     @cached_method
....:     def f(self, x):
....:         print("computing cached method")
....:         return x*2
sage: c = C()
sage: type(C.__hash__)
<class 'sage.misc.cachefunc.CachedMethodCallerNoArgs'>
sage: hash(c)
compute hash
5
```

When calling a cached method for the second time with the same arguments, the value is gotten from the cache, so that a new computation is not needed:

```
sage: hash(c)
5
sage: c.f(4)
computing cached method
8
sage: c.f(4) is c.f(4)
True
```

Different instances have distinct caches:

```
sage: d = C()
sage: d.f(4) is c.f(4)
computing cached method
False
sage: d.f.clear_cache()
sage: c.f(4)
8
sage: d.f(4)
computing cached method
8
```

Using cached methods for the hash and other special methods was implemented in github issue #12601, by means of `CachedSpecialMethod`. We show that it is used behind the scenes:

```
sage: cached_method(c.__hash__)
<sage.misc.cachefunc.CachedSpecialMethod object at ...>
sage: cached_method(c.f)
<sage.misc.cachefunc.CachedMethod object at ...>
```

The parameter `do_pickle` can be used if the contents of the cache should be stored in a pickle of the cached method. This can be dangerous with special methods such as `__hash__`:

```
sage: class C:
....:     @cached_method(do_pickle=True)
....:     def __hash__(self):
....:         return id(self)

sage: import __main__
sage: __main__.C = C
sage: c = C()
sage: hash(c) # random output
sage: d = loads(dumps(c))
sage: hash(d) == hash(c)
True
```

However, the contents of a method's cache are not pickled unless `do_pickle` is set:

```
sage: class C:
....:     @cached_method
....:     def __hash__(self):
....:         return id(self)

sage: __main__.C = C
sage: c = C()
sage: hash(c) # random output
sage: d = loads(dumps(c))
sage: hash(d) == hash(c)
False
```

cardinality()

Return the number of elements in `self`.

EXAMPLES:

```
sage: MatrixSpace(GF(3), 2, 3).cardinality()
729
sage: MatrixSpace(ZZ, 2).cardinality()
+Infinity
sage: MatrixSpace(ZZ, 0, 3).cardinality()
1
```

change_ring(*R*)

Return matrix space over R with otherwise same parameters as `self`.

INPUT:

- R - ring

OUTPUT: a matrix space

EXAMPLES:

```
sage: Mat(QQ,3,5).change_ring(GF(7))
Full MatrixSpace of 3 by 5 dense matrices over Finite Field of size 7
```

characteristic()

Return the characteristic.

EXAMPLES:

```
sage: MatrixSpace(ZZ, 2).characteristic()
0
sage: MatrixSpace(GF(9), 0).characteristic()
3
```

column_space()

Return the module spanned by all columns of matrices in this matrix space. This is a free module of rank the number of columns. It will be sparse or dense as this matrix space is sparse or dense.

EXAMPLES:

```
sage: M = Mat(GF(9,'a'),20,5,sparse=True); M.column_space()
Sparse vector space of dimension 20 over Finite Field in a of size 3^2
```

construction()

EXAMPLES:

```
sage: A = matrix(ZZ, 2, [1..4], sparse=True)
sage: A.parent().construction()
(MatrixFunctor, Integer Ring)
sage: A.parent().construction()[0](QQ['x'])
Full MatrixSpace of 2 by 2 sparse matrices over Univariate Polynomial Ring in x␣
↪over Rational Field
sage: parent(A/2)
Full MatrixSpace of 2 by 2 sparse matrices over Rational Field
```

diagonal_matrix(*entries*)

Create a diagonal matrix in `self` using the specified elements

INPUT:

- `entries` – the elements to use as the diagonal entries

`self` must be a space of square matrices. The length of `entries` must be less than or equal to the matrix dimensions. If the length of `entries` is less than the matrix dimensions, `entries` is padded with zeroes at the end.

EXAMPLES:

```
sage: MS1 = MatrixSpace(ZZ,4)
sage: MS2 = MatrixSpace(QQ,3,4)
sage: I = MS1.diagonal_matrix([1, 2, 3, 4])
sage: I
[1 0 0 0]
[0 2 0 0]
```

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```
[0 0 3 0]
[0 0 0 4]
sage: MS2.diagonal_matrix([1, 2])
Traceback (most recent call last):
...
TypeError: diagonal matrix must be square
sage: MS1.diagonal_matrix([1, 2, 3, 4, 5])
Traceback (most recent call last):
...
ValueError: number of diagonal matrix entries (5) exceeds the matrix size (4)
sage: MS1.diagonal_matrix([1/2, 2, 3, 4])
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

Check different implementations:

```
sage: M1 = MatrixSpace(ZZ, 2, implementation='flint')
sage: M2 = MatrixSpace(ZZ, 2, implementation='generic')

sage: type(M1.diagonal_matrix([1, 2]))
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
sage: type(M2.diagonal_matrix([1, 2]))
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>
```

dimension()

Return (m rows) * (n cols) of `self` as `Integer`.

EXAMPLES:

```
sage: MS = MatrixSpace(ZZ,4,6)
sage: u = MS.dimension()
sage: u - 24 == 0
True
```

dims()

Return (m row, n col) representation of `self` dimension.

EXAMPLES:

```
sage: MS = MatrixSpace(ZZ,4,6)
sage: MS.dims()
(4, 6)
```

gen(*n*)

Return the n-th generator of this matrix space.

This does not compute all basis matrices, so it is reasonably intelligent.

EXAMPLES:

```
sage: M = Mat(GF(7),10000,5); M.ngens()
50000
sage: a = M.10
```

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```
sage: a[:4]
[0 0 0 0 0]
[0 0 0 0 0]
[1 0 0 0 0]
[0 0 0 0 0]
```

identity_matrix()

Return the identity matrix in `self`.

`self` must be a space of square matrices. The returned matrix is immutable. Please use `copy` if you want a modified copy.

EXAMPLES:

```
sage: MS1 = MatrixSpace(ZZ,4)
sage: MS2 = MatrixSpace(QQ,3,4)
sage: I = MS1.identity_matrix()
sage: I
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: Er = MS2.identity_matrix()
Traceback (most recent call last):
...
TypeError: identity matrix must be square
```

is_dense()

Return whether matrices in `self` are dense.

EXAMPLES:

```
sage: Mat(RDF,2,3).is_sparse()
False
sage: Mat(RR,123456,22,sparse=True).is_sparse()
True
```

is_finite()

Return whether this matrix space is finite.

EXAMPLES:

```
sage: MatrixSpace(GF(101), 10000).is_finite()
True
sage: MatrixSpace(QQ, 2).is_finite()
False
```

is_sparse()

Return whether matrices in `self` are sparse.

EXAMPLES:

```
sage: Mat(GF(2011),10000).is_sparse()
False
```

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```
sage: Mat(GF(2011),10000,sparse=True).is_sparse()
True
```

matrix(*x=None*, ***kwds*)

Create a matrix in `self`.

INPUT:

- `x` – data to construct a new matrix from. See *matrix()*
- `coerce` – (default: `True`) if False, assume without checking that the values in `x` lie in the base ring

OUTPUT:

- a matrix in `self`.

EXAMPLES:

```
sage: M = MatrixSpace(ZZ, 2)
sage: M.matrix([[1,0],[0,-1]])
[ 1  0]
[ 0 -1]
sage: M.matrix([1,0,0,-1])
[ 1  0]
[ 0 -1]
sage: M.matrix([1,2,3,4])
[1 2]
[3 4]
```

Note that the last "flip" cannot be performed if `x` is a matrix, no matter what is `rows` (it used to be possible but was fixed by github issue #10793):

```
sage: projection = matrix(ZZ,[[1,0,0],[0,1,0]])
sage: projection
[1 0 0]
[0 1 0]
sage: projection.parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
sage: M = MatrixSpace(ZZ, 3 , 2)
sage: M
Full MatrixSpace of 3 by 2 dense matrices over Integer Ring
sage: M(projection)
Traceback (most recent call last):
...
ValueError: inconsistent number of rows: should be 3 but got 2
```

If you really want to make from a matrix another matrix of different dimensions, use either transpose method or explicit conversion to a list:

```
sage: M(projection.list())
[1 0]
[0 0]
[1 0]
```

matrix_space(*nrows=None*, *ncols=None*, *sparse=False*)

Return the matrix space with given number of rows, columns and sparsity over the same base ring as self, and defaults the same as self.

EXAMPLES:

```
sage: M = Mat(GF(7),100,200)
sage: M.matrix_space(5000)
Full MatrixSpace of 5000 by 200 dense matrices over Finite Field of size 7
sage: M.matrix_space(ncols=5000)
Full MatrixSpace of 100 by 5000 dense matrices over Finite Field of size 7
sage: M.matrix_space(sparse=True)
Full MatrixSpace of 100 by 200 sparse matrices over Finite Field of size 7
```

ncols()

Return the number of columns of matrices in this space.

EXAMPLES:

```
sage: M = Mat(ZZ['x'],200000,500000,sparse=True)
sage: M.ncols()
500000
```

ngens()

Return the number of generators of this matrix space.

This is the number of entries in the matrices in this space.

EXAMPLES:

```
sage: M = Mat(GF(7),100,200); M.ngens()
20000
```

nrows()

Return the number of rows of matrices in this space.

EXAMPLES:

```
sage: M = Mat(ZZ,200000,500000)
sage: M.nrows()
200000
```

one()

Return the identity matrix in `self`.

`self` must be a space of square matrices. The returned matrix is immutable. Please use `copy` if you want a modified copy.

EXAMPLES:

```
sage: MS1 = MatrixSpace(ZZ,4)
sage: MS2 = MatrixSpace(QQ,3,4)
sage: I = MS1.identity_matrix()
sage: I
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
```

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```
[0 0 0 1]
sage: Er = MS2.identity_matrix()
Traceback (most recent call last):
...
TypeError: identity matrix must be square
```

random_element(*density=None*, **args*, ***kwds*)

Return a random element from this matrix space.

INPUT:

- `density` - `float` or `None` (default: `None`); rough measure of the proportion of nonzero entries in the random matrix; if set to `None`, all entries of the matrix are randomized, allowing for any element of the underlying ring, but if set to a `float`, a proportion of entries is selected and randomized to non-zero elements of the ring
- `*args, **kwds` - remaining parameters, which may be passed to the random_element function of the base ring. ("may be", since this function calls the `randomize` function on the zero matrix, which need not call the `random_element` function of the base ring at all in general.)

OUTPUT:

- Matrix

Note: This method will randomize a proportion of roughly `density` entries in a newly allocated zero matrix.

By default, if the user sets the value of `density` explicitly, this method will enforce that these entries are set to non-zero values. However, if the test for equality with zero in the base ring is too expensive, the user can override this behaviour by passing the argument `nonzero=False` to this method.

Otherwise, if the user does not set the value of `density`, the default value is taken to be 1, and the option `nonzero=False` is passed to the `randomize` method.

EXAMPLES:

```
sage: M = Mat(ZZ, 2, 5).random_element()
sage: TestSuite(M).run()

sage: M = Mat(QQ, 2, 5).random_element(density=0.5)
sage: TestSuite(M).run()

sage: M = Mat(QQ, 3, sparse=True).random_element()
sage: TestSuite(M).run()

sage: M = Mat(GF(9,'a'), 3, sparse=True).random_element()
sage: TestSuite(M).run()
```

row_space()

Return the module spanned by all rows of matrices in this matrix space. This is a free module of rank the number of rows. It will be sparse or dense as this matrix space is sparse or dense.

EXAMPLES:

```
sage: M = Mat(ZZ,20,5,sparse=False); M.row_space()
Ambient free module of rank 5 over the principal ideal domain Integer Ring
```

some_elements()

Return some elements of this matrix space.

See TestSuite for a typical use case.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: M = MatrixSpace(ZZ, 2, 2)
sage: tuple(M.some_elements())
(
[ 0  1]  [1 0]  [0 1]  [0 0]  [0 0]
[-1  2], [0 0], [0 0], [1 0], [0 1]
)
sage: M = MatrixSpace(QQ, 2, 3)
sage: tuple(M.some_elements())
(
[ 1/2 -1/2    2]  [1 0 0]  [0 1 0]  [0 0 1]  [0 0 0]  [0 0 0]  [0 0 0]
[  -2    0    1], [0 0 0], [0 0 0], [0 0 0], [1 0 0], [0 1 0], [0 0 1]
)
sage: M = MatrixSpace(SR, 2, 2)
sage: tuple(M.some_elements())
(
[some_variable some_variable]  [1 0]  [0 1]  [0 0]  [0 0]
[some_variable some_variable], [0 0], [0 0], [1 0], [0 1]
)
```

submodule(*gens*, *check=True*, *already_echelonized=False*, *unitriangular=False*, *support_order=None*, *category=None*, **args*, ***opts*)

The submodule spanned by a finite set of matrices.

INPUT:

- gens – a list or family of elements of self
- **check – (default: True) whether to verify that the** elements of gens are in self
- **already_echelonized – (default: False) whether** the elements of gens are already in (not necessarily reduced) echelon form
- unitriangular – (default: False) whether the lift morphism is unitriangular
- support_order – (optional) either something that can be converted into a tuple or a key function

If already_echelonized is False, then the generators are put in reduced echelon form using echelonize(), and reindexed by $0, 1, \ldots$.

Warning: At this point, this method only works for finite dimensional submodules and if matrices can be echelonized over the base ring.

If in addition `unitriangular` is `True`, then the generators are made such that the coefficients of the pivots are 1, so that lifting map is unitriangular.

The basis of the submodule uses the same index set as the generators, and the lifting map sends y_i to $gens[i]$.

See also:

`ModulesWithBasis.ParentMethods.submodule()`

EXAMPLES:

```
sage: M = MatrixSpace(QQ, 2)
sage: mat = M.matrix([[1, 2], [3, 4]])
sage: X = M.submodule([mat], already_echelonized=True); X
Free module generated by {0} over Rational Field

sage: mat2 = M.matrix([[1, 0], [-3, 2]])
sage: X = M.submodule([mat, mat2])
sage: [X.lift(b) for b in X.basis()]
[
[ 1  0]  [0 1]
[-3  2], [3 1]
]

sage: A = matrix([[1, 1], [0, -1]])
sage: B = matrix([[0, 1], [0, 2]])
sage: X = M.submodule([A, B])
sage: Xp = M.submodule([A, B], support_order=[(0,1), (1,1), (0,0)])
sage: [X.lift(b) for b in X.basis()]
[
[ 1  0]  [0 1]
[ 0 -3], [0 2]
]
sage: [Xp.lift(b) for b in Xp.basis()]
[
[2/3   1]  [-1/3    0]
[  0   0], [   0    1]
]
```

transposed()

The transposed matrix space, having the same base ring and sparseness, but number of columns and rows is swapped.

EXAMPLES:

```
sage: MS = MatrixSpace(GF(3), 7, 10)
sage: MS.transposed
Full MatrixSpace of 10 by 7 dense matrices over Finite Field of size 3
sage: MS = MatrixSpace(GF(3), 7, 7)
sage: MS.transposed is MS
True

sage: M = MatrixSpace(ZZ, 2, 3)
sage: M.transposed
Full MatrixSpace of 3 by 2 dense matrices over Integer Ring
```

zero()

Return the zero matrix in `self`.

`self` must be a space of square matrices. The returned matrix is immutable. Please use `copy` if you want a modified copy.

EXAMPLES:

```
sage: z = MatrixSpace(GF(7),2,4).zero_matrix(); z
[0 0 0 0]
[0 0 0 0]
sage: z.is_mutable()
False
```

zero_matrix()

Return the zero matrix in `self`.

`self` must be a space of square matrices. The returned matrix is immutable. Please use `copy` if you want a modified copy.

EXAMPLES:

```
sage: z = MatrixSpace(GF(7),2,4).zero_matrix(); z
[0 0 0 0]
[0 0 0 0]
sage: z.is_mutable()
False
```

sage.matrix.matrix_space.**dict_to_list**(*entries*, *nrows*, *ncols*)

Given a dictionary of coordinate tuples, return the list given by reading off the nrows*ncols matrix in row order.

EXAMPLES:

```
sage: from sage.matrix.matrix_space import dict_to_list
sage: d = {}
sage: d[(0,0)] = 1
sage: d[(1,1)] = 2
sage: dict_to_list(d, 2, 2)
[1, 0, 0, 2]
sage: dict_to_list(d, 2, 3)
[1, 0, 0, 0, 2, 0]
```

sage.matrix.matrix_space.**get_matrix_class**(*R*, *nrows*, *ncols*, *sparse*, *implementation*)

Return a matrix class according to the input.

Note: This returns the base class without the category.

INPUT:

- `R` – a base ring
- `nrows` – number of rows
- `ncols` – number of columns
- `sparse` – (boolean) whether the matrix class should be sparse

- `implementation` – (None or string or a matrix class) a possible implementation. See the documentation of the constructor of *MatrixSpace*.

EXAMPLES:

```
sage: from sage.matrix.matrix_space import get_matrix_class

sage: get_matrix_class(ZZ, 4, 5, False, None)
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
sage: get_matrix_class(ZZ, 4, 5, True, None)
<class 'sage.matrix.matrix_integer_sparse.Matrix_integer_sparse'>

sage: get_matrix_class(ZZ, 3, 3, False, 'flint')
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
sage: get_matrix_class(ZZ, 3, 3, False, 'gap')
<class 'sage.matrix.matrix_gap.Matrix_gap'>
sage: get_matrix_class(ZZ, 3, 3, False, 'generic')
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>

sage: get_matrix_class(GF(2^15), 3, 3, False, None)
<class 'sage.matrix.matrix_gf2e_dense.Matrix_gf2e_dense'>
sage: get_matrix_class(GF(2^17), 3, 3, False, None)
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>

sage: get_matrix_class(GF(2), 2, 2, False, 'm4ri')
<class 'sage.matrix.matrix_mod2_dense.Matrix_mod2_dense'>
sage: get_matrix_class(GF(4), 2, 2, False, 'm4ri')
<class 'sage.matrix.matrix_gf2e_dense.Matrix_gf2e_dense'>
sage: get_matrix_class(GF(7), 2, 2, False, 'linbox-float')
<class 'sage.matrix.matrix_modn_dense_float.Matrix_modn_dense_float'>
sage: get_matrix_class(GF(7), 2, 2, False, 'linbox-double')
<class 'sage.matrix.matrix_modn_dense_double.Matrix_modn_dense_double'>

sage: get_matrix_class(RDF, 2, 2, False, 'numpy')
<class 'sage.matrix.matrix_real_double_dense.Matrix_real_double_dense'>
sage: get_matrix_class(CDF, 2, 3, False, 'numpy')
<class 'sage.matrix.matrix_complex_double_dense.Matrix_complex_double_dense'>

sage: get_matrix_class(GF(25,'x'), 4, 4, False, 'meataxe')         # optional -␣
↪meataxe
<class 'sage.matrix.matrix_gfpn_dense.Matrix_gfpn_dense'>
sage: get_matrix_class(IntegerModRing(3), 4, 4, False, 'meataxe')  # optional -␣
↪meataxe
<class 'sage.matrix.matrix_gfpn_dense.Matrix_gfpn_dense'>
sage: get_matrix_class(IntegerModRing(4), 4, 4, False, 'meataxe')
Traceback (most recent call last):
...
ValueError: 'meataxe' matrix can only deal with finite fields of order < 256
sage: get_matrix_class(GF(next_prime(255)), 4, 4, False, 'meataxe')
Traceback (most recent call last):
...
ValueError: 'meataxe' matrix can only deal with finite fields of order < 256

sage: get_matrix_class(ZZ, 3, 5, False, 'crazy_matrix')
```

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```
Traceback (most recent call last):
...
ValueError: unknown matrix implementation 'crazy_matrix' over Integer Ring
sage: get_matrix_class(GF(3), 2, 2, False, 'm4ri')
Traceback (most recent call last):
...
ValueError: 'm4ri' matrices are only available for fields of characteristic 2 and␣
↪order <= 65536
sage: get_matrix_class(Zmod(2**30), 2, 2, False, 'linbox-float')
Traceback (most recent call last):
...
ValueError: 'linbox-float' matrices can only deal with order < 256
sage: get_matrix_class(Zmod(2**30), 2, 2, False, 'linbox-double')
Traceback (most recent call last):
...
ValueError: 'linbox-double' matrices can only deal with order < 8388608

sage: type(matrix(SR, 2, 2, 0))
<class 'sage.matrix.matrix_symbolic_dense.Matrix_symbolic_dense'>
sage: type(matrix(GF(7), 2, range(4)))
<class 'sage.matrix.matrix_modn_dense_float.Matrix_modn_dense_float'>
sage: type(matrix(GF(16007), 2, range(4)))
<class 'sage.matrix.matrix_modn_dense_double.Matrix_modn_dense_double'>
sage: type(matrix(CBF, 2, range(4)))
<class 'sage.matrix.matrix_complex_ball_dense.Matrix_complex_ball_dense'>
sage: type(matrix(GF(2), 2, range(4)))
<class 'sage.matrix.matrix_mod2_dense.Matrix_mod2_dense'>
sage: type(matrix(GF(64,'z'), 2, range(4)))
<class 'sage.matrix.matrix_gf2e_dense.Matrix_gf2e_dense'>
sage: type(matrix(GF(125,'z'), 2, range(4)))     # optional - meataxe
<class 'sage.matrix.matrix_gfpn_dense.Matrix_gfpn_dense'>
```

sage.matrix.matrix_space.**is_MatrixSpace**(*x*)

Return whether `self` is an instance of `MatrixSpace`.

EXAMPLES:

```
sage: from sage.matrix.matrix_space import is_MatrixSpace
sage: MS = MatrixSpace(QQ,2)
sage: A = MS.random_element()
sage: is_MatrixSpace(MS)
True
sage: is_MatrixSpace(A)
False
sage: is_MatrixSpace(5)
False
```

CHAPTER

TWO

GENERAL MATRIX CONSTRUCTOR AND DISPLAY OPTIONS

sage.matrix.constructor.**Matrix**(**args*, ***kwds*)

Create a matrix.

This implements the `matrix` constructor:

```
sage: matrix([[1,2],[3,4]])
[1 2]
[3 4]
```

It also contains methods to create special types of matrices, see `matrix.[tab]` for more options. For example:

```
sage: matrix.identity(2)
[1 0]
[0 1]
```

INPUT:

The matrix command takes the entries of a matrix, optionally preceded by a ring and the dimensions of the matrix, and returns a matrix.

The entries of a matrix can be specified as a flat list of elements, a list of lists (i.e., a list of rows), a list of Sage vectors, a callable object, or a dictionary having positions as keys and matrix entries as values (see the examples). If you pass in a callable object, then you must specify the number of rows and columns. You can create a matrix of zeros by passing an empty list or the integer zero for the entries. To construct a multiple of the identity (cI), you can specify square dimensions and pass in c. Calling matrix() with a Sage object may return something that makes sense. Calling matrix() with a NumPy array will convert the array to a matrix.

All arguments (even the positional) are optional.

Positional and keyword arguments:

- `ring` – parent of the entries of the matrix (despite the name, this is not a priori required to be a ring). By default, determine this from the given entries, falling back to ZZ if no entries are given.
- `nrows` – the number of rows in the matrix.
- `ncols` – the number of columns in the matrix.
- `entries` – see examples below.

If either `nrows` or `ncols` is given as keyword argument, then no positional arguments `nrows` and `ncols` may be given.

Keyword-only arguments:

- `sparse` – (boolean) create a sparse matrix. This defaults to `True` when the entries are given as a dictionary, otherwise defaults to `False`.

- space – matrix space which will be the parent of the output matrix. This determines ring, nrows, ncols and sparse.
- immutable – (boolean) make the matrix immutable. By default, the output matrix is mutable.

OUTPUT:

a matrix

EXAMPLES:

```
sage: m = matrix(2); m; m.parent()
[0 0]
[0 0]
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
```

```
sage: m = matrix(2,3); m; m.parent()
[0 0 0]
[0 0 0]
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

```
sage: m = matrix(QQ,[[1,2,3],[4,5,6]]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

```
sage: m = matrix(QQ, 3, 3, lambda i, j: i+j); m
[0 1 2]
[1 2 3]
[2 3 4]
sage: m = matrix(3, lambda i,j: i-j); m
[ 0 -1 -2]
[ 1  0 -1]
[ 2  1  0]
```

```
sage: matrix(QQ, 2, 3, lambda x, y: x+y)
[0 1 2]
[1 2 3]
sage: matrix(QQ, 5, 5, lambda x, y: (x+1) / (y+1))
[  1 1/2 1/3 1/4 1/5]
[  2   1 2/3 1/2 2/5]
[  3 3/2   1 3/4 3/5]
[  4   2 4/3   1 4/5]
[  5 5/2 5/3 5/4   1]
```

```
sage: v1=vector((1,2,3))
sage: v2=vector((4,5,6))
sage: m = matrix([v1,v2]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

```
sage: m = matrix(QQ,2,[1,2,3,4,5,6]); m; m.parent()
[1 2 3]
```

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```
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

```
sage: m = matrix(QQ,2,3,[1,2,3,4,5,6]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

```
sage: m = matrix({(0,1): 2, (1,1):2/5}); m; m.parent()
[  0   2]
[  0 2/5]
Full MatrixSpace of 2 by 2 sparse matrices over Rational Field
```

```
sage: m = matrix(QQ,2,3,{(1,1): 2}); m; m.parent()
[0 0 0]
[0 2 0]
Full MatrixSpace of 2 by 3 sparse matrices over Rational Field
```

```
sage: import numpy
sage: n = numpy.array([[1,2],[3,4]],float)
sage: m = matrix(n); m; m.parent()
[1.0 2.0]
[3.0 4.0]
Full MatrixSpace of 2 by 2 dense matrices over Real Double Field
```

```
sage: v = vector(ZZ, [1, 10, 100])
sage: m = matrix(v); m; m.parent()
[  1  10 100]
Full MatrixSpace of 1 by 3 dense matrices over Integer Ring
sage: m = matrix(GF(7), v); m; m.parent()
[1 3 2]
Full MatrixSpace of 1 by 3 dense matrices over Finite Field of size 7
sage: m = matrix(GF(7), 3, 1, v); m; m.parent()
[1]
[3]
[2]
Full MatrixSpace of 3 by 1 dense matrices over Finite Field of size 7
```

```
sage: matrix(pari.mathilbert(3))
[  1 1/2 1/3]
[1/2 1/3 1/4]
[1/3 1/4 1/5]
```

```
sage: g = graphs.PetersenGraph()
sage: m = matrix(g); m; m.parent()
[0 1 0 0 1 1 0 0 0 0]
[1 0 1 0 0 0 1 0 0 0]
[0 1 0 1 0 0 0 1 0 0]
[0 0 1 0 1 0 0 0 1 0]
[1 0 0 1 0 0 0 0 0 1]
```

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```
[1 0 0 0 0 0 0 1 1 0]
[0 1 0 0 0 0 0 0 1 1]
[0 0 1 0 0 1 0 0 0 1]
[0 0 0 1 0 1 1 0 0 0]
[0 0 0 0 1 0 1 1 0 0]
Full MatrixSpace of 10 by 10 dense matrices over Integer Ring
```

```
sage: matrix(ZZ, 10, 10, range(100), sparse=True).parent()
Full MatrixSpace of 10 by 10 sparse matrices over Integer Ring
```

```
sage: R = PolynomialRing(QQ, 9, 'x')
sage: A = matrix(R, 3, 3, R.gens()); A
[x0 x1 x2]
[x3 x4 x5]
[x6 x7 x8]
sage: det(A)
-x2*x4*x6 + x1*x5*x6 + x2*x3*x7 - x0*x5*x7 - x1*x3*x8 + x0*x4*x8
```

```
sage: M = Matrix([[1,2,3],[4,5,6],[7,8,9]], immutable=True)
sage: M[0] = [9,9,9]
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use copy(M) to␣
↪change a copy of M).
```

AUTHORS:

- William Stein: Initial implementation
- Jason Grout (2008-03): almost a complete rewrite, with bits and pieces from the original implementation
- Jeroen Demeyer (2016-02-05): major clean up, see github issue #20015 and github issue #20016
- Jeroen Demeyer (2018-02-20): completely rewritten using `MatrixArgs`, see github issue #24742

`sage.matrix.constructor.`**`matrix`**(**args*, ***kwds*)

Create a matrix.

This implements the `matrix` constructor:

```
sage: matrix([[1,2],[3,4]])
[1 2]
[3 4]
```

It also contains methods to create special types of matrices, see `matrix.[tab]` for more options. For example:

```
sage: matrix.identity(2)
[1 0]
[0 1]
```

INPUT:

The matrix command takes the entries of a matrix, optionally preceded by a ring and the dimensions of the matrix, and returns a matrix.

The entries of a matrix can be specified as a flat list of elements, a list of lists (i.e., a list of rows), a list of Sage vectors, a callable object, or a dictionary having positions as keys and matrix entries as values (see the examples). If you pass in a callable object, then you must specify the number of rows and columns. You can create a matrix of zeros by passing an empty list or the integer zero for the entries. To construct a multiple of the identity (cI), you can specify square dimensions and pass in c. Calling matrix() with a Sage object may return something that makes sense. Calling matrix() with a NumPy array will convert the array to a matrix.

All arguments (even the positional) are optional.

Positional and keyword arguments:

- `ring` – parent of the entries of the matrix (despite the name, this is not a priori required to be a ring). By default, determine this from the given entries, falling back to ZZ if no entries are given.
- `nrows` – the number of rows in the matrix.
- `ncols` – the number of columns in the matrix.
- `entries` – see examples below.

If either `nrows` or `ncols` is given as keyword argument, then no positional arguments `nrows` and `ncols` may be given.

Keyword-only arguments:

- `sparse` – (boolean) create a sparse matrix. This defaults to `True` when the entries are given as a dictionary, otherwise defaults to `False`.
- `space` – matrix space which will be the parent of the output matrix. This determines `ring`, `nrows`, `ncols` and `sparse`.
- `immutable` – (boolean) make the matrix immutable. By default, the output matrix is mutable.

OUTPUT:

a matrix

EXAMPLES:

```
sage: m = matrix(2); m; m.parent()
[0 0]
[0 0]
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
```

```
sage: m = matrix(2,3); m; m.parent()
[0 0 0]
[0 0 0]
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

```
sage: m = matrix(QQ,[[1,2,3],[4,5,6]]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

```
sage: m = matrix(QQ, 3, 3, lambda i, j: i+j); m
[0 1 2]
[1 2 3]
[2 3 4]
sage: m = matrix(3, lambda i,j: i-j); m
```

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```
[ 0 -1 -2]
[ 1  0 -1]
[ 2  1  0]
```

```
sage: matrix(QQ, 2, 3, lambda x, y: x+y)
[0 1 2]
[1 2 3]
sage: matrix(QQ, 5, 5, lambda x, y: (x+1) / (y+1))
[  1 1/2 1/3 1/4 1/5]
[  2   1 2/3 1/2 2/5]
[  3 3/2   1 3/4 3/5]
[  4   2 4/3   1 4/5]
[  5 5/2 5/3 5/4   1]
```

```
sage: v1=vector((1,2,3))
sage: v2=vector((4,5,6))
sage: m = matrix([v1,v2]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

```
sage: m = matrix(QQ,2,[1,2,3,4,5,6]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

```
sage: m = matrix(QQ,2,3,[1,2,3,4,5,6]); m; m.parent()
[1 2 3]
[4 5 6]
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

```
sage: m = matrix({(0,1): 2, (1,1):2/5}); m; m.parent()
[  0   2]
[  0 2/5]
Full MatrixSpace of 2 by 2 sparse matrices over Rational Field
```

```
sage: m = matrix(QQ,2,3,{(1,1): 2}); m; m.parent()
[0 0 0]
[0 2 0]
Full MatrixSpace of 2 by 3 sparse matrices over Rational Field
```

```
sage: import numpy
sage: n = numpy.array([[1,2],[3,4]],float)
sage: m = matrix(n); m; m.parent()
[1.0 2.0]
[3.0 4.0]
Full MatrixSpace of 2 by 2 dense matrices over Real Double Field
```

```
sage: v = vector(ZZ, [1, 10, 100])
sage: m = matrix(v); m; m.parent()
```

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```
[  1  10 100]
Full MatrixSpace of 1 by 3 dense matrices over Integer Ring
sage: m = matrix(GF(7), v); m; m.parent()
[1 3 2]
Full MatrixSpace of 1 by 3 dense matrices over Finite Field of size 7
sage: m = matrix(GF(7), 3, 1, v); m; m.parent()
[1]
[3]
[2]
Full MatrixSpace of 3 by 1 dense matrices over Finite Field of size 7
```

```
sage: matrix(pari.mathilbert(3))
[  1 1/2 1/3]
[1/2 1/3 1/4]
[1/3 1/4 1/5]
```

```
sage: g = graphs.PetersenGraph()
sage: m = matrix(g); m; m.parent()
[0 1 0 0 1 1 0 0 0 0]
[1 0 1 0 0 0 1 0 0 0]
[0 1 0 1 0 0 0 1 0 0]
[0 0 1 0 1 0 0 0 1 0]
[1 0 0 1 0 0 0 0 0 1]
[1 0 0 0 0 0 0 1 1 0]
[0 1 0 0 0 0 0 0 1 1]
[0 0 1 0 0 1 0 0 0 1]
[0 0 0 1 0 1 1 0 0 0]
[0 0 0 0 1 0 1 1 0 0]
Full MatrixSpace of 10 by 10 dense matrices over Integer Ring
```

```
sage: matrix(ZZ, 10, 10, range(100), sparse=True).parent()
Full MatrixSpace of 10 by 10 sparse matrices over Integer Ring
```

```
sage: R = PolynomialRing(QQ, 9, 'x')
sage: A = matrix(R, 3, 3, R.gens()); A
[x0 x1 x2]
[x3 x4 x5]
[x6 x7 x8]
sage: det(A)
-x2*x4*x6 + x1*x5*x6 + x2*x3*x7 - x0*x5*x7 - x1*x3*x8 + x0*x4*x8
```

```
sage: M = Matrix([[1,2,3],[4,5,6],[7,8,9]], immutable=True)
sage: M[0] = [9,9,9]
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use copy(M) to␣
↪change a copy of M).
```

AUTHORS:

- William Stein: Initial implementation

- Jason Grout (2008-03): almost a complete rewrite, with bits and pieces from the original implementation
- Jeroen Demeyer (2016-02-05): major clean up, see github issue #20015 and github issue #20016
- Jeroen Demeyer (2018-02-20): completely rewritten using `MatrixArgs`, see github issue #24742

`sage.matrix.constructor.`**`options`**(**get_value*, ***set_value*)

Global options for matrices.

OPTIONS:

- `format_numeric` – (default: `{:.{prec}}`) string used for formatting floating point numbers of an (optional) precision `prec`; only supported for entry types implementing `__format__`
- `max_cols` – (default: `49`) maximum number of columns to display
- `max_rows` – (default: `19`) maximum number of rows to display
- `precision` – (default: `None`) number of digits to display for floating point entries; if `None`, the exact representation is used instead. This option is also set by the IPython magic `%precision`.

EXAMPLES:

```
sage: matrix.options.max_cols = 6
sage: matrix.options.max_rows = 3
sage: matrix(ZZ, 3, 6)
[0 0 0 0 0 0]
[0 0 0 0 0 0]
[0 0 0 0 0 0]
sage: matrix(ZZ, 3, 7)
3 x 7 dense matrix over Integer Ring...
sage: matrix(ZZ, 4, 6)
4 x 6 dense matrix over Integer Ring...
sage: matrix.options._reset()
```

The precision can also be set via the IPython magic:

```
sage: from sage.repl.interpreter import get_test_shell
sage: shell = get_test_shell()
sage: shell.run_cell('%precision 5')
'%.5f'
sage: matrix.options.precision
5
sage: A = matrix(RR, [[200/3]]); A
[66.667]
```

The number format can be specified as well:

```
sage: matrix.options.format_numeric = '{:.{prec}e}'
sage: A
[6.66667e+1]
sage: matrix.options.format_numeric = '{:.{prec}f}'
sage: A
[66.66667]
sage: matrix.options.format_numeric = '{:+.{prec}g}'
sage: A
```

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```
[+66.667]
sage: matrix.options._reset()
```

See GlobalOptions for more features of these options.

CHAPTER

THREE

CONSTRUCTORS FOR SPECIAL MATRICES

This module gathers several constructors for special, commonly used or interesting matrices. These can be reached through `matrix.<tab>`.

For example, here is a circulant matrix of order five:

```
sage: matrix.circulant(SR.var('a b c d e'))
[a b c d e]
[e a b c d]
[d e a b c]
[c d e a b]
[b c d e a]
```

The following constructions are available:

block_diagonal_matrix()
block_matrix()
circulant()
column_matrix()
companion_matrix()
diagonal_matrix()
elementary_matrix()
hankel()
hilbert()
identity_matrix()
ith_to_zero_rotation_matrix()
jordan_block()
lehmer()
ones_matrix()
random_matrix()
random_diagonalizable_matrix()
random_echelonizable_matrix()
random_rref_matrix()
random_subspaces_matrix()
random_unimodular_matrix()
toeplitz()
vandermonde()
vector_on_axis_rotation_matrix()
zero_matrix()

The Combinatorics module provides further matrix constructors, such as Hadamard matrices and Latin squares. See:

- sage.combinat.matrices.hadamard_matrix
- sage.combinat.matrices.latin

sage.matrix.special.**block_diagonal_matrix**(**sub_matrices*, ***kwds*)

This function is available as block_diagonal_matrix(...) and matrix.block_diagonal(...).

Create a block matrix whose diagonal block entries are given by sub_matrices, with zero elsewhere.

See also *block_matrix()*.

EXAMPLES:

```
sage: A = matrix(ZZ, 2, [1,2,3,4])
sage: block_diagonal_matrix(A, A)
[1 2|0 0]
[3 4|0 0]
[---+---]
[0 0|1 2]
[0 0|3 4]
```

The sub-matrices need not be square:

```
sage: B = matrix(QQ, 2, 3, range(6))
sage: block_diagonal_matrix(~A, B)
[  -2    1|   0    0    0]
[ 3/2 -1/2|   0    0    0]
[---------+--------------]
[   0    0|   0    1    2]
[   0    0|   3    4    5]
```

sage.matrix.special.**block_matrix**(**args*, ***kwds*)

This function is available as block_matrix(...) and matrix.block(...).

Return a larger matrix made by concatenating submatrices (rows first, then columns). For example, the matrix

```
[ A B ]
[ C D ]
```

is made up of submatrices A, B, C, and D.

INPUT:

The block_matrix command takes a list of submatrices to add as blocks, optionally preceded by a ring and the number of block rows and block columns, and returns a matrix.

The submatrices can be specified as a list of matrices (using `nrows` and `ncols` to determine their layout), or a list of lists of matrices, where each list forms a row.

- `ring` - the base ring
- `nrows` - the number of block rows
- `ncols` - the number of block cols
- `sub_matrices` - matrices (see below for syntax)
- `subdivide` - boolean, whether or not to add subdivision information to the matrix
- `sparse` - boolean, whether to make the resulting matrix sparse

EXAMPLES:

```
sage: A = matrix(QQ, 2, 2, [3,9,6,10])
sage: block_matrix([ [A, -A], [~A, 100*A] ])
[    3     9|   -3    -9]
[    6    10|   -6   -10]
[-----------+-----------]
[-5/12   3/8|  300   900]
[  1/4  -1/8|  600  1000]
```

If the number of submatrices in each row is the same, you can specify the submatrices as a single list too:

```
sage: block_matrix(2, 2, [ A, A, A, A ])
[ 3  9| 3  9]
[ 6 10| 6 10]
[-----+-----]
[ 3  9| 3  9]
[ 6 10| 6 10]
```

One can use constant entries:

```
sage: block_matrix([ [1, A], [0, 1] ])
[ 1  0| 3  9]
[ 0  1| 6 10]
[-----+-----]
[ 0  0| 1  0]
[ 0  0| 0  1]
```

A zero entry may represent any square or non-square zero matrix:

```
sage: B = matrix(QQ, 1, 1, [ 1 ] )
sage: C = matrix(QQ, 2, 2, [ 2, 3, 4, 5 ] )
sage: block_matrix([ [B, 0], [0, C] ])
[1|0 0]
[-+---]
[0|2 3]
[0|4 5]
```

One can specify the number of rows or columns as keywords too:

```
sage: block_matrix([A, -A, ~A, 100*A], ncols=4)
[    3     9|   -3    -9|-5/12   3/8|  300   900]
[    6    10|   -6   -10|  1/4  -1/8|  600  1000]

sage: block_matrix([A, -A, ~A, 100*A], nrows=1)
[    3     9|   -3    -9|-5/12   3/8|  300   900]
[    6    10|   -6   -10|  1/4  -1/8|  600  1000]
```

It handles base rings nicely too:

```
sage: R.<x> = ZZ['x']
sage: block_matrix(2, 2, [1/2, A, 0, x-1])
[  1/2     0|    3     9]
[    0   1/2|    6    10]
[-----------+-----------]
[    0     0|x - 1     0]
```

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```
[    0     0|    0 x - 1]

sage: block_matrix(2, 2, [1/2, A, 0, x-1]).parent()
Full MatrixSpace of 4 by 4 dense matrices over Univariate Polynomial Ring in x over␣
↪Rational Field
```

Subdivisions are optional. If they are disabled, the columns need not line up:

```
sage: B = matrix(QQ, 2, 3, range(6))
sage: block_matrix([ [~A, B], [B, ~A] ], subdivide=False)
[-5/12   3/8     0     1     2]
[  1/4  -1/8     3     4     5]
[    0     1     2 -5/12   3/8]
[    3     4     5   1/4  -1/8]
```

Without subdivisions it also deduces dimensions for scalars if possible:

```
sage: C = matrix(ZZ, 1, 2, range(2))
sage: block_matrix([ [ C, 0 ], [ 3, 4 ], [ 5, 6, C ] ], subdivide=False )
[0 1 0 0]
[3 0 4 0]
[0 3 0 4]
[5 6 0 1]
```

If all submatrices are sparse (unless there are none at all), the result will be a sparse matrix. Otherwise it will be dense by default. The `sparse` keyword can be used to override this:

```
sage: A = Matrix(ZZ, 2, 2, [0, 1, 0, 0], sparse=True)
sage: block_matrix([ [ A ], [ A ] ]).parent()
Full MatrixSpace of 4 by 2 sparse matrices over Integer Ring
sage: block_matrix([ [ A ], [ A ] ], sparse=False).parent()
Full MatrixSpace of 4 by 2 dense matrices over Integer Ring
```

Consecutive zero submatrices are consolidated.

```
sage: B = matrix(2, range(4))
sage: C = matrix(2, 8, range(16))
sage: block_matrix(2, [[B,0,0,B],[C]], subdivide=False)
[ 0  1  0  0  0  0  0  1]
[ 2  3  0  0  0  0  2  3]
[ 0  1  2  3  4  5  6  7]
[ 8  9 10 11 12 13 14 15]
```

Ambiguity is not tolerated.

```
sage: B = matrix(2, range(4))
sage: C = matrix(2, 8, range(16))
sage: block_matrix(2, [[B,0,B,0],[C]], subdivide=False)
Traceback (most recent call last):
...
ValueError: insufficient information to determine submatrix widths
```

Giving only a flat list of submatrices does not work:

```
sage: A = matrix(2, 3, range(6))
sage: B = matrix(3, 3, range(9))
sage: block_matrix([A, A, B, B])
Traceback (most recent call last):
...
ValueError: must specify either nrows or ncols
```

sage.matrix.special.**circulant**(*v*, *sparse=None*)

This function is available as circulant(...) and matrix.circulant(...).

Return the circulant matrix specified by its 1st row v

A circulant $n \times n$ matrix specified by the 1st row $v = (v_0 ... v_{n-1})$ is the matrix $(c_{ij})_{0 \le i,j \le n-1}$, where $c_{ij} = v_{j-i \mod b}$.

INPUT:

- v – a list or a vector of values
- sparse – None by default; if sparse is set to True, the output will be sparse. Respectively, setting it to False produces dense output. If sparse is not set, and if v is a vector, the output sparsity is determined by the sparsity of v; else, the output will be dense.

EXAMPLES:

```
sage: v=[1,2,3,4,8]
sage: matrix.circulant(v)
[1 2 3 4 8]
[8 1 2 3 4]
[4 8 1 2 3]
[3 4 8 1 2]
[2 3 4 8 1]
sage: m = matrix.circulant(vector(GF(3),[0,1,-1],sparse=True)); m
[0 1 2]
[2 0 1]
[1 2 0]
sage: m.is_sparse()
True
```

sage.matrix.special.**column_matrix**(**args*, ***kwds*)

This function is available as column_matrix(...) and matrix.column(...).

Construct a matrix, and then swap rows for columns and columns for rows.

Note: Linear algebra in Sage favors rows over columns. So, generally, when creating a matrix, input vectors and lists are treated as rows. This function is a convenience that turns around this convention when creating a matrix. If you are not familiar with the usual *matrix()* constructor, you might want to consider it first.

INPUT:

Inputs are almost exactly the same as for the *matrix()* constructor, which are documented there. But see examples below for how dimensions are handled.

OUTPUT:

Output is exactly the transpose of what the *matrix()* constructor would return. In other words, the matrix constructor builds a matrix and then this function exchanges rows for columns, and columns for rows.

EXAMPLES:

The most compelling use of this function is when you have a collection of lists or vectors that you would like to become the columns of a matrix. In almost any other situation, the `matrix`() constructor can probably do the job just as easily, or easier.

```
sage: col_1 = [1,2,3]
sage: col_2 = [4,5,6]
sage: column_matrix([col_1, col_2])
[1 4]
[2 5]
[3 6]

sage: v1 = vector(QQ, [10, 20])
sage: v2 = vector(QQ, [30, 40])
sage: column_matrix(QQ, [v1, v2])
[10 30]
[20 40]
```

If you only specify one dimension along with a flat list of entries, then it will be the number of columns in the result (which is different from the behavior of the `matrix` constructor).

```
sage: column_matrix(ZZ, 8, range(24))
[ 0  3  6  9 12 15 18 21]
[ 1  4  7 10 13 16 19 22]
[ 2  5  8 11 14 17 20 23]
```

And when you specify two dimensions, then they should be number of columns first, then the number of rows, which is the reverse of how they would be specified for the `matrix` constructor.

```
sage: column_matrix(QQ, 5, 3, range(15))
[ 0  3  6  9 12]
[ 1  4  7 10 13]
[ 2  5  8 11 14]
```

And a few unproductive, but illustrative, examples.

```
sage: A = matrix(ZZ, 3, 4, range(12))
sage: B = column_matrix(ZZ, 3, 4, range(12))
sage: A == B.transpose()
True

sage: A = matrix(QQ, 7, 12, range(84))
sage: A == column_matrix(A.columns())
True

sage: A = column_matrix(QQ, matrix(ZZ, 3, 2, range(6)) )
sage: A
[0 2 4]
[1 3 5]
sage: A.parent()
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

sage.matrix.special.**companion_matrix**(*poly*, *format='right'*)

This function is available as companion_matrix(...) and matrix.companion(...).

Create a companion matrix from a monic polynomial.

INPUT:

- `poly` – a univariate polynomial, or an iterable containing the coefficients of a polynomial, with low-degree coefficients first. The polynomial (or the polynomial implied by the coefficients) must be monic. In other words, the leading coefficient must be one. A symbolic expression that might also be a polynomial is not proper input, see examples below.
- `format` – default: 'right' - specifies one of four variations of a companion matrix. Allowable values are 'right', 'left', 'top' and 'bottom', which indicates which border of the matrix contains the negatives of the coefficients.

OUTPUT:

A square matrix with a size equal to the degree of the polynomial. The returned matrix has ones above, or below the diagonal, and the negatives of the coefficients along the indicated border of the matrix (excepting the leading one coefficient). See the first examples below for precise illustrations.

EXAMPLES:

Each of the four possibilities. Notice that the coefficients are specified and their negatives become the entries of the matrix. The leading one must be given, but is not used. The permutation matrix `P` is the identity matrix, with the columns reversed. The last three statements test the general relationships between the four variants.

```
sage: poly = [-2, -3, -4, -5, -6, 1]
sage: R = companion_matrix(poly, format='right'); R
[0 0 0 0 2]
[1 0 0 0 3]
[0 1 0 0 4]
[0 0 1 0 5]
[0 0 0 1 6]
sage: L = companion_matrix(poly, format='left'); L
[6 1 0 0 0]
[5 0 1 0 0]
[4 0 0 1 0]
[3 0 0 0 1]
[2 0 0 0 0]
sage: B = companion_matrix(poly, format='bottom'); B
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[2 3 4 5 6]
sage: T = companion_matrix(poly, format='top'); T
[6 5 4 3 2]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]

sage: perm = Permutation([5, 4, 3, 2, 1])
sage: P = perm.to_matrix()
sage: L == P*R*P
True
sage: B == R.transpose()
True
```

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```
sage: T == P*R.transpose()*P
True
```

A polynomial may be used as input, however a symbolic expression, even if it looks like a polynomial, is not regarded as such when used as input to this routine. Obtaining the list of coefficients from a symbolic polynomial is one route to the companion matrix.

```
sage: x = polygen(QQ, 'x')
sage: p = x^3 - 4*x^2 + 8*x - 12
sage: companion_matrix(p)
[ 0  0 12]
[ 1  0 -8]
[ 0  1  4]

sage: y = var('y')
sage: q = y^3 -2*y + 1
sage: companion_matrix(q)
Traceback (most recent call last):
...
TypeError: input must be a polynomial (not a symbolic expression, see docstring),␣
↪or other iterable, not y^3 - 2*y + 1

sage: coeff_list = [q(y=0)] + [q.coefficient(y^k) for k in range(1, q.degree(y)+1)]
sage: coeff_list
[1, -2, 0, 1]
sage: companion_matrix(coeff_list)
[ 0  0 -1]
[ 1  0  2]
[ 0  1  0]
```

The minimal polynomial of a companion matrix is equal to the polynomial used to create it. Used in a block diagonal construction, they can be used to create matrices with any desired minimal polynomial, or characteristic polynomial.

```
sage: t = polygen(QQ, 't')
sage: p = t^12 - 7*t^4 + 28*t^2 - 456
sage: C = companion_matrix(p, format='top')
sage: q = C.minpoly(var='t'); q
t^12 - 7*t^4 + 28*t^2 - 456
sage: p == q
True

sage: p = t^3 + 3*t - 8
sage: q = t^5 + t - 17
sage: A = block_diagonal_matrix( companion_matrix(p),
....:                            companion_matrix(p^2),
....:                            companion_matrix(q),
....:                            companion_matrix(q) )
sage: A.charpoly(var='t').factor()
(t^3 + 3*t - 8)^3 * (t^5 + t - 17)^2
sage: A.minpoly(var='t').factor()
(t^3 + 3*t - 8)^2 * (t^5 + t - 17)
```

AUTHOR:

- Rob Beezer (2011-05-19)

sage.matrix.special.**diagonal_matrix**(*arg0=None*, *arg1=None*, *arg2=None*, *sparse=True*)

This function is available as diagonal_matrix(...) and matrix.diagonal(...).

Return a square matrix with specified diagonal entries, and zeros elsewhere.

FORMATS:

1. diagonal_matrix(entries)
2. diagonal_matrix(nrows, entries)
3. diagonal_matrix(ring, entries)
4. diagonal_matrix(ring, nrows, entries)

INPUT:

- `entries` - the values to place along the diagonal of the returned matrix. This may be a flat list, a flat tuple, a vector or free module element, or a one-dimensional NumPy array.
- `nrows` - the size of the returned matrix, which will have an equal number of columns
- `ring` - the ring containing the entries of the diagonal entries. This may not be specified in combination with a NumPy array.
- `sparse` - default: `True` - whether or not the result has a sparse implementation.

OUTPUT:

A square matrix over the given `ring` with a size given by `nrows`. If the ring is not given it is inferred from the given entries. The values on the diagonal of the returned matrix come from `entries`. If the number of entries is not enough to fill the whole diagonal, it is padded with zeros.

EXAMPLES:

We first demonstrate each of the input formats with various different ways to specify the entries.

Format 1: a flat list of entries.

```
sage: A = diagonal_matrix([2, 1.3, 5]); A
[ 2.00000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000  1.30000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000  5.00000000000000]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Real Field with 53 bits of precision
```

Format 2: size specified, a tuple with initial entries. Note that a short list of entries is effectively padded with zeros.

```
sage: A = diagonal_matrix(3, (4, 5)); A
[4 0 0]
[0 5 0]
[0 0 0]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
```

Format 3: ring specified, a vector of entries.

```
sage: A = diagonal_matrix(QQ, vector(ZZ, [1,2,3])); A
[1 0 0]
[0 2 0]
[0 0 3]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Rational Field
```

Format 4: ring, size and list of entries.

```
sage: A = diagonal_matrix(FiniteField(3), 3, [2, 16]); A
[2 0 0]
[0 1 0]
[0 0 0]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Finite Field of size 3
```

NumPy arrays may be used as input.

```
sage: import numpy
sage: entries = numpy.array([1.2, 5.6]); entries
array([1.2, 5.6])
sage: A = diagonal_matrix(3, entries); A
[1.2 0.0 0.0]
[0.0 5.6 0.0]
[0.0 0.0 0.0]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Real Double Field

sage: j = complex(0,1)
sage: entries = numpy.array([2.0+j, 8.1, 3.4+2.6*j]); entries
array([2. +1.j , 8.1+0.j , 3.4+2.6j])
sage: A = diagonal_matrix(entries); A
[2.0 + 1.0*I         0.0         0.0]
[        0.0         8.1         0.0]
[        0.0         0.0 3.4 + 2.6*I]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Complex Double Field

sage: entries = numpy.array([4, 5, 6])
sage: A = diagonal_matrix(entries); A
[4 0 0]
[0 5 0]
[0 0 6]
sage: A.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring

sage: entries = numpy.array([4.1, 5.2, 6.3])
sage: A = diagonal_matrix(ZZ, entries); A
Traceback (most recent call last):
...
TypeError: unable to convert 4.1 to an element of Integer Ring
```

By default returned matrices have a sparse implementation. This can be changed when using any of the formats.

```
sage: A = diagonal_matrix([1,2,3], sparse=False)
sage: A.parent()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
```

An empty list and no ring specified defaults to the integers.

```
sage: A = diagonal_matrix([])
sage: A.parent()
Full MatrixSpace of 0 by 0 sparse matrices over Integer Ring
```

Giving the entries improperly may first complain about not being iterable:

```
sage: diagonal_matrix(QQ, 5, 10)
Traceback (most recent call last):
...
TypeError: 'sage.rings.integer.Integer' object is not iterable
```

Giving too many entries will raise an error.

```
sage: diagonal_matrix(QQ, 3, [1,2,3,4])
Traceback (most recent call last):
...
ValueError: number of diagonal matrix entries (4) exceeds the requested matrix size␣
↪(3)
```

A negative size sometimes causes the error that there are too many elements.

```
sage: diagonal_matrix(-2, [2])
Traceback (most recent call last):
...
ValueError: number of diagonal matrix entries (1) exceeds the requested matrix size␣
↪(-2)
```

Types for the entries need to be iterable (tuple, list, vector, NumPy array, etc):

```
sage: diagonal_matrix(x^2)
Traceback (most recent call last):
...
TypeError: 'sage.symbolic.expression.Expression' object is not iterable
```

AUTHOR:

- Rob Beezer (2011-01-11): total rewrite

sage.matrix.special.**elementary_matrix**(*arg0*, *arg1=None*, ***kwds*)

This function is available as elementary_matrix(...) and matrix.elementary(...).

Creates a square matrix that corresponds to a row operation or a column operation.

FORMATS:

In each case, `R` is the base ring, and is optional. `n` is the size of the square matrix created. Any call may include the `sparse` keyword to determine the representation used. The default is `False` which leads to a dense representation. We describe the matrices by their associated row operation, see the output description for more.

- `elementary_matrix(R, n, row1=i, row2=j)`

  The matrix which swaps rows `i` and `j`.

- `elementary_matrix(R, n, row1=i, scale=s)`

  The matrix which multiplies row `i` by `s`.

- `elementary_matrix(R, n, row1=i, row2=j, scale=s)`

  The matrix which multiplies row `j` by `s` and adds it to row `i`.

Elementary matrices representing column operations are created in an entirely analogous way, replacing `row1` by `col1` and replacing `row2` by `col2`.

Specifying the ring for entries of the matrix is optional. If it is not given, and a scale parameter is provided, then a ring containing the value of `scale` will be used. Otherwise, the ring defaults to the integers.

OUTPUT:

An elementary matrix is a square matrix that is very close to being an identity matrix. If `E` is an elementary matrix and `A` is any matrix with the same number of rows, then `E*A` is the result of applying a row operation to `A`. This is how the three types created by this function are described. Similarly, an elementary matrix can be associated with a column operation, so if `E` has the same number of columns as `A` then `A*E` is the result of performing a column operation on `A`.

An elementary matrix representing a row operation is created if `row1` is specified, while an elementary matrix representing a column operation is created if `col1` is specified.

EXAMPLES:

Over the integers, creating row operations. Recall that row and column numbering begins at zero.

```
sage: A = matrix(ZZ, 4, 10, range(40)); A
[ 0  1  2  3  4  5  6  7  8  9]
[10 11 12 13 14 15 16 17 18 19]
[20 21 22 23 24 25 26 27 28 29]
[30 31 32 33 34 35 36 37 38 39]

sage: E = elementary_matrix(4, row1=1, row2=3); E
[1 0 0 0]
[0 0 0 1]
[0 0 1 0]
[0 1 0 0]
sage: E*A
[ 0  1  2  3  4  5  6  7  8  9]
[30 31 32 33 34 35 36 37 38 39]
[20 21 22 23 24 25 26 27 28 29]
[10 11 12 13 14 15 16 17 18 19]

sage: E = elementary_matrix(4, row1=2, scale=10); E
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0 10  0]
[ 0  0  0  1]
sage: E*A
[  0   1   2   3   4   5   6   7   8   9]
[ 10  11  12  13  14  15  16  17  18  19]
[200 210 220 230 240 250 260 270 280 290]
[ 30  31  32  33  34  35  36  37  38  39]

sage: E = elementary_matrix(4, row1=2, row2=1, scale=10); E
```

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```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0 10  1  0]
[ 0  0  0  1]
sage: E*A
[  0   1   2   3   4   5   6   7   8   9]
[ 10  11  12  13  14  15  16  17  18  19]
[120 131 142 153 164 175 186 197 208 219]
[ 30  31  32  33  34  35  36  37  38  39]
```

Over the rationals, now as column operations. Recall that row and column numbering begins at zero. Checks now have the elementary matrix on the right.

```
sage: A = matrix(QQ, 5, 4, range(20)); A
[ 0  1  2  3]
[ 4  5  6  7]
[ 8  9 10 11]
[12 13 14 15]
[16 17 18 19]

sage: E = elementary_matrix(QQ, 4, col1=1, col2=3); E
[1 0 0 0]
[0 0 0 1]
[0 0 1 0]
[0 1 0 0]
sage: A*E
[ 0  3  2  1]
[ 4  7  6  5]
[ 8 11 10  9]
[12 15 14 13]
[16 19 18 17]

sage: E = elementary_matrix(QQ, 4, col1=2, scale=1/2); E
[  1   0   0   0]
[  0   1   0   0]
[  0   0 1/2   0]
[  0   0   0   1]
sage: A*E
[ 0  1  1  3]
[ 4  5  3  7]
[ 8  9  5 11]
[12 13  7 15]
[16 17  9 19]

sage: E = elementary_matrix(QQ, 4, col1=2, col2=1, scale=10); E
[ 1  0  0  0]
[ 0  1 10  0]
[ 0  0  1  0]
[ 0  0  0  1]
sage: A*E
[  0   1  12   3]
[  4   5  56   7]
```

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```
[  8   9 100  11]
[ 12  13 144  15]
[ 16  17 188  19]
```

An elementary matrix is always nonsingular. Then repeated row operations can be represented by products of elementary matrices, and this product is again nonsingular. If row operations are to preserve fundamental properties of a matrix (like rank), we do not allow scaling a row by zero. Similarly, the corresponding elementary matrix is not constructed. Also, we do not allow adding a multiple of a row to itself, since this could also lead to a new zero row.

```
sage: A = matrix(QQ, 4, 10, range(40)); A
[ 0  1  2  3  4  5  6  7  8  9]
[10 11 12 13 14 15 16 17 18 19]
[20 21 22 23 24 25 26 27 28 29]
[30 31 32 33 34 35 36 37 38 39]

sage: E1 = elementary_matrix(QQ, 4, row1=0, row2=1)
sage: E2 = elementary_matrix(QQ, 4, row1=3, row2=0, scale=100)
sage: E = E2*E1
sage: E.is_singular()
False
sage: E*A
[  10   11   12   13   14   15   16   17   18   19]
[   0    1    2    3    4    5    6    7    8    9]
[  20   21   22   23   24   25   26   27   28   29]
[1030 1131 1232 1333 1434 1535 1636 1737 1838 1939]

sage: E3 = elementary_matrix(QQ, 4, row1=3, scale=0)
Traceback (most recent call last):
...
ValueError: scale parameter of row of elementary matrix must be non-zero

sage: E4 = elementary_matrix(QQ, 4, row1=3, row2=3, scale=12)
Traceback (most recent call last):
...
ValueError: cannot add a multiple of a row to itself
```

If the ring is not specified, and a scale parameter is given, the base ring for the matrix is chosen to contain the scale parameter. Otherwise, if no ring is given, the default is the integers.

```
sage: E = elementary_matrix(4, row1=1, row2=3)
sage: E.parent()
Full MatrixSpace of 4 by 4 dense matrices over Integer Ring

sage: E = elementary_matrix(4, row1=1, scale=4/3)
sage: E.parent()
Full MatrixSpace of 4 by 4 dense matrices over Rational Field

sage: E = elementary_matrix(4, row1=1, scale=I)
sage: E.parent()
Full MatrixSpace of 4 by 4 dense matrices over Number Field in I with defining␣
↪polynomial x^2 + 1 with I = 1*I
```

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```
sage: E = elementary_matrix(4, row1=1, scale=CDF(I))
sage: E.parent()
Full MatrixSpace of 4 by 4 dense matrices over Complex Double Field

sage: E = elementary_matrix(4, row1=1, scale=QQbar(I))
sage: E.parent()
Full MatrixSpace of 4 by 4 dense matrices over Algebraic Field
```

Returned matrices have a dense implementation by default, but a sparse implementation may be requested.

```
sage: E = elementary_matrix(4, row1=0, row2=1)
sage: E.is_dense()
True

sage: E = elementary_matrix(4, row1=0, row2=1, sparse=True)
sage: E.is_sparse()
True
```

And the ridiculously small cases. The zero-row matrix cannot be built since then there are no rows to manipulate.

```
sage: elementary_matrix(QQ, 1, row1=0, row2=0)
[1]
sage: elementary_matrix(QQ, 0, row1=0, row2=0)
Traceback (most recent call last):
...
ValueError: size of elementary matrix must be 1 or greater, not 0
```

AUTHOR:

- Rob Beezer (2011-03-04)

sage.matrix.special.**hankel**(*c*, *r=None*, *ring=None*)

This function is available as hankel(...) and matrix.hankel(...).

Return a Hankel matrix of given first column and whose elements are zero below the first anti-diagonal.

The Hankel matrix is symmetric and constant across the anti-diagonals, with elements

$$H_{ij} = v_{i+j-1}, \qquad i = 1, \ldots, m,\ j = 1, \ldots, n,$$

where the vector $v_i = c_i$ for $i = 1, \ldots, m$ and $v_{m+i} = r_i$ for $i = 1, \ldots, n-1$ completely determines the Hankel matrix. If the last row, r, is not given, the Hankel matrix is square by default and $r = 0$. For more information see the Wikipedia article Hankel_matrix.

INPUT:

- `c` – vector, first column of the Hankel matrix
- `r` – vector (optional, default: None), last row of the Hankel matrix, from the second to the last column
- `ring` – base ring (optional, default: None) of the resulting matrix

EXAMPLES:

A Hankel matrix with symbolic entries:

```
sage: matrix.hankel(SR.var('a, b, c, d, e'))
[a b c d e]
[b c d e 0]
[c d e 0 0]
[d e 0 0 0]
[e 0 0 0 0]
```

We can also pass the elements of the last row, starting at the second column:

```
sage: matrix.hankel(SR.var('a, b, c, d, e'), SR.var('f, g, h, i'))
[a b c d e]
[b c d e f]
[c d e f g]
[d e f g h]
[e f g h i]
```

A third order Hankel matrix in the integers:

```
sage: matrix.hankel([1, 2, 3])
[1 2 3]
[2 3 0]
[3 0 0]
```

The second argument allows to customize the last row:

```
sage: matrix.hankel([1..3], [7..10])
[ 1  2  3  7  8]
[ 2  3  7  8  9]
[ 3  7  8  9 10]
```

sage.matrix.special.**hilbert**(*dim*, *ring=Rational Field*)

This function is available as hilbert(...) and matrix.hilbert(...).

Return a Hilbert matrix of the given dimension.

The n dimensional Hilbert matrix is a square matrix with entries being unit fractions,

$$H_{ij} = \frac{1}{i+j-1}, \qquad i,j = 1,\ldots,n.$$

For more information see the Wikipedia article Hilbert_matrix.

INPUT:

- `dim` – integer, the dimension of the Hilbert matrix
- `ring` – base ring (optional, default: \QQ) of the resulting matrix

EXAMPLES:

```
sage: matrix.hilbert(5)
[  1 1/2 1/3 1/4 1/5]
[1/2 1/3 1/4 1/5 1/6]
[1/3 1/4 1/5 1/6 1/7]
[1/4 1/5 1/6 1/7 1/8]
[1/5 1/6 1/7 1/8 1/9]
```

sage.matrix.special.**identity_matrix**(*ring*, *n=0*, *sparse=False*)

This function is available as identity_matrix(...) and matrix.identity(...).

Return the $n \times n$ identity matrix over the given ring.

The default ring is the integers.

EXAMPLES:

```
sage: M = identity_matrix(QQ, 2); M
[1 0]
[0 1]
sage: M.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: M = identity_matrix(2); M
[1 0]
[0 1]
sage: M.parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: M.is_mutable()
True
sage: M = identity_matrix(3, sparse=True); M
[1 0 0]
[0 1 0]
[0 0 1]
sage: M.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
sage: M.is_mutable()
True
```

sage.matrix.special.**ith_to_zero_rotation_matrix**(*v*, *i*, *ring=None*)

This function is available as ith_to_zero_rotation_matrix(...) and matrix.ith_to_zero_rotation(...).

Return a rotation matrix that sends the i-th coordinates of the vector v to zero by doing a rotation with the (i-1)-th coordinate.

INPUT:

- `v` – vector
- `i` – integer
- `ring` – ring (optional, default: `None`) of the resulting matrix

OUTPUT:

A matrix

EXAMPLES:

```
sage: from sage.matrix.constructor import ith_to_zero_rotation_matrix
sage: v = vector((1,2,3))
sage: ith_to_zero_rotation_matrix(v, 2)
[             1              0              0]
[             0  2/13*sqrt(13)  3/13*sqrt(13)]
[             0 -3/13*sqrt(13)  2/13*sqrt(13)]
sage: ith_to_zero_rotation_matrix(v, 2) * v
(1, sqrt(13), 0)
```

```
sage: ith_to_zero_rotation_matrix(v, 0)
[ 3/10*sqrt(10)              0 -1/10*sqrt(10)]
[             0              1              0]
[ 1/10*sqrt(10)              0  3/10*sqrt(10)]
sage: ith_to_zero_rotation_matrix(v, 1)
[ 1/5*sqrt(5)  2/5*sqrt(5)            0]
[-2/5*sqrt(5)  1/5*sqrt(5)            0]
[           0            0            1]
sage: ith_to_zero_rotation_matrix(v, 2)
[             1              0              0]
[             0  2/13*sqrt(13)  3/13*sqrt(13)]
[             0 -3/13*sqrt(13)  2/13*sqrt(13)]
```

```
sage: ith_to_zero_rotation_matrix(v, 0) * v
(0, 2, sqrt(10))
sage: ith_to_zero_rotation_matrix(v, 1) * v
(sqrt(5), 0, 3)
sage: ith_to_zero_rotation_matrix(v, 2) * v
(1, sqrt(13), 0)
```

Other ring:

```
sage: ith_to_zero_rotation_matrix(v, 2, ring=RR)
[  1.00000000000000  0.000000000000000  0.000000000000000]
[ 0.000000000000000  0.554700196225229  0.832050294337844]
[ 0.000000000000000 -0.832050294337844  0.554700196225229]
sage: ith_to_zero_rotation_matrix(v, 2, ring=RDF)
[                1.0                 0.0                 0.0]
[                0.0  0.5547001962252291  0.8320502943378437]
[                0.0 -0.8320502943378437  0.5547001962252291]
```

On the symbolic ring:

```
sage: x,y,z = var('x,y,z')
sage: v = vector((x,y,z))
sage: ith_to_zero_rotation_matrix(v, 2)
[                1                 0                 0]
[                0  y/sqrt(y^2 + z^2)  z/sqrt(y^2 + z^2)]
[                0 -z/sqrt(y^2 + z^2)  y/sqrt(y^2 + z^2)]
sage: ith_to_zero_rotation_matrix(v, 2) * v
(x, y^2/sqrt(y^2 + z^2) + z^2/sqrt(y^2 + z^2), 0)
```

AUTHORS:

Sébastien Labbé (April 2010)

sage.matrix.special.**jordan_block**(*eigenvalue*, *size*, *sparse=False*)

This function is available as jordan_block(...) and matrix.jordan_block(...).

Return the Jordan block for the given eigenvalue with given size.

INPUT:

- `eigenvalue` – eigenvalue for the diagonal entries of the block
- `size` – size of the square matrix

- sparse – (default: False) - if True, return a sparse matrix

EXAMPLES:

```
sage: jordan_block(5, 3)
[5 1 0]
[0 5 1]
[0 0 5]
```

sage.matrix.special.**lehmer**(*ring*, *n=0*)

This function is available as lehmer(...) and matrix.lehmer(...).

Return the $n \times n$ Lehmer matrix.

The default ring is the rationals.

Element (i, j) in the Lehmer matrix is $min(i, j)/max(i, j)$.

See Wikipedia article Lehmer_matrix.

EXAMPLES:

```
sage: matrix.lehmer(3)
[  1 1/2 1/3]
[1/2   1 2/3]
[1/3 2/3   1]
```

sage.matrix.special.**matrix_method**(*func=None*, *name=None*)

Allows a function to be tab-completed on the global matrix constructor object.

INPUT:

- *function – a single argument. The function that is being decorated.
- **kwds – a single optional keyword argument name=<string>. The name of the corresponding method in the global matrix constructor object. If not given, it is derived from the function name.

EXAMPLES:

```
sage: from sage.matrix.constructor import matrix_method
sage: def foo_matrix(n): return matrix.diagonal(range(n))
sage: matrix_method(foo_matrix)
<function foo_matrix at ...>
sage: matrix.foo(5)
[0 0 0 0 0]
[0 1 0 0 0]
[0 0 2 0 0]
[0 0 0 3 0]
[0 0 0 0 4]
sage: matrix_method(foo_matrix, name='bar')
<function foo_matrix at ...>
sage: matrix.bar(3)
[0 0 0]
[0 1 0]
[0 0 2]
```

sage.matrix.special.**ones_matrix**(*ring*, *nrows=None*, *ncols=None*, *sparse=False*)

This function is available as ones_matrix(...) and matrix.ones(...).

Return a matrix with all entries equal to 1.

CALL FORMATS:

In each case, the optional keyword `sparse` can be used.

1. ones_matrix(ring, nrows, ncols)
2. ones_matrix(ring, nrows)
3. ones_matrix(nrows, ncols)
4. ones_matrix(nrows)

INPUT:

- `ring` - default: `ZZ` - base ring for the matrix.
- `nrows` - number of rows in the matrix.
- `ncols` - number of columns in the matrix. If omitted, defaults to the number of rows, producing a square matrix.
- `sparse` - default: `False` - if `True` creates a sparse representation.

OUTPUT:

A matrix of size `nrows` by `ncols` over the `ring` with every entry equal to 1. While the result is far from sparse, you may wish to choose a sparse representation when mixing this matrix with other sparse matrices.

EXAMPLES:

A call specifying the ring and the size.

```
sage: M= ones_matrix(QQ, 2, 5); M
[1 1 1 1 1]
[1 1 1 1 1]
sage: M.parent()
Full MatrixSpace of 2 by 5 dense matrices over Rational Field
```

Without specifying the number of columns, the result is square.

```
sage: M = ones_matrix(RR, 2); M
[1.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000]
sage: M.parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Field with 53 bits of precision
```

The ring defaults to the integers if not given.

```
sage: M = ones_matrix(2, 3); M
[1 1 1]
[1 1 1]
sage: M.parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

A lone integer input produces a square matrix over the integers.

```
sage: M = ones_matrix(3); M
[1 1 1]
[1 1 1]
```

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```
[1 1 1]
sage: M.parent()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
```

The result can have a sparse implementation.

```
sage: M = ones_matrix(3, 1, sparse=True); M
[1]
[1]
[1]
sage: M.parent()
Full MatrixSpace of 3 by 1 sparse matrices over Integer Ring
```

Giving just a ring will yield an error.

```
sage: ones_matrix(CC)
Traceback (most recent call last):
...
ValueError: constructing an all ones matrix requires at least one dimension
```

sage.matrix.special.**random_diagonalizable_matrix**(*parent*, *eigenvalues=None*, *dimensions=None*)

This function is available as random_diagonalizable_matrix(...) and matrix.random_diagonalizable(...).

Create a random matrix that diagonalizes nicely.

To be used as a teaching tool. Return matrices have only real eigenvalues.

INPUT:

If eigenvalues and dimensions are not specified in a list, they will be assigned randomly.

- `parent` - the desired size of the square matrix.
- `eigenvalues` - the list of desired eigenvalues (default=None).
- `dimensions` - the list of dimensions corresponding to each eigenspace (default=None).

OUTPUT:

A square, diagonalizable, matrix with only integer entries. The eigenspaces of this matrix, if computed by hand, give basis vectors with only integer entries.

Note: It is easiest to use this function via a call to the `random_matrix()` function with the `algorithm='diagonalizable'` keyword. We provide one example accessing this function directly, while the remainder will use this more general function.

EXAMPLES:

A diagonalizable matrix, size 5.

```
sage: from sage.matrix.constructor import random_diagonalizable_matrix
sage: matrix_space = sage.matrix.matrix_space.MatrixSpace(QQ, 5)
sage: A = random_diagonalizable_matrix(matrix_space)
sage: eigenvalues = A.eigenvalues()
sage: S = A.right_eigenmatrix()[1]
sage: eigenvalues2 = (S.inverse()*A*S).diagonal()
```

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```
sage: sorted(eigenvalues) == sorted(eigenvalues2)
True
```

A diagonalizable matrix with eigenvalues and dimensions designated, with a check that if eigenvectors were calculated by hand entries would all be integers.

```
sage: eigenvalues = [ZZ.random_element() for _ in range(3)]
sage: B = random_matrix(QQ, 6, algorithm='diagonalizable', eigenvalues=eigenvalues,␣
↪dimensions=[2,3,1])
sage: all(x in ZZ for x in (B-(-12*identity_matrix(6))).rref().list())
True
sage: all(x in ZZ for x in (B-(4*identity_matrix(6))).rref().list())
True
sage: all(x in ZZ for x in (B-(6*identity_matrix(6))).rref().list())
True
sage: S = B.right_eigenmatrix()[1]
sage: eigenvalues2 = (S.inverse()*B*S).diagonal()
sage: all(e in eigenvalues for e in eigenvalues2)
True
```

Todo: Modify the routine to allow for complex eigenvalues.

AUTHOR:

Billy Wonderly (2010-07)

sage.matrix.special.**random_echelonizable_matrix**(*parent*, *rank*, *upper_bound=None*, *max_tries=100*)

This function is available as random_echelonizable_matrix(...) and matrix.random_echelonizable(...).

Generate a matrix of a desired size and rank, over a desired ring, whose reduced row-echelon form has only integral values.

INPUT:

- `parent` – A matrix space specifying the base ring, dimensions and representation (dense/sparse) for the result. The base ring must be exact.
- `rank` – Rank of result, i.e the number of non-zero rows in the reduced row echelon form.
- `upper_bound` – If designated, size control of the matrix entries is desired. Set `upper_bound` to 1 more than the maximum value entries can achieve. If None, no size control occurs. But see the warning below. (default: None)
- `max_tries` - If designated, number of tries used to generate each new random row; only matters when upper_bound!=None. Used to prevent endless looping. (default: 100)

OUTPUT:

A matrix not in reduced row-echelon form with the desired dimensions and properties.

Warning: When `upper_bound` is set, it is possible for this constructor to fail with a `ValueError`. This may happen when the `upper_bound`, `rank` and/or matrix dimensions are all so small that it becomes infeasible or unlikely to create the requested matrix. If you *must* have this routine return successfully, do not set `upper_bound`.

Note: It is easiest to use this function via a call to the `random_matrix()` function with the `algorithm='echelonizable'` keyword. We provide one example accessing this function directly, while the remainder will use this more general function.

EXAMPLES:

Generated matrices have the desired dimensions, rank and entry size. The matrix in reduced row-echelon form has only integer entries.

```
sage: from sage.matrix.constructor import random_echelonizable_matrix
sage: matrix_space = sage.matrix.matrix_space.MatrixSpace(QQ, 5, 6)
sage: A = random_echelonizable_matrix(matrix_space, rank=4, upper_bound=40)
sage: A.rank()
4
sage: max(map(abs,A.list()))<40
True
sage: A.rref() == A.rref().change_ring(ZZ)
True
```

An example with default settings (i.e. no entry size control).

```
sage: C=random_matrix(QQ, 6, 7, algorithm='echelonizable', rank=5)
sage: C.rank()
5
sage: C.rref() == C.rref().change_ring(ZZ)
True
```

A matrix without size control may have very large entry sizes.

```
sage: D=random_matrix(ZZ, 7, 8, algorithm='echelonizable', rank=6); D  # random
[    1     2     8   -35  -178  -239  -284   778]
[    4     9    37  -163  -827 -1111 -1324  3624]
[    5     6    21   -88  -454  -607  -708  1951]
[   -4    -5   -22    97   491   656   779 -2140]
[    4     4    13   -55  -283  -377  -436  1206]
[    4    11    43  -194  -982 -1319 -1576  4310]
[   -1    -2   -13    59   294   394   481 -1312]
```

Matrices can be generated over any exact ring.

```
sage: F.<a>=GF(2^3)
sage: B = random_matrix(F, 4, 5, algorithm='echelonizable', rank=4, upper_
↪bound=None)
sage: B.rank()
4
sage: B.base_ring() is F
True
```

Square matrices over ZZ or QQ with full rank are always unimodular.

```
sage: E = random_matrix(QQ, 7, 7, algorithm='echelonizable', rank=7)
sage: det(E)
1
```

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```
sage: E = random_matrix(ZZ, 7, 7, algorithm='echelonizable', rank=7)
sage: det(E)
1
```

AUTHOR:

Billy Wonderly (2010-07)

sage.matrix.special.**random_matrix**(*ring*, *nrows*, *ncols=None*, *algorithm='randomize'*, *implementation=None*, **args*, ***kwds*)

This function is available as random_matrix(...) and matrix.random(...).

Return a random matrix with entries in a specified ring, and possibly with additional properties.

INPUT:

- `ring` – base ring for entries of the matrix
- `nrows` – Integer; number of rows
- `ncols` – (default: `None`); number of columns; if `None` defaults to `nrows`
- `algorithm` – (default: `randomize`); determines what properties the matrix will have. See examples below for possible additional arguments.
  - `randomize` – create a matrix of random elements from the base ring, possibly controlling the density of non-zero entries.
  - `echelon_form` – creates a matrix in echelon form
  - `echelonizable` – creates a matrix that has a predictable echelon form
  - `subspaces` – creates a matrix whose four subspaces, when explored, have reasonably sized, integral valued, entries.
  - `unimodular` – creates a matrix of determinant 1.
  - `diagonalizable` – creates a diagonalizable matrix whose eigenvectors, if computed by hand, will have only integer entries.
- `implementation` – (`None` or string or a matrix class) a possible implementation. See the documentation of the constructor of *MatrixSpace*.
- `*args, **kwds` – arguments and keywords to describe additional properties. See more detailed documentation below.

Warning: Matrices generated are not uniformly distributed. For unimodular matrices over finite field this function does not even generate all of them: for example `Matrix.random(GF(3), 2, algorithm='unimodular')` never generates `[[2,0],[0,2]]`. This function is made for teaching purposes.

Warning: An upper bound on the absolute value of the entries may be set when the `algorithm` is `echelonizable` or `unimodular`. In these cases it is possible for this constructor to fail with a `ValueError`. If you *must* have this routine return successfully, do not set `upper_bound`. This behavior can be partially controlled by a `max_tries` keyword.

Note: When constructing matrices with random entries and no additional properties (i.e. when `algorithm='randomize'`), most of the randomness is controlled by the `random_element` method for elements of the base ring of the matrix, so the documentation of that method may be relevant or useful.

EXAMPLES:

Random integer matrices. With no arguments, the majority of the entries are zero, -1, and 1, and rarely "large."

```
sage: from collections import defaultdict
sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: def add_samples(*args, **kwds):
....:     global dic, total_count
....:     for _ in range(100):
....:         A = random_matrix(*args, **kwds)
....:         for a in A.list():
....:             dic[a] += 1
....:             total_count += 1.0

sage: expected = lambda n : 2 / (5*abs(n)*(abs(n) + 1)) if n != 0 else 1/5
sage: expected(0)
1/5
sage: expected(0) == expected(1) == expected(-1)
True
sage: expected(100)
1/25250
sage: add_samples(ZZ, 5, 5)
sage: while not all(abs(dic[a]/total_count - expected(a)) < 0.001 for a in dic):
....:     add_samples(ZZ, 5, 5)
```

The `distribution` keyword set to `uniform` will limit values between -2 and 2.

```
sage: expected = lambda n : 1/5 if n in range(-2, 3) else 0
sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: add_samples(ZZ, 5, 5, distribution='uniform')
sage: while not all(abs(dic[a]/total_count - expected(a)) < 0.001 for a in dic):
....:     add_samples(ZZ, 5, 5, distribution='uniform')
```

The `x` and `y` keywords can be used to distribute entries uniformly. When both are used `x` is the minimum and `y` is one greater than the maximum.

```
sage: expected = lambda n : 1/30 if n in range(70, 100) else 0
sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: add_samples(ZZ, 4, 8, x=70, y=100)
sage: while not all(abs(dic[a]/total_count - expected(a)) < 0.001 for a in dic):
....:     add_samples(ZZ, 4, 8, x=70, y=100)

sage: expected = lambda n : 1/10 if n in range(-5, 5) else 0
sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: add_samples(ZZ, 3, 7, x=-5, y=5)
```

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```
sage: while not all(abs(dic[a]/total_count - expected(a)) < 0.001 for a in dic):
....:     add_samples(ZZ, 3, 7, x=-5, y=5)
```

If only `x` is given, then it is used as the upper bound of a range starting at 0.

```
sage: expected = lambda n : 1/25 if n in range(25) else 0
sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: add_samples(ZZ, 5, 5, x=25)
sage: while not all(abs(dic[a]/total_count - expected(a)) < 0.001 for a in dic):
....:     add_samples(ZZ, 5, 5, x=25)
```

To control the number of nonzero entries, use the `density` keyword at a value strictly below the default of 1.0. The `density` keyword is used to compute the number of entries per row that will be nonzero, but the same entry may be selected more than once. So the value provided will be an upper bound for the density of the created matrix. Note that for a square matrix it is only necessary to set a single dimension.

```
sage: def add_sample(*args, **kwds):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     A = random_matrix(*args, **kwds)
....:     density_sum += float(A.density())

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(ZZ, 5, x=-10, y=10, density=0.75)
sage: expected_density = (1 - (4/5)^3)
sage: float(expected_density)
0.488
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(ZZ, 5, x=-10, y=10, density=0.75)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(ZZ, 5, x=20, y=30, density=0.75)
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(ZZ, 5, x=20, y=30, density=0.75)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(ZZ, 100, x=20, y=30, density=0.75)
sage: expected_density = (1 - (99/100)^75)
sage: float(expected_density)
0.529...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(ZZ, 100, x=20, y=30, density=0.75)
```

For a matrix with low density it may be advisable to insist on a sparse representation, as this representation is not selected automatically.

```
sage: A=random_matrix(ZZ, 5, 5)
sage: A.is_sparse()
False
```

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```
sage: A = random_matrix(ZZ, 5, 5, sparse=True)
sage: A.is_sparse()
True
```

For algorithm testing you might want to control the number of bits, say 10,000 entries, each limited to 16 bits.

```
sage: A = random_matrix(ZZ, 100, 100, x=2^16); A
100 x 100 dense matrix over Integer Ring (use the '.str()' method to see the␣
↪entries)
```

One can prescribe a specific matrix implementation:

```
sage: K.<a> = FiniteField(2^8)
sage: type(random_matrix(K, 2, 5))
<class 'sage.matrix.matrix_gf2e_dense.Matrix_gf2e_dense'>
sage: type(random_matrix(K, 2, 5, implementation="generic"))
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>
```

Random rational matrices. Now num_bound and den_bound control the generation of random elements, by specifying limits on the absolute value of numerators and denominators (respectively). Entries will be positive and negative (map the absolute value function through the entries to get all positive values). If either the numerator or denominator bound (or both) is not used, then the values default to 2:

```
sage: A = random_matrix(QQ, 2, 8, num_bound=20, den_bound=4)
sage: A.dimensions()
(2, 8)
sage: type(A)
<class 'sage.matrix.matrix_rational_dense.Matrix_rational_dense'>
sage: all(a.numerator() in range(-20, 21) and
....:     a.denominator() in range(1, 5)
....:     for a in A.list())
True

sage: A = random_matrix(QQ, 4, density = 0.5, sparse=True)
sage: type(A)
<class 'sage.matrix.matrix_rational_sparse.Matrix_rational_sparse'>
sage: A.density() <= 0.5
True

sage: A = random_matrix(QQ, 3, 10, num_bound = 99, den_bound = 99)
sage: positives = list(map(abs, A.list()))
sage: A1 = matrix(QQ, 3, 10, positives)
sage: all(abs(A.list()[i]) == A1.list()[i] for i in range(30))
True
sage: all(a.numerator() in range(100) and
....:     a.denominator() in range(1, 100)
....:     for a in A1.list())
True

sage: A = random_matrix(QQ, 4, 10, den_bound = 10)
sage: all(a.numerator() in range(-2, 3) and
....:     a.denominator() in range(1, 11)
....:     for a in A.list())
```

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```
True

sage: A = random_matrix(QQ, 4, 10)
sage: all(a.numerator() in range(-2, 3) and
....:     a.denominator() in range(1, 3)
....:     for a in A.list())
True
```

Random matrices over other rings. Several classes of matrices have specialized `randomize()` methods. You can locate these with the Sage command:

```
search_def('randomize')
```

The default implementation of `randomize()` relies on the `random_element()` method for the base ring. The `density` and `sparse` keywords behave as described above. Since we have a different randomisation when using the optional meataxe package, we have to make sure that we use the default implementation in this test:

```
sage: K.<a>=FiniteField(3^2)
sage: A = random_matrix(K, 2, 5, implementation='generic')
sage: type(A)
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>
sage: A.base_ring() is K
True
sage: TestSuite(A).run()

sage: A = random_matrix(RR, 3, 4, density=0.66)
sage: type(A)
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>
sage: A.base_ring() is RR
True
sage: TestSuite(A).run()

sage: A = random_matrix(ComplexField(32), 3, density=0.8, sparse=True)
sage: A.is_sparse()
True
sage: type(A)
<class 'sage.matrix.matrix_generic_sparse.Matrix_generic_sparse'>
sage: A.base_ring() is ComplexField(32)
True
sage: TestSuite(A).run()
```

Random matrices in echelon form. The `algorithm='echelon_form'` keyword, along with a requested number of non-zero rows (`num_pivots`) will return a random matrix in echelon form. When the base ring is `QQ` the result has integer entries. Other exact rings may be also specified.

```
sage: A = random_matrix(QQ, 4, 8, algorithm='echelon_form', num_pivots=3)
sage: A.base_ring()
Rational Field
sage: (A.nrows(), A.ncols())
(4, 8)
sage: A in sage.matrix.matrix_space.MatrixSpace(ZZ, 4, 8)
True
sage: A.rank()
```

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```
3
sage: A == A.rref()
True
```

For more, see the documentation of the `random_rref_matrix()` function. In the notebook or at the Sage command-line, first execute the following to make this further documentation available:

```
from sage.matrix.constructor import random_rref_matrix
```

Random matrices with predictable echelon forms. The `algorithm='echelonizable'` keyword, along with a requested rank (`rank`) and optional size control (`upper_bound`) will return a random matrix in echelon form. When the base ring is `ZZ` or `QQ` the result has integer entries, whose magnitudes can be limited by the value of `upper_bound`, and the echelon form of the matrix also has integer entries. Other exact rings may be also specified, but there is no notion of controlling the size. Square matrices of full rank generated by this function always have determinant one, and can be constructed with the `unimodular` keyword.

```
sage: A = random_matrix(QQ, 4, 8, algorithm='echelonizable', rank=3, upper_bound=60)
sage: A.base_ring()
Rational Field
sage: (A.nrows(), A.ncols())
(4, 8)
sage: A in sage.matrix.matrix_space.MatrixSpace(ZZ, 4, 8)
True
sage: A.rank()
3
sage: all(abs(x)<60 for x in A.list())
True
sage: A.rref() in sage.matrix.matrix_space.MatrixSpace(ZZ, 4, 8)
True
```

For more, see the documentation of the `random_echelonizable_matrix()` function. In the notebook or at the Sage command-line, first execute the following to make this further documentation available:

```
from sage.matrix.constructor import random_echelonizable_matrix
```

Random diagonalizable matrices. The `algorithm='diagonalizable'` keyword, along with a requested matrix size (`size`) and optional lists of eigenvalues (`eigenvalues`) and the corresponding eigenspace dimensions (`dimensions`) will return a random diagonalizable matrix. When the eigenvalues and dimensions are not specified the result will have randomly generated values for both that fit with the designated size.

```
sage: A = random_matrix(QQ, 5, algorithm='diagonalizable', eigenvalues=[2,3,-1],␣
↪dimensions=[1,2,2]); A # random
sage: all(x in ZZ for x in (A-(2*identity_matrix(5))).rref().list())
True
sage: all(x in ZZ for x in (A-(3*identity_matrix(5))).rref().list())
True
sage: all(x in ZZ for x in (A-(-1*identity_matrix(5))).rref().list())
True
sage: A.jordan_form()
[ 2| 0| 0| 0| 0]
[--+--+--+--+--]
[ 0| 3| 0| 0| 0]
[--+--+--+--+--]
```

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```
[ 0| 0| 3| 0| 0]
[--+--+--+--+--]
[ 0| 0| 0|-1| 0]
[--+--+--+--+--]
[ 0| 0| 0| 0|-1]
```

For more, see the documentation of the `random_diagonalizable_matrix()` function. In the notebook or at the Sage command-line, first execute the following to make this further documentation available:

```
from sage.matrix.constructor import random_diagonalizable_matrix
```

Random matrices with predictable subspaces. The `algorithm='subspaces'` keyword, along with an optional rank (`rank`) will return a matrix whose natural basis vectors for its four fundamental subspaces, if computed as described in the documentation of the `random_subspaces_matrix()` contain only integer entries. If `rank`, is not set, the rank of the matrix will be generated randomly.

```
sage: B = random_matrix(QQ, 5, 6, algorithm='subspaces', rank=3); B #random
sage: B_expanded=B.augment(identity_matrix(5)).rref()
sage: (B.nrows(), B.ncols())
(5, 6)
sage: all(x in ZZ for x in B_expanded.list())
True
sage: C=B_expanded.submatrix(0,0,B.nrows()-B.nullity(),B.ncols())
sage: L=B_expanded.submatrix(B.nrows()-B.nullity(),B.ncols())
sage: B.right_kernel() == C.right_kernel()
True
sage: B.row_space() == C.row_space()
True
sage: B.column_space() == L.right_kernel()
True
sage: B.left_kernel() == L.row_space()
True
```

For more, see the documentation of the `random_subspaces_matrix()` function. In the notebook or at the Sage command-line, first execute the following to make this further documentation available:

```
from sage.matrix.constructor import random_subspaces_matrix
```

Random unimodular matrices. The `algorithm='unimodular'` keyword, along with an optional entry size control (`upper_bound`) will return a matrix of determinant 1. When the base ring is `ZZ` or `QQ` the result has integer entries, whose magnitudes can be limited by the value of `upper_bound`.

```
sage: C=random_matrix(QQ, 5, algorithm='unimodular', upper_bound=70); C # random
sage: det(C)
1
sage: C.base_ring()
Rational Field
sage: (C.nrows(), C.ncols())
(5, 5)
sage: all(abs(x)<70 for x in C.list())
True
```

For more, see the documentation of the `random_unimodular_matrix()` function. In the notebook or at the Sage command-line, first execute the following to make this further documentation available:

```
from sage.matrix.constructor import random_unimodular_matrix
```

AUTHOR:

- William Stein (2007-02-06)
- Rob Beezer (2010-08-25) Documentation, code to allow additional types of output

sage.matrix.special.**random_rref_matrix**(*parent*, *num_pivots*)

This function is available as random_rref_matrix(...) and matrix.random_rref(...).

Generate a matrix in reduced row-echelon form with a specified number of non-zero rows.

INPUT:

- `parent` – A matrix space specifying the base ring, dimensions and representation (dense/sparse) for the result. The base ring must be exact.
- `num_pivots` – The number of non-zero rows in the result, i.e. the rank.

OUTPUT:

A matrix in reduced row echelon form with `num_pivots` non-zero rows. If the base ring is ZZ or QQ then the entries are all integers.

Note: It is easiest to use this function via a call to the `random_matrix()` function with the `algorithm='echelon_form'` keyword. We provide one example accessing this function directly, while the remainder will use this more general function.

EXAMPLES:

Matrices generated are in reduced row-echelon form with specified rank. If the base ring is QQ the result has only integer entries.

```
sage: from sage.matrix.constructor import random_rref_matrix
sage: matrix_space = sage.matrix.matrix_space.MatrixSpace(QQ, 5, 6)
sage: A = random_rref_matrix(matrix_space, num_pivots=4); A # random
[ 1  0  0 -6  0 -3]
[ 0  1  0  2  0  3]
[ 0  0  1 -4  0 -2]
[ 0  0  0  0  1  3]
[ 0  0  0  0  0  0]
sage: A.base_ring()
Rational Field
sage: (A.nrows(), A.ncols())
(5, 6)
sage: A in sage.matrix.matrix_space.MatrixSpace(ZZ, 5, 6)
True
sage: A.rank()
4
sage: A == A.rref()
True
```

Matrices can be generated over other exact rings.

```
sage: B = random_matrix(FiniteField(7), 4, 4, algorithm='echelon_form', num_
↪pivots=3); B # random
```

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```
[1 0 0 0]
[0 1 0 6]
[0 0 1 4]
[0 0 0 0]
sage: B.rank() == 3
True
sage: B.base_ring()
Finite Field of size 7
sage: B == B.rref()
True
```

AUTHOR:

Billy Wonderly (2010-07)

sage.matrix.special.**random_subspaces_matrix**(*parent*, *rank=None*)

This function is available as random_subspaces_matrix(...) and matrix.random_subspaces(...).

Create a matrix of the designated size and rank whose right and left null spaces, column space, and row space have desirable properties that simplify the subspaces.

INPUT:

- `parent` - A matrix space specifying the base ring, dimensions, and representation (dense/sparse) for the result. The base ring must be exact.
- `rank` - The desired rank of the return matrix (default: None).

OUTPUT:

A matrix whose natural basis vectors for its four subspaces, when computed, have reasonably sized, integral valued, entries.

Note: It is easiest to use this function via a call to the `random_matrix()` function with the `algorithm='subspaces'` keyword. We provide one example accessing this function directly, while the remainder will use this more general function.

EXAMPLES:

A 6x8 matrix with designated rank of 3. The four subspaces are determined using one simple routine in which we augment the original matrix with the equal row dimension identity matrix. The resulting matrix is then put in reduced row-echelon form and the subspaces can then be determined by analyzing subdivisions of this matrix. See the four subspaces routine in [Bee] for more.

```
sage: from sage.matrix.constructor import random_subspaces_matrix
sage: matrix_space = sage.matrix.matrix_space.MatrixSpace(QQ, 6, 8)
sage: B = random_subspaces_matrix(matrix_space, rank=3)
sage: B.rank()
3
sage: B.nullity()
3
sage: (B.nrows(), B.ncols())
(6, 8)
sage: all(x in ZZ for x in B.list())
True
```

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```
sage: B_expanded = B.augment(identity_matrix(6)).rref()
sage: all(x in ZZ for x in B_expanded.list())
True
```

Check that we fixed github issue #10543 (echelon forms should be immutable):

```
sage: B_expanded.is_immutable()
True
```

We want to modify B_expanded, so replace it with a copy:

```
sage: B_expanded = copy(B_expanded)
sage: B_expanded.subdivide(B.nrows()-B.nullity(), B.ncols())
sage: C = B_expanded.subdivision(0, 0)
sage: L = B_expanded.subdivision(1, 1)
sage: B.right_kernel() == C.right_kernel()
True
sage: B.row_space() == C.row_space()
True
sage: B.column_space() == L.right_kernel()
True
sage: B.left_kernel() == L.row_space()
True
```

A matrix to show that the null space of the L matrix is the column space of the starting matrix.

```
sage: A = random_matrix(QQ, 5, 7, algorithm='subspaces', rank=None)
sage: (A.nrows(), A.ncols())
(5, 7)
sage: all(x in ZZ for x in A.list())
True
sage: A_expanded=A.augment(identity_matrix(5)).rref()
sage: all(x in ZZ for x in A_expanded.list())
True
sage: C = A_expanded.submatrix(0,0,A.nrows()-A.nullity(), A.ncols())
sage: L = A_expanded.submatrix(A.nrows()-A.nullity(), A.ncols())
sage: A.right_kernel() == C.right_kernel()
True
sage: A.row_space() == C.row_space()
True
sage: A.column_space() == L.right_kernel()
True
sage: A.left_kernel() == L.row_space()
True
```

AUTHOR:

Billy Wonderly (2010-07)

sage.matrix.special.**random_unimodular_matrix**(*parent*, *upper_bound=None*, *max_tries=100*)

This function is available as random_unimodular_matrix(...) and matrix.random_unimodular(...).

Generate a random unimodular (determinant 1) matrix of a desired size over a desired ring.

INPUT:

- `parent` - A matrix space specifying the base ring, dimensions and representation (dense/sparse) for the result. The base ring must be exact.
- `upper_bound` - For large matrices over QQ or ZZ, `upper_bound` is the largest value matrix entries can achieve. But see the warning below.
- `max_tries` - If designated, number of tries used to generate each new random row; only matters when upper_bound!=None. Used to prevent endless looping. (default: 100)

A matrix not in reduced row-echelon form with the desired dimensions and properties.

OUTPUT:

An invertible matrix with the desired properties and determinant 1.

Warning: When `upper_bound` is set, it is possible for this constructor to fail with a `ValueError`. This may happen when the `upper_bound`, `rank` and/or matrix dimensions are all so small that it becomes infeasible or unlikely to create the requested matrix. If you *must* have this routine return successfully, do not set `upper_bound`.

Note: It is easiest to use this function via a call to the `random_matrix()` function with the `algorithm='unimodular'` keyword. We provide one example accessing this function directly, while the remainder will use this more general function.

EXAMPLES:

A matrix size 5 over QQ.

```
sage: from sage.matrix.constructor import random_unimodular_matrix
sage: matrix_space = sage.matrix.matrix_space.MatrixSpace(QQ, 5)
sage: A = random_unimodular_matrix(matrix_space)
sage: det(A)
1
```

A matrix size 6 with entries no larger than 50.

```
sage: B = random_matrix(ZZ, 7, algorithm='unimodular', upper_bound=50)
sage: det(B)
1
sage: all(abs(b) < 50 for b in B.list())
True
```

A matrix over the number Field in y with defining polynomial $y^2 - 2y - 2$.

```
sage: y = var('y')
sage: K = NumberField(y^2-2*y-2, 'y')
sage: C = random_matrix(K, 3, algorithm='unimodular')
sage: det(C)
1
sage: C.base_ring() is K
True
```

AUTHOR:

Billy Wonderly (2010-07)

sage.matrix.special.**toeplitz**(*c*, *r*, *ring=None*)

This function is available as toeplitz(...) and matrix.toeplitz(...).

Return a Toeplitz matrix of given first column and first row.

In a Toeplitz matrix, each descending diagonal from left to right is constant, such that:

$$T_{i,j} = T_{i+1,j+1}.$$

For more information see the Wikipedia article Toeplitz_matrix.

INPUT:

- `c` – vector, first column of the Toeplitz matrix
- `r` – vector, first row of the Toeplitz matrix, counting from the second column
- `ring` – base ring (optional, default: None) of the resulting matrix

EXAMPLES:

A rectangular Toeplitz matrix:

```
sage: matrix.toeplitz([1..4], [5..6])
[1 5 6]
[2 1 5]
[3 2 1]
[4 3 2]
```

The following $N \times N$ Toeplitz matrix arises in the discretization of boundary value problems:

```
sage: N = 4
sage: matrix.toeplitz([-2, 1] + [0]*(N-2), [1] + [0]*(N-2))
[-2  1  0  0]
[ 1 -2  1  0]
[ 0  1 -2  1]
[ 0  0  1 -2]
```

sage.matrix.special.**vandermonde**(*v*, *ring=None*)

This function is available as vandermonde(...) and matrix.vandermonde(...).

Return a Vandermonde matrix of the given vector.

The n dimensional Vandermonde matrix is a square matrix with columns being the powers of a given vector v,

$$V_{ij} = v_i^{j-1}, \qquad i,j = 1,\ldots,n.$$

For more information see the Wikipedia article Vandermonde_matrix.

INPUT:

- `v` – vector, the second column of the Vandermonde matrix
- `ring` – base ring (optional, default: None) of the resulting matrix

EXAMPLES:

A Vandermonde matrix of order three over the symbolic ring:

```
sage: matrix.vandermonde(SR.var(['x0', 'x1', 'x2']))
[   1   x0 x0^2]
[   1   x1 x1^2]
[   1   x2 x2^2]
```

sage.matrix.special.**vector_on_axis_rotation_matrix**(*v*, *i*, *ring=None*)

This function is available as vector_on_axis_rotation_matrix(...) and matrix.vector_on_axis_rotation(...).

Return a rotation matrix M such that $det(M) = 1$ sending the vector v on the i-th axis so that all other coordinates of Mv are zero.

Note: Such a matrix is not uniquely determined. This function returns one such matrix.

INPUT:

- `v` – vector
- `i` – integer
- `ring` – ring (optional, default: `None`) of the resulting matrix

OUTPUT:

A matrix

EXAMPLES:

```
sage: from sage.matrix.constructor import vector_on_axis_rotation_matrix
sage: v = vector((1,2,3))
sage: vector_on_axis_rotation_matrix(v, 2) * v
(0, 0, sqrt(14))
sage: vector_on_axis_rotation_matrix(v, 1) * v
(0, sqrt(14), 0)
sage: vector_on_axis_rotation_matrix(v, 0) * v
(sqrt(14), 0, 0)
```

```
sage: x,y = var('x,y')
sage: v = vector((x,y))
sage: vector_on_axis_rotation_matrix(v, 1)
[ y/sqrt(x^2 + y^2) -x/sqrt(x^2 + y^2)]
[ x/sqrt(x^2 + y^2)  y/sqrt(x^2 + y^2)]
sage: vector_on_axis_rotation_matrix(v, 0)
[ x/sqrt(x^2 + y^2)  y/sqrt(x^2 + y^2)]
[-y/sqrt(x^2 + y^2)  x/sqrt(x^2 + y^2)]
sage: vector_on_axis_rotation_matrix(v, 0) * v
(x^2/sqrt(x^2 + y^2) + y^2/sqrt(x^2 + y^2), 0)
sage: vector_on_axis_rotation_matrix(v, 1) * v
(0, x^2/sqrt(x^2 + y^2) + y^2/sqrt(x^2 + y^2))
```

```
sage: v = vector((1,2,3,4))
sage: vector_on_axis_rotation_matrix(v, 0) * v
(sqrt(30), 0, 0, 0)
sage: vector_on_axis_rotation_matrix(v, 0, ring=RealField(10))
[ 0.18  0.37  0.55  0.73]
[-0.98 0.068  0.10  0.14]
[ 0.00 -0.93  0.22  0.30]
[ 0.00  0.00 -0.80  0.60]
sage: vector_on_axis_rotation_matrix(v, 0, ring=RealField(10)) * v
(5.5, 0.00..., 0.00..., 0.00...)
```

AUTHORS:

Sébastien Labbé (April 2010)

sage.matrix.special.**zero_matrix**(*ring*, *nrows=None*, *ncols=None*, *sparse=False*)

This function is available as zero_matrix(...) and matrix.zero(...).

Return the $nrows \times ncols$ zero matrix over the given ring.

The default ring is the integers.

EXAMPLES:

```
sage: M = zero_matrix(QQ, 2); M
[0 0]
[0 0]
sage: M.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: M = zero_matrix(2, 3); M
[0 0 0]
[0 0 0]
sage: M.parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
sage: M.is_mutable()
True
sage: M = zero_matrix(3, 1, sparse=True); M
[0]
[0]
[0]
sage: M.parent()
Full MatrixSpace of 3 by 1 sparse matrices over Integer Ring
sage: M.is_mutable()
True
sage: matrix.zero(5)
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
```

CHAPTER

FOUR

HELPERS FOR CREATING MATRICES

class sage.matrix.args.**MatrixArgs**

Bases: `object`

Collect arguments for constructing a matrix.

This class is meant to pass around arguments, for example from the global *matrix()* constructor to the matrix space or to the element class constructor.

A typical use case is first creating a `MatrixArgs` instance, possibly adjusting the attributes. This instance can then be passed around and a matrix can be constructed from it using the *matrix()* method. Also, a flat list can be constructed using *list()* or a sparse dict using *dict()*. It is safe to construct multiple objects (of the same or a different kind) from the same `MatrixArgs` instance.

`MatrixArgs` also supports iteration using the *iter()* method. This is a more low-level interface.

When `MatrixArgs` produces output, it is first *finalized*. This means that all missing attributes are derived or guessed. After finalization, you should no longer change the attributes or it will end up in an inconsistent state. You can also finalize explicitly by calling the *finalized()* method.

A `MatrixArgs` can contain invalid input. This is not checked when constructing the `MatrixArgs` instance, but it is checked either when finalizing or when constructing an object from it.

Warning: Instances of this class should only be temporary, they are not meant to be stored long-term.

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs
sage: ma = MatrixArgs(2, 2, (x for x in range(4))); ma
<MatrixArgs for None; typ=UNKNOWN; entries=<generator ...>>
sage: ma.finalized()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=SEQ_FLAT; entries=[0, 1, 2, 3]>
```

Many types of input are possible:

```
sage: ma = MatrixArgs(2, 2, entries=None); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=ZERO; entries=None>
[0 0]
[0 0]
sage: ma = MatrixArgs(2, 2, entries={}); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 sparse matrices over Integer Ring;␣
```

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```
↪typ=SEQ_SPARSE; entries=[]>
[0 0]
[0 0]
sage: ma = MatrixArgs(2, 2, entries=[1,2,3,4]); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=SEQ_FLAT; entries=[1, 2, 3, 4]>
[1 2]
[3 4]
sage: ma = MatrixArgs(2, 2, entries=math.pi); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Real Double Field;␣
↪typ=SCALAR; entries=3.141592653589793>
[3.141592653589793               0.0]
[              0.0 3.141592653589793]
sage: ma = MatrixArgs(2, 2, entries=pi); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Symbolic Ring;␣
↪typ=SCALAR; entries=pi>
[pi  0]
[ 0 pi]
sage: ma = MatrixArgs(ZZ, 2, 2, entries={(0,0):7}); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 sparse matrices over Integer Ring;␣
↪typ=SEQ_SPARSE; entries=[SparseEntry(0, 0, 7)]>
[7 0]
[0 0]
sage: ma = MatrixArgs(ZZ, 2, 2, entries=((1,2),(3,4))); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=SEQ_SEQ; entries=((1, 2), (3, 4))>
[1 2]
[3 4]
sage: ma = MatrixArgs(ZZ, 2, 2, entries=(1,2,3,4)); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=SEQ_FLAT; entries=(1, 2, 3, 4)>
[1 2]
[3 4]
sage: ma = MatrixArgs(QQ, entries=pari("[1,2;3,4]")); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Rational Field;␣
↪typ=SEQ_FLAT; entries=[1, 2, 3, 4]>
[1 2]
[3 4]
sage: ma = MatrixArgs(QQ, 2, 2, entries=pari("[1,2,3,4]")); ma.finalized(); ma.
↪matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Rational Field;␣
↪typ=SEQ_FLAT; entries=[1, 2, 3, 4]>
[1 2]
[3 4]
sage: ma = MatrixArgs(QQ, 2, 2, entries=pari("3/5")); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Rational Field;␣
↪typ=SCALAR; entries=3/5>
[3/5   0]
[  0 3/5]
sage: ma = MatrixArgs(entries=matrix(2,2)); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=MATRIX; entries=[0 0]
```

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```
[0 0]>
[0 0]
[0 0]
sage: ma = MatrixArgs(2, 2, entries=lambda i,j: 1+2*i+j); ma.finalized(); ma.
↪matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=SEQ_FLAT; entries=[1, 2, 3, 4]>
[1 2]
[3 4]
sage: ma = MatrixArgs(ZZ, 2, 2, entries=lambda i,j: 1+2*i+j); ma.finalized(); ma.
↪matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=CALLABLE; entries=<function ...>>
[1 2]
[3 4]
sage: from numpy import array
sage: ma = MatrixArgs(array([[1,2],[3,4]])); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Integer Ring;␣
↪typ=SEQ_SEQ; entries=array([[1, 2],
       [3, 4]])>
[1 2]
[3 4]
sage: ma = MatrixArgs(array([[1.,2.],[3.,4.]])); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Real Double Field;␣
↪typ=MATRIX; entries=[1.0 2.0]
[3.0 4.0]>
[1.0 2.0]
[3.0 4.0]
sage: ma = MatrixArgs(RealField(20), array([[1.,2.],[3.,4.]])); ma.finalized(); ma.
↪matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 dense matrices over Real Field with 20␣
↪bits of precision; typ=MATRIX; entries=[1.0 2.0]
[3.0 4.0]>
[1.0000 2.0000]
[3.0000 4.0000]
sage: ma = MatrixArgs(graphs.CycleGraph(3)); ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 3 by 3 dense matrices over Integer Ring;␣
↪typ=MATRIX; entries=[0 1 1]
[1 0 1]
[1 1 0]>
[0 1 1]
[1 0 1]
[1 1 0]

sage: ma = MatrixArgs([vector([0,1], sparse=True), vector([0,0], sparse=True)],␣
↪sparse=True)
sage: ma.finalized(); ma.matrix()
<MatrixArgs for Full MatrixSpace of 2 by 2 sparse matrices over Integer Ring;␣
↪typ=SEQ_SPARSE; entries=[SparseEntry(0, 1, 1)]>
[0 1]
[0 0]
```

Test invalid input:

```
sage: MatrixArgs(ZZ, 2, 2, entries="abcd").finalized()
Traceback (most recent call last):
...
TypeError: unable to convert 'abcd' to a matrix
sage: MatrixArgs(ZZ, 2, 2, entries=MatrixArgs()).finalized()
Traceback (most recent call last):
...
TypeError: unable to convert <MatrixArgs for None; typ=UNKNOWN; entries=None> to a␣
↪matrix
```

base

dict(*convert=True*)

Return the entries of the matrix as a dict. The keys of this dict are the non-zero positions `(i,j)`. The corresponding value is the entry at that position. Zero values are skipped.

If `convert` is True, the entries are converted to the base ring. Otherwise, the entries are returned as given.

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs
sage: L = list(range(6))
sage: MatrixArgs(2, 3, L).dict()
{(0, 1): 1, (0, 2): 2, (1, 0): 3, (1, 1): 4, (1, 2): 5}
```

```
sage: ma = MatrixArgs(GF(2), 2, 3, L)
sage: ma.dict(convert=False)
{(0, 1): 1, (0, 2): 2, (1, 0): 3, (1, 1): 4, (1, 2): 5}
sage: ma.dict()
{(0, 1): 1, (1, 0): 1, (1, 2): 1}
```

entries

finalized()

Determine all missing values.

Depending on the input, this might be a non-trivial operation. In some cases, this will do most of the work of constructing the matrix. That is actually not a problem since we eventually want to construct the matrix anyway. However, care is taken to avoid double work or needless checking or copying.

OUTPUT: `self`

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs
sage: MatrixArgs(pi).finalized()
Traceback (most recent call last):
...
TypeError: the dimensions of the matrix must be specified
sage: MatrixArgs(RR, pi).finalized()
Traceback (most recent call last):
...
TypeError: the dimensions of the matrix must be specified
sage: MatrixArgs(2, 3, 0.0).finalized()
<MatrixArgs for Full MatrixSpace of 2 by 3 dense matrices over Real Field with␣
```

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```
↪53 bits of precision; typ=ZERO; entries=0.000000000000000>
sage: MatrixArgs(RR, 2, 3, 1.0).finalized()
Traceback (most recent call last):
...
TypeError: nonzero scalar matrix must be square
```

Check github issue #19134:

```
sage: matrix(2, 3, [])
Traceback (most recent call last):
...
ValueError: sequence too short (expected length 6, got 0)
sage: matrix(ZZ, 2, 3, [])
Traceback (most recent call last):
...
ValueError: sequence too short (expected length 6, got 0)
sage: matrix(2, 3, [1])
Traceback (most recent call last):
...
ValueError: sequence too short (expected length 6, got 1)
```

iter(*convert=True*, *sparse=False*)

Iteration over the entries in the matrix

INPUT:

- `convert` – If `True`, the entries are converted to the base right. If `False`, the entries are returned as given.
- `sparse` – See OUTPUT below.

OUTPUT: iterator

- If `sparse` is False: yield all entries of the matrix in the following order:

```
[1 2 3]
[4 5 6]
```

- If `sparse` is True: yield instances of *SparseEntry*. The indices `(i, j)` are guaranteed to lie within the matrix. Zero entries in the input are *not* skipped.

Warning: If an iterator is given as input to *MatrixArgs*, it may be exhausted breaking any further usage. Otherwise, it is safe to iterate multiple times.

EXAMPLES:

```
sage: from sage.matrix.args import SparseEntry, MatrixArgs
sage: ma = MatrixArgs(ZZ, 2, 3, iter(range(6)))
sage: list(ma.iter())
[0, 1, 2, 3, 4, 5]
sage: ma = MatrixArgs(ZZ, 3, 3, [SparseEntry(0, 0, 0)])
sage: list(ma.iter())
Traceback (most recent call last):
```

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```
...
TypeError: dense iteration is not supported for sparse input
```

Sparse examples:

```
sage: ma = MatrixArgs(3, 3, pi)
sage: list(ma.iter(sparse=True))
[SparseEntry(0, 0, pi), SparseEntry(1, 1, pi), SparseEntry(2, 2, pi)]
sage: ma = MatrixArgs(2, 3)
sage: list(ma.iter(sparse=True))
[]
sage: ma = MatrixArgs(2, 2, lambda i, j: i > j)
sage: list(ma.iter(convert=False, sparse=True))
[SparseEntry(0, 0, False),
 SparseEntry(0, 1, False),
 SparseEntry(1, 0, True),
 SparseEntry(1, 1, False)]
sage: ma = MatrixArgs(2, 2, {(1,0):88, (0,1):89})
sage: sorted(tuple(x) for x in ma.iter(sparse=True))
[(0, 1, 89), (1, 0, 88)]
sage: ma = MatrixArgs(QQ, 2, 1, {(1,0):88, (0,1):89})
sage: ma.finalized()
Traceback (most recent call last):
...
IndexError: invalid column index 1
sage: ma = MatrixArgs(QQ, 1, 2, {(1,0):88, (0,1):89})
sage: ma.finalized()
Traceback (most recent call last):
...
IndexError: invalid row index 1
```

kwds

list(*convert=True*)

Return the entries of the matrix as a flat list of scalars.

If possible and `convert=False`, this returns a direct reference to `self.entries` without copying.

INPUT:

- `convert` – If True, the entries are converted to the base ring. Otherwise, the entries are returned as given.

Note: This changes `self.entries` to the returned list. This means that it is unsafe to access the `self.entries` attribute after calling this method.

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs
sage: L = list(range(6))
sage: MatrixArgs(2, 3, L).list()
[0, 1, 2, 3, 4, 5]
```

```
sage: ma = MatrixArgs(RDF, 2, 3, L)
sage: ma.list(convert=False)
[0, 1, 2, 3, 4, 5]
sage: ma.list()
[0.0, 1.0, 2.0, 3.0, 4.0, 5.0]
```

If we remove our reference to the input `L` and `convert=False`, no copy is made:

```
sage: idL = id(L)
sage: ma = MatrixArgs(2, 3, L)
sage: del L
sage: L = ma.list(convert=False)
sage: id(L) == idL
True
```

matrix(*convert=True*)

Return the entries of the matrix as a Sage Matrix.

If possible, this returns a direct reference to `self.entries` without copying.

INPUT:

- `convert` – if True, the matrix is guaranteed to have the correct parent matrix space. If False, the input matrix may be returned even if it lies in the wrong space.

Note: This changes `self.entries` to the returned matrix. This means that it is unsafe to access the `self.entries` attribute after calling this method.

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs
sage: M = matrix(2, 3, range(6), sparse=True)
```

```
sage: ma = MatrixArgs(M); ma.finalized()
<MatrixArgs for Full MatrixSpace of 2 by 3 sparse matrices over Integer Ring;␣
↪typ=MATRIX; entries=[0 1 2]
[3 4 5]>
sage: ma.matrix()
[0 1 2]
[3 4 5]
```

```
sage: ma = MatrixArgs(M, sparse=False); ma.finalized()
<MatrixArgs for Full MatrixSpace of 2 by 3 dense matrices over Integer Ring;␣
↪typ=MATRIX; entries=[0 1 2]
[3 4 5]>
sage: ma.matrix()
[0 1 2]
[3 4 5]
```

```
sage: ma = MatrixArgs(RDF, M); ma.finalized()
<MatrixArgs for Full MatrixSpace of 2 by 3 sparse matrices over Real Double␣
↪Field; typ=MATRIX; entries=[0 1 2]
```

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```
[3 4 5]>
sage: ma.matrix(convert=False)
[0 1 2]
[3 4 5]
sage: ma.matrix()
[0.0 1.0 2.0]
[3.0 4.0 5.0]
```

If we remove our reference to the input `M`, no copy is made:

```
sage: idM = id(M)
sage: ma = MatrixArgs(M)
sage: del M
sage: M = ma.matrix()
sage: id(M) == idM
True
```

Immutable matrices are never copied:

```
sage: M.set_immutable()
sage: matrix(M) is M
True
```

ncols

nrows

set_space(*space*)

Set inputs from a given matrix space.

INPUT:

- `space` – a *MatrixSpace*

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs
sage: ma = MatrixArgs()
sage: S = MatrixSpace(QQ, 3, 2, sparse=True)
sage: _ = ma.set_space(S)
sage: ma.finalized()
<MatrixArgs for Full MatrixSpace of 3 by 2 sparse matrices over Rational Field;␣
↪typ=ZERO; entries=None>
sage: M = ma.matrix(); M
[0 0]
[0 0]
[0 0]
sage: M.parent() is S
True
```

space

sparse

sage.matrix.args.**MatrixArgs_init**(*space*, *entries*)

Construct a *MatrixArgs* object from a matrix space and entries. This is the typical use in a matrix constructor.

If the given entries is already an instance of *MatrixArgs*, then just set the space and return the same object.

EXAMPLES:

```
sage: from sage.matrix.args import MatrixArgs_init
sage: S = MatrixSpace(GF(2), 2, 4)
sage: ma = MatrixArgs_init(S, {(1,3):7})
sage: M = ma.matrix(); M
[0 0 0 0]
[0 0 0 1]
sage: parent(M) is S
True
```

class sage.matrix.args.**SparseEntry**

Bases: `object`

Specialized class for dealing with sparse input in *MatrixArgs*. An instance of `SparseEntry` represents one position in a matrix to be constructed. To construct a sparse matrix, one would typically make a list of such.

Previous versions of Sage used a `dict` as data structure for sparse input, but that is not so suitable because the keys are not guaranteed to be of the correct format. There is also the performance cost of creating tuples of Python integers.

Users of this class are expected to know what they are doing, so the indices are not checked when constructing a matrix.

INPUT:

- `i`, `j` – row and column index
- `entry` – value to be put at position (i, j)

EXAMPLES:

```
sage: from sage.matrix.args import SparseEntry
sage: SparseEntry(123, 456, "abc")
SparseEntry(123, 456, 'abc')
sage: SparseEntry(1/3, 2/3, x)
Traceback (most recent call last):
...
TypeError: unable to convert rational 1/3 to an integer
```

entry

i

j

CHAPTER

FIVE

MATRICES OVER AN ARBITRARY RING

AUTHORS:

- William Stein
- Martin Albrecht: conversion to Pyrex
- Jaap Spies: various functions
- Gary Zablackis: fixed a sign bug in generic determinant.
- William Stein and Robert Bradshaw - complete restructuring.
- Rob Beezer - refactor kernel functions.

Elements of matrix spaces are of class `Matrix` (or a class derived from Matrix). They can be either sparse or dense, and can be defined over any base ring.

EXAMPLES:

We create the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

as an element of a matrix space over $\mathbf{Q}$:

```
sage: M = MatrixSpace(QQ,2,3)
sage: A = M([1,2,3, 4,5,6]); A
[1 2 3]
[4 5 6]
sage: A.parent()
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

Alternatively, we could create A more directly as follows (which would completely avoid having to create the matrix space):

```
sage: A = matrix(QQ, 2, [1,2,3, 4,5,6]); A
[1 2 3]
[4 5 6]
```

We next change the top-right entry of A. Note that matrix indexing is 0-based in Sage, so the top right entry is $(0, 2)$, which should be thought of as "row number 0, column number 2".

```
sage: A[0,2] = 389
sage: A
[  1   2 389]
[  4   5   6]
```

Also notice how matrices print. All columns have the same width and entries in a given column are right justified. Next we compute the reduced row echelon form of A.

```
sage: A.rref()
[      1       0 -1933/3]
[      0       1  1550/3]
```

5.1 Indexing

Sage has quite flexible ways of extracting elements or submatrices from a matrix:

```
sage: m=[(1, -2, -1, -1,9), (1, 8, 6, 2,2), (1, 1, -1, 1,4), (-1, 2, -2, -1,4)] ; M =␣
↪matrix(m)
sage: M
[ 1 -2 -1 -1  9]
[ 1  8  6  2  2]
[ 1  1 -1  1  4]
[-1  2 -2 -1  4]
```

Get the 2 x 2 submatrix of M, starting at row index and column index 1:

```
sage: M[1:3,1:3]
[ 8  6]
[ 1 -1]
```

Get the 2 x 3 submatrix of M starting at row index and column index 1:

```
sage: M[1:3,[1..3]]
[ 8  6  2]
[ 1 -1  1]
```

Get the second column of M:

```
sage: M[:,1]
[-2]
[ 8]
[ 1]
[ 2]
```

Get the first row of M:

```
sage: M[0,:]
[ 1 -2 -1 -1  9]
```

Get the last row of M (negative numbers count from the end):

```
sage: M[-1,:]
[-1  2 -2 -1  4]
```

More examples:

```
sage: M[range(2),:]
[ 1 -2 -1 -1  9]
[ 1  8  6  2  2]
sage: M[range(2),4]
[9]
[2]
sage: M[range(3),range(5)]
[ 1 -2 -1 -1  9]
[ 1  8  6  2  2]
[ 1  1 -1  1  4]

sage: M[3,range(5)]
[-1  2 -2 -1  4]
sage: M[3,:]
[-1  2 -2 -1  4]
sage: M[3,4]
4

sage: M[-1,:]
[-1  2 -2 -1  4]

sage: A = matrix(ZZ,3,4, [3, 2, -5, 0, 1, -1, 1, -4, 1, 0, 1, -3]); A
[ 3  2 -5  0]
[ 1 -1  1 -4]
[ 1  0  1 -3]
```

A series of three numbers, separated by colons, like `n:m:s`, means numbers from `n` up to (but not including) `m`, in steps of `s`. So `0:5:2` means the sequence `[0,2,4]`:

```
sage: A[:,0:4:2]
[ 3 -5]
[ 1  1]
[ 1  1]

sage: A[1:,0:4:2]
[1 1]
[1 1]

sage: A[2::-1,:]
[ 1  0  1 -3]
[ 1 -1  1 -4]
[ 3  2 -5  0]

sage: A[1:,3::-1]
[-4  1 -1  1]
[-3  1  0  1]

sage: A[1:,3::-2]
[-4 -1]
[-3  0]

sage: A[2::-1,3:1:-1]
[-3  1]
```

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```
[-4  1]
[ 0 -5]
```

We can also change submatrices using these indexing features:

```
sage: M=matrix([(1, -2, -1, -1,9), (1, 8, 6, 2,2), (1, 1, -1, 1,4), (-1, 2, -2, -1,4)]);␣
↪M
[ 1 -2 -1 -1  9]
[ 1  8  6  2  2]
[ 1  1 -1  1  4]
[-1  2 -2 -1  4]
```

Set the 2 x 2 submatrix of M, starting at row index and column index 1:

```
sage: M[1:3,1:3] = [[1,0],[0,1]]; M
[ 1 -2 -1 -1  9]
[ 1  1  0  2  2]
[ 1  0  1  1  4]
[-1  2 -2 -1  4]
```

Set the 2 x 3 submatrix of M starting at row index and column index 1:

```
sage: M[1:3,[1..3]] = M[2:4,0:3]; M
[ 1 -2 -1 -1  9]
[ 1  1  0  1  2]
[ 1 -1  2 -2  4]
[-1  2 -2 -1  4]
```

Set part of the first column of M:

```
sage: M[1:,0]=[[2],[3],[4]]; M
[ 1 -2 -1 -1  9]
[ 2  1  0  1  2]
[ 3 -1  2 -2  4]
[ 4  2 -2 -1  4]
```

Or do a similar thing with a vector:

```
sage: M[1:,0]=vector([-2,-3,-4]); M
[ 1 -2 -1 -1  9]
[-2  1  0  1  2]
[-3 -1  2 -2  4]
[-4  2 -2 -1  4]
```

Or a constant:

```
sage: M[1:,0]=30; M
[ 1 -2 -1 -1  9]
[30  1  0  1  2]
[30 -1  2 -2  4]
[30  2 -2 -1  4]
```

Set the first row of M:

```
sage: M[0,:]=[[20,21,22,23,24]]; M
[20 21 22 23 24]
[30  1  0  1  2]
[30 -1  2 -2  4]
[30  2 -2 -1  4]
sage: M[0,:]=vector([0,1,2,3,4]); M
[ 0  1  2  3  4]
[30  1  0  1  2]
[30 -1  2 -2  4]
[30  2 -2 -1  4]
sage: M[0,:]=-3; M
[-3 -3 -3 -3 -3]
[30  1  0  1  2]
[30 -1  2 -2  4]
[30  2 -2 -1  4]


sage: A = matrix(ZZ,3,4, [3, 2, -5, 0, 1, -1, 1, -4, 1, 0, 1, -3]); A
[ 3  2 -5  0]
[ 1 -1  1 -4]
[ 1  0  1 -3]
```

We can use the step feature of slices to set every other column:

```
sage: A[:,0:3:2] = 5; A
[ 5  2  5  0]
[ 5 -1  5 -4]
[ 5  0  5 -3]

sage: A[1:,0:4:2] = [[100,200],[300,400]]; A
[  5   2   5   0]
[100  -1 200  -4]
[300   0 400  -3]
```

We can also count backwards to flip the matrix upside down:

```
sage: A[::-1,:]=A; A
[300   0 400  -3]
[100  -1 200  -4]
[  5   2   5   0]


sage: A[1:,3::-1]=[[2,3,0,1],[9,8,7,6]]; A
[300   0 400  -3]
[  1   0   3   2]
[  6   7   8   9]

sage: A[1:,::-2] = A[1:,::2]; A
[300   0 400  -3]
[  1   3   3   1]
[  6   8   8   6]

sage: A[::-1,3:1:-1] = [[4,3],[1,2],[-1,-2]]; A
```

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```
[300   0  -2  -1]
[  1   3   2   1]
[  6   8   3   4]
```

We save and load a matrix:

```
sage: A = matrix(Integers(8),3,range(9))
sage: loads(dumps(A)) == A
True
```

MUTABILITY: Matrices are either immutable or not. When initially created, matrices are typically mutable, so one can change their entries. Once a matrix A is made immutable using `A.set_immutable()` the entries of A cannot be changed, and A can never be made mutable again. However, properties of A such as its rank, characteristic polynomial, etc., are all cached so computations involving A may be more efficient. Once A is made immutable it cannot be changed back. However, one can obtain a mutable copy of A using `copy(A)`.

EXAMPLES:

```
sage: A = matrix(RR,2,[1,10,3.5,2])
sage: A.set_immutable()
sage: copy(A) is A
False
```

The echelon form method always returns immutable matrices with known rank.

EXAMPLES:

```
sage: A = matrix(Integers(8),3,range(9))
sage: A.determinant()
0
sage: A[0,0] = 5
sage: A.determinant()
1
sage: A.set_immutable()
sage: A[0,0] = 5
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use copy(M) to␣
↪change a copy of M).
```

5.1.1 Implementation and Design

Class Diagram (an x means that class is currently supported):

```
x Matrix
x   Matrix_sparse
x     Matrix_generic_sparse
x     Matrix_integer_sparse
x     Matrix_rational_sparse
      Matrix_cyclo_sparse
x     Matrix_modn_sparse
      Matrix_RR_sparse
```

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```
      Matrix_CC_sparse
      Matrix_RDF_sparse
      Matrix_CDF_sparse

x  Matrix_dense
x     Matrix_generic_dense
x     Matrix_integer_dense
x     Matrix_rational_dense
      Matrix_cyclo_dense    -- idea: restrict scalars to QQ, compute charpoly there,␣
↪then factor
x     Matrix_modn_dense
      Matrix_RR_dense
      Matrix_CC_dense
x     Matrix_real_double_dense
x     Matrix_complex_double_dense
x     Matrix_complex_ball_dense
```

The corresponding files in the sage/matrix library code directory are named

```
[matrix] [base ring] [dense or sparse].
```

```
New matrices types can only be implemented in Cython.

********** LEVEL 1  **********
NON-OPTIONAL
For each base field it is *absolutely* essential to completely
implement the following functionality for that base ring:

   * __cinit__     -- should use check_allocarray from cysignals.memory
                      (only needed if allocate memory)
   * __init__      -- this signature: 'def __init__(self, parent, entries, copy, coerce)'
   * __dealloc__   -- use sig_free (only needed if allocate memory)
   * set_unsafe(self, size_t i, size_t j, x) -- doesn't do bounds or any other checks;␣
↪assumes x is in self._base_ring
   * get_unsafe(self, size_t i, size_t j) -- doesn't do checks
   * __richcmp__    -- always the same (I don't know why its needed -- bug in PYREX).

Note that the __init__ function must construct the all zero matrix if ``entries ==␣
↪None``.

********** LEVEL 2  **********

IMPORTANT (and *highly* recommended):

After getting the special class with all level 1 functionality to
work, implement all of the following (they should not change
functionality, except speed (always faster!) in any way):

   * def _pickle(self):
         return data, version
   * def _unpickle(self, data, int version)
         reconstruct matrix from given data and version; may assume _parent, _nrows,␣
```

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```
↪and _ncols are set.
         Use version numbers >= 0 so if you change the pickle strategy then
         old objects still unpickle.
   * cdef _list -- list of underlying elements (need not be a copy)
   * cdef _dict -- sparse dictionary of underlying elements
   * cdef _add_ -- add two matrices with identical parents
   * _matrix_times_matrix_c_impl -- multiply two matrices with compatible dimensions and
                                    identical base rings (both sparse or both dense)
   * cpdef _richcmp_ -- compare two matrices with identical parents
   * cdef _lmul_c_impl -- multiply this matrix on the right by a scalar, i.e., self *␣
↪scalar
   * cdef _rmul_c_impl -- multiply this matrix on the left by a scalar, i.e., scalar *␣
↪self
   * __copy__
   * __neg__

The list and dict returned by _list and _dict will *not* be changed
by any internal algorithms and are not accessible to the user.

********** LEVEL 3  **********
OPTIONAL:

   * cdef _sub_
   * __invert__
   * _multiply_classical
   * __deepcopy__

Further special support:
   * Matrix windows -- to support Strassen multiplication for a given base ring.
   * Other functions, e.g., transpose, for which knowing the
     specific representation can be helpful.

.. note::

   - For caching, use self.fetch and self.cache.

   - Any method that can change the matrix should call
     ``check_mutability()`` first.  There are also many fast cdef'd bounds checking␣
↪methods.

   - Kernels of matrices
     Implement only a left_kernel() or right_kernel() method, whichever requires
     the least overhead (usually meaning little or no transposing).  Let the
     methods in the matrix2 class handle left, right, generic kernel distinctions.
```

CHAPTER

SIX

BASE CLASS FOR MATRICES, PART 0

Note: For design documentation see matrix/docs.py.

EXAMPLES:

```
sage: matrix(2,[1,2,3,4])
[1 2]
[3 4]
```

class sage.matrix.matrix0.**Matrix**

Bases: Matrix

A generic matrix.

The Matrix class is the base class for all matrix classes. To create a Matrix, first create a MatrixSpace, then coerce a list of elements into the MatrixSpace. See the documentation of MatrixSpace for more details.

EXAMPLES:

We illustrate matrices and matrix spaces. Note that no actual matrix that you make should have class Matrix; the class should always be derived from Matrix.

```
sage: M = MatrixSpace(CDF,2,3); M
Full MatrixSpace of 2 by 3 dense matrices over Complex Double Field
sage: a = M([1,2,3,  4,5,6]); a
[1.0 2.0 3.0]
[4.0 5.0 6.0]
sage: type(a)
<class 'sage.matrix.matrix_complex_double_dense.Matrix_complex_double_dense'>
sage: parent(a)
Full MatrixSpace of 2 by 3 dense matrices over Complex Double Field
```

```
sage: matrix(CDF, 2,3, [1,2,3, 4,5,6])
[1.0 2.0 3.0]
[4.0 5.0 6.0]
sage: Mat(CDF,2,3)(range(1,7))
[1.0 2.0 3.0]
[4.0 5.0 6.0]
```

```
sage: Q.<i,j,k> = QuaternionAlgebra(QQ, -1,-1)
sage: matrix(Q,2,1,[1,2])
```

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```
[1]
[2]
```

act_on_polynomial(*f*)

Return the polynomial f(self*x).

INPUT:

- `self` - an nxn matrix
- `f` - a polynomial in n variables x=(x1,...,xn)

OUTPUT: The polynomial f(self*x).

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: x, y = R.gens()
sage: f = x**2 - y**2
sage: M = MatrixSpace(QQ, 2)
sage: A = M([1,2,3,4])
sage: A.act_on_polynomial(f)
-8*x^2 - 20*x*y - 12*y^2
```

add_multiple_of_column(*i*, *j*, *s*, *start_row=0*)

Add s times column j to column i.

EXAMPLES: We add -1 times the third column to the second column of an integer matrix, remembering to start numbering cols at zero:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.add_multiple_of_column(1,2,-1)
sage: a
[ 0 -1  2]
[ 3 -1  5]
```

To add a rational multiple, we first need to change the base ring:

```
sage: a = a.change_ring(QQ)
sage: a.add_multiple_of_column(1,0,1/3)
sage: a
[ 0 -1  2]
[ 3  0  5]
```

If not, we get an error message:

```
sage: a.add_multiple_of_column(1,0,SR.I())
Traceback (most recent call last):
...
TypeError: Multiplying column by Symbolic Ring element cannot be done over␣
↪Rational Field, use change_ring or with_added_multiple_of_column instead.
```

add_multiple_of_row(*i*, *j*, *s*, *start_col=0*)

Add s times row j to row i.

EXAMPLES: We add -3 times the first row to the second row of an integer matrix, remembering to start numbering rows at zero:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.add_multiple_of_row(1,0,-3)
sage: a
[ 0  1  2]
[ 3  1 -1]
```

To add a rational multiple, we first need to change the base ring:

```
sage: a = a.change_ring(QQ)
sage: a.add_multiple_of_row(1,0,1/3)
sage: a
[   0    1    2]
[   3  4/3 -1/3]
```

If not, we get an error message:

```
sage: a.add_multiple_of_row(1,0,SR.I())
Traceback (most recent call last):
...
TypeError: Multiplying row by Symbolic Ring element cannot be done over␣
↪Rational Field, use change_ring or with_added_multiple_of_row instead.
```

add_to_entry(*i*, *j*, *elt*)

Add `elt` to the entry at position `(i, j)`.

EXAMPLES:

```
sage: m = matrix(QQ['x,y'], 2, 2)
sage: m.add_to_entry(0, 1, 2)
sage: m
[0 2]
[0 0]
```

anticommutator(*other*)

Return the anticommutator `self` and `other`.

The *anticommutator* of two $n \times n$ matrices A and B is defined as $\{A, B\} := AB + BA$ (sometimes this is written as $[A, B]_+$).

EXAMPLES:

```
sage: A = Matrix(ZZ, 2, 2, range(4))
sage: B = Matrix(ZZ, 2, 2, [0, 1, 0, 0])
sage: A.anticommutator(B)
[2 3]
[0 2]
sage: A.anticommutator(B) == B.anticommutator(A)
True
sage: A.commutator(B) + B.anticommutator(A) == 2*A*B
True
```

base_ring()

Return the base ring of the matrix.

EXAMPLES:

```
sage: m=matrix(QQ,2,[1,2,3,4])
sage: m.base_ring()
Rational Field
```

change_ring(*ring*)

Return the matrix obtained by coercing the entries of this matrix into the given ring.

Always returns a copy (unless self is immutable, in which case returns self).

EXAMPLES:

```
sage: A = Matrix(QQ, 2, 2, [1/2, 1/3, 1/3, 1/4])
sage: A.parent()
 Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: A.change_ring(GF(25,'a'))
[3 2]
[2 4]
sage: A.change_ring(GF(25,'a')).parent()
 Full MatrixSpace of 2 by 2 dense matrices over Finite Field in a of size 5^2
sage: A.change_ring(ZZ)
Traceback (most recent call last):
...
TypeError: matrix has denominators so can...t change to ZZ
```

Changing rings preserves subdivisions:

```
sage: A.subdivide([1], []); A
[1/2 1/3]
[-------]
[1/3 1/4]
sage: A.change_ring(GF(25,'a'))
[3 2]
[---]
[2 4]
```

commutator(*other*)

Return the commutator self*other - other*self.

EXAMPLES:

```
sage: A = Matrix(ZZ, 2, 2, range(4))
sage: B = Matrix(ZZ, 2, 2, [0, 1, 0, 0])
sage: A.commutator(B)
[-2 -3]
[ 0  2]
sage: A.commutator(B) == -B.commutator(A)
True
```

dict(*copy=True*)

Dictionary of the elements of `self` with keys pairs `(i,j)` and values the nonzero entries of `self`.

INPUT:

- copy – (default: True) make a copy of the dict corresponding to self

If copy=True, then is safe to change the returned dictionary. Otherwise, this can cause undesired behavior by mutating the dict.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: a = matrix(R,2,[x,y,0, 0,0,2*x+y]); a
[      x       y       0]
[      0       0 2*x + y]
sage: d = a.dict(); d
{(0, 0): x, (0, 1): y, (1, 2): 2*x + y}
```

Notice that changing the returned list does not change a (the list is a copy):

```
sage: d[0,0] = 25
sage: a
[      x       y       0]
[      0       0 2*x + y]
```

dimensions()

Return the dimensions of this matrix as the tuple (nrows, ncols).

EXAMPLES:

```
sage: M = matrix([[1,2,3],[4,5,6]])
sage: N = M.transpose()
sage: M.dimensions()
(2, 3)
sage: N.dimensions()
(3, 2)
```

AUTHORS:

- Benjamin Lundell (2012-02-09): examples

inverse_of_unit(*algorithm=None*)

Return the inverse of this matrix in the same matrix space.

The matrix must be invertible on the base ring. Otherwise, an ArithmeticError is raised.

The computation goes through the matrix of cofactors and avoids division. In particular the base ring does not need to have a fraction field.

INPUT:

- algorithm – (default: None) either None or "df" (for division free)

EXAMPLES:

```
sage: R.<a,b,c,d> = ZZ[]
sage: RR = R.quotient(a*d-b*c-1)
sage: a,b,c,d = RR.gens()
sage: m = matrix(2, [a,b,c,d])
sage: n = m.inverse_of_unit()
sage: m * n
[1 0]
```

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```
[0 1]

sage: matrix(RR, 2, 1, [a,b]).inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: self must be a square matrix
sage: matrix(RR, 1, 1, [2]).inverse_of_unit()
Traceback (most recent call last):
...
NotImplementedError: Lifting of multivariate polynomials over non-fields is not␣
↪implemented.

sage: R = ZZ.cartesian_product(ZZ)
sage: m = matrix(R, 2, [R((2,1)), R((1,1)), R((1,1)), R((1,2))])
sage: m * m.inverse_of_unit()
[(1, 1) (0, 0)]
[(0, 0) (1, 1)]
```

Tests for github issue #28570:

```
sage: P = posets.TamariLattice(7)
sage: M = P._hasse_diagram._leq_matrix
sage: M.inverse_of_unit()   # this was very slow, now 1s
429 x 429 sparse matrix over Integer Ring...

sage: m = matrix(Zmod(2**2), 1, 1, [1], sparse=True)
sage: mi = ~m; mi
[1]
sage: mi.parent()
Full MatrixSpace of 1 by 1 sparse matrices over Ring of integers modulo 4
```

is_alternating()

Return `True` if `self` is an alternating matrix.

Here, "alternating matrix" means a square matrix A satisfying $A^T = -A$ and such that the diagonal entries of A are 0. Notice that the condition that the diagonal entries be 0 is not redundant for matrices over arbitrary ground rings (but it is redundant when 2 is invertible in the ground ring). A square matrix A only required to satisfy $A^T = -A$ is said to be "skew-symmetric", and this property is checked by the *is_skew_symmetric()* method.

EXAMPLES:

```
sage: m = matrix(QQ, [[0,2], [-2,0]])
sage: m.is_alternating()
True
sage: m = matrix(QQ, [[1,2], [2,1]])
sage: m.is_alternating()
False
```

In contrast to the property of being skew-symmetric, the property of being alternating does not tolerate nonzero entries on the diagonal even if they are their own negatives:

```
sage: n = matrix(Zmod(4), [[0, 1], [-1, 2]])
sage: n.is_alternating()
False
```

is_dense()

Return True if this is a dense matrix.

In Sage, being dense is a property of the underlying representation, not the number of nonzero entries.

EXAMPLES:

```
sage: matrix(QQ,2,2,range(4)).is_dense()
True
sage: matrix(QQ,2,2,range(4),sparse=True).is_dense()
False
```

is_hermitian()

Return True if the matrix is equal to its conjugate-transpose.

OUTPUT:

True if the matrix is square and equal to the transpose with every entry conjugated, and False otherwise.

Note that if conjugation has no effect on elements of the base ring (such as for integers), then the *is_symmetric()* method is equivalent and faster.

This routine is for matrices over exact rings and so may not work properly for matrices over RR or CC. For matrices with approximate entries, the rings of double-precision floating-point numbers, RDF and CDF, are a better choice since the *sage.matrix.matrix_double_dense.Matrix_double_dense.is_hermitian()* method has a tolerance parameter. This provides control over allowing for minor discrepancies between entries when checking equality.

The result is cached.

EXAMPLES:

```
sage: A = matrix(QQbar, [[ 1 + I,  1 - 6*I, -1 - I],
....:                    [-3 - I,     -4*I,     -2],
....:                    [-1 + I, -2 - 8*I,  2 + I]])
sage: A.is_hermitian()
False
sage: B = A*A.conjugate_transpose()
sage: B.is_hermitian()
True
```

Sage has several fields besides the entire complex numbers where conjugation is non-trivial.

```
sage: F.<b> = QuadraticField(-7)
sage: C = matrix(F, [[-2*b - 3,  7*b - 6, -b + 3],
....:                [-2*b - 3, -3*b + 2,   -2*b],
....:                [   b + 1,        0,     -2]])
sage: C.is_hermitian()
False
sage: C = C*C.conjugate_transpose()
sage: C.is_hermitian()
True
```

A matrix that is nearly Hermitian, but for a non-real diagonal entry.

```
sage: A = matrix(QQbar, [[    2,   2-I, 1+4*I],
....:                    [  2+I,   3+I, 2-6*I],
....:                    [1-4*I, 2+6*I,     5]])
sage: A.is_hermitian()
False
sage: A[1,1] = 132
sage: A.is_hermitian()
True
```

Rectangular matrices are never Hermitian.

```
sage: A = matrix(QQbar, 3, 4)
sage: A.is_hermitian()
False
```

A square, empty matrix is trivially Hermitian.

```
sage: A = matrix(QQ, 0, 0)
sage: A.is_hermitian()
True
```

is_immutable()

Return True if this matrix is immutable.

See the documentation for self.set_immutable for more details about mutability.

EXAMPLES:

```
sage: A = Matrix(QQ['t','s'], 2, 2, range(4))
sage: A.is_immutable()
False
sage: A.set_immutable()
sage: A.is_immutable()
True
```

is_invertible()

Return True if this matrix is invertible.

EXAMPLES: The following matrix is invertible over $\mathbf{Q}$ but not over $\mathbf{Z}$.

```
sage: A = MatrixSpace(ZZ, 2)(range(4))
sage: A.is_invertible()
False
sage: A.matrix_over_field().is_invertible()
True
```

The inverse function is a constructor for matrices over the fraction field, so it can work even if A is not invertible.

```
sage: ~A   # inverse of A
[-3/2  1/2]
[   1    0]
```

The next matrix is invertible over $\mathbf{Z}$.

```
sage: A = MatrixSpace(IntegerRing(),2)([1,10,0,-1])
sage: A.is_invertible()
True
sage: ~A                # compute the inverse
[ 1 10]
[ 0 -1]
```

The following nontrivial matrix is invertible over $\mathbf{Z}[x]$.

```
sage: R.<x> = PolynomialRing(IntegerRing())
sage: A = MatrixSpace(R,2)([1,x,0,-1])
sage: A.is_invertible()
True
sage: ~A
[ 1  x]
[ 0 -1]
```

is_mutable()

Return True if this matrix is mutable.

See the documentation for self.set_immutable for more details about mutability.

EXAMPLES:

```
sage: A = Matrix(QQ['t','s'], 2, 2, range(4))
sage: A.is_mutable()
True
sage: A.set_immutable()
sage: A.is_mutable()
False
```

is_singular()

Return `True` if `self` is singular.

OUTPUT:

A square matrix is singular if it has a zero determinant and this method will return `True` in exactly this case. When the entries of the matrix come from a field, this is equivalent to having a nontrivial kernel, or lacking an inverse, or having linearly dependent rows, or having linearly dependent columns.

For square matrices over a field the methods *is_invertible()* and *is_singular()* are logical opposites. However, it is an error to apply *is_singular()* to a matrix that is not square, while *is_invertible()* will always return `False` for a matrix that is not square.

EXAMPLES:

A singular matrix over the field `QQ`.

```
sage: A = matrix(QQ, 4, [-1,2,-3,6,0,-1,-1,0,-1,1,-5,7,-1,6,5,2])
sage: A.is_singular()
True
sage: A.right_kernel().dimension()
1
```

A matrix that is not singular, i.e. nonsingular, over a field.

```
sage: B = matrix(QQ, 4, [1,-3,-1,-5,2,-5,-2,-7,-2,5,3,4,-1,4,2,6])
sage: B.is_singular()
False
sage: B.left_kernel().dimension()
0
```

For *rectangular* matrices, invertibility is always `False`, but asking about singularity will give an error.

```
sage: C = matrix(QQ, 5, range(30))
sage: C.is_invertible()
False
sage: C.is_singular()
Traceback (most recent call last):
...
ValueError: self must be a square matrix
```

When the base ring is not a field, then a matrix may be both not invertible and not singular.

```
sage: D = matrix(ZZ, 4, [2,0,-4,8,2,1,-2,7,2,5,7,0,0,1,4,-6])
sage: D.is_invertible()
False
sage: D.is_singular()
False
sage: d = D.determinant(); d
2
sage: d.is_unit()
False
```

is_skew_hermitian()

Return `True` if the matrix is equal to the negative of its conjugate transpose.

OUTPUT:

`True` if the matrix is square and equal to the negative of its conjugate transpose, and `False` otherwise.

Note that if conjugation has no effect on elements of the base ring (such as for integers), then the *is_skew_symmetric()* method is equivalent and faster.

This routine is for matrices over exact rings and so may not work properly for matrices over `RR` or `CC`. For matrices with approximate entries, the rings of double-precision floating-point numbers, `RDF` and `CDF`, are a better choice since the *sage.matrix.matrix_double_dense.Matrix_double_dense.is_skew_hermitian()* method has a tolerance parameter. This provides control over allowing for minor discrepancies between entries when checking equality.

The result is cached.

EXAMPLES:

```
sage: A = matrix(QQbar, [[0, -1],
....:                    [1,  0]])
sage: A.is_skew_hermitian()
True
```

A matrix that is nearly skew-Hermitian, but for a non-real diagonal entry.

```
sage: A = matrix(QQbar, [[  -I, -1, 1-I],
....:                    [   1,  1,  -1],
....:                    [-1-I,  1,  -I]])
sage: A.is_skew_hermitian()
False
sage: A[1,1] = -I
sage: A.is_skew_hermitian()
True
```

Rectangular matrices are never skew-Hermitian.

```
sage: A = matrix(QQbar, 3, 4)
sage: A.is_skew_hermitian()
False
```

A square, empty matrix is trivially Hermitian.

```
sage: A = matrix(QQ, 0, 0)
sage: A.is_skew_hermitian()
True
```

is_skew_symmetric()

Return `True` if `self` is a skew-symmetric matrix.

Here, "skew-symmetric matrix" means a square matrix A satisfying $A^T = -A$. It does not require that the diagonal entries of A are 0 (although this automatically follows from $A^T = -A$ when 2 is invertible in the ground ring over which the matrix is considered). Skew-symmetric matrices A whose diagonal entries are 0 are said to be "alternating", and this property is checked by the *is_alternating()* method.

EXAMPLES:

```
sage: m = matrix(QQ, [[0,2], [-2,0]])
sage: m.is_skew_symmetric()
True
sage: m = matrix(QQ, [[1,2], [2,1]])
sage: m.is_skew_symmetric()
False
```

Skew-symmetric is not the same as alternating when 2 is a zero-divisor in the ground ring:

```
sage: n = matrix(Zmod(4), [[0, 1], [-1, 2]])
sage: n.is_skew_symmetric()
True
```

but yet the diagonal cannot be completely arbitrary in this case:

```
sage: n = matrix(Zmod(4), [[0, 1], [-1, 3]])
sage: n.is_skew_symmetric()
False
```

is_skew_symmetrizable(*return_diag=False*, *positive=True*)

This function takes a square matrix over an *ordered integral domain* and checks if it is skew-symmetrizable. A matrix B is skew-symmetrizable iff there exists an invertible diagonal matrix D such that DB is skew-symmetric.

Warning: Expects `self` to be a matrix over an *ordered integral domain*.

INPUT:

- `return_diag` – bool(default:False) if True and `self` is skew-symmetrizable the diagonal entries of the matrix D are returned.
- `positive` – bool(default:True) if True, the condition that D has positive entries is added.

OUTPUT:

- True – if `self` is skew-symmetrizable and `return_diag` is False
- the diagonal entries of a matrix D such that DB is skew-symmetric – iff `self` is skew-symmetrizable and `return_diag` is True
- False – iff `self` is not skew-symmetrizable

EXAMPLES:

```
sage: matrix([[0,6],[3,0]]).is_skew_symmetrizable(positive=False)
True
sage: matrix([[0,6],[3,0]]).is_skew_symmetrizable(positive=True)
False

sage: M = matrix(4,[0,1,0,0,-1,0,-1,0,0,2,0,1,0,0,-1,0]); M
[ 0  1  0  0]
[-1  0 -1  0]
[ 0  2  0  1]
[ 0  0 -1  0]

sage: M.is_skew_symmetrizable(return_diag=True)
[1, 1, 1/2, 1/2]

sage: M2 = diagonal_matrix([1,1,1/2,1/2])*M; M2
[   0    1    0    0]
[  -1    0   -1    0]
[   0    1    0  1/2]
[   0    0 -1/2    0]

sage: M2.is_skew_symmetric()
True
```

REFERENCES:

- [FZ2001]

is_sparse()

Return True if this is a sparse matrix.

In Sage, being sparse is a property of the underlying representation, not the number of nonzero entries.

EXAMPLES:

```
sage: matrix(QQ,2,2,range(4)).is_sparse()
False
sage: matrix(QQ,2,2,range(4),sparse=True).is_sparse()
True
```

is_square()

Return True precisely if this matrix is square, i.e., has the same number of rows and columns.

EXAMPLES:

```
sage: matrix(QQ,2,2,range(4)).is_square()
True
sage: matrix(QQ,2,3,range(6)).is_square()
False
```

is_symmetric()

Return True if this is a symmetric matrix.

A symmetric matrix is necessarily square.

EXAMPLES:

```
sage: m=Matrix(QQ,2,range(0,4))
sage: m.is_symmetric()
False

sage: m=Matrix(QQ,2,(1,1,1,1,1,1))
sage: m.is_symmetric()
False

sage: m=Matrix(QQ,1,(2,))
sage: m.is_symmetric()
True
```

is_symmetrizable(*return_diag=False*, *positive=True*)

This function takes a square matrix over an *ordered integral domain* and checks if it is symmetrizable. A matrix B is symmetrizable iff there exists an invertible diagonal matrix D such that DB is symmetric.

Warning: Expects `self` to be a matrix over an *ordered integral domain*.

INPUT:

- `return_diag` – bool(default:False) if True and `self` is symmetrizable the diagonal entries of the matrix D are returned.
- `positive` – bool(default:True) if True, the condition that D has positive entries is added.

OUTPUT:

- True – if `self` is symmetrizable and `return_diag` is False
- the diagonal entries of a matrix D such that DB is symmetric – iff `self` is symmetrizable and `return_diag` is True
- False – iff `self` is not symmetrizable

EXAMPLES:

```
sage: matrix([[0,6],[3,0]]).is_symmetrizable(positive=False)
True

sage: matrix([[0,6],[3,0]]).is_symmetrizable(positive=True)
```

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```
True

sage: matrix([[0,6],[0,0]]).is_symmetrizable(return_diag=True)
False

sage: matrix([2]).is_symmetrizable(positive=True)
True

sage: matrix([[1,2],[3,4]]).is_symmetrizable(return_diag=true)
[1, 2/3]
```

REFERENCES:

- [FZ2001]

is_unit()

Return True if this matrix is invertible.

EXAMPLES: The following matrix is invertible over $\mathbf{Q}$ but not over $\mathbf{Z}$.

```
sage: A = MatrixSpace(ZZ, 2)(range(4))
sage: A.is_invertible()
False
sage: A.matrix_over_field().is_invertible()
True
```

The inverse function is a constructor for matrices over the fraction field, so it can work even if A is not invertible.

```
sage: ~A   # inverse of A
[-3/2  1/2]
[   1    0]
```

The next matrix is invertible over $\mathbf{Z}$.

```
sage: A = MatrixSpace(IntegerRing(),2)([1,10,0,-1])
sage: A.is_invertible()
True
sage: ~A                # compute the inverse
[ 1 10]
[ 0 -1]
```

The following nontrivial matrix is invertible over $\mathbf{Z}[x]$.

```
sage: R.<x> = PolynomialRing(IntegerRing())
sage: A = MatrixSpace(R,2)([1,x,0,-1])
sage: A.is_invertible()
True
sage: ~A
[ 1  x]
[ 0 -1]
```

items()

Return an iterable of `((i,j), value)` elements.

This may (but is not guaranteed to) suppress zero values.

EXAMPLES:

```
sage: a = matrix(QQ['x,y'], 2, range(6), sparse=True); a
[0 1 2]
[3 4 5]
sage: list(a.items())
[((0, 1), 1), ((0, 2), 2), ((1, 0), 3), ((1, 1), 4), ((1, 2), 5)]
```

iterates(*v*, *n*, *rows=True*)

Let A be this matrix and v be a free module element. If rows is True, return a matrix whose rows are the entries of the following vectors:

$$v, vA, vA^2, \ldots, vA^{n-1}.$$

If rows is False, return a matrix whose columns are the entries of the following vectors:

$$v, Av, A^2v, \ldots, A^{n-1}v.$$

INPUT:

- `v` - free module element
- `n` - nonnegative integer

EXAMPLES:

```
sage: A = matrix(ZZ,2, [1,1,3,5]); A
[1 1]
[3 5]
sage: v = vector([1,0])
sage: A.iterates(v,0)
[]
sage: A.iterates(v,5)
[  1   0]
[  1   1]
[  4   6]
[ 22  34]
[124 192]
```

Another example:

```
sage: a = matrix(ZZ,3,range(9)); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: v = vector([1,0,0])
sage: a.iterates(v,4)
[  1   0   0]
[  0   1   2]
[ 15  18  21]
[180 234 288]
sage: a.iterates(v,4,rows=False)
[  1   0  15 180]
[  0   3  42 558]
[  0   6  69 936]
```

linear_combination_of_columns(*v*)

Return the linear combination of the columns of `self` given by the coefficients in the list `v`.

INPUT:

- `v` - a list of scalars. The length can be less than the number of columns of `self` but not greater.

OUTPUT:

The vector (or free module element) that is a linear combination of the columns of `self`. If the list of scalars has fewer entries than the number of columns, additional zeros are appended to the list until it has as many entries as the number of columns.

EXAMPLES:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.linear_combination_of_columns([1,1,1])
(3, 12)

sage: a.linear_combination_of_columns([0,0,0])
(0, 0)

sage: a.linear_combination_of_columns([1/2,2/3,3/4])
(13/6, 95/12)
```

The list `v` can be anything that is iterable. Perhaps most naturally, a vector may be used.

```
sage: v = vector(ZZ, [1,2,3])
sage: a.linear_combination_of_columns(v)
(8, 26)
```

We check that a matrix with no columns behaves properly.

```
sage: matrix(QQ,2,0).linear_combination_of_columns([])
(0, 0)
```

The object returned is a vector, or a free module element.

```
sage: B = matrix(ZZ, 4, 3, range(12))
sage: w = B.linear_combination_of_columns([-1,2,-3])
sage: w
(-4, -10, -16, -22)
sage: w.parent()
Ambient free module of rank 4 over the principal ideal domain Integer Ring
sage: x = B.linear_combination_of_columns([1/2,1/3,1/4])
sage: x
(5/6, 49/12, 22/3, 127/12)
sage: x.parent()
Vector space of dimension 4 over Rational Field
```

The length of v can be less than the number of columns, but not greater.

```
sage: A = matrix(QQ,3,5, range(15))
sage: A.linear_combination_of_columns([1,-2,3,-4])
```

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```
(-8, -18, -28)
sage: A.linear_combination_of_columns([1,2,3,4,5,6])
Traceback (most recent call last):
...
ValueError: length of v must be at most the number of columns of self
```

linear_combination_of_rows(*v*)

Return the linear combination of the rows of `self` given by the coefficients in the list `v`.

INPUT:

- `v` - a list of scalars. The length can be less than the number of rows of `self` but not greater.

OUTPUT:

The vector (or free module element) that is a linear combination of the rows of `self`. If the list of scalars has fewer entries than the number of rows, additional zeros are appended to the list until it has as many entries as the number of rows.

EXAMPLES:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.linear_combination_of_rows([1,2])
(6, 9, 12)

sage: a.linear_combination_of_rows([0,0])
(0, 0, 0)

sage: a.linear_combination_of_rows([1/2,2/3])
(2, 19/6, 13/3)
```

The list `v` can be anything that is iterable. Perhaps most naturally, a vector may be used.

```
sage: v = vector(ZZ, [1,2])
sage: a.linear_combination_of_rows(v)
(6, 9, 12)
```

We check that a matrix with no rows behaves properly.

```
sage: matrix(QQ,0,2).linear_combination_of_rows([])
(0, 0)
```

The object returned is a vector, or a free module element.

```
sage: B = matrix(ZZ, 4, 3, range(12))
sage: w = B.linear_combination_of_rows([-1,2,-3,4])
sage: w
(24, 26, 28)
sage: w.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: x = B.linear_combination_of_rows([1/2,1/3,1/4,1/5])
sage: x
(43/10, 67/12, 103/15)
```

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```
sage: x.parent()
Vector space of dimension 3 over Rational Field
```

The length of v can be less than the number of rows, but not greater.

```
sage: A = matrix(QQ,3,4,range(12))
sage: A.linear_combination_of_rows([2,3])
(12, 17, 22, 27)
sage: A.linear_combination_of_rows([1,2,3,4])
Traceback (most recent call last):
...
ValueError: length of v must be at most the number of rows of self
```

list()

List of the elements of `self` ordered by elements in each row. It is safe to change the returned list.

Warning: This function returns a list of the entries in the matrix `self`. It does not return a list of the rows of `self`, so it is different than the output of `list(self)`, which returns `[self[0],self[1],...]`.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: a = matrix(R,2,[x,y,x*y, y,x,2*x+y]); a
[      x       y     x*y]
[      y       x 2*x + y]
sage: v = a.list(); v
[x, y, x*y, y, x, 2*x + y]
```

Note that list(a) is different than a.list():

```
sage: a.list()
[x, y, x*y, y, x, 2*x + y]
sage: list(a)
[(x, y, x*y), (y, x, 2*x + y)]
```

Notice that changing the returned list does not change a (the list is a copy):

```
sage: v[0] = 25
sage: a
[      x       y     x*y]
[      y       x 2*x + y]
```

mod(p)

Return matrix mod p, over the reduced ring.

EXAMPLES:

```
sage: M = matrix(ZZ, 2, 2, [5, 9, 13, 15])
sage: M.mod(7)
[5 2]
[6 1]
```

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```
sage: parent(M.mod(7))
Full MatrixSpace of 2 by 2 dense matrices over Ring of integers modulo 7
```

monomial_coefficients(*copy=True*)

Dictionary of the elements of `self` with keys pairs `(i,j)` and values the nonzero entries of `self`.

INPUT:

- `copy` – (default: `True`) make a copy of the `dict` corresponding to `self`

If `copy=True`, then is safe to change the returned dictionary. Otherwise, this can cause undesired behavior by mutating the `dict`.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: a = matrix(R,2,[x,y,0, 0,0,2*x+y]); a
[      x       y       0]
[      0       0 2*x + y]
sage: d = a.dict(); d
{(0, 0): x, (0, 1): y, (1, 2): 2*x + y}
```

Notice that changing the returned list does not change a (the list is a copy):

```
sage: d[0,0] = 25
sage: a
[      x       y       0]
[      0       0 2*x + y]
```

multiplicative_order()

Return the multiplicative order of this matrix, which must therefore be invertible.

Only implemented over finite fields and over $\mathbf{Z}$.

EXAMPLES:

Over finite fields:

```
sage: A = matrix(GF(59),3,[10,56,39,53,56,33,58,24,55])
sage: A.multiplicative_order()
580
sage: (A^580).is_one()
True

sage: B = matrix(GF(10007^3,'b'),0)
sage: B.multiplicative_order()
1

sage: M = MatrixSpace(GF(11^2,'e'),5)
sage: E = M.random_element()
sage: while E.det() == 0:
....:     E = M.random_element()
sage: (E^E.multiplicative_order()).is_one()
True
```

Over $\mathbf{Z}$:

```
sage: m = matrix(ZZ,2,2,[-1,1,-1,0])
sage: m.multiplicative_order()
3

sage: m = posets.ChainPoset(6).coxeter_transformation()
sage: m.multiplicative_order()
7

sage: P = posets.TamariLattice(4).coxeter_transformation()
sage: P.multiplicative_order()
10

sage: M = matrix(ZZ, 2, 2, [1, 1, 0, 1])
sage: M.multiplicative_order()
+Infinity

sage: for k in range(600):
....:     m = SL2Z.random_element()
....:     o = m.multiplicative_order()
....:     if o != Infinity and m**o != SL2Z.one():
....:         raise RuntimeError

sage: m24 = matrix.companion(cyclotomic_polynomial(24))
sage: def val(i, j):
....:     if i < j:
....:         return 0
....:     elif i == j:
....:         return 1
....:     else:
....:         return ZZ.random_element(-100,100)
sage: rnd = matrix(ZZ, 8, 8, val)
sage: (rnd * m24 * rnd.inverse_of_unit()).multiplicative_order()
24
```

REFERENCES:

- [CLG1997]
- [KP2002b]

mutate(*k*)

Mutates `self` at row and column index `k`.

Warning: Only makes sense if `self` is skew-symmetrizable.

INPUT:

- `k` – integer at which row/column `self` is mutated.

EXAMPLES:

Mutation of the B-matrix of the quiver of type A_3:

```
sage: M = matrix(ZZ,3,[0,1,0,-1,0,-1,0,1,0]); M
[ 0  1  0]
[-1  0 -1]
[ 0  1  0]

sage: M.mutate(0); M
[ 0 -1  0]
[ 1  0 -1]
[ 0  1  0]

sage: M.mutate(1); M
[ 0  1 -1]
[-1  0  1]
[ 1 -1  0]

sage: M = matrix(ZZ,6,[0,1,0,-1,0,-1,0,1,0,1,0,0,0,1,0,0,0,1]); M
[ 0  1  0]
[-1  0 -1]
[ 0  1  0]
[ 1  0  0]
[ 0  1  0]
[ 0  0  1]

sage: M.mutate(0); M
[ 0 -1  0]
[ 1  0 -1]
[ 0  1  0]
[-1  1  0]
[ 0  1  0]
[ 0  0  1]
```

REFERENCES:

- [FZ2001]

ncols()

Return the number of columns of this matrix.

EXAMPLES:

```
sage: M = MatrixSpace(QQ, 2, 3)
sage: A = M([1,2,3, 4,5,6])
sage: A
[1 2 3]
[4 5 6]
sage: A.ncols()
3
sage: A.nrows()
2
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples

nonpivots()

Return the list of i such that the i-th column of self is NOT a pivot column of the reduced row echelon form

of self.

OUTPUT: sorted tuple of (Python) integers

EXAMPLES:

```
sage: a = matrix(QQ,3,3,range(9)); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: a.echelon_form()
[ 1  0 -1]
[ 0  1  2]
[ 0  0  0]
sage: a.nonpivots()
(2,)
```

nonzero_positions(*copy=True*, *column_order=False*)

Return the sorted list of pairs `(i,j)` such that `self[i,j] != 0`.

INPUT:

- `copy` – (default: `True`) it is safe to change the resulting list (unless you give the option `copy=False`)
- `column_order` – (default: `False`) If `True`, returns the list of pairs `(i,j)` such that `self[i,j] != 0`, but sorted by columns, i.e., column `j=0` entries occur first, then column `j=1` entries, etc.

EXAMPLES:

```
sage: a = matrix(QQ, 2,3, [1,2,0,2,0,0]); a
[1 2 0]
[2 0 0]
sage: a.nonzero_positions()
[(0, 0), (0, 1), (1, 0)]
sage: a.nonzero_positions(copy=False)
[(0, 0), (0, 1), (1, 0)]
sage: a.nonzero_positions(column_order=True)
[(0, 0), (1, 0), (0, 1)]
sage: a = matrix(QQ, 2,3, [1,2,0,2,0,0], sparse=True); a
[1 2 0]
[2 0 0]
sage: a.nonzero_positions()
[(0, 0), (0, 1), (1, 0)]
sage: a.nonzero_positions(copy=False)
[(0, 0), (0, 1), (1, 0)]
sage: a.nonzero_positions(column_order=True)
[(0, 0), (1, 0), (0, 1)]
```

nonzero_positions_in_column(*i*)

Return a sorted list of the integers `j` such that `self[j,i]` is nonzero, i.e., such that the `j`-th position of the `i`-th column is nonzero.

INPUT:

- `i` – an integer

OUTPUT: list

EXAMPLES:

```
sage: a = matrix(QQ, 3,2, [1,2,0,2,0,0]); a
[1 2]
[0 2]
[0 0]
sage: a.nonzero_positions_in_column(0)
[0]
sage: a.nonzero_positions_in_column(1)
[0, 1]
```

You will get an IndexError if you select an invalid column:

```
sage: a.nonzero_positions_in_column(2)
Traceback (most recent call last):
...
IndexError: matrix column index out of range
```

nonzero_positions_in_row(*i*)

Return the integers j such that self[i,j] is nonzero, i.e., such that the j-th position of the i-th row is nonzero.

INPUT:

- i – an integer

OUTPUT: list

EXAMPLES:

```
sage: a = matrix(QQ, 3,2, [1,2,0,2,0,0]); a
[1 2]
[0 2]
[0 0]
sage: a.nonzero_positions_in_row(0)
[0, 1]
sage: a.nonzero_positions_in_row(1)
[1]
sage: a.nonzero_positions_in_row(2)
[]
```

nrows()

Return the number of rows of this matrix.

EXAMPLES:

```
sage: M = MatrixSpace(QQ,6,7)
sage: A = M([1,2,3,4,5,6,7, 22,3/4,34,11,7,5,3, 99,65,1/2,2/3,3/5,4/5,5/6, 9,8/
↪9, 9/8,7/6,6/7,76,4, 0,9,8,7,6,5,4, 123,99,91,28,6,1024,1])
sage: A
[   1    2    3    4    5    6    7]
[  22  3/4   34   11    7    5    3]
[  99   65  1/2  2/3  3/5  4/5  5/6]
[   9  8/9  9/8  7/6  6/7   76    4]
[   0    9    8    7    6    5    4]
[ 123   99   91   28    6 1024    1]
sage: A.ncols()
```

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```
7
sage: A.nrows()
6
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples

permute_columns(*permutation*)

Permute the columns of `self` by applying the permutation group element `permutation`.

As permutation group elements act on integers $\{1, \ldots, n\}$, columns are considered numbered from 1 for this operation.

INPUT:

- `permutation` – a `PermutationGroupElement`.

EXAMPLES: We create a matrix:

```
sage: M = matrix(ZZ,[[1,0,0,0,0],[0,2,0,0,0],[0,0,3,0,0],[0,0,0,4,0],[0,0,0,0,
↪5]])
sage: M
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

Next of all, create a permutation group element and act on `M` with it:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma, tau = G.gens()
sage: sigma
(1,2,3)(4,5)
sage: M.permute_columns(sigma)
sage: M
[0 0 1 0 0]
[2 0 0 0 0]
[0 3 0 0 0]
[0 0 0 0 4]
[0 0 0 5 0]
```

permute_rows(*permutation*)

Permute the rows of `self` by applying the permutation group element `permutation`.

As permutation group elements act on integers $\{1, \ldots, n\}$, rows are considered numbered from 1 for this operation.

INPUT:

- `permutation` – a `PermutationGroupElement`

EXAMPLES: We create a matrix:

```
sage: M = matrix(ZZ,[[1,0,0,0,0],[0,2,0,0,0],[0,0,3,0,0],[0,0,0,4,0],[0,0,0,0,
↪5]])
```

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```
sage: M
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

Next of all, create a permutation group element and act on `M`:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma, tau = G.gens()
sage: sigma
(1,2,3)(4,5)
sage: M.permute_rows(sigma)
sage: M
[0 2 0 0 0]
[0 0 3 0 0]
[1 0 0 0 0]
[0 0 0 0 5]
[0 0 0 4 0]
```

permute_rows_and_columns(*row_permutation*, *column_permutation*)

Permute the rows and columns of `self` by applying the permutation group elements `row_permutation` and `column_permutation` respectively.

As permutation group elements act on integers $\{1, \ldots, n\}$, rows and columns are considered numbered from 1 for this operation.

INPUT:

- `row_permutation` – a `PermutationGroupElement`
- `column_permutation` – a `PermutationGroupElement`

OUTPUT:

- A matrix.

EXAMPLES: We create a matrix:

```
sage: M = matrix(ZZ,[[1,0,0,0,0],[0,2,0,0,0],[0,0,3,0,0],[0,0,0,4,0],[0,0,0,0,
↪5]])
sage: M
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

Next of all, create a permutation group element and act on `M`:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma, tau = G.gens()
sage: sigma
(1,2,3)(4,5)
sage: M.permute_rows_and_columns(sigma,tau)
```

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```
sage: M
[2 0 0 0 0]
[0 3 0 0 0]
[0 0 0 0 1]
[0 0 0 5 0]
[0 0 4 0 0]
```

pivots()

Return the pivot column positions of this matrix.

OUTPUT: a tuple of Python integers: the position of the first nonzero entry in each row of the echelon form.

This returns a tuple so it is immutable; see github issue #10752.

EXAMPLES:

```
sage: A = matrix(QQ, 2, 2, range(4))
sage: A.pivots()
(0, 1)
```

rank()

Return the rank of this matrix.

EXAMPLES:

```
sage: m = matrix(GF(7),5,range(25))
sage: m.rank()
2
```

Rank is not implemented over the integers modulo a composite yet.:

```
sage: m = matrix(Integers(4), 2, [2,2,2,2])
sage: m.rank()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪4'.
```

rescale_col(*i*, *s*, *start_row=0*)

Replace i-th col of self by s times i-th col of self.

INPUT:

- `i` - ith column
- `s` - scalar
- `start_row` - only rescale entries at this row and lower

EXAMPLES: We rescale the last column of a matrix over the rational numbers:

```
sage: a = matrix(QQ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.rescale_col(2,1/2); a
```

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```
[  0   1   1]
[  3   4 5/2]
sage: R.<x> = QQ[]
```

We rescale the last column of a matrix over a polynomial ring:

```
sage: a = matrix(R,2,3,[1,x,x^2,x^3,x^4,x^5]); a
[  1   x x^2]
[x^3 x^4 x^5]
sage: a.rescale_col(2,1/2); a
[      1       x 1/2*x^2]
[    x^3     x^4 1/2*x^5]
```

We try and fail to rescale a matrix over the integers by a non-integer:

```
sage: a = matrix(ZZ,2,3,[0,1,2, 3,4,4]); a
[0 1 2]
[3 4 4]
sage: a.rescale_col(2,1/2)
Traceback (most recent call last):
...
TypeError: Rescaling column by Rational Field element cannot be done over␣
↪Integer Ring, use change_ring or with_rescaled_col instead.
```

To rescale the matrix by 1/2, you must change the base ring to the rationals:

```
sage: a = a.change_ring(QQ); a
[0 1 2]
[3 4 4]
sage: a.rescale_col(2,1/2); a
[0 1 1]
[3 4 2]
```

rescale_row(*i*, *s*, *start_col=0*)

Replace i-th row of self by s times i-th row of self.

INPUT:

- `i` - ith row
- `s` - scalar
- `start_col` - only rescale entries at this column and to the right

EXAMPLES: We rescale the second row of a matrix over the rational numbers:

```
sage: a = matrix(QQ,3,range(6)); a
[0 1]
[2 3]
[4 5]
sage: a.rescale_row(1,1/2); a
[ 0   1]
[ 1 3/2]
[ 4   5]
```

We rescale the second row of a matrix over a polynomial ring:

```
sage: R.<x> = QQ[]
sage: a = matrix(R,3,[1,x,x^2,x^3,x^4,x^5]);a
[  1   x]
[x^2 x^3]
[x^4 x^5]
sage: a.rescale_row(1,1/2); a
[      1       x]
[1/2*x^2 1/2*x^3]
[    x^4     x^5]
```

We try and fail to rescale a matrix over the integers by a non-integer:

```
sage: a = matrix(ZZ,2,3,[0,1,2, 3,4,4]); a
[0 1 2]
[3 4 4]
sage: a.rescale_row(1,1/2)
Traceback (most recent call last):
...
TypeError: Rescaling row by Rational Field element cannot be done over Integer␣
↪Ring, use change_ring or with_rescaled_row instead.
```

To rescale the matrix by 1/2, you must change the base ring to the rationals:

```
sage: a = a.change_ring(QQ); a
[0 1 2]
[3 4 4]
sage: a.rescale_col(1,1/2); a
[  0 1/2   2]
[  3   2   4]
```

reverse_rows_and_columns()

Reverse the row order and column order of this matrix.

This method transforms a matrix $m_{i,j}$ with $0 \leq i < nrows$ and $0 \leq j < ncols$ into $m_{nrows-i-1,ncols-j-1}$.

EXAMPLES:

```
sage: m = matrix(ZZ, 2, 2, range(4))
sage: m.reverse_rows_and_columns()
sage: m
[3 2]
[1 0]

sage: m = matrix(ZZ, 2, 3, range(6), sparse=True)
sage: m.reverse_rows_and_columns()
sage: m
[5 4 3]
[2 1 0]
sage: m = matrix(ZZ, 3, 2, range(6), sparse=True)
sage: m.reverse_rows_and_columns()
sage: m
[5 4]
[3 2]
```

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```
[1 0]
sage: m.reverse_rows_and_columns()
sage: m
[0 1]
[2 3]
[4 5]

sage: m = matrix(QQ, 3, 2, [1/i for i in range(1,7)])
sage: m.reverse_rows_and_columns()
sage: m
[1/6 1/5]
[1/4 1/3]
[1/2   1]

sage: R.<x,y> = ZZ['x,y']
sage: m = matrix(R, 3, 3, lambda i,j: x**i*y**j, sparse=True)
sage: m.reverse_rows_and_columns()
sage: m
[x^2*y^2   x^2*y     x^2]
[  x*y^2     x*y       x]
[    y^2       y       1]
```

If the matrix is immutable, the method raises an error:

```
sage: m = matrix(ZZ, 2, [1, 3, -2, 4])
sage: m.set_immutable()
sage: m.reverse_rows_and_columns()
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy
instead (i.e., use copy(M) to change a copy of M).
```

set_col_to_multiple_of_col(*i*, *j*, *s*)

Set column i equal to s times column j.

EXAMPLES: We change the second column to -3 times the first column.

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.set_col_to_multiple_of_col(1,0,-3)
sage: a
[ 0  0  2]
[ 3 -9  5]
```

If we try to multiply a column by a rational number, we get an error message:

```
sage: a.set_col_to_multiple_of_col(1,0,1/2)
Traceback (most recent call last):
...
TypeError: Multiplying column by Rational Field element cannot be done over␣
↪Integer Ring, use change_ring or with_col_set_to_multiple_of_col instead.
```

set_immutable()

Call this function to set the matrix as immutable.

Matrices are always mutable by default, i.e., you can change their entries using `A[i,j] = x`. However, mutable matrices aren't hashable, so can't be used as keys in dictionaries, etc. Also, often when implementing a class, you might compute a matrix associated to it, e.g., the matrix of a Hecke operator. If you return this matrix to the user you're really returning a reference and the user could then change an entry; this could be confusing. Thus you should set such a matrix immutable.

EXAMPLES:

```
sage: A = Matrix(QQ, 2, 2, range(4))
sage: A.is_mutable()
True
sage: A[0,0] = 10
sage: A
[10   1]
[ 2   3]
```

Mutable matrices are not hashable, so can't be used as keys for dictionaries:

```
sage: hash(A)
Traceback (most recent call last):
...
TypeError: mutable matrices are unhashable
sage: v = {A:1}
Traceback (most recent call last):
...
TypeError: mutable matrices are unhashable
```

If we make A immutable it suddenly is hashable.

```
sage: A.set_immutable()
sage: A.is_mutable()
False
sage: A[0,0] = 10
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
↪copy(M) to change a copy of M).
sage: hash(A) #random
12
sage: v = {A:1}; v
{[10  1]
 [ 2  3]: 1}
```

set_row_to_multiple_of_row(*i*, *j*, *s*)

Set row i equal to s times row j.

EXAMPLES: We change the second row to -3 times the first row:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.set_row_to_multiple_of_row(1,0,-3)
```

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```
sage: a
[ 0  1  2]
[ 0 -3 -6]
```

If we try to multiply a row by a rational number, we get an error message:

```
sage: a.set_row_to_multiple_of_row(1,0,1/2)
Traceback (most recent call last):
...
TypeError: Multiplying row by Rational Field element cannot be done over␣
↪Integer Ring, use change_ring or with_row_set_to_multiple_of_row instead.
```

str(*rep_mapping=None*, *zero=None*, *plus_one=None*, *minus_one=None*, *unicode=False*, *shape=None*, *character_art=False*)

Return a nice string representation of the matrix.

INPUT:

- `rep_mapping` - a dictionary or callable used to override the usual representation of elements.

  If `rep_mapping` is a dictionary then keys should be elements of the base ring and values the desired string representation. Values sent in via the other keyword arguments will override values in the dictionary. Use of a dictionary can potentially take a very long time due to the need to hash entries of the matrix. Matrices with entries from `QQbar` are one example.

  If `rep_mapping` is callable then it will be called with elements of the matrix and must return a string. Simply call `repr()` on elements which should have the default representation.

- `zero` - string (default: `None`); if not `None` use the value of `zero` as the representation of the zero element.

- `plus_one` - string (default: `None`); if not `None` use the value of `plus_one` as the representation of the one element.

- `minus_one` - string (default: `None`); if not `None` use the value of `minus_one` as the representation of the negative of the one element.

- `unicode` - boolean (default: `False`). Whether to use Unicode symbols instead of ASCII symbols for brackets and subdivision lines.

- `shape` - one of `"square"` or `"round"` (default: `None`). Switches between round and square brackets. The default depends on the setting of the `unicode` keyword argument. For Unicode symbols, the default is round brackets in accordance with the TeX rendering, while the ASCII rendering defaults to square brackets.

- `character_art` – boolean (default: `False`); if `True`, the result will be of type `AsciiArt` or `UnicodeArt` which support line breaking of wide matrices that exceed the window width

EXAMPLES:

```
sage: R = PolynomialRing(QQ,6,'z')
sage: a = matrix(2,3, R.gens())
sage: a.__repr__()
'[z0 z1 z2]\n[z3 z4 z5]'

sage: M = matrix([[1,0],[2,-1]])
sage: M.str()
```

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```
'[ 1  0]\n[ 2 -1]'
sage: M.str(plus_one='+',minus_one='-',zero='.')
'[+ .]\n[2 -]'
sage: M.str({1:"not this one",2:"II"},minus_one="*",plus_one="I")
'[ I  0]\n[II  *]'

sage: def print_entry(x):
....:   if x>0:
....:       return '+'
....:   elif x<0:
....:       return '-'
....:   else: return '.'
...
sage: M.str(print_entry)
'[+ .]\n[+ -]'
sage: M.str(repr)
'[ 1  0]\n[ 2 -1]'

sage: M = matrix([[1,2,3],[4,5,6],[7,8,9]])
sage: M.subdivide(None, 2)
sage: print(M.str(unicode=True))
⎛1 2|3⎞
⎜    ⎟
⎜4 5|6⎟
⎝7 8|9⎠
sage: M.subdivide([0,1,1,3], [0,2,3,3])
sage: print(M.str(unicode=True, shape="square"))
⎡+-+-++⎤
⎢      ⎥
⎢|1 2|3||⎥
⎢+-+-++⎥
⎢+-+-++⎥
⎢|4 5|6||⎥
⎢|7 8|9||⎥
⎣+-+-++⎦
```

If `character_art` is set, the lines of large matrices are wrapped in a readable way:

```
sage: set_random_seed(0)
sage: matrix.random(RDF, 3, 5).str(unicode=True, character_art=True)
⎛ -0.27440062056807446    0.5031965950979831 -0.001975438590219314
⎜
⎜ -0.05461130074681608 -0.033673314214051286   -0.9401270875197381
⎜  0.19906256610645512    0.3242250183948632    0.6026443545751128
⎝

  -0.9467802263760512    0.5056889961514748⎞
⎟
 -0.35104242112828943    0.5084492941557279⎟
  -0.9541798283979341   -0.8948790563276592⎠
```

The number of floating point digits to display is controlled by *matrix.options.precision* and can also be set by the IPython magic `%precision`. This does not affect the internal precision of the represented data, but only the textual display of matrices:

```
sage: matrix.options.precision = 4
sage: A = matrix(RR, [[1/3, 200/3], [-3, 1e6]]); A
[  0.3333    66.67]
[  -3.000 1.000E+6]
sage: unicode_art(A)
⎛  0.3333    66.67⎞
⎝  -3.000 1.000E+6⎠
sage: matrix.options.precision = None
sage: A
[ 0.333333333333333   66.6666666666667]
[ -3.00000000000000 1.00000000000000e6]
```

swap_columns(*c1*, *c2*)

Swap columns c1 and c2 of self.

EXAMPLES: We create a rational matrix:

```
sage: M = MatrixSpace(QQ,3,3)
sage: A = M([1,9,-7,4/5,4,3,6,4,3])
sage: A
[  1   9  -7]
[4/5   4   3]
[  6   4   3]
```

Since the first column is numbered zero, this swaps the second and third columns:

```
sage: A.swap_columns(1,2); A
[  1  -7   9]
[4/5   3   4]
[  6   3   4]
```

swap_rows(*r1*, *r2*)

Swap rows r1 and r2 of self.

EXAMPLES: We create a rational matrix:

```
sage: M = MatrixSpace(QQ,3,3)
sage: A = M([1,9,-7,4/5,4,3,6,4,3])
sage: A
[  1   9  -7]
[4/5   4   3]
[  6   4   3]
```

Since the first row is numbered zero, this swaps the first and third rows:

```
sage: A.swap_rows(0,2); A
[  6   4   3]
[4/5   4   3]
[  1   9  -7]
```

with_added_multiple_of_column(*i*, *j*, *s*, *start_row=0*)

Add s times column j to column i, returning new matrix.

EXAMPLES: We add -1 times the third column to the second column of an integer matrix, remembering to start numbering cols at zero:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: b = a.with_added_multiple_of_column(1,2,-1); b
[ 0 -1  2]
[ 3 -1  5]
```

The original matrix is unchanged:

```
sage: a
[0 1 2]
[3 4 5]
```

Adding a rational multiple is okay, and reassigning a variable is okay:

```
sage: a = a.with_added_multiple_of_column(0,1,1/3); a
[ 1/3    1    2]
[13/3    4    5]
```

with_added_multiple_of_row(*i*, *j*, *s*, *start_col=0*)

Add s times row j to row i, returning new matrix.

EXAMPLES: We add -3 times the first row to the second row of an integer matrix, remembering to start numbering rows at zero:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: b = a.with_added_multiple_of_row(1,0,-3); b
[ 0  1  2]
[ 3  1 -1]
```

The original matrix is unchanged:

```
sage: a
[0 1 2]
[3 4 5]
```

Adding a rational multiple is okay, and reassigning a variable is okay:

```
sage: a = a.with_added_multiple_of_row(0,1,1/3); a
[   1  7/3 11/3]
[   3    4    5]
```

with_col_set_to_multiple_of_col(*i*, *j*, *s*)

Set column i equal to s times column j, returning a new matrix.

EXAMPLES: We change the second column to -3 times the first column.

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
```

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```
sage: b = a.with_col_set_to_multiple_of_col(1,0,-3); b
[ 0  0  2]
[ 3 -9  5]
```

Note that the original matrix is unchanged:

```
sage: a
[0 1 2]
[3 4 5]
```

Adding a rational multiple is okay, and reassigning a variable is okay:

```
sage: a = a.with_col_set_to_multiple_of_col(1,0,1/2); a
[  0   0   2]
[  3 3/2   5]
```

with_permuted_columns(*permutation*)

Return the matrix obtained from permuting the columns of `self` by applying the permutation group element `permutation`.

As permutation group elements act on integers $\{1, \ldots, n\}$, columns are considered numbered from 1 for this operation.

INPUT:

- `permutation`, a `PermutationGroupElement`

OUTPUT:

- A matrix.

EXAMPLES: We create some matrix:

```
sage: M = matrix(ZZ,[[1,0,0,0,0],[0,2,0,0,0],[0,0,3,0,0],[0,0,0,4,0],[0,0,0,0,
↪5]])
sage: M
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

Next of all, create a permutation group element and act on `M`:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma, tau = G.gens()
sage: sigma
(1,2,3)(4,5)
sage: M.with_permuted_columns(sigma)
[0 0 1 0 0]
[2 0 0 0 0]
[0 3 0 0 0]
[0 0 0 0 4]
[0 0 0 5 0]
```

with_permuted_rows(*permutation*)

Return the matrix obtained from permuting the rows of `self` by applying the permutation group element `permutation`.

As permutation group elements act on integers $\{1, \ldots, n\}$, rows are considered numbered from 1 for this operation.

INPUT:

- `permutation` – a `PermutationGroupElement`

OUTPUT:

- A matrix.

EXAMPLES: We create a matrix:

```
sage: M = matrix(ZZ,[[1,0,0,0,0],[0,2,0,0,0],[0,0,3,0,0],[0,0,0,4,0],[0,0,0,0,
↪5]])
sage: M
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

Next of all, create a permutation group element and act on `M`:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma, tau = G.gens()
sage: sigma
(1,2,3)(4,5)
sage: M.with_permuted_rows(sigma)
[0 2 0 0 0]
[0 0 3 0 0]
[1 0 0 0 0]
[0 0 0 0 5]
[0 0 0 4 0]
```

with_permuted_rows_and_columns(*row_permutation*, *column_permutation*)

Return the matrix obtained from permuting the rows and columns of `self` by applying the permutation group elements `row_permutation` and `column_permutation`.

As permutation group elements act on integers $\{1, \ldots, n\}$, rows and columns are considered numbered from 1 for this operation.

INPUT:

- `row_permutation` – a `PermutationGroupElement`
- `column_permutation` – a `PermutationGroupElement`

OUTPUT:

- A matrix.

EXAMPLES: We create a matrix:

```
sage: M = matrix(ZZ,[[1,0,0,0,0],[0,2,0,0,0],[0,0,3,0,0],[0,0,0,4,0],[0,0,0,0,
↪5]])
sage: M
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

Next of all, create a permutation group element and act on `M`:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma, tau = G.gens()
sage: sigma
(1,2,3)(4,5)
sage: M.with_permuted_rows_and_columns(sigma,tau)
[2 0 0 0 0]
[0 3 0 0 0]
[0 0 0 0 1]
[0 0 0 5 0]
[0 0 4 0 0]
```

with_rescaled_col(*i*, *s*, *start_row=0*)

Replaces i-th col of self by s times i-th col of self, returning new matrix.

EXAMPLES: We rescale the last column of a matrix over the integers:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: b = a.with_rescaled_col(2,-2); b
[  0   1  -4]
[  3   4 -10]
```

The original matrix is unchanged:

```
sage: a
[0 1 2]
[3 4 5]
```

Adding a rational multiple is okay, and reassigning a variable is okay:

```
sage: a = a.with_rescaled_col(1,1/3); a
[  0 1/3   2]
[  3 4/3   5]
```

with_rescaled_row(*i*, *s*, *start_col=0*)

Replaces i-th row of self by s times i-th row of self, returning new matrix.

EXAMPLES: We rescale the second row of a matrix over the integers:

```
sage: a = matrix(ZZ,3,2,range(6)); a
[0 1]
[2 3]
```

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```
[4 5]
sage: b = a.with_rescaled_row(1,-2); b
[ 0  1]
[-4 -6]
[ 4  5]
```

The original matrix is unchanged:

```
sage: a
[0 1]
[2 3]
[4 5]
```

Adding a rational multiple is okay, and reassigning a variable is okay:

```
sage: a = a.with_rescaled_row(2,1/3); a
[  0   1]
[  2   3]
[4/3 5/3]
```

with_row_set_to_multiple_of_row(*i*, *j*, *s*)

Set row i equal to s times row j, returning a new matrix.

EXAMPLES: We change the second row to -3 times the first row:

```
sage: a = matrix(ZZ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: b = a.with_row_set_to_multiple_of_row(1,0,-3); b
[ 0  1  2]
[ 0 -3 -6]
```

Note that the original matrix is unchanged:

```
sage: a
[0 1 2]
[3 4 5]
```

Adding a rational multiple is okay, and reassigning a variable is okay:

```
sage: a = a.with_row_set_to_multiple_of_row(1,0,1/2); a
[  0   1   2]
[  0 1/2   1]
```

with_swapped_columns(*c1*, *c2*)

Swap columns `c1` and `c2` of `self` and return a new matrix.

INPUT:

- `c1`, `c2` - integers specifying columns of `self` to interchange

OUTPUT:

A new matrix, identical to `self` except that columns `c1` and `c2` are swapped.

EXAMPLES:

Remember that columns are numbered starting from zero.

```
sage: A = matrix(QQ, 4, range(20))
sage: A.with_swapped_columns(1, 2)
[ 0  2  1  3  4]
[ 5  7  6  8  9]
[10 12 11 13 14]
[15 17 16 18 19]
```

Trying to swap a column with itself will succeed, but still return a new matrix.

```
sage: A = matrix(QQ, 4, range(20))
sage: B = A.with_swapped_columns(2, 2)
sage: A == B
True
sage: A is B
False
```

The column specifications are checked.

```
sage: A = matrix(4, range(20))
sage: A.with_swapped_columns(-1, 2)
Traceback (most recent call last):
...
IndexError: matrix column index out of range

sage: A.with_swapped_columns(2, 5)
Traceback (most recent call last):
...
IndexError: matrix column index out of range
```

with_swapped_rows(*r1*, *r2*)

Swap rows `r1` and `r2` of `self` and return a new matrix.

INPUT:

- `r1`, `r2` - integers specifying rows of `self` to interchange

OUTPUT:

A new matrix, identical to `self` except that rows `r1` and `r2` are swapped.

EXAMPLES:

Remember that rows are numbered starting from zero.

```
sage: A = matrix(QQ, 4, range(20))
sage: A.with_swapped_rows(1, 2)
[ 0  1  2  3  4]
[10 11 12 13 14]
[ 5  6  7  8  9]
[15 16 17 18 19]
```

Trying to swap a row with itself will succeed, but still return a new matrix.

```
sage: A = matrix(QQ, 4, range(20))
sage: B = A.with_swapped_rows(2, 2)
```

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```
sage: A == B
True
sage: A is B
False
```

The row specifications are checked.

```
sage: A = matrix(4, range(20))
sage: A.with_swapped_rows(-1, 2)
Traceback (most recent call last):
...
IndexError: matrix row index out of range

sage: A.with_swapped_rows(2, 5)
Traceback (most recent call last):
...
IndexError: matrix row index out of range
```

sage.matrix.matrix0.**set_max_cols**(*n*)

Sets the global variable max_cols (which is used in deciding how to output a matrix).

EXAMPLES:

```
sage: from sage.matrix.matrix0 import set_max_cols
sage: set_max_cols(50)
doctest:...: DeprecationWarning: 'set_max_cols' is replaced by 'matrix.options.max_
↪cols'
See https://github.com/sagemath/sage/issues/30552 for details.
```

sage.matrix.matrix0.**set_max_rows**(*n*)

Sets the global variable max_rows (which is used in deciding how to output a matrix).

EXAMPLES:

```
sage: from sage.matrix.matrix0 import set_max_rows
sage: set_max_rows(20)
doctest:...: DeprecationWarning: 'set_max_rows' is replaced by 'matrix.options.max_
↪rows'
See https://github.com/sagemath/sage/issues/30552 for details.
```

sage.matrix.matrix0.**unpickle**(*cls*, *parent*, *immutability*, *cache*, *data*, *version*)

Unpickle a matrix. This is only used internally by Sage. Users should never call this function directly.

EXAMPLES: We illustrating saving and loading several different types of matrices.

OVER $\mathbf{Z}$:

```
sage: A = matrix(ZZ,2,range(4))
sage: loads(dumps(A)) # indirect doctest
[0 1]
[2 3]
```

Sparse OVER $\mathbf{Q}$:

Dense over $\mathbf{Q}[x, y]$:

Dense over finite field.

CHAPTER

SEVEN

BASE CLASS FOR MATRICES, PART 1

For design documentation see *sage.matrix.docs*.

class sage.matrix.matrix1.**Matrix**

Bases: *Matrix*

augment(*right*, *subdivide=False*)

Returns a new matrix formed by appending the matrix (or vector) `right` on the right side of `self`.

INPUT:

- `right` - a matrix, vector or free module element, whose dimensions are compatible with `self`.
- `subdivide` - default: `False` - request the resulting matrix to have a new subdivision, separating `self` from `right`.

OUTPUT:

A new matrix formed by appending `right` onto the right side of `self`. If `right` is a vector (or free module element) then in this context it is appropriate to consider it as a column vector. (The code first converts a vector to a 1-column matrix.)

If `subdivide` is `True` then any column subdivisions for the two matrices are preserved, and a new subdivision is added between `self` and `right`. If the row divisions are identical, then they are preserved, otherwise they are discarded. When `subdivide` is `False` there is no subdivision information in the result.

Warning: If `subdivide` is `True` then unequal row subdivisions will be discarded, since it would be ambiguous how to interpret them. If the subdivision behavior is not what you need, you can manage subdivisions yourself with methods like *get_subdivisions()* and *subdivide()*. You might also find `block_matrix()` or `block_diagonal_matrix()` useful and simpler in some instances.

EXAMPLES:

Augmenting with a matrix.

```
sage: A = matrix(QQ, 3, range(12))
sage: B = matrix(QQ, 3, range(9))
sage: A.augment(B)
[ 0  1  2  3  0  1  2]
[ 4  5  6  7  3  4  5]
[ 8  9 10 11  6  7  8]
```

Augmenting with a vector.

```
sage: A = matrix(QQ, 2, [0, 2, 4, 6, 8, 10])
sage: v = vector(QQ, 2, [100, 200])
sage: A.augment(v)
[  0   2   4 100]
[  6   8  10 200]
```

Errors are raised if the sizes are incompatible.

```
sage: A = matrix(RR, [[1, 2],[3, 4]])
sage: B = matrix(RR, [[10, 20], [30, 40], [50, 60]])
sage: A.augment(B)
Traceback (most recent call last):
...
TypeError: number of rows must be the same, 2 != 3

sage: v = vector(RR, [100, 200, 300])
sage: A.augment(v)
Traceback (most recent call last):
...
TypeError: number of rows must be the same, 2 != 3
```

Setting `subdivide` to `True` will, in its simplest form, add a subdivision between `self` and `right`.

```
sage: A = matrix(QQ, 3, range(12))
sage: B = matrix(QQ, 3, range(15))
sage: A.augment(B, subdivide=True)
[ 0  1  2  3| 0  1  2  3  4]
[ 4  5  6  7| 5  6  7  8  9]
[ 8  9 10 11|10 11 12 13 14]
```

Column subdivisions are preserved by augmentation, and enriched, if subdivisions are requested. (So multiple augmentations can be recorded.)

```
sage: A = matrix(QQ, 3, range(6))
sage: A.subdivide(None, [1])
sage: B = matrix(QQ, 3, range(9))
sage: B.subdivide(None, [2])
sage: A.augment(B, subdivide=True)
[0|1|0 1|2]
[2|3|3 4|5]
[4|5|6 7|8]
```

Row subdivisions can be preserved, but only if they are identical. Otherwise, this information is discarded and must be managed separately.

```
sage: A = matrix(QQ, 3, range(6))
sage: A.subdivide([1,3], None)
sage: B = matrix(QQ, 3, range(9))
sage: B.subdivide([1,3], None)
sage: A.augment(B, subdivide=True)
[0 1|0 1 2]
[---+-----]
[2 3|3 4 5]
```

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```
[4 5|6 7 8]
[---+-----]

sage: A.subdivide([1,2], None)
sage: A.augment(B, subdivide=True)
[0 1|0 1 2]
[2 3|3 4 5]
[4 5|6 7 8]
```

The result retains the base ring of `self` by coercing the elements of `right` into the base ring of `self`.

```
sage: A = matrix(QQ, 2, [1,2])
sage: B = matrix(RR, 2, [sin(1.1), sin(2.2)])
sage: C = A.augment(B); C
[                  1 183017397/205358938]
[                  2 106580492/131825561]
sage: C.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field

sage: D = B.augment(A); D
[0.89120736006...  1.00000000000000]
[0.80849640381...  2.00000000000000]
sage: D.parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Field with 53 bits of␣
↪precision
```

Sometimes it is not possible to coerce into the base ring of `self`. A solution is to change the base ring of `self` to a more expansive ring. Here we mix the rationals with a ring of polynomials with rational coefficients.

```
sage: R.<y> = PolynomialRing(QQ)
sage: A = matrix(QQ, 1, [1,2])
sage: B = matrix(R, 1, [y, y^2])

sage: C = B.augment(A); C
[  y y^2   1   2]
sage: C.parent()
Full MatrixSpace of 1 by 4 dense matrices over Univariate Polynomial Ring in y␣
↪over Rational Field

sage: D = A.augment(B)
Traceback (most recent call last):
...
TypeError: not a constant polynomial

sage: E = A.change_ring(R)
sage: F = E.augment(B); F
[  1   2   y y^2]
sage: F.parent()
Full MatrixSpace of 1 by 4 dense matrices over Univariate Polynomial Ring in y␣
↪over Rational Field
```

AUTHORS:

- Naqi Jaffery (2006-01-24): examples
- Rob Beezer (2010-12-07): vector argument, docstring, subdivisions

block_sum(*other*)

Return the block matrix that has self and other on the diagonal:

```
[ self     0 ]
[    0 other ]
```

EXAMPLES:

```
sage: A = matrix(QQ[['t']], 2, range(1, 5))
sage: A.block_sum(100*A)
[  1   2   0   0]
[  3   4   0   0]
[  0   0 100 200]
[  0   0 300 400]
```

column(*i*, *from_list=False*)

Return the `i`'th column of this matrix as a vector.

This column is a dense vector if and only if the matrix is a dense matrix.

INPUT:

- `i` - integer
- `from_list` - bool (default: False); if true, returns the `i`'th element of `self.columns()` (see *columns()*), which may be faster, but requires building a list of all columns the first time it is called after an entry of the matrix is changed.

EXAMPLES:

```
sage: a = matrix(2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.column(1)
(1, 4)
```

If the column is negative, it wraps around, just like with list indexing, e.g., -1 gives the right-most column:

```
sage: a.column(-1)
(2, 5)
```

column_ambient_module(*base_ring=None*, *sparse=None*)

Return the free module that contains the columns of the matrix.

EXAMPLES:

```
sage: M = matrix(Zmod(5), 2, 3)
sage: M.column_ambient_module()
Vector space of dimension 2 over Ring of integers modulo 5
sage: M.column(1).parent() == M.column_ambient_module()
True

sage: M = Matrix(ZZ, 3, 4)
```

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```
sage: M.column_ambient_module()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: M.column_ambient_module(QQ)
Vector space of dimension 3 over Rational Field

sage: M = Matrix(QQ, 4, 5)
sage: M.column_ambient_module()
Vector space of dimension 4 over Rational Field
sage: M.column_ambient_module(ZZ)
Ambient free module of rank 4 over the principal ideal domain Integer Ring
```

columns(*copy=True*)

Return a list of the columns of self.

INPUT:

- copy - (default: True) if True, return a copy of the list of columns which is safe to change.

If self is a sparse matrix, columns are returned as sparse vectors, otherwise returned vectors are dense.

EXAMPLES:

```
sage: matrix(3, [1..9]).columns()
[(1, 4, 7), (2, 5, 8), (3, 6, 9)]
sage: matrix(RR, 2, [sqrt(2), pi, exp(1), 0]).columns()
[(1.41421356237310, 2.71828182845905), (3.14159265358979, 0.000000000000000)]
sage: matrix(RR, 0, 2, []).columns()
[(), ()]
sage: matrix(RR, 2, 0, []).columns()
[]
sage: m = matrix(RR, 3, 3, {(1,2): pi, (2, 2): -1, (0,1): sqrt(2)})
sage: parent(m.columns()[0])
Sparse vector space of dimension 3 over Real Field with 53 bits of precision
```

Sparse matrices produce sparse columns.

```
sage: A = matrix(QQ, 2, range(4), sparse=True)
sage: v = A.columns()[0]
sage: v.is_sparse()
True
```

delete_columns(*dcols*, *check=True*)

Return the matrix constructed from deleting the columns with indices in the dcols list.

INPUT:

- dcols - list of indices of columns to be deleted from self.
- check - checks whether any index in dcols is out of range. Defaults to True.

See also:

The methods *delete_rows()* and *matrix_from_columns()* are related.

EXAMPLES:

```
sage: A = Matrix(3,4,range(12)); A
[ 0  1  2  3]
[ 4  5  6  7]
[ 8  9 10 11]
sage: A.delete_columns([0,2])
[ 1  3]
[ 5  7]
[ 9 11]
```

dcols can be a tuple. But only the underlying set of indices matters.

```
sage: A.delete_columns((2,0,2))
[ 1  3]
[ 5  7]
[ 9 11]
```

The default is to check whether any index in dcols is out of range.

```
sage: A.delete_columns([-1,2,4])
Traceback (most recent call last):
...
IndexError: [-1, 4] contains invalid indices
sage: A.delete_columns([-1,2,4], check=False)
[ 0  1  3]
[ 4  5  7]
[ 8  9 11]
```

AUTHORS:

- Wai Yan Pong (2012-03-05)

delete_rows(*drows*, *check=True*)

Return the matrix constructed from deleting the rows with indices in the drows list.

INPUT:

- drows - list of indices of rows to be deleted from self.
- check - checks whether any index in drows is out of range. Defaults to True.

See also:

The methods *delete_columns()* and *matrix_from_rows()* are related.

EXAMPLES:

```
sage: A = Matrix(4,3,range(12)); A
[ 0  1  2]
[ 3  4  5]
[ 6  7  8]
[ 9 10 11]
sage: A.delete_rows([0,2])
[ 3  4  5]
[ 9 10 11]
```

drows can be a tuple. But only the underlying set of indices matters.

```
sage: A.delete_rows((2,0,2))
[ 3  4  5]
[ 9 10 11]
```

The default is to check whether the any index in drows is out of range.

```
sage: A.delete_rows([-1,2,4])
Traceback (most recent call last):
...
IndexError: [-1, 4] contains invalid indices
sage: A.delete_rows([-1,2,4], check=False)
[ 0  1  2]
[ 3  4  5]
[ 9 10 11]
```

dense_columns(*copy=True*)

Return list of the dense columns of self.

INPUT:

- copy - (default: True) if True, return a copy so you can modify it safely

EXAMPLES:

An example over the integers:

```
sage: a = matrix(3,3,range(9)); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: a.dense_columns()
[(0, 3, 6), (1, 4, 7), (2, 5, 8)]
```

We do an example over a polynomial ring:

```
sage: R.<x> = QQ[ ]
sage: a = matrix(R, 2, [x,x^2, 2/3*x,1+x^5]); a
[      x     x^2]
[  2/3*x x^5 + 1]
sage: a.dense_columns()
[(x, 2/3*x), (x^2, x^5 + 1)]
sage: a = matrix(R, 2, [x,x^2, 2/3*x,1+x^5], sparse=True)
sage: c = a.dense_columns(); c
[(x, 2/3*x), (x^2, x^5 + 1)]
sage: parent(c[1])
Ambient free module of rank 2 over the principal ideal domain Univariate␣
↪Polynomial Ring in x over Rational Field
```

dense_matrix()

If this matrix is sparse, return a dense matrix with the same entries. If this matrix is dense, return this matrix (not a copy).

Note: The definition of "dense" and "sparse" in Sage have nothing to do with the number of nonzero entries. Sparse and dense are properties of the underlying representation of the matrix.

EXAMPLES:

```
sage: A = MatrixSpace(QQ,2, sparse=True)([1,2,0,1])
sage: A.is_sparse()
True
sage: B = A.dense_matrix()
sage: B.is_sparse()
False
sage: A == B
True
sage: B.dense_matrix() is B
True
sage: A*B
[1 4]
[0 1]
sage: A.parent()
Full MatrixSpace of 2 by 2 sparse matrices over Rational Field
sage: B.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

In Sage, the product of a sparse and a dense matrix is always dense:

```
sage: (A*B).parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: (B*A).parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

dense_rows(*copy=True*)

Return list of the dense rows of self.

INPUT:

- copy - (default: True) if True, return a copy so you can modify it safely (note that the individual vectors in the copy should not be modified since they are mutable!)

EXAMPLES:

```
sage: m = matrix(3, range(9)); m
[0 1 2]
[3 4 5]
[6 7 8]
sage: v = m.dense_rows(); v
[(0, 1, 2), (3, 4, 5), (6, 7, 8)]
sage: v is m.dense_rows()
False
sage: m.dense_rows(copy=False) is m.dense_rows(copy=False)
True
sage: m[0,0] = 10
sage: m.dense_rows()
[(10, 1, 2), (3, 4, 5), (6, 7, 8)]
```

lift()

Return lift of self to the covering ring of the base ring R, which is by definition the ring returned by calling cover_ring() on R, or just R itself if the cover_ring method is not defined.

EXAMPLES:

```
sage: M = Matrix(Integers(7), 2, 2, [5, 9, 13, 15]) ; M
[5 2]
[6 1]
sage: M.lift()
[5 2]
[6 1]
sage: parent(M.lift())
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
```

The field QQ doesn't have a cover_ring method:

```
sage: hasattr(QQ, 'cover_ring')
False
```

So lifting a matrix over QQ gives back the same exact matrix.

```
sage: B = matrix(QQ, 2, [1..4])
sage: B.lift()
[1 2]
[3 4]
sage: B.lift() is B
True
```

lift_centered()

Apply the lift_centered method to every entry of self.

OUTPUT:

If self is a matrix over the Integers mod n, this method returns the unique matrix m such that m is congruent to self mod n and for every entry $m[i,j]$ we have $-n/2 < m[i,j] \leq n/2$. If the coefficient ring does not have a cover_ring method, return self.

EXAMPLES:

```
sage: M = Matrix(Integers(8), 2, 4, range(8)) ; M
[0 1 2 3]
[4 5 6 7]
sage: L = M.lift_centered(); L
[ 0  1  2  3]
[ 4 -3 -2 -1]
sage: parent(L)
Full MatrixSpace of 2 by 4 dense matrices over Integer Ring
```

The returned matrix is congruent to M modulo 8.:

```
sage: L.mod(8)
[0 1 2 3]
[4 5 6 7]
```

The field QQ doesn't have a cover_ring method:

```
sage: hasattr(QQ, 'cover_ring')
False
```

So lifting a matrix over QQ gives back the same exact matrix.

```
sage: B = matrix(QQ, 2, [1..4])
sage: B.lift_centered()
[1 2]
[3 4]
sage: B.lift_centered() is B
True
```

matrix_from_columns(*columns*)

Return the matrix constructed from self using columns with indices in the columns list.

EXAMPLES:

```
sage: M = MatrixSpace(Integers(8),3,3)
sage: A = M(range(9)); A
[0 1 2]
[3 4 5]
[6 7 0]
sage: A.matrix_from_columns([2,1])
[2 1]
[5 4]
[0 7]
```

matrix_from_rows(*rows*)

Return the matrix constructed from self using rows with indices in the rows list.

EXAMPLES:

```
sage: M = MatrixSpace(Integers(8),3,3)
sage: A = M(range(9)); A
[0 1 2]
[3 4 5]
[6 7 0]
sage: A.matrix_from_rows([2,1])
[6 7 0]
[3 4 5]
```

matrix_from_rows_and_columns(*rows*, *columns*)

Return the matrix constructed from self from the given rows and columns.

EXAMPLES:

```
sage: M = MatrixSpace(Integers(8),3,3)
sage: A = M(range(9)); A
[0 1 2]
[3 4 5]
[6 7 0]
sage: A.matrix_from_rows_and_columns([1], [0,2])
[3 5]
sage: A.matrix_from_rows_and_columns([1,2], [1,2])
[4 5]
[7 0]
```

Note that row and column indices can be reordered or repeated:

```
sage: A.matrix_from_rows_and_columns([2,1], [2,1])
[0 7]
[5 4]
```

For example here we take from row 1 columns 2 then 0 twice, and do this 3 times.

```
sage: A.matrix_from_rows_and_columns([1,1,1],[2,0,0])
[5 3 3]
[5 3 3]
[5 3 3]
```

AUTHORS:

- Jaap Spies (2006-02-18)
- Didier Deshommes: some Pyrex speedups implemented

matrix_over_field()

Return copy of this matrix, but with entries viewed as elements of the fraction field of the base ring (assuming it is defined).

EXAMPLES:

```
sage: A = MatrixSpace(IntegerRing(),2)([1,2,3,4])
sage: B = A.matrix_over_field()
sage: B
[1 2]
[3 4]
sage: B.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

matrix_space(*nrows=None*, *ncols=None*, *sparse=None*)

Return the ambient matrix space of self.

INPUT:

- `nrows`, `ncols` - (optional) number of rows and columns in returned matrix space.
- `sparse` - whether the returned matrix space uses sparse or dense matrices.

EXAMPLES:

```
sage: m = matrix(3, [1..9])
sage: m.matrix_space()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
sage: m.matrix_space(ncols=2)
Full MatrixSpace of 3 by 2 dense matrices over Integer Ring
sage: m.matrix_space(1)
Full MatrixSpace of 1 by 3 dense matrices over Integer Ring
sage: m.matrix_space(1, 2, True)
Full MatrixSpace of 1 by 2 sparse matrices over Integer Ring

sage: M = MatrixSpace(QQ, 3, implementation='generic')
sage: m = M.an_element()
sage: m.matrix_space()
Full MatrixSpace of 3 by 3 dense matrices over Rational Field (using Matrix_
↪generic_dense)
```

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```
sage: m.matrix_space(nrows=2, ncols=12)
Full MatrixSpace of 2 by 12 dense matrices over Rational Field (using Matrix_
↪generic_dense)
sage: m.matrix_space(nrows=2, sparse=True)
Full MatrixSpace of 2 by 3 sparse matrices over Rational Field
```

new_matrix(*nrows=None*, *ncols=None*, *entries=None*, *coerce=True*, *copy=True*, *sparse=None*)

Create a matrix in the parent of this matrix with the given number of rows, columns, etc. The default parameters are the same as for self.

INPUT:

These three variables get sent to *matrix_space()*:

- `nrows`, `ncols` - number of rows and columns in returned matrix. If not specified, defaults to `None` and will give a matrix of the same size as self.
- `sparse` - whether returned matrix is sparse or not. Defaults to same value as self.

The remaining three variables (`coerce`, `entries`, and `copy`) are used by *sage.matrix.matrix_space.MatrixSpace()* to construct the new matrix.

Warning: This function called with no arguments returns the zero matrix of the same dimension and sparseness of self.

EXAMPLES:

```
sage: A = matrix(ZZ,2,2,[1,2,3,4]); A
[1 2]
[3 4]
sage: A.new_matrix()
[0 0]
[0 0]
sage: A.new_matrix(1,1)
[0]
sage: A.new_matrix(3,3).parent()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
```

```
sage: A = matrix(RR,2,3,[1.1,2.2,3.3,4.4,5.5,6.6]); A
[1.10000000000000 2.20000000000000 3.30000000000000]
[4.40000000000000 5.50000000000000 6.60000000000000]
sage: A.new_matrix()
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: A.new_matrix().parent()
Full MatrixSpace of 2 by 3 dense matrices over Real Field with 53 bits of␣
↪precision
```

```
sage: M = MatrixSpace(ZZ, 2, 3, implementation='generic')
sage: m = M.an_element()
sage: m.new_matrix().parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring (using Matrix_
```

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```
↪generic_dense)
sage: m.new_matrix(3,3).parent()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring (using Matrix_
↪generic_dense)
sage: m.new_matrix(3,3, sparse=True).parent()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
```

numpy(*dtype=None*)

Return the Numpy matrix associated to this matrix.

INPUT:

- `dtype` - The desired data-type for the array. If not given, then the type will be determined as the minimum type required to hold the objects in the sequence.

EXAMPLES:

```
sage: a = matrix(3,range(12))
sage: a.numpy()
array([[ 0,  1,  2,  3],
       [ 4,  5,  6,  7],
       [ 8,  9, 10, 11]])
sage: a.numpy('f')
array([[  0.,   1.,   2.,   3.],
       [  4.,   5.,   6.,   7.],
       [  8.,   9.,  10.,  11.]], dtype=float32)
sage: a.numpy('d')
array([[  0.,   1.,   2.,   3.],
       [  4.,   5.,   6.,   7.],
       [  8.,   9.,  10.,  11.]])
sage: a.numpy('B')
array([[ 0,  1,  2,  3],
       [ 4,  5,  6,  7],
       [ 8,  9, 10, 11]], dtype=uint8)
```

Type `numpy.typecodes` for a list of the possible typecodes:

```
sage: import numpy
sage: sorted(numpy.typecodes.items())
[('All', '?bhilqpBHILQPefdgFDGSUVOMm'), ('AllFloat', 'efdgFDG'), ('AllInteger',
↪'bBhHiIlLqQpP'), ('Character', 'c'), ('Complex', 'FDG'), ('Datetime', 'Mm'), (
↪'Float', 'efdg'), ('Integer', 'bhilqp'), ('UnsignedInteger', 'BHILQP')]
```

Alternatively, numpy automatically calls this function (via the magic `__array__()` method) to convert Sage matrices to numpy arrays:

```
sage: import numpy
sage: b=numpy.array(a); b
array([[ 0,  1,  2,  3],
       [ 4,  5,  6,  7],
       [ 8,  9, 10, 11]])
sage: b.dtype
dtype('int32')  # 32-bit
dtype('int64')  # 64-bit
```

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```
sage: b.shape
(3, 4)
```

row(*i*, *from_list=False*)

Return the `i`'th row of this matrix as a vector.

This row is a dense vector if and only if the matrix is a dense matrix.

INPUT:

- `i` - integer
- `from_list` - bool (default: False); if true, returns the `i`'th element of `self.rows()` (see *rows()*), which may be faster, but requires building a list of all rows the first time it is called after an entry of the matrix is changed.

EXAMPLES:

```
sage: a = matrix(2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.row(0)
(0, 1, 2)
sage: a.row(1)
(3, 4, 5)
sage: a.row(-1)  # last row
(3, 4, 5)
```

row_ambient_module(*base_ring=None*, *sparse=None*)

Return the free module that contains the rows of the matrix.

EXAMPLES:

```
sage: M = matrix(Zmod(5), 2, 3)
sage: M.row_ambient_module()
Vector space of dimension 3 over Ring of integers modulo 5
sage: M.row(1).parent() == M.row_ambient_module()
True

sage: M = Matrix(ZZ, 3, 4)
sage: M.row_ambient_module()
Ambient free module of rank 4 over the principal ideal domain Integer Ring
sage: M.row_ambient_module(QQ)
Vector space of dimension 4 over Rational Field

sage: M = Matrix(QQ, 4, 5)
sage: M.row_ambient_module()
Vector space of dimension 5 over Rational Field
sage: M.row_ambient_module(ZZ)
Ambient free module of rank 5 over the principal ideal domain Integer Ring
```

rows(*copy=True*)

Return a list of the rows of self.

INPUT:

- copy - (default: True) if True, return a copy of the list of rows which is safe to change.

If self is a sparse matrix, rows are returned as sparse vectors, otherwise returned vectors are dense.

EXAMPLES:

```
sage: matrix(3, [1..9]).rows()
[(1, 2, 3), (4, 5, 6), (7, 8, 9)]
sage: matrix(RR, 2, [sqrt(2), pi, exp(1), 0]).rows()
[(1.41421356237310, 3.14159265358979), (2.71828182845905, 0.000000000000000)]
sage: matrix(RR, 0, 2, []).rows()
[]
sage: matrix(RR, 2, 0, []).rows()
[(), ()]
sage: m = matrix(RR, 3, 3, {(1,2): pi, (2, 2): -1, (0,1): sqrt(2)})
sage: parent(m.rows()[0])
Sparse vector space of dimension 3 over Real Field with 53 bits of precision
```

Sparse matrices produce sparse rows.

```
sage: A = matrix(QQ, 2, range(4), sparse=True)
sage: v = A.rows()[0]
sage: v.is_sparse()
True
```

set_column(*col*, *v*)

Sets the entries of column col to the entries of v.

INPUT:

- col - index of column to be set.
- v - a list or vector of the new entries.

OUTPUT:

Changes the matrix in-place, so there is no output.

EXAMPLES:

New entries may be contained in a vector.:

```
sage: A = matrix(QQ, 5, range(25))
sage: u = vector(QQ, [0, -1, -2, -3, -4])
sage: A.set_column(2, u)
sage: A
[ 0  1  0  3  4]
[ 5  6 -1  8  9]
[10 11 -2 13 14]
[15 16 -3 18 19]
[20 21 -4 23 24]
```

New entries may be in any sort of list.:

```
sage: A = matrix([[1, 2], [3, 4]]); A
[1 2]
[3 4]
sage: A.set_column(0, [0, 0]); A
```

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```
[0 2]
[0 4]
sage: A.set_column(1, (0, 0)); A
[0 0]
[0 0]
```

set_row(*row*, *v*)

Sets the entries of row `row` to the entries of `v`.

INPUT:

- `row` - index of row to be set.
- `v` - a list or vector of the new entries.

OUTPUT:

Changes the matrix in-place, so there is no output.

EXAMPLES:

New entries may be contained in a vector.:

```
sage: A = matrix(QQ, 5, range(25))
sage: u = vector(QQ, [0, -1, -2, -3, -4])
sage: A.set_row(2, u)
sage: A
[ 0  1  2  3  4]
[ 5  6  7  8  9]
[ 0 -1 -2 -3 -4]
[15 16 17 18 19]
[20 21 22 23 24]
```

New entries may be in any sort of list.:

```
sage: A = matrix([[1, 2], [3, 4]]); A
[1 2]
[3 4]
sage: A.set_row(0, [0, 0]); A
[0 0]
[3 4]
sage: A.set_row(1, (0, 0)); A
[0 0]
[0 0]
```

sparse_columns(*copy=True*)

Return a list of the columns of `self` as sparse vectors (or free module elements).

INPUT:

- **copy - (default: True) if True, return a copy so you can**
  modify it safely

EXAMPLES:

```
sage: a = matrix(2,3,range(6)); a
[0 1 2]
```

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```
[3 4 5]
sage: v = a.sparse_columns(); v
[(0, 3), (1, 4), (2, 5)]
sage: v[1].is_sparse()
True
```

sparse_matrix()

If this matrix is dense, return a sparse matrix with the same entries. If this matrix is sparse, return this matrix (not a copy).

Note: The definition of "dense" and "sparse" in Sage have nothing to do with the number of nonzero entries. Sparse and dense are properties of the underlying representation of the matrix.

EXAMPLES:

```
sage: A = MatrixSpace(QQ,2, sparse=False)([1,2,0,1])
sage: A.is_sparse()
False
sage: B = A.sparse_matrix()
sage: B.is_sparse()
True
sage: A == B
True
sage: B.sparse_matrix() is B
True
sage: A*B
[1 4]
[0 1]
sage: A.parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: B.parent()
Full MatrixSpace of 2 by 2 sparse matrices over Rational Field
sage: (A*B).parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: (B*A).parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

sparse_rows(*copy=True*)

Return a list of the rows of `self` as sparse vectors (or free module elements).

INPUT:

- **copy - (default: True) if True, return a copy so you can**
  modify it safely

EXAMPLES:

```
sage: m = Mat(ZZ,3,3,sparse=True)(range(9)); m
[0 1 2]
[3 4 5]
[6 7 8]
sage: v = m.sparse_rows(); v
```

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```
[(0, 1, 2), (3, 4, 5), (6, 7, 8)]
sage: m.sparse_rows(copy=False) is m.sparse_rows(copy=False)
True
sage: v[1].is_sparse()
True
sage: m[0,0] = 10
sage: m.sparse_rows()
[(10, 1, 2), (3, 4, 5), (6, 7, 8)]
```

stack(*bottom*, *subdivide=False*)

Return a new matrix formed by appending the matrix (or vector) `bottom` below `self`:

```
[  self  ]
[ bottom ]
```

INPUT:

- `bottom` - a matrix, vector or free module element, whose dimensions are compatible with `self`.
- `subdivide` - default: `False` - request the resulting matrix to have a new subdivision, separating `self` from `bottom`.

OUTPUT:

A new matrix formed by appending `bottom` beneath `self`. If `bottom` is a vector (or free module element) then in this context it is appropriate to consider it as a row vector. (The code first converts a vector to a 1-row matrix.)

If `subdivide` is `True` then any row subdivisions for the two matrices are preserved, and a new subdivision is added between `self` and `bottom`. If the column divisions are identical, then they are preserved, otherwise they are discarded. When `subdivide` is `False` there is no subdivision information in the result.

Warning: If `subdivide` is `True` then unequal column subdivisions will be discarded, since it would be ambiguous how to interpret them. If the subdivision behavior is not what you need, you can manage subdivisions yourself with methods like *subdivisions()* and *subdivide()*. You might also find `block_matrix()` or `block_diagonal_matrix()` useful and simpler in some instances.

EXAMPLES:

Stacking with a matrix.

```
sage: A = matrix(QQ, 4, 3, range(12))
sage: B = matrix(QQ, 3, 3, range(9))
sage: A.stack(B)
[ 0  1  2]
[ 3  4  5]
[ 6  7  8]
[ 9 10 11]
[ 0  1  2]
[ 3  4  5]
[ 6  7  8]
```

Stacking with a vector.

```
sage: A = matrix(QQ, 3, 2, [0, 2, 4, 6, 8, 10])
sage: v = vector(QQ, 2, [100, 200])
sage: A.stack(v)
[  0   2]
[  4   6]
[  8  10]
[100 200]
```

Errors are raised if the sizes are incompatible.

```
sage: A = matrix(RR, [[1, 2],[3, 4]])
sage: B = matrix(RR, [[10, 20, 30], [40, 50, 60]])
sage: A.stack(B)
Traceback (most recent call last):
...
TypeError: number of columns must be the same, not 2 and 3

sage: v = vector(RR, [100, 200, 300])
sage: A.stack(v)
Traceback (most recent call last):
...
TypeError: number of columns must be the same, not 2 and 3
```

Setting `subdivide` to `True` will, in its simplest form, add a subdivision between `self` and `bottom`.

```
sage: A = matrix(QQ, 2, 5, range(10))
sage: B = matrix(QQ, 3, 5, range(15))
sage: A.stack(B, subdivide=True)
[ 0  1  2  3  4]
[ 5  6  7  8  9]
[--------------]
[ 0  1  2  3  4]
[ 5  6  7  8  9]
[10 11 12 13 14]
```

Row subdivisions are preserved by stacking, and enriched, if subdivisions are requested. (So multiple stackings can be recorded.)

```
sage: A = matrix(QQ, 2, 4, range(8))
sage: A.subdivide([1], None)
sage: B = matrix(QQ, 3, 4, range(12))
sage: B.subdivide([2], None)
sage: A.stack(B, subdivide=True)
[ 0  1  2  3]
[-----------]
[ 4  5  6  7]
[-----------]
[ 0  1  2  3]
[ 4  5  6  7]
[-----------]
[ 8  9 10 11]
```

Column subdivisions can be preserved, but only if they are identical. Otherwise, this information is discarded and must be managed separately.

```
sage: A = matrix(QQ, 2, 5, range(10))
sage: A.subdivide(None, [2,4])
sage: B = matrix(QQ, 3, 5, range(15))
sage: B.subdivide(None, [2,4])
sage: A.stack(B, subdivide=True)
[ 0  1| 2  3| 4]
[ 5  6| 7  8| 9]
[-----+-----+--]
[ 0  1| 2  3| 4]
[ 5  6| 7  8| 9]
[10 11|12 13|14]

sage: A.subdivide(None, [1,2])
sage: A.stack(B, subdivide=True)
[ 0  1  2  3  4]
[ 5  6  7  8  9]
[--------------]
[ 0  1  2  3  4]
[ 5  6  7  8  9]
[10 11 12 13 14]
```

The base ring of the result is the common parent for the base rings of `self` and `bottom`. In particular, the parent for `A.stack(B)` and `B.stack(A)` should be equal:

```
sage: A = matrix(QQ, 1, 2, [1,2])
sage: B = matrix(RR, 1, 2, [sin(1.1), sin(2.2)])
sage: C = A.stack(B); C
[ 1.00000000000000  2.00000000000000]
[0.891207360061435 0.808496403819590]
sage: C.parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Field with 53 bits of␣
↪precision

sage: D = B.stack(A); D
[0.891207360061435 0.808496403819590]
[ 1.00000000000000  2.00000000000000]
sage: D.parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Field with 53 bits of␣
↪precision
```

```
sage: R.<y> = PolynomialRing(ZZ)
sage: A = matrix(QQ, 1, 2, [1, 2/3])
sage: B = matrix(R, 1, 2, [y, y^2])

sage: C = A.stack(B); C
[  1 2/3]
[  y y^2]
sage: C.parent()
Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in y␣
↪over Rational Field
```

Stacking a dense matrix atop a sparse one returns a sparse matrix:

```
sage: M = Matrix(ZZ, 2, 3, range(6), sparse=False)
sage: N = diagonal_matrix([10,11,12], sparse=True)
sage: P = M.stack(N); P
[ 0  1  2]
[ 3  4  5]
[10  0  0]
[ 0 11  0]
[ 0  0 12]
sage: P.is_sparse()
True
sage: P = N.stack(M); P
[10  0  0]
[ 0 11  0]
[ 0  0 12]
[ 0  1  2]
[ 3  4  5]
sage: P.is_sparse()
True
```

One can stack matrices over different rings (github issue #16399).

```
sage: M = Matrix(ZZ, 2, 3, range(6))
sage: N = Matrix(QQ, 1, 3, [10,11,12])
sage: M.stack(N)
[ 0  1  2]
[ 3  4  5]
[10 11 12]
sage: N.stack(M)
[10 11 12]
[ 0  1  2]
[ 3  4  5]
```

AUTHORS:

- Rob Beezer (2011-03-19): rewritten to mirror code for *augment()*
- Jeroen Demeyer (2015-01-06): refactor, see github issue #16399. Put all boilerplate in one place (here) and put the actual type-dependent implementation in `_stack_impl`.

submatrix(*row=0*, *col=0*, *nrows=-1*, *ncols=-1*)

Return the matrix constructed from self using the specified range of rows and columns.

INPUT:

- `row`, `col` - index of the starting row and column. Indices start at zero.
- `nrows`, `ncols` - (optional) number of rows and columns to take. If not provided, take all rows below and all columns to the right of the starting entry.

See also:

The functions *matrix_from_rows()*, *matrix_from_columns()*, and *matrix_from_rows_and_columns()* allow one to select arbitrary subsets of rows and/or columns.

EXAMPLES:

Take the 3×3 submatrix starting from entry (1,1) in a 4×4 matrix:

```
sage: m = matrix(4, [1..16])
sage: m.submatrix(1, 1)
[ 6  7  8]
[10 11 12]
[14 15 16]
```

Same thing, except take only two rows:

```
sage: m.submatrix(1, 1, 2)
[ 6  7  8]
[10 11 12]
```

And now take only one column:

```
sage: m.submatrix(1, 1, 2, 1)
[ 6]
[10]
```

You can take zero rows or columns if you want:

```
sage: m.submatrix(1, 1, 0)
[]
sage: parent(m.submatrix(1, 1, 0))
Full MatrixSpace of 0 by 3 dense matrices over Integer Ring
```

zero_pattern_matrix(*ring=None*)

Return a matrix that contains one for corresponding zero entries.

All other entries are zero.

INPUT:

- `ring` – (optional); base ring of the output; default is ZZ

OUTPUT:

A new dense matrix with same dimensions as `self` and with base ring `ring`.

EXAMPLES:

```
sage: M = Matrix(ZZ, 2, [1,2,-2,0])
sage: M.zero_pattern_matrix()
[0 0]
[0 1]

sage: M = Matrix(QQ, 2, [1,2/3,-2,0])
sage: M.zero_pattern_matrix()
[0 0]
[0 1]
```

Default base ring for the output is ZZ:

```
sage: M.zero_pattern_matrix().base_ring()
Integer Ring
```

Specify a different base ring for the output:

```
sage: M.zero_pattern_matrix(GF(2)).base_ring()
Finite Field of size 2
```

Examples for different base rings for `self`:

```
sage: M = Matrix(Zmod(8), 3, 2, [2, 3, 9, 8, 1, 0]); M
[2 3]
[1 0]
[1 0]
sage: M.zero_pattern_matrix()
[0 0]
[0 1]
[0 1]
```

```
sage: W.<a> = CyclotomicField(100)
sage: M = Matrix(2, 3, [a, a/2, 0, a^2, a^100-1, a^2 - a]); M
[      a   1/2*a       0]
[    a^2       0 a^2 - a]
sage: M.zero_pattern_matrix()
[0 0 1]
[0 1 0]
```

```
sage: K.<a> = GF(2^4)
sage: l = [a^2 + 1, a^3 + 1, 0, 0, a, a^3 + a + 1, a + 1,
....:      a + 1, a^2, a^3 + a + 1, a^3 + a, a^3 + a]
sage: M = Matrix(K, 3, 4, l); M
[    a^2 + 1     a^3 + 1           0           0]
[          a a^3 + a + 1       a + 1       a + 1]
[        a^2 a^3 + a + 1     a^3 + a     a^3 + a]
sage: M.zero_pattern_matrix()
[0 0 1 1]
[0 0 0 0]
[0 0 0 0]
```

```
sage: K.<a> = GF(25)
sage: M = Matrix(K, 2, 3, [0, 2, 3, 5, a, a^2])
sage: M
[    0     2     3]
[    0     a a + 3]
sage: M.zero_pattern_matrix()
[1 0 0]
[1 0 0]
```

Note: This method can be optimized by improving `get_is_zero_unsafe()` for derived matrix classes.

CHAPTER

EIGHT

BASE CLASS FOR MATRICES, PART 2

For design documentation see matrix/docs.py.

AUTHORS:

- William Stein: initial version
- Jaap Spies (2006-02-24): added `prod_of_row_sums`, `permanent`, `permanental_minor`, `rook_vector` methods
- Robert Bradshaw (2007-06-14): added `subdivide` method
- Jaap Spies (2007-11-14): implemented `_binomial`, `_choose` auxiliary functions
- William Stein (2007-11-18): added `_gram_schmidt_noscale` method
- David Loeffler (2008-12-05): added `smith_form` method
- David Loeffler (2009-06-01): added `_echelon_form_PID` method
- Sebastian Pancratz (2009-06-25): implemented `adjoint` and `charpoly` methods; fixed `adjoint` reflecting the change that `_adjoint` is now implemented in *Matrix*; used the division-free algorithm for `charpoly`
- Rob Beezer (2009-07-13): added `elementwise_product` method
- Miguel Marco (2010-06-19): modified eigenvalues and eigenvectors functions to allow the option `extend=False`
- Thierry Monteil (2010-10-05): bugfix for github issue #10063, so that the determinant is computed even for rings for which the `is_field` method is not implemented.
- Rob Beezer (2010-12-13): added `conjugate_transpose` method
- Rob Beezer (2011-02-05): refactored all of the matrix kernel routines; added `extended_echelon_form`, `right_kernel_matrix`, `QR`, `_gram_schmidt_noscale`, `is_similar` methods
- Moritz Minzlaff (2011-03-17): corrected `_echelon_form_PID` method for matrices of one row, fixed in github issue #9053
- Rob Beezer (2011-06-09): added `is_normal`, `is_diagonalizable`, `LU`, `cyclic_subspace`, `zigzag_form`, `rational_form` methods
- Rob Beezer (2012-05-27): added `indefinite_factorization`, `is_positive_definite`, `cholesky` methods
- Darij Grinberg (2013-10-01): added first (slow) pfaffian implementation
- Mario Pernici (2014-07-01): modified `rook_vector` method
- Rob Beezer (2015-05-25): modified `is_similar` method

- Samuel Lelièvre (2020-09-18): improved method `LLL_gram` based on a patch by William Stein posted at github issue #5178, moving the method from its initial location in `sage.matrix.integer_matrix_dense`
- Michael Jung (2020-10-02): added Bär-Faddeev-LeVerrier algorithm for the Pfaffian
- Moritz Firsching(2020-10-05): added `quantum_determinant`
- Dima Pasechnik (2022-11-08): fixed `echelonize` for inexact matrices

class sage.matrix.matrix2.**Matrix**

Bases: *Matrix*

Base class for matrices, part 2

C

Returns the conjugate matrix.

EXAMPLES:

```
sage: A = matrix(QQbar, [[     -3,  5 - 3*I, 7 - 4*I],
....:                    [7 + 3*I, -1 + 6*I, 3 + 5*I],
....:                    [3 + 3*I, -3 + 6*I, 5 +   I]])
sage: A.C
[      -3  5 + 3*I  7 + 4*I]
[ 7 - 3*I -1 - 6*I  3 - 5*I]
[ 3 - 3*I -3 - 6*I  5 - 1*I]
```

H

Returns the conjugate-transpose (Hermitian) matrix.

EXAMPLES:

```
sage: A = matrix(QQbar, [[     -3,  5 - 3*I, 7 - 4*I],
....:                    [7 + 3*I, -1 + 6*I, 3 + 5*I],
....:                    [3 + 3*I, -3 + 6*I, 5 +   I]])
sage: A.H
[      -3  7 - 3*I  3 - 3*I]
[ 5 + 3*I -1 - 6*I -3 - 6*I]
[ 7 + 4*I  3 - 5*I  5 - 1*I]
```

LLL_gram(*flag=0*)

Return the LLL transformation matrix for this Gram matrix.

That is, the transformation matrix U over ZZ of determinant 1 that transforms the lattice with this matrix as Gram matrix to a lattice that is LLL-reduced.

Always works when `self` is positive definite, might work in some semidefinite and indefinite cases.

INPUT:

- `self` – the Gram matrix of a quadratic form or of a lattice equipped with a bilinear form
- `flag` – an optional flag passed to `qflllgram`. According to pari:qflllgram's documentation the options are:
  - `0` – (default), assume that `self` has either exact (integral or rational) or real floating point entries. The matrix is rescaled, converted to integers and the behavior is then as in `flag=1`.
  - 1 – assume that G is integral. Computations involving Gram-Schmidt vectors are approximate, with precision varying as needed.

OUTPUT:

A dense matrix `U` over the integers with determinant 1 such that `U.T * M * U` is LLL-reduced.

ALGORITHM:

Calls PARI's pari:qflllgram.

EXAMPLES:

Create a Gram matrix and LLL-reduce it:

```
sage: M = Matrix(ZZ, 2, 2, [5, 3, 3, 2])
sage: U = M.LLL_gram()
sage: MM = U.transpose() * M * U
sage: M, U, MM
(
[5 3]  [-1  1]  [1 0]
[3 2], [ 1 -2], [0 1]
)
```

For a Gram matrix over RR with a length one first vector and a very short second vector, the LLL-reduced basis is obtained by swapping the two basis vectors (and changing sign to preserve orientation).

```
sage: M = Matrix(RDF, 2, 2, [1, 0, 0, 1e-5])
sage: M.LLL_gram()
[ 0 -1]
[ 1  0]
```

The algorithm might work for some semidefinite and indefinite forms:

```
sage: Matrix(ZZ, 2, 2, [2, 6, 6, 3]).LLL_gram()
[-3 -1]
[ 1  0]
sage: Matrix(ZZ, 2, 2, [1, 0, 0, -1]).LLL_gram()
[ 0 -1]
[ 1  0]
```

However, it might fail for others, either raising a `ValueError`:

```
sage: Matrix(ZZ, 1, 1, [0]).LLL_gram()
Traceback (most recent call last):
...
ValueError: qflllgram did not return a square matrix,
perhaps the matrix is not positive definite

sage: Matrix(ZZ, 2, 2, [0, 1, 1, 0]).LLL_gram()
Traceback (most recent call last):
...
ValueError: qflllgram did not return a square matrix,
perhaps the matrix is not positive definite
```

or running forever:

```
sage: Matrix(ZZ, 2, 2, [-5, -1, -1, -5]).LLL_gram()  # not tested
Traceback (most recent call last):
```

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```
...
RuntimeError: infinite loop while calling qflllgram
```

Nonreal input leads to a value error:

```
sage: Matrix(2, 2, [CDF(1, 1), 0, 0, 1]).LLL_gram()
Traceback (most recent call last):
...
ValueError: qflllgram failed, perhaps the matrix is not positive definite
```

LU(*pivot=None*, *format='plu'*)

Finds a decomposition into a lower-triangular matrix and an upper-triangular matrix.

INPUT:

- `pivot` - pivoting strategy
  - 'auto' (default) - see if the matrix entries are ordered (i.e. if they have an absolute value method), and if so, use a the partial pivoting strategy. Otherwise, fall back to the nonzero strategy. This is the best choice for general routines that may call this for matrix entries of a variety of types.
  - 'partial' - each column is examined for the element with the largest absolute value and the row containing this element is swapped into place.
  - 'nonzero' - the first nonzero element in a column is located and the row with this element is used.
- `format` - contents of output, see more discussion below about output.
  - 'plu' (default) - a triple; matrices P, L and U such that A = P*L*U.
  - 'compact' - a pair; row permutation as a tuple, and the matrices L and U combined into one matrix.

OUTPUT:

Suppose that A is an $m \times n$ matrix, then an LU decomposition is a lower-triangular $m \times m$ matrix L with every diagonal element equal to 1, and an upper-triangular $m \times n$ matrix, U such that the product LU, after a permutation of the rows, is then equal to A. For the 'plu' format the permutation is returned as an $m \times m$ permutation matrix P such that

$$A = PLU$$

It is more common to place the permutation matrix just to the left of A. If you desire this version, then use the inverse of P which is computed most efficiently as its transpose.

If the 'partial' pivoting strategy is used, then the non-diagonal entries of L will be less than or equal to 1 in absolute value. The 'nonzero' pivot strategy may be faster, but the growth of data structures for elements of the decomposition might counteract the advantage.

By necessity, returned matrices have a base ring equal to the fraction field of the base ring of the original matrix.

In the 'compact' format, the first returned value is a tuple that is a permutation of the rows of LU that yields A. See the doctest for how you might employ this permutation. Then the matrices L and U are merged into one matrix – remove the diagonal of ones in L and the remaining nonzero entries can replace the entries of U beneath the diagonal.

The results are cached, only in the compact format, separately for each pivot strategy called. Repeated requests for the 'plu' format will require just a small amount of overhead in each call to bust out the compact format to the three matrices. Since only the compact format is cached, the components of the compact format are immutable, while the components of the 'plu' format are regenerated, and hence are mutable.

Notice that while U is similar to row-echelon form and the rows of U span the row space of A, the rows of U are not generally linearly independent. Nor are the pivot columns (or rank) immediately obvious. However for rings without specialized echelon form routines, this method is about twice as fast as the generic echelon form routine since it only acts "below the diagonal", as would be predicted from a theoretical analysis of the algorithms.

Note: This is an exact computation, so limited to exact rings. If you need numerical results, convert the base ring to the field of real double numbers, `RDF` or the field of complex double numbers, `CDF`, which will use a faster routine that is careful about numerical subtleties.

ALGORITHM:

"Gaussian Elimination with Partial Pivoting," Algorithm 21.1 of [TB1997].

EXAMPLES:

Notice the difference in the L matrix as a result of different pivoting strategies. With partial pivoting, every entry of L has absolute value 1 or less.

```
sage: A = matrix(QQ, [[1, -1,  0,  2,  4,  7, -1],
....:                 [2, -1,  0,  6,  4,  8, -2],
....:                 [2,  0,  1,  4,  2,  6,  0],
....:                 [1,  0, -1,  8, -1, -1, -3],
....:                 [1,  1,  2, -2, -1,  1,  3]])
sage: P, L, U = A.LU(pivot='partial')
sage: P
[0 0 0 0 1]
[1 0 0 0 0]
[0 0 0 1 0]
[0 0 1 0 0]
[0 1 0 0 0]
sage: L
[   1    0    0    0    0]
[ 1/2    1    0    0    0]
[ 1/2  1/3    1    0    0]
[   1  2/3  1/5    1    0]
[ 1/2 -1/3 -2/5    0    1]
sage: U
[    2    -1     0     6     4     8    -2]
[    0   3/2     2    -5    -3    -3     4]
[    0     0  -5/3  20/3    -2    -4 -10/3]
[    0     0     0     0   2/5   4/5     0]
[    0     0     0     0   1/5   2/5     0]
sage: A == P*L*U
True
sage: P, L, U = A.LU(pivot='nonzero')
sage: P
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
sage: L
[ 1  0  0  0  0]
```

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```
[ 2  1  0  0  0]
[ 2  2  1  0  0]
[ 1  1 -1  1  0]
[ 1  2  2  0  1]
sage: U
[ 1 -1  0  2  4  7 -1]
[ 0  1  0  2 -4 -6  0]
[ 0  0  1 -4  2  4  2]
[ 0  0  0  0  1  2  0]
[ 0  0  0  0 -1 -2  0]
sage: A == P*L*U
True
```

An example of the compact format.

```
sage: B = matrix(QQ, [[ 1,  3,  5,  5],
....:                 [ 1,  4,  7,  8],
....:                 [-1, -4, -6, -6],
....:                 [ 0, -2, -5, -8],
....:                 [-2, -6, -6, -2]])
sage: perm, M = B.LU(format='compact')
sage: perm
(4, 3, 0, 1, 2)
sage: M
[  -2   -6   -6   -2]
[   0   -2   -5   -8]
[-1/2    0    2    4]
[-1/2 -1/2  3/4    0]
[ 1/2  1/2 -1/4    0]
```

We can easily illustrate the relationships between the two formats with a square matrix.

```
sage: C = matrix(QQ, [[-2,  3, -2, -5],
....:                 [ 1, -2,  1,  3],
....:                 [-4,  7, -3, -8],
....:                 [-3,  8, -1, -5]])
sage: P, L, U = C.LU(format='plu')
sage: perm, M = C.LU(format='compact')
sage: (L - identity_matrix(4)) + U == M
True
sage: p = [perm[i]+1 for i in range(len(perm))]
sage: PP = Permutation(p).to_matrix()
sage: PP == P
True
```

For a nonsingular matrix, and the 'nonzero' pivot strategy there is no need to permute rows, so the permutation matrix will be the identity. Furthermore, it can be shown that then the L and U matrices are uniquely determined by requiring L to have ones on the diagonal.

```
sage: D = matrix(QQ, [[ 1,  0,  2,  0, -2, -1],
....:                 [ 3, -2,  3, -1,  0,  6],
....:                 [-4,  2, -3,  1, -1, -8],
....:                 [-2,  2, -3,  2,  1,  0],
```

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```
....:                 [ 0, -1, -1,  0,  2,  5],
....:                 [-1,  2, -4, -1,  5, -3]])
sage: P, L, U = D.LU(pivot='nonzero')
sage: P
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
sage: L
[   1    0    0    0    0    0]
[   3    1    0    0    0    0]
[  -4   -1    1    0    0    0]
[  -2   -1   -1    1    0    0]
[   0  1/2  1/4  1/2    1    0]
[  -1   -1 -5/2   -2   -6    1]
sage: U
[   1    0    2    0   -2   -1]
[   0   -2   -3   -1    6    9]
[   0    0    2    0   -3   -3]
[   0    0    0    1    0    4]
[   0    0    0    0 -1/4 -3/4]
[   0    0    0    0    0    1]
sage: D == L*U
True
```

The base ring of the matrix may be any field, or a ring which has a fraction field implemented in Sage. The ring needs to be exact (there is a numerical LU decomposition for matrices over RDF and CDF). Matrices returned are over the original field, or the fraction field of the ring. If the field is not ordered (i.e. the absolute value function is not implemented), then the pivot strategy needs to be 'nonzero'.

```
sage: A = matrix(RealField(100), 3, 3, range(9))
sage: P, L, U = A.LU()
Traceback (most recent call last):
...
TypeError: base ring of the matrix must be exact, not Real Field with 100 bits␣
↪of precision

sage: A = matrix(Integers(6), 3, 2, range(6))
sage: A.LU()
Traceback (most recent call last):
...
TypeError: base ring of the matrix needs a field of fractions, not Ring of␣
↪integers modulo 6

sage: R.<y> = PolynomialRing(QQ, 'y')
sage: B = matrix(R, [[y+1, y^2+y], [y^2, y^3]])
sage: P, L, U = B.LU(pivot='partial')
Traceback (most recent call last):
...
TypeError: cannot take absolute value of matrix entries, try 'pivot=nonzero'
```

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```
sage: P, L, U = B.LU(pivot='nonzero')
sage: P
[1 0]
[0 1]
sage: L
[          1           0]
[y^2/(y + 1)           1]
sage: U
[  y + 1 y^2 + y]
[      0       0]
sage: L.base_ring()
Fraction Field of Univariate Polynomial Ring in y over Rational Field
sage: B == P*L*U
True

sage: F.<a> = FiniteField(5^2)
sage: C = matrix(F, [[a + 3, 4*a + 4, 2, 4*a + 2],
....:                [3, 2*a + 4, 2*a + 4, 2*a + 1],
....:                [3*a + 1, a + 3, 2*a + 4, 4*a + 3],
....:                [a, 3, 3*a + 1, a]])
sage: P, L, U = C.LU(pivot='nonzero')
sage: P
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: L
[      1       0       0       0]
[3*a + 3       1       0       0]
[    2*a 4*a + 2       1       0]
[2*a + 3       2 2*a + 4       1]
sage: U
[  a + 3 4*a + 4       2 4*a + 2]
[      0   a + 1   a + 3 2*a + 4]
[      0       0       1 4*a + 2]
[      0       0       0       0]
sage: L.base_ring()
Finite Field in a of size 5^2
sage: C == P*L*U
True
```

With no pivoting strategy given (i.e. `pivot=None`) the routine will try to use partial pivoting, but then fall back to the nonzero strategy. For the nonsingular matrix below, we see evidence of pivoting when viewed over the rationals, and no pivoting over the integers mod 29.

```
sage: entries = [3, 20, 11, 7, 16, 28, 5, 15, 21, 23, 22, 18, 8, 23, 15, 2]
sage: A = matrix(Integers(29), 4, 4, entries)
sage: perm, _ = A.LU(format='compact'); perm
(0, 1, 2, 3)
sage: B = matrix(QQ, 4, 4, entries)
sage: perm, _ = B.LU(format='compact'); perm
(2, 0, 1, 3)
```

The U matrix is only guaranteed to be upper-triangular. The rows are not necessarily linearly independent, nor are the pivots columns or rank in evidence.

```
sage: A = matrix(QQ, [[ 1, -4,  1,  0, -2,  1, 3,  3,  2],
....:                 [-1,  4,  0, -4,  0, -4, 5, -7, -7],
....:                 [ 0,  0,  1, -4, -1, -3, 6, -5, -6],
....:                 [-2,  8, -1, -4,  2, -4, 1, -8, -7],
....:                 [ 1, -4,  2, -4, -3,  2, 5,  6,  4]])
sage: P, L, U = A.LU()
sage: U
[   -2     8    -1    -4     2    -4     1    -8    -7]
[    0     0   1/2    -2    -1    -2   9/2    -3  -7/2]
[    0     0   3/2    -6    -2     0  11/2     2   1/2]
[    0     0     0     0  -1/3    -1   5/3  -5/3  -5/3]
[    0     0     0     0   1/3    -3   7/3 -19/3 -19/3]
sage: A.rref()
[ 1 -4  0  4  0  0 -1 -1 -1]
[ 0  0  1 -4  0  0  1  0 -1]
[ 0  0  0  0  1  0 -2 -1 -1]
[ 0  0  0  0  0  1 -1  2  2]
[ 0  0  0  0  0  0  0  0  0]
sage: A.pivots()
(0, 2, 4, 5)
```

QR(*full=True*)

Returns a factorization of `self` as a unitary matrix and an upper-triangular matrix.

INPUT:

- `full` - default: `True` - if `True` then the returned matrices have dimensions as described below. If `False` the `R` matrix has no zero rows and the columns of `Q` are a basis for the column space of `self`.

OUTPUT:

If `self` is an $m \times n$ matrix and `full=True` then this method returns a pair of matrices: Q is an $m \times m$ unitary matrix (meaning its inverse is its conjugate-transpose) and R is an $m \times n$ upper-triangular matrix with non-negative entries on the diagonal. For a matrix of full rank this factorization is unique (due to the restriction to positive entries on the diagonal).

If `full=False` then Q has m rows and the columns form an orthonormal basis for the column space of `self`. So, in particular, the conjugate-transpose of Q times Q will be an identity matrix. The matrix R will still be upper-triangular but will also have full rank, in particular it will lack the zero rows present in a full factorization of a rank-deficient matrix.

The results obtained when `full=True` are cached, hence Q and R are immutable matrices in this case.

Note: This is an exact computation, so limited to exact rings. Also the base ring needs to have a fraction field implemented in Sage and this field must contain square roots. One example is the field of algebraic numbers, `QQbar`, as used in the examples below. If you need numerical results, convert the base ring to the field of complex double numbers, `CDF`, which will use a faster routine that is careful about numerical subtleties.

ALGORITHM:

"Modified Gram-Schmidt," Algorithm 8.1 of [TB1997].

EXAMPLES:

For a nonsingular matrix, the QR decomposition is unique.

```
sage: A = matrix(QQbar, [[-2, 0, -4, -1, -1],
....:                    [-2, 1, -6, -3, -1],
....:                    [1, 1, 7, 4, 5],
....:                    [3, 0, 8, 3, 3],
....:                    [-1, 1, -6, -6, 5]])
sage: Q, R = A.QR()
sage: Q
[ -0.4588314677411235?  -0.1260506983326509?   0.3812120831224489?   -0.
↪394573711338418?     -0.6874400625964?]
[ -0.4588314677411235?   0.4726901187474409? -0.05198346588033394?   0.
↪7172941251646595?     -0.2209628772631?]
[  0.2294157338705618?   0.6617661662464172?   0.6619227988762521?  -0.
↪1808720937375480?      0.1964114464561?]
[  0.6882472016116853?   0.1890760474989764?  -0.2044682991293135?   0.
↪0966302966543065?     -0.6628886317894?]
[ -0.2294157338705618?   0.5357154679137663?   -0.6099393329959191?   -0.
↪5364220314271112?      0.0245514308070?]
sage: R
[  4.358898943540674? -0.4588314677411235?   13.076696830622002?   6.
↪1942248145051680?   2.9824045403173033?]
[                   0   1.6701717529076250?  0.59874081708009170?  -1.
↪2920196570096720?   6.2079968928830570?]
[                   0                   0   5.4444016598669740?   5.
↪4686606106111300? -0.6827161852283857?]
[                   0                   0                   0   1.
↪0276260394198360?  -3.6193001496866200?]
[                   0                   0                   0                ␣
↪   0   0.0245514308070120?]
sage: Q.conjugate_transpose()*Q
[1.0000000000000000?            0.?e-18            0.?e-17            0.?e-16   ␣
↪        0.?e-13]
[          0.?e-18 1.0000000000000000?            0.?e-17            0.?e-16   ␣
↪        0.?e-13]
[          0.?e-17            0.?e-17 1.0000000000000000?            0.?e-16   ␣
↪        0.?e-13]
[          0.?e-16            0.?e-16            0.?e-16 1.0000000000000000?   ␣
↪        0.?e-13]
[          0.?e-13            0.?e-13            0.?e-13            0.?e-13  ␣
↪1.0000000000000000?]
sage: Q*R == A
True
```

An example with complex numbers in QQbar, the field of algebraic numbers.

```
sage: A = matrix(QQbar, [[-8, 4*I + 1, -I + 2, 2*I + 1],
....:                    [1, -2*I - 1, -I + 3, -I + 1],
....:                    [I + 7, 2*I + 1, -2*I + 7, -I + 1],
....:                    [I + 2, 0, I + 12, -1]])
sage: Q, R = A.QR()
sage: Q
[                          -0.7302967433402215?    0.2070566455055649? + 0.
↪5383472783144687?*I    0.2463049809998642? - 0.0764456358723292?*I    0.
```

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```
↪2381617683194332? - 0.1036596032779695?*I]
[                          0.0912870929175277?   -0.2070566455055649? - 0.
↪3778783780476559?*I    0.3786559533863033? - 0.19522214955246667?*I     0.
↪7012444502144697 - 0.3643711650986595?*I]
[   0.6390096504226938? + 0.0912870929175277?*I    0.1708217325420910? + 0.
↪6677576817554466?*I -0.03411475806452072? + 0.040901987417671437*I    0.
↪3140171085506764? - 0.08251917187054127*I]
[   0.1825741858350554? + 0.0912870929175277?*I  -0.03623491296347385? + 0.
↪0724698259269477?*I   0.8632284069415110? + 0.06322839976356195?*I   -0.
↪4499694867611521? - 0.01161191812089187*I]
sage: R
[                         10.954451150103337               0.?e-18 - 1.
↪9170289512680827*I    5.3859384821341337 - 2.190890230020665?*I  -0.
↪2738612787525831? - 2.190890230020665?*I]
[                                          0               4.829596256417300?␣
↪+ 0.?e-18*I   -0.869637911123373? - 5.864879483945125?*I   0.9938718984267127␣
↪- 0.30540855212070827*I]
[                                          0                                  ␣
↪         0               12.001607609358147 + 0.?e-16*I -0.27095334022972737␣
↪+ 0.44206296444863237*I]
[                                          0                                  ␣
↪         0                                          0               1.
↪942963944258992? + 0.?e-16*I]
sage: Q.conjugate_transpose()*Q
[1.000000000000000? + 0.?e-19*I            0.?e-18 + 0.?e-17*I            0.?e-
↪17 + 0.?e-17*I            0.?e-16 + 0.?e-16*I]
[           0.?e-18 + 0.?e-17*I 1.000000000000000? + 0.?e-17*I            0.?e-
↪17 + 0.?e-17*I            0.?e-16 + 0.?e-16*I]
[           0.?e-17 + 0.?e-17*I            0.?e-17 + 0.?e-17*I 1.
↪000000000000000? + 0.?e-17*I           0.?e-16 + 0.?e-16*I]
[           0.?e-16 + 0.?e-16*I            0.?e-16 + 0.?e-16*I            0.?e-
↪16 + 0.?e-16*I 1.000000000000000? + 0.?e-16*I]
sage: Q*R - A
[            0.?e-17 0.?e-17 + 0.?e-17*I 0.?e-16 + 0.?e-16*I 0.?e-16 + 0.?e-
↪16*I]
[            0.?e-18 0.?e-17 + 0.?e-17*I 0.?e-16 + 0.?e-16*I 0.?e-16 + 0.?e-
↪16*I]
[0.?e-17 + 0.?e-18*I 0.?e-17 + 0.?e-17*I 0.?e-16 + 0.?e-16*I 0.?e-16 + 0.?e-
↪16*I]
[0.?e-18 + 0.?e-18*I 0.?e-18 + 0.?e-18*I 0.?e-16 + 0.?e-16*I 0.?e-16 + 0.?e-
↪16*I]
```

A rank-deficient rectangular matrix, with both values of the `full` keyword.

```
sage: A = matrix(QQbar, [[2, -3, 3],
....:                    [-1, 1, -1],
....:                    [-1, 3, -3],
....:                    [-5, 1, -1]])
sage: Q, R = A.QR()
sage: Q
[  0.3592106040535498?  -0.5693261797050169?   0.7239227659930268?   0.
↪1509015305256380?]
```

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```
[ -0.1796053020267749?   0.1445907757980996?                    0   0.
↪97305469683773417]
[ -0.1796053020267749?   0.7048800320157352?    0.6722139969935257  -0.
↪13789277789411747]
[ -0.8980265101338745?  -0.3976246334447737?   0.1551263069985058? -0.
↪106671771578468187]
sage: R
[ 5.567764362830022? -2.694079530401624?  2.694079530401624?]
[                  0  3.569584777515583? -3.569584777515583?]
[                  0                  0                  0]
[                  0                  0                  0]
sage: Q.conjugate_transpose()*Q
[                 1            0.?e-18            0.?e-18            0.?e-18]
[           0.?e-18                  1            0.?e-18            0.?e-18]
[           0.?e-18            0.?e-18 1.000000000000000?            0.?e-18]
[           0.?e-18            0.?e-18            0.?e-18 1.000000000000000?]

sage: Q, R = A.QR(full=False)
sage: Q
[ 0.3592106040535498? -0.5693261797050169?]
[-0.1796053020267749?  0.1445907757980996?]
[-0.1796053020267749?  0.7048800320157352?]
[-0.8980265101338745? -0.3976246334447737?]
sage: R
[ 5.567764362830022? -2.694079530401624?  2.694079530401624?]
[                  0  3.569584777515583? -3.569584777515583?]
sage: Q.conjugate_transpose()*Q
[      1 0.?e-18]
[0.?e-18       1]
```

Another rank-deficient rectangular matrix, with complex entries, as a reduced decomposition.

```
sage: A = matrix(QQbar, [[-3*I - 3, I - 3, -12*I + 1, -2],
....:                    [-I - 1, -2, 5*I - 1, -I - 2],
....:                    [-4*I - 4, I - 5, -7*I, -I - 4]])
sage: Q, R = A.QR(full=False)
sage: Q
[ -0.4160251471689219? - 0.4160251471689219?*I   0.5370861555295747? + 0.
↪17902871850985837?*I]
[ -0.1386750490563073? - 0.1386750490563073?*I  -0.7519206177414046? - 0.
↪25064020591380157?*I]
[ -0.5547001962252291? - 0.5547001962252291?*I -0.2148344622118299? - 0.
↪07161148740394329?*I]
sage: R
[                     7.211102550927979?  3.328201177351375? - 5.
↪269651864139676?*I   7.904477796209515? + 8.45917799243475?*I  4.
↪021576422632911? - 2.6348259320698387?*I]
[                                      0                         1.
↪074172311059150?  -1.611258466588724? - 9.130464644002777?*I 1.611258466588724?
↪ + 0.5370861555295747?*I]
sage: Q.conjugate_transpose()*Q
[1 0]
```

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```
[0 1]
sage: Q*R-A
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

Results of full decompositions are cached and thus returned immutable.

```
sage: A = random_matrix(QQbar, 2, 2)
sage: Q, R = A.QR()
sage: Q.is_mutable()
False
sage: R.is_mutable()
False
```

Trivial cases return trivial results of the correct size, and we check Q itself in one case.

```
sage: A = zero_matrix(QQbar, 0, 10)
sage: Q, R = A.QR()
sage: Q.nrows(), Q.ncols()
(0, 0)
sage: R.nrows(), R.ncols()
(0, 10)
sage: A = zero_matrix(QQbar, 3, 0)
sage: Q, R = A.QR()
sage: Q.nrows(), Q.ncols()
(3, 3)
sage: R.nrows(), R.ncols()
(3, 0)
sage: Q
[1 0 0]
[0 1 0]
[0 0 1]
```

T

Returns the transpose of a matrix.

EXAMPLES:

```
sage: A = matrix(QQ, 5, range(25))
sage: A.T
[ 0  5 10 15 20]
[ 1  6 11 16 21]
[ 2  7 12 17 22]
[ 3  8 13 18 23]
[ 4  9 14 19 24]
```

adjoint(**args*, ***kwds*)

Deprecated: Use *adjugate()* instead. See github issue #10501 for details.

adjoint_classical()

Return the adjugate matrix of `self` (that is, the transpose of the matrix of cofactors).

Let M be an $n \times n$-matrix. The adjugate matrix of M is the $n \times n$-matrix N whose (i, j)-th entry is

$(-1)^{i+j}\det(M_{j,i})$, where $M_{j,i}$ is the matrix M with its j-th row and i-th column removed. It is known to satisfy $NM = MN = \det(M)I$.

EXAMPLES:

```
sage: M = Matrix(ZZ,2,2,[5,2,3,4]) ; M
[5 2]
[3 4]
sage: N = M.adjugate() ; N
[ 4 -2]
[-3  5]
sage: M * N
[14  0]
[ 0 14]
sage: N * M
[14  0]
[ 0 14]
sage: M = Matrix(QQ,2,2,[5/3,2/56,33/13,41/10]) ; M
[  5/3  1/28]
[33/13 41/10]
sage: N = M.adjugate() ; N
[ 41/10  -1/28]
[-33/13    5/3]
sage: M * N
[7363/1092         0]
[        0 7363/1092]
```

An alias is *adjoint_classical()*, which replaces the deprecated *adjoint()* method:

```
sage: M.adjoint()
...: DeprecationWarning: adjoint is deprecated. Please use adjugate instead.
See https://github.com/sagemath/sage/issues/10501 for details.
[ 41/10  -1/28]
[-33/13    5/3]
sage: M.adjoint_classical()
[ 41/10  -1/28]
[-33/13    5/3]
```

ALGORITHM:

Use PARI whenever the method `self._adjugate` is included to do so in an inheriting class. Otherwise, use a generic division-free algorithm that computes the adjugate matrix from the characteristic polynomial.

The result is cached.

adjugate()

Return the adjugate matrix of `self` (that is, the transpose of the matrix of cofactors).

Let M be an $n \times n$-matrix. The adjugate matrix of M is the $n \times n$-matrix N whose (i, j)-th entry is $(-1)^{i+j}\det(M_{j,i})$, where $M_{j,i}$ is the matrix M with its j-th row and i-th column removed. It is known to satisfy $NM = MN = \det(M)I$.

EXAMPLES:

```
sage: M = Matrix(ZZ,2,2,[5,2,3,4]) ; M
[5 2]
[3 4]
```

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```
sage: N = M.adjugate() ; N
[ 4 -2]
[-3  5]
sage: M * N
[14  0]
[ 0 14]
sage: N * M
[14  0]
[ 0 14]
sage: M = Matrix(QQ,2,2,[5/3,2/56,33/13,41/10]) ; M
[  5/3  1/28]
[33/13 41/10]
sage: N = M.adjugate() ; N
[ 41/10  -1/28]
[-33/13    5/3]
sage: M * N
[7363/1092         0]
[        0 7363/1092]
```

An alias is *adjoint_classical()*, which replaces the deprecated *adjoint()* method:

```
sage: M.adjoint()
...: DeprecationWarning: adjoint is deprecated. Please use adjugate instead.
See https://github.com/sagemath/sage/issues/10501 for details.
[ 41/10  -1/28]
[-33/13    5/3]
sage: M.adjoint_classical()
[ 41/10  -1/28]
[-33/13    5/3]
```

ALGORITHM:

Use PARI whenever the method `self._adjugate` is included to do so in an inheriting class. Otherwise, use a generic division-free algorithm that computes the adjugate matrix from the characteristic polynomial.

The result is cached.

apply_map(*phi*, *R=None*, *sparse=None*)

Apply the given map phi (an arbitrary Python function or callable object) to this dense matrix. If R is not given, automatically determine the base ring of the resulting matrix.

INPUT:

- `sparse` – True to make the output a sparse matrix; default False
- `phi` - arbitrary Python function or callable object
- `R` - (optional) ring

OUTPUT: a matrix over R

EXAMPLES:

```
sage: m = matrix(ZZ, 3, 3, range(9))
sage: k.<a> = GF(9)
sage: f = lambda x: k(x)
sage: n = m.apply_map(f); n
```

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```
[0 1 2]
[0 1 2]
[0 1 2]
sage: n.parent()
Full MatrixSpace of 3 by 3 dense matrices over Finite Field in a of size 3^2
```

In this example, we explicitly specify the codomain.

```
sage: s = GF(3)
sage: f = lambda x: s(x)
sage: n = m.apply_map(f, k); n
[0 1 2]
[0 1 2]
[0 1 2]
sage: n.parent()
Full MatrixSpace of 3 by 3 dense matrices over Finite Field in a of size 3^2
```

If self is subdivided, the result will be as well:

```
sage: m = matrix(2, 2, srange(4))
sage: m.subdivide(None, 1); m
[0|1]
[2|3]
sage: m.apply_map(lambda x: x*x)
[0|1]
[4|9]
```

If the matrix is sparse, the result will be as well:

```
sage: m = matrix(ZZ,100,100,sparse=True)
sage: m[18,32] = -6
sage: m[1,83] = 19
sage: n = m.apply_map(abs, R=ZZ)
sage: n.dict()
{(1, 83): 19, (18, 32): 6}
sage: n.is_sparse()
True
```

If the map sends most of the matrix to zero, then it may be useful to get the result as a sparse matrix.

```
sage: m = matrix(ZZ, 3, 3, range(1, 10))
sage: n = m.apply_map(lambda x: 1//x, sparse=True); n
[1 0 0]
[0 0 0]
[0 0 0]
sage: n.parent()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
```

apply_morphism(*phi*)

Apply the morphism phi to the coefficients of this dense matrix.

The resulting matrix is over the codomain of phi.

INPUT:

- phi - a morphism, so phi is callable and phi.domain() and phi.codomain() are defined. The codomain must be a ring.

OUTPUT: a matrix over the codomain of phi

EXAMPLES:

```
sage: m = matrix(ZZ, 3, 3, range(9))
sage: phi = ZZ.hom(GF(5))
sage: m.apply_morphism(phi)
[0 1 2]
[3 4 0]
[1 2 3]
sage: parent(m.apply_morphism(phi))
Full MatrixSpace of 3 by 3 dense matrices over Finite Field of size 5
```

We apply a morphism to a matrix over a polynomial ring:

```
sage: R.<x,y> = QQ[]
sage: m = matrix(2, [x,x^2 + y, 2/3*y^2-x, x]); m
[          x     x^2 + y]
[2/3*y^2 - x           x]
sage: phi = R.hom([y,x])
sage: m.apply_morphism(phi)
[          y     y^2 + x]
[2/3*x^2 - y           y]
```

as_bipartite_graph()

Construct a bipartite graph `B` representing the matrix uniquely.

Vertices are labeled 1 to `nrows` on the left and `nrows` + 1 to `nrows` + `ncols` on the right, representing rows and columns correspondingly. Each row is connected to each column with an edge weighted by the value of the corresponding matrix entry.

This graph is a helper for calculating automorphisms of a matrix under row and column permutations. See *automorphisms_of_rows_and_columns()*.

OUTPUT:

- A bipartite graph.

EXAMPLES:

```
sage: M = matrix(QQ, [[1/3, 7], [6, 1/4], [8, -5]])
sage: M
[1/3   7]
[  6 1/4]
[  8  -5]

sage: B = M.as_bipartite_graph()
sage: B
Bipartite graph on 5 vertices
sage: B.edges(sort=True)
[(1, 4, 1/3), (1, 5, 7), (2, 4, 6), (2, 5, 1/4), (3, 4, 8), (3, 5, -5)]
sage: len(B.left) == M.nrows()
True
```

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```
sage: len(B.right) == M.ncols()
True
```

as_sum_of_permutations()

Returns the current matrix as a sum of permutation matrices

According to the Birkhoff-von Neumann Theorem, any bistochastic matrix can be written as a positive sum of permutation matrices, which also means that the polytope of bistochastic matrices is integer.

As a non-bistochastic matrix can obviously not be written as a sum of permutations, this theorem is an equivalence.

This function, given a bistochastic matrix, returns the corresponding decomposition.

See also:

- bistochastic_as_sum_of_permutations – for more information on this method.
- Birkhoff_polytope()

EXAMPLES:

We create a bistochastic matrix from a convex sum of permutations, then try to deduce the decomposition from the matrix

```
sage: L = []
sage: L.append((9,Permutation([4, 1, 3, 5, 2])))
sage: L.append((6,Permutation([5, 3, 4, 1, 2])))
sage: L.append((3,Permutation([3, 1, 4, 2, 5])))
sage: L.append((2,Permutation([1, 4, 2, 3, 5])))
sage: M = sum([c * p.to_matrix() for (c,p) in L])
sage: decomp = sage.combinat.permutation.bistochastic_as_sum_of_permutations(M)
sage: print(decomp)
2*B[[1, 4, 2, 3, 5]] + 3*B[[3, 1, 4, 2, 5]] + 9*B[[4, 1, 3, 5, 2]] + 6*B[[5, 3,␣
↪4, 1, 2]]
```

An exception is raised when the matrix is not bistochastic:

```
sage: M = Matrix([[2,3],[2,2]])
sage: decomp = sage.combinat.permutation.bistochastic_as_sum_of_permutations(M)
Traceback (most recent call last):
...
ValueError: The matrix is not bistochastic
```

automorphisms_of_rows_and_columns()

Return the automorphisms of `self` under permutations of rows and columns as a list of pairs of `PermutationGroupElement` objects.

EXAMPLES:

```
sage: M = matrix(ZZ,[[1,0],[1,0],[0,1]])
sage: M
[1 0]
[1 0]
[0 1]
```

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```
sage: A = M.automorphisms_of_rows_and_columns()
sage: A
[((), ()), ((1,2), ())]
sage: M = matrix(ZZ,[[1,1,1,1],[1,1,1,1]])
sage: A = M.automorphisms_of_rows_and_columns()
sage: len(A)
48
```

One can now apply these automorphisms to `M` to show that it leaves it invariant:

```
sage: all(M.with_permuted_rows_and_columns(*i) == M for i in A)
True
```

Check that github issue #25426 is fixed:

```
sage: j = matrix([(3, 2, 1, 0, 0),
....:             (2, 2, 0, 1, 0),
....:             (1, 0, 3, 0, 2),
....:             (0, 1, 0, 2, 1),
....:             (0, 0, 2, 1, 2)])
sage: j.automorphisms_of_rows_and_columns()
[((), ()), ((1,3)(2,5), (1,3)(2,5))]
```

block_ldlt(*classical=False*)

Compute a block-LDL^T factorization of a Hermitian matrix.

The standard LDL^T factorization of a positive-definite matrix A factors it as $A = LDL^T$ where L is unit-lower-triangular and D is diagonal. If one allows row/column swaps via a permutation matrix P, then this factorization can be extended to many positive-semidefinite matrices A via the factorization $P^TAP = LDL^T$ that places the zeros at the bottom of D to avoid division by zero. These factorizations extend easily to complex Hermitian matrices when one replaces the transpose by the conjugate-transpose.

However, we can go one step further. If, in addition, we allow D to potentially contain 2×2 blocks on its diagonal, then every real or complex Hermitian matrix A can be factored as $A = PLDL^*P^T$. When the row/column swaps are made intelligently, this process is numerically stable over inexact rings like `RDF`. Bunch and Kaufman describe such a "pivot" scheme that is suitable for the solution of Hermitian systems, and that is how we choose our row and column swaps.

INPUT:

- `classical` – (default: `False`) whether or not to attempt a classical non-block LDL^T factorization with no row/column swaps.

Warning: Not all matrices have a classical LDL^T factorization. Set `classical=True` at your own risk, preferably after verifying that your matrix is positive-definite and (over inexact rings) not ill-conditioned.

OUTPUT:

If the input matrix is not Hermitian, the output from this function is undefined. Otherwise, we return a triple (P, L, D) such that $A = PLDL^*P^T$ and

- P is a permutation matrix,
- L is unit lower-triangular,

- D is a block-diagonal matrix whose blocks are of size one or two.

With `classical=True`, the permutation matrix P is always an identity matrix and the diagonal blocks are always one-by-one. A `ValueError` is raised if the matrix has no classical LDL^T factorization.

ALGORITHM:

We essentially follow "Algorithm A" in the paper by Bunch and Kaufman [BK1977] that describes the stable pivoting strategy. The same scheme is described by Higham [Hig2002].

See also:

indefinite_factorization()

REFERENCES:

- [BK1977]
- [Hig2002]

EXAMPLES:

This three-by-three real symmetric matrix has one positive, one negative, and one zero eigenvalue – so it is not any flavor of (semi)definite, yet we can still factor it:

```
sage: A =  matrix(QQ, [[0, 1, 0],
....:                  [1, 1, 2],
....:                  [0, 2, 0]])
sage: P,L,D = A.block_ldlt()
sage: P
[0 0 1]
[1 0 0]
[0 1 0]
sage: L
[  1   0   0]
[  2   1   0]
[  1 1/2   1]
sage: D
[ 1| 0| 0]
[--+--+--]
[ 0|-4| 0]
[--+--+--]
[ 0| 0| 0]
sage: P.transpose()*A*P == L*D*L.transpose()
True
```

This two-by-two matrix has no classical factorization, but it constitutes its own block-factorization:

```
sage: A = matrix(QQ, [ [0,1],
....:                  [1,0] ])
sage: A.block_ldlt(classical=True)
Traceback (most recent call last):
...
ValueError: matrix has no classical LDL^T factorization
sage: A.block_ldlt()
(
[1 0]  [1 0]  [0 1]
[0 1], [0 1], [1 0]
)
```

The same is true of the following complex Hermitian matrix:

```
sage: A = matrix(QQbar, [ [ 0,I],
....:                     [-I,0] ])
sage: A.block_ldlt(classical=True)
Traceback (most recent call last):
...
ValueError: matrix has no classical LDL^T factorization
sage: A.block_ldlt()
(
[1 0]  [1 0]  [ 0  I]
[0 1], [0 1], [-I  0]
)
```

Complete diagonal pivoting could cause problems for the following matrix, since the diagonal entries are small compared to the off-diagonals that must be zeroed; however, the block algorithm refuses to factor it:

```
sage: A = matrix(RDF, 2, 2, [ [1e-10, 1    ],
....:                         [1    , 2e-10] ])
sage: _,L,D = A.block_ldlt(classical=True)
sage: L*D*L.T
[1e-10   1.0]
[  1.0   0.0]
sage: A.block_ldlt()
(
[1.0 0.0]  [1.0 0.0]  [1e-10   1.0]
[0.0 1.0], [0.0 1.0], [  1.0 2e-10]
)
```

The factorization over an inexact ring is necessarily inexact, but $P^T AP$ will ideally be close to LDL^* in the metric induced by the norm:

```
sage: A = matrix(CDF, 2, 2, [ [-1.1933, -0.3185 - 1.3553*I],
....:                         [-0.3185 + 1.3553*I, 1.5729 ] ])
sage: P,L,D = A.block_ldlt()
sage: P.T*A*P == L*D*L.H
False
sage: (P.T*A*P - L*D*L.H).norm() < 1e-10
True
```

This matrix has a singular three-by-three leading principal submatrix, and therefore has no classical factorization:

```
sage: A = matrix(QQ, [[21, 15, 12, -2],
....:                 [15, 12,  9,  6],
....:                 [12,  9,  7,  3],
....:                 [-2,  6,  3,  8]])
sage: A[0:3,0:3].det() == 0
True
sage: A.block_ldlt(classical=True)
Traceback (most recent call last):
...
ValueError: matrix has no classical LDL^T factorization
sage: A.block_ldlt()
```

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```
(
[1 0 0 0]  [     1      0      0      0]
[0 0 1 0]  [ -2/21      1      0      0]
[0 0 0 1]  [   5/7  39/41      1      0]
[0 1 0 0], [   4/7 87/164  48/79      1],

[     21|      0|      0|      0]
[-------+-------+-------+-------]
[      0| 164/21|      0|      0]
[-------+-------+-------+-------]
[      0|      0|-237/41|      0]
[-------+-------+-------+-------]
[      0|      0|      0| 25/316]
)
```

An indefinite symmetric matrix that happens to have a classical factorization:

```
sage: A = matrix(QQ, [[ 3,  -6,   9,   6,  -9],
....:                 [-6,  11, -16, -11,  17],
....:                 [ 9, -16,  28,  16, -40],
....:                 [ 6, -11,  16,   9, -19],
....:                 [-9,  17, -40, -19,  68]])
sage: A.block_ldlt(classical=True)[1:]
(
                  [ 3| 0| 0| 0| 0]
                  [--+--+--+--+--]
                  [ 0|-1| 0| 0| 0]
                  [--+--+--+--+--]
[ 1  0  0  0  0]  [ 0| 0| 5| 0| 0]
[-2  1  0  0  0]  [--+--+--+--+--]
[ 3 -2  1  0  0]  [ 0| 0| 0|-2| 0]
[ 2 -1  0  1  0]  [--+--+--+--+--]
[-3  1 -3  1  1], [ 0| 0| 0| 0|-1]
)
```

An indefinite Hermitian matrix that happens to have a classical factorization:

```
sage: F.<I> = QuadraticField(-1)
sage: A = matrix(F, [[      2, 4 - 2*I, 2 + 2*I],
....:                [4 + 2*I,       8,    10*I],
....:                [2 - 2*I,  -10*I,      -3]])
sage: A.block_ldlt(classical=True)[1:]
(
                           [ 2| 0| 0]
                           [--+--+--]
[      1       0       0]  [ 0|-2| 0]
[  I + 2       1       0]  [--+--+--]
[ -I + 1 2*I + 1       1], [ 0| 0| 3]
)
```

characteristic_polynomial(**args*, ***kwds*)

Synonym for self.charpoly(...).

EXAMPLES:

```
sage: a = matrix(QQ, 2,2, [1,2,3,4]); a
[1 2]
[3 4]
sage: a.characteristic_polynomial('T')
T^2 - 5*T - 2
```

charpoly(*var='x'*, *algorithm=None*)

Returns the characteristic polynomial of self, as a polynomial over the base ring.

ALGORITHM:

If the base ring has a method $_matrix_charpoly$, we use it.

In the generic case of matrices over a ring (commutative and with unity), there is a division-free algorithm, which can be accessed using `"df"`, with complexity $O(n^4)$. Alternatively, by specifying `"hessenberg"`, this method computes the Hessenberg form of the matrix and then reads off the characteristic polynomial. Moreover, for matrices over number fields, this method can use PARI's charpoly implementation instead.

The method's logic is as follows: If no algorithm is specified, first check if the base ring is a number field (and then use PARI), otherwise check if the base ring is the ring of integers modulo n (in which case compute the characteristic polynomial of a lift of the matrix to the integers, and then coerce back to the base), next check if the base ring is an exact field (and then use the Hessenberg form), or otherwise, use the generic division-free algorithm. If an algorithm is specified explicitly, if `algorithm == "hessenberg"`, use the Hessenberg form, or otherwise use the generic division-free algorithm.

The result is cached.

INPUT:

- `var` - a variable name (default: 'x')
- **`algorithm` - string:**
  - `"df"` - Generic $O(n^4)$ division-free algorithm
  - `"hessenberg"` - Use the Hessenberg form of the matrix

EXAMPLES:

First a matrix over $\mathbf{Z}$:

```
sage: A = MatrixSpace(ZZ,2)( [1,2,  3,4] )
sage: f = A.charpoly('x')
sage: f
x^2 - 5*x - 2
sage: f.parent()
Univariate Polynomial Ring in x over Integer Ring
sage: f(A)
[0 0]
[0 0]
```

An example over $\mathbf{Q}$:

```
sage: A = MatrixSpace(QQ,3)(range(9))
sage: A.charpoly('x')
x^3 - 12*x^2 - 18*x
sage: A.trace()
```

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```
12
sage: A.determinant()
0
```

We compute the characteristic polynomial of a matrix over the polynomial ring $\mathbf{Z}[a]$:

```
sage: R.<a> = PolynomialRing(ZZ)
sage: M = MatrixSpace(R,2)([a,1,  a,a+1]); M
[    a     1]
[    a a + 1]
sage: f = M.charpoly('x'); f
x^2 + (-2*a - 1)*x + a^2
sage: f.parent()
Univariate Polynomial Ring in x over Univariate Polynomial Ring in a over␣
↪Integer Ring
sage: M.trace()
2*a + 1
sage: M.determinant()
a^2
```

We compute the characteristic polynomial of a matrix over the multi-variate polynomial ring $\mathbf{Z}[x, y]$:

```
sage: R.<x,y> = PolynomialRing(ZZ,2)
sage: A = MatrixSpace(R,2)([x, y, x^2, y^2])
sage: f = A.charpoly('x'); f
x^2 + (-y^2 - x)*x - x^2*y + x*y^2
```

It's a little difficult to distinguish the variables. To fix this, we temporarily view the indeterminate as Z:

```
sage: with localvars(f.parent(), 'Z'): print(f)
Z^2 + (-y^2 - x)*Z - x^2*y + x*y^2
```

We could also compute f in terms of Z from the start:

```
sage: A.charpoly('Z')
Z^2 + (-y^2 - x)*Z - x^2*y + x*y^2
```

Here is an example over a number field:

```
sage: x = QQ['x'].gen()
sage: K.<a> = NumberField(x^2 - 2)
sage: m = matrix(K, [[a-1, 2], [a, a+1]])
sage: m.charpoly('Z')
Z^2 - 2*a*Z - 2*a + 1
sage: m.charpoly('a')(m) == 0
True
```

Over integers modulo n with composite n:

```
sage: A = Mat(Integers(6),3,3)(range(9))
sage: A.charpoly()
x^3
```

Here is an example over a general commutative ring, that is to say, as of version 4.0.2, Sage does not even positively determine that S in the following example is an integral domain. But the computation of the characteristic polynomial succeeds as follows:

```
sage: R.<a,b> = QQ[]
sage: S.<x,y> = R.quo((b^3))
sage: A = matrix(S, [[x*y^2,2*x],[2,x^10*y]])
sage: A
[ x*y^2    2*x]
[     2 x^10*y]
sage: A.charpoly('T')
T^2 + (-x^10*y - x*y^2)*T - 4*x
```

cholesky()

Returns the Cholesky decomposition of a Hermitian matrix.

INPUT:

A positive-definite matrix. Generally, the base ring for the entries of the matrix needs to be a subfield of the algebraic numbers (QQbar). Examples include the rational numbers (QQ), some number fields, and real algebraic numbers and the algebraic numbers themselves. Symbolic matrices can also occasionally be factored.

OUTPUT:

For a matrix A the routine returns a lower triangular matrix L such that,

$$A = LL^*$$

where L^* is the conjugate-transpose. If the matrix is not positive-definite (for example, if it is not Hermitian) then a ValueError results.

If possible, the output matrix will be over the fraction field of the base ring of the input matrix. If that fraction field is missing the requisite square roots but if no imaginaries are encountered, then the algebraic-reals will be used. Otherwise, the algebraic closure of the fraction field (typically QQbar) will be used.

ALGORITHM:

First we ensure that the matrix A *is_hermitian()*. Afterwards, we attempt to compute a classical *block_ldlt()* factorization, $A = LDL^*$, of the matrix. If that fails, then the matrix was not positive-definite and an error is raised. Otherwise we take the entrywise square-root $\sqrt{D}$ of the diagonal matrix D (whose entries are the positive eigenvalues of the original matrix) to obtain the Cholesky factorization $A = \left(L\sqrt{D}\right)\left(L\sqrt{D}\right)^*$. If the necessary square roots cannot be taken in the fraction field of original base ring, then we move to either its algebraic closure or the algebraic reals, depending on whether or not imaginary numbers are required.

EXAMPLES:

This simple example has a result with entries that remain in the field of rational numbers:

```
sage: A = matrix(QQ, [[ 4, -2,  4,  2],
....:                 [-2, 10, -2, -7],
....:                 [ 4, -2,  8,  4],
....:                 [ 2, -7,  4,  7]])
sage: A.is_symmetric()
True
sage: L = A.cholesky()
sage: L
```

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```
[ 2  0  0  0]
[-1  3  0  0]
[ 2  0  2  0]
[ 1 -2  1  1]
sage: L.parent()
Full MatrixSpace of 4 by 4 dense matrices over Rational Field
sage: L*L.transpose() == A
True
```

This seemingly simple example requires first moving to the rational numbers for field operations, and then square roots necessitate that the result has entries in the field of algebraic numbers:

```
sage: A = matrix(ZZ, [[ 78, -30, -37,  -2],
....:                 [-30, 102, 179, -18],
....:                 [-37, 179, 326, -38],
....:                 [ -2, -18, -38,  15]])
sage: A.is_symmetric()
True
sage: L = A.cholesky()
sage: L
[   8.83176086632785?                    0                    0                ␣
↪   0]
[ -3.396831102433787?    9.51112708681461?                    0                ␣
↪   0]
[ -4.189425026335004?   17.32383862241232?   2.886751345948129?                ␣
↪   0]
[-0.2264554068289192?  -1.973397116652010?  -1.649572197684645?   2.
↪886751345948129?]
sage: L.parent()
Full MatrixSpace of 4 by 4 dense matrices over Algebraic Real Field
sage: L*L.transpose() == A
True
```

Some subfields of the complex numbers, such as this number field of complex numbers with rational real and imaginary parts, allow for this computation:

```
sage: C.<I> = QuadraticField(-1)
sage: A = matrix(C, [[        23,  17*I + 3,  24*I + 25,     21*I],
....:                [ -17*I + 3,        38, -69*I + 89, 7*I + 15],
....:                [-24*I + 25, 69*I + 89,        976, 24*I + 6],
....:                [     -21*I, -7*I + 15,  -24*I + 6,       28]])
sage: A.is_hermitian()
True
sage: L = A.cholesky()
sage: L
[               4.79...?                          0                          0     ␣
↪   0]
[   0.62...? - 3.54...?*I                  5.00...?                          0     ␣
↪   0]
[   5.21...? - 5.00...?*I   13.58...? + 10.72...?*I               24.98...?     ␣
↪   0]
[            -4.37...?*I   -0.10...? -  0.85...?*I  -0.21...? + 0.37...?*I 2.
```

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```
↪81...?]
sage: L.parent()
Full MatrixSpace of 4 by 4 dense matrices over Algebraic Field
sage: (L*L.conjugate_transpose() - A.change_ring(QQbar)).norm() < 10^-10
True
```

The field of algebraic numbers is an ideal setting for this computation:

```
sage: A = matrix(QQbar, [[        2,   4 + 2*I,   6 - 4*I],
....:                    [ -2*I + 4,         11, 10 - 12*I],
....:                    [  4*I + 6, 10 + 12*I,         37]])
sage: A.is_hermitian()
True
sage: L = A.cholesky()
sage: L
[                       1.414213562373095?                0                       0]
[2.828427124746190? - 1.414213562373095?*I                1                       0]
[4.242640687119285? + 2.828427124746190?*I   -2*I + 2   1.732050807568878?]
sage: L.parent()
Full MatrixSpace of 3 by 3 dense matrices over Algebraic Field
sage: L*L.conjugate_transpose() == A
True
```

Results are cached, hence immutable. Use the `copy` function if you need to make a change:

```
sage: A = matrix(QQ, [[ 4, -2,  4,  2],
....:                 [-2, 10, -2, -7],
....:                 [ 4, -2,  8,  4],
....:                 [ 2, -7,  4,  7]])
sage: L = A.cholesky()
sage: L.is_immutable()
True
sage: from copy import copy
sage: LC = copy(L)
sage: LC[0,0] = 1000
sage: LC
[1000    0    0    0]
[  -1    3    0    0]
[   2    0    2    0]
[   1   -2    1    1]
```

The base ring need not be exact, although you should expect the result to be inexact (correct only in the norm) as well in that case:

```
sage: F = RealField(100)
sage: A = A = matrix(F, [[1.0, 2.0], [2.0, 6.0]])
sage: L = A.cholesky(); L
[ 1.000... 0.000...]
[ 2.000... 1.414...]
sage: (L*L.transpose() - A).norm() < 1e-10
True
```

Even symbolic matrices can sometimes be factored:

```
sage: A = matrix(SR, [[pi,0], [0,pi]])      # optional - sage.symbolic
sage: A.cholesky()                          # optional - sage.symbolic
[sqrt(pi)        0]
[       0 sqrt(pi)]
```

There are a variety of situations which will prevent the computation of a Cholesky decomposition.

The base ring may not be able to be viewed as a subset of the complex numbers, implying that "Hermitian" is meaningless:

```
sage: A = matrix(Integers(6), [[2, 0], [0, 4]])
sage: A.cholesky()
Traceback (most recent call last):
...
AttributeError: 'sage.rings.finite_rings.integer_mod.IntegerMod_int'
object has no attribute 'conjugate'
```

The matrix may not be Hermitian:

```
sage: F.<a> = FiniteField(5^4)
sage: A = matrix(F, [[2+a^3, 3], [3, 3]])
sage: A.cholesky()
Traceback (most recent call last):
...
ValueError: matrix is not Hermitian
```

The matrix may not be positive-definite:

```
sage: C.<I> = QuadraticField(-1)
sage: B = matrix(C, [[      2, 4 - 2*I, 2 + 2*I],
....:                [4 + 2*I,       8,    10*I],
....:                [2 - 2*I,   -10*I,      -3]])
sage: B.is_positive_definite()
False
sage: B.cholesky()
Traceback (most recent call last):
...
ValueError: matrix is not positive definite
```

```
sage: A = matrix(QQ, [[21, 15, 12, -3],
....:                 [15, 12,  9,  12],
....:                 [12,  9,  7,  3],
....:                 [-3,  12,  3,  8]])
sage: A.is_positive_definite()
False
sage: A.cholesky()
Traceback (most recent call last):
...
ValueError: matrix is not positive definite
```

column_module()

Return the free module over the base ring spanned by the columns of this matrix.

EXAMPLES:

```
sage: t = matrix(QQ, 3, 3, range(9)); t
[0 1 2]
[3 4 5]
[6 7 8]
sage: t.column_module()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
```

column_space()

Return the vector space over the base ring spanned by the columns of this matrix.

EXAMPLES:

```
sage: M = MatrixSpace(QQ,3,3)
sage: A = M([1,9,-7,4/5,4,3,6,4,3])
sage: A.column_space()
Vector space of degree 3 and dimension 3 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]
sage: W = MatrixSpace(CC,2,2)
sage: B = W([1, 2+3*I,4+5*I,9]); B
[                    1.00000000000000 2.00000000000000 + 3.00000000000000*I]
[4.00000000000000 + 5.00000000000000*I                     9.00000000000000]
sage: B.column_space()
Vector space of degree 2 and dimension 2 over Complex Field with 53 bits of␣
↪precision
Basis matrix:
[ 1.00000000000000 0.000000000000000]
[0.000000000000000  1.00000000000000]
```

conjugate()

Return the conjugate of self, i.e. the matrix whose entries are the conjugates of the entries of self.

EXAMPLES:

```
sage: A = matrix(CDF, [[1+I,1],[0,2*I]])
sage: A.conjugate()
[1.0 - 1.0*I         1.0]
[        0.0      -2.0*I]
```

A matrix over a not-totally-real number field:

```
sage: K.<j> = NumberField(x^2+5)
sage: M = matrix(K, [[1+j,1], [0,2*j]])
sage: M.conjugate()
[-j + 1      1]
[     0   -2*j]
```

There is a shortcut for the conjugate:

```
sage: M.C
[-j + 1      1]
[     0   -2*j]
```

There is also a shortcut for the conjugate transpose, or "Hermitian transpose":

```
sage: M.H
[-j + 1      0]
[     1   -2*j]
```

Conjugates work (trivially) for matrices over rings that embed canonically into the real numbers:

```
sage: M = random_matrix(ZZ, 2)
sage: M == M.conjugate()
True
sage: M = random_matrix(QQ, 3)
sage: M == M.conjugate()
True
sage: M = random_matrix(RR, 2)
sage: M == M.conjugate()
True
```

conjugate_transpose()

Return the transpose of `self` after each entry has been converted to its complex conjugate.

Note: This function is sometimes known as the "adjoint" of a matrix, though there is substantial variation and some confusion with the use of that term.

OUTPUT:

A matrix formed by taking the complex conjugate of every entry of `self` and then transposing the resulting matrix.

Complex conjugation is implemented for many subfields of the complex numbers. See the examples below, or more at *conjugate()*.

EXAMPLES:

```
sage: M = matrix(SR, 2, 2, [[2-I, 3+4*I], [9-6*I, 5*I]])    # optional - sage.
↪symbolic
sage: M.base_ring()                                          # optional - sage.
↪symbolic
Symbolic Ring
sage: M.conjugate_transpose()                                # optional - sage.
↪symbolic
[   I + 2  6*I + 9]
[-4*I + 3     -5*I]

sage: P = matrix(CC, 3, 2, [0.95-0.63*I, 0.84+0.13*I, 0.94+0.23*I, 0.23+0.59*I,␣
↪0.52-0.41*I, -0.50+0.90*I])
sage: P.base_ring()
Complex Field with 53 bits of precision
sage: P.conjugate_transpose()
```

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```
[ 0.950... + 0.630...*I  0.940... - 0.230...*I  0.520... + 0.410...*I]
[ 0.840... - 0.130...*I  0.230... - 0.590...*I -0.500... - 0.900...*I]
```

There is also a shortcut for the conjugate transpose, or "Hermitian transpose":

```
sage: M.H
[   I + 2  6*I + 9]
[-4*I + 3     -5*I]
```

Matrices over base rings that can be embedded in the real numbers will behave as expected.

```
sage: P = random_matrix(QQ, 3, 4)
sage: P.conjugate_transpose() == P.transpose()
True
```

The conjugate of a matrix is formed by taking conjugates of all the entries. Some specialized subfields of the complex numbers are implemented in Sage and complex conjugation can be applied. (Matrices over quadratic number fields are another class of examples.)

```
sage: C = CyclotomicField(5)
sage: a = C.gen(); a
zeta5
sage: CC(a)
0.309016994374947 + 0.951056516295154*I
sage: M = matrix(C, 1, 2, [a^2, a+a^3])
sage: M.conjugate_transpose()
[             zeta5^3]
[-zeta5^3 - zeta5 - 1]
```

Furthermore, this method can be applied to matrices over quadratic extensions of finite fields:

```
sage: F.<a> = GF(9,'a')
sage: N = matrix(F, 2, [0,a,-a,1]); N
[  0   a]
[2*a   1]
sage: N.conjugate_transpose()
[      0   a + 2]
[2*a + 1       1]
```

Conjugation does not make sense over rings not containing complex numbers or finite fields which are not a quadratic extension:

```
sage: N = matrix(GF(5), 2, [0,1,2,3])
sage: N.conjugate_transpose()
Traceback (most recent call last):
...
AttributeError: 'sage.rings.finite_rings.integer_mod.IntegerMod_int' object has␣
↪no attribute 'conjugate'
```

cyclic_subspace(*v*, *var=None*, *basis='echelon'*)

Create a cyclic subspace for a vector, and optionally, a minimal polynomial for the iterated powers.

These subspaces are also known as Krylov subspaces. They are spanned by the vectors

$$\{v, Av, A^2v, A^3v, \dots\}$$

INPUT:

- `self` - a square matrix with entries from a field.
- `v` - a vector with a degree equal to the size of the matrix and entries compatible with the entries of the matrix.
- `var` - default: `None` - if specified as a string or a generator of a polynomial ring, then this will be used to construct a polynomial reflecting a relation of linear dependence on the powers $A^i v$ *and* this will cause the polynomial to be returned along with the subspace. A generator must create polynomials with coefficients from the same field as the matrix entries.
- `basis` - default: `echelon` - the basis for the subspace is "echelonized" by default, but the keyword 'iterates' will return a subspace with a user basis equal to the largest linearly independent set $\{v, Av, A^2v, A^3v, \ldots, A^{k-1}v\}$.

OUTPUT:

Suppose k is the smallest power such that $\{v, Av, A^2v, A^3v, \ldots, A^kv\}$ is linearly dependent. Then the subspace returned will have dimension k and be spanned by the powers 0 through $k-1$.

If a polynomial is requested through the use of the `var` keyword, then a pair is returned, with the polynomial first and the subspace second. The polynomial is the unique monic polynomial whose coefficients provide a relation of linear dependence on the first k powers.

For less convenient, but more flexible output, see the helper method "_cyclic_subspace" in this module.

EXAMPLES:

```
sage: A = matrix(QQ, [[5,4,2,1],[0,1,-1,-1],[-1,-1,3,0],[1,1,-1,2]])
sage: v = vector(QQ, [0,1,0,0])
sage: E = A.cyclic_subspace(v); E
Vector space of degree 4 and dimension 3 over Rational Field
Basis matrix:
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  1 -1]
sage: F = A.cyclic_subspace(v, basis='iterates'); F
Vector space of degree 4 and dimension 3 over Rational Field
User basis matrix:
[ 0  1  0  0]
[ 4  1 -1  1]
[23  1 -8  8]
sage: E == F
True
sage: p, S = A.cyclic_subspace(v, var='T'); p
T^3 - 9*T^2 + 24*T - 16
sage: p.degree() == E.dimension()
True
```

The polynomial has coefficients that yield a non-trivial relation of linear dependence on the iterates. Or, equivalently, evaluating the polynomial with the matrix will create a matrix that annihilates the vector.

```
sage: A = matrix(QQ, [[15, 37/3, -16, -104/3, -29, -7/3, 35, 2/3, -29/3, -1/3],
....:                 [ 2, 9, -1, -6, -6, 0, 7, 0, -2, 0],
....:                 [24, 74/3, -29, -208/3, -58, -14/3, 70, 4/3, -58/3, -2/3],
....:                 [-6, -19, 3, 21, 19, 0, -21, 0, 6, 0],
....:                 [2, 6, -1, -6, -3, 0, 7, 0, -2, 0],
```

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```
....:                 [-96, -296/3, 128, 832/3, 232, 65/3, -279, -16/3, 232/3,␣
↪8/3],
....:                 [0, 0, 0, 0, 0, 0, 3, 0, 0, 0],
....:                 [20, 26/3, -30, -199/3, -42, -14/3, 70, 13/3, -55/3, -2/
↪3],
....:                 [18, 57, -9, -54, -57, 0, 63, 0, -15, 0],
....:                 [0, 0, 0, 0, 0, 0, 0, 0, 0, 3]])
sage: u = zero_vector(QQ, 10); u[0] = 1
sage: p, S = A.cyclic_subspace(u, var='t', basis='iterates')
sage: S
Vector space of degree 10 and dimension 3 over Rational Field
User basis matrix:
[   1    0    0    0    0    0    0    0    0    0]
[  15    2   24   -6    2  -96    0   20   18    0]
[  79   12  140  -36   12 -560    0  116  108    0]
sage: p
t^3 - 9*t^2 + 27*t - 27
sage: k = p.degree()
sage: coeffs = p.list()
sage: iterates = S.basis() + [A^k*u]
sage: sum(coeffs[i]*iterates[i] for i in range(k+1))
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
sage: u in p(A).right_kernel()
True
```

decomposition(*algorithm='spin'*, *is_diagonalizable=False*, *dual=False*)

Returns the decomposition of the free module on which this matrix A acts from the right (i.e., the action is x goes to x A), along with whether this matrix acts irreducibly on each factor. The factors are guaranteed to be sorted in the same way as the corresponding factors of the characteristic polynomial.

Let A be the matrix acting from the on the vector space V of column vectors. Assume that A is square. This function computes maximal subspaces W_1, ..., W_n corresponding to Galois conjugacy classes of eigenvalues of A. More precisely, let $f(X)$ be the characteristic polynomial of A. This function computes the subspace $W_i = ker(g_(A)^n)$, where $g_i(X)$ is an irreducible factor of $f(X)$ and $g_i(X)$ exactly divides $f(X)$. If the optional parameter is_diagonalizable is True, then we let $W_i = ker(g(A))$, since then we know that $ker(g(A)) = ker(g(A)^n)$.

INPUT:

- `self` - a matrix
- `algorithm` - 'spin' (default): algorithm involves iterating the action of self on a vector. 'kernel': naively just compute $ker(f_i(A))$ for each factor f_i.
- `dual` - bool (default: False): If True, also returns the corresponding decomposition of V under the action of the transpose of A. The factors are guaranteed to correspond.
- `is_diagonalizable` - if the matrix is known to be diagonalizable, set this to True, which might speed up the algorithm in some cases.

Note: If the base ring is not a field, the kernel algorithm is used.

OUTPUT:

- Sequence - list of pairs (V,t), where V is a vector spaces and t is a bool, and t is True exactly when the charpoly of self on V is irreducible.
- (optional) list - list of pairs (W,t), where W is a vector space and t is a bool, and t is True exactly when the charpoly of the transpose of self on W is irreducible.

EXAMPLES:

```
sage: A = matrix(ZZ, 4, [3,4,5,6,7,3,8,10,14,5,6,7,2,2,10,9])
sage: B = matrix(QQ, 6, 6, range(36))
sage: B*11
[  0  11  22  33  44  55]
[ 66  77  88  99 110 121]
[132 143 154 165 176 187]
[198 209 220 231 242 253]
[264 275 286 297 308 319]
[330 341 352 363 374 385]
sage: A.decomposition()
[
(Ambient free module of rank 4 over the principal ideal domain Integer Ring,␣
↪True)
]
sage: B.decomposition()
[
(Vector space of degree 6 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1 -2 -3 -4]
[ 0  1  2  3  4  5], True),
(Vector space of degree 6 and dimension 4 over Rational Field
Basis matrix:
[ 1  0  0  0 -5  4]
[ 0  1  0  0 -4  3]
[ 0  0  1  0 -3  2]
[ 0  0  0  1 -2  1], False)
]
```

decomposition_of_subspace(*M*, *check_restrict=True*, ***kwds*)

Suppose the right action of self on M leaves M invariant. Return the decomposition of M as a list of pairs (W, is_irred) where is_irred is True if the charpoly of self acting on the factor W is irreducible.

Additional inputs besides M are passed onto the decomposition command.

INPUT:

- *M* – A subspace of the free module `self` acts on.
- **check_restrict – A boolean (default: True); Call restrict** with or without check.
- `kwds` – Keywords that will be forwarded to *decomposition()*.

EXAMPLES:

```
sage: t = matrix(QQ, 3, [3, 0, -2, 0, -2, 0, 0, 0, 0]); t
[ 3  0 -2]
[ 0 -2  0]
[ 0  0  0]
```

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```
sage: t.fcp('X')   # factored charpoly
(X - 3) * X * (X + 2)
sage: v = kernel(t*(t+2)); v   # an invariant subspace
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[0 1 0]
[0 0 1]
sage: D = t.decomposition_of_subspace(v); D
[
(Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 0 1], True),
(Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0], True)
]
sage: t.restrict(D[0][0])
[0]
sage: t.restrict(D[1][0])
[-2]
```

We do a decomposition over ZZ:

```
sage: a = matrix(ZZ,6,[0, 0, -2, 0, 2, 0, 2, -4, -2, 0, 2, 0, 0, 0, -2, -2, 0,␣
↪0, 2, 0, -2, -4, 2, -2, 0, 2, 0, -2, -2, 0, 0, 2, 0, -2, 0, 0])
sage: a.decomposition_of_subspace(ZZ^6)
[
(Free module of degree 6 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1  0  1 -1  1 -1]
[ 0  1  0 -1  2 -1], False),
(Free module of degree 6 and rank 4 over Integer Ring
Echelon basis matrix:
[ 1  0 -1  0  1  0]
[ 0  1  0  0  0  0]
[ 0  0  0  1  0  0]
[ 0  0  0  0  0  1], False)
]
```

denominator()

Return the least common multiple of the denominators of the elements of self.

If there is no denominator function for the base field, or no LCM function for the denominators, raise a TypeError.

EXAMPLES:

```
sage: A = MatrixSpace(QQ,2)([1/2, 1/3, 1/5, 1/7])
sage: A.denominator()
210
```

A trivial example:

```
sage: A = matrix(QQ, 0,2)
sage: A.denominator()
1
```

Denominators are not defined for real numbers:

```
sage: A = MatrixSpace(RealField(),2)([1,2,3,4])
sage: A.denominator()
Traceback (most recent call last):
...
TypeError: denominator not defined for elements of the base ring
```

We can even compute the denominator of matrix over the fraction field of $\mathbf{Z}[x]$.

```
sage: K.<x> = Frac(ZZ['x'])
sage: A = MatrixSpace(K,2)([1/x, 2/(x+1), 1, 5/(x^3)])
sage: A.denominator()
x^4 + x^3
```

Here's an example involving a cyclotomic field:

```
sage: K.<z> = CyclotomicField(3)
sage: M = MatrixSpace(K,3,sparse=True)
sage: A = M([(1+z)/3,(2+z)/3,z/3,1,1+z,-2,1,5,-1+z])
sage: print(A)
[1/3*z + 1/3 1/3*z + 2/3       1/3*z]
[          1       z + 1          -2]
[          1           5       z - 1]
sage: print(A.denominator())
3
```

density()

Return the density of the matrix.

By density we understand the ratio of the number of nonzero positions and the self.nrows() * self.ncols(), i.e. the number of possible nonzero positions.

EXAMPLES:

First, note that the density parameter does not ensure the density of a matrix, it is only an upper bound.

```
sage: A = random_matrix(GF(127),200,200,density=0.3)
sage: A.density() <= 0.3
True
```

```
sage: A = matrix(QQ,3,3,[0,1,2,3,0,0,6,7,8])
sage: A.density()
2/3
```

```
sage: a = matrix([[],[],[],[]])
sage: a.density()
0
```

derivative(**args*)

Derivative with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

EXAMPLES:

```
sage: v = vector([1,x,x^2])
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v = vector([1,x,x^2], sparse=True)
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v.derivative(x,x)
(0, 0, 2)
```

det(**args*, ***kwds*)

Synonym for self.determinant(...).

EXAMPLES:

```
sage: A = MatrixSpace(Integers(8),3)([1,7,3, 1,1,1, 3,4,5])
sage: A.det()
6
```

determinant(*algorithm=None*)

Return the determinant of `self`.

ALGORITHM:

If the base ring has a method `_matrix_determinant()`, we call it.

Otherwise, for small matrices (n less than 4), this is computed using the naive formula. In the specific case of matrices over the integers modulo a non-prime, the determinant of a lift is computed over the integers. In general, the characteristic polynomial is computed either using the Hessenberg form (specified by "`hessenberg`") or the generic division-free algorithm (specified by "`df`"). When the base ring is an exact field, the default choice is "`hessenberg`", otherwise it is "`df`". Note that for matrices over most rings, more sophisticated algorithms can be used. (Type `A.determinant?` to see what is done for a specific matrix `A`.)

INPUT:

- **algorithm - string:**
  - "`df`" - Generic O(n^4) division-free algorithm
  - "`hessenberg`" - Use the Hessenberg form of the matrix

EXAMPLES:

```
sage: A = MatrixSpace(Integers(8),3)([1,7,3, 1,1,1, 3,4,5])
sage: A.determinant()
6
sage: A.determinant() is A.determinant()
True
sage: A[0,0] = 10
```

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```
sage: A.determinant()
7
```

We compute the determinant of the arbitrary 3x3 matrix:

```
sage: R = PolynomialRing(QQ,9,'x')
sage: A = matrix(R,3,R.gens())
sage: A
[x0 x1 x2]
[x3 x4 x5]
[x6 x7 x8]
sage: A.determinant()
-x2*x4*x6 + x1*x5*x6 + x2*x3*x7 - x0*x5*x7 - x1*x3*x8 + x0*x4*x8
```

We create a matrix over $\mathbf{Z}[x, y]$ and compute its determinant.

```
sage: R.<x,y> = PolynomialRing(IntegerRing(),2)
sage: A = MatrixSpace(R,2)([x, y, x**2, y**2])
sage: A.determinant()
-x^2*y + x*y^2
```

A matrix over a non-domain:

```
sage: m = matrix(Integers(4), 2, [1,2,2,3])
sage: m.determinant()
3
```

diagonal()

Return the diagonal entries of `self`.

OUTPUT:

A list containing the entries of the matrix that have equal row and column indices, in order of the indices. Behavior is not limited to square matrices.

EXAMPLES:

```
sage: A = matrix([[2,5],[3,7]]); A
[2 5]
[3 7]
sage: A.diagonal()
[2, 7]
```

Two rectangular matrices.

```
sage: B = matrix(3, 7, range(21)); B
[ 0  1  2  3  4  5  6]
[ 7  8  9 10 11 12 13]
[14 15 16 17 18 19 20]
sage: B.diagonal()
[0, 8, 16]

sage: C = matrix(3, 2, range(6)); C
[0 1]
```

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```
[2 3]
[4 5]
sage: C.diagonal()
[0, 3]
```

Empty matrices behave properly.

```
sage: E = matrix(0, 5, []); E
[]
sage: E.diagonal()
[]
```

diagonalization(*base_field=None*)

Return a diagonal matrix similar to `self` along with the transformation matrix.

INPUT:

- `base_field` – if given, `self` is regarded as a matrix over it

OUTPUT: a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$, if `self` is a diagonalizable matrix A.

EXAMPLES:

```
sage: A = matrix(QQ, 4, [-4, 6, 3, 3, -3, 5, 3, 3, 3, -6, -4, -3, -3, 6, 3, 2])
sage: A
[-4  6  3  3]
[-3  5  3  3]
[ 3 -6 -4 -3]
[-3  6  3  2]
sage: A.is_diagonalizable()
True
sage: A.diagonalization()
(
[ 2  0  0  0]  [ 1  1  0  0]
[ 0 -1  0  0]  [ 1  0  1  0]
[ 0  0 -1  0]  [-1  0  0  1]
[ 0  0  0 -1], [ 1  1 -2 -1]
)
sage: D, P = A.diagonalization()
sage: P^-1*A*P == D
True

sage: A = matrix(QQ, 2, [0, 2, 1, 0])
sage: A.is_diagonalizable()
False
sage: A.is_diagonalizable(QQbar)
True
sage: D, P = A.diagonalization(QQbar)
sage: P^-1*A*P == D
True
```

Matrices may fail to be diagonalizable for various reasons:

```
sage: A = matrix(QQ, 2, [1,2,3,4,5,6])
sage: A
[1 2 3]
[4 5 6]
sage: A.diagonalization()
Traceback (most recent call last):
...
TypeError: not a square matrix

sage: B = matrix(ZZ, 2, [1, 2, 3, 4])
sage: B
[1 2]
[3 4]
sage: B.diagonalization()
Traceback (most recent call last):
...
ValueError: matrix entries must be from a field

sage: C = matrix(RR, 2, [1., 2., 3., 4.])
sage: C
[1.00000000000000 2.00000000000000]
[3.00000000000000 4.00000000000000]
sage: C.diagonalization()
Traceback (most recent call last):
...
ValueError: base field must be exact, but Real Field with 53 bits of precision␣
↪is not

sage: D = matrix(QQ, 2, [0, 2, 1, 0])
sage: D
[0 2]
[1 0]
sage: D.diagonalization()
Traceback (most recent call last):
...
ValueError: not diagonalizable over Rational Field

sage: E = matrix(QQ, 2, [3, 1, 0, 3])
sage: E
[3 1]
[0 3]
sage: E.diagonalization()
Traceback (most recent call last):
...
ValueError: not diagonalizable
sage: E.jordan_form()
[3 1]
[0 3]
```

echelon_form(*algorithm='default'*, *cutoff=0*, ***kwds*)

Return the echelon form of `self`.

Note: This row reduction does not use division if the matrix is not over a field (e.g., if the matrix is over

the integers). If you want to calculate the echelon form using division, then use *rref()*, which assumes that the matrix entries are in a field (specifically, the field of fractions of the base ring of the matrix).

INPUT:

- `algorithm` – string. Which algorithm to use. Choices are
  - `'default'`: Let Sage choose an algorithm (default).
  - `'classical'`: Gauss elimination.
  - `'partial_pivoting'`: Gauss elimination, using partial pivoting (if base ring has absolute value)
  - `'scaled_partial_pivoting'`: Gauss elimination, using scaled partial pivoting (if base ring has absolute value)
  - `'scaled_partial_pivoting_valuation'`: Gauss elimination, using scaled partial pivoting (if base ring has valuation)
  - `'strassen'`: use a Strassen divide and conquer algorithm (if available)
- `cutoff` – integer. Only used if the Strassen algorithm is selected.
- `transformation` – boolean. Whether to also return the transformation matrix. Some matrix backends do not provide this information, in which case this option is ignored.

OUTPUT:

The reduced row echelon form of `self`, as an immutable matrix. Note that self is *not* changed by this command. Use *echelonize()* to change `self` in place.

If the optional parameter `transformation=True` is specified, the output consists of a pair (E, T) of matrices where E is the echelon form of `self` and T is the transformation matrix.

EXAMPLES:

```
sage: MS = MatrixSpace(GF(19),2,3)
sage: C = MS.matrix([1,2,3,4,5,6])
sage: C.rank()
2
sage: C.nullity()
0
sage: C.echelon_form()
[ 1  0 18]
[ 0  1  2]
```

The matrix library used for $\mathbf{Z}/p$-matrices does not return the transformation matrix, so the `transformation` option is ignored:

```
sage: C.echelon_form(transformation=True)
[ 1  0 18]
[ 0  1  2]

sage: D = matrix(ZZ, 2, 3, [1,2,3,4,5,6])
sage: D.echelon_form(transformation=True)
(
[1 2 3]  [ 1  0]
[0 3 6], [ 4 -1]
)
```

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```
sage: E, T = D.echelon_form(transformation=True)
sage: T*D == E
True
```

echelonize(*algorithm='default'*, *cutoff=0*, ***kwds*)

Transform `self` into a matrix in echelon form over the same base ring as self.

Note: This row reduction does not use division if the matrix is not over a field (e.g., if the matrix is over the integers). If you want to calculate the echelon form using division, then use *rref()*, which assumes that the matrix entries are in a field (specifically, the field of fractions of the base ring of the matrix).

INPUT:

- `algorithm` – string. Which algorithm to use. Choices are
  - `'default'`: Let Sage choose an algorithm (default).
  - `'classical'`: Gauss elimination.
  - `'partial_pivoting'`: Gauss elimination, using partial pivoting (if base ring has absolute value)
  - `'scaled_partial_pivoting'`: Gauss elimination, using scaled partial pivoting (if base ring has absolute value)
  - `'scaled_partial_pivoting_valuation'`: Gauss elimination, using scaled partial pivoting (if base ring has valuation)
  - `'strassen'`: use a Strassen divide and conquer algorithm (if available)
- `cutoff` – integer. Only used if the Strassen algorithm is selected.
- `transformation` – boolean. Whether to also return the transformation matrix. Some matrix backends do not provide this information, in which case this option is ignored.

OUTPUT:

The matrix `self` is put into echelon form. Nothing is returned unless the keyword option `transformation=True` is specified, in which case the transformation matrix is returned.

EXAMPLES:

```
sage: a = matrix(QQ,3,3,range(9)); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: a.echelonize()
sage: a
[ 1  0 -1]
[ 0  1  2]
[ 0  0  0]
```

An immutable matrix cannot be transformed into echelon form. Use `self.echelon_form()` instead:

```
sage: a = matrix(QQ,3,3,range(9)); a.set_immutable()
sage: a.echelonize()
Traceback (most recent call last):
...
```

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```
ValueError: matrix is immutable; please change a copy instead
(i.e., use copy(M) to change a copy of M).
sage: a.echelon_form()
[ 1  0 -1]
[ 0  1  2]
[ 0  0  0]
```

Echelon form over the integers is what is also classically often known as Hermite normal form:

```
sage: a = matrix(ZZ,3,3,range(9))
sage: a.echelonize(); a
[ 3  0 -3]
[ 0  1  2]
[ 0  0  0]
```

We compute an echelon form both over a domain and fraction field:

```
sage: R.<x,y> = QQ[]
sage: a = matrix(R, 2, [x,y,x,y])
sage: a.echelon_form()              # not very useful? -- why two copies of␣
↪the same row?
[x y]
[x y]
```

```
sage: b = a.change_ring(R.fraction_field())
sage: b.echelon_form()              # potentially useful
[  1 y/x]
[  0   0]
```

We check that the echelon form works for matrices over p-adics. See github issue #17272:

```
sage: R = ZpCA(5,5,print_mode='val-unit')
sage: A = matrix(R,3,3,[250,2369,1147,106,927,362,90,398,2483])
sage: A
[5^3 * 2 + O(5^5)    2369 + O(5^5)    1147 + O(5^5)]
[    106 + O(5^5)     927 + O(5^5)     362 + O(5^5)]
[ 5 * 18 + O(5^5)     398 + O(5^5)    2483 + O(5^5)]
sage: K = R.fraction_field()
sage: A.change_ring(K).augment(identity_matrix(K,3)).echelon_form()
[      1 + O(5^5)           O(5^5)           O(5^5) 5 * 212 + O(5^5)    3031 +␣
↪O(5^5)    2201 + O(5^5)]
[          O(5^5)       1 + O(5^5)           O(5^5)    1348 + O(5^5) 5 * 306 +␣
↪O(5^5)    2648 + O(5^5)]
[          O(5^5)           O(5^5)       1 + O(5^5)    1987 + O(5^5) 5 * 263 +␣
↪O(5^5)     154 + O(5^5)]
```

Echelon form is not defined over arbitrary rings:

```
sage: a = matrix(Integers(9),3,3,range(9))
sage: a.echelon_form()
Traceback (most recent call last):
...
```

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```
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪9'.
```

Involving a sparse matrix:

```
sage: m = matrix(3,[1, 1, 1, 1, 0, 2, 1, 2, 0], sparse=True); m
[1 1 1]
[1 0 2]
[1 2 0]
sage: m.echelon_form()
[ 1  0  2]
[ 0  1 -1]
[ 0  0  0]
sage: m.echelonize(); m
[ 1  0  2]
[ 0  1 -1]
[ 0  0  0]
```

The transformation matrix is optionally returned:

```
sage: m_original = m
sage: transformation_matrix = m.echelonize(transformation=True)
sage: m == transformation_matrix * m_original
True
```

eigenmatrix_left(*other=None*)

Return matrices D and P, where D is a diagonal matrix of eigenvalues and the rows of P are corresponding eigenvectors (or zero vectors).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved

OUTPUT:

If `self` is a square matrix A, then the output is a diagonal matrix D and a matrix P such that

$$PA = DP,$$

where the rows of P are eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$PA = DPB.$$

The ordinary eigenvalue problem is equivalent to the generalized one if B is the identity matrix.

The generalized eigenvector decomposition is currently only implemented for matrices over `RDF` and `CDF`.

EXAMPLES:

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
```

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```
[6 7 8]
sage: D, P = A.eigenmatrix_left()
sage: D
[                  0                   0                   0]
[                  0 -1.348469228349535?                   0]
[                  0                   0  13.348469228349547?]
sage: P
[                   1                   -2                    1]
[                   1  0.3101020514433644? -0.3797958971132713?]
[                   1   1.289897948556636?   1.579795897113272?]
sage: P*A == D*P
True
```

Because P is invertible, A is diagonalizable.

```
sage: A == (~P)*D*P
True
```

The matrix P may contain zero rows corresponding to eigenvalues for which the algebraic multiplicity is greater than the geometric multiplicity. In these cases, the matrix is not diagonalizable.

```
sage: A = jordan_block(2,3); A
[2 1 0]
[0 2 1]
[0 0 2]
sage: D, P = A.eigenmatrix_left()
sage: D
[2 0 0]
[0 2 0]
[0 0 2]
sage: P
[0 0 1]
[0 0 0]
[0 0 0]
sage: P*A == D*P
True
```

A generalized eigenvector decomposition:

```
sage: A = matrix(RDF, [[1, -2], [3, 4]])
sage: B = matrix(RDF, [[0, 7], [2, -3]])
sage: D, P = A.eigenmatrix_left(B)
sage: (P * A - D * P * B).norm() < 1e-14
True
```

The matrix B in a generalized eigenvalue problem may be singular:

```
sage: A = matrix.identity(CDF, 2)
sage: B = matrix(CDF, [[2, 1+I], [4, 2+2*I]])
sage: D, P = A.eigenmatrix_left(B)
sage: D.diagonal()  # tol 1e-14
[0.2 - 0.1*I, +infinity]
```

In this case, we can still verify the eigenvector equation for the first eigenvalue and first eigenvector:

```
sage: l = D[0, 0]
sage: v = P[0, :]
sage: (v * A - l * v * B).norm() < 1e-14
True
```

The second eigenvector is contained in the left kernel of B:

```
sage: (P[1, :] * B).norm() < 1e-14
True
```

See also:

eigenvalues(), *eigenvectors_left()*, *Matrix_double_dense.eigenvectors_left()*, *eigenmatrix_right()*.

eigenmatrix_right(*other=None*)

Return matrices D and P, where D is a diagonal matrix of eigenvalues and the columns of P are corresponding eigenvectors (or zero vectors).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved

OUTPUT:

If `self` is a square matrix A, then the output is a diagonal matrix D and a matrix P such that

$$AP = PD,$$

where the columns of P are eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$AP = BPD.$$

The ordinary eigenvalue problem is equivalent to the generalized one if B is the identity matrix.

The generalized eigenvector decomposition is currently only implemented for matrices over `RDF` and `CDF`.

EXAMPLES:

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: D, P = A.eigenmatrix_right()
sage: D
[                  0                   0                   0]
[                  0 -1.348469228349535?                   0]
[                  0                   0  13.34846922834954?]
sage: P
[                  1                   1                   1]
[                 -2  0.1303061543300932?   3.069693845669907?]
[                  1 -0.7393876913398137?   5.139387691339814?]
sage: A*P == P*D
True
```

Because P is invertible, A is diagonalizable.

```
sage: A == P*D*(~P)
True
```

The matrix P may contain zero columns corresponding to eigenvalues for which the algebraic multiplicity is greater than the geometric multiplicity. In these cases, the matrix is not diagonalizable.

```
sage: A = jordan_block(2,3); A
[2 1 0]
[0 2 1]
[0 0 2]
sage: D, P = A.eigenmatrix_right()
sage: D
[2 0 0]
[0 2 0]
[0 0 2]
sage: P
[1 0 0]
[0 0 0]
[0 0 0]
sage: A*P == P*D
True
```

A generalized eigenvector decomposition:

```
sage: A = matrix(RDF, [[1, -2], [3, 4]])
sage: B = matrix(RDF, [[0, 7], [2, -3]])
sage: D, P = A.eigenmatrix_right(B)
sage: (A * P - B * P * D).norm() < 1e-14
True
```

The matrix B in a generalized eigenvalue problem may be singular:

```
sage: A = matrix.identity(RDF, 2)
sage: B = matrix(RDF, [[3, 5], [6, 10]])
sage: D, P = A.eigenmatrix_right(B); D   # tol 1e-14
[0.07692307692307694                 0.0]
[                0.0           +infinity]
```

In this case, we can still verify the eigenvector equation for the first eigenvalue and first eigenvector:

```
sage: l = D[0, 0]
sage: v = P[:, 0]
sage: (A * v  - B * v * l).norm() < 1e-14
True
```

The second eigenvector is contained in the right kernel of B:

```
sage: (B * P[:, 1]).norm() < 1e-14
True
```

See also:

eigenvalues(), *eigenvectors_right()*, *Matrix_double_dense.eigenvectors_right()*, *eigenmatrix_left()*.

eigenspaces_left(*format='all'*, *var='a'*, *algebraic_multiplicity=False*)

Compute the left eigenspaces of a matrix.

Note that `eigenspaces_left()` and `left_eigenspaces()` are identical methods. Here "left" refers to the eigenvectors being placed to the left of the matrix.

INPUT:

- `self` - a square matrix over an exact field. For inexact matrices consult the numerical or symbolic matrix classes.
- `format` - default: `None`
  - `'all'` - attempts to create every eigenspace. This will always be possible for matrices with rational entries.
  - `'galois'` - for each irreducible factor of the characteristic polynomial, a single eigenspace will be output for a single root/eigenvalue for the irreducible factor.
  - `None` - Uses the 'all' format if the base ring is contained in an algebraically closed field which is implemented. Otherwise, uses the 'galois' format.
- `var` - default: 'a' - variable name used to represent elements of the root field of each irreducible factor of the characteristic polynomial. If var='a', then the root fields will be in terms of a0, a1, a2, ..., where the numbering runs across all the irreducible factors of the characteristic polynomial, even for linear factors.
- `algebraic_multiplicity` - default: False - whether or not to include the algebraic multiplicity of each eigenvalue in the output. See the discussion below.

OUTPUT:

If algebraic_multiplicity=False, return a list of pairs (e, V) where e is an eigenvalue of the matrix, and V is the corresponding left eigenspace. For Galois conjugates of eigenvalues, there may be just one representative eigenspace, depending on the `format` keyword.

If algebraic_multiplicity=True, return a list of triples (e, V, n) where e and V are as above and n is the algebraic multiplicity of the eigenvalue.

Warning: Uses a somewhat naive algorithm (simply factors the characteristic polynomial and computes kernels directly over the extension field).

EXAMPLES:

We compute the left eigenspaces of a 3×3 rational matrix. First, we request *all* of the eigenvalues, so the results are in the field of algebraic numbers, $QQbar$. Then we request just one eigenspace per irreducible factor of the characteristic polynomial with the *galois* keyword.

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: es = A.eigenspaces_left(format='all'); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(-1.348469228349535?, Vector space of degree 3 and dimension 1 over Algebraic␣
```

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```
↪Field
User basis matrix:
[                   1  0.3101020514433644? -0.3797958971132713?]),
(13.34846922834954?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
User basis matrix:
[                 1 1.289897948556636? 1.579795897113272?])
]

sage: es = A.eigenspaces_left(format='galois'); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[            1 1/15*a1 + 2/5 2/15*a1 - 1/5])
]
sage: es = A.eigenspaces_left(format='galois', algebraic_multiplicity=True); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1], 1),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[            1 1/15*a1 + 2/5 2/15*a1 - 1/5], 1)
]
sage: e, v, n = es[0]; v = v.basis()[0]
sage: delta = e*v - v*A
sage: abs(abs(delta)) < 1e-10
True
```

The same computation, but with implicit base change to a field.

```
sage: A = matrix(ZZ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.eigenspaces_left(format='galois')
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[            1 1/15*a1 + 2/5 2/15*a1 - 1/5])
]
```

We compute the left eigenspaces of the matrix of the Hecke operator T_2 on level 43 modular symbols, both with all eigenvalues (the default) and with one subspace per factor.

```
sage: A = ModularSymbols(43).T(2).matrix(); A
[ 3  0  0  0  0  0 -1]
[ 0 -2  1  0  0  0  0]
[ 0 -1  1  1  0 -1  0]
[ 0 -1  0 -1  2 -1  1]
[ 0 -1  0  1  1 -1  1]
[ 0  0 -2  0  2 -2  1]
[ 0  0 -1  0  1  0 -1]
sage: A.base_ring()
Rational Field
sage: f = A.charpoly(); f
x^7 + x^6 - 12*x^5 - 16*x^4 + 36*x^3 + 52*x^2 - 32*x - 48
sage: factor(f)
(x - 3) * (x + 2)^2 * (x^2 - 2)^2
sage: A.eigenspaces_left(algebraic_multiplicity=True)
[
(3, Vector space of degree 7 and dimension 1 over Rational Field
User basis matrix:
[   1    0  1/7    0 -1/7    0 -2/7], 1),
(-2, Vector space of degree 7 and dimension 2 over Rational Field
User basis matrix:
[ 0  1  0  1 -1  1 -1]
[ 0  0  1  0 -1  2 -1], 2),
(-1.4142135623730951?, Vector space of degree 7 and dimension 2 over Algebraic␣
↪Field
User basis matrix:
[                  0                   1                   0                   -
↪1 0.4142135623730951?                   1                  -1]
[                  0                   0                   1                  ␣
↪0                  -1                   0  2.4142135623730951?], 2),
(1.4142135623730951?, Vector space of degree 7 and dimension 2 over Algebraic␣
↪Field
User basis matrix:
[                  0                   1                   0                  ␣
↪  -1  -2.4142135623730951?                   1                  -1]
[                  0                   0                   1                  ␣
↪   0                  -1                   0 -0.4142135623730951?], 2)
]
sage: A.eigenspaces_left(format='galois', algebraic_multiplicity=True)
[
(3, Vector space of degree 7 and dimension 1 over Rational Field
User basis matrix:
[   1    0  1/7    0 -1/7    0 -2/7], 1),
(-2, Vector space of degree 7 and dimension 2 over Rational Field
User basis matrix:
[ 0  1  0  1 -1  1 -1]
[ 0  0  1  0 -1  2 -1], 2),
(a2, Vector space of degree 7 and dimension 2 over Number Field in a2 with␣
↪defining polynomial x^2 - 2
User basis matrix:
[      0       1       0      -1 -a2 - 1       1      -1]
[      0       0       1       0      -1       0 -a2 + 1], 2)
]
```

Next we compute the left eigenspaces over the finite field of order 11.

```
sage: A = ModularSymbols(43, base_ring=GF(11), sign=1).T(2).matrix(); A
[ 3  0  9  0]
[ 0  9  0 10]
[ 0  0 10  1]
[ 0  0  1  1]
sage: A.base_ring()
Finite Field of size 11
sage: A.charpoly()
x^4 + 10*x^3 + 3*x^2 + 2*x + 1
sage: A.eigenspaces_left(format='galois', var = 'beta')
[
(9, Vector space of degree 4 and dimension 1 over Finite Field of size 11
User basis matrix:
[0 1 5 6]),
(3, Vector space of degree 4 and dimension 1 over Finite Field of size 11
User basis matrix:
[1 0 1 6]),
(beta2, Vector space of degree 4 and dimension 1 over Univariate Quotient␣
↪Polynomial Ring in beta2 over Finite Field of size 11 with modulus x^2 + 9
User basis matrix:
[        0         0         1 beta2 + 1])
]
```

This method is only applicable to exact matrices. The "eigenmatrix" routines for matrices with double-precision floating-point entries (RDF, CDF) are the best alternative. (Since some platforms return eigenvectors that are the negatives of those given here, this one example is not tested here.) There are also "eigenmatrix" routines for matrices with symbolic entries.

```
sage: A = matrix(QQ, 3, 3, range(9))
sage: A.change_ring(RR).eigenspaces_left()
Traceback (most recent call last):
...
NotImplementedError: eigenspaces cannot be computed reliably for inexact rings␣
↪such as Real Field with 53 bits of precision,
consult numerical or symbolic matrix classes for other options

sage: em = A.change_ring(RDF).eigenmatrix_left()
sage: eigenvalues = em[0]; eigenvalues.dense_matrix() # abs tol 1e-13
[13.348469228349522                0.0                 0.0]
[               0.0 -1.348469228349534                 0.0]
[               0.0                0.0                 0.0]
sage: eigenvectors = em[1]; eigenvectors # not tested
[ 0.440242867...  0.567868371...  0.695493875...]
[ 0.897878732...  0.278434036... -0.341010658...]
[ 0.408248290... -0.816496580...  0.408248290...]

sage: x, y = var('x y')
sage: S = matrix([[x, y], [y, 3*x^2]])
sage: em = S.eigenmatrix_left()
sage: eigenvalues = em[0]; eigenvalues
[3/2*x^2 + 1/2*x - 1/2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)                       ␣
↪                                0]
```

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```
[                                                          0 3/2*x^2 + 1/2*x + 1/
↪2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)]
sage: eigenvectors = em[1]; eigenvectors
[                                                      1 1/2*(3*x^2 - x -␣
↪sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y]
[                                                      1 1/2*(3*x^2 - x +␣
↪sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y]
```

A request for `'all'` the eigenvalues, when it is not possible, will raise an error. Using the `'galois'` format option is more likely to be successful.

```
sage: F.<b> = FiniteField(11^2)
sage: A = matrix(F, [[b + 1, b + 1], [10*b + 4, 5*b + 4]])
sage: A.eigenspaces_left(format='all')
Traceback (most recent call last):
...
NotImplementedError: unable to construct eigenspaces for eigenvalues outside␣
↪the base field,
try the keyword option: format='galois'

sage: A.eigenspaces_left(format='galois')
[
(a0, Vector space of degree 2 and dimension 1 over Univariate Quotient␣
↪Polynomial Ring in a0 over Finite Field in b of size 11^2 with modulus x^2 +␣
↪(5*b + 6)*x + 8*b + 10
User basis matrix:
[               1 6*b*a0 + 3*b + 1])
]
```

eigenspaces_right(*format='all'*, *var='a'*, *algebraic_multiplicity=False*)

Compute the right eigenspaces of a matrix.

Note that `eigenspaces_right()` and `right_eigenspaces()` are identical methods. Here "right" refers to the eigenvectors being placed to the right of the matrix.

INPUT:

- `self` - a square matrix over an exact field. For inexact matrices consult the numerical or symbolic matrix classes.
- `format` - default: `None`
  - `'all'` - attempts to create every eigenspace. This will always be possible for matrices with rational entries.
  - `'galois'` - for each irreducible factor of the characteristic polynomial, a single eigenspace will be output for a single root/eigenvalue for the irreducible factor.
  - `None` - Uses the 'all' format if the base ring is contained in an algebraically closed field which is implemented. Otherwise, uses the 'galois' format.
- `var` - default: 'a' - variable name used to represent elements of the root field of each irreducible factor of the characteristic polynomial. If var='a', then the root fields will be in terms of a0, a1, a2, ..., where the numbering runs across all the irreducible factors of the characteristic polynomial, even for linear factors.

- `algebraic_multiplicity` - default: False - whether or not to include the algebraic multiplicity of each eigenvalue in the output. See the discussion below.

OUTPUT:

If algebraic_multiplicity=False, return a list of pairs (e, V) where e is an eigenvalue of the matrix, and V is the corresponding left eigenspace. For Galois conjugates of eigenvalues, there may be just one representative eigenspace, depending on the `format` keyword.

If algebraic_multiplicity=True, return a list of triples (e, V, n) where e and V are as above and n is the algebraic multiplicity of the eigenvalue.

Warning: Uses a somewhat naive algorithm (simply factors the characteristic polynomial and computes kernels directly over the extension field).

EXAMPLES:

Right eigenspaces are computed from the left eigenspaces of the transpose of the matrix. As such, there is a greater collection of illustrative examples at the *eigenspaces_left()*.

We compute the right eigenspaces of a 3×3 rational matrix.

```
sage: A = matrix(QQ, 3 ,3, range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.eigenspaces_right()
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(-1.348469228349535?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
User basis matrix:
[                   1  0.1303061543300932? -0.7393876913398137?]),
(13.34846922834954?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
User basis matrix:
[                 1 3.069693845669907? 5.139387691339814?])
]
sage: es = A.eigenspaces_right(format='galois'); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[           1 1/5*a1 + 2/5 2/5*a1 - 1/5])
]
sage: es = A.eigenspaces_right(format='galois', algebraic_multiplicity=True); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
```

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```
[ 1 -2  1], 1),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[           1 1/5*a1 + 2/5 2/5*a1 - 1/5], 1)
]
sage: e, v, n = es[0]; v = v.basis()[0]
sage: delta = v*e - A*v
sage: abs(abs(delta)) < 1e-10
True
```

The same computation, but with implicit base change to a field:

```
sage: A = matrix(ZZ, 3, range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.eigenspaces_right(format='galois')
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[           1 1/5*a1 + 2/5 2/5*a1 - 1/5])
]
```

This method is only applicable to exact matrices. The "eigenmatrix" routines for matrices with double-precision floating-point entries (RDF, CDF) are the best alternative. (Since some platforms return eigenvectors that are the negatives of those given here, this one example is not tested here.) There are also "eigenmatrix" routines for matrices with symbolic entries.

```
sage: B = matrix(RR, 3, 3, range(9))
sage: B.eigenspaces_right()
Traceback (most recent call last):
...
NotImplementedError: eigenspaces cannot be computed reliably for inexact rings␣
↪such as Real Field with 53 bits of precision,
consult numerical or symbolic matrix classes for other options

sage: em = B.change_ring(RDF).eigenmatrix_right()
sage: eigenvalues = em[0]; eigenvalues.dense_matrix() # abs tol 1e-13
[13.348469228349522                0.0                0.0]
[               0.0 -1.348469228349534                0.0]
[               0.0                0.0                0.0]
sage: eigenvectors = em[1]; eigenvectors # not tested
[ 0.164763817...  0.799699663...  0.408248290...]
[ 0.505774475...  0.104205787... -0.816496580...]
[ 0.846785134... -0.591288087...  0.408248290...]

sage: x, y = var('x y')
```

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```
sage: S = matrix([[x, y], [y, 3*x^2]])
sage: em = S.eigenmatrix_right()
sage: eigenvalues = em[0]; eigenvalues
[3/2*x^2 + 1/2*x - 1/2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)                       ␣
↪                                 0]
[                                                          0 3/2*x^2 + 1/2*x + 1/
↪2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)]
sage: eigenvectors = em[1]; eigenvectors
[                                                      1                       ␣
↪                             1]
[1/2*(3*x^2 - x - sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y 1/2*(3*x^2 - x +␣
↪sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y]
```

eigenvalue_multiplicity(*s*)

Return the multiplicity of `s` as a generalized eigenvalue of the matrix.

EXAMPLES:

```
sage: M = Matrix(QQ, [[0,1],[0,0]])
sage: M.eigenvalue_multiplicity(0)
2
sage: M.eigenvalue_multiplicity(1)
0

sage: M = posets.DiamondPoset(5).coxeter_transformation()
sage: [M.eigenvalue_multiplicity(x) for x in [-1, 1]]
[3, 2]
```

eigenvalues(*extend=True*)

Return a sequence of the eigenvalues of a matrix, with multiplicity. If the eigenvalues are roots of polynomials in QQ, then QQbar elements are returned that represent each separate root.

If the option extend is set to False, only eigenvalues in the base ring are considered.

EXAMPLES:

```
sage: a = matrix(ZZ, 4, range(16)); a
[ 0  1  2  3]
[ 4  5  6  7]
[ 8  9 10 11]
[12 13 14 15]
sage: sorted(a.eigenvalues(), reverse=True)
[32.46424919657298?, 0, 0, -2.464249196572981?]
```

```
sage: a=matrix([(1, 9, -1, -1), (-2, 0, -10, 2), (-1, 0, 15, -2), (0, 1, 0, -
↪1)])
sage: a.eigenvalues()
[-0.9386318578049146?, 15.50655435353258?, 0.2160387521361705? - 4.
↪713151979747493?*I, 0.2160387521361705? + 4.713151979747493?*I]
```

A symmetric matrix a+a.transpose() should have real eigenvalues

```
sage: b=a+a.transpose()
sage: ev = b.eigenvalues(); ev
[-8.350660860579572, -1.107247901349379?, 5.718651326708515?, 33.73925743522043?
↪]
```

The eigenvalues are elements of QQbar, so they really represent exact roots of polynomials, not just approximations.

```
sage: e = ev[0]; e
-8.350660860579572
sage: p = e.minpoly(); p
x^4 - 30*x^3 - 171*x^2 + 1460*x + 1784
sage: p(e) == 0
True
```

To perform computations on the eigenvalue as an element of a number field, you can always convert back to a number field element.

```
sage: e.as_number_field_element()
(Number Field in a with defining polynomial y^4 - 2*y^3 - 507*y^2 - 3972*y -␣
↪4264,
a + 7,
Ring morphism:
  From: Number Field in a with defining polynomial y^4 - 2*y^3 - 507*y^2 -␣
↪3972*y - 4264
  To:   Algebraic Real Field
  Defn: a |--> -15.350660860579572?)
```

Notice the effect of the extend option.

```
sage: M=matrix(QQ,[[0,-1,0],[1,0,0],[0,0,2]])
sage: M.eigenvalues()
[2, -1*I, 1*I]
sage: M.eigenvalues(extend=False)
[2]
```

The method also works for matrices over finite fields:

```
sage: M = matrix(GF(3), [[0,1,1],[1,2,0],[2,0,1]])
sage: ev = sorted(M.eigenvalues()); ev
[2*z3, 2*z3 + 1, 2*z3 + 2]
```

Similarly as in the case of QQbar, the eigenvalues belong to some algebraic closure but they can be converted to elements of a finite field:

```
sage: e = ev[0]
sage: e.parent()
Algebraic closure of Finite Field of size 3
sage: e.as_finite_field_element()
(Finite Field in z3 of size 3^3, 2*z3, Ring morphism:
  From: Finite Field in z3 of size 3^3
  To:   Algebraic closure of Finite Field of size 3
  Defn: z3 |--> z3)
```

eigenvectors_left(*other=None*, *extend=True*)

Compute the left eigenvectors of a matrix.

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved (currently supported only if the base ring of `self` is `RDF` or `CDF`)
- `extend` – boolean (default: `True`)

OUTPUT:

For each distinct eigenvalue, returns a list of the form (e,V,n) where e is the eigenvalue, V is a list of eigenvectors forming a basis for the corresponding left eigenspace, and n is the algebraic multiplicity of the eigenvalue.

If the option extend is set to False, then only the eigenvalues that live in the base ring are considered.

EXAMPLES:

We compute the left eigenvectors of a 3×3 rational matrix.

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: es = A.eigenvectors_left(); es
[(0, [
(1, -2, 1)
], 1),
(-1.348469228349535?, [(1, 0.3101020514433644?, -0.3797958971132713?)], 1),
(13.34846922834954?, [(1, 1.289897948556636?, 1.579795897113272?)], 1)]
sage: eval, [evec], mult = es[0]
sage: delta = eval*evec - evec*A
sage: abs(abs(delta)) < 1e-10
True
```

Notice the difference between considering ring extensions or not.

```
sage: M=matrix(QQ,[[0,-1,0],[1,0,0],[0,0,2]])
sage: M.eigenvectors_left()
[(2, [
(0, 0, 1)
], 1), (-1*I, [(1, -1*I, 0)], 1), (1*I, [(1, 1*I, 0)], 1)]
sage: M.eigenvectors_left(extend=False)
[(2, [
(0, 0, 1)
], 1)]
```

eigenvectors_right(*other=None*, *extend=True*)

Compute the right eigenvectors of a matrix.

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved (currently supported only if the base ring of `self` is `RDF` or `CDF`)
- `extend` – boolean (default: `True`)

OUTPUT:

For each distinct eigenvalue, returns a list of the form (e,V,n) where e is the eigenvalue, V is a list of eigenvectors forming a basis for the corresponding right eigenspace, and n is the algebraic multiplicity of the eigenvalue. If `extend = True` (the default), this will return eigenspaces over the algebraic closure of the base field where this is implemented; otherwise it will restrict to eigenvalues in the base field.

EXAMPLES:

We compute the right eigenvectors of a 3×3 rational matrix.

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: es = A.eigenvectors_right(); es
[(0, [
(1, -2, 1)
], 1),
(-1.348469228349535?, [(1, 0.1303061543300932?, -0.7393876913398137?)], 1),
(13.34846922834954?, [(1, 3.069693845669907?, 5.1393876913398147?)], 1)]
sage: A.eigenvectors_right(extend=False)
[(0, [
(1, -2, 1)
], 1)]
sage: eval, [evec], mult = es[0]
sage: delta = eval*evec - A*evec
sage: abs(abs(delta)) < 1e-10
True
```

elementary_divisors()

If self is a matrix over a principal ideal domain R, return elements d_i for $1 \leq i \leq k = \min(r, s)$ where r and s are the number of rows and columns of self, such that the cokernel of self is isomorphic to

$$R/(d_1) \oplus R/(d_2) \oplus R/(d_k)$$

with $d_i \mid d_{i+1}$ for all i. These are the diagonal entries of the Smith form of self (see *smith_form()*).

EXAMPLES:

```
sage: OE.<w> = EquationOrder(x^2 - x + 2)
sage: m = Matrix([ [1, w],[w,7]])
sage: m.elementary_divisors()
[1, -w + 9]
```

See also:

smith_form()

elementwise_product(*right*)

Returns the elementwise product of two matrices of the same size (also known as the Hadamard product).

INPUT:

- `right` - the right operand of the product. A matrix of the same size as `self` such that multiplication of elements of the base rings of `self` and `right` is defined, once Sage's coercion model is applied. If the matrices have different sizes, or if multiplication of individual entries cannot be achieved, a `TypeError` will result.

OUTPUT:

A matrix of the same size as `self` and `right`. The entry in location (i, j) of the output is the product of the two entries in location (i, j) of `self` and `right` (in that order).

The parent of the result is determined by Sage's coercion model. If the base rings are identical, then the result is dense or sparse according to this property for the left operand. If the base rings must be adjusted for one, or both, matrices then the result will be sparse only if both operands are sparse. No subdivisions are present in the result.

If the type of the result is not to your liking, or the ring could be "tighter," adjust the operands with *change_ring()*. Adjust sparse versus dense inputs with the methods *sparse_matrix()* and *dense_matrix()*.

EXAMPLES:

```
sage: A = matrix(ZZ, 2, 3, range(6))
sage: B = matrix(QQ, 2, 3, [5, 1/3, 2/7, 11/2, -3/2, 8])
sage: C = A.elementwise_product(B)
sage: C
[   0  1/3  4/7]
[33/2   -6   40]
sage: C.parent()
Full MatrixSpace of 2 by 3 dense matrices over Rational Field
```

Notice the base ring of the results in the next two examples.

```
sage: D = matrix(ZZ['x'],2,[1+x^2,2,3,4-x])
sage: E = matrix(QQ,2,[1,2,3,4])
sage: F = D.elementwise_product(E)
sage: F
[  x^2 + 1         4]
[        9 -4*x + 16]
sage: F.parent()
Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x␣
↪over Rational Field
```

```
sage: G = matrix(GF(3),2,[0,1,2,2])
sage: H = matrix(ZZ,2,[1,2,3,4])
sage: J = G.elementwise_product(H)
sage: J
[0 2]
[0 2]
sage: J.parent()
Full MatrixSpace of 2 by 2 dense matrices over Finite Field of size 3
```

Non-commutative rings behave as expected. These are the usual quaternions.

```
sage: R.<i,j,k> = QuaternionAlgebra(-1, -1)
sage: A = matrix(R, 2, [1,i,j,k])
sage: B = matrix(R, 2, [i,i,i,i])
sage: A.elementwise_product(B)
[ i -1]
[-k  j]
sage: B.elementwise_product(A)
```

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```
[ i -1]
[ k -j]
```

Input that is not a matrix will raise an error.

```
sage: A = random_matrix(ZZ,5,10,x=20)
sage: A.elementwise_product(vector(ZZ, [1,2,3,4]))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Full␣
↪MatrixSpace of 5 by 10 dense matrices over Integer Ring' and 'Ambient free␣
↪module of rank 4 over the principal ideal domain Integer Ring'

sage: A = matrix(2, 2, range(4))
sage: A.elementwise_product(polygen(parent(A)))
Traceback (most recent call last):
...
TypeError: elementwise_product() argument should be a matrix or coercible to a␣
↪matrix
```

Matrices of different sizes for operands will raise an error.

```
sage: A = random_matrix(ZZ,5,10,x=20)
sage: B = random_matrix(ZZ,10,5,x=40)
sage: A.elementwise_product(B)
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Full␣
↪MatrixSpace of 5 by 10 dense matrices over Integer Ring' and 'Full␣
↪MatrixSpace of 10 by 5 dense matrices over Integer Ring'
```

Some pairs of rings do not have a common parent where multiplication makes sense. This will raise an error.

```
sage: A = matrix(QQ, 3, 2, range(6))
sage: B = matrix(GF(3), 3, [2]*6)
sage: A.elementwise_product(B)
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Full␣
↪MatrixSpace of 3 by 2 dense matrices over Rational Field' and 'Full␣
↪MatrixSpace of 3 by 2 dense matrices over Finite Field of size 3'
```

We illustrate various combinations of sparse and dense matrices. The usual coercion rules apply:

```
sage: A = matrix(ZZ, 5, 6, range(30), sparse=False)
sage: B = matrix(ZZ, 5, 6, range(30), sparse=True)
sage: C = matrix(QQ, 5, 6, range(30), sparse=True)
sage: A.elementwise_product(C).is_sparse()
True
sage: B.elementwise_product(C).is_sparse()
True
sage: A.elementwise_product(B).is_dense()
```

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```
True
sage: B.elementwise_product(A).is_dense()
True
```

exp()

Calculate the exponential of this matrix X, which is the matrix

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

This function depends on maxima's matrix exponentiation function, which does not deal well with floating point numbers. If the matrix has floating point numbers, they will be rounded automatically to rational numbers during the computation. If you want approximations to the exponential that are calculated numerically, you may get better results by first converting your matrix to RDF or CDF, as shown in the last example.

EXAMPLES:

```
sage: a=matrix([[1,2],[3,4]])
sage: a.exp()
[-1/22*((sqrt(33) - 11)*e^sqrt(33) - sqrt(33) - 11)*e^(-1/2*sqrt(33) + 5/2)    ␣
↪         2/33*(sqrt(33)*e^sqrt(33) - sqrt(33))*e^(-1/2*sqrt(33) + 5/2)]
[             1/11*(sqrt(33)*e^sqrt(33) - sqrt(33))*e^(-1/2*sqrt(33) + 5/2)  1/
↪22*((sqrt(33) + 11)*e^sqrt(33) - sqrt(33) + 11)*e^(-1/2*sqrt(33) + 5/2)]

sage: type(a.exp())
<class 'sage.matrix.matrix_symbolic_dense.Matrix_symbolic_dense'>

sage: a=matrix([[1/2,2/3],[3/4,4/5]])
sage: a.exp()
[-1/418*((3*sqrt(209) - 209)*e^(1/10*sqrt(209)) - 3*sqrt(209) - 209)*e^(-1/
↪20*sqrt(209) + 13/20)                    20/627*(sqrt(209)*e^(1/10*sqrt(209)) -
↪ sqrt(209))*e^(-1/20*sqrt(209) + 13/20)]
[                  15/418*(sqrt(209)*e^(1/10*sqrt(209)) - sqrt(209))*e^(-1/
↪20*sqrt(209) + 13/20)  1/418*((3*sqrt(209) + 209)*e^(1/10*sqrt(209)) -␣
↪3*sqrt(209) + 209)*e^(-1/20*sqrt(209) + 13/20)]

sage: a=matrix(RR,[[1,pi.n()],[1e2,1e-2]])
sage: a.exp()
[ 1/11882424341266*((11*sqrt(227345670387496707609) + 5941212170633)*e^(3/
↪1275529100*sqrt(227345670387496707609)) - 11*sqrt(227345670387496707609) +␣
↪5941212170633)*e^(-3/2551058200*sqrt(227345670387496707609) + 101/200)       ␣
↪                    445243650/
↪75781890129165569203*(sqrt(227345670387496707609)*e^(3/
↪1275529100*sqrt(227345670387496707609)) - sqrt(227345670387496707609))*e^(-3/
↪2551058200*sqrt(227345670387496707609) + 101/200)]
[                                    10000/
↪53470909535697*(sqrt(227345670387496707609)*e^(3/
↪1275529100*sqrt(227345670387496707609)) - sqrt(227345670387496707609))*e^(-3/
↪2551058200*sqrt(227345670387496707609) + 101/200) -1/
↪11882424341266*((11*sqrt(227345670387496707609) - 5941212170633)*e^(3/
↪1275529100*sqrt(227345670387496707609)) - 11*sqrt(227345670387496707609) -␣
↪5941212170633)*e^(-3/2551058200*sqrt(227345670387496707609) + 101/200)]
```

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```
sage: a.change_ring(RDF).exp()  # rel tol 1e-14
[42748127.31532951 7368259.244159399]
[234538976.1381042 40426191.45156228]
```

extended_echelon_form(*subdivide=False*, ***kwds*)

Returns the echelon form of `self` augmented with an identity matrix.

INPUT:

- `subdivide` - default: `False` - determines if the returned matrix is subdivided. See the description of the (output) below for details.
- `kwds` - additional keywords that can be passed to the method that computes the echelon form.

OUTPUT:

If A is an $m \times n$ matrix, add the m columns of an $m \times m$ identity matrix to the right of `self`. Then row-reduce this $m \times (n + m)$ matrix. This matrix is returned as an immutable matrix.

If `subdivide` is `True` then the returned matrix has a single division among the columns and a single division among the rows. The column subdivision has n columns to the left and m columns to the right. The row division separates the non-zero rows from the zero rows, when restricted to the first n columns.

For a nonsingular matrix the final m columns of the extended echelon form are the inverse of `self`. For a matrix of any size, the final m columns provide a matrix that transforms `self` to echelon form when it multiplies `self` from the left. When the base ring is a field, the uniqueness of reduced row-echelon form implies that this transformation matrix can be taken as the coefficients giving a canonical set of linear combinations of the rows of `self` that yield reduced row-echelon form.

When subdivided as described above, and again over a field, the parts of the subdivision in the upper-left corner and lower-right corner satisfy several interesting relationships with the row space, column space, left kernel and right kernel of `self`. See the examples below.

Note: This method returns an echelon form. If the base ring is not a field, no attempt is made to move to the fraction field. See an example below where the base ring is changed manually.

EXAMPLES:

The four relationships at the end of this example hold in general.

```
sage: A = matrix(QQ, [[2, -1, 7, -1, 0, -3],
....:                 [-1, 1, -5, 3, 4, 4],
....:                 [2, -1, 7, 0, 2, -2],
....:                 [2, 0, 4, 3, 6, 1],
....:                 [2, -1, 7, 0, 2, -2]])
sage: E = A.extended_echelon_form(subdivide=True); E
[   1    0    2    0    0   -1|   0   -1    0    1   -1]
[   0    1   -3    0   -2    0|   0   -2    0    2   -3]
[   0    0    0    1    2    1|   0  2/3    0 -1/3  2/3]
[------------------------------+------------------------]
[   0    0    0    0    0    0|   1  2/3    0 -1/3 -1/3]
[   0    0    0    0    0    0|   0    0    1    0   -1]
sage: J = E.matrix_from_columns(range(6,11)); J
[   0   -1    0    1   -1]
[   0   -2    0    2   -3]
```

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```
[   0  2/3    0 -1/3  2/3]
[   1  2/3    0 -1/3 -1/3]
[   0    0    1    0   -1]
sage: J*A == A.rref()
True
sage: C = E.subdivision(0,0); C
[ 1  0  2  0  0 -1]
[ 0  1 -3  0 -2  0]
[ 0  0  0  1  2  1]
sage: L = E.subdivision(1,1); L
[   1  2/3    0 -1/3 -1/3]
[   0    0    1    0   -1]
sage: A.right_kernel() == C.right_kernel()
True
sage: A.row_space() == C.row_space()
True
sage: A.column_space() == L.right_kernel()
True
sage: A.left_kernel() == L.row_space()
True
```

For a nonsingular matrix, the right half of the extended echelon form is the inverse matrix.

```
sage: B = matrix(QQ, [[1,3,4], [1,4,4], [0,-2,-1]])
sage: E = B.extended_echelon_form()
sage: J = E.matrix_from_columns(range(3,6)); J
[-4  5  4]
[-1  1  0]
[ 2 -2 -1]
sage: J == B.inverse()
True
```

The result is in echelon form, so if the base ring is not a field, the leading entry of each row may not be 1. But you can easily change to the fraction field if necessary.

```
sage: A = matrix(ZZ, [[16, 20, 4, 5, -4, 13, 5],
....:                 [10, 13, 3, -3, 7, 11, 6],
....:                 [-12, -15, -3, -3, 2, -10, -4],
....:                 [10, 13, 3, 3, -1, 9, 4],
....:                 [4, 5, 1, 8, -10, 1, -1]])
sage: E = A.extended_echelon_form(subdivide=True); E
[ 2  0 -2  2 -9 -3 -4| 0  4 -3 -9  4]
[ 0  1  1  2  0  1  1| 0  1  2  1  1]
[ 0  0  0  3 -4 -1 -1| 0  3  1 -3  3]
[--------------------+--------------]
[ 0  0  0  0  0  0  0| 1  6  3 -6  5]
[ 0  0  0  0  0  0  0| 0  7  2 -7  6]
sage: J = E.matrix_from_columns(range(7,12)); J
[ 0  4 -3 -9  4]
[ 0  1  2  1  1]
[ 0  3  1 -3  3]
[ 1  6  3 -6  5]
```

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```
[ 0  7  2 -7  6]
sage: J*A == A.echelon_form()
True
sage: B = A.change_ring(QQ)
sage: B.extended_echelon_form(subdivide=True)
[     1      0     -1      0  -19/6   -7/6   -5/3|     0      0 -89/42   -5/2  ␣
↪ 1/7]
[     0      1      1      0    8/3    5/3    5/3|     0      0  34/21      2  ␣
↪-1/7]
[     0      0      0      1   -4/3   -1/3   -1/3|     0      0   1/21      0  ␣
↪ 1/7]
[-------------------------------------------------+-------------------------------
↪----]
[     0      0      0      0      0      0      0|     1      0    9/7      0  ␣
↪-1/7]
[     0      0      0      0      0      0      0|     0      1    2/7     -1  ␣
↪ 6/7]
```

Subdivided, or not, the result is immutable, so make a copy if you want to make changes.

```
sage: A = matrix(FiniteField(7), [[2,0,3], [5,5,3], [5,6,5]])
sage: E = A.extended_echelon_form()
sage: E.is_mutable()
False
sage: F = A.extended_echelon_form(subdivide=True)
sage: F
[1 0 0|0 4 6]
[0 1 0|4 2 2]
[0 0 1|5 2 3]
[-----+-----]
sage: F.is_mutable()
False
sage: G = copy(F)
sage: G.subdivide([],[]); G
[1 0 0 0 4 6]
[0 1 0 4 2 2]
[0 0 1 5 2 3]
```

If you want to determine exactly which algorithm is used to compute the echelon form, you can add additional keywords to pass on to the `echelon_form()` routine employed on the augmented matrix. Sending the flag `include_zero_rows` is a bit silly, since the extended echelon form will never have any zero rows.

```
sage: A = matrix(ZZ, [[1,2], [5,0], [5,9]])
sage: E = A.extended_echelon_form(algorithm='padic', include_zero_rows=False)
sage: E
[  1   0  36   1  -8]
[  0   1   5   0  -1]
[  0   0  45   1 -10]
```

fcp(*var='x'*)

Return the factorization of the characteristic polynomial of self.

INPUT:

- `var` - (default: 'x') name of variable of charpoly

EXAMPLES:

```
sage: M = MatrixSpace(QQ,3,3)
sage: A = M([1,9,-7,4/5,4,3,6,4,3])
sage: A.fcp()
x^3 - 8*x^2 + 209/5*x - 286
sage: A = M([3, 0, -2, 0, -2, 0, 0, 0, 0])
sage: A.fcp('T')
(T - 3) * T * (T + 2)
```

find(*f*, *indices=False*)

Find elements in this matrix satisfying the constraints in the function f. The function is evaluated on each element of the matrix .

INPUT:

- `f` - a function that is evaluated on each element of this matrix.
- `indices` - whether or not to return the indices and elements of this matrix that satisfy the function.

OUTPUT: If `indices` is not specified, return a matrix with 1 where f is satisfied and 0 where it is not. If `indices` is specified, return a dictionary containing the elements of this matrix satisfying f.

EXAMPLES:

```
sage: M = matrix(4,3,[1, -1/2, -1, 1, -1, -1/2, -1, 0, 0, 2, 0, 1])
sage: M.find(lambda entry:entry==0)
[0 0 0]
[0 0 0]
[0 1 1]
[0 1 0]
```

```
sage: M.find(lambda u:u<0)
[0 1 1]
[0 1 1]
[1 0 0]
[0 0 0]
```

```
sage: M = matrix(4,3,[1, -1/2, -1, 1, -1, -1/2, -1, 0, 0, 2, 0, 1])
sage: len(M.find(lambda u:u<1 and u>-1,indices=True))
5
```

```
sage: M.find(lambda u:u!=1/2)
[1 1 1]
[1 1 1]
[1 1 1]
[1 1 1]
```

```
sage: M.find(lambda u:u>1.2)
[0 0 0]
[0 0 0]
[0 0 0]
[1 0 0]
```

```
sage: sorted(M.find(lambda u:u!=0,indices=True).keys()) == M.nonzero_positions()
True
```

get_subdivisions()

Returns the current subdivision of self.

EXAMPLES:

```
sage: M = matrix(5, 5, range(25))
sage: M.subdivisions()
([], [])
sage: M.subdivide(2,3)
sage: M.subdivisions()
([2], [3])
sage: N = M.parent()(1)
sage: N.subdivide(M.subdivisions()); N
[1 0 0|0 0]
[0 1 0|0 0]
[-----+---]
[0 0 1|0 0]
[0 0 0|1 0]
[0 0 0|0 1]
```

gram_schmidt(*orthonormal=False*)

Performs Gram-Schmidt orthogonalization on the rows of the matrix, returning a new matrix and a matrix accomplishing the transformation.

INPUT:

- `self` - a matrix whose rows are to be orthogonalized.
- `orthonormal` - default: `False` - if `True` the returned orthogonal vectors are unit vectors. This keyword is ignored if the matrix is over `RDF` or `CDF` and the results are always orthonormal.

OUTPUT:

A pair of matrices, `G` and `M` such that if `A` represents `self`, where the parenthetical properties occur when `orthonormal = True`:

- `A = M*G`
- The rows of `G` are an orthogonal (resp. orthonormal) set of vectors.
- `G` times the conjugate-transpose of `G` is a diagonal (resp. identity) matrix.
- The row space of `G` equals the row space of `A`.
- `M` is a full-rank matrix with zeros above the diagonal.

For exact rings, any zero vectors produced (when the original vectors are linearly dependent) are not output, thus the orthonormal set is linearly independent, and thus a basis for the row space of the original matrix.

Any notion of a Gram-Schmidt procedure requires that the base ring of the matrix has a fraction field implemented. In order to arrive at an orthonormal set, it must be possible to construct square roots of the elements of the base field. In Sage, your best option is the field of algebraic numbers, `QQbar`, which properly contains the rationals and number fields.

If you have an approximate numerical matrix, then this routine requires that your base field be the real and complex double-precision floating point numbers, `RDF` and `CDF`. In this case, the matrix is treated as having full rank, as no attempt is made to recognize linear dependence with approximate calculations.

EXAMPLES:

Inexact Rings, Numerical Matrices:

First, the inexact rings, CDF and RDF.

```
sage: A = matrix(CDF, [[ 0.6454 + 0.7491*I, -0.8662 + 0.1489*I,  0.7656 - 0.
↪00344*I],
....:                  [-0.2913 + 0.8057*I,  0.8321 + 0.8170*I, -0.6744 + 0.
↪9248*I],
....:                  [ 0.2554 + 0.3517*I, -0.4454 - 0.1715*I,  0.8325 - 0.
↪6282*I]])
sage: G, M = A.gram_schmidt()
sage: G.round(6)  # random signs
[-0.422243 - 0.490087*I  0.566698 - 0.097416*I -0.500882 + 0.002251*I]
[-0.057002 - 0.495035*I  -0.35059 - 0.625323*I  0.255514 - 0.415284*I]
[ 0.394105 - 0.421778*I -0.392266 - 0.039345*I  -0.352905 + 0.62195*I]
sage: M.round(6)  # random
[             -1.528503                    0.0                    0.0]
[  0.459974 - 0.40061*I              -1.741233                    0.0]
[-0.934304 + 0.148868*I   0.54833 + 0.073202*I              -0.550725]
sage: (A - M*G).zero_at(10^-12)
[0.0 0.0 0.0]
[0.0 0.0 0.0]
[0.0 0.0 0.0]
sage: (G*G.conjugate_transpose())  # random
[0.9999999999999999                0.0                0.0]
[               0.0 0.9999999999999997                0.0]
[               0.0                0.0                1.0]
```

A rectangular matrix. Note that the `orthonormal` keyword is ignored in these cases.

```
sage: A = matrix(RDF, [[-0.978325, -0.751994, 0.925305, -0.200512, 0.420458],
....:                  [-0.474877, -0.983403, 0.089836,  0.132218, 0.672965]])
sage: G, M = A.gram_schmidt(orthonormal=False)
sage: G.round(6).zero_at(10^-6)
[-0.607223 -0.466745  0.574315 -0.124453  0.260968]
[ 0.123203 -0.617909 -0.530578  0.289773  0.487368]
sage: M.round(6).zero_at(10^-6)
[1.611147      0.0]
[0.958116 0.867778]
sage: (A-M*G).zero_at(10^-12)
[0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0]
sage: (G*G.transpose()).round(6).zero_at(10^-6)
[1.0 0.0]
[0.0 1.0]
```

Even though a set of vectors may be linearly dependent, no effort is made to decide when a zero vector is really the result of a relation of linear dependence. So in this regard, input matrices are treated as being of full rank. Try one of the base rings that provide exact results if you need exact results.

```
sage: entries = [[1,1,2], [2,1,3], [3,1,4]]
sage: A = matrix(QQ, entries)
sage: A.rank()
```

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```
2
sage: B = matrix(RDF, entries)
sage: G, M = B.gram_schmidt()
sage: G.round(6)  # random signs
[-0.408248 -0.408248 -0.816497]
[ 0.707107 -0.707107      -0.0]
[ -0.57735  -0.57735   0.57735]
sage: M.round(10)  # random
[-2.4494897428           0.0           0.0]
[-3.6742346142  0.7071067812           0.0]
[-4.8989794856  1.4142135624           0.0]
sage: (A - M*G).zero_at(1e-14)
[0.0 0.0 0.0]
[0.0 0.0 0.0]
[0.0 0.0 0.0]
sage: (G*G.transpose())  # abs tol 1e-14
[0.9999999999999997                0.0                0.0]
[               0.0 0.9999999999999998                0.0]
[               0.0                0.0                1.0]
```

Exact Rings, Orthonormalization:

To scale a vector to unit length requires taking a square root, which often takes us outside the base ring. For the integers and the rationals, the field of algebraic numbers (`QQbar`) is big enough to contain what we need, but the price is that the computations are very slow, hence mostly of value for small cases or instruction. Now we need to use the `orthonormal` keyword.

```
sage: A = matrix(QQbar, [[6, -8,  1],
....:                    [4,  1,  3],
....:                    [6,  3,  3],
....:                    [7,  1, -5],
....:                    [7, -3,  5]])
sage: G, M = A.gram_schmidt(orthonormal=True)
sage: G
[ 0.5970223141259934? -0.7960297521679913? 0.09950371902099891?]
[ 0.6063218341690895?  0.5289635311888953?  0.5937772444966257?]
[ 0.5252981913594170?  0.2941669871612735?  -0.7984532508663147?]
sage: M
[ 10.04987562112089?                    0                    0]
[ 1.8905706613989800?  4.735582601355131?                    0]
[ 1.4925557853149840?  7.006153332071100?  1.6389303570413817?]
[ 2.8856078516089690?  1.804330147889395?  7.963520581008761?]
[ 7.0647640504909230?  5.626248468100069? -1.197679876299471?]
sage: M*G-A
[0 0 0]
[0 0 0]
[0 0 0]
[0 0 0]
[0 0 0]
sage: (G*G.transpose()-identity_matrix(3)).norm() < 10^-10
True
sage: G.row_space() == A.row_space()
True
```

After github issue #14047, the matrix can also be over the algebraic reals AA:

```
sage: A = matrix(AA, [[6, -8,  1],
....:                 [4,  1,  3],
....:                 [6,  3,  3],
....:                 [7,  1, -5],
....:                 [7, -3,  5]])
sage: G, M = A.gram_schmidt(orthonormal=True)
sage: G
[ 0.5970223141259934? -0.7960297521679913? 0.09950371902099891?]
[ 0.6063218341690895?  0.5289635311888953?  0.5937724449662257?]
[ 0.5252981913594170?  0.2941669871612735?  -0.7984532508663147?]
sage: M
[ 10.04987562112089?                    0                    0]
[ 1.8905706613989800?  4.735582601355131?                    0]
[ 1.4925557853149840?  7.006153332071100?  1.6389303570413810?]
[ 2.8856078516089690?  1.804330147889395?  7.9635205810087610?]
[ 7.0647640504909230?  5.626248468100069? -1.1976798762994710?]
```

Starting with complex numbers with rational real and imaginary parts. Note the use of the conjugate-transpose when checking the orthonormality.

```
sage: A = matrix(QQbar, [[  -2,    -I - 1, 4*I + 2,       -1],
....:                    [-4*I, -2*I + 17,       0,  9*I + 1],
....:                    [   1,  -2*I - 6, -I + 11, -5*I + 1]])
sage: G, M = A.gram_schmidt(orthonormal=True)
sage: (M*G-A).norm() < 10^-10
True
sage: id3 = G*G.conjugate().transpose()
sage: (id3 - identity_matrix(3)).norm() < 10^-10
True
sage: G.row_space() == A.row_space()  # long time
True
```

A square matrix with small rank. The zero vectors produced as a result of linear dependence get eliminated, so the rows of G are a basis for the row space of A.

```
sage: A = matrix(QQbar, [[2, -6, 3, 8],
....:                    [1, -3, 2, 5],
....:                    [0,  0, 2, 4],
....:                    [2, -6, 3, 8]])
sage: A.change_ring(QQ).rank()
2
sage: G, M = A.gram_schmidt(orthonormal=True)
sage: G
[ 0.1881441736767195? -0.5644325210301583?  0.2822162605150792?  0.
↪7525766947068779?]
[-0.2502818123591464?   0.7508454370774397?  0.3688363550555841?  0.
↪4873908977520218?]
sage: M
[10.630145812734649?                    0]
[ 6.208757731331742? 0.6718090752798139?]
[ 3.574739299857670?  2.687236301119256?]
[10.630145812734649?                    0]
```

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```
sage: M*G-A
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
sage: (G*G.transpose()-identity_matrix(2)).norm() < 10^-10
True
sage: G.row_space() == A.row_space()
True
```

Exact Rings, Orthogonalization:

If we forego scaling orthogonal vectors to unit vectors, we can apply Gram-Schmidt to a much greater variety of rings. Use the `orthonormal=False` keyword (or assume it as the default). Note that now the orthogonality check creates a diagonal matrix whose diagonal entries are the squares of the lengths of the vectors.

First, in the rationals, without involving `QQbar`.

```
sage: A = matrix(QQ, [[-1,  3,  2,  2],
....:                 [-1,  0, -1,  0],
....:                 [-1, -2, -3, -1],
....:                 [ 1,  1,  2,  0]])
sage: A.rank()
3
sage: G, M = A.gram_schmidt()
sage: G
[    -1      3      2      2]
[-19/18    1/6   -8/9    1/9]
[  2/35  -4/35  -2/35   9/35]
sage: M
[     1      0      0]
[ -1/18      1      0]
[-13/18  59/35      1]
[   1/3 -48/35     -2]
sage: M*G-A
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
sage: G*G.transpose()
[   18     0     0]
[    0 35/18     0]
[    0     0  3/35]
sage: G.row_space() == A.row_space()
True
```

A complex subfield of the complex numbers.

```
sage: C.<z> = CyclotomicField(5)
sage: A = matrix(C, [[              -z^3 - 2*z,              -z^3 - 1, 2*z^3 -␣
↪2*z^2 + 2*z,             1],
....:                [         z^3 - 2*z^2 + 1, -z^3 + 2*z^2 - z - 1,          ␣
```

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```
↪       -1,       z^2 + z],
....:                [-1/2*z^3 - 2*z^2 + z + 1,         -z^3 + z - 2,    -2*z^3␣
↪+ 1/2*z^2, 2*z^2 - z + 2]])
sage: G, M = A.gram_schmidt(orthonormal=False)
sage: G
[                                                          -z^3 - 2*z              ␣
↪                                              -z^3 - 1                            ␣
↪                  2*z^3 - 2*z^2 + 2*z                                             ␣
↪                       1]
[                  155/139*z^3 - 161/139*z^2 + 31/139*z + 13/139                   ␣
↪  -175/139*z^3 + 180/139*z^2 - 125/139*z - 142/139                     230/
↪139*z^3 + 124/139*z^2 + 6/139*z + 19/139                      -14/139*z^3 +␣
↪92/139*z^2 - 6/139*z - 95/139]
[-10359/19841*z^3 - 36739/39682*z^2 + 24961/39682*z - 11879/39682  -28209/
↪39682*z^3 - 3671/19841*z^2 + 51549/39682*z - 38613/39682    -42769/39682*z^3 -
↪ 615/39682*z^2 - 1252/19841*z - 14392/19841   4895/19841*z^3 + 57885/39682*z^
↪2 - 46094/19841*z + 65747/39682]
sage: M
[                                                                 1                ␣
↪                                          0                                       ␣
↪                        0]
[                14/139*z^3 + 47/139*z^2 + 145/139*z + 95/139                      ␣
↪                                          1                                       ␣
↪                        0]
[              -7/278*z^3 + 199/278*z^2 + 183/139*z + 175/278 -3785/39682*z^3 +␣
↪3346/19841*z^2 - 3990/19841*z + 2039/19841                                       ␣
↪                        1]
sage: M*G - A
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
sage: G*G.conjugate().transpose()
[                                 15*z^3 + 15*z^2 + 28                            ␣
↪                       0                                                      0]
[                                                  0                463/139*z^3␣
↪+ 463/139*z^2 + 1971/139                                                         0]
[                                                  0                              ␣
↪                       0 230983/19841*z^3 + 230983/19841*z^2 + 1003433/39682]
sage: G.row_space() == A.row_space()
True
```

A slightly edited legacy example.

```
sage: A = matrix(ZZ, 3, [-1, 2, 5, -11, 1, 1, 1, -1, -3]); A
[ -1   2   5]
[-11   1   1]
[  1  -1  -3]
sage: G, mu = A.gram_schmidt()
sage: G
[     -1       2       5]
[  -52/5    -1/5      -2]
[  2/187  36/187 -14/187]
```

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```
sage: mu
[     1      0      0]
[   3/5      1      0]
[  -3/5 -7/187      1]
sage: G.row(0) * G.row(1)
0
sage: G.row(0) * G.row(2)
0
sage: G.row(1) * G.row(2)
0
```

The relation between mu and A is as follows.

```
sage: mu*G == A
True
```

hadamard_bound()

Return an int n such that the absolute value of the determinant of this matrix is at most 10^n.

This is got using both the row norms and the column norms.

This function only makes sense when the base field can be coerced to the real double field RDF or the MPFR Real Field with 53-bits precision.

EXAMPLES:

```
sage: a = matrix(ZZ, 3, [1,2,5,7,-3,4,2,1,123])
sage: a.hadamard_bound()
4
sage: a.det()
-2014
sage: 10^4
10000
```

In this example the Hadamard bound has to be computed (automatically) using MPFR instead of doubles, since doubles overflow:

```
sage: a = matrix(ZZ, 2, [2^10000,3^10000,2^50,3^19292])
sage: a.hadamard_bound()
12215
sage: len(str(a.det()))
12215
```

hermite_form(*include_zero_rows=True*, *transformation=False*)

Return the Hermite form of self, if it is defined.

INPUT:

- `include_zero_rows` – bool (default: True); if False the zero rows in the output matrix are deleted.
- `transformation` – bool (default: False) a matrix U such that U*self == H.

OUTPUT:

- matrix H
- (optional) transformation matrix U such that U*self == H, possibly with zero rows deleted…

EXAMPLES:

```
sage: M = FunctionField(GF(7),'x').maximal_order()
sage: K.<x> = FunctionField(GF(7)); M = K.maximal_order()
sage: A = matrix(M, 2, 3, [x, 1, 2*x, x, 1+x, 2])
sage: A.hermite_form()
[      x       1     2*x]
[      0       x 5*x + 2]
sage: A.hermite_form(transformation=True)
(
[      x       1     2*x]  [1 0]
[      0       x 5*x + 2], [6 1]
)
sage: A = matrix(M, 2, 3, [x, 1, 2*x, 2*x, 2, 4*x])
sage: A.hermite_form(transformation=True, include_zero_rows=False)
([  x   1 2*x], [1 0])
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=True); H, U
(
[  x   1 2*x]  [1 0]
[  0   0   0], [5 1]
)
sage: U*A == H
True
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=False)
sage: U*A
[  x   1 2*x]
sage: U*A == H
True
```

hessenberg_form()

Return Hessenberg form of self.

If the base ring is merely an integral domain (and not a field), the Hessenberg form will (in general) only be defined over the fraction field of the base ring.

EXAMPLES:

```
sage: A = matrix(ZZ,4,[2, 1, 1, -2, 2, 2, -1, -1, -1,1,2,3,4,5,6,7])
sage: h = A.hessenberg_form(); h
[    2  -7/2 -19/5    -2]
[    2   1/2 -17/5    -1]
[    0  25/4  15/2   5/2]
[    0     0  58/5     3]
sage: parent(h)
Full MatrixSpace of 4 by 4 dense matrices over Rational Field
sage: A.hessenbergize()
Traceback (most recent call last):
...
TypeError: Hessenbergize only possible for matrices over a field
```

hessenbergize()

Transform self to Hessenberg form.

The hessenberg form of a matrix A is a matrix that is similar to A, so has the same characteristic polynomial as A, and is upper triangular except possible for entries right below the diagonal.

ALGORITHM: See Henri Cohen's first book.

EXAMPLES:

```
sage: A = matrix(QQ,3, [2, 1, 1, -2, 2, 2, -1, -1, -1])
sage: A.hessenbergize(); A
[  2 3/2   1]
[ -2   3   2]
[  0  -3  -2]
```

```
sage: A = matrix(QQ,4, [2, 1, 1, -2, 2, 2, -1, -1, -1,1,2,3,4,5,6,7])
sage: A.hessenbergize(); A
[    2  -7/2 -19/5    -2]
[    2   1/2 -17/5    -1]
[    0  25/4  15/2   5/2]
[    0     0  58/5     3]
```

You can't Hessenbergize an immutable matrix:

```
sage: A = matrix(QQ, 3, [1..9])
sage: A.set_immutable()
sage: A.hessenbergize()
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
↪copy(M) to change a copy of M).
```

image()

Return the image of the homomorphism on rows defined by right multiplication by this matrix: that is, the row-space.

EXAMPLES:

```
sage: MS1 = MatrixSpace(ZZ,4)
sage: MS2 = MatrixSpace(QQ,6)
sage: A = MS1.matrix([3,4,5,6,7,3,8,10,14,5,6,7,2,2,10,9])
sage: B = MS2.random_element()
```

```
sage: image(A)
Free module of degree 4 and rank 4 over Integer Ring
Echelon basis matrix:
[  1   0   0 426]
[  0   1   0 518]
[  0   0   1 293]
[  0   0   0 687]
```

```
sage: image(B) == B.row_module()
True
sage: image(B) == B.transpose().column_module()
True
```

See also:

row_module(), *column_module()*

indefinite_factorization(*algorithm='symmetric', check=True*)

Decomposes a symmetric or Hermitian matrix into a lower triangular matrix and a diagonal matrix.

INPUT:

- `self` - a square matrix over a ring. The base ring must have an implemented fraction field.
- `algorithm` - default: `'symmetric'`. Either `'symmetric'` or `'hermitian'`, according to if the input matrix is symmetric or hermitian.
- `check` - default: `True` - if `True` then performs the check that the matrix is consistent with the `algorithm` keyword.

OUTPUT:

A lower triangular matrix L with each diagonal element equal to 1 and a vector of entries that form a diagonal matrix D. The vector of diagonal entries can be easily used to form the matrix, as demonstrated below in the examples.

For a symmetric matrix, A, these will be related by

$$A = LDL^T$$

If A is Hermitian matrix, then the transpose of L should be replaced by the conjugate-transpose of L.

If any leading principal submatrix (a square submatrix in the upper-left corner) is singular then this method will fail with a `ValueError`.

ALGORITHM:

The algorithm employed only uses field operations, but the computation of each diagonal entry has the potential for division by zero. The number of operations is of order $n^3/3$, which is half the count for an LU decomposition. This makes it an appropriate candidate for solving systems with symmetric (or Hermitian) coefficient matrices.

See also:

block_ldlt()

EXAMPLES:

There is no requirement that a matrix be positive definite, as indicated by the negative entries in the resulting diagonal matrix. The default is that the input matrix is symmetric.

```
sage: A = matrix(QQ, [[ 3,  -6,   9,   6,  -9],
....:                 [-6,  11, -16, -11,  17],
....:                 [ 9, -16,  28,  16, -40],
....:                 [ 6, -11,  16,   9, -19],
....:                 [-9,  17, -40, -19,  68]])
sage: A.is_symmetric()
True
sage: L, d = A.indefinite_factorization()
sage: D = diagonal_matrix(d)
sage: L
[ 1  0  0  0  0]
[-2  1  0  0  0]
[ 3 -2  1  0  0]
[ 2 -1  0  1  0]
[-3  1 -3  1  1]
sage: D
```

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```
[ 3  0  0  0  0]
[ 0 -1  0  0  0]
[ 0  0  5  0  0]
[ 0  0  0 -2  0]
[ 0  0  0  0 -1]
sage: A == L*D*L.transpose()
True
```

Optionally, Hermitian matrices can be factored and the result has a similar property (but not identical). Here, the field is all complex numbers with rational real and imaginary parts. As theory predicts, the diagonal entries will be real numbers.

```
sage: C.<I> = QuadraticField(-1)
sage: B = matrix(C, [[      2, 4 - 2*I, 2 + 2*I],
....:                [4 + 2*I,       8,    10*I],
....:                [2 - 2*I,   -10*I,      -3]])
sage: B.is_hermitian()
True
sage: L, d = B.indefinite_factorization(algorithm='hermitian')
sage: D = diagonal_matrix(d)
sage: L
[      1       0       0]
[  I + 2       1       0]
[ -I + 1 2*I + 1       1]
sage: D
[ 2  0  0]
[ 0 -2  0]
[ 0  0  3]
sage: B == L*D*L.conjugate_transpose()
True
```

If a leading principal submatrix has zero determinant, this algorithm will fail. This will never happen with a positive definite matrix.

```
sage: A = matrix(QQ, [[21, 15, 12, -2],
....:                 [15, 12,  9,  6],
....:                 [12,  9,  7,  3],
....:                 [-2,  6,  3,  8]])
sage: A.is_symmetric()
True
sage: A[0:3,0:3].det() == 0
True
sage: A.indefinite_factorization()
Traceback (most recent call last):
...
ValueError: 3x3 leading principal submatrix is singular,
so cannot create indefinite factorization
```

This algorithm only depends on field operations, so outside of the singular submatrix situation, any matrix may be factored. This provides a reasonable alternative to the Cholesky decomposition.

```
sage: F.<a> = FiniteField(5^3)
sage: A = matrix(F,
```

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```
....:     [[      a^2 + 2*a, 4*a^2 + 3*a + 4,       3*a^2 + a, 2*a^2 + 2*a + 1],
....:      [4*a^2 + 3*a + 4,       4*a^2 + 2,             3*a, 2*a^2 + 4*a + 2],
....:      [      3*a^2 + a,             3*a,       3*a^2 + 2, 3*a^2 + 2*a + 3],
....:      [2*a^2 + 2*a + 1, 2*a^2 + 4*a + 2, 3*a^2 + 2*a + 3, 3*a^2 + 2*a +␣
↪4]])
sage: A.is_symmetric()
True
sage: L, d = A.indefinite_factorization()
sage: D = diagonal_matrix(d)
sage: L
[              1               0               0               0]
[4*a^2 + 4*a + 3               1               0               0]
[              3   4*a^2 + a + 2               1               0]
[      4*a^2 + 4 2*a^2 + 3*a + 3 2*a^2 + 3*a + 1               1]
sage: D
[      a^2 + 2*a               0               0               0]
[              0 2*a^2 + 2*a + 4               0               0]
[              0               0 3*a^2 + 4*a + 3               0]
[              0               0               0       a^2 + 3*a]
sage: A == L*D*L.transpose()
True
```

This works correctly for the 0x0 matrix:

```
sage: Matrix(0).indefinite_factorization()
([], ())
```

integer_kernel(*ring='ZZ'*)

Return the kernel of this matrix over the given ring (which should be either the base ring, or a PID whose fraction field is the base ring).

Assume that the base field of this matrix has a numerator and denominator functions for its elements, e.g., it is the rational numbers or a fraction field. This function computes a basis over the integers for the kernel of self.

If the matrix is not coercible into QQ, then the PID itself should be given as a second argument, as in the third example below.

EXAMPLES:

```
sage: A = MatrixSpace(QQ, 4)(range(16))
sage: A.integer_kernel()
Free module of degree 4 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1  0 -3  2]
[ 0  1 -2  1]
```

The integer kernel even makes sense for matrices with fractional entries:

```
sage: A = MatrixSpace(QQ, 2)([1/2, 0, 0, 0])
sage: A.integer_kernel()
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1]
```

An example over a bigger ring:

```
sage: L.<w> = NumberField(x^2 - x + 2)
sage: OL = L.ring_of_integers()
sage: A = matrix(L, 2, [1, w/2])
sage: A.integer_kernel(OL)
Free module of degree 2 and rank 1 over Maximal Order in Number Field in w with␣
↪defining polynomial x^2 - x + 2
Echelon basis matrix:
[    -1 -w + 1]
```

inverse()

Returns the inverse of self, without changing self.

Note that one can use the Python inverse operator to obtain the inverse as well.

See also:

inverse_positive_definite()

EXAMPLES:

```
sage: m = matrix([[1,2],[3,4]])
sage: m^(-1)
[  -2    1]
[ 3/2 -1/2]
sage: m.inverse()
[  -2    1]
[ 3/2 -1/2]
sage: ~m
[  -2    1]
[ 3/2 -1/2]
```

```
sage: m = matrix([[1,2],[3,4]], sparse=True)
sage: m^(-1)
[  -2    1]
[ 3/2 -1/2]
sage: m.inverse()
[  -2    1]
[ 3/2 -1/2]
sage: ~m
[  -2    1]
[ 3/2 -1/2]
```

inverse_positive_definite()

Compute the inverse of a positive-definite matrix.

In accord with *is_positive_definite()*, only Hermitian matrices are considered positive-definite. Positive-definite matrices have several factorizations (Cholesky, LDLT, et cetera) that allow them to be inverted in a fast, numerically-stable way. This method uses an appropriate factorization, and is akin to the `cholinv` and `chol2inv` functions available in R, Octave, and Stata.

You should ensure that your matrix is positive-definite before using this method. When in doubt, use the generic *inverse()* method instead.

OUTPUT:

If the given matrix is positive-definite, the return value is the same as that of the *inverse()* method. If the matrix is not positive-definite, the behavior of this function is undefined.

See also:

inverse(), *is_positive_definite()*, *cholesky()*, *indefinite_factorization()*

EXAMPLES:

A simple two-by-two matrix with rational entries:

```
sage: A = matrix(QQ, [[ 2, -1],
....:                 [-1,  2]])
sage: A.is_positive_definite()
True
sage: A.inverse_positive_definite()
[2/3 1/3]
[1/3 2/3]
sage: A.inverse_positive_definite() == A.inverse()
True
```

A matrix containing real roots:

```
sage: A = matrix(AA, [ [1,       0,       sqrt(2)],
....:                  [0,       sqrt(3), 0      ],
....:                  [sqrt(2), 0,       sqrt(5)] ])
sage: A.is_positive_definite()
True
sage: B = matrix(AA, [ [2*sqrt(5) + 5, 0, -sqrt(8*sqrt(5) + 18)],
....:                  [0,             sqrt(1/3),             0],
....:                  [-sqrt(8*sqrt(5) + 18), 0, sqrt(5) + 2] ])
sage: A.inverse_positive_definite() == B
True
sage: A*B == A.matrix_space().identity_matrix()
True
```

A Hermitian (but not symmetric) matrix with complex entries:

```
sage: A = matrix(QQbar, [ [ 1,  0,        I  ],
....:                     [ 0,  sqrt(5),  0  ],
....:                     [-I,  0,        3  ] ])
sage: A.is_positive_definite()
True
sage: B = matrix(QQbar, [ [ 3/2, 0,         -I/2 ],
....:                     [ 0,   sqrt(1/5), 0    ],
....:                     [ I/2, 0,          1/2 ] ])
sage: A.inverse_positive_definite() == B
True
sage: A*B == A.matrix_space().identity_matrix()
True
```

is_Z_operator_on(K)

Determine if this matrix is a Z-operator on a cone.

We say that a matrix L is a Z-operator on a closed convex cone K if the inner product of Lx and s is nonpositive for all pairs of orthogonal vectors x in K and s in the dual of K. This property need only be checked for generators of K and its dual.

A matrix is a Z-operator on K if and only if its negation is a cross-positive operator on K.

To reliably check whether or not this matrix is a Z operator, its base ring must be either exact (for example, the rationals) or the symbolic ring. An exact ring is more reliable, but in some cases a matrix whose entries contain symbolic constants like e and π will work.

INPUT:

- `K` – a polyhedral closed convex cone.

OUTPUT:

If the base ring of this matrix is exact, then `True` will be returned if and only if this matrix is a Z-operator on `K`.

If the base ring of this matrix is symbolic, then the situation is more complicated:

- `True` will be returned if it can be proven that this matrix is a Z-operator on `K`.
- `False` will be returned if it can be proven that this matrix is not a Z-operator on `K`.
- `False` will also be returned if we can't decide; specifically if we arrive at a symbolic inequality that cannot be resolved.

See also:

is_positive_operator_on(), *is_cross_positive_on()*, *is_lyapunov_like_on()*

REFERENCES:

A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. SIAM, Philadelphia, 1994.

EXAMPLES:

Z-matrices are Z-operators on the nonnegative orthant:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = matrix(SR, [ [-1, -2,  0],
....:                  [ 0,  2, -7],
....:                  [-3,  0,  3] ])
sage: L.is_Z_operator_on(K)
True
```

Symbolic entries also work in some easy cases:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = matrix(SR, [ [-1,      -e,  0 ],
....:                  [ 0,       2, -pi],
....:                  [-sqrt(2), 0,  3 ] ])
sage: L.is_Z_operator_on(K)
True
```

`is_bistochastic`(*normalized=True*)

Returns `True` if this matrix is bistochastic.

A matrix is said to be bistochastic if both the sums of the entries of each row and the sum of the entries of each column are equal to 1 and all entries are nonnegative.

INPUT:

- `normalized` – if set to `True` (default), checks that the sums are equal to 1. When set to `False`, checks that the row sums and column sums are all equal to some constant possibly different from 1.

EXAMPLES:

The identity matrix is clearly bistochastic:

```
sage: Matrix(5,5,1).is_bistochastic()
True
```

The same matrix, multiplied by 2, is not bistochastic anymore, though is verifies the constraints of `normalized == False`:

```
sage: (2 * Matrix(5,5,1)).is_bistochastic()
False
sage: (2 * Matrix(5,5,1)).is_bistochastic(normalized = False)
True
```

Here is a matrix whose row and column sums is 1, but not all entries are nonnegative:

```
sage: m = matrix([[-1,2],[2,-1]])
sage: m.is_bistochastic()
False
```

is_cross_positive_on(K)

Determine if this matrix is cross-positive on a cone.

We say that a matrix L is cross-positive on a closed convex cone K if the inner product of Lx and s is nonnegative for all pairs of orthogonal vectors x in K and s in the dual of K. This property need only be checked for generators of K and its dual.

To reliably check whether or not this matrix is cross-positive, its base ring must be either exact (for example, the rationals) or the symbolic ring. An exact ring is more reliable, but in some cases a matrix whose entries contain symbolic constants like e and π will work.

INPUT:

- `K` – a polyhedral closed convex cone.

OUTPUT:

If the base ring of this matrix is exact, then `True` will be returned if and only if this matrix is cross-positive on `K`.

If the base ring of this matrix is symbolic, then the situation is more complicated:

- `True` will be returned if it can be proven that this matrix is cross-positive on `K`.
- `False` will be returned if it can be proven that this matrix is not cross-positive on `K`.
- `False` will also be returned if we can't decide; specifically if we arrive at a symbolic inequality that cannot be resolved.

See also:

is_positive_operator_on(), *is_Z_operator_on()*, *is_lyapunov_like_on()*

REFERENCES:

H. Schneider and M. Vidyasagar. Cross-positive matrices. SIAM Journal on Numerical Analysis, 7:508-519, 1970.

EXAMPLES:

Negative Z-matrices are cross-positive operators on the nonnegative orthant:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = matrix(SR, [ [-1, 2, 0],
....:                  [ 0, 2, 7],
....:                  [ 3, 0, 3] ])
sage: L.is_cross_positive_on(K)
True
```

Symbolic entries also work in some easy cases:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = matrix(SR, [ [-1,       e, 0 ],
....:                  [ 0,       2, pi],
....:                  [ sqrt(2), 0, 3 ] ])
sage: L.is_cross_positive_on(K)
True
```

is_diagonal()

Return `True` if this matrix is a diagonal matrix.

OUTPUT:

boolean

EXAMPLES:

```
sage: m = matrix(QQ,2,2,range(4))
sage: m.is_diagonal()
False
sage: m = matrix(QQ,2,[5,0,0,5])
sage: m.is_diagonal()
True
sage: m = matrix(QQ,2,[1,0,0,1])
sage: m.is_diagonal()
True
sage: m = matrix(QQ,2,[1,1,1,1])
sage: m.is_diagonal()
False
```

is_diagonalizable(*base_field=None*)

Determine if the matrix is similar to a diagonal matrix.

INPUT:

- `base_field` - a new field to use for entries of the matrix.

OUTPUT:

If `self` is the matrix A, then it is diagonalizable if there is an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D$$

This routine returns `True` if `self` is diagonalizable. The diagonal entries of the matrix D are the eigenvalues of A.

A matrix not diagonalizable over the base field may become diagonalizable by extending the base field to contain all of the eigenvalues. Over the rationals, the field of algebraic numbers, `sage.rings.qqbar` is a good choice.

To obtain the matrices D and P, use the *diagonalization()* method.

ALGORITHM:

For each eigenvalue, this routine checks that the algebraic multiplicity (number of occurrences as a root of the characteristic polynomial) is equal to the geometric multiplicity (dimension of the eigenspace), which is sufficient to ensure a basis of eigenvectors for the columns of P.

EXAMPLES:

A matrix that is diagonalizable over the rationals:

```
sage: A = matrix(QQ, [[-7, 16, 12,  0,    6],
....:                 [-9, 15,  0,  12, -27],
....:                 [ 9, -8, 11, -12,  51],
....:                 [ 3, -4,  0,  -1,   9],
....:                 [-1,  0, -4,   4, -12]])
sage: A.is_diagonalizable()
True
sage: A.diagonalization()
(
[ 2  0  0  0  0]  [    1     1     0     1     0]
[ 0  3  0  0  0]  [  1/2     0     1     0     1]
[ 0  0  3  0  0]  [  1/6     1  -3/2   2/3 -14/9]
[ 0  0  0 -1  0]  [ -1/6     0  -1/4     0  -1/3]
[ 0  0  0  0 -1], [ -1/6  -1/3   1/3  -1/3   4/9]
)
```

This is a matrix not diagonalizable over the rationals, but you can still get its Jordan form.

```
sage: A = matrix(QQ, [[-3, -14, 2, -1, 15],
....:                 [4, 6, -2, 3, -8],
....:                 [-2, -14, 0, 0, 10],
....:                 [3, 13, -2, 0, -11],
....:                 [-1, 6, 1, -3, 1]])
sage: A.is_diagonalizable()
False
sage: A.jordan_form(subdivide=False)
[-1  1  0  0  0]
[ 0 -1  0  0  0]
[ 0  0  2  1  0]
[ 0  0  0  2  1]
[ 0  0  0  0  2]
```

If any eigenvalue of a matrix is outside the base ring, then this routine raises an error. However, the ring can be extended to contain the eigenvalues.

```
sage: A = matrix(QQ, [[1,  0,  1,  1, -1],
....:                 [0,  1,  0,  4,  8],
....:                 [2,  1,  3,  5,  1],
....:                 [2, -1,  1,  0, -2],
....:                 [0, -1, -1, -5, -8]])

sage: [e in QQ for e in A.eigenvalues()]
[False, False, False, False, False]
sage: A.is_diagonalizable()
```

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```
False
sage: A.diagonalization()
Traceback (most recent call last):
...
ValueError: not diagonalizable over Rational Field

sage: [e in QQbar for e in A.eigenvalues()]
[True, True, True, True, True]
sage: A.is_diagonalizable(base_field=QQbar)
True
```

Other exact fields may be employed, though it will not always be possible to extend their base fields to contain all the eigenvalues.

```
sage: F.<b> = FiniteField(5^2)
sage: A = matrix(F, [[      4, 3*b + 2, 3*b + 1, 3*b + 4],
....:                [2*b + 1,      4*b,        0,        2],
....:                [    4*b,    b + 2, 2*b + 3,        3],
....:                [    2*b,      3*b, 4*b + 4, 3*b + 3]])
sage: A.is_diagonalizable()
False
sage: A.jordan_form()
[      4       1|      0       0]
[      0       4|      0       0]
[---------------+---------------]
[      0       0|2*b + 1       1]
[      0       0|      0 2*b + 1]

sage: F.<c> = QuadraticField(-7)
sage: A = matrix(F, [[   c + 3,   2*c - 2,   -2*c + 2,     c - 1],
....:                [2*c + 10, 13*c + 15, -13*c - 17, 11*c + 31],
....:                [2*c + 10, 14*c + 10, -14*c - 12, 12*c + 30],
....:                [       0,   2*c - 2,   -2*c + 2,   2*c + 2]])
sage: A.is_diagonalizable()
True
sage: A.diagonalization()
(
[    4     0     0     0]  [   1    0    1    0]
[    0    -2     0     0]  [   4    1    0    1]
[    0     0 c + 3     0]  [   5    1 -2/9 10/9]
[    0     0     0 c + 3], [   1    0 -4/9  2/9]
)
```

A trivial matrix is diagonalizable, trivially.

```
sage: A = matrix(QQ, 0, 0)
sage: A.is_diagonalizable()
True
```

A matrix must be square to be diagonalizable.

```
sage: A = matrix(QQ, 3, 4)
sage: A.is_diagonalizable()
```

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```
Traceback (most recent call last):
...
TypeError: not a square matrix
```

The matrix must have entries from a field, and it must be an exact field.

```
sage: A = matrix(ZZ, 4, range(16))
sage: A.is_diagonalizable()
Traceback (most recent call last):
...
ValueError: matrix entries must be from a field

sage: A = matrix(RDF, 4, range(16))
sage: A.is_diagonalizable()
Traceback (most recent call last):
...
ValueError: base field must be exact, but Real Double Field is not
```

is_lyapunov_like_on(K)

Determine if this matrix is Lyapunov-like on a cone.

We say that a matrix L is Lyapunov-like on a closed convex cone K if the inner product of Lx and s is zero for all pairs of orthogonal vectors x in K and s in the dual of K. This property need only be checked for generators of K and its dual.

An operator is Lyapunov-like on K if and only if both the operator itself and its negation are cross-positive on K.

To reliably check whether or not this matrix is Lyapunov-like, its base ring must be either exact (for example, the rationals) or the symbolic ring. An exact ring is more reliable, but in some cases a matrix whose entries contain symbolic constants like e and π will work.

INPUT:

- `K` – a polyhedral closed convex cone.

OUTPUT:

If the base ring of this matrix is exact, then `True` will be returned if and only if this matrix is Lyapunov-like on `K`.

If the base ring of this matrix is symbolic, then the situation is more complicated:

- `True` will be returned if it can be proven that this matrix is Lyapunov-like on `K`.
- `False` will be returned if it can be proven that this matrix is not Lyapunov-like on `K`.
- `False` will also be returned if we can't decide; specifically if we arrive at a symbolic inequality that cannot be resolved.

See also:

is_positive_operator_on(), *is_cross_positive_on()*, *is_Z_operator_on()*

REFERENCES:

- [Or2017]

EXAMPLES:

Diagonal matrices are Lyapunov-like operators on the nonnegative orthant:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = diagonal_matrix(random_vector(QQ,3))
sage: L.is_lyapunov_like_on(K)
True
```

Symbolic entries also work in some easy cases:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = matrix(SR, [ [e, 0,  0      ],
....:                  [0, pi, 0      ],
....:                  [0, 0,  sqrt(2)] ])
sage: L.is_lyapunov_like_on(K)
True
```

is_normal()

Returns `True` if the matrix commutes with its conjugate-transpose.

OUTPUT:

`True` if the matrix is square and commutes with its conjugate-transpose, and `False` otherwise.

Normal matrices are precisely those that can be diagonalized by a unitary matrix.

This routine is for matrices over exact rings and so may not work properly for matrices over `RR` or `CC`. For matrices with approximate entries, the rings of double-precision floating-point numbers, `RDF` and `CDF`, are a better choice since the *sage.matrix.matrix_double_dense.Matrix_double_dense.is_normal()* method has a tolerance parameter. This provides control over allowing for minor discrepancies between entries when checking equality.

The result is cached.

EXAMPLES:

Hermitian matrices are normal.

```
sage: A = matrix(QQ, 5, 5, range(25)) + I*matrix(QQ, 5, 5, range(0, 50, 2))
sage: B = A*A.conjugate_transpose()
sage: B.is_hermitian()
True
sage: B.is_normal()
True
```

Circulant matrices are normal.

```
sage: G = graphs.CirculantGraph(20, [3, 7])
sage: D = digraphs.Circuit(20)
sage: A = 3*D.adjacency_matrix() - 5*G.adjacency_matrix()
sage: A.is_normal()
True
```

Skew-symmetric matrices are normal.

```
sage: A = matrix(QQ, 5, 5, range(25))
sage: B = A - A.transpose()
sage: B.is_skew_symmetric()
True
```

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```
sage: B.is_normal()
True
```

A small matrix that does not fit into any of the usual categories of normal matrices.

```
sage: A = matrix(ZZ, [[1, -1],
....:                 [1,  1]])
sage: A.is_normal()
True
sage: not A.is_hermitian() and not A.is_skew_symmetric()
True
```

Sage has several fields besides the entire complex numbers where conjugation is non-trivial.

```
sage: F.<b> = QuadraticField(-7)
sage: C = matrix(F, [[-2*b - 3,  7*b - 6, -b + 3],
....:                [-2*b - 3, -3*b + 2,   -2*b],
....:                [   b + 1,        0,     -2]])
sage: C = C*C.conjugate_transpose()
sage: C.is_normal()
True
```

A matrix that is nearly normal, but for a non-real diagonal entry.

```
sage: A = matrix(QQbar, [[    2,   2-I, 1+4*I],
....:                    [  2+I,   3+I, 2-6*I],
....:                    [1-4*I, 2+6*I,     5]])
sage: A.is_normal()
False
sage: A[1,1] = 132
sage: A.is_normal()
True
```

Rectangular matrices are never normal.

```
sage: A = matrix(QQbar, 3, 4)
sage: A.is_normal()
False
```

A square, empty matrix is trivially normal.

```
sage: A = matrix(QQ, 0, 0)
sage: A.is_normal()
True
```

is_one()

Return True if this matrix is the identity matrix.

EXAMPLES:

```
sage: m = matrix(QQ,2,2,range(4))
sage: m.is_one()
False
```

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```
sage: m = matrix(QQ,2,[5,0,0,5])
sage: m.is_one()
False
sage: m = matrix(QQ,2,[1,0,0,1])
sage: m.is_one()
True
sage: m = matrix(QQ,2,[1,1,1,1])
sage: m.is_one()
False
```

is_permutation_of(*N*, *check=False*)

Return `True` if there exists a permutation of rows and columns sending `self` to `N` and `False` otherwise.

INPUT:

- `N` – a matrix.
- **check – boolean (default: `False`). If `False`** return Boolean indicating whether there exists a permutation of rows and columns sending `self` to `N` and False otherwise. If `True` return a tuple of a Boolean and a permutation mapping `self` to `N` if such a permutation exists, and (`False`, `None`) if it does not.

OUTPUT:

A Boolean or a tuple of a Boolean and a permutation.

EXAMPLES:

```
sage: M = matrix(ZZ,[[1,2,3],[3,5,3],[2,6,4]])
sage: M
[1 2 3]
[3 5 3]
[2 6 4]
sage: N = matrix(ZZ,[[1,2,3],[2,6,4],[3,5,3]])
sage: N
[1 2 3]
[2 6 4]
[3 5 3]
sage: M.is_permutation_of(N)
True
```

Some examples that are not permutations of each other:

```
sage: N = matrix(ZZ,[[1,2,3],[4,5,6],[7,8,9]])
sage: N
[1 2 3]
[4 5 6]
[7 8 9]
sage: M.is_permutation_of(N)
False
sage: N = matrix(ZZ,[[1,2],[3,4]])
sage: N
[1 2]
[3 4]
```

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```
sage: M.is_permutation_of(N)
False
```

And for when check is True:

```
sage: N = matrix(ZZ,[[3,5,3],[2,6,4],[1,2,3]])
sage: N
[3 5 3]
[2 6 4]
[1 2 3]
sage: r = M.is_permutation_of(N, check=True)
sage: r
(True, ((1,2,3), ()))
sage: p = r[1]
sage: M.with_permuted_rows_and_columns(*p) == N
True
```

is_positive_definite(*certificate=False*)

Determine if a matrix is positive-definite.

A matrix A is positive definite if it *is_hermitian()* and if, for every non-zero vector x,

$$\langle Ax, x\rangle > 0.$$

ALGORITHM:

A Hermitian matrix is positive-definite if and only if the diagonal blocks in its *block_ldlt()* factorization are all 1-by-1 and have positive entries. We first check that the matrix *is_hermitian()*, and then compute this factorization.

INPUT:

- `self` – a matrix
- `certificate` – (default: `False`) return the lower-triangular and diagonal parts of the *block_ldlt()* factorization when the matrix is positive-definite. Deprecated.

OUTPUT:

This routine will return `True` if the matrix is Hermitian and meets the condition above for the quadratic form.

The base ring for the elements of the matrix must

1. Have a fraction field implemented; and
2. Be a subring of the real numbers, complex numbers, or symbolic ring.

If `certificate` is `True`, a triplet `(b, L, d)` will be returned instead, with `b` containing the result (true or false). If the matrix is positive-definite, then `L` and `d` will contain the lower-triangular and diagonal parts of the *block_ldlt()* factorization, respectively. Or if the matrix is not positive-definite (that is, if `b` is `False`), then both `L` and `d` will be `None`.

See also:

block_ldlt(), *is_hermitian()*, *is_positive_semidefinite()*

EXAMPLES:

A real symmetric matrix that is positive-definite, as evidenced by the positive determinants of its leading principal submatrices:

```
sage: A = matrix(QQ, [[ 4, -2,  4,  2],
....:                 [-2, 10, -2, -7],
....:                 [ 4, -2,  8,  4],
....:                 [ 2, -7,  4,  7]])
sage: A.is_positive_definite()
True
sage: [A[:i,:i].determinant() for i in range(1,A.nrows()+1)]
[4, 36, 144, 144]
```

A real symmetric matrix that is not positive-definite and a vector u that makes the corresponding quadratic form negative:

```
sage: A = matrix(QQ, [[ 3,  -6,   9,   6,  -9],
....:                 [-6,  11, -16, -11,  17],
....:                 [ 9, -16,  28,  16, -40],
....:                 [ 6, -11,  16,   9, -19],
....:                 [-9,  17, -40, -19,  68]])
sage: A.is_positive_definite()
False
sage: u = vector(QQ, [2, 2, 0, 1, 0])
sage: (A*u).inner_product(u)
-3
```

Another real symmetric matrix that is not positive-definite and a vector u that makes the corresponding quadratic form zero:

```
sage: A = matrix(QQ, [[21, 15, 12, -2],
....:                 [15, 12,  9,  6],
....:                 [12,  9,  7,  3],
....:                 [-2,  6,  3,  8]])
sage: A.is_positive_definite()
False
sage: u = vector(QQ, [1,1,-3,0])
sage: (A*u).inner_product(u)
0
```

A complex Hermitian matrix that is positive-definite, confirmed by the positive determinants of its leading principal submatrices:

```
sage: C.<I> = NumberField(x^2 + 1, embedding=CC(0,1))
sage: A = matrix(C, [[        23,  17*I + 3,  24*I + 25,     21*I],
....:                [ -17*I + 3,         38, -69*I + 89, 7*I + 15],
....:                [-24*I + 25, 69*I + 89,         976, 24*I + 6],
....:                [     -21*I, -7*I + 15,  -24*I + 6,        28]])
sage: A.is_positive_definite()
True
sage: [A[:i,:i].determinant() for i in range(1,A.nrows()+1)]
[23, 576, 359540, 2842600]
```

An Hermitian matrix that is not positive-definite and a vector u that makes the corresponding quadratic form negative:

```
sage: C.<I> = QuadraticField(-1)
sage: B = matrix(C, [[      2, 4 - 2*I, 2 + 2*I],
```

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```
....:                [4 + 2*I,        8,     10*I],
....:                [2 - 2*I,   -10*I,       -3]])
sage: B.is_positive_definite()
False
sage: u = vector(C, [-5 + 10*I, 4 - 3*I, 0])
sage: (B*u).hermitian_inner_product(u)
-50
```

A positive-definite matrix over an algebraically-closed field, confirmed by the positive determinants of its leading principal submatrices:

```
sage: A = matrix(QQbar, [[        2,   4 + 2*I,   6 - 4*I],
....:                    [ -2*I + 4,         11, 10 - 12*I],
....:                    [  4*I + 6, 10 + 12*I,         37]])
sage: A.is_positive_definite()
True
sage: [A[:i,:i].determinant() for i in range(1,A.nrows()+1)]
[2, 2, 6]
```

is_positive_operator_on(*K1*, *K2=None*)

Determine if this matrix is a positive operator on a cone.

A matrix is a positive operator on a cone if the image of the cone under the matrix is itself a subset of the cone. That concept can be extended to two cones: a matrix is a positive operator on a pair of cones if the image of the first cone is contained in the second cone.

To reliably check whether or not this matrix is a positive operator, its base ring must be either exact (for example, the rationals) or the symbolic ring. An exact ring is more reliable, but in some cases a matrix whose entries contain symbolic constants like e and π will work. Performance is best for integer or rational matrices, for which we can check the "is a subset of the other cone" condition quickly.

INPUT:

- `K1` – a polyhedral closed convex cone.
- `K2` – (default: `K1`) the codomain cone; this matrix is a positive operator if the image of `K1` is a subset of `K2`.

OUTPUT:

If the base ring of this matrix is exact, then `True` will be returned if and only if this matrix is a positive operator.

If the base ring of this matrix is symbolic, then the situation is more complicated:

- `True` will be returned if it can be proven that this matrix is a positive operator.
- `False` will be returned if it can be proven that this matrix is not a positive operator.
- `False` will also be returned if we can't decide; specifically if we arrive at a symbolic inequality that cannot be resolved.

See also:

is_cross_positive_on(), *is_Z_operator_on()*, *is_lyapunov_like_on()*

REFERENCES:

A. Berman and P. Gaiha. A generalization of irreducible monotonicity. Linear Algebra and its Applications, 5:29-38, 1972.

A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. SIAM, Philadelphia, 1994.

EXAMPLES:

Nonnegative matrices are positive operators on the nonnegative orthant:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = random_matrix(QQ,3).apply_map(abs)
sage: L.is_positive_operator_on(K)
True
```

Symbolic entries also work in some easy cases:

```
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: L = matrix(SR, [ [0,       e, 0 ],
....:                  [0,       2, pi],
....:                  [sqrt(2), 0, 0 ] ])
sage: L.is_positive_operator_on(K)
True
```

Your matrix can be over any exact ring, for example the ring of univariate polynomials with rational coefficients:

```
sage: K = Cone([(1,0),(-1,0),(0,1),(0,-1)])
sage: K.is_full_space()
True
sage: L = matrix(QQ[x], [[x,0],[0,1]])
sage: L.is_positive_operator_on(K)
True
```

is_positive_semidefinite()

Returns whether or not this matrix is positive-semidefinite.

By SageMath convention, positive (semi)definite matrices must be either real symmetric or complex Hermitian.

ALGORITHM:

Bunch and Kaufman [BK1977] describe a fast, numerically-stable scheme for computing the "inertia" of a matrix by way Sylvester's inertia theorem and a block-LDL^T factorization. We perform this factorization, and read off the signs of the eigenvalues from the resulting diagonal blocks.

REFERENCES:

- [BK1977]

See also:

block_ldlt(), *is_positive_definite()*

EXAMPLES:

A positive-definite matrix:

```
sage: A = matrix(QQ, [ [2,1],
....:                  [1,2] ] )
sage: A.eigenvalues()
[3, 1]
```

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```
sage: A.is_positive_semidefinite()
True
```

A positive-semidefinite (but not positive-definite) matrix:

```
sage: A = matrix(QQ, [ [1,1],
....:                  [1,1] ] )
sage: A.eigenvalues()
[2, 0]
sage: A.is_positive_semidefinite()
True
```

And finally, an indefinite matrix:

```
sage: A = matrix(QQ, [ [0,1],
....:                  [1,0] ] )
sage: A.eigenvalues()
[1, -1]
sage: A.is_positive_semidefinite()
False
```

A non-Hermitian matrix cannot be positive-semidefinite, regardless of its eigenvalues:

```
sage: A = matrix(QQ, [ [2,1],
....:                  [0,0] ])
sage: A.eigenvalues()
[2, 0]
sage: A.is_positive_semidefinite()
False
```

Any of the preceding examples are valid over inexact rings and with complex numbers as well:

```
sage: A = matrix(CDF, [ [ 2, I],
....:                   [-I, 2] ] )
sage: A.is_positive_semidefinite()
True

sage: A = matrix(CDF, [ [ 1, I],
....:                   [-I, 1] ] )
sage: A.is_positive_semidefinite()
True

sage: A = matrix(CDF, [ [0,I],
....:                   [I,0] ] )
sage: A.is_positive_semidefinite()
False

sage: A = matrix(CDF, [ [2,I],
....:                   [0,0] ])
sage: A.is_positive_semidefinite()
False
```

is_scalar(*a=None*)

Return True if this matrix is a scalar matrix.

INPUT:

- base_ring element a, which is chosen as self[0][0] if a = None

OUTPUT:

- whether self is a scalar matrix (in fact the scalar matrix aI if a is input)

EXAMPLES:

```
sage: m = matrix(QQ,2,2,range(4))
sage: m.is_scalar(5)
False
sage: m = matrix(QQ,2,[5,0,0,5])
sage: m.is_scalar(5)
True
sage: m = matrix(QQ,2,[1,0,0,1])
sage: m.is_scalar(1)
True
sage: m = matrix(QQ,2,[1,1,1,1])
sage: m.is_scalar(1)
False
```

is_similar(*other*, *transformation=False*)

Return `True` if `self` and `other` are similar, i.e. related by a change-of-basis matrix.

INPUT:

- `other` – a matrix, which should be square, and of the same size as `self`.
- `transformation` – default: `False` - if `True`, the output may include the change-of-basis matrix (also known as the similarity transformation). See below for an exact description.

OUTPUT:

Two matrices, A and B are similar if they are square matrices of the same size and there is an invertible matrix S such that $A = S^{-1}BS$. S can be interpreted as a change-of-basis matrix if A and B are viewed as matrix representations of the same linear transformation from a vector space to itself.

When `transformation=False` this method will return `True` if such a matrix S exists, otherwise it will return `False`. When `transformation=True` the method returns a pair. The first part of the pair is `True` or `False` depending on if the matrices are similar. The second part of the pair is the change-of-basis matrix when the matrices are similar and `None` when the matrices are not similar.

When a similarity transformation matrix `S` is requested, it will satisfy `self = S.inverse()*other*S`.

rings and coefficients

Inexact rings are not supported. Only rings having a fraction field can be used as coefficients.

The base rings for the matrices are promoted to a common field for the similarity check using rational form over this field.

If the fraction fields of both matrices are the same, this field is used. Otherwise, if the fraction fields are only related by a canonical coercion, the common coercion field is used.

In all cases, the result is about similarity over a common field.

similarity transformation

For computation of the similarity transformation, the matrices are first checked to be similar over their common field.

In this case, a similarity transformation is then searched for over the common field. If this fails, the matrices are promoted to the algebraic closure of their common field (whenever it is available) and a similarity transformation is looked for over the algebraic closure.

For example, matrices over the rationals may be promoted to the field of algebraic numbers (`QQbar`) for computation of the similarity transformation.

Warning: When the two matrices are similar, this routine may fail to find the similarity transformation. A technical explanation follows.

The similarity check is accomplished with rational form, which will be successful for any pair of matrices over the same field. However, the computation of rational form does not provide a transformation. So we instead compute Jordan form, which does provide a transformation. But Jordan form will require that the eigenvalues of the matrix can be represented within Sage, requiring the existence of the appropriate extension field. When this is not possible, a `RuntimeError` is raised, as demonstrated in an example below.

EXAMPLES:

The two matrices in this example were constructed to be similar. The computations happen in the field of algebraic numbers, but we are able to convert the change-of-basis matrix back to the rationals (which may not always be possible).

```
sage: A = matrix(ZZ, [[-5, 2, -11],
....:                 [-6, 7, -42],
....:                 [0, 1, -6]])
sage: B = matrix(ZZ, [[ 1, 12,  3],
....:                 [-1, -6, -1],
....:                 [ 0,  6,  1]])
sage: A.is_similar(B)
True
sage: _, T = A.is_similar(B, transformation=True)
sage: T
[ 1.00000000000000? + 0.?e-14*I            0.?e-14 + 0.?e-14*I            0.?e-
↪14 + 0.?e-14*I]
[-0.66666666666667? + 0.?e-15*I 0.166666666666667? + 0.?e-15*I -0.
↪83333333333334? + 0.?e-14*I]
[ 0.66666666666667? + 0.?e-14*I            0.?e-14 + 0.?e-14*I -0.
↪33333333333333? + 0.?e-14*I]
sage: T.change_ring(QQ)
[   1    0    0]
[-2/3  1/6 -5/6]
[ 2/3    0 -1/3]
sage: A == T.inverse()*B*T
True
```

Other exact fields are supported.

```
sage: F.<a> = FiniteField(7^2)
sage: A = matrix(F,[[2*a + 5, 6*a + 6,   a + 3],
....:               [  a + 3, 2*a + 2, 4*a + 2],
....:               [2*a + 6, 5*a + 5,     3*a]])
sage: B = matrix(F,[[5*a + 5, 6*a + 4,   a + 1],
....:               [  a + 5, 4*a + 3, 3*a + 3],
....:               [3*a + 5,   a + 4, 5*a + 6]])
sage: A.is_similar(B)
True
sage: B.is_similar(A)
True
sage: _, T = A.is_similar(B, transformation=True)
sage: T
[      1       0       0]
[6*a + 1 4*a + 3 4*a + 2]
[6*a + 3 3*a + 5 3*a + 6]
sage: A == T.inverse()*B*T
True
```

Two matrices with different sets of eigenvalues, so they cannot possibly be similar.

```
sage: A = matrix(QQ, [[ 2,  3, -3, -6],
....:                 [ 0,  1, -2, -8],
....:                 [-3, -3,  4,  3],
....:                 [-1, -2,  2,  6]])
sage: B = matrix(QQ, [[ 1,  1,  2,  4],
....:                 [-1,  2, -3, -7],
....:                 [-2,  3, -4, -7],
....:                 [ 0, -1,  0,  0]])
sage: A.eigenvalues() == B.eigenvalues()
False
sage: A.is_similar(B, transformation=True)
(False, None)
```

Similarity is an equivalence relation, so this routine computes a representative of the equivalence class for each matrix, the rational form, as provided by *rational_form()*. The matrices below have identical eigenvalues (as evidenced by equal characteristic polynomials), but slightly different rational forms, and hence are not similar.

```
sage: A = matrix(QQ, [[ 19, -7, -29],
....:                 [-16, 11,  30],
....:                 [ 15, -7, -25]])
sage: B = matrix(QQ, [[-38, -63,  42],
....:                 [ 14,  25, -14],
....:                 [-14, -21,  18]])
sage: A.charpoly() == B.charpoly()
True
sage: A.rational_form()
[  0   0 -48]
[  1   0   8]
[  0   1   5]
sage: B.rational_form()
[ 4| 0  0]
```

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```
[--+-----]
[ 0| 0 12]
[ 0| 1  1]
sage: A.is_similar(B)
False
```

Obtaining the transformation between two similar matrices requires the Jordan form, which requires computing the eigenvalues of the matrix, which may not lie in the field used for entries of the matrix. In this unfortunate case, the computation of the transformation may fail with a `RuntimeError`, EVEN when the matrices are similar. This is not the case for matrices over the integers, rationals or algebraic numbers, since the computations are done in the algebraically closed field of algebraic numbers. Here is an example where the similarity is obvious by design, but we are not able to resurrect a similarity transformation.

```
sage: F.<a> = FiniteField(7^2)
sage: C = matrix(F,[[  a + 2, 5*a + 4],
....:               [6*a + 6, 6*a + 4]])
sage: S = matrix(F, [[0, 1],
....:                [1, 0]])
sage: D = S.inverse()*C*S
sage: C.is_similar(D)
True
sage: C.is_similar(D, transformation=True)
Traceback (most recent call last):
...
RuntimeError: unable to compute transformation for similar matrices
sage: C.jordan_form()
Traceback (most recent call last):
...
RuntimeError: Some eigenvalue does not exist in
Finite Field in a of size 7^2.
```

An example over a finite field of prime order, which uses the algebraic closure of this field to find the change-of-basis matrix:

```
sage: cox = posets.TamariLattice(3).coxeter_transformation()
sage: M = cox.change_ring(GF(3))
sage: M.is_similar(M**3, True)  # long time
(
      [1 0 0 0 0]
      [0 1 1 0 2]
      [0 0 0 0 1]
      [1 2 0 2 1]
True, [0 0 1 0 0]
)
```

Inexact rings and fields are not supported.

```
sage: A = matrix(CDF, 2, 2, range(4))
sage: B = copy(A)
sage: A.is_similar(B)
Traceback (most recent call last):
...
```

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```
TypeError: matrix entries must come from an exact field,
not Complex Double Field
```

Base rings for the matrices need to have a fraction field. So in particular, the ring needs to be at least an integral domain.

```
sage: Z6 = Integers(6)
sage: A = matrix(Z6, 2, 2, range(4))
sage: A.is_similar(A)
Traceback (most recent call last):
...
ValueError: base ring of a matrix needs a fraction field,
maybe the ring is not an integral domain
```

If the fraction fields of the entries are unequal and do not coerce in a common field, it is an error.

```
sage: A = matrix(GF(3), 2, 2, range(4))
sage: B = matrix(GF(2), 2, 2, range(4))
sage: A.is_similar(B, transformation=True)
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents:
'Full MatrixSpace of 2 by 2 dense matrices over Finite Field
of size 3' and
'Full MatrixSpace of 2 by 2 dense matrices over Finite Field
of size 2'
```

A matrix over the integers and a matrix over the algebraic numbers will be compared over the algebraic numbers (by coercion of `QQ` in `QQbar`).

```
sage: A = matrix(ZZ, 2, 2, range(4))
sage: B = matrix(QQbar, 2, 2, range(4))
sage: A.is_similar(B)
True
```

is_triangular(*side='lower'*)

Return `True` if this matrix is a triangular matrix.

INPUT:

- `side` – either `"lower"` (default) or `"upper"`

OUTPUT:

boolean

EXAMPLES:

```
sage: m = matrix(QQ, 2, 2, range(4))
sage: m.is_triangular()
False
sage: m = matrix(QQ, 2, [5, 0, 0, 5])
sage: m.is_triangular()
True
sage: m = matrix(QQ, 2, [1, 2, 0, 1])
```

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```
sage: m.is_triangular("upper")
True
sage: m.is_triangular("lower")
False
```

is_unitary()

Return True if the columns of the matrix are an orthonormal basis.

For a matrix with real entries this determines if a matrix is "orthogonal" and for a matrix with complex entries this determines if the matrix is "unitary."

OUTPUT:

True if the matrix is square and its conjugate-transpose is its inverse, and False otherwise. In other words, a matrix is orthogonal or unitary if the product of its conjugate-transpose times the matrix is the identity matrix.

For numerical matrices a specialized routine available over RDF and CDF is a good choice.

EXAMPLES:

```
sage: A = matrix(QQbar, [[(1/sqrt(5))*(1+i), (1/sqrt(55))*(3+2*I), (1/
↪sqrt(22))*(2+2*I)],
....:                    [(1/sqrt(5))*(1-i), (1/sqrt(55))*(2+2*I),  (1/
↪sqrt(22))*(-3+I)],
....:                    [    (1/sqrt(5))*I, (1/sqrt(55))*(3-5*I),    (1/
↪sqrt(22))*(-2)]])
sage: A.is_unitary()
True
```

A permutation matrix is always orthogonal.

```
sage: sigma = Permutation([1,3,4,5,2])
sage: P = sigma.to_matrix(); P
[1 0 0 0 0]
[0 0 0 0 1]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
sage: P.is_unitary()
True
sage: P.change_ring(GF(3)).is_unitary()
True
sage: P.change_ring(GF(3)).is_unitary()
True
```

A square matrix far from unitary.

```
sage: A = matrix(QQ, 4, range(16))
sage: A.is_unitary()
False
```

Rectangular matrices are never unitary.

```
sage: A = matrix(QQbar, 3, 4)
sage: A.is_unitary()
False
```

jordan_form(*base_ring=None*, *sparse=False*, *subdivide=True*, *transformation=False*, *eigenvalues=None*, *check_input=True*)

Compute the Jordan normal form of this square matrix A, if it exists.

This computation is performed in a naive way using the ranks of powers of $A - xI$, where x is an eigenvalue of the matrix A. If desired, a transformation matrix P can be returned, which is such that the Jordan canonical form is given by $P^{-1}AP$; if the matrix is diagonalizable, this equals to *eigendecomposition* or *spectral decomposition*.

INPUT:

- `base_ring` - Ring in which to compute the Jordan form.
- `sparse` - (default `False`) If `sparse=True`, return a sparse matrix.
- `subdivide` - (default `True`) If `subdivide=True`, the subdivisions for the Jordan blocks in the matrix are shown.
- `transformation` - (default `False`) If `transformation=True`, computes also the transformation matrix.
- `eigenvalues` - (default `None`) A complete set of roots, with multiplicity, of the characteristic polynomial of A, encoded as a list of pairs, each having the form (r, m) with r a root and m its multiplicity. If this is `None`, then Sage computes this list itself, but this is only possible over base rings in whose quotient fields polynomial factorization is implemented. Over all other rings, providing this list manually is the only way to compute Jordan normal forms.
- `check_input` - (default `True`) A Boolean specifying whether the list `eigenvalues` (if provided) has to be checked for correctness. Set this to `False` for a speedup if the eigenvalues are known to be correct.

Note: Currently, the Jordan normal form is not computed over inexact rings in any but the trivial cases when the matrix is either 0×0 or 1×1.

In the case of exact rings, this method does not compute any generalized form of the Jordan normal form, but is only able to compute the result if the characteristic polynomial of the matrix splits over the specific base ring.

Note that the base ring must be a field or a ring with an implemented fraction field.

EXAMPLES:

```
sage: a = matrix(ZZ,4,[1, 0, 0, 0, 0, 1, 0, 0,
....:                  1, -1, 1, 0, 1, -1, 1, 2]); a
[ 1  0  0  0]
[ 0  1  0  0]
[ 1 -1  1  0]
[ 1 -1  1  2]
sage: a.jordan_form()
[2|0 0|0]
[-+---+-]
[0|1 1|0]
```

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```
[0|0 1|0]
[-+---+-]
[0|0 0|1]
sage: a.jordan_form(subdivide=False)
[2 0 0 0]
[0 1 1 0]
[0 0 1 0]
[0 0 0 1]
sage: b = matrix(ZZ,3,3,range(9)); b
[0 1 2]
[3 4 5]
[6 7 8]
sage: b.jordan_form()
Traceback (most recent call last):
...
RuntimeError: Some eigenvalue does not exist in Rational Field.
sage: b.jordan_form(RealField(15))
Traceback (most recent call last):
...
ValueError: Jordan normal form not implemented over inexact rings.
```

Here we need to specify a field, since the eigenvalues are not defined in the smallest ring containing the matrix entries (github issue #14508):

```
sage: c = matrix([[0,1,0],[0,0,1],[1,0,0]])
sage: c.jordan_form(CyclotomicField(3))
[         1|         0|         0]
[----------+----------+----------]
[         0|     zeta3|         0]
[----------+----------+----------]
[         0|         0|-zeta3 - 1]
```

If you need the transformation matrix as well as the Jordan form of `self`, then pass the option `transformation=True`. For example:

```
sage: m = matrix([[5,4,2,1],[0,1,-1,-1],[-1,-1,3,0],[1,1,-1,2]]); m
[ 5  4  2  1]
[ 0  1 -1 -1]
[-1 -1  3  0]
[ 1  1 -1  2]
sage: jf, p = m.jordan_form(transformation=True)
sage: jf
[2|0|0 0]
[-+-+---]
[0|1|0 0]
[-+-+---]
[0|0|4 1]
[0|0|0 4]
sage: ~p * m * p
[2 0 0 0]
[0 1 0 0]
[0 0 4 1]
[0 0 0 4]
```

Note that for matrices over inexact rings, we do not attempt to compute the Jordan normal form, since it is not numerically stable:

```
sage: b = matrix(ZZ,3,3,range(9))
sage: jf, p = b.jordan_form(RealField(15), transformation=True)
Traceback (most recent call last):
...
ValueError: Jordan normal form not implemented over inexact rings.
```

kernel(**args*, ***kwds*)

Returns the left kernel of this matrix, as a vector space or free module. This is the set of vectors `x` such that `x*self = 0`.

Note: For the right kernel, use *right_kernel()*. The method *kernel()* is exactly equal to *left_kernel()*.

For inexact rings use *right_kernel_matrix()* with `basis='computed'` (on the transpose of the matrix) to avoid echolonizing.

INPUT:

- `algorithm` - default: 'default' - a keyword that selects the algorithm employed. Allowable values are:
  - 'default' - allows the algorithm to be chosen automatically
  - 'generic' - naive algorithm usable for matrices over any field
  - 'flint' - FLINT library code for matrices over the rationals or the integers
  - 'pari' - PARI library code for matrices over number fields or the integers
  - 'padic' - padic algorithm from IML library for matrices over the rationals and integers
  - 'pluq' - PLUQ matrix factorization for matrices mod 2
- `basis` - default: 'echelon' - a keyword that describes the format of the basis used to construct the left kernel. Allowable values are:
  - 'echelon': the basis matrix is returned in echelon form
  - 'pivot' : each basis vector is computed from the reduced row-echelon form of `self` by placing a single one in a non-pivot column and zeros in the remaining non-pivot columns. Only available for matrices over fields.
  - 'LLL': an LLL-reduced basis. Only available for matrices over the integers.

OUTPUT:

A vector space or free module whose degree equals the number of rows in `self` and which contains all the vectors `x` such that `x*self = 0`.

If `self` has 0 rows, the kernel has dimension 0, while if `self` has 0 columns the kernel is the entire ambient vector space.

The result is cached. Requesting the left kernel a second time, but with a different basis format, will return the cached result with the format from the first computation.

Note: For much more detailed documentation of the various options see *right_kernel()*, since this method just computes the right kernel of the transpose of `self`.

EXAMPLES:

Over the rationals with a basis matrix in echelon form.

```
sage: A = matrix(QQ, [[1, 2, 4, -7, 4],
....:                 [1, 1, 0, 2, -1],
....:                 [1, 0, 3, -3, 1],
....:                 [0, -1, -1, 3, -2],
....:                 [0, 0, -1, 2, -1]])
sage: A.left_kernel()
Vector space of degree 5 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1  2 -1]
[ 0  1 -1  1 -4]
```

Over a finite field, with a basis matrix in "pivot" format.

```
sage: A = matrix(FiniteField(7), [[5, 0, 5, 2, 4],
....:                             [1, 3, 2, 3, 6],
....:                             [1, 1, 6, 5, 3],
....:                             [2, 5, 6, 0, 0]])
sage: A.kernel(basis='pivot')
Vector space of degree 4 and dimension 2 over Finite Field of size 7
User basis matrix:
[5 2 1 0]
[6 3 0 1]
```

The left kernel of a zero matrix is the entire ambient vector space whose degree equals the number of rows of `self` (i.e. everything).

```
sage: A = MatrixSpace(QQ, 3, 4)(0)
sage: A.kernel()
Vector space of degree 3 and dimension 3 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]
```

We test matrices with no rows or columns.

```
sage: A = matrix(QQ, 2, 0)
sage: A.left_kernel()
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: A = matrix(QQ, 0, 2)
sage: A.left_kernel()
Vector space of degree 0 and dimension 0 over Rational Field
Basis matrix:
[]
```

The results are cached. Note that requesting a new format for the basis is ignored and the cached copy is returned. Work with a copy if you need a new left kernel, or perhaps investigate the

right_kernel_matrix() method on the transpose, which does not cache its results and is more flexible.

```
sage: A = matrix(QQ, [[1,1],[2,2]])
sage: K1 = A.left_kernel()
sage: K1
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[   1 -1/2]
sage: K2 = A.left_kernel()
sage: K1 is K2
True
sage: K3 = A.left_kernel(basis='pivot')
sage: K3
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[   1 -1/2]
sage: B = copy(A)
sage: K3 = B.left_kernel(basis='pivot')
sage: K3
Vector space of degree 2 and dimension 1 over Rational Field
User basis matrix:
[-2  1]
sage: K3 is K1
False
sage: K3 == K1
True
```

kernel_on(*V*, *poly=None*, *check=True*)

Return the kernel of self restricted to the invariant subspace V. The result is a vector subspace of V, which is also a subspace of the ambient space.

INPUT:

- `V` - vector subspace
- `check` - (optional) default: True; whether to check that V is invariant under the action of self.
- `poly` - (optional) default: None; if not None, compute instead the kernel of poly(self) on V.

OUTPUT:

- a subspace

Warning: This function does *not* check that V is in fact invariant under self if check is False. With check False this function is much faster.

EXAMPLES:

```
sage: t = matrix(QQ, 4, [39, -10, 0, -12, 0, 2, 0, -1, 0, 1, -2, 0, 0, 2, 0, -
↪2]); t
[ 39 -10   0 -12]
[  0   2   0  -1]
[  0   1  -2   0]
[  0   2   0  -2]
```

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```
sage: t.fcp()
(x - 39) * (x + 2) * (x^2 - 2)
sage: s = (t-39)*(t^2-2)
sage: V = s.kernel(); V
Vector space of degree 4 and dimension 3 over Rational Field
Basis matrix:
[1 0 0 0]
[0 1 0 0]
[0 0 0 1]
sage: s.restrict(V)
[0 0 0]
[0 0 0]
[0 0 0]
sage: s.kernel_on(V)
Vector space of degree 4 and dimension 3 over Rational Field
Basis matrix:
[1 0 0 0]
[0 1 0 0]
[0 0 0 1]
sage: k = t-39
sage: k.restrict(V)
[  0 -10 -12]
[  0 -37  -1]
[  0   2 -41]
sage: ker = k.kernel_on(V); ker
Vector space of degree 4 and dimension 1 over Rational Field
Basis matrix:
[   1 -2/7    0 -2/7]
sage: ker.0 * k
(0, 0, 0, 0)
```

Test that github issue #9425 is fixed.

```
sage: V = span([[1/7,0,0] ,[0,1,0]], ZZ); V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1/7   0   0]
[  0   1   0]
sage: T = matrix(ZZ,3,[1,0,0,0,0,0,0,0,0]); T
[1 0 0]
[0 0 0]
[0 0 0]
sage: W = T.kernel_on(V); W.basis()
[
(0, 1, 0)
]
sage: W.is_submodule(V)
True
```

left_eigenmatrix(*other=None*)

Return matrices D and P, where D is a diagonal matrix of eigenvalues and the rows of P are corresponding eigenvectors (or zero vectors).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved

OUTPUT:

If `self` is a square matrix A, then the output is a diagonal matrix D and a matrix P such that

$$PA = DP,$$

where the rows of P are eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$PA = DPB.$$

The ordinary eigenvalue problem is equivalent to the generalized one if B is the identity matrix.

The generalized eigenvector decomposition is currently only implemented for matrices over `RDF` and `CDF`.

EXAMPLES:

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: D, P = A.eigenmatrix_left()
sage: D
[                  0                   0                   0]
[                  0 -1.348469228349535?                   0]
[                  0                   0  13.348469228349547?]
sage: P
[                   1                   -2                    1]
[                   1  0.3101020514433644? -0.3797958971132713?]
[                   1   1.289897948556636?   1.579795897113272?]
sage: P*A == D*P
True
```

Because P is invertible, A is diagonalizable.

```
sage: A == (~P)*D*P
True
```

The matrix P may contain zero rows corresponding to eigenvalues for which the algebraic multiplicity is greater than the geometric multiplicity. In these cases, the matrix is not diagonalizable.

```
sage: A = jordan_block(2,3); A
[2 1 0]
[0 2 1]
[0 0 2]
sage: D, P = A.eigenmatrix_left()
sage: D
[2 0 0]
[0 2 0]
[0 0 2]
```

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```
sage: P
[0 0 1]
[0 0 0]
[0 0 0]
sage: P*A == D*P
True
```

A generalized eigenvector decomposition:

```
sage: A = matrix(RDF, [[1, -2], [3, 4]])
sage: B = matrix(RDF, [[0, 7], [2, -3]])
sage: D, P = A.eigenmatrix_left(B)
sage: (P * A - D * P * B).norm() < 1e-14
True
```

The matrix B in a generalized eigenvalue problem may be singular:

```
sage: A = matrix.identity(CDF, 2)
sage: B = matrix(CDF, [[2, 1+I], [4, 2+2*I]])
sage: D, P = A.eigenmatrix_left(B)
sage: D.diagonal()  # tol 1e-14
[0.2 - 0.1*I, +infinity]
```

In this case, we can still verify the eigenvector equation for the first eigenvalue and first eigenvector:

```
sage: l = D[0, 0]
sage: v = P[0, :]
sage: (v * A - l * v * B).norm() < 1e-14
True
```

The second eigenvector is contained in the left kernel of B:

```
sage: (P[1, :] * B).norm() < 1e-14
True
```

See also:

eigenvalues(), *eigenvectors_left()*, *Matrix_double_dense.eigenvectors_left()*, *eigenmatrix_right()*.

left_eigenspaces(*format='all'*, *var='a'*, *algebraic_multiplicity=False*)

Compute the left eigenspaces of a matrix.

Note that `eigenspaces_left()` and `left_eigenspaces()` are identical methods. Here "left" refers to the eigenvectors being placed to the left of the matrix.

INPUT:

- `self` - a square matrix over an exact field. For inexact matrices consult the numerical or symbolic matrix classes.
- `format` - default: `None`
  - `'all'` - attempts to create every eigenspace. This will always be possible for matrices with rational entries.

  - `'galois'` - for each irreducible factor of the characteristic polynomial, a single eigenspace will be output for a single root/eigenvalue for the irreducible factor.
  - `None` - Uses the 'all' format if the base ring is contained in an algebraically closed field which is implemented. Otherwise, uses the 'galois' format.

- `var` - default: 'a' - variable name used to represent elements of the root field of each irreducible factor of the characteristic polynomial. If var='a', then the root fields will be in terms of a0, a1, a2, ..., where the numbering runs across all the irreducible factors of the characteristic polynomial, even for linear factors.
- `algebraic_multiplicity` - default: False - whether or not to include the algebraic multiplicity of each eigenvalue in the output. See the discussion below.

OUTPUT:

If algebraic_multiplicity=False, return a list of pairs (e, V) where e is an eigenvalue of the matrix, and V is the corresponding left eigenspace. For Galois conjugates of eigenvalues, there may be just one representative eigenspace, depending on the `format` keyword.

If algebraic_multiplicity=True, return a list of triples (e, V, n) where e and V are as above and n is the algebraic multiplicity of the eigenvalue.

Warning: Uses a somewhat naive algorithm (simply factors the characteristic polynomial and computes kernels directly over the extension field).

EXAMPLES:

We compute the left eigenspaces of a 3×3 rational matrix. First, we request all of the eigenvalues, so the results are in the field of algebraic numbers, $QQbar$. Then we request just one eigenspace per irreducible factor of the characteristic polynomial with the $galois$ keyword.

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: es = A.eigenspaces_left(format='all'); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(-1.348469228349535?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
User basis matrix:
[                   1  0.3101020514433644? -0.3797958971132713?]),
(13.34846922834954?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
User basis matrix:
[                 1 1.289897948556636? 1.579795897113272?])
]

sage: es = A.eigenspaces_left(format='galois'); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
```

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```
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[            1 1/15*a1 + 2/5 2/15*a1 - 1/5])
]
sage: es = A.eigenspaces_left(format='galois', algebraic_multiplicity=True); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1], 1),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[            1 1/15*a1 + 2/5 2/15*a1 - 1/5], 1)
]
sage: e, v, n = es[0]; v = v.basis()[0]
sage: delta = e*v - v*A
sage: abs(abs(delta)) < 1e-10
True
```

The same computation, but with implicit base change to a field.

```
sage: A = matrix(ZZ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.eigenspaces_left(format='galois')
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[            1 1/15*a1 + 2/5 2/15*a1 - 1/5])
]
```

We compute the left eigenspaces of the matrix of the Hecke operator T_2 on level 43 modular symbols, both with all eigenvalues (the default) and with one subspace per factor.

```
sage: A = ModularSymbols(43).T(2).matrix(); A
[ 3  0  0  0  0  0 -1]
[ 0 -2  1  0  0  0  0]
[ 0 -1  1  1  0 -1  0]
[ 0 -1  0 -1  2 -1  1]
[ 0 -1  0  1  1 -1  1]
[ 0  0 -2  0  2 -2  1]
[ 0  0 -1  0  1  0 -1]
sage: A.base_ring()
Rational Field
sage: f = A.charpoly(); f
```

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```
x^7 + x^6 - 12*x^5 - 16*x^4 + 36*x^3 + 52*x^2 - 32*x - 48
sage: factor(f)
(x - 3) * (x + 2)^2 * (x^2 - 2)^2
sage: A.eigenspaces_left(algebraic_multiplicity=True)
[
(3, Vector space of degree 7 and dimension 1 over Rational Field
User basis matrix:
[   1    0  1/7    0 -1/7    0 -2/7], 1),
(-2, Vector space of degree 7 and dimension 2 over Rational Field
User basis matrix:
[ 0  1  0  1 -1  1 -1]
[ 0  0  1  0 -1  2 -1], 2),
(-1.4142135623730951?, Vector space of degree 7 and dimension 2 over Algebraic␣
↪Field
User basis matrix:
[                  0                   1                   0                   -
↪1 0.4142135623730951?                   1                  -1]
[                  0                   0                   1                  ␣
↪0                  -1                   0  2.414213562373095?], 2),
(1.414213562373095?, Vector space of degree 7 and dimension 2 over Algebraic␣
↪Field
User basis matrix:
[                   0                   1                   0                  ␣
↪  -1  -2.414213562373095?                   1                  -1]
[                   0                   0                   1                  ␣
↪   0                  -1                   0 -0.4142135623730951?], 2)
]
sage: A.eigenspaces_left(format='galois', algebraic_multiplicity=True)
[
(3, Vector space of degree 7 and dimension 1 over Rational Field
User basis matrix:
[   1    0  1/7    0 -1/7    0 -2/7], 1),
(-2, Vector space of degree 7 and dimension 2 over Rational Field
User basis matrix:
[ 0  1  0  1 -1  1 -1]
[ 0  0  1  0 -1  2 -1], 2),
(a2, Vector space of degree 7 and dimension 2 over Number Field in a2 with␣
↪defining polynomial x^2 - 2
User basis matrix:
[      0       1       0      -1 -a2 - 1       1      -1]
[      0       0       1       0      -1       0 -a2 + 1], 2)
]
```

Next we compute the left eigenspaces over the finite field of order 11.

```
sage: A = ModularSymbols(43, base_ring=GF(11), sign=1).T(2).matrix(); A
[ 3  0  9  0]
[ 0  9  0 10]
[ 0  0 10  1]
[ 0  0  1  1]
sage: A.base_ring()
Finite Field of size 11
```

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```
sage: A.charpoly()
x^4 + 10*x^3 + 3*x^2 + 2*x + 1
sage: A.eigenspaces_left(format='galois', var = 'beta')
[
(9, Vector space of degree 4 and dimension 1 over Finite Field of size 11
User basis matrix:
[0 1 5 6]),
(3, Vector space of degree 4 and dimension 1 over Finite Field of size 11
User basis matrix:
[1 0 1 6]),
(beta2, Vector space of degree 4 and dimension 1 over Univariate Quotient␣
↪Polynomial Ring in beta2 over Finite Field of size 11 with modulus x^2 + 9
User basis matrix:
[        0         0         1 beta2 + 1])
]
```

This method is only applicable to exact matrices. The "eigenmatrix" routines for matrices with double-precision floating-point entries (RDF, CDF) are the best alternative. (Since some platforms return eigenvectors that are the negatives of those given here, this one example is not tested here.) There are also "eigenmatrix" routines for matrices with symbolic entries.

```
sage: A = matrix(QQ, 3, 3, range(9))
sage: A.change_ring(RR).eigenspaces_left()
Traceback (most recent call last):
...
NotImplementedError: eigenspaces cannot be computed reliably for inexact rings␣
↪such as Real Field with 53 bits of precision,
consult numerical or symbolic matrix classes for other options

sage: em = A.change_ring(RDF).eigenmatrix_left()
sage: eigenvalues = em[0]; eigenvalues.dense_matrix() # abs tol 1e-13
[13.348469228349522                0.0                0.0]
[               0.0 -1.348469228349534                0.0]
[               0.0                0.0                0.0]
sage: eigenvectors = em[1]; eigenvectors # not tested
[ 0.440242867...  0.567868371...  0.695493875...]
[ 0.897878732...  0.278434036... -0.341010658...]
[ 0.408248290... -0.816496580...  0.408248290...]

sage: x, y = var('x y')
sage: S = matrix([[x, y], [y, 3*x^2]])
sage: em = S.eigenmatrix_left()
sage: eigenvalues = em[0]; eigenvalues
[3/2*x^2 + 1/2*x - 1/2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)                      ␣
↪                                 0]
[                                                    0 3/2*x^2 + 1/2*x + 1/
↪2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)]
sage: eigenvectors = em[1]; eigenvectors
[                                                  1 1/2*(3*x^2 - x -␣
↪sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y]
[                                                  1 1/2*(3*x^2 - x +␣
↪sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y]
```

A request for 'all' the eigenvalues, when it is not possible, will raise an error. Using the 'galois' format option is more likely to be successful.

```
sage: F.<b> = FiniteField(11^2)
sage: A = matrix(F, [[b + 1, b + 1], [10*b + 4, 5*b + 4]])
sage: A.eigenspaces_left(format='all')
Traceback (most recent call last):
...
NotImplementedError: unable to construct eigenspaces for eigenvalues outside␣
↪the base field,
try the keyword option: format='galois'

sage: A.eigenspaces_left(format='galois')
[
(a0, Vector space of degree 2 and dimension 1 over Univariate Quotient␣
↪Polynomial Ring in a0 over Finite Field in b of size 11^2 with modulus x^2 +␣
↪(5*b + 6)*x + 8*b + 10
User basis matrix:
[               1 6*b*a0 + 3*b + 1])
]
```

left_eigenvectors(*other=None*, *extend=True*)

Compute the left eigenvectors of a matrix.

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved (currently supported only if the base ring of `self` is `RDF` or `CDF`)
- `extend` – boolean (default: `True`)

OUTPUT:

For each distinct eigenvalue, returns a list of the form (e,V,n) where e is the eigenvalue, V is a list of eigenvectors forming a basis for the corresponding left eigenspace, and n is the algebraic multiplicity of the eigenvalue.

If the option extend is set to False, then only the eigenvalues that live in the base ring are considered.

EXAMPLES:

We compute the left eigenvectors of a 3×3 rational matrix.

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: es = A.eigenvectors_left(); es
[(0, [
(1, -2, 1)
], 1),
(-1.348469228349535?, [(1, 0.31010205144336442?, -0.37979589711327137?)], 1),
(13.34846922834954?, [(1, 1.2898979485566362?, 1.5797958971132722?)], 1)]
sage: eval, [evec], mult = es[0]
sage: delta = eval*evec - evec*A
sage: abs(abs(delta)) < 1e-10
True
```

Notice the difference between considering ring extensions or not.

```
sage: M=matrix(QQ,[[0,-1,0],[1,0,0],[0,0,2]])
sage: M.eigenvectors_left()
[(2, [
(0, 0, 1)
], 1), (-1*I, [(1, -1*I, 0)], 1), (1*I, [(1, 1*I, 0)], 1)]
sage: M.eigenvectors_left(extend=False)
[(2, [
(0, 0, 1)
], 1)]
```

left_kernel(**args*, ***kwds*)

Returns the left kernel of this matrix, as a vector space or free module. This is the set of vectors `x` such that `x*self = 0`.

Note: For the right kernel, use *right_kernel()*. The method *kernel()* is exactly equal to *left_kernel()*.

For inexact rings use *right_kernel_matrix()* with `basis='computed'` (on the transpose of the matrix) to avoid echolonizing.

INPUT:

- `algorithm` - default: 'default' - a keyword that selects the algorithm employed. Allowable values are:
  - 'default' - allows the algorithm to be chosen automatically
  - 'generic' - naive algorithm usable for matrices over any field
  - 'flint' - FLINT library code for matrices over the rationals or the integers
  - 'pari' - PARI library code for matrices over number fields or the integers
  - 'padic' - padic algorithm from IML library for matrices over the rationals and integers
  - 'pluq' - PLUQ matrix factorization for matrices mod 2
- `basis` - default: 'echelon' - a keyword that describes the format of the basis used to construct the left kernel. Allowable values are:
  - 'echelon': the basis matrix is returned in echelon form
  - 'pivot' : each basis vector is computed from the reduced row-echelon form of `self` by placing a single one in a non-pivot column and zeros in the remaining non-pivot columns. Only available for matrices over fields.
  - 'LLL': an LLL-reduced basis. Only available for matrices over the integers.

OUTPUT:

A vector space or free module whose degree equals the number of rows in `self` and which contains all the vectors `x` such that `x*self = 0`.

If `self` has 0 rows, the kernel has dimension 0, while if `self` has 0 columns the kernel is the entire ambient vector space.

The result is cached. Requesting the left kernel a second time, but with a different basis format, will return the cached result with the format from the first computation.

Note: For much more detailed documentation of the various options see *right_kernel()*, since this method just computes the right kernel of the transpose of `self`.

EXAMPLES:

Over the rationals with a basis matrix in echelon form.

```
sage: A = matrix(QQ, [[1, 2, 4, -7, 4],
....:                 [1, 1, 0, 2, -1],
....:                 [1, 0, 3, -3, 1],
....:                 [0, -1, -1, 3, -2],
....:                 [0, 0, -1, 2, -1]])
sage: A.left_kernel()
Vector space of degree 5 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1  2 -1]
[ 0  1 -1  1 -4]
```

Over a finite field, with a basis matrix in "pivot" format.

```
sage: A = matrix(FiniteField(7), [[5, 0, 5, 2, 4],
....:                             [1, 3, 2, 3, 6],
....:                             [1, 1, 6, 5, 3],
....:                             [2, 5, 6, 0, 0]])
sage: A.kernel(basis='pivot')
Vector space of degree 4 and dimension 2 over Finite Field of size 7
User basis matrix:
[5 2 1 0]
[6 3 0 1]
```

The left kernel of a zero matrix is the entire ambient vector space whose degree equals the number of rows of `self` (i.e. everything).

```
sage: A = MatrixSpace(QQ, 3, 4)(0)
sage: A.kernel()
Vector space of degree 3 and dimension 3 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]
```

We test matrices with no rows or columns.

```
sage: A = matrix(QQ, 2, 0)
sage: A.left_kernel()
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: A = matrix(QQ, 0, 2)
sage: A.left_kernel()
Vector space of degree 0 and dimension 0 over Rational Field
```

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```
Basis matrix:
[]
```

The results are cached. Note that requesting a new format for the basis is ignored and the cached copy is returned. Work with a copy if you need a new left kernel, or perhaps investigate the *right_kernel_matrix()* method on the transpose, which does not cache its results and is more flexible.

```
sage: A = matrix(QQ, [[1,1],[2,2]])
sage: K1 = A.left_kernel()
sage: K1
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[   1 -1/2]
sage: K2 = A.left_kernel()
sage: K1 is K2
True
sage: K3 = A.left_kernel(basis='pivot')
sage: K3
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[   1 -1/2]
sage: B = copy(A)
sage: K3 = B.left_kernel(basis='pivot')
sage: K3
Vector space of degree 2 and dimension 1 over Rational Field
User basis matrix:
[-2  1]
sage: K3 is K1
False
sage: K3 == K1
True
```

left_kernel_matrix(**args*, ***kwds*)

Returns a matrix whose rows form a basis for the left kernel of `self`.

This method is a thin wrapper around *right_kernel_matrix()*. For supported parameters and input/output formats, see there.

EXAMPLES:

```
sage: M = matrix([[1,2],[3,4],[5,6]])
sage: K = M.left_kernel_matrix(); K
[ 1 -2  1]
sage: K * M
[0 0]
```

left_nullity()

Return the (left) nullity of this matrix, which is the dimension of the (left) kernel of this matrix acting from the right on row vectors.

EXAMPLES:

```
sage: M = Matrix(QQ,[[1,0,0,1],[0,1,1,0],[1,1,1,0]])
sage: M.nullity()
0
sage: M.left_nullity()
0
```

```
sage: A = M.transpose()
sage: A.nullity()
1
sage: A.left_nullity()
1
```

```
sage: M = M.change_ring(ZZ)
sage: M.nullity()
0
sage: A = M.transpose()
sage: A.nullity()
1
```

matrix_window(*row=0*, *col=0*, *nrows=-1*, *ncols=-1*, *check=1*)

Return the requested matrix window.

EXAMPLES:

```
sage: A = matrix(QQ, 3, 3, range(9))
sage: A.matrix_window(1,1, 2, 1)
Matrix window of size 2 x 1 at (1,1):
[0 1 2]
[3 4 5]
[6 7 8]
```

We test the optional check flag.

```
sage: matrix([1]).matrix_window(0,1,1,1, check=False)
Matrix window of size 1 x 1 at (0,1):
[1]
sage: matrix([1]).matrix_window(0,1,1,1)
Traceback (most recent call last):
...
IndexError: matrix window index out of range
```

Another test of bounds checking:

```
sage: matrix([1]).matrix_window(1,1,1,1)
Traceback (most recent call last):
...
IndexError: matrix window index out of range
```

maxspin(v)

Computes the largest integer n such that the list of vectors $S = [v, v*A, ..., v*A^n]$ are linearly independent, and returns that list.

INPUT:

- `self` - Matrix

- v - Vector

OUTPUT:

- list - list of Vectors

ALGORITHM: The current implementation just adds vectors to a vector space until the dimension doesn't grow. This could be optimized by directly using matrices and doing an efficient Echelon form. Also, when the base is Q, maybe we could simultaneously keep track of what is going on in the reduction modulo p, which might make things much faster.

EXAMPLES:

```
sage: t = matrix(QQ, 3, 3, range(9)); t
[0 1 2]
[3 4 5]
[6 7 8]
sage: v = (QQ^3).0
sage: t.maxspin(v)
[(1, 0, 0), (0, 1, 2), (15, 18, 21)]
sage: k = t.kernel(); k
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -2  1]
sage: t.maxspin(k.0)
[(1, -2, 1)]
```

minimal_polynomial(*var='x'*, ***kwds*)

This is a synonym for self.minpoly

EXAMPLES:

```
sage: a = matrix(QQ, 4, 4, range(16))
sage: a.minimal_polynomial('z')
z^3 - 30*z^2 - 80*z
sage: a.minpoly()
x^3 - 30*x^2 - 80*x
```

minors(k)

Return the list of all $k \times k$ minors of self.

Let A be an $m \times n$ matrix and k an integer with $0 \leq k$, $k \leq m$ and $k \leq n$. A $k \times k$ minor of A is the determinant of a $k \times k$ matrix obtained from A by deleting $m - k$ rows and $n - k$ columns. There are no $k \times k$ minors of A if k is larger than either m or n.

The returned list is sorted in lexicographical row major ordering, e.g., if A is a 3×3 matrix then the minors returned are with these rows/columns: [[0, 1]x[0, 1], [0, 1]x[0, 2], [0, 1]x[1, 2], [0, 2]x[0, 1], [0, 2]x[0, 2], [0, 2]x[1, 2], [1, 2]x[0, 1], [1, 2]x[0, 2], [1, 2]x[1, 2]].

INPUT:

- k – integer

EXAMPLES:

```
sage: A = Matrix(ZZ,2,3,[1,2,3,4,5,6]); A
[1 2 3]
[4 5 6]
```

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```
sage: A.minors(2)
[-3, -6, -3]
sage: A.minors(1)
[1, 2, 3, 4, 5, 6]
sage: A.minors(0)
[1]
sage: A.minors(5)
[]
```

```
sage: k = GF(37)
sage: P.<x0,x1,x2> = PolynomialRing(k)
sage: A = Matrix(P,2,3,[x0*x1, x0, x1, x2, x2 + 16, x2 + 5*x1 ])
sage: A.minors(2)
[x0*x1*x2 + 16*x0*x1 - x0*x2, 5*x0*x1^2 + x0*x1*x2 - x1*x2, 5*x0*x1 + x0*x2 -␣
↪x1*x2 - 16*x1]
```

This test addresses an issue raised at github issue #20512:

```
sage: A.minors(0)[0].parent() == P
True
```

minpoly(*var='x'*, ***kwds*)

Return the minimal polynomial of self.

This uses a simplistic - and potentially very very slow - algorithm that involves computing kernels to determine the powers of the factors of the charpoly that divide the minpoly.

EXAMPLES:

```
sage: A = matrix(GF(9,'c'), 4, [1, 1, 0,0, 0,1,0,0, 0,0,5,0, 0,0,0,5])
sage: factor(A.minpoly())
(x + 1) * (x + 2)^2
sage: A.minpoly()(A) == 0
True
sage: factor(A.charpoly())
(x + 1)^2 * (x + 2)^2
```

The default variable name is x, but you can specify another name:

```
sage: factor(A.minpoly('y'))
(y + 1) * (y + 2)^2
```

norm(*p=2*)

Return the p-norm of this matrix, where p can be 1, 2, ∞, or the Frobenius norm.

INPUT:

- `self` - a matrix whose entries are coercible into CDF
- `p` - one of the following options:
- `1` - the largest column-sum norm
- `2 (default)` - the Euclidean norm
- `Infinity` - the largest row-sum norm

- `'frob'` - the Frobenius (sum of squares) norm

OUTPUT: RDF number

See also:

- sage.misc.functional.norm()

EXAMPLES:

```
sage: A = matrix(ZZ, [[1,2,4,3],[-1,0,3,-10]])
sage: A.norm(1)
13.0
sage: A.norm(Infinity)
14.0
sage: B = random_matrix(QQ, 20, 21)
sage: B.norm(Infinity) == (B.transpose()).norm(1)
True
```

```
sage: Id = identity_matrix(12)
sage: Id.norm(2)
1.0
sage: A = matrix(RR, 2, 2, [13,-4,-4,7])
sage: A.norm()  # rel tol 2e-16
14.999999999999998
```

Norms of numerical matrices over high-precision reals are computed by this routine. Faster routines for double precision entries from RDF or CDF are provided by the *Matrix_double_dense* class.

```
sage: A = matrix(CC, 2, 3, [3*I,4,1-I,1,2,0])
sage: A.norm('frob')
5.656854249492381
sage: A.norm(2)
5.470684443210...
sage: A.norm(1)
6.0
sage: A.norm(Infinity)
8.414213562373096
sage: a = matrix([[],[],[],[]])
sage: a.norm()
0.0
sage: a.norm(Infinity) == a.norm(1)
True
```

nullity()

Return the (left) nullity of this matrix, which is the dimension of the (left) kernel of this matrix acting from the right on row vectors.

EXAMPLES:

```
sage: M = Matrix(QQ,[[1,0,0,1],[0,1,1,0],[1,1,1,0]])
sage: M.nullity()
0
sage: M.left_nullity()
0
```

```
sage: A = M.transpose()
sage: A.nullity()
1
sage: A.left_nullity()
1
```

```
sage: M = M.change_ring(ZZ)
sage: M.nullity()
0
sage: A = M.transpose()
sage: A.nullity()
1
```

numerical_approx(*prec=None*, *digits=None*, *algorithm=None*)

Return a numerical approximation of `self` with `prec` bits (or decimal `digits`) of precision.

INPUT:

- `prec` – precision in bits
- `digits` – precision in decimal digits (only used if `prec` is not given)
- `algorithm` – ignored for matrices

OUTPUT: A matrix converted to a real or complex field

EXAMPLES:

```
sage: d = matrix([[3, 0],[0,sqrt(2)]])
sage: b = matrix([[1, -1], [2, 2]]) ; e = b * d * b.inverse();e
[ 1/2*sqrt(2) + 3/2 -1/4*sqrt(2) + 3/4]
[      -sqrt(2) + 3  1/2*sqrt(2) + 3/2]
```

```
sage: e.numerical_approx(53)
[ 2.20710678118655 0.396446609406726]
[ 1.58578643762690  2.20710678118655]
```

```
sage: e.numerical_approx(20)
[ 2.2071 0.39645]
[ 1.5858  2.2071]
```

```
sage: (e-I).numerical_approx(20)
[2.2071 - 1.0000*I           0.39645]
[          1.5858 2.2071 - 1.0000*I]
```

```
sage: M=matrix(QQ,4,[i/(i+1) for i in range(12)]);M
[    0   1/2   2/3]
[  3/4   4/5   5/6]
[  6/7   7/8   8/9]
[ 9/10 10/11 11/12]
```

```
sage: M.numerical_approx()
[0.000000000000000 0.500000000000000 0.666666666666667]
[0.750000000000000 0.800000000000000 0.833333333333333]
```

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```
[0.857142857142857 0.875000000000000 0.888888888888889]
[0.900000000000000 0.909090909090909 0.916666666666667]
```

```
sage: matrix(SR, 2, 2, range(4)).n()                    # optional - sage.
↪symbolic
[0.000000000000000  1.00000000000000]
[ 2.00000000000000  3.00000000000000]
```

```
sage: numerical_approx(M)
[0.000000000000000 0.500000000000000 0.666666666666667]
[0.750000000000000 0.800000000000000 0.833333333333333]
[0.857142857142857 0.875000000000000 0.888888888888889]
[0.900000000000000 0.909090909090909 0.916666666666667]
```

We check that github issue #29700 is fixed:

```
sage: M = matrix(3,[1,1,1,1,0,0,0,1,0])
sage: A,B = M.diagonalization(QQbar)
sage: _ = A.n()
```

permanent(*algorithm='Ryser'*)

Return the permanent of this matrix.

Let $A = (a_{i,j})$ be an $m \times n$ matrix over any commutative ring with $m \leq n$. The permanent of A is

$$\operatorname{per}(A) = \sum_{\pi} a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{m,\pi(m)}$$

where the summation extends over all one-to-one functions π from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.

The product $a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{m,\pi(m)}$ is called *diagonal product*. So the permanent of an $m \times n$ matrix A is the sum of all the diagonal products of A.

By default, this method uses Ryser's algorithm, but setting `algorithm` to "ButeraPernici" you can use the algorithm of Butera and Pernici (which is well suited for band matrices, i.e. matrices whose entries are concentrated near the diagonal).

INPUT:

- `A` – matrix of size $m \times n$ with $m \leq n$
- `algorithm` – either "Ryser" (default) or "ButeraPernici". The Butera-Pernici algorithm takes advantage of presence of zeros and is very well suited for sparse matrices.

ALGORITHM:

The Ryser algorithm is implemented in the method `_permanent_ryser()`. It is a modification of theorem 7.1.1. from Brualdi and Ryser: Combinatorial Matrix Theory. Instead of deleting columns from A, we choose columns from A and calculate the product of the row sums of the selected submatrix.

The Butera-Pernici algorithm is implemented in the function *permanental_minor_polynomial()*. It takes advantage of cancellations that may occur in the computations.

EXAMPLES:

```
sage: A = ones_matrix(4,4)
sage: A.permanent()
```

(continues on next page)

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```
24

sage: A = matrix(3,6,[1,1,1,1,0,0,0,1,1,1,1,0,0,0,1,1,1,1])
sage: A.permanent()
36
sage: B = A.change_ring(RR)
sage: B.permanent()
36.0000000000000
```

The permanent above is directed to the Sloane's sequence OEIS sequence A079908 ("The Dancing School Problems") for which the third term is 36:

```
sage: oeis(79908)                              # optional -- internet
A079908: Solution to the Dancing School Problem with 3 girls and n+3 boys: f(3,
↪n).
sage: _(3)                                     # optional -- internet
36
```

```
sage: A = matrix(4,5,[1,1,0,1,1,0,1,1,1,1,1,0,1,0,1,1,1,0,1,0])
sage: A.permanent()
32
```

A huge permanent that cannot be reasonably computed with the Ryser algorithm (a 50×50 band matrix with width 5):

```
sage: n, w = 50, 5
sage: A = matrix(ZZ, n, n, lambda i,j: (i+j)%5 + 1 if abs(i-j) <= w else 0)
sage: A.permanent(algorithm="ButeraPernici")
577669727355110970369624817108922684046701056046769329O8
```

See Minc: Permanents, Example 2.1, p. 5.

```
sage: A = matrix(QQ,2,2,[1/5,2/7,3/2,4/5])
sage: A.permanent()
103/175
```

```
sage: R.<a> = PolynomialRing(ZZ)
sage: A = matrix(R,2,2,[a,1,a,a+1])
sage: A.permanent()
a^2 + 2*a
```

```
sage: R.<x,y> = PolynomialRing(ZZ,2)
sage: A = matrix(R,2,2,[x, y, x^2, y^2])
sage: A.permanent()
x^2*y + x*y^2
```

permanental_minor(*k*, *algorithm='Ryser'*)

Return the permanental k-minor of this matrix.

The *permanental k-minor* of a matrix A is the sum of the permanents of all possible k by k submatrices of A. Note that the maximal permanental minor is just the permanent.

For a (0,1)-matrix A the permanental k-minor counts the number of different selections of k 1's of A with no two of the 1's on the same row and no two of the 1's on the same column.

See Brualdi and Ryser: Combinatorial Matrix Theory, p. 203. Note the typo $p_0(A) = 0$ in that reference! For applications see Theorem 7.2.1 and Theorem 7.2.4.

See also:

The method *rook_vector()* returns the list of all permanental minors.

INPUT:

- `k` – the size of the minor
- `algorithm` – either "Ryser" (default) or "ButeraPernici". The Butera-Pernici algorithm is well suited for band matrices.

EXAMPLES:

```
sage: A = matrix(4,[1,0,1,0,1,0,1,0,1,0,10,10,1,0,1,1])
sage: A.permanental_minor(2)
114
```

```
sage: A = matrix(3,6,[1,1,1,1,0,0,0,1,1,1,1,0,0,0,1,1,1,1])
sage: A.permanental_minor(0)
1
sage: A.permanental_minor(1)
12
sage: A.permanental_minor(2)
40
sage: A.permanental_minor(3)
36
```

Note that if $k = m = n$, the permanental k-minor equals $\mathrm{per}(A)$:

```
sage: A.permanent()
36
```

The permanental minors of the "complement" matrix of A is related to the permanent of A:

```
sage: m, n = 3, 6
sage: C = matrix(m, n, lambda i,j: 1 - A[i,j])
sage: sum((-1)^k * C.permanental_minor(k)*factorial(n-k)/factorial(n-m) for k␣
↪in range(m+1))
36
```

See Theorem 7.2.1 of Brualdi and Ryser: Combinatorial Matrix Theory: per(A)

permutation_normal_form(*check=False*)

Take the set of matrices that are `self` permuted by any row and column permutation, and return the maximal one of the set where matrices are ordered lexicographically going along each row.

INPUT:

- **check – (default: `False`) If `True` return a tuple of**
  the maximal matrix and the permutations taking `self` to the maximal matrix. If `False`, return only the maximal matrix.

OUTPUT:

The maximal matrix.

EXAMPLES:

```
sage: M = matrix(ZZ, [[0, 0, 1], [1, 0, 2], [0, 0, 0]])
sage: M
[0 0 1]
[1 0 2]
[0 0 0]

sage: M.permutation_normal_form()
[2 1 0]
[1 0 0]
[0 0 0]

sage: M = matrix(ZZ, [[-1, 3], [-1, 5], [2, 4]])
sage: M
[-1  3]
[-1  5]
[ 2  4]

sage: M.permutation_normal_form(check=True)
(
[ 5 -1]
[ 4  2]
[ 3 -1],
((1,2,3), (1,2))
)
```

pfaffian(*algorithm=None*, *check=True*)

Return the Pfaffian of `self`, assuming that `self` is an alternating matrix.

The result is cached.

INPUT:

- `algorithm` (default: `None`) – string, the algorithm to use; currently the following algorithms have been implemented:
  - `'bfl'` - using the Bär-Faddeev-LeVerrier algorithm
  - `'definition'` - using the definition given by perfect matchings
- `check` (default: `True`) – boolean determining whether to check `self` for alternatingness and squareness. This has to be set to `False` if `self` is defined over a non-discrete ring.

The Pfaffian of an alternating matrix is defined as follows:

Let A be an alternating $k \times k$ matrix over a commutative ring. (Here, "alternating" means that $A^T = -A$ and that the diagonal entries of A are zero.) If k is odd, then the Pfaffian of the matrix A is defined to be 0. Let us now define it when k is even. In this case, set $n = k/2$ (this is an integer). For every i and j, we denote the (i, j)-th entry of A by $a_{i,j}$. Let M denote the set of all perfect matchings of the set $\{1, 2, \ldots, 2n\}$ (see `sage.combinat.perfect_matching.PerfectMatchings`). For every matching $m \in M$, define the sign $\mathrm{sign}(m)$ of m by writing m as $\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\}$ with $i_k < j_k$ for all k, and setting $\mathrm{sign}(m)$ to be the sign of the permutation $(i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$ (written here in one-line notation). For every matching $m \in M$, define the weight $w(m)$ of m by writing m as $\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\}$ with $i_k < j_k$ for all k, and setting $w(m) = a_{i_1,j_1} a_{i_2,j_2} \cdots a_{i_n,j_n}$. Now, the Pfaffian of the matrix A is defined to be the sum

$$\sum_{m \in M} \mathrm{sign}(m) w(m).$$

The Pfaffian of A is commonly denoted by $\mathrm{Pf}(A)$. It is well-known that $(\mathrm{Pf}(A))^2 = \det A$ for every alternating matrix A, and that $\mathrm{Pf}(U^T AU) = \det U \cdot \mathrm{Pf}(A)$ for any $n \times n$ matrix U and any alternating $n \times n$ matrix A.

See [Knu1995], [DW1995] and [Rot2001], [Baer2020], just to name a few sources, for further properties of Pfaffians.

ALGORITHM:

If the matrix is small, namely up to size 4×4, the naive formulas are always used.

The Bär-Faddeev-LeVerrier algorithm can be accessed using `'bfl'`. It works over any $\mathbf{Q}$-algebra or ring whose fraction field is an $\mathbf{Q}$-algebra (see [Baer2020] for details). If that check fails, the implementation raises an error because correct results cannot be guaranteed.

To access the algorithm using the above definition, use `'definition'`. However, notice that this algorithm is usually very slow.

By default, i.e. if no options are set, the implementation tries to apply the BFL algorithm first. If BFL is not applicable, it uses the definition by perfect matchings.

The alternatingness of the matrix `self` is checked only if `check` is `True` (this is important because even if `self` is alternating, a non-discrete base ring might prevent Sage from being able to check this).

EXAMPLES:

A 3×3 alternating matrix has Pfaffian 0 independently of its entries:

```
sage: MSp = MatrixSpace(Integers(27), 3)
sage: A = MSp([0, 2, -3,  -2, 0, 8,  3, -8, 0])
sage: A.pfaffian()
0
sage: parent(A.pfaffian())
Ring of integers modulo 27
```

The Pfaffian of a 2×2 alternating matrix is just its northeast entry:

```
sage: MSp = MatrixSpace(QQ, 2)
sage: A = MSp([0, 4,  -4, 0])
sage: A.pfaffian()
4
sage: parent(A.pfaffian())
Rational Field
```

The Pfaffian of a 0×0 alternating matrix is 1:

```
sage: MSp = MatrixSpace(ZZ, 0)
sage: A = MSp([])
sage: A.pfaffian()
1
sage: parent(A.pfaffian())
Integer Ring
```

Let us compute the Pfaffian of a generic 4×4 alternating matrix:

```
sage: R = PolynomialRing(QQ, 'x12,x13,x14,x23,x24,x34')
sage: x12, x13, x14, x23, x24, x34 = R.gens()
sage: A = matrix(R, [[   0,  x12,  x13,  x14],
```

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```
....:                [-x12,    0,  x23,  x24],
....:                [-x13, -x23,    0,  x34],
....:                [-x14, -x24, -x34,    0]])
sage: A.pfaffian()
x14*x23 - x13*x24 + x12*x34
sage: parent(A.pfaffian())
Multivariate Polynomial Ring in x12, x13, x14, x23, x24, x34 over Rational Field
```

The Pfaffian of an alternating matrix squares to its determinant:

```
sage: A = [[0] * 6 for i in range(6)]
sage: for i in range(6):
....:     for j in range(i):
....:         u = floor(random() * 10)
....:         A[i][j] = u
....:         A[j][i] = -u
....:     A[i][i] = 0
sage: AA = Matrix(ZZ, A)
sage: AA.pfaffian() ** 2 == AA.det()
True
```

In order to use the Bär-Faddeev-LeVerrier algorithm, the base ring must have characteristic zero:

```
sage: A = matrix(GF(5), [(0, 3, 4, 1, 3, 4),
....:                    (2, 0, 2, 0, 1, 0),
....:                    (1, 3, 0, 4, 1, 0),
....:                    (4, 0, 1, 0, 2, 0),
....:                    (2, 4, 4, 3, 0, 0),
....:                    (1, 0, 0, 0, 0, 0)])
sage: A.pfaffian(algorithm='bfl')
Traceback (most recent call last):
...
TypeError: Bär-Faddeev-LeVerrier algorithm not applicable,
 use another algorithm instead
```

In that case, the definition by perfect matchings is used instead:

```
sage: A.pfaffian()
2
```

pivot_rows()

Return the pivot row positions for this matrix, which are a topmost subset of the rows that span the row space and are linearly independent.

OUTPUT: a tuple of integers

EXAMPLES:

```
sage: A = matrix(QQ,3,3, [0,0,0,1,2,3,2,4,6]); A
[0 0 0]
[1 2 3]
[2 4 6]
sage: A.pivot_rows()
(1,)
```

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```
sage: A.pivot_rows() # testing cached value
(1,)
```

plot(**args*, ***kwds*)

A plot of this matrix.

Each (ith, jth) matrix element is given a different color value depending on its relative size compared to the other elements in the matrix.

The tick marks drawn on the frame axes denote the (ith, jth) element of the matrix.

This method just calls `matrix_plot`. `*args` and `**kwds` are passed to `matrix_plot`.

EXAMPLES:

A matrix over ZZ colored with different grey levels:

```
sage: A = matrix([[1,3,5,1],[2,4,5,6],[1,3,5,7]])
sage: A.plot()
Graphics object consisting of 1 graphics primitive
```

Here we make a random matrix over RR and use cmap='hsv' to color the matrix elements different RGB colors (see documentation for `matrix_plot` for more information on cmaps):

```
sage: A = random_matrix(RDF, 50)
sage: plot(A, cmap='hsv')
Graphics object consisting of 1 graphics primitive
```

Another random plot, but over GF(389):

```
sage: A = random_matrix(GF(389), 10)
sage: A.plot(cmap='Oranges')
Graphics object consisting of 1 graphics primitive
```

principal_square_root(*check_positivity=True*)

Return the principal square root of a positive definite matrix.

A positive definite matrix A has a unique positive definite matrix M such that $M^2 = A$.

See Wikipedia article Square_root_of_a_matrix.

EXAMPLES:

```
sage: A = Matrix([[1,-1/2,0],[-1/2,1,-1/2],[0,-1/2,1]])
sage: B = A.principal_square_root()
sage: A == B^2
True
```

prod_of_row_sums(*cols*)

Calculate the product of all row sums of a submatrix of A for a list of selected columns `cols`.

EXAMPLES:

```
sage: a = matrix(QQ, 2,2, [1,2,3,2]); a
[1 2]
[3 2]
```

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```
sage: a.prod_of_row_sums([0,1])
15
```

Another example:

```
sage: a = matrix(QQ, 2,3, [1,2,3,2,5,6]); a
[1 2 3]
[2 5 6]
sage: a.prod_of_row_sums([1,2])
55
```

pseudoinverse(*algorithm=None*)

Return the Moore-Penrose pseudoinverse of this matrix.

INPUT:

- `algorithm` (default: guess) – one of the following:
  - `"numpy"` – Use numpy's `linalg.pinv()` which is suitable over real or complex fields.
  - `"exact"` – Use a simple algorithm which is not numerically stable but useful over exact fields. Assume that no conjugation is needed, that the conjugate transpose is just the transpose.
  - `"exactconj"` – Like `exact` but use the conjugate transpose.

OUTPUT: a matrix

EXAMPLES:

```
sage: M = diagonal_matrix(CDF, [0, I, 1+I])
sage: M
[        0.0         0.0         0.0]
[        0.0       1.0*I         0.0]
[        0.0         0.0 1.0 + 1.0*I]
sage: M.pseudoinverse()  # tol 1e-15
[        0.0         0.0         0.0]
[        0.0      -1.0*I         0.0]
[        0.0         0.0 0.5 - 0.5*I]
```

We check the properties of the pseudoinverse over an exact field:

```
sage: M = random_matrix(QQ, 6, 3) * random_matrix(QQ, 3, 5)
sage: Mx = M.pseudoinverse()
sage: M * Mx * M == M
True
sage: Mx * M * Mx == Mx
True
sage: (M * Mx).is_symmetric()
True
sage: (Mx * M).is_symmetric()
True
```

Beware that the `exact` algorithm is not numerically stable, but the default `numpy` algorithm is:

```
sage: M = matrix.hilbert(12,ring=RR)
sage: (~M*M).norm()  # a considerable error
```

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```
1.3...
sage: Mx = M.pseudoinverse(algorithm="exact")
sage: (Mx*M).norm()  # huge error
11.5...
sage: Mx = M.pseudoinverse(algorithm="numpy")
sage: (Mx*M).norm()  # still OK
1.00...
```

When multiplying the given matrix with the pseudoinverse, the result is symmetric for the `exact` algorithm or hermitian for the `exactconj` algorithm:

```
sage: M = matrix(QQbar, 2, 2, [1, sqrt(-3), -sqrt(-3), 3])
sage: M * M.pseudoinverse()
[   0.2500000000000000?  0.4330127018922193?*I]
[-0.4330127018922193?*I     0.7500000000000000?]
sage: M * M.pseudoinverse(algorithm="exactconj")
[                   1/4  0.4330127018922193?*I]
[-0.4330127018922193?*I                   3/4]
sage: M * M.pseudoinverse(algorithm="exact")
[                -1/2 0.866025403784439?*I]
[0.866025403784439?*I                 3/2]
```

For an invertible matrix, the pseudoinverse is just the inverse:

```
sage: M = matrix([[1,2], [3,4]])
sage: ~M
[  -2    1]
[ 3/2 -1/2]
sage: M.pseudoinverse()
[  -2    1]
[ 3/2 -1/2]
```

Numpy gives a strange answer due to rounding errors:

```
sage: M.pseudoinverse(algorithm="numpy")  # random
[-1286742750677287/643371375338643 1000799917193445/1000799917193444]
[  519646110850445/346430740566963  -300239975158034/600479950316067]
```

Although it is not too far off:

```
sage: (~M-M.pseudoinverse(algorithm="numpy")).norm() < 1e-14
True
```

qdet(*q=None*)

Return the quantum deteminant of `self`.

The quantum determinant of a matrix $M = (m_{ij})_{i,j=1}^{n}$ is defined by

$$\det_q(M) = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} M_{\sigma(i),j},$$

where S_n is the symmetric group on $\{1, \ldots, n\}$ and $\ell(\sigma)$ denotes the length of σ written as simple transpositions (equivalently the number of inversions when written in one-line notation).

INPUT:

- q – the parameter q; the default is $q \in F[q]$, where F is the base ring of self

EXAMPLES:

```
sage: A = matrix(SR, 2, lambda i, j: f'a{i}{j}'); A              # optional -␣
↪sage.symbolic
[a00 a01]
[a10 a11]
sage: A.quantum_determinant()                                    # optional -␣
↪sage.symbolic
-a01*a10*q + a00*a11

sage: A = matrix(SR, 3, lambda i, j: f'a{i}{j}')                 # optional -␣
↪sage.symbolic
sage: A.quantum_determinant()                                    # optional -␣
↪sage.symbolic
-a02*a11*a20*q^3 + (a01*a12*a20 + a02*a10*a21)*q^2
 + (-a00*a12*a21 - a01*a10*a22)*q + a00*a11*a22

sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: MS = MatrixSpace(Integers(8), 3)
sage: A = MS([1,7,3, 1,1,1, 3,4,5])
sage: A.det()
6
sage: A.quantum_determinant(q^-2)
7*q^-6 + q^-4 + q^-2 + 5

sage: S.<x,y> = PolynomialRing(GF(7))
sage: R.<q> = LaurentPolynomialRing(S)
sage: MS = MatrixSpace(S, 3, sparse=True)
sage: A = MS([[x, y, 3], [4, 2+y, x^2], [0, 1-x, x+y]])
sage: A.det()
x^4 - x^3 + x^2*y + x*y^2 + 2*x^2 - 2*x*y + 3*y^2 + 2*x - 2
sage: A.quantum_determinant()
(2*x - 2)*q^2 + (x^4 - x^3 + 3*x*y + 3*y^2)*q + x^2*y + x*y^2 + 2*x^2 + 2*x*y
sage: A.quantum_determinant(int(2))
2*x^4 - 2*x^3 + x^2*y + x*y^2 + 2*x^2 + x*y - y^2 + x - 1
sage: A.quantum_determinant(q*x + q^-1*y)
(2*x*y^2 - 2*y^2)*q^-2 + (x^4*y - x^3*y + 3*x*y^2 + 3*y^3)*q^-1
 + (-2*x^2*y + x*y^2 + 2*x^2 - 2*x*y)
 + (x^5 - x^4 + 3*x^2*y + 3*x*y^2)*q + (2*x^3 - 2*x^2)*q^2
```

quantum_determinant(*q=None*)

Return the quantum deteminant of self.

The quantum determinant of a matrix $M = (m_{ij})_{i,j=1}^{n}$ is defined by

$$\det_q(M) = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} M_{\sigma(i),j},$$

where S_n is the symmetric group on $\{1, \ldots, n\}$ and $\ell(\sigma)$ denotes the length of σ written as simple transpositions (equivalently the number of inversions when written in one-line notation).

INPUT:

- q – the parameter q; the default is $q \in F[q]$, where F is the base ring of self

EXAMPLES:

```
sage: A = matrix(SR, 2, lambda i, j: f'a{i}{j}'); A              # optional -␣
↪sage.symbolic
[a00 a01]
[a10 a11]
sage: A.quantum_determinant()                                    # optional -␣
↪sage.symbolic
-a01*a10*q + a00*a11

sage: A = matrix(SR, 3, lambda i, j: f'a{i}{j}')                 # optional -␣
↪sage.symbolic
sage: A.quantum_determinant()                                    # optional -␣
↪sage.symbolic
-a02*a11*a20*q^3 + (a01*a12*a20 + a02*a10*a21)*q^2
 + (-a00*a12*a21 - a01*a10*a22)*q + a00*a11*a22

sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: MS = MatrixSpace(Integers(8), 3)
sage: A = MS([1,7,3, 1,1,1, 3,4,5])
sage: A.det()
6
sage: A.quantum_determinant(q^-2)
7*q^-6 + q^-4 + q^-2 + 5

sage: S.<x,y> = PolynomialRing(GF(7))
sage: R.<q> = LaurentPolynomialRing(S)
sage: MS = MatrixSpace(S, 3, sparse=True)
sage: A = MS([[x, y, 3], [4, 2+y, x^2], [0, 1-x, x+y]])
sage: A.det()
x^4 - x^3 + x^2*y + x*y^2 + 2*x^2 - 2*x*y + 3*y^2 + 2*x - 2
sage: A.quantum_determinant()
(2*x - 2)*q^2 + (x^4 - x^3 + 3*x*y + 3*y^2)*q + x^2*y + x*y^2 + 2*x^2 + 2*x*y
sage: A.quantum_determinant(int(2))
2*x^4 - 2*x^3 + x^2*y + x*y^2 + 2*x^2 + x*y - y^2 + x - 1
sage: A.quantum_determinant(q*x + q^-1*y)
(2*x*y^2 - 2*y^2)*q^-2 + (x^4*y - x^3*y + 3*x*y^2 + 3*y^3)*q^-1
 + (-2*x^2*y + x*y^2 + 2*x^2 - 2*x*y)
 + (x^5 - x^4 + 3*x^2*y + 3*x*y^2)*q + (2*x^3 - 2*x^2)*q^2
```

randomize(*density=1*, *nonzero=False*, **args*, ***kwds*)

Replace a proportion of the entries of a matrix by random elements, leaving the remaining entries unchanged.

Note: The locations of the entries of the matrix to change are determined randomly, with the total number of locations determined by the `density` keyword. These locations are not guaranteed to be distinct. So it is possible that the same position can be chosen multiple times, especially for a very small matrix. The exception is when `density = 1`, in which case every entry of the matrix will be changed.

INPUT:

- `density` - `float` (default: `1`); upper bound for the proportion of entries that are changed
- `nonzero` - Bool (default: `False`); if `True`, then new entries will be nonzero

- `*args, **kwds` - Remaining parameters may be passed to the `random_element` function of the base ring

EXAMPLES:

We construct the zero matrix over a polynomial ring.

```
sage: a = matrix(QQ['x'], 3); a
[0 0 0]
[0 0 0]
[0 0 0]
```

We then randomize roughly half the entries:

```
sage: a.randomize(0.5)
sage: a.density() <= 0.5
True
```

Now we randomize all the entries of the resulting matrix:

```
sage: while a.density() < 0.9:
....:     a = matrix(QQ['x'], 3)
....:     a.randomize()
```

We create the zero matrix over the integers:

```
sage: a = matrix(ZZ, 2); a
[0 0]
[0 0]
```

Then we randomize it; the `x` and `y` keywords, which determine the size of the random elements, are passed on to the `random_element` method for `ZZ`.

```
sage: a.randomize(x=-2^64, y=2^64)
sage: while all(abs(b) < 2^63 for b in a.list()):
....:     a.randomize(x=-2^64, y=2^64)
sage: all(abs(b) < 2^64 for b in a.list())
True
```

rational_form(*format='right'*, *subdivide=True*)

Returns the rational canonical form, also known as Frobenius form.

INPUT:

- `self` - a square matrix with entries from an exact field.
- `format` - default: 'right' - one of 'right', 'bottom', 'left', 'top' or 'invariants'. The first four will cause a matrix to be returned with companion matrices dictated by the keyword. The value 'invariants' will cause a list of lists to be returned, where each list contains coefficients of a polynomial associated with a companion matrix.
- `subdivide` - default: 'True' - if 'True' and a matrix is returned, then it contains subdivisions delineating the companion matrices along the diagonal.

OUTPUT:

The rational form of a matrix is a similar matrix composed of submatrices ("blocks") placed on the main diagonal. Each block is a companion matrix. Associated with each companion matrix is a polynomial.

In rational form, the polynomial of one block will divide the polynomial of the next block (and thus, the polynomials of all subsequent blocks).

Rational form, also known as Frobenius form, is a canonical form. In other words, two matrices are similar if and only if their rational canonical forms are equal. The algorithm used does not provide the similarity transformation matrix (also known as the change-of-basis matrix).

Companion matrices may be written in one of four styles, and any such style may be selected with the `format` keyword. See the companion matrix constructor, `sage.matrix.constructor.companion_matrix()`, for more information about companion matrices.

If the 'invariants' value is used for the `format` keyword, then the return value is a list of lists, where each list is the coefficients of the polynomial associated with one of the companion matrices on the diagonal. These coefficients include the leading one of the monic polynomial and are ready to be coerced into any polynomial ring over the same field (see examples of this below). This return value is intended to be the most compact representation and the easiest to use for testing equality of rational forms.

Because the minimal and characteristic polynomials of a companion matrix are the associated polynomial, it is easy to see that the product of the polynomials of the blocks will be the characteristic polynomial and the final polynomial will be the minimal polynomial of the entire matrix.

ALGORITHM:

We begin with ZigZag form, which is due to Arne Storjohann and is documented at *zigzag_form()*. Then we eliminate ''corner" entries enroute to rational form via an additional algorithm of Storjohann's [Sto2011].

EXAMPLES:

The lists of coefficients returned with the `invariants` keyword are designed to easily convert to the polynomials associated with the companion matrices. This is illustrated by the construction below of the `polys` list. Then we can test the divisibility condition on the list of polynomials. Also the minimal and characteristic polynomials are easy to determine from this list.

```
sage: A = matrix(QQ, [[ 11,  14, -15,  -4, -38, -29,  1,  23,  14, -63,  17, ␣
↪24,  36,  32],
....:                 [ 18,   6, -17, -11, -31, -43, 12,  26,   0, -69,  11, ␣
↪13,  17,  24],
....:                 [ 11,  16, -22,  -8, -48, -34,  0,  31,  16, -82,  26, ␣
↪31,  39,  37],
....:                 [ -8, -18,  22,  10,  46,  33,  3, -27, -12,  70, -19, -
↪20, -42, -31],
....:                 [-13, -21,  16,  10,  52,  43,  4, -28, -25,  89, -37, -
↪20, -53, -62],
....:                 [ -2,  -6,   0,   0,   6,  10,  1,   1,  -7,  14, -11,  -
↪3, -10, -18],
....:                 [ -9, -19,  -3,   4,  23,  30,  8,  -3, -27,  55, -40,  -
↪5, -40, -69],
....:                 [  4,  -8,  -1,  -1,   5,  -4,  9,   5, -11,   4, -14,  -
↪2, -13, -17],
....:                 [  1,  -2,  16,  -1,  19,  -2, -1, -17,   2,  19,   5, -
↪25,  -7,  14],
....:                 [  7,   7, -13,  -4, -26,  -21, 3,  18,   5, -40,   7, ␣
↪15,  20,  14],
....:                 [ -6,  -7, -12,   4,  -1,  18,  3,   8, -11,  15, -18, ␣
↪17, -15, -41],
....:                 [  5,  11, -11,  -3, -26, -19, -1,  14,  10, -42,  14, ␣
```

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```
↪17,  25,  23],
....:                [-16, -15,   3,  10,  29,  45, -1, -13, -19,  71, -35,  -
↪2, -35, -65],
....:                [  4,   2,   3,  -2,  -2, -10,  1,   0,   3, -11,   6,  -
↪4,   6,  17]])
sage: A.rational_form()
[   0   -4|   0    0    0    0|   0    0    0    0    0    0    0    0]
[   1    4|   0    0    0    0|   0    0    0    0    0    0    0    0]
[---------+--------------------+---------------------------------------]
[   0    0|   0    0    0   12|   0    0    0    0    0    0    0    0]
[   0    0|   1    0    0   -4|   0    0    0    0    0    0    0    0]
[   0    0|   0    1    0   -9|   0    0    0    0    0    0    0    0]
[   0    0|   0    0    1    6|   0    0    0    0    0    0    0    0]
[---------+--------------------+---------------------------------------]
[   0    0|   0    0    0    0|   0    0    0    0    0    0    0 -216]
[   0    0|   0    0    0    0|   1    0    0    0    0    0    0  108]
[   0    0|   0    0    0    0|   0    1    0    0    0    0    0  306]
[   0    0|   0    0    0    0|   0    0    1    0    0    0    0 -271]
[   0    0|   0    0    0    0|   0    0    0    1    0    0    0  -41]
[   0    0|   0    0    0    0|   0    0    0    0    1    0    0  134]
[   0    0|   0    0    0    0|   0    0    0    0    0    1    0  -64]
[   0    0|   0    0    0    0|   0    0    0    0    0    0    1   13]

sage: R = PolynomialRing(QQ, 'x')
sage: invariants = A.rational_form(format='invariants')
sage: invariants
[[4, -4, 1], [-12, 4, 9, -6, 1], [216, -108, -306, 271, 41, -134, 64, -13, 1]]
sage: polys = [R(p) for p in invariants]
sage: [p.factor() for p in polys]
[(x - 2)^2, (x - 3) * (x + 1) * (x - 2)^2, (x + 1)^2 * (x - 3)^3 * (x - 2)^3]
sage: all(polys[i].divides(polys[i+1]) for i in range(len(polys)-1))
True
sage: polys[-1] == A.minimal_polynomial(var='x')
True
sage: prod(polys) == A.characteristic_polynomial(var='x')
True
```

Rational form is a canonical form. Any two matrices are similar if and only if their rational forms are equal. By starting with Jordan canonical forms, the matrices `C` and `D` below were built as similar matrices, while `E` was built to be just slightly different. All three matrices have equal characteristic polynomials though `E`'s minimal polynomial differs.

```
sage: C = matrix(QQ, [[2,  31, -10,  -9, -125,  13,  62, -12],
....:                 [0,  48, -16, -16, -188,  20,  92, -16],
....:                 [0,   9,  -1,   2,  -33,   5,  18,   0],
....:                 [0,  15,  -5,   0,  -59,   7,  30,  -4],
....:                 [0, -21,   7,   2,   84, -10, -42,   5],
....:                 [0, -42,  14,   8,  167, -17, -84,  13],
....:                 [0, -50,  17,  10,  199, -23, -98,  14],
....:                 [0,  15,  -5,  -2,  -59,   7,  30, -2]])
sage: C.minimal_polynomial().factor()
(x - 2)^2
```

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```
sage: C.characteristic_polynomial().factor()
(x - 2)^8
sage: C.rational_form()
[ 0 -4| 0  0| 0  0| 0  0]
[ 1  4| 0  0| 0  0| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0 -4| 0  0| 0  0]
[ 0  0| 1  4| 0  0| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  0| 0 -4| 0  0]
[ 0  0| 0  0| 1  4| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  0| 0  0| 0 -4]
[ 0  0| 0  0| 0  0| 1  4]

sage: D = matrix(QQ, [[ -4,   3,    7,   2,  -4,   5,    7,   -3],
....:                 [ -6,   5,    7,   2,  -4,   5,    7,   -3],
....:                 [ 21, -12,   89,  25,   8,  27,   98,  -95],
....:                 [ -9,   5,  -44, -11,  -3, -13,  -48,   47],
....:                 [ 23, -13,   74,  21,  12,  22,   85,  -84],
....:                 [ 31, -18,  135,  38,  12,  47,  155, -147],
....:                 [-33,  19, -138, -39, -13, -45, -156,  151],
....:                 [ -7,   4,  -29,  -8,  -3, -10,  -34,   34]])
sage: D.minimal_polynomial().factor()
(x - 2)^2
sage: D.characteristic_polynomial().factor()
(x - 2)^8
sage: D.rational_form()
[ 0 -4| 0  0| 0  0| 0  0]
[ 1  4| 0  0| 0  0| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0 -4| 0  0| 0  0]
[ 0  0| 1  4| 0  0| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  0| 0 -4| 0  0]
[ 0  0| 0  0| 1  4| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  0| 0  0| 0 -4]
[ 0  0| 0  0| 0  0| 1  4]

sage: E = matrix(QQ, [[ 0, -8,   4, -6, -2,   5, -3,  11],
....:                 [-2, -4,   2, -4, -2,   4, -2,   6],
....:                 [ 5, 14,  -7, 12,  3,  -8,  6, -27],
....:                 [-3, -8,   7, -5,  0,   2, -6,  17],
....:                 [ 0,  5,   0,  2,  4,  -4,  1,   2],
....:                 [-3, -7,   5, -6, -1,   5, -4,  14],
....:                 [ 6, 18, -10, 14,  4, -10, 10, -28],
....:                 [-2, -6,   4, -5, -1,   3,  -3, 13]])
sage: E.minimal_polynomial().factor()
(x - 2)^3
sage: E.characteristic_polynomial().factor()
(x - 2)^8
```

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```
sage: E.rational_form()
[  2|  0   0|  0   0|  0   0   0]
[---+-------+-------+-----------]
[  0|  0  -4|  0   0|  0   0   0]
[  0|  1   4|  0   0|  0   0   0]
[---+-------+-------+-----------]
[  0|  0   0|  0  -4|  0   0   0]
[  0|  0   0|  1   4|  0   0   0]
[---+-------+-------+-----------]
[  0|  0   0|  0   0|  0   0   8]
[  0|  0   0|  0   0|  1   0 -12]
[  0|  0   0|  0   0|  0   1   6]
```

The principal feature of rational canonical form is that it can be computed over any field using only field operations. Other forms, such as Jordan canonical form, are complicated by the need to determine the eigenvalues of the matrix, which can lie outside the field. The following matrix has all of its eigenvalues outside the rationals - some are irrational ($\pm\sqrt{2}$) and the rest are complex ($-1 \pm 2i$).

```
sage: A = matrix(QQ,
....: [[-154,  -3,  -54,   44,   48, -244,  -19,   67, -326,   85,   355,
↪581],
....:  [ 504,  25,  156, -145, -171,  793,   99, -213, 1036, -247, -1152, -
↪1865],
....:  [ 294,  -1,  112,  -89,  -90,  469,   36, -128,  634, -160,  -695, -
↪1126],
....:  [ -49,  -32,  25,    7,   37,  -64,  -58,   12,  -42,  -14,    72,
↪106],
....:  [-261, -123,  65,   47,  169, -358, -254,   70, -309,  -29,   454,
↪673],
....:  [-448, -123, -10,  109,  227, -668, -262,  163, -721,   95,   896,
↪1410],
....:  [  38,    7,   8,  -14,  -17,   66,    6,  -23,   73,  -29,   -78,  -
↪143],
....:  [ -96,   10, -55,   37,   24, -168,   17,   56, -231,   88,   237,
↪412],
....:  [ 310,   67,  31,  -81, -143,  473,  143, -122,  538,  -98,  -641, -
↪1029],
....:  [ 139,  -35,  99,  -49,  -18,  236,  -41,  -70,  370, -118,  -377,  -
↪619],
....:  [ 243,    9,  81,  -72,  -81,  386,   43, -105,  508, -124,  -564,  -
↪911],
....:  [-155,   -3, -55,   45,   50, -245,  -27,   65, -328,   77,   365,
↪583]])
sage: A.characteristic_polynomial().factor()
(x^2 - 2)^2 * (x^2 + 2*x + 5)^4
sage: A.eigenvalues(extend=False)
[]
sage: A.rational_form()
[  0  -5|  0   0   0   0|  0   0   0   0   0   0]
[  1  -2|  0   0   0   0|  0   0   0   0   0   0]
[-------+---------------+-----------------------]
[  0   0|  0   0   0  10|  0   0   0   0   0   0]
```

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```
[  0   0|  1   0   0   4|  0   0   0   0   0   0]
[  0   0|  0   1   0  -3|  0   0   0   0   0   0]
[  0   0|  0   0   1  -2|  0   0   0   0   0   0]
[-------+----------------+------------------------]
[  0   0|  0   0   0   0|  0   0   0   0   0  50]
[  0   0|  0   0   0   0|  1   0   0   0   0  40]
[  0   0|  0   0   0   0|  0   1   0   0   0   3]
[  0   0|  0   0   0   0|  0   0   1   0   0 -12]
[  0   0|  0   0   0   0|  0   0   0   1   0 -12]
[  0   0|  0   0   0   0|  0   0   0   0   1  -4]
sage: F.<x> = QQ[]
sage: polys = A.rational_form(format='invariants')
sage: [F(p).factor() for p in polys]
[x^2 + 2*x + 5, (x^2 - 2) * (x^2 + 2*x + 5), (x^2 - 2) * (x^2 + 2*x + 5)^2]
```

Rational form may be computed over any field. The matrix below is an example where the eigenvalues lie outside the field.

```
sage: F.<a> = FiniteField(7^2)
sage: A = matrix(F,
....: [[5*a + 3, 4*a + 1, 6*a + 2, 2*a + 5,       6, 4*a + 5, 4*a + 5,       5,␣
↪  a + 6,      5,  4*a + 4],
....:  [6*a + 3, 2*a + 4,       0,       6, 5*a + 5,     2*a, 5*a + 1,       1,␣
↪5*a + 2,     4*a, 5*a + 6],
....:  [3*a + 1, 6*a + 6,   a + 6,       2,       0, 3*a + 6, 5*a + 4, 5*a + 6,␣
↪5*a + 2,       3, 4*a + 2],
....:  [    3*a,     6*a,     3*a,     4*a, 4*a + 4, 3*a + 6,     6*a,       4,␣
↪3*a + 4, 6*a + 2,     4*a],
....:  [4*a + 5,   a + 1, 4*a + 3, 6*a + 5, 5*a + 2, 5*a + 2,     6*a, 4*a + 6,␣
↪6*a + 4, 5*a + 3, 3*a + 1],
....:  [    3*a,     6*a, 4*a + 1, 6*a + 2, 2*a + 5, 4*a + 6,       2,   a + 5,␣
↪2*a + 4, 2*a + 1, 2*a + 1],
....:  [4*a + 5, 3*a + 3,       6, 4*a + 1, 4*a + 3, 6*a + 3,       6, 3*a + 3,␣
↪      3,   a + 3,       0],
....:  [6*a + 6,   a + 4, 2*a + 6, 3*a + 5, 4*a + 3,       2,       a, 3*a + 4,␣
↪    5*a, 2*a + 5, 4*a + 3],
....:  [3*a + 5, 6*a + 2,     4*a,   a + 5,       0,     5*a, 6*a + 5, 2*a + 1,␣
↪3*a + 1, 3*a + 5, 4*a + 2],
....:  [3*a + 2,   a + 3, 3*a + 6,       a, 3*a + 5, 5*a + 1, 3*a + 2,   a + 3,␣
↪  a + 2, 6*a + 1, 3*a + 3],
....:  [6*a + 6, 5*a + 1,     4*a,       2, 5*a + 5, 3*a + 5, 3*a + 1,     2*a,␣
↪    2*a, 2*a + 4, 4*a + 2]])
sage: A.rational_form()
[  a + 2|      0       0       0|      0       0       0       0       0      ␣
↪0       0]
[-------+------------------------+-----------------------------------------------
↪--------]
[      0|      0       0   a + 6|      0       0       0       0       0      ␣
↪0       0]
[      0|      1       0 6*a + 4|      0       0       0       0       0      ␣
↪0       0]
[      0|      0       1 6*a + 4|      0       0       0       0       0      ␣
```

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```
↪0       0]
[-------+-----------------------+-----------------------------------------
↪--------]
[      0|      0       0       0|      0       0       0       0       0  ␣
↪0     2*a]
[      0|      0       0       0|      1       0       0       0       0  ␣
↪0 6*a + 3]
[      0|      0       0       0|      0       1       0       0       0  ␣
↪0 6*a + 1]
[      0|      0       0       0|      0       0       1       0       0  ␣
↪0   a + 2]
[      0|      0       0       0|      0       0       0       1       0  ␣
↪0   a + 6]
[      0|      0       0       0|      0       0       0       0       1  ␣
↪0 2*a + 1]
[      0|      0       0       0|      0       0       0       0       0  ␣
↪1 2*a + 1]
sage: invariants = A.rational_form(format='invariants')
sage: invariants
[[6*a + 5, 1], [6*a + 1, a + 3, a + 3, 1], [5*a, a + 4, a + 6, 6*a + 5, 6*a + 1,
↪ 5*a + 6, 5*a + 6, 1]]
sage: R.<x> = F[]
sage: polys = [R(p) for p in invariants]
sage: [p.factor() for p in polys]
[x + 6*a + 5, (x + 6*a + 5) * (x^2 + (2*a + 5)*x + 5*a), (x + 6*a + 5) * (x^2 +␣
↪(2*a + 5)*x + 5*a)^3]
sage: polys[-1] == A.minimal_polynomial()
True
sage: prod(polys) == A.characteristic_polynomial()
True
sage: A.eigenvalues()
Traceback (most recent call last):
...
NotImplementedError: algebraic closures of finite fields are only implemented␣
↪for prime fields
```

Companion matrices may be selected as any one of four different types. See the documentation for the companion matrix constructor, `sage.matrix.constructor.companion_matrix()`, for more information.

```
sage: A = matrix(QQ, [[35, -18, -2, -45],
....:                 [22, -22, 12, -16],
....:                 [ 5, -12, 12,   4],
....:                 [16,  -6, -4, -23]])
sage: A.rational_form(format='right')
[ 2| 0  0  0]
[--+--------]
[ 0| 0  0 10]
[ 0| 1  0 -1]
[ 0| 0  1  0]
sage: A.rational_form(format='bottom')
[ 2| 0  0  0]
```

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```
[--+--------]
[ 0| 0  1  0]
[ 0| 0  0  1]
[ 0|10 -1  0]
sage: A.rational_form(format='left')
[ 2| 0  0  0]
[--+--------]
[ 0| 0  1  0]
[ 0|-1  0  1]
[ 0|10  0  0]
sage: A.rational_form(format='top')
[ 2| 0  0  0]
[--+--------]
[ 0| 0 -1 10]
[ 0| 1  0  0]
[ 0| 0  1  0]
```

restrict(*V*, *check=True*)

Returns the matrix that defines the action of self on the chosen basis for the invariant subspace V. If V is an ambient, returns self (not a copy of self).

INPUT:

- `V` - vector subspace
- `check` - (optional) default: True; if False may not check that V is invariant (hence can be faster).

OUTPUT: a matrix

Warning: This function returns an nxn matrix, where V has dimension n. It does *not* check that V is in fact invariant under self, unless check is True.

EXAMPLES:

```
sage: V = VectorSpace(QQ, 3)
sage: M = MatrixSpace(QQ, 3)
sage: A = M([1,2,0, 3,4,0, 0,0,0])
sage: W = V.subspace([[1,0,0], [0,1,0]])
sage: A.restrict(W)
[1 2]
[3 4]
sage: A.restrict(W, check=True)
[1 2]
[3 4]
```

We illustrate the warning about invariance not being checked by default, by giving a non-invariant subspace. With the default check=False this function returns the 'restriction' matrix, which is meaningless as check=True reveals.

```
sage: W2 = V.subspace([[1,0,0], [0,1,1]])
sage: A.restrict(W2, check=False)
[1 2]
[3 4]
```

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```
sage: A.restrict(W2, check=True)
Traceback (most recent call last):
...
ArithmeticError: subspace is not invariant under matrix
```

restrict_codomain(*V*)

Suppose that self defines a linear map from some domain to a codomain that contains V and that the image of self is contained in V. This function returns a new matrix A that represents this linear map but as a map to V, in the sense that if x is in the domain, then xA is the linear combination of the elements of the basis of V that equals v*self.

INPUT:

- `V` - vector space (space of degree `self.ncols()`) that contains the image of self.

See also:

restrict(), *restrict_domain()*

EXAMPLES:

```
sage: A = matrix(QQ,3,[1..9])
sage: V = (QQ^3).span([[1,2,3], [7,8,9]]); V
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: z = vector(QQ,[1,2,5])
sage: B = A.restrict_codomain(V); B
[1 2]
[4 5]
[7 8]
sage: z*B
(44, 52)
sage: z*A
(44, 52, 60)
sage: 44*V.0 + 52*V.1
(44, 52, 60)
```

restrict_domain(*V*)

Compute the matrix relative to the basis for V on the domain obtained by restricting self to V, but not changing the codomain of the matrix. This is the matrix whose rows are the images of the basis for V.

INPUT:

- `V` - vector space (subspace of ambient space on which self acts)

See also:

restrict()

EXAMPLES:

```
sage: V = QQ^3
sage: A = matrix(QQ,3,[1,2,0, 3,4,0, 0,0,0])
sage: W = V.subspace([[1,0,0], [1,2,3]])
sage: A.restrict_domain(W)
```

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```
[1 2 0]
[3 4 0]
sage: W2 = V.subspace_with_basis([[1,0,0], [1,2,3]])
sage: A.restrict_domain(W2)
[ 1  2  0]
[ 7 10  0]
```

right_eigenmatrix(*other=None*)

Return matrices D and P, where D is a diagonal matrix of eigenvalues and the columns of P are corresponding eigenvectors (or zero vectors).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved

OUTPUT:

If `self` is a square matrix A, then the output is a diagonal matrix D and a matrix P such that

$$AP = PD,$$

where the columns of P are eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$AP = BPD.$$

The ordinary eigenvalue problem is equivalent to the generalized one if B is the identity matrix.

The generalized eigenvector decomposition is currently only implemented for matrices over `RDF` and `CDF`.

EXAMPLES:

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: D, P = A.eigenmatrix_right()
sage: D
[                  0                   0                   0]
[                  0 -1.348469228349535?                   0]
[                  0                   0  13.34846922834954?]
sage: P
[                  1                   1                   1]
[                 -2  0.1303061543300932?   3.069693845669907?]
[                  1 -0.7393876913398137?   5.139387691339814?]
sage: A*P == P*D
True
```

Because P is invertible, A is diagonalizable.

```
sage: A == P*D*(~P)
True
```

The matrix P may contain zero columns corresponding to eigenvalues for which the algebraic multiplicity is greater than the geometric multiplicity. In these cases, the matrix is not diagonalizable.

```
sage: A = jordan_block(2,3); A
[2 1 0]
[0 2 1]
[0 0 2]
sage: D, P = A.eigenmatrix_right()
sage: D
[2 0 0]
[0 2 0]
[0 0 2]
sage: P
[1 0 0]
[0 0 0]
[0 0 0]
sage: A*P == P*D
True
```

A generalized eigenvector decomposition:

```
sage: A = matrix(RDF, [[1, -2], [3, 4]])
sage: B = matrix(RDF, [[0, 7], [2, -3]])
sage: D, P = A.eigenmatrix_right(B)
sage: (A * P - B * P * D).norm() < 1e-14
True
```

The matrix B in a generalized eigenvalue problem may be singular:

```
sage: A = matrix.identity(RDF, 2)
sage: B = matrix(RDF, [[3, 5], [6, 10]])
sage: D, P = A.eigenmatrix_right(B); D   # tol 1e-14
[0.07692307692307694                 0.0]
[                0.0           +infinity]
```

In this case, we can still verify the eigenvector equation for the first eigenvalue and first eigenvector:

```
sage: l = D[0, 0]
sage: v = P[:, 0]
sage: (A * v  - B * v * l).norm() < 1e-14
True
```

The second eigenvector is contained in the right kernel of B:

```
sage: (B * P[:, 1]).norm() < 1e-14
True
```

See also:

eigenvalues(), *eigenvectors_right()*, *Matrix_double_dense.eigenvectors_right()*, *eigenmatrix_left()*.

right_eigenspaces(*format='all'*, *var='a'*, *algebraic_multiplicity=False*)

Compute the right eigenspaces of a matrix.

Note that `eigenspaces_right()` and `right_eigenspaces()` are identical methods. Here "right" refers to the eigenvectors being placed to the right of the matrix.

INPUT:

- `self` - a square matrix over an exact field. For inexact matrices consult the numerical or symbolic matrix classes.
- `format` - default: `None`
  - `'all'` - attempts to create every eigenspace. This will always be possible for matrices with rational entries.
  - `'galois'` - for each irreducible factor of the characteristic polynomial, a single eigenspace will be output for a single root/eigenvalue for the irreducible factor.
  - `None` - Uses the 'all' format if the base ring is contained in an algebraically closed field which is implemented. Otherwise, uses the 'galois' format.
- `var` - default: 'a' - variable name used to represent elements of the root field of each irreducible factor of the characteristic polynomial. If var='a', then the root fields will be in terms of a0, a1, a2, ..., where the numbering runs across all the irreducible factors of the characteristic polynomial, even for linear factors.
- `algebraic_multiplicity` - default: False - whether or not to include the algebraic multiplicity of each eigenvalue in the output. See the discussion below.

OUTPUT:

If algebraic_multiplicity=False, return a list of pairs (e, V) where e is an eigenvalue of the matrix, and V is the corresponding left eigenspace. For Galois conjugates of eigenvalues, there may be just one representative eigenspace, depending on the `format` keyword.

If algebraic_multiplicity=True, return a list of triples (e, V, n) where e and V are as above and n is the algebraic multiplicity of the eigenvalue.

Warning: Uses a somewhat naive algorithm (simply factors the characteristic polynomial and computes kernels directly over the extension field).

EXAMPLES:

Right eigenspaces are computed from the left eigenspaces of the transpose of the matrix. As such, there is a greater collection of illustrative examples at the *eigenspaces_left()*.

We compute the right eigenspaces of a 3×3 rational matrix.

```
sage: A = matrix(QQ, 3 ,3, range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.eigenspaces_right()
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(-1.348469228349535?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
User basis matrix:
[                   1  0.1303061543300932? -0.7393876913398137?]),
(13.34846922834954?, Vector space of degree 3 and dimension 1 over Algebraic␣
↪Field
```

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```
User basis matrix:
[                1 3.069693845669907? 5.139387691339814?])
]
sage: es = A.eigenspaces_right(format='galois'); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[           1 1/5*a1 + 2/5 2/5*a1 - 1/5])
]
sage: es = A.eigenspaces_right(format='galois', algebraic_multiplicity=True); es
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1], 1),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[           1 1/5*a1 + 2/5 2/5*a1 - 1/5], 1)
]
sage: e, v, n = es[0]; v = v.basis()[0]
sage: delta = v*e - A*v
sage: abs(abs(delta)) < 1e-10
True
```

The same computation, but with implicit base change to a field:

```
sage: A = matrix(ZZ, 3, range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.eigenspaces_right(format='galois')
[
(0, Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]),
(a1, Vector space of degree 3 and dimension 1 over Number Field in a1 with␣
↪defining polynomial x^2 - 12*x - 18
User basis matrix:
[           1 1/5*a1 + 2/5 2/5*a1 - 1/5])
]
```

This method is only applicable to exact matrices. The "eigenmatrix" routines for matrices with double-precision floating-point entries (RDF, CDF) are the best alternative. (Since some platforms return eigenvectors that are the negatives of those given here, this one example is not tested here.) There are also "eigenmatrix" routines for matrices with symbolic entries.

```
sage: B = matrix(RR, 3, 3, range(9))
sage: B.eigenspaces_right()
```

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```
Traceback (most recent call last):
...
NotImplementedError: eigenspaces cannot be computed reliably for inexact rings␣
↪such as Real Field with 53 bits of precision,
consult numerical or symbolic matrix classes for other options

sage: em = B.change_ring(RDF).eigenmatrix_right()
sage: eigenvalues = em[0]; eigenvalues.dense_matrix() # abs tol 1e-13
[13.348469228349522                0.0                0.0]
[               0.0 -1.348469228349534                0.0]
[               0.0                0.0                0.0]
sage: eigenvectors = em[1]; eigenvectors # not tested
[ 0.164763817...  0.799699663...  0.408248290...]
[ 0.505774475...  0.104205787... -0.816496580...]
[ 0.846785134... -0.591288087...  0.408248290...]

sage: x, y = var('x y')
sage: S = matrix([[x, y], [y, 3*x^2]])
sage: em = S.eigenmatrix_right()
sage: eigenvalues = em[0]; eigenvalues
[3/2*x^2 + 1/2*x - 1/2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)                       ␣
↪                                  0]
[                                                    0 3/2*x^2 + 1/2*x + 1/
↪2*sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2)]
sage: eigenvectors = em[1]; eigenvectors
[                                                   1                        ␣
↪                             1]
[1/2*(3*x^2 - x - sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y 1/2*(3*x^2 - x +␣
↪sqrt(9*x^4 - 6*x^3 + x^2 + 4*y^2))/y]
```

right_eigenvectors(*other=None*, *extend=True*)

Compute the right eigenvectors of a matrix.

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved (currently supported only if the base ring of `self` is `RDF` or `CDF`)
- `extend` – boolean (default: `True`)

OUTPUT:

For each distinct eigenvalue, returns a list of the form (e,V,n) where e is the eigenvalue, V is a list of eigenvectors forming a basis for the corresponding right eigenspace, and n is the algebraic multiplicity of the eigenvalue. If `extend = True` (the default), this will return eigenspaces over the algebraic closure of the base field where this is implemented; otherwise it will restrict to eigenvalues in the base field.

EXAMPLES:

We compute the right eigenvectors of a 3×3 rational matrix.

```
sage: A = matrix(QQ,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
```

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```
sage: es = A.eigenvectors_right(); es
[(0, [
(1, -2, 1)
], 1),
(-1.348469228349535?, [(1, 0.1303061543300932?, -0.7393876913398137?)], 1),
(13.34846922834954?, [(1, 3.069693845669907?, 5.139387691339814?)], 1)]
sage: A.eigenvectors_right(extend=False)
[(0, [
(1, -2, 1)
], 1)]
sage: eval, [evec], mult = es[0]
sage: delta = eval*evec - A*evec
sage: abs(abs(delta)) < 1e-10
True
```

right_kernel(**args*, ***kwds*)

Returns the right kernel of this matrix, as a vector space or free module. This is the set of vectors `x` such that `self*x = 0`.

Note: For the left kernel, use *left_kernel()*. The method *kernel()* is exactly equal to *left_kernel()*.

For inexact rings use *right_kernel_matrix()* with `basis='computed'` to avoid echolonizing.

INPUT:

- `algorithm` - default: 'default' - a keyword that selects the algorithm employed. Allowable values are:
  - 'default' - allows the algorithm to be chosen automatically
  - 'generic' - naive algorithm usable for matrices over any field
  - 'flint' - FLINT library code for matrices over the rationals or the integers
  - 'pari' - PARI library code for matrices over number fields or the integers
  - 'padic' - padic algorithm from IML library for matrices over the rationals and integers
  - 'pluq' - PLUQ matrix factorization for matrices mod 2
- `basis` - default: 'echelon' - a keyword that describes the format of the basis used to construct the right kernel. Allowable values are:
  - 'echelon': the basis matrix is returned in echelon form
  - 'pivot' : each basis vector is computed from the reduced row-echelon form of `self` by placing a single one in a non-pivot column and zeros in the remaining non-pivot columns. Only available for matrices over fields.
  - 'LLL': an LLL-reduced basis. Only available for matrices over the integers.

OUTPUT:

A vector space or free module whose degree equals the number of columns in `self` and which contains all the vectors `x` such that `self*x = 0`.

If `self` has 0 columns, the kernel has dimension 0, while if `self` has 0 rows the kernel is the entire ambient vector space.

The result is cached. Requesting the right kernel a second time, but with a different basis format, will return the cached result with the format from the first computation.

Note: For more detailed documentation on the selection of algorithms used and a more flexible method for computing a basis matrix for a right kernel (rather than computing a vector space), see *right_kernel_matrix()*, which powers the computations for this method.

EXAMPLES:

```
sage: A = matrix(QQ, [[0, 0, 1, 2, 2, -5, 3],
....:                 [-1, 5, 2, 2, 1, -7, 5],
....:                 [0, 0, -2, -3, -3, 8, -5],
....:                 [-1, 5, 0, -1, -2, 1, 0]])
sage: K = A.right_kernel(); K
Vector space of degree 7 and dimension 4 over Rational Field
Basis matrix:
[ 1  0  0  0 -1 -1 -1]
[ 0  1  0  0  5  5  5]
[ 0  0  1  0 -1 -2 -3]
[ 0  0  0  1  0  1  1]
sage: A*K.basis_matrix().transpose() == zero_matrix(QQ, 4, 4)
True
```

The default is basis vectors that form a matrix in echelon form. A "pivot basis" instead has a basis matrix where the columns of an identity matrix are in the locations of the non-pivot columns of the original matrix. This alternate format is available whenever the base ring is a field.

```
sage: A = matrix(QQ, [[0, 0, 1, 2, 2, -5, 3],
....:                 [-1, 5, 2, 2, 1, -7, 5],
....:                 [0, 0, -2, -3, -3, 8, -5],
....:                 [-1, 5, 0, -1, -2, 1, 0]])
sage: A.rref()
[ 1 -5  0  0  1  1 -1]
[ 0  0  1  0  0 -1  1]
[ 0  0  0  1  1 -2  1]
[ 0  0  0  0  0  0  0]
sage: A.nonpivots()
(1, 4, 5, 6)
sage: K = A.right_kernel(basis='pivot'); K
Vector space of degree 7 and dimension 4 over Rational Field
User basis matrix:
[ 5  1  0  0  0  0  0]
[-1  0  0 -1  1  0  0]
[-1  0  1  2  0  1  0]
[ 1  0 -1 -1  0  0  1]
sage: A*K.basis_matrix().transpose() == zero_matrix(QQ, 4, 4)
True
```

Matrices may have any field as a base ring. Number fields are computed by PARI library code, matrices over $GF(2)$ are computed by the M4RI library, and matrices over the rationals are computed by the IML library. For any of these specialized cases, general-purpose code can be called instead with the keyword setting `algorithm='generic'`.

Over an arbitrary field, with two basis formats. Same vector space, different bases.

```
sage: F.<a> = FiniteField(5^2)
sage: A = matrix(F, 3, 4, [[  1,   a,     1+a,  a^3+a^5],
....:                      [  a, a^4,   a+a^4,  a^4+a^8],
....:                      [a^2, a^6, a^2+a^6, a^5+a^10]])
sage: K = A.right_kernel(); K
Vector space of degree 4 and dimension 2 over Finite Field in a of size 5^2
Basis matrix:
[      1       0 3*a + 4 2*a + 2]
[      0       1     2*a 3*a + 3]
sage: A*K.basis_matrix().transpose() == zero_matrix(F, 3, 2)
True
```

In the following test, we have to force usage of *Matrix_generic_dense*, since the option `basis = 'pivot'` would simply yield the same result as the previous test, if the optional meataxe package is installed.

```
sage: from sage.matrix.matrix_generic_dense import Matrix_generic_dense
sage: B = Matrix_generic_dense(A.parent(), A.list(), False, False)
sage: P = B.right_kernel(basis = 'pivot'); P
Vector space of degree 4 and dimension 2 over Finite Field in a of size 5^2
User basis matrix:
[      4       4       1       0]
[  a + 2 3*a + 3       0       1]
```

If the optional meataxe package is installed, we again have to make sure to work with a copy of B that has the same type as `P.basis_matrix()`:

```
sage: B.parent()(B.list())*P.basis_matrix().transpose() == zero_matrix(F, 3, 2)
True
sage: K == P
True
```

Over number fields, PARI is used by default, but general-purpose code can be requested. Same vector space, same bases, different code.:

```
sage: Q = QuadraticField(-7)
sage: a = Q.gen(0)
sage: A = matrix(Q, [[  2, 5-a,     15-a, 16+4*a],
....:                [2+a,   a, -7 + 5*a, -3+3*a]])
sage: K = A.right_kernel(algorithm='default'); K
Vector space of degree 4 and dimension 2 over Number Field in a with defining␣
↪polynomial x^2 + 7 with a = 2.645751311064591?*I
Basis matrix:
[                1                 0     7/88*a + 3/88 -3/176*a - 39/176]
[                0                 1   -1/88*a - 13/88  13/176*a - 7/176]
sage: A*K.basis_matrix().transpose() == zero_matrix(Q, 2, 2)
True
sage: B = copy(A)
sage: G = A.right_kernel(algorithm='generic'); G
Vector space of degree 4 and dimension 2 over Number Field in a with defining␣
↪polynomial x^2 + 7 with a = 2.645751311064591?*I
Basis matrix:
[                1                 0     7/88*a + 3/88 -3/176*a - 39/176]
```

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```
[               0                1   -1/88*a - 13/88  13/176*a - 7/176]
sage: B*G.basis_matrix().transpose() == zero_matrix(Q, 2, 2)
True
sage: K == G
True
```

For matrices over the integers, several options are possible. The basis can be an LLL-reduced basis or an echelon basis. The pivot basis isnot available. A heuristic will decide whether to use a p-adic algorithm from the IML library or an algorithm from the PARI library. Note how specifying the algorithm can mildly influence the LLL basis.

```
sage: A = matrix(ZZ, [[0, -1, -1, 2, 9, 4, -4],
....:                 [-1, 1, 0, -2, -7, -1, 6],
....:                 [2, 0, 1, 0, 1, -5, -2],
....:                 [-1, -1, -1, 3, 10, 10, -9],
....:                 [-1, 2, 0, -3, -7, 1, 6]])
sage: A.right_kernel(basis='echelon')
Free module of degree 7 and rank 2 over Integer Ring
Echelon basis matrix:
[  1   5  -8   3  -1  -1  -1]
[  0  11 -19   5  -2  -3  -3]
sage: B = copy(A)
sage: B.right_kernel(basis='LLL')
Free module of degree 7 and rank 2 over Integer Ring
User basis matrix:
[ 2 -1  3  1  0  1  1]
[-5 -3  2 -5  1 -1 -1]
sage: C = copy(A)
sage: C.right_kernel(basis='pivot')
Traceback (most recent call last):
...
ValueError: pivot basis only available over a field, not over Integer Ring
sage: D = copy(A)
sage: D.right_kernel(algorithm='pari')
Free module of degree 7 and rank 2 over Integer Ring
Echelon basis matrix:
[  1   5  -8   3  -1  -1  -1]
[  0  11 -19   5  -2  -3  -3]
sage: E = copy(A)
sage: E.right_kernel(algorithm='padic', basis='LLL')
Free module of degree 7 and rank 2 over Integer Ring
User basis matrix:
[-2  1 -3 -1  0 -1 -1]
[ 5  3 -2  5 -1  1  1]
```

Besides the integers, rings may be as general as principal ideal domains. Results are then free modules.

```
sage: R.<y> = QQ[]
sage: A = matrix(R, [[  1,   y, 1+y^2],
....:                [y^3, y^2, 2*y^3]])
sage: K = A.right_kernel(algorithm='default', basis='echelon'); K
Free module of degree 3 and rank 1 over Univariate Polynomial Ring in y over␣
```

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```
↪Rational Field
Echelon basis matrix:
[-1 -y  1]
sage: A*K.basis_matrix().transpose() == zero_matrix(ZZ, 2, 1)
True
```

It is possible to compute a kernel for a matrix over an integral domain which is not a PID, but usually this will fail.

```
sage: D.<x> = ZZ[]
sage: A = matrix(D, 2, 2, [[x^2 - x, -x + 5],
....:                      [x^2 - 8, -x + 2]])
sage: A.right_kernel()
Traceback (most recent call last):
...
ArithmeticError: Ideal Ideal (x^2 - x, x^2 - 8) of Univariate Polynomial Ring␣
↪in x over Integer Ring not principal
```

Matrices over non-commutative rings are not a good idea either. These are the "usual" quaternions.

```
sage: Q.<i,j,k> = QuaternionAlgebra(-1,-1)
sage: A = matrix(Q, 2, [i,j,-1,k])
sage: A.right_kernel()
Traceback (most recent call last):
...
NotImplementedError: Cannot compute a matrix kernel over Quaternion Algebra (-1,
↪ -1) with base ring Rational Field
```

Sparse matrices, over the rationals and the integers, use the same routines as the dense versions.

```
sage: A = matrix(ZZ, [[0, -1, 1, 1, 2],
....:                 [1, -2, 0, 1, 3],
....:                 [-1, 2, 0, -1, -3]],
....:            sparse=True)
sage: A.right_kernel()
Free module of degree 5 and rank 3 over Integer Ring
Echelon basis matrix:
[ 1  0  0  2 -1]
[ 0  1  0 -1  1]
[ 0  0  1 -3  1]
sage: B = A.change_ring(QQ)
sage: B.is_sparse()
True
sage: B.right_kernel()
Vector space of degree 5 and dimension 3 over Rational Field
Basis matrix:
[ 1  0  0  2 -1]
[ 0  1  0 -1  1]
[ 0  0  1 -3  1]
```

With no columns, the kernel can only have dimension zero. With no rows, every possible vector is in the kernel.

```
sage: A = matrix(QQ, 2, 0)
sage: A.right_kernel()
Vector space of degree 0 and dimension 0 over Rational Field
Basis matrix:
[]
sage: A = matrix(QQ, 0, 2)
sage: A.right_kernel()
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
```

Every vector is in the kernel of a zero matrix, the dimension is the number of columns.

```
sage: A = zero_matrix(QQ, 10, 20)
sage: A.right_kernel()
Vector space of degree 20 and dimension 20 over Rational Field
Basis matrix:
20 x 20 dense matrix over Rational Field
```

Results are cached as the right kernel of the matrix. Subsequent requests for the right kernel will return the cached result, without regard for new values of the algorithm or format keyword. Work with a copy if you need a new right kernel, or perhaps investigate the *right_kernel_matrix()* method, which does not cache its results and is more flexible.

```
sage: A = matrix(QQ, 3, 3, range(9))
sage: K1 = A.right_kernel(basis='echelon')
sage: K1
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -2  1]
sage: K2 = A.right_kernel(basis='pivot')
sage: K2
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -2  1]
sage: K1 is K2
True
sage: B = copy(A)
sage: K3 = B.kernel(basis='pivot')
sage: K3
Vector space of degree 3 and dimension 1 over Rational Field
User basis matrix:
[ 1 -2  1]
sage: K3 is K1
False
sage: K3 == K1
True
```

right_kernel_matrix(**args*, ***kwds*)

Returns a matrix whose rows form a basis for the right kernel of `self`.

INPUT:

- `algorithm` - default: 'default' - a keyword that selects the algorithm employed. Allowable values are:

  - 'default' - allows the algorithm to be chosen automatically
  - 'generic' - naive algorithm usable for matrices over any field
  - 'flint' - FLINT library code for matrices over the rationals or the integers
  - 'pari' - PARI library code for matrices over number fields or the integers
  - 'padic' - padic algorithm from IML library for matrices over the rationals and integers
  - 'pluq' - PLUQ matrix factorization for matrices mod 2
- `basis` - default: 'default' - a keyword that describes the format of the basis returned. Allowable values are:
  - 'default': uses 'echelon' over fields; 'computed' otherwise.
  - 'echelon': the basis matrix is returned in echelon form.
  - 'pivot' : each basis vector is computed from the reduced row-echelon form of `self` by placing a single one in a non-pivot column and zeros in the remaining non-pivot columns. Only available for matrices over fields.
  - 'LLL': an LLL-reduced basis. Only available for matrices over the integers.
  - 'computed': no work is done to transform the basis, it is returned exactly as provided by whichever routine actually computed the basis. Request this for the least possible computation possible, but with no guarantees about the format of the basis. This option is recommended for inexact rings.

OUTPUT:

A matrix `X` whose rows are an independent set spanning the right kernel of `self`. So `self*X.transpose()` is a zero matrix.

The output varies depending on the choice of `algorithm` and the format chosen by `basis`.

The results of this routine are not cached, so you can call it again with different options to get possibly different output (like the basis format). Conversely, repeated calls on the same matrix will always start from scratch.

Note: If you want to get the most basic description of a kernel, with a minimum of overhead, then ask for the right kernel matrix with the basis format requested as 'computed'. You are then free to work with the output for whatever purpose. For a left kernel, call this method on the transpose of your matrix.

For greater convenience, plus cached results, request an actual vector space or free module with *right_kernel()* or *left_kernel()*.

EXAMPLES:

Over the Rational Numbers:

Kernels are computed by the IML library in `_right_kernel_matrix()`. Setting the *algorithm* keyword to 'default', 'padic' or unspecified will yield the same result, as there is no optional behavior. The 'computed' format of the basis vectors are exactly the negatives of the vectors in the 'pivot' format.

```
sage: A = matrix(QQ, [[1, 0, 1, -3, 1],
....:                 [-5, 1, 0, 7, -3],
....:                 [0, -1, -4, 6, -2],
....:                 [4, -1, 0, -6, 2]])
sage: C = A.right_kernel_matrix(algorithm='default', basis='computed'); C
[-1  2 -2 -1  0]
```

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```
[ 1  2  0  0 -1]
sage: A*C.transpose() == zero_matrix(QQ, 4, 2)
True
sage: P = A.right_kernel_matrix(algorithm='padic', basis='pivot'); P
[ 1 -2  2  1  0]
[-1 -2  0  0  1]
sage: A*P.transpose() == zero_matrix(QQ, 4, 2)
True
sage: C == -P
True
sage: E = A.right_kernel_matrix(algorithm='default', basis='echelon'); E
[   1    0    1  1/2 -1/2]
[   0    1 -1/2 -1/4 -1/4]
sage: A*E.transpose() == zero_matrix(QQ, 4, 2)
True
```

Since the rationals are a field, we can call the general code available for any field by using the 'generic' keyword.

```
sage: A = matrix(QQ, [[1, 0, 1, -3, 1],
....:                 [-5, 1, 0, 7, -3],
....:                 [0, -1, -4, 6, -2],
....:                 [4, -1, 0, -6, 2]])
sage: G = A.right_kernel_matrix(algorithm='generic', basis='echelon'); G
[   1    0    1  1/2 -1/2]
[   0    1 -1/2 -1/4 -1/4]
sage: A*G.transpose() == zero_matrix(QQ, 4, 2)
True
```

We verify that the rational matrix code is called for both dense and sparse rational matrices, with equal result.

```
sage: A = matrix(QQ, [[1, 0, 1, -3, 1],
....:                 [-5, 1, 0, 7, -3],
....:                 [0, -1, -4, 6, -2],
....:                 [4, -1, 0, -6, 2]],
....:            sparse=False)
sage: B = copy(A).sparse_matrix()
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(1)
sage: D = A.right_kernel(); D
verbose 1 (<module>) computing a right kernel for 4x5 matrix over Rational Field
...
Vector space of degree 5 and dimension 2 over Rational Field
Basis matrix:
[   1    0    1  1/2 -1/2]
[   0    1 -1/2 -1/4 -1/4]
sage: S = B.right_kernel(); S
verbose 1 (<module>) computing a right kernel for 4x5 matrix over Rational Field
...
Vector space of degree 5 and dimension 2 over Rational Field
Basis matrix:
```

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```
[   1    0    1  1/2 -1/2]
[   0    1 -1/2 -1/4 -1/4]
sage: set_verbose(0)
sage: D == S
True
```

Over Number Fields:

Kernels are by default computed by PARI, (except for exceptions like the rationals themselves). The raw results from PARI are a pivot basis, so the *basis* keywords 'computed' and 'pivot' will return the same results.

```
sage: Q = QuadraticField(-7)
sage: a = Q.gen(0)
sage: A = matrix(Q, [[2, 5-a, 15-a, 16+4*a],
....:                [2+a, a, -7 + 5*a, -3+3*a]])
sage: C = A.right_kernel_matrix(algorithm='default', basis='computed'); C
[    -a     -3      1      0]
[    -2 -a - 1      0      1]
sage: A*C.transpose() == zero_matrix(Q, 2, 2)
True
sage: P = A.right_kernel_matrix(algorithm='pari', basis='pivot'); P
[    -a     -3      1      0]
[    -2 -a - 1      0      1]
sage: A*P.transpose() == zero_matrix(Q, 2, 2)
True
sage: E = A.right_kernel_matrix(algorithm='default', basis='echelon'); E
[               1                0     7/88*a + 3/88 -3/176*a - 39/176]
[               0                1   -1/88*a - 13/88  13/176*a - 7/176]
sage: A*E.transpose() == zero_matrix(Q, 2, 2)
True
```

We can bypass using PARI for number fields and use Sage's general code for matrices over any field. The basis vectors as computed are in pivot format.

```
sage: Q = QuadraticField(-7)
sage: a = Q.gen(0)
sage: A = matrix(Q, [[2, 5-a, 15-a, 16+4*a],[2+a, a, -7 + 5*a, -3+3*a]])
sage: G = A.right_kernel_matrix(algorithm='generic', basis='computed'); G
[    -a     -3      1      0]
[    -2 -a - 1      0      1]
sage: A*G.transpose() == zero_matrix(Q, 2, 2)
True
```

We check that number fields are handled by the right routine as part of typical right kernel computation.

```
sage: Q = QuadraticField(-7)
sage: a = Q.gen(0)
sage: A = matrix(Q, [[2, 5-a, 15-a, 16+4*a],[2+a, a, -7 + 5*a, -3+3*a]])
sage: set_verbose(1)
sage: A.right_kernel(algorithm='default')
verbose ...
verbose 1 (<module>) computing right kernel matrix over a number field for 2x4␣
```

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```
↪matrix
verbose 1 (<module>) done computing right kernel matrix over a number field for␣
↪2x4 matrix
...
Vector space of degree 4 and dimension 2 over Number Field in a with defining␣
↪polynomial x^2 + 7 with a = 2.645751311064591?*I
Basis matrix:
[                1                 0     7/88*a + 3/88 -3/176*a - 39/176]
[                0                 1   -1/88*a - 13/88  13/176*a - 7/176]
sage: set_verbose(0)
```

Over the Finite Field of Order 2:

Kernels are computed by the M4RI library using PLUQ matrix decomposition in the `_right_kernel_matrix()` method. There are no options for the algorithm used.

```
sage: A = matrix(GF(2),[[0, 1, 1, 0, 0, 0],
....:                   [1, 0, 0, 0, 1, 1,],
....:                   [1, 0, 0, 0, 1, 1]])
sage: E = A.right_kernel_matrix(algorithm='default', format='echelon'); E
[1 0 0 0 0 1]
[0 1 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 1]
sage: A*E.transpose() == zero_matrix(GF(2), 3, 4)
True
```

Since GF(2) is a field we can route this computation to the generic code and obtain the 'pivot' form of the basis. The `algorithm` keywords, 'pluq', 'default' and unspecified, all have the same effect as there is no optional behavior.

```
sage: A = matrix(GF(2),[[0, 1, 1, 0, 0, 0],
....:                   [1, 0, 0, 0, 1, 1,],
....:                   [1, 0, 0, 0, 1, 1]])
sage: P = A.right_kernel_matrix(algorithm='generic', basis='pivot'); P
[0 1 1 0 0 0]
[0 0 0 1 0 0]
[1 0 0 0 1 0]
[1 0 0 0 0 1]
sage: A*P.transpose() == zero_matrix(GF(2), 3, 4)
True
sage: DP = A.right_kernel_matrix(algorithm='default', basis='pivot'); DP
[0 1 1 0 0 0]
[0 0 0 1 0 0]
[1 0 0 0 1 0]
[1 0 0 0 0 1]
sage: A*DP.transpose() == zero_matrix(GF(2), 3, 4)
True
sage: A.right_kernel_matrix(algorithm='pluq', basis='echelon')
[1 0 0 0 0 1]
[0 1 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 1]
```

We test that the mod 2 code is called for matrices over GF(2).

```
sage: A = matrix(GF(2),[[0, 1, 1, 0, 0, 0],
....:                   [1, 0, 0, 0, 1, 1,],
....:                   [1, 0, 0, 0, 1, 1]])
sage: set_verbose(1)
sage: A.right_kernel(algorithm='default')
verbose ...
verbose 1 (<module>) computing right kernel matrix over integers mod 2 for 3x6␣
↪matrix
verbose 1 (<module>) done computing right kernel matrix over integers mod 2 for␣
↪3x6 matrix
...
Vector space of degree 6 and dimension 4 over Finite Field of size 2
Basis matrix:
[1 0 0 0 0 1]
[0 1 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 1]
sage: set_verbose(0)
```

Over Arbitrary Fields:

For kernels over fields not listed above, totally general code will compute a set of basis vectors in the pivot format. These could be returned as a basis in echelon form.

```
sage: F.<a> = FiniteField(5^2)
sage: A = matrix(F, 3, 4, [[  1,   a,     1+a,  a^3+a^5],
....:                      [  a, a^4,   a+a^4,  a^4+a^8],
....:                      [a^2, a^6, a^2+a^6, a^5+a^10]])
sage: P = A.right_kernel_matrix(algorithm='default', basis='pivot'); P
[      4       4       1       0]
[  a + 2 3*a + 3       0       1]
sage: A*P.transpose() == zero_matrix(F, 3, 2)
True
sage: E = A.right_kernel_matrix(algorithm='default', basis='echelon'); E
[      1       0 3*a + 4 2*a + 2]
[      0       1     2*a 3*a + 3]
sage: A*E.transpose() == zero_matrix(F, 3, 2)
True
```

This general code can be requested for matrices over any field with the `algorithm` keyword 'generic'. Normally, matrices over the rationals would be handled by specific routines from the IML library. The default format is an echelon basis, but a pivot basis may be requested, which is identical to the computed basis.

```
sage: A = matrix(QQ, 3, 4, [[1,3,-2,4],
....:                       [2,0,2,2],
....:                       [-1,1,-2,0]])
sage: G = A.right_kernel_matrix(algorithm='generic'); G
[   1    0 -1/2 -1/2]
[   0    1  1/2 -1/2]
sage: A*G.transpose() == zero_matrix(QQ, 3, 2)
True
```

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```
sage: C = A.right_kernel_matrix(algorithm='generic', basis='computed'); C
[-1  1  1  0]
[-1 -1  0  1]
sage: A*C.transpose() == zero_matrix(QQ, 3, 2)
True
```

We test that the generic code is called for matrices over fields, lacking any more specific routine.

```
sage: F.<a> = FiniteField(5^2)
sage: A = matrix(F, 3, 4, [[  1,   a,     1+a,  a^3+a^5],
....:                      [  a, a^4,   a+a^4,  a^4+a^8],
....:                      [a^2, a^6, a^2+a^6, a^5+a^10]])
sage: set_verbose(1)
sage: A.right_kernel(algorithm='default')
verbose ...
verbose 1 (<module>) computing right kernel matrix over an arbitrary field for␣
↪3x4 matrix
...
Vector space of degree 4 and dimension 2 over Finite Field in a of size 5^2
Basis matrix:
[      1       0 3*a + 4 2*a + 2]
[      0       1     2*a 3*a + 3]
sage: set_verbose(0)
```

Over the Integers:

Either the IML or PARI libraries are used to provide a set of basis vectors. The `algorithm` keyword can be used to select either, or when set to 'default' a heuristic will choose between the two. Results can be returned in the 'compute' format, straight out of the libraries. Unique to the integers, the basis vectors can be returned as an LLL basis. Note the similarities and differences in the results. The 'pivot' format is not available, since the integers are not a field.

```
sage: A = matrix(ZZ, [[8, 0, 7, 1, 3, 4, 6],
....:                 [4, 0, 3, 4, 2, 7, 7],
....:                 [1, 4, 6, 1, 2, 8, 5],
....:                 [0, 3, 1, 2, 3, 6, 2]])

sage: X = A.right_kernel_matrix(algorithm='default', basis='echelon'); X
[  1  12   3  14  -3 -10   1]
[  0  35   0  25  -1 -31  17]
[  0   0   7  12  -3  -1  -8]
sage: A*X.transpose() == zero_matrix(ZZ, 4, 3)
True

sage: X = A.right_kernel_matrix(algorithm='padic', basis='LLL'); X
[ -3  -1   5   7   2  -3  -2]
[  3   1   2   5  -5   2  -6]
[ -4 -13   2  -7   5   7  -3]
sage: A*X.transpose() == zero_matrix(ZZ, 4, 3)
True

sage: X = A.right_kernel_matrix(algorithm='pari', basis='computed'); X
[ 3  1 -5 -7 -2  3  2]
```

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```
[ 3  1  2  5 -5  2 -6]
[ 4 13 -2  7 -5 -7  3]
sage: A*X.transpose() == zero_matrix(ZZ, 4, 3)
True

sage: X = A.right_kernel_matrix(algorithm='padic', basis='computed'); X
[ 265  345 -178   17 -297    0    0]
[-242 -314  163  -14  271   -1    0]
[ -36  -47   25   -1   40    0   -1]
sage: A*X.transpose() == zero_matrix(ZZ, 4, 3)
True
```

We test that the code for integer matrices is called for matrices defined over the integers, both dense and sparse, with equal result.

```
sage: A = matrix(ZZ, [[8, 0, 7, 1, 3, 4, 6],
....:                 [4, 0, 3, 4, 2, 7, 7],
....:                 [1, 4, 6, 1, 2, 8, 5],
....:                 [0, 3, 1, 2, 3, 6, 2]],
....:            sparse=False)
sage: B = copy(A).sparse_matrix()
sage: set_verbose(1)
sage: D = A.right_kernel(); D
verbose 1 (<module>) computing a right kernel for 4x7 matrix over Integer Ring
verbose 1 (<module>) computing right kernel matrix over the integers for 4x7␣
↪matrix
...
verbose 1 (<module>) done computing right kernel matrix over the integers for␣
↪4x7 matrix
...
Free module of degree 7 and rank 3 over Integer Ring
Echelon basis matrix:
[  1  12   3  14  -3 -10   1]
[  0  35   0  25  -1 -31  17]
[  0   0   7  12  -3  -1  -8]
sage: S = B.right_kernel(); S
verbose 1 (<module>) computing a right kernel for 4x7 matrix over Integer Ring
verbose 1 (<module>) computing right kernel matrix over the integers for 4x7␣
↪matrix
...
verbose 1 (<module>) done computing right kernel matrix over the integers for␣
↪4x7 matrix
...
Free module of degree 7 and rank 3 over Integer Ring
Echelon basis matrix:
[  1  12   3  14  -3 -10   1]
[  0  35   0  25  -1 -31  17]
[  0   0   7  12  -3  -1  -8]
sage: set_verbose(0)
sage: D == S
True
```

Over Principal Ideal Domains:

Kernels can be computed using Smith normal form. Only the default algorithm is available, and the 'pivot' basis format is not available.

```
sage: R.<y> = QQ[]
sage: A = matrix(R, [[  1,   y, 1+y^2],
....:                [y^3, y^2, 2*y^3]])
sage: E = A.right_kernel_matrix(algorithm='default', basis='echelon'); E
[-1 -y  1]
sage: A*E.transpose() == zero_matrix(ZZ, 2, 1)
True
```

It can be computationally expensive to determine if an integral domain is a principal ideal domain. The Smith normal form routine can fail for non-PIDs, as in this example.

```
sage: D.<x> = ZZ[]
sage: A = matrix(D, 2, 2, [[x^2 - x, -x + 5],
....:                      [x^2 - 8, -x + 2]])
sage: A.right_kernel_matrix()
Traceback (most recent call last):
...
ArithmeticError: Ideal Ideal (x^2 - x, x^2 - 8) of Univariate Polynomial Ring␣
↪in x over Integer Ring not principal
```

We test that the domain code is called for domains that lack any extra structure.

```
sage: R.<y> = QQ[]
sage: A = matrix(R, [[  1,   y, 1+y^2],
....:                [y^3, y^2, 2*y^3]])
sage: set_verbose(1)
sage: A.right_kernel(algorithm='default', basis='echelon')
verbose ...
verbose 1 (<module>) computing right kernel matrix over a domain for 2x3 matrix
verbose 1 (<module>) done computing right kernel matrix over a domain for 2x3␣
↪matrix
...
Free module of degree 3 and rank 1 over Univariate Polynomial Ring in y over␣
↪Rational Field
Echelon basis matrix:
[-1 -y  1]
sage: set_verbose(0)
```

Over inexact rings:

For inexact rings one should avoid echolonizing if possible:

```
sage: A = Matrix(
....: [[          0.0,           0.5,  0.8090169944],
....:  [          0.0,           0.5, -0.8090169944],
....:  [          0.0,          -0.5,  0.8090169944],
....:  [          0.0,          -0.5, -0.8090169944],
....:  [          0.5,  0.8090169944,           0.0],
....:  [          0.5, -0.8090169944,           0.0],
....:  [         -0.5,  0.8090169944,           0.0],
....:  [         -0.5, -0.8090169944,           0.0],
....:  [ 0.8090169944,           0.0,           0.5],
```

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```
....:  [-0.8090169944,           0.0,           0.5],
....:  [ 0.8090169944,           0.0,          -0.5],
....:  [-0.8090169944,           0.0,          -0.5]]).transpose()
sage: (A*A.right_kernel_matrix().transpose()).norm() > 2
True
sage: (A*A.right_kernel_matrix(basis='computed').transpose()).norm() < 1e-15
True
```

Trivial Cases:

We test two trivial cases. Any possible values for the keywords (`algorithm`, `basis`) will return identical results.

```
sage: A = matrix(ZZ, 0, 2)
sage: A.right_kernel_matrix()
[1 0]
[0 1]
sage: A = matrix(FiniteField(7), 2, 0)
sage: A.right_kernel_matrix().parent()
Full MatrixSpace of 0 by 0 dense matrices over Finite Field of size 7
```

right_nullity()

Return the right nullity of this matrix, which is the dimension of the right kernel.

EXAMPLES:

```
sage: A = MatrixSpace(QQ,3,2)(range(6))
sage: A.right_nullity()
0
```

```
sage: A = matrix(ZZ,3,3,range(9))
sage: A.right_nullity()
1
```

rook_vector(*algorithm=None*, *complement=False*, *use_complement=None*)

Return the rook vector of this matrix.

Let A be an m by n (0,1)-matrix. We identify A with a chessboard where rooks can be placed on the fields (i, j) with $A_{i,j} = 1$. The number $r_k = p_k(A)$ (the permanental k-minor) counts the number of ways to place k rooks on this board so that no rook can attack another.

The *rook vector* of the matrix A is the list consisting of $r_0, r_1, \ldots, r_h$, where $h = min(m, n)$. The *rook polynomial* is defined by $r(x) = \sum_{k=0}^{h} r_k x^k$.

The rook vector can be generalized to matrices defined over any rings using permanental minors. Among the available algorithms, only "Godsil" needs the condition on the entries to be either 0 or 1.

See Wikipedia article Rook_polynomial for more information and also the method *permanental_minor()* to compute individual permanental minor.

See also `sage.matrix.matrix2.permanental_minor_polynomial` and the graph method `matching_polynomial`.

INPUT:

- `self` – an m by n matrix

- `algorithm` – a string which must be either "Ryser" or "ButeraPernici" (default) or "Godsil"; Ryser one might be faster on simple and small instances. Godsil only accepts input in 0,1.
- `complement` – boolean (default: `False`) whether we consider the rook vector of the complement matrix. If set to `True` then the matrix must have entries in {0, 1} and the complement matrix is the one for which the 0's are replaced by 1's and 1's by 0's.
- `use_complement` – Boolean (default: `None`) whether to compute the rook vector of a (0,1)-matrix from its complement. By default this is determined by the density of ones in the matrix.

EXAMPLES:

The standard chessboard is an 8 by 8 grid in which any positions is allowed. In that case one gets that the number of ways to position 4 non-attacking rooks is 117600 while for 8 rooks it is 40320:

```
sage: ones_matrix(8,8).rook_vector()
[1, 64, 1568, 18816, 117600, 376320, 564480, 322560, 40320]
```

These numbers are the coefficients of a modified Laguerre polynomial:

```
sage: x = polygen(QQ)
sage: factorial(8) * laguerre(8,-x)
x^8 + 64*x^7 + 1568*x^6 + 18816*x^5 + 117600*x^4 + 376320*x^3 +
564480*x^2 + 322560*x + 40320
```

The number of derangements of length n is the permanent of a matrix with 0 on the diagonal and 1 elsewhere; for $n = 21$ it is 18795307255050944540 (see OEIS sequence A000166):

sage: A = identity_matrix(21) sage: A.rook_vector(complement=True)[-1] 18795307255050944540 sage: Derangements(21).cardinality() 18795307255050944540

An other example that we convert into a rook polynomial:

```
sage: A = matrix(3,6, [1,1,1,1,0,0,0,1,1,1,1,0,0,0,1,1,1,1])
sage: A
[1 1 1 1 0 0]
[0 1 1 1 1 0]
[0 0 1 1 1 1]
sage: A.rook_vector()
[1, 12, 40, 36]

sage: R = PolynomialRing(ZZ, 'x')
sage: R(A.rook_vector())
36*x^3 + 40*x^2 + 12*x + 1
```

Different algorithms are available:

```
sage: A = matrix([[1,0,0,1],[0,1,1,0],[0,1,1,0],[1,0,0,1]])
sage: A.rook_vector(algorithm="ButeraPernici")
[1, 8, 20, 16, 4]
sage: A.rook_vector(algorithm="Ryser")
[1, 8, 20, 16, 4]
sage: A.rook_vector(algorithm="Godsil")
[1, 8, 20, 16, 4]
```

When the matrix A has more ones then zeroes it is usually faster to compute the rook polynomial of the complementary matrix, with zeroes and ones interchanged, and use the inclusion-exclusion theorem, giving

for a $m \times n$ matrix A with complementary matrix B

$$r_k(A) = \sum_{j=0}^{k} (-1)^j \binom{m-j}{k-j} \binom{n-j}{k-j} (k-j)! r_j(B)$$

see [Rio1958] or the introductory text [AS2011]. This can be done setting the argument `use_complement` to `True`.

An example with an exotic matrix (for which only Butera-Pernici and Ryser algorithms are available):

```
sage: R.<x,y> = PolynomialRing(GF(5))
sage: A = matrix(R,[[1,x,y],[x*y,x**2+y,0]])
sage: A.rook_vector(algorithm="ButeraPernici")
[1, x^2 + x*y + x + 2*y + 1, 2*x^2*y + x*y^2 + x^2 + y^2 + y]
sage: A.rook_vector(algorithm="Ryser")
[1, x^2 + x*y + x + 2*y + 1, 2*x^2*y + x*y^2 + x^2 + y^2 + y]
sage: A.rook_vector(algorithm="Godsil")
Traceback (most recent call last):
...
ValueError: coefficients must be zero or one, but we have 'x' in position (0,1).
sage: B = A.transpose()
sage: B.rook_vector(algorithm="ButeraPernici")
[1, x^2 + x*y + x + 2*y + 1, 2*x^2*y + x*y^2 + x^2 + y^2 + y]
sage: B.rook_vector(algorithm="Ryser")
[1, x^2 + x*y + x + 2*y + 1, 2*x^2*y + x*y^2 + x^2 + y^2 + y]
```

row_module(*base_ring=None*)

Return the free module over the base ring spanned by the rows of self.

EXAMPLES:

```
sage: A = MatrixSpace(IntegerRing(), 2)([1,2,3,4])
sage: A.row_module()
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 2]
```

row_space(*base_ring=None*)

Return the row space of this matrix. (Synonym for self.row_module().)

EXAMPLES:

```
sage: t = matrix(QQ, 3, 3, range(9)); t
[0 1 2]
[3 4 5]
[6 7 8]
sage: t.row_space()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
```

```
sage: m = Matrix(Integers(5),2,2,[2,2,2,2])
sage: m.row_space()
```

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```
Vector space of degree 2 and dimension 1 over Ring of integers modulo 5
Basis matrix:
[1 1]
```

rref(**args*, ***kwds*)

Return the reduced row echelon form of the matrix, considered as a matrix over a field.

If the matrix is over a ring, then an equivalent matrix is constructed over the fraction field, and then row reduced.

All arguments are passed on to *echelon_form()*.

Note: Because the matrix is viewed as a matrix over a field, every leading coefficient of the returned matrix will be one and will be the only nonzero entry in its column.

EXAMPLES:

```
sage: A=matrix(3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.rref()
[ 1  0 -1]
[ 0  1  2]
[ 0  0  0]
```

Note that there is a difference between *rref()* and *echelon_form()* when the matrix is not over a field (in this case, the integers instead of the rational numbers):

```
sage: A.base_ring()
Integer Ring
sage: A.echelon_form()
[ 3  0 -3]
[ 0  1  2]
[ 0  0  0]

sage: B = random_matrix(QQ, 3, num_bound=10)
sage: while B.rank() != 3:
....:     B = random_matrix(QQ, 3, num_bound=10)
sage: B.rref()
[1 0 0]
[0 1 0]
[0 0 1]
```

In this case, since B is a matrix over a field (the rational numbers), *rref()* and *echelon_form()* are exactly the same:

```
sage: B.echelon_form()
[1 0 0]
[0 1 0]
[0 0 1]
sage: B.echelon_form() is B.rref()
True
```

Since *echelon_form()* is not implemented for every ring, sometimes behavior varies, as here:

```
sage: R.<x>=ZZ[]
sage: C = matrix(3,[2,x,x^2,x+1,3-x,-1,3,2,1])
sage: C.rref()
[1 0 0]
[0 1 0]
[0 0 1]
sage: C.base_ring()
Univariate Polynomial Ring in x over Integer Ring
sage: C.echelon_form()
Traceback (most recent call last):
...
NotImplementedError: Ideal Ideal (2, x + 1) of Univariate Polynomial Ring in x␣
↪over Integer Ring not principal
Echelon form not implemented over 'Univariate Polynomial Ring in x over Integer␣
↪Ring'.
sage: C = matrix(3,[2,x,x^2,x+1,3-x,-1,3,2,1/2])
sage: C.echelon_form()
[                                2                                 x              ␣
↪                x^2]
[                                0                                 1           ␣
↪15*x^2 - 3/2*x - 31/2]
[                                0                                 0 5/2*x^3 - 15/
↪4*x^2 - 9/4*x + 7/2]
sage: C.rref()
[1 0 0]
[0 1 0]
[0 0 1]
sage: C = matrix(3,[2,x,x^2,x+1,3-x,-1/x,3,2,1/2])
sage: C.echelon_form()
[1 0 0]
[0 1 0]
[0 0 1]
```

set_block(*row*, *col*, *block*)

Sets the sub-matrix of self, with upper left corner given by row, col to block.

EXAMPLES:

```
sage: A = matrix(QQ, 3, 3, range(9))/2
sage: B = matrix(ZZ, 2, 1, [100,200])
sage: A.set_block(0, 1, B)
sage: A
[  0 100   1]
[3/2 200 5/2]
[  3 7/2   4]
```

We test that an exception is raised when the block is out of bounds:

```
sage: matrix([1]).set_block(0,1,matrix([1]))
Traceback (most recent call last):
...
IndexError: matrix window index out of range
```

smith_form(*transformation=True*, *integral=None*, *exact=True*)

Return a Smith normal form of this matrix.

For a matrix M, a Smith normal form is a matrix $S = UMV$ such that:

- U and V are invertible matrices
- the only non-vanishing entries of S are located on the diagonal (though S might not be a square matrix)
- if d_i denotes the entry of S at (i, i), then d_i divides d_{i+1} for all i, i.e., the d_i are the ordered *elementary_divisors()* of M

Note that the matrices U and V are not uniquely determined and the d_i are only uniquely determined up to units. For some base rings, such as local rings, the d_i might be further normalized, see `LOCAL RINGS` below.

If the base ring is not a PID, the routine might work, or else it will fail having found an example of a non-principal ideal. Note that we do not call any methods to check whether or not the base ring is a PID, since this might be quite expensive (e.g. for rings of integers of number fields of large degree).

INPUT:

- `transformation` – a boolean (default: `True`); whether the matrices U and V should be returned
- `integral` – a subring of the base ring, boolean or `None` (default: `None`); the entries of U and V are taken from this subring. If `True`, the ring is taken to be the ring of integers of the base ring; if `False` the fraction field of the base ring; if `None` the base ring itself. When a subring is specified, multiplying by the denominator must map the entries into the subring; in this case the transformation matrices will have entries in this subring.
- `exact` – a boolean (default: `True`), only used for local rings/fields. See `LOCAL RINGS` for more details.

OUTPUT:

The matrices S, U, V or the matrix S depending on `transformation`.

ALGORITHM:

If the base ring has a method `_matrix_smith_form`, use it; note that `_matrix_smith_form` might choose to further normalize the output.

Otherwise, use the algorithm from Wikipedia article Smith_normal_form.

LOCAL RINGS:

Over local rings, we normalize S to only contain powers of the uniformizer.

In order to simplify the precision handling, we truncate the absolute precision of the input matrix to the minimum absolute precision of any of its entries. As long as all of the elementary divisors are nonzero modulo this precision, they can be determined exactly since they are defined to be powers of the uniformizer. In this case, which is specified by the keyword `exact=True`, one of the transformation matrices will be inexact: U in the case that the number of rows is at least the number of columns, and V otherwise.

If `exact=False`, we instead return an inexact Smith form. Now the transformation matrices are exact and we can deal gracefully with elementary divisors that are zero modulo the working precision. However, the off-diagonal entries of the smith form carry a precision that can affect the precision of future calculations.

See `_matrix_smith_form` on the base ring for more detail.

EXAMPLES:

An example over the ring of integers of a number field (of class number 1):

```
sage: OE.<w> = EquationOrder(x^2 - x + 2)
sage: m = Matrix([ [1, w],[w,7]])
sage: d, u, v = m.smith_form()
sage: (d, u, v)
(
[     1      0]  [ 1  0]  [ 1 -w]
[     0 -w + 9], [-w  1], [ 0  1]
)
sage: u * m * v == d
True
sage: u.base_ring() == v.base_ring() == d.base_ring() == OE
True
sage: u.det().is_unit() and v.det().is_unit()
True
```

An example over the polynomial ring QQ[x]:

```
sage: R.<x> = QQ[]; m=x*matrix(R,2,2,1) - matrix(R, 2,2,[3,-4,1,-1]); m.smith_
↪form()
(
[            1             0]  [    0    -1]  [    1 x + 1]
[            0 x^2 - 2*x + 1], [    1 x - 3], [    0     1]
)
```

An example over a field:

```
sage: m = matrix( GF(17), 3, 3, [11,5,1,3,6,8,1,16,0]); d,u,v = m.smith_form()
sage: d
[1 0 0]
[0 1 0]
[0 0 0]
sage: u*m*v == d
True
```

When the base ring has a `ring_of_integers` method and supports denominators, you can get an integral version of the smith form:

```
sage: m = matrix(QQ, 2, 2, [17/6, 47/6, 25/6, 23/2])
sage: m.smith_form()
(
[1 0]  [6/17    0]  [     1 -47/17]
[0 1], [  75  -51], [     0      1]
)
sage: m.smith_form(integral=True)
(
[1/6   0]  [  3  -2]  [ 1  3]
[  0 1/3], [-25  17], [ 0 -1]
)
```

Some examples over non-PID's work anyway:

```
sage: R.<s> = EquationOrder(x^2 + 5) # class number 2
sage: A = matrix(R, 2, 2, [s-1,-s,-s,2*s+1])
```

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```
sage: D, U, V = A.smith_form()
sage: D, U, V
(
[     1      0]  [    4 s + 4]  [       1 -5*s + 6]
[     0 -s - 6], [    s s - 1], [       0        1]
)
sage: D == U*A*V
True
```

Others don't, but they fail quite constructively:

```
sage: matrix(R,2,2,[s-1,-s-2,-2*s,-s-2]).smith_form()
Traceback (most recent call last):
...
ArithmeticError: Ideal Fractional ideal (2, s + 1) not principal
```

Empty matrices are handled safely:

```
sage: m = MatrixSpace(OE, 2,0)(0); d,u,v=m.smith_form(); u*m*v == d
True
sage: m = MatrixSpace(OE, 0,2)(0); d,u,v=m.smith_form(); u*m*v == d
True
sage: m = MatrixSpace(OE, 0,0)(0); d,u,v=m.smith_form(); u*m*v == d
True
```

Some pathological cases that crashed earlier versions:

```
sage: m = Matrix(OE, [[2*w,2*w-1,-w+1],[2*w+2,-2*w-1,w-1],[-2*w-1,-2*w-2,2*w-
↪1]]); d, u, v = m.smith_form(); u * m * v == d
True
sage: m = matrix(OE, 3, 3, [-5*w-1,-2*w-2,4*w-10,8*w,-w,w-1,-1,1,-8]); d,u,v =␣
↪m.smith_form(); u*m*v == d
True
```

Over local fields, we can request the transformation matrices to be integral:;

sage: K = Qp(2, 5, print_mode='terse') sage: M = matrix(K, 2, 3, [1/2, 1, 2, 1/3, 1, 3]) sage: M.smith_form(integral=True) ([1/2 + O(2^4) 0 0] [1 + O(2^5) 0] [0 1 + O(2^5) 0], [42 + O(2^6) 1 + O(2^5)], <BLANKLINE> [1 + O(2^5) 26 + O(2^5) 6 + O(2^5)] [O(2^4) 3 + O(2^4) 11 + O(2^4)] [0 0 1 + O(2^5)])

solve_left(*B*, *check=True*)

Try to find a solution X to the equation $XA = B$.

If `self` is a matrix A, then this function returns a vector or matrix X such that $XA = B$. If B is a vector then X is a vector and if B is a matrix, then X is a matrix.

Over inexact rings, the output of this function may not be an exact solution. For example, over the real or complex double field, this method computes a least-squares solution if the system is not square.

Note: In Sage one can also write `B / A` for `A.solve_left(B)`, that is, Sage implements "the MATLAB/Octave slash operator".

INPUT:

- B – a matrix or vector
- check – boolean (default: True); verify the answer if the system is non-square or rank-deficient, and if its entries lie in an exact ring. Meaningless over most inexact rings, or when the system is square and of full rank.

OUTPUT:

If the system is square and has full rank, the unique solution is returned, and no check is done on the answer. Over inexact rings, you should expect this answer to be inexact. Moreover, due to the numerical issues involved, an error may be thrown in this case – specifically if the system is singular but if SageMath fails to notice that.

If the system is not square or does not have full rank, then a solution is attempted via other means. For example, over RDF or CDF a least-squares solution is returned, as with MATLAB's "backslash" operator. For most inexact rings, the check parameter is ignored because an approximate solution will be returned in any case. Over exact rings, on the other hand, setting the check parameter results in an additional test to determine whether or not the answer actually solves the system exactly. If a symbolic system involves only exact elements, its solution can still be checked.

If B is a vector, the result is returned as a vector, as well, and as a matrix, otherwise.

See also:

solve_right()

EXAMPLES:

```
sage: A = matrix(QQ,4,2, [0, -1, 1, 0, -2, 2, 1, 0])
sage: B = matrix(QQ,2,2, [1, 0, 1, -1])
sage: X = A.solve_left(B)
sage: X*A == B
True
sage: X == B / A
True
```

```
sage: A = matrix([(3, -1, 0, 0), (1, 1, -2, 0), (0, 0, 0, -3)])
sage: B = matrix(QQ, 3, 1, [0, 0, -1])
sage: A.solve_left(B)
Traceback (most recent call last):
...
ValueError: number of columns of self must equal number of columns
of right-hand side
```

Over the reals:

```
sage: A = matrix(RDF, 3,3, [1,2,5,7.6,2.3,1,1,2,-1]); A
[ 1.0  2.0  5.0]
[ 7.6  2.3  1.0]
[ 1.0  2.0 -1.0]
sage: b = vector(RDF,[1,2,3])
sage: x = A.solve_left(b); x.zero_at(2e-17) # fix noisy zeroes
(0.666666666..., 0.0, 0.333333333...)
sage: x.parent()
Vector space of dimension 3 over Real Double Field
sage: x*A  # tol 1e-14
(0.9999999999999999, 1.9999999999999998, 3.0)
```

Over the complex numbers:

```
sage: A = matrix(CDF, [[      0, -1 + 2*I,  1 - 3*I,        I],
....:                  [2 + 4*I, -2 + 3*I, -1 + 2*I,   -1 - I],
....:                  [  2 + I,    1 - I,       -1,        5],
....:                  [    3*I,   -1 - I,   -1 + I,   -3 + I]])
sage: b = vector(CDF, [2 -3*I, 3, -2 + 3*I, 8])
sage: x = A.solve_left(b); x
(-1.55765124... - 0.644483985...*I, 0.183274021... + 0.286476868...*I, 0.
↪270818505... + 0.246619217...*I, -1.69003558... - 0.828113879...*I)
sage: x.parent()
Vector space of dimension 4 over Complex Double Field
sage: abs(x*A - b) < 1e-14
True
```

If `b` is given as a matrix, the result will be a matrix, as well:

```
sage: A = matrix(RDF, 3, 3, [2, 5, 0, 7, 7, -2, -4.3, 0, 1])
sage: b = matrix(RDF, 2, 3, [2, -4, -5, 1, 1, 0.1])
sage: A.solve_left(b) # tol 1e-14
[  -6.495454545454545    4.068181818181818   3.1363636363636354]
[  0.5277272727272727  -0.2340909090909091 -0.36818181818181817]
```

If A is a non-square matrix, the result is a least-squares solution. For a tall matrix, this may give a solution with a least-squares error of almost zero:

```
sage: A = matrix(RDF, 3, 2, [1, 3, 4, 2, 0, -3])
sage: b = vector(RDF, [5, 6])
sage: x = A.solve_left(b)
sage: (x * A - b).norm() < 1e-14
True
```

For a wide matrix A, the error is usually not small:

```
sage: A = matrix(RDF, 2, 3, [1, 3, 4, 2, 0, -3])
sage: b = vector(RDF, [5, 6, 1])
sage: x = A.solve_left(b)
sage: (x * A - b).norm()  # tol 1e-14
0.9723055853282466
```

solve_right(*B*, *check=True*)

Try to find a solution X to the equation $AX = B$.

If `self` is a matrix A, then this function returns a vector or matrix X such that $AX = B$. If B is a vector then X is a vector and if B is a matrix, then X is a matrix.

Over inexact rings, the output of this function may not be an exact solution. For example, over the real or complex double field, this method computes a least-squares solution if the system is not square.

Note: In Sage one can also write `A \ B` for `A.solve_right(B)`, that is, Sage implements "the MATLAB/Octave backslash operator".

INPUT:

- `B` – a matrix or vector

- check – boolean (default: True); verify the answer if the system is non-square or rank-deficient, and if its entries lie in an exact ring. Meaningless over most inexact rings, or when the system is square and of full rank.

OUTPUT:

If the system is square and has full rank, the unique solution is returned, and no check is done on the answer. Over inexact rings, you should expect this answer to be inexact. Moreover, due to the numerical issues involved, an error may be thrown in this case – specifically if the system is singular but if SageMath fails to notice that.

If the system is not square or does not have full rank, then a solution is attempted via other means. For example, over RDF or CDF a least-squares solution is returned, as with MATLAB's "backslash" operator. For most inexact rings, the check parameter is ignored because an approximate solution will be returned in any case. Over exact rings, on the other hand, setting the check parameter results in an additional test to determine whether or not the answer actually solves the system exactly. If a symbolic system involves only exact elements, its solution can still be checked.

If B is a vector, the result is returned as a vector, as well, and as a matrix, otherwise.

See also:

solve_left()

EXAMPLES:

```
sage: A = matrix(QQ, 3, [1,2,3,-1,2,5,2,3,1])
sage: b = vector(QQ,[1,2,3])
sage: x = A \ b; x
(-13/12, 23/12, -7/12)
sage: A * x
(1, 2, 3)
```

We solve with A nonsquare:

```
sage: A = matrix(QQ,2,4, [0, -1, 1, 0, -2, 2, 1, 0]); B = matrix(QQ,2,2, [1, 0,␣
↪1, -1])
sage: X = A.solve_right(B); X
[-3/2  1/2]
[  -1    0]
[   0    0]
[   0    0]
sage: A*X == B
True
```

Another nonsingular example:

```
sage: A = matrix(QQ,2,3, [1,2,3,2,4,6]); v = vector([-1/2,-1])
sage: x = A \ v; x
(-1/2, 0, 0)
sage: A*x == v
True
```

Same example but over **Z**:

```
sage: A = matrix(ZZ,2,3, [1,2,3,2,4,6]); v = vector([-1,-2])
sage: A \ v
(-1, 0, 0)
```

An example in which there is no solution:

```
sage: A = matrix(QQ,2,3, [1,2,3,2,4,6]); v = vector([1,1])
sage: A \ v
Traceback (most recent call last):
...
ValueError: matrix equation has no solutions
```

A ValueError is raised if the input is invalid:

```
sage: A = matrix(QQ,4,2, [0, -1, 1, 0, -2, 2, 1, 0])
sage: B = matrix(QQ,2,2, [1, 0, 1, -1])
sage: X = A.solve_right(B)
Traceback (most recent call last):
...
ValueError: number of rows of self must equal number of rows of
right-hand side
```

We solve with A singular:

```
sage: A = matrix(QQ,2,3, [1,2,3,2,4,6]); B = matrix(QQ,2,2, [6, -6, 12, -12])
sage: X = A.solve_right(B); X
[ 6 -6]
[ 0  0]
[ 0  0]
sage: A*X == B
True
```

We illustrate left associativity, etc., of the backslash operator.

```
sage: A = matrix(QQ, 2, [1,2,3,4])
sage: A \ A
[1 0]
[0 1]
sage: A \ A \ A
[1 2]
[3 4]
sage: A.parent()(1) \ A
[1 2]
[3 4]
sage: A \ (A \ A)
[  -2    1]
[ 3/2 -1/2]
sage: X = A \ (A - 2); X
[ 5 -2]
[-3  2]
sage: A * X
[-1  2]
[ 3  2]
```

Solving over a polynomial ring:

```
sage: x = polygen(QQ, 'x')
sage: A = matrix(2, [x,2*x,-5*x^2+1,3])
```

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```
sage: v = vector([3,4*x - 2])
sage: X = A \ v
sage: X
((-4/5*x^2 + 2/5*x + 9/10)/(x^3 + 1/10*x), (19/10*x^2 - 1/5*x - 3/10)/(x^3 + 1/
↪10*x))
sage: A * X == v
True
```

Solving some systems over $\mathbf{Z}/n\mathbf{Z}$:

```
sage: A = Matrix(Zmod(6), 3, 2, [1,2,3,4,5,6])
sage: B = vector(Zmod(6), [1,1,1])
sage: A.solve_right(B)
(5, 1)
sage: B = vector(Zmod(6), [5,1,1])
sage: A.solve_right(B)
Traceback (most recent call last):
...
ValueError: matrix equation has no solutions
sage: A = Matrix(Zmod(128), 2, 3, [23,11,22,4,1,0])
sage: B = Matrix(Zmod(128), 2, 1, [1,0])
sage: A.solve_right(B)
[  5]
[108]
[127]
sage: B = B.column(0)
sage: A.solve_right(B)
(5, 108, 127)
sage: A = Matrix(Zmod(15), 3,4, range(12))
sage: B = Matrix(Zmod(15), 3,3, range(3,12))
sage: X = A.solve_right(B)
sage: A*X == B
True
```

Solving a system over the p-adics:

```
sage: k = Qp(5,4)
sage: a = matrix(k, 3, [1,7,3,2,5,4,1,1,2]); a
[    1 + O(5^4) 2 + 5 + O(5^4)     3 + O(5^4)]
[    2 + O(5^4)     5 + O(5^5)     4 + O(5^4)]
[    1 + O(5^4)     1 + O(5^4)     2 + O(5^4)]
sage: v = vector(k, 3, [1,2,3])
sage: x = a \ v; x
(4 + 5 + 5^2 + 3*5^3 + O(5^4), 2 + 5 + 3*5^2 + 5^3 + O(5^4), 1 + 5 + O(5^4))
sage: a * x == v
True
```

Solving a system of linear equations symbolically using symbolic matrices:

```
sage: var('a,b,c,d,x,y')                                    # optional -␣
↪sage.symbolic
(a, b, c, d, x, y)
sage: A = matrix(SR, 2, [a,b,c,d]); A                       # optional -␣
```

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```
↪sage.symbolic
[a b]
[c d]
sage: result = vector(SR, [3,5]); result                          # optional -␣
↪sage.symbolic
(3, 5)
sage: soln = A.solve_right(result); soln                          # optional -␣
↪sage.symbolic
(-b*(3*c/a - 5)/(a*(b*c/a - d)) + 3/a, (3*c/a - 5)/(b*c/a - d))
sage: (a*x+b*y).subs(x=soln[0], y=soln[1]).simplify_full()        # optional -␣
↪sage.symbolic
3
sage: (c*x+d*y).subs(x=soln[0], y=soln[1]).simplify_full()        # optional -␣
↪sage.symbolic
5
sage: (A*soln).apply_map(lambda x: x.simplify_full())             # optional -␣
↪sage.symbolic
(3, 5)
```

Over inexact rings, the output of this function may not be an exact solution. For example, over the real or complex double field, this computes a least-squares solution:

```
sage: A = matrix(RDF, 3, 2, [1, 3, 4, 2, 0, -3])
sage: b = vector(RDF, [5, 6, 1])
sage: A.solve_right(b)  # tol 1e-14
(1.4782608695652177, 0.35177865612648235)
sage: ~(A.T * A) * A.T * b  # closed form solution, tol 1e-14
(1.4782608695652177, 0.35177865612648235)
```

Over the reals:

```
sage: A = matrix(RDF, 3,3, [1,2,5,7.6,2.3,1,1,2,-1]); A
[ 1.0  2.0  5.0]
[ 7.6  2.3  1.0]
[ 1.0  2.0 -1.0]
sage: b = vector(RDF,[1,2,3])
sage: x = A.solve_right(b); x  # tol 1e-14
(-0.1136950904392765, 1.3901808785529717, -0.33333333333333337)
sage: x.parent()
Vector space of dimension 3 over Real Double Field
sage: A*x  # tol 1e-14
(1.0, 1.9999999999999996, 3.0000000000000004)
```

Over the complex numbers:

```
sage: A = matrix(CDF, [[      0, -1 + 2*I,  1 - 3*I,         I],
....:                  [2 + 4*I, -2 + 3*I, -1 + 2*I,   -1 - I],
....:                  [  2 + I,     1 - I,        -1,        5],
....:                  [    3*I,   -1 - I,   -1 + I,   -3 + I]])
sage: b = vector(CDF, [2 -3*I, 3, -2 + 3*I, 8])
sage: x = A.solve_right(b); x
(1.96841637... - 1.07606761...*I, -0.614323843... + 1.68416370...*I, 0.
↪0733985765... + 1.73487544...*I, -1.6018683... + 0.524021352...*I)
```

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```
sage: x.parent()
Vector space of dimension 4 over Complex Double Field
sage: abs(A*x - b) < 1e-14
True
```

If b is given as a matrix, the result will be a matrix, as well:

```
sage: A = matrix(RDF, 3, 3, [1, 2, 2, 3, 4, 5, 2, 2, 2])
sage: b = matrix(RDF, 3, 2, [3, 2, 3, 2, 3, 2])
sage: A.solve_right(b) # tol 1e-14
[ 0.0  0.0]
[ 4.5  3.0]
[-3.0 -2.0]
```

If A is a non-square matrix, the result is a least-squares solution. For a wide matrix, this may give a solution with a least-squares error of almost zero:

```
sage: A = matrix(RDF, 2, 3, [1, 3, 4, 2, 0, -3])
sage: b = vector(RDF, [5, 6])
sage: x = A.solve_right(b)
sage: (A * x - b).norm() < 1e-14
True
```

For a tall matrix A, the error is usually not small:

```
sage: A = matrix(RDF, 3, 2, [1, 3, 4, 2, 0, -3])
sage: b = vector(RDF, [5, 6, 1])
sage: x = A.solve_right(b)
sage: (A * x - b).norm()  # tol 1e-14
3.2692119900020438
```

subdivide(*row_lines=None*, *col_lines=None*)

Divides `self` into logical submatrices which can then be queried and extracted.

If a subdivision already exists, this method forgets the previous subdivision and flushes the cache.

INPUT:

- `row_lines` – `None`, an integer, or a list of integers (lines at which self must be split)
- `col_lines` – `None`, an integer, or a list of integers (columns at which self must be split)

OUTPUT: `None` but changes `self`

Note: One may also pass a tuple into the first argument which will be interpreted as `(row_lines, col_lines)`.

EXAMPLES:

```
sage: M = matrix(5, 5, prime_range(100))
sage: M.subdivide(2,3); M
[ 2  3  5| 7 11]
[13 17 19|23 29]
[--------+-----]
```

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```
[31 37 41|43 47]
[53 59 61|67 71]
[73 79 83|89 97]
sage: M.subdivision(0,0)
[ 2  3  5]
[13 17 19]
sage: M.subdivision(1,0)
[31 37 41]
[53 59 61]
[73 79 83]
sage: M.subdivision_entry(1,0,0,0)
31
sage: M.subdivisions()
([2], [3])
sage: M.subdivide(None, [1,3]); M
[ 2| 3  5| 7 11]
[13|17 19|23 29]
[31|37 41|43 47]
[53|59 61|67 71]
[73|79 83|89 97]
```

Degenerate cases work too:

```
sage: M.subdivide([2,5], [0,1,3]); M
[| 2| 3  5| 7 11]
[|13|17 19|23 29]
[+--+-----+-----]
[|31|37 41|43 47]
[|53|59 61|67 71]
[|73|79 83|89 97]
[+--+-----+-----]
sage: M.subdivision(0,0)
[]
sage: M.subdivision(0,1)
[ 2]
[13]
sage: M.subdivide([2,2,3], [0,0,1,1]); M
[|| 2|| 3  5  7 11]
[||13||17 19 23 29]
[++--++-----------]
[++--++-----------]
[||31||37 41 43 47]
[++--++-----------]
[||53||59 61 67 71]
[||73||79 83 89 97]
sage: M.subdivision(0,0)
[]
sage: M.subdivision(2,4)
[37 41 43 47]
```

Indices do not need to be in the right order (github issue #14064):

```
sage: M.subdivide([4, 2], [3, 1]); M
[ 2| 3  5| 7 11]
[13|17 19|23 29]
[--+-----+-----]
[31|37 41|43 47]
[53|59 61|67 71]
[--+-----+-----]
[73|79 83|89 97]
```

subdivision(*i*, *j*)

Returns an immutable copy of the (i,j)th submatrix of self, according to a previously set subdivision.

Before a subdivision is set, the only valid arguments are (0,0) which returns self.

EXAMPLES:

```
sage: M = matrix(3, 4, range(12))
sage: M.subdivide(1,2); M
[ 0  1| 2  3]
[-----+-----]
[ 4  5| 6  7]
[ 8  9|10 11]
sage: M.subdivision(0,0)
[0 1]
sage: M.subdivision(0,1)
[2 3]
sage: M.subdivision(1,0)
[4 5]
[8 9]
```

It handles size-zero subdivisions as well.

```
sage: M = matrix(3, 4, range(12))
sage: M.subdivide([0],[0,2,2,4]); M
[+-----++-----+]
[| 0  1|| 2  3|]
[| 4  5|| 6  7|]
[| 8  9||10 11|]
sage: M.subdivision(0,0)
[]
sage: M.subdivision(1,1)
[0 1]
[4 5]
[8 9]
sage: M.subdivision(1,2)
[]
sage: M.subdivision(1,0)
[]
sage: M.subdivision(0,1)
[]
```

subdivision_entry(*i*, *j*, *x*, *y*)

Returns the x,y entry of the i,j submatrix of self.

EXAMPLES:

```
sage: M = matrix(5, 5, range(25))
sage: M.subdivide(3,3); M
[ 0  1  2| 3  4]
[ 5  6  7| 8  9]
[10 11 12|13 14]
[--------+-----]
[15 16 17|18 19]
[20 21 22|23 24]
sage: M.subdivision_entry(0,0,1,2)
7
sage: M.subdivision(0,0)[1,2]
7
sage: M.subdivision_entry(0,1,0,0)
3
sage: M.subdivision_entry(1,0,0,0)
15
sage: M.subdivision_entry(1,1,1,1)
24
```

Even though this entry exists in the matrix, the index is invalid for the submatrix.

```
sage: M.subdivision_entry(0,0,4,0)
Traceback (most recent call last):
...
IndexError: Submatrix 0,0 has no entry 4,0
```

subdivisions()

Returns the current subdivision of self.

EXAMPLES:

```
sage: M = matrix(5, 5, range(25))
sage: M.subdivisions()
([], [])
sage: M.subdivide(2,3)
sage: M.subdivisions()
([2], [3])
sage: N = M.parent()(1)
sage: N.subdivide(M.subdivisions()); N
[1 0 0|0 0]
[0 1 0|0 0]
[-----+---]
[0 0 1|0 0]
[0 0 0|1 0]
[0 0 0|0 1]
```

subs(**args*, ***kwds*)

Substitute values to the variables in that matrix.

All the arguments are transmitted unchanged to the method `subs` of the coefficients.

EXAMPLES:

```
sage: var('a,b,d,e')
(a, b, d, e)
sage: m = matrix([[a,b], [d,e]])
sage: m.substitute(a=1)
[1 b]
[d e]
sage: m.subs(a=b, b=d)
[b d]
[d e]
sage: m.subs({a: 3, b:2, d:1, e:-1})
[ 3  2]
[ 1 -1]
```

The parent of the newly created matrix might be different from the initial one. It depends on what the method `.subs` does on coefficients (see github issue #19045):

```
sage: x = polygen(ZZ)
sage: m = matrix([[x]])
sage: m2 = m.subs(x=2)
sage: m2.parent()
Full MatrixSpace of 1 by 1 dense matrices over Integer Ring
sage: m1 = m.subs(x=RDF(1))
sage: m1.parent()
Full MatrixSpace of 1 by 1 dense matrices over Real Double Field
```

However, sparse matrices remain sparse:

```
sage: m = matrix({(3,2): -x, (59,38): x^2+2}, nrows=1000, ncols=1000)
sage: m1 = m.subs(x=1)
sage: m1.is_sparse()
True
```

symplectic_form()

Find a symplectic form for self if self is an anti-symmetric, alternating matrix defined over a field.

Returns a pair (F, C) such that the rows of C form a symplectic basis for self and F = C * self * C.transpose().

Raises a ValueError if not over a field, or self is not anti-symmetric, or self is not alternating.

Anti-symmetric means that $M = -M^t$. Alternating means that the diagonal of M is identically zero.

A symplectic basis is a basis of the form $e_1, \ldots, e_j, f_1, \ldots f_j, z_1, \ldots, z_k$ such that

- $z_i M v^t = 0$ for all vectors v
- $e_i M {e_j}^t = 0$ for all i, j
- $f_i M {f_j}^t = 0$ for all i, j
- $e_i M {f_i}^t = 1$ for all i
- $e_i M {f_j}^t = 0$ for all i not equal j.

See the example for a pictorial description of such a basis.

EXAMPLES:

```
sage: E = matrix(QQ, 8, 8, [0, -1/2, -2, 1/2, 2, 0, -2, 1, 1/2, 0, -1, -3, 0, 2,
↪ 5/2, -3, 2, 1, 0, 3/2, -1, 0, -1, -2, -1/2, 3, -3/2, 0, 1, 3/2, -1/2, -1/2, -
↪2, 0, 1, -1, 0, 0, 1, -1, 0, -2, 0, -3/2, 0, 0, 1/2, -2, 2, -5/2, 1, 1/2, -1,␣
↪-1/2, 0, -1, -1, 3, 2, 1/2, 1, 2, 1, 0]); E
[   0 -1/2   -2  1/2    2    0   -2    1]
[ 1/2    0   -1   -3    0    2  5/2   -3]
[   2    1    0  3/2   -1    0   -1   -2]
[-1/2    3 -3/2    0    1  3/2 -1/2 -1/2]
[  -2    0    1   -1    0    0    1   -1]
[   0   -2    0 -3/2    0    0  1/2   -2]
[   2 -5/2    1  1/2   -1 -1/2    0   -1]
[  -1    3    2  1/2    1    2    1    0]
sage: F, C = E.symplectic_form(); F
[ 0  0  0  0  1  0  0  0]
[ 0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  1  0]
[ 0  0  0  0  0  0  0  1]
[-1  0  0  0  0  0  0  0]
[ 0 -1  0  0  0  0  0  0]
[ 0  0 -1  0  0  0  0  0]
[ 0  0  0 -1  0  0  0  0]
sage: F == C * E * C.transpose()
True
```

tensor_product(*A*, *subdivide=True*)

Returns the tensor product of two matrices.

INPUT:

- `A` - a matrix
- `subdivide` - default: True - whether or not to return natural subdivisions with the matrix

OUTPUT:

Replace each element of `self` by a copy of `A`, but first create a scalar multiple of `A` by the element it replaces. So if `self` is an $m \times n$ matrix and `A` is a $p \times q$ matrix, then the tensor product is an $mp \times nq$ matrix. By default, the matrix will be subdivided into submatrices of size $p \times q$.

EXAMPLES:

```
sage: M1=Matrix(QQ,[[-1,0],[-1/2,-1]])
sage: M2=Matrix(ZZ,[[1,-1,2],[-2,4,8]])
sage: M1.tensor_product(M2)
[  -1    1   -2|   0    0    0]
[   2   -4   -8|   0    0    0]
[--------------+--------------]
[-1/2  1/2   -1|  -1    1   -2]
[   1   -2   -4|   2   -4   -8]
sage: M2.tensor_product(M1)
[  -1    0|   1    0|  -2    0]
[-1/2   -1| 1/2    1|  -1   -2]
[---------+---------+---------]
[   2    0|  -4    0|  -8    0]
[   1    2|  -2   -4|  -4   -8]
```

Subdivisions can be optionally suppressed.

```
sage: M1.tensor_product(M2, subdivide=False)
[  -1    1   -2    0    0    0]
[   2   -4   -8    0    0    0]
[-1/2  1/2   -1   -1    1   -2]
[   1   -2   -4    2   -4   -8]
```

Different base rings are handled sensibly.

```
sage: A = matrix(ZZ, 2, 3, range(6))
sage: B = matrix(FiniteField(23), 3, 4, range(12))
sage: C = matrix(FiniteField(29), 4, 5, range(20))
sage: D = A.tensor_product(B)
sage: D.parent()
Full MatrixSpace of 6 by 12 dense matrices over Finite Field of size 23
sage: E = C.tensor_product(B)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'Finite Field of size 29' and
↪'Full MatrixSpace of 3 by 4 dense matrices over Finite Field of size 23'
```

The input is checked to be sure it is a matrix.

```
sage: A = matrix(QQ, 2, 2, range(4))
sage: A.tensor_product('junk')
Traceback (most recent call last):
...
TypeError: tensor product requires a second matrix, not junk
```

trace()

Return the trace of self, which is the sum of the diagonal entries of self.

INPUT:

- `self` - a square matrix

OUTPUT: element of the base ring of self

EXAMPLES:

```
sage: a = matrix(3,3,range(9)); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: a.trace()
12
sage: a = matrix({(1,1):10, (2,1):-3, (2,2):4/3}); a
[  0   0   0]
[  0  10   0]
[  0  -3 4/3]
sage: a.trace()
34/3
```

trace_of_product(*other*)

Returns the trace of self * other without computing the entire product.

EXAMPLES:

```
sage: M = random_matrix(ZZ, 10, 20)
sage: N = random_matrix(ZZ, 20, 10)
sage: M.trace_of_product(N) == (M*N).trace()
True
```

visualize_structure(*maxsize=512*)

Visualize the non-zero entries

White pixels are put at positions with zero entries. If 'maxsize' is given, then the maximal dimension in either x or y direction is set to 'maxsize' depending on which is bigger. If the image is scaled, the darkness of the pixel reflects how many of the represented entries are nonzero. So if e.g. one image pixel actually represents a 2x2 submatrix, the dot is darker the more of the four values are nonzero.

INPUT:

- `maxsize` - integer (default: `512`). Maximal dimension in either x or y direction of the resulting image. If `None` or a maxsize larger than `max(self.nrows(),self.ncols())` is given the image will have the same pixelsize as the matrix dimensions.

OUTPUT:

Bitmap image as an instance of `Image`.

EXAMPLES:

```
sage: M = random_matrix(CC, 5, 7)
sage: for i in range(5):  M[i,i] = 0
sage: M[4, 0] = M[0, 6] = M[4, 6] = 0
sage: img = M.visualize_structure();  img
7x5px 24-bit RGB image
```

You can use `save()` to save the resulting image:

```
sage: filename = tmp_filename(ext='.png')
sage: img.save(filename)
sage: with open(filename, 'rb') as fobj:
....:     fobj.read().startswith(b'\x89PNG')
True
```

wiedemann(*i*, *t=0*)

Application of Wiedemann's algorithm to the i-th standard basis vector.

INPUT:

- `i` - an integer
- `t` - an integer (default: 0) if t is nonzero, use only the first t linear recurrence relations.

IMPLEMENTATION: This is a toy implementation.

EXAMPLES:

```
sage: t = matrix(QQ, 3, 3, range(9)); t
[0 1 2]
[3 4 5]
[6 7 8]
sage: t.wiedemann(0)
```

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```
x^2 - 12*x - 18
sage: t.charpoly()
x^3 - 12*x^2 - 18*x
```

zigzag_form(*subdivide=True*, *transformation=False*)

Find a matrix in ZigZag form that is similar to `self`.

INPUT:

- `self` - a square matrix with entries from an exact field.
- `transformation` - default: False - if `True` return a change-of-basis matrix relating the matrix and its ZigZag form.
- `subdivide` - default: `True` - if `True` the ZigZag form matrix is subdivided according to the companion matrices described in the output section below.

OUTPUT:

A matrix in ZigZag form has blocks on the main diagonal that are companion matrices. The first companion matrix has ones just below the main diagonal. The last column has the negatives of coefficients of a monic polynomial, but not the leading one. Low degree monomials have their coefficients in the earlier rows. The second companion matrix is like the first only transposed. The third is like the first. The fourth is like the second. And so on.

These blocks on the main diagonal define blocks just off the diagonal. To the right of the first companion matrix, and above the second companion matrix is a block that is totally zero, except the entry of the first row and first column may be a one. Below the second block and to the left of the third block is a block that is totally zero, except the entry of the first row and first column may be one. This alternating pattern continues. It may now be apparent how this form gets its name. Any other entry of the matrix is zero. So this form is reminiscent of rational canonical form and is a good precursor to that form.

If `transformation` is `True`, then the output is a pair of matrices. The first is the form `Z` and the second is an invertible matrix `U` such that `U.inverse()*self*U` equals `Z`. In other words, the representation of `self` with respect to the columns of `U` will be `Z`.

If subdivide is `True` then the matrix returned as the form is partitioned according to the companion matrices and these may be manipulated by several different matrix methods.

For output that may be more useful as input to other routines, see the helper method `_zigzag_form()`.

Note: An effort has been made to optimize computation of the form, but no such work has been done for the computation of the transformation matrix, so for fastest results do not request the transformation matrix.

ALGORITHM:

ZigZag form, and its computation, are due to Arne Storjohann and are described in [Sto2000] and [Sto1998], where the former is more representative of the code here.

EXAMPLES:

Two examples that illustrate ZigZag form well. Notice that this is *not* a canonical form. The two matrices below are similar, since they have equal Jordan canonical forms, yet their ZigZag forms are quite different. In other words, while the computation of the form is deterministic, the final result, when viewed as a property of a linear transformation, is dependent on the basis used for the matrix representation.

```
sage: A = matrix(QQ, [[-68,   69, -27, -11, -65,   9, -181, -32],
....:                 [-52,   52, -27,  -8, -52, -16, -133, -14],
....:                 [ 92,  -97,  47,  14,  90,  32,  241,  18],
....:                 [139, -144,  60,  18, 148, -10,  362,  77],
....:                 [ 40,  -41,  12,   6,  45, -24,  105,  42],
....:                 [-46,   48, -20,  -7, -47,   0, -122, -22],
....:                 [-26,   27, -13,  -4, -29,  -6,  -66, -14],
....:                 [-33,   34, -13,  -5, -35,   7,  -87, -23]])
sage: Z, U = A.zigzag_form(transformation=True)
sage: Z
[  0   0   0  40|  1   0|  0   0]
[  1   0   0  52|  0   0|  0   0]
[  0   1   0  18|  0   0|  0   0]
[  0   0   1  -1|  0   0|  0   0]
[---------------+-------+-------]
[  0   0   0   0|  0   1|  0   0]
[  0   0   0   0|-25  10|  0   0]
[---------------+-------+-------]
[  0   0   0   0|  1   0|  0  -4]
[  0   0   0   0|  0   0|  1  -4]
sage: U.inverse()*A*U == Z
True

sage: B = matrix(QQ, [[ 16,  69, -13,   2, -52,  143,   90,  -3],
....:                 [ 26,  54,   6,  -5, -28,   73,   73, -48],
....:                 [-16, -79,  12, -10,  64, -142, -115,  41],
....:                 [ 27,  -7,  21, -33,  39,  -20,  -42,  43],
....:                 [  8, -75,  34, -32,  86, -156, -130,  42],
....:                 [  2, -17,   7,  -8,  20,  -33,  -31,  16],
....:                 [-24, -80,   7,  -3,  56, -136, -112,  42],
....:                 [ -6, -19,   0,  -1,  13,  -28,  -27,  15]])
sage: Z, U = B.zigzag_form(transformation=True)
sage: Z
[   0    0    0    0    0 1000|   0|   0]
[   1    0    0    0    0  900|   0|   0]
[   0    1    0    0    0  -30|   0|   0]
[   0    0    1    0    0 -153|   0|   0]
[   0    0    0    1    0    3|   0|   0]
[   0    0    0    0    1    9|   0|   0]
[-----------------------------+----+----]
[   0    0    0    0    0    0|  -2|   0]
[-----------------------------+----+----]
[   0    0    0    0    0    0|   1|  -2]
sage: U.inverse()*B*U == Z
True

sage: A.jordan_form() == B.jordan_form()
True
```

Two more examples, illustrating the two extremes of the zig-zag nature of this form. The first has a one in each of the off-diagonal blocks, the second has all zeros in each off-diagonal block. Notice again that the two matrices are similar, since their Jordan canonical forms are equal.

```
sage: C = matrix(QQ, [[2,  31, -10,  -9, -125,  13,  62, -12],
....:                 [0,  48, -16, -16, -188,  20,  92, -16],
....:                 [0,   9,  -1,   2,  -33,   5,  18,   0],
....:                 [0,  15,  -5,   0,  -59,   7,  30,  -4],
....:                 [0, -21,   7,   2,   84, -10, -42,   5],
....:                 [0, -42,  14,   8,  167, -17, -84,  13],
....:                 [0, -50,  17,  10,  199, -23, -98,  14],
....:                 [0,  15,  -5,  -2,  -59,   7,  30, -2]])
sage: Z, U = C.zigzag_form(transformation=True)
sage: Z
[2|1|0|0|0|0|0|0]
[-+-+-+-+-+-+-+-]
[0|2|0|0|0|0|0|0]
[-+-+-+-+-+-+-+-]
[0|1|2|1|0|0|0|0]
[-+-+-+-+-+-+-+-]
[0|0|0|2|0|0|0|0]
[-+-+-+-+-+-+-+-]
[0|0|0|1|2|1|0|0]
[-+-+-+-+-+-+-+-]
[0|0|0|0|0|2|0|0]
[-+-+-+-+-+-+-+-]
[0|0|0|0|0|1|2|1]
[-+-+-+-+-+-+-+-]
[0|0|0|0|0|0|0|2]
sage: U.inverse()*C*U == Z
True

sage: D = matrix(QQ, [[ -4,   3,    7,   2,  -4,   5,    7,   -3],
....:                 [ -6,   5,    7,   2,  -4,   5,    7,   -3],
....:                 [ 21, -12,   89,  25,   8,  27,   98,  -95],
....:                 [ -9,   5,  -44, -11,  -3, -13,  -48,   47],
....:                 [ 23, -13,   74,  21,  12,  22,   85,  -84],
....:                 [ 31, -18,  135,  38,  12,  47,  155, -147],
....:                 [-33,  19, -138, -39, -13, -45, -156,  151],
....:                 [ -7,   4,  -29,  -8,  -3, -10,  -34,   34]])
sage: Z, U = D.zigzag_form(transformation=True)
sage: Z
[ 0 -4| 0  0| 0  0| 0  0]
[ 1  4| 0  0| 0  0| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  1| 0  0| 0  0]
[ 0  0|-4  4| 0  0| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  0| 0 -4| 0  0]
[ 0  0| 0  0| 1  4| 0  0]
[-----+-----+-----+-----]
[ 0  0| 0  0| 0  0| 0  1]
[ 0  0| 0  0| 0  0|-4  4]
sage: U.inverse()*D*U == Z
True

sage: C.jordan_form() == D.jordan_form()
```

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```
True
```

ZigZag form is achieved entirely with the operations of the field, so while the eigenvalues may lie outside the field, this does not impede the computation of the form.

```
sage: F.<a> = GF(5^4)
sage: A = matrix(F, [[     a,      0,  0, a + 3],
....:                [     0,a^2 + 1,  0,     0],
....:                [     0,      0,a^3,     0],
....:                [a^2 +4 ,     0,   0,a + 2]])
sage: A.zigzag_form()
[                    0 a^3 + 2*a^2 + 2*a + 2|                     0|           ␣
↪       0]
[                    1               2*a + 2|                     0|           ␣
↪       0]
[-----------------------------------------+----------------------+-------------
↪--------]
[                    0                    0|                   a^3|           ␣
↪       0]
[-----------------------------------------+----------------------+-------------
↪--------]
[                    0                    0|                     0|           ␣
↪ a^2 + 1]
sage: A.eigenvalues()
Traceback (most recent call last):
...
NotImplementedError: algebraic closures of finite fields are only implemented␣
↪for prime fields
```

Subdivisions are optional.

```
sage: F.<a> = GF(5^4)
sage: A = matrix(F, [[     a,      0,  0, a + 3],
....:                [     0,a^2 + 1,  0,     0],
....:                [     0,      0,a^3,     0],
....:                [a^2 +4 ,     0,   0,a + 2]])
sage: A.zigzag_form(subdivide=False)
[                    0 a^3 + 2*a^2 + 2*a + 2                     0           ␣
↪       0]
[                    1               2*a + 2                     0           ␣
↪       0]
[                    0                    0                   a^3           ␣
↪       0]
[                    0                    0                     0           ␣
↪ a^2 + 1]
```

exception sage.matrix.matrix2.**NotFullRankError**

Bases: ValueError

An error that indicates that a matrix is not of full rank.

The fact that a square system is rank-deficient sometimes only becomes apparent while attempting to solve it. The methods *Matrix.solve_left()* and *Matrix.solve_right()* defer to `Matrix._solve_right_nonsingular_square()` for square systems, and that method raises this error if the system

turns out to be singular.

sage.matrix.matrix2.**decomp_seq**(*v*)

This function is used internally be the decomposition matrix method. It takes a list of tuples and produces a sequence that is correctly sorted and prints with carriage returns.

EXAMPLES:

```
sage: from sage.matrix.matrix2 import decomp_seq
sage: V = [(QQ^3, 2), (QQ^2, 1)]
sage: decomp_seq(V)
[
(Vector space of dimension 2 over Rational Field, 1),
(Vector space of dimension 3 over Rational Field, 2)
]
```

sage.matrix.matrix2.**ideal_or_fractional**(*R*, **args*)

CHAPTER

NINE

GENERIC ASYMPTOTICALLY FAST STRASSEN ALGORITHMS

Sage implements asymptotically fast echelon form and matrix multiplication algorithms.

class sage.matrix.strassen.**int_range**(*indices=None*, *range=None*)

Bases: object

Represent a list of integers as a list of integer intervals.

Note: Repetitions are not considered.

Useful class for dealing with pivots in the strassen echelon, could have much more general application

INPUT:

It can be one of the following:

- indices - integer, start of the unique interval
- range - integer, length of the unique interval

OR

- indices - list of integers, the integers to wrap into intervals

OR

- indices - None (default), shortcut for an empty list

OUTPUT:

An instance of int_range, i.e. a list of pairs (start, length).

EXAMPLES:

From a pair of integers:

```
sage: from sage.matrix.strassen import int_range
sage: int_range(2, 4)
[(2, 4)]
```

Default:

```
sage: int_range()
[]
```

From a list of integers:

```
sage: int_range([1,2,3,4])
[(1, 4)]
sage: int_range([1,2,3,4,6,7,8])
[(1, 4), (6, 3)]
sage: int_range([1,2,3,4,100,101,102])
[(1, 4), (100, 3)]
sage: int_range([1,1000,2,101,3,4,100,102])
[(1, 4), (100, 3), (1000, 1)]
```

Repetitions are not considered:

```
sage: int_range([1,2,3])
[(1, 3)]
sage: int_range([1,1,1,1,2,2,2,3])
[(1, 3)]
```

AUTHORS:

- Robert Bradshaw

intervals()

Return the list of intervals.

OUTPUT:

A list of pairs of integers.

EXAMPLES:

```
sage: from sage.matrix.strassen import int_range
sage: I = int_range([4,5,6,20,21,22,23])
sage: I.intervals()
[(4, 3), (20, 4)]
sage: type(I.intervals())
<... 'list'>
```

to_list()

Return the (sorted) list of integers represented by this object.

OUTPUT:

A list of integers.

EXAMPLES:

```
sage: from sage.matrix.strassen import int_range
sage: I = int_range([6,20,21,4,5,22,23])
sage: I.to_list()
[4, 5, 6, 20, 21, 22, 23]
```

```
sage: I = int_range(34, 9)
sage: I.to_list()
[34, 35, 36, 37, 38, 39, 40, 41, 42]
```

Repetitions are not considered:

```
sage: I = int_range([1,1,1,1,2,2,2,3])
sage: I.to_list()
[1, 2, 3]
```

sage.matrix.strassen.**strassen_echelon**(*A*, *cutoff*)

Compute echelon form, in place. Internal function, call with M.echelonize(algorithm="strassen") Based on work of Robert Bradshaw and David Harvey at MSRI workshop in 2006.

INPUT:

- `A` - matrix window
- `cutoff` - size at which algorithm reverts to naive Gaussian elimination and multiplication must be at least 1.

OUTPUT: The list of pivot columns

EXAMPLES:

```
sage: A = matrix(QQ, 7, [5, 0, 0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, -1, 3, 1, 0,␣
↪-1, 0, 0, -1, 0, 1, 2, -1, 1, 0, -1, 0, 1, 3, -1, 1, 0, 0, -2, 0, 2, 0, 1, 0, 0, -
↪1, 0, 1, 0, 1])
sage: B = A.__copy__(); B._echelon_strassen(1); B
[ 1  0  0  0  0  0  0]
[ 0  1  0 -1  0  1  0]
[ 0  0  1  0  0  0  0]
[ 0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  1]
[ 0  0  0  0  0  0  0]
[ 0  0  0  0  0  0  0]
sage: C = A.__copy__(); C._echelon_strassen(2); C == B
True
sage: C = A.__copy__(); C._echelon_strassen(4); C == B
True
```

```
sage: n = 32; A = matrix(Integers(389),n,range(n^2))
sage: B = A.__copy__(); B._echelon_in_place_classical()
sage: C = A.__copy__(); C._echelon_strassen(2)
sage: B == C
True
```

AUTHORS:

- Robert Bradshaw

sage.matrix.strassen.**strassen_window_multiply**(*C*, *A*, *B*, *cutoff*)

Multiplies the submatrices specified by A and B, places result in C. Assumes that A and B have compatible dimensions to be multiplied, and that C is the correct size to receive the product, and that they are all defined over the same ring.

Uses strassen multiplication at high levels and then uses MatrixWindow methods at low levels. EXAMPLES: The following matrix dimensions are chosen especially to exercise the eight possible parity combinations that could occur while subdividing the matrix in the strassen recursion. The base case in both cases will be a (4x5) matrix times a (5x6) matrix.

```
sage: A = MatrixSpace(Integers(2^65), 64, 83).random_element()
sage: B = MatrixSpace(Integers(2^65), 83, 101).random_element()
sage: A._multiply_classical(B) == A._multiply_strassen(B, 3) #indirect doctest
True
```

AUTHORS:

- David Harvey
- Simon King (2011-07): Improve memory efficiency; github issue #11610

sage.matrix.strassen.**test**(*n*, *m*, *R*, *c=2*)

INPUT:

- n - integer
- m - integer
- R - ring
- c - integer (optional, default:2)

EXAMPLES:

```
sage: from sage.matrix.strassen import test
sage: for n in range(5):
....:     print("{} {}".format(n, test(2*n,n,Frac(QQ['x']),2)))
0 True
1 True
2 True
3 True
4 True
```

CHAPTER

TEN

MINIMAL POLYNOMIALS OF LINEAR RECURRENCE SEQUENCES

AUTHORS:

- William Stein

sage.matrix.berlekamp_massey.**berlekamp_massey**(a)

Use the Berlekamp-Massey algorithm to find the minimal polynomial of a linear recurrence sequence a.

The minimal polynomial of a linear recurrence $\{a_r\}$ is by definition the unique monic polynomial g, such that if $\{a_r\}$ satisfies a linear recurrence $a_{j+k} + b_{j-1}a_{j-1+k} + \cdots + b_0 a_k = 0$ (for all $k \geq 0$), then g divides the polynomial $x^j + \sum_{i=0}^{j-1} b_i x^i$.

INPUT:

- a – a list of even length of elements of a field (or domain)

OUTPUT:

the minimal polynomial of the sequence, as a polynomial over the field in which the entries of a live

Warning: The result is only guaranteed to be correct on the full sequence if there exists a linear recurrence of length less than half the length of a.

EXAMPLES:

```
sage: from sage.matrix.berlekamp_massey import berlekamp_massey
sage: berlekamp_massey([1,2,1,2,1,2])
x^2 - 1
sage: berlekamp_massey([GF(7)(1),19,1,19])
x^2 + 6
sage: berlekamp_massey([2,2,1,2,1,191,393,132])
x^4 - 36727/11711*x^3 + 34213/5019*x^2 + 7024942/35133*x - 335813/1673
sage: berlekamp_massey(prime_range(2,38))
x^6 - 14/9*x^5 - 7/9*x^4 + 157/54*x^3 - 25/27*x^2 - 73/18*x + 37/9
```

CHAPTER

ELEVEN

BASE CLASS FOR DENSE MATRICES

class sage.matrix.matrix_dense.**Matrix_dense**

Bases: *Matrix*

antitranspose()

Returns the antitranspose of self, without changing self.

EXAMPLES:

```
sage: A = matrix(2,3,range(6)); A
[0 1 2]
[3 4 5]
sage: A.antitranspose()
[5 2]
[4 1]
[3 0]
```

```
sage: A.subdivide(1,2); A
[0 1|2]
[---+-]
[3 4|5]
sage: A.antitranspose()
[5|2]
[-+-]
[4|1]
[3|0]
```

transpose()

Returns the transpose of self, without changing self.

EXAMPLES: We create a matrix, compute its transpose, and note that the original matrix is not changed.

```
sage: M = MatrixSpace(QQ,  2)
sage: A = M([1,2,3,4])
sage: B = A.transpose()
sage: print(B)
[1 3]
[2 4]
sage: print(A)
[1 2]
[3 4]
```

`.T` is a convenient shortcut for the transpose:

```
sage: A.T
[1 3]
[2 4]
```

```
sage: A.subdivide(None, 1); A
[1|2]
[3|4]
sage: A.transpose()
[1 3]
[---]
[2 4]
```

CHAPTER

TWELVE

BASE CLASS FOR SPARSE MATRICES

class sage.matrix.matrix_sparse.**Matrix_sparse**

Bases: *Matrix*

antitranspose()

Return the antitranspose of `self`, without changing `self`.

This is the mirror image along the other diagonal.

EXAMPLES:

```
sage: M = MatrixSpace(QQ, 2, sparse=True)
sage: A = M([1,2,3,4]); A
[1 2]
[3 4]
sage: A.antitranspose()
[4 2]
[3 1]
```

See also:

transpose()

apply_map(*phi*, *R=None*, *sparse=True*)

Apply the given map `phi` (an arbitrary Python function or callable object) to this matrix.

If `R` is not given, automatically determine the base ring of the resulting matrix.

INPUT:

- `phi` – arbitrary Python function or callable object
- `R` – (optional) ring
- `sparse` – (optional, default `True`) whether to return a sparse or a dense matrix

OUTPUT: a matrix over `R`

EXAMPLES:

```
sage: m = matrix(ZZ, 10000, {(1,2): 17}, sparse=True)
sage: k.<a> = GF(9)
sage: f = lambda x: k(x)
sage: n = m.apply_map(f)
sage: n.parent()
Full MatrixSpace of 10000 by 10000 sparse matrices over Finite Field in a of␣
↪size 3^2
```

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```
sage: n[1,2]
2
```

An example where the codomain is explicitly specified.

```
sage: n = m.apply_map(lambda x:x%3, GF(3))
sage: n.parent()
Full MatrixSpace of 10000 by 10000 sparse matrices over Finite Field of size 3
sage: n[1,2]
2
```

If we did not specify the codomain, the resulting matrix in the above case ends up over **Z** again:

```
sage: n = m.apply_map(lambda x:x%3)
sage: n.parent()
Full MatrixSpace of 10000 by 10000 sparse matrices over Integer Ring
sage: n[1,2]
2
```

If self is subdivided, the result will be as well:

```
sage: m = matrix(2, 2, [0, 0, 3, 0])
sage: m.subdivide(None, 1); m
[0|0]
[3|0]
sage: m.apply_map(lambda x: x*x)
[0|0]
[9|0]
```

If the map sends zero to a non-zero value, then it may be useful to get the result as a dense matrix.

```
sage: m = matrix(ZZ, 3, 3, [0] * 7 + [1,2], sparse=True); m
[0 0 0]
[0 0 0]
[0 1 2]
sage: parent(m)
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
sage: n = m.apply_map(lambda x: x+polygen(QQ), sparse=False); n
[    x     x     x]
[    x     x     x]
[    x x + 1 x + 2]
sage: parent(n)
Full MatrixSpace of 3 by 3 dense matrices over Univariate Polynomial Ring in x␣
↪over Rational Field
```

apply_morphism(*phi*)

Apply the morphism `phi` to the coefficients of this sparse matrix.

The resulting matrix is over the codomain of `phi`.

INPUT:

- **`phi` – a morphism, so `phi` is callable and**
  `phi.domain()` and `phi.codomain()` are defined. The codomain must be a ring.

OUTPUT: a matrix over the codomain of phi

EXAMPLES:

```
sage: m = matrix(ZZ, 3, range(9), sparse=True)
sage: phi = ZZ.hom(GF(5))
sage: m.apply_morphism(phi)
[0 1 2]
[3 4 0]
[1 2 3]
sage: m.apply_morphism(phi).parent()
Full MatrixSpace of 3 by 3 sparse matrices over Finite Field of size 5
```

augment(*right*, *subdivide=False*)

Return the augmented matrix of the form:

```
[self | right].
```

EXAMPLES:

```
sage: M = MatrixSpace(QQ, 2, 2, sparse=True)
sage: A = M([1,2, 3,4])
sage: A
[1 2]
[3 4]
sage: N = MatrixSpace(QQ, 2, 1, sparse=True)
sage: B = N([9,8])
sage: B
[9]
[8]
sage: A.augment(B)
[1 2 9]
[3 4 8]
sage: B.augment(A)
[9 1 2]
[8 3 4]
```

A vector may be augmented to a matrix.

```
sage: A = matrix(QQ, 3, 4, range(12), sparse=True)
sage: v = vector(QQ, 3, range(3), sparse=True)
sage: A.augment(v)
[ 0  1  2  3  0]
[ 4  5  6  7  1]
[ 8  9 10 11  2]
```

The subdivide option will add a natural subdivision between self and right. For more details about how subdivisions are managed when augmenting, see *sage.matrix.matrix1.Matrix.augment()*.

```
sage: A = matrix(QQ, 3, 5, range(15), sparse=True)
sage: B = matrix(QQ, 3, 3, range(9), sparse=True)
sage: A.augment(B, subdivide=True)
[ 0  1  2  3  4| 0  1  2]
[ 5  6  7  8  9| 3  4  5]
[10 11 12 13 14| 6  7  8]
```

change_ring(*ring*)

Return the matrix obtained by coercing the entries of this matrix into the given ring.

Always returns a copy (unless `self` is immutable, in which case returns `self`).

EXAMPLES:

```
sage: A = matrix(QQ['x,y'], 2, [0,-1,2*x,-2], sparse=True); A
[  0  -1]
[2*x  -2]
sage: A.change_ring(QQ['x,y,z'])
[  0  -1]
[2*x  -2]
```

Subdivisions are preserved when changing rings:

```
sage: A.subdivide([2],[]); A
[  0  -1]
[2*x  -2]
[-------]
sage: A.change_ring(RR['x,y'])
[                  0  -1.000000000000000]
[2.000000000000000*x  -2.000000000000000]
[------------------------------------]
```

charpoly(*var='x'*, ***kwds*)

Return the characteristic polynomial of this matrix.

Note: the generic sparse charpoly implementation in Sage is to just compute the charpoly of the corresponding dense matrix, so this could use a lot of memory. In particular, for this matrix, the charpoly will be computed using a dense algorithm.

EXAMPLES:

```
sage: A = matrix(ZZ, 4, range(16), sparse=True)
sage: A.charpoly()
x^4 - 30*x^3 - 80*x^2
sage: A.charpoly('y')
y^4 - 30*y^3 - 80*y^2
sage: A.charpoly()
x^4 - 30*x^3 - 80*x^2
```

density()

Return the density of the matrix.

By density we understand the ratio of the number of nonzero positions and the number `self.nrows() * self.ncols()`, i.e. the number of possible nonzero positions.

EXAMPLES:

```
sage: a = matrix([[],[],[],[]], sparse=True); a.density()
0
sage: a = matrix(5000,5000,{(1,2): 1}); a.density()
1/25000000
```

determinant(***kwds*)

Return the determinant of this matrix.

Note: the generic sparse determinant implementation in Sage is to just compute the determinant of the corresponding dense matrix, so this could use a lot of memory. In particular, for this matrix, the determinant will be computed using a dense algorithm.

EXAMPLES:

```
sage: A = matrix(ZZ, 4, range(16), sparse=True)
sage: B = A + identity_matrix(ZZ, 4, sparse=True)
sage: B.det()
-49
```

matrix_from_rows_and_columns(*rows*, *columns*)

Return the matrix constructed from `self` from the given rows and columns.

EXAMPLES:

```
sage: M = MatrixSpace(Integers(8),3,3, sparse=True)
sage: A = M(range(9)); A
[0 1 2]
[3 4 5]
[6 7 0]
sage: A.matrix_from_rows_and_columns([1], [0,2])
[3 5]
sage: A.matrix_from_rows_and_columns([1,2], [1,2])
[4 5]
[7 0]
```

Note that row and column indices can be reordered or repeated:

```
sage: A.matrix_from_rows_and_columns([2,1], [2,1])
[0 7]
[5 4]
```

For example here we take from row 1 columns 2 then 0 twice, and do this 3 times.

```
sage: A.matrix_from_rows_and_columns([1,1,1],[2,0,0])
[5 3 3]
[5 3 3]
[5 3 3]
```

We can efficiently extract large submatrices:

```
sage: A = random_matrix(ZZ, 100000, density=.00005, sparse=True)  # long time␣
↪(4s on sage.math, 2012)
sage: B = A[50000:,:50000]        # long time
sage: count = 0
sage: for i, j in A.nonzero_positions():  # long time
....:     if i >= 50000 and j < 50000:
....:         assert B[i-50000, j] == A[i, j]
....:         count += 1
```

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```
sage: count == sum(1 for _ in B.nonzero_positions())  # long time
True
```

We must pass in a list of indices:

```
sage: A = random_matrix(ZZ,100,density=.02,sparse=True)
sage: A.matrix_from_rows_and_columns(1,[2,3])
Traceback (most recent call last):
...
TypeError: 'sage.rings.integer.Integer' object is not iterable

sage: A.matrix_from_rows_and_columns([1,2],3)
Traceback (most recent call last):
...
TypeError: 'sage.rings.integer.Integer' object is not iterable
```

AUTHORS:

- Jaap Spies (2006-02-18)
- Didier Deshommes: some Pyrex speedups implemented
- Jason Grout: sparse matrix optimizations

transpose()

Return the transpose of `self`, without changing `self`.

EXAMPLES: We create a matrix, compute its transpose, and note that the original matrix is not changed.

```
sage: M = MatrixSpace(QQ, 2, sparse=True)
sage: A = M([1,2,3,4]); A
[1 2]
[3 4]
sage: B = A.transpose(); B
[1 3]
[2 4]
```

`.T` is a convenient shortcut for the transpose:

```
sage: A.T
[1 3]
[2 4]
```

See also:

antitranspose()

CHAPTER

THIRTEEN

DENSE MATRICES OVER A GENERAL RING

class sage.matrix.matrix_generic_dense.**Matrix_generic_dense**

Bases: *Matrix_dense*

The `Matrix_generic_dense` class derives from `Matrix`, and defines functionality for dense matrices over any base ring. Matrices are represented by a list of elements in the base ring, and element access operations are implemented in this class.

EXAMPLES:

```
sage: A = random_matrix(Integers(25)['x'], 2)
sage: type(A)
<class 'sage.matrix.matrix_generic_dense.Matrix_generic_dense'>
sage: TestSuite(A).run(skip='_test_minpoly')
```

Test comparisons:

```
sage: A = random_matrix(Integers(25)['x'], 2)
sage: A == A
True
sage: A < A + 1 or A[0, 0].coefficients()[0] == 24
True
sage: A+1 < A and A[0, 0].coefficients()[0] != 24
False
```

Test hashing:

```
sage: A = random_matrix(Integers(25)['x'], 2)
sage: hash(A)
Traceback (most recent call last):
...
TypeError: mutable matrices are unhashable
sage: A.set_immutable()
sage: H = hash(A)
```

CHAPTER

FOURTEEN

SPARSE MATRICES OVER A GENERAL RING

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: M = MatrixSpace(QQ['x'],2,3,sparse=True); M
Full MatrixSpace of 2 by 3 sparse matrices over Univariate Polynomial Ring in x over␣
↪Rational Field
sage: a = M(range(6)); a
[0 1 2]
[3 4 5]
sage: b = M([x^n for n in range(6)]); b
[  1   x x^2]
[x^3 x^4 x^5]
sage: a * b.transpose()
[            2*x^2 + x           2*x^5 + x^4]
[      5*x^2 + 4*x + 3 5*x^5 + 4*x^4 + 3*x^3]
sage: pari(a)*pari(b.transpose())
[2*x^2 + x, 2*x^5 + x^4; 5*x^2 + 4*x + 3, 5*x^5 + 4*x^4 + 3*x^3]
sage: c = copy(b); c
[  1   x x^2]
[x^3 x^4 x^5]
sage: c[0,0] = 5; c
[  5   x x^2]
[x^3 x^4 x^5]
sage: b[0,0]
1
sage: c.dict()
{(0, 0): 5, (0, 1): x, (0, 2): x^2, (1, 0): x^3, (1, 1): x^4, (1, 2): x^5}
sage: c.list()
[5, x, x^2, x^3, x^4, x^5]
sage: c.rows()
[(5, x, x^2), (x^3, x^4, x^5)]
sage: TestSuite(c).run()
sage: d = c.change_ring(CC['x']); d
[5.00000000000000                x             x^2]
[            x^3            x^4             x^5]
sage: latex(c)
\left(\begin{array}{rrr}
5 & x & x^{2} \\
x^{3} & x^{4} & x^{5}
\end{array}\right)
```

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```
sage: c.sparse_rows()
[(5, x, x^2), (x^3, x^4, x^5)]
sage: d = c.dense_matrix(); d
[  5   x x^2]
[x^3 x^4 x^5]
sage: parent(d)
Full MatrixSpace of 2 by 3 dense matrices over Univariate Polynomial Ring in x over␣
↪Rational Field
sage: c.sparse_matrix() is c
True
sage: c.is_sparse()
True
```

class sage.matrix.matrix_generic_sparse.**Matrix_generic_sparse**

Bases: *Matrix_sparse*

Generic sparse matrix.

The `Matrix_generic_sparse` class derives from *Matrix_sparse*, and defines functionality for sparse matrices over any base ring. A generic sparse matrix is represented using a dictionary whose keys are pairs of integers (i, j) and values in the base ring. The values of the dictionary must never be zero.

EXAMPLES:

```
sage: R.<a,b> = PolynomialRing(ZZ,'a,b')
sage: M = MatrixSpace(R,5,5,sparse=True)
sage: M({(0,0):5*a+2*b, (3,4): -a})
[5*a + 2*b         0         0         0         0]
[         0         0         0         0         0]
[         0         0         0         0         0]
[         0         0         0         0        -a]
[         0         0         0         0         0]
sage: M(3)
[3 0 0 0 0]
[0 3 0 0 0]
[0 0 3 0 0]
[0 0 0 3 0]
[0 0 0 0 3]
sage: V = FreeModule(ZZ, 5,sparse=True)
sage: m = M([V({0:3}), V({2:2, 4:-1}), V(0), V(0), V({1:2})])
sage: m
[ 3  0  0  0  0]
[ 0  0  2  0 -1]
[ 0  0  0  0  0]
[ 0  0  0  0  0]
[ 0  2  0  0  0]
```

Note: The datastructure can potentially be optimized. Firstly, as noticed in github issue #17663, we lose time in using 2-tuples to store indices. Secondly, there is no fast way to access non-zero elements in a given row/column.

sage.matrix.matrix_generic_sparse.**Matrix_sparse_from_rows**(X)

INPUT:

- X - nonempty list of SparseVector rows

OUTPUT: Sparse_matrix with those rows.

EXAMPLES:

```
sage: V = VectorSpace(QQ,20,sparse=True)
sage: v = V(0)
sage: v[9] = 4
sage: from sage.matrix.matrix_generic_sparse import Matrix_sparse_from_rows
sage: Matrix_sparse_from_rows([v])
[0 0 0 0 0 0 0 0 0 4 0 0 0 0 0 0 0 0 0 0]
sage: Matrix_sparse_from_rows([v, v, v, V(0)])
[0 0 0 0 0 0 0 0 0 4 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 4 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 4 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
```

CHAPTER

FIFTEEN

DENSE MATRICES OVER THE INTEGER RING

AUTHORS:

- William Stein
- Robert Bradshaw
- Marc Masdeu (August 2014). Implemented using FLINT, see github issue #16803.
- Jeroen Demeyer (October 2014): lots of fixes, see github issue #17090 and github issue #17094.
- Vincent Delecroix (February 2015): make it faster, see github issue #17822.
- Vincent Delecroix (May 2017): removed duplication of entries and cleaner linbox interface

EXAMPLES:

```
sage: a = matrix(ZZ, 3,3, range(9)); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: a.det()
0
sage: a[0,0] = 10; a.det()
-30
sage: a.charpoly()
x^3 - 22*x^2 + 102*x + 30
sage: b = -3*a
sage: a == b
False
sage: b < a
True
```

class sage.matrix.matrix_integer_dense.**Matrix_integer_dense**

Bases: *Matrix_dense*

Matrix over the integers, implemented using FLINT.

On a 32-bit machine, they can have at most $2^{32} - 1$ rows or columns. On a 64-bit machine, matrices can have at most $2^{64} - 1$ rows or columns.

EXAMPLES:

```
sage: a = MatrixSpace(ZZ,3)(2); a
[2 0 0]
[0 2 0]
```

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```
[0 0 2]
sage: a = matrix(ZZ,1,3, [1,2,-3]); a
[ 1  2 -3]
sage: a = MatrixSpace(ZZ,2,4)(2); a
Traceback (most recent call last):
...
TypeError: nonzero scalar matrix must be square
```

BKZ(*delta=None*, *algorithm='fpLLL'*, *fp=None*, *block_size=10*, *prune=0*, *use_givens=False*, *precision=0*, *proof=None*, ***kwds*)

Block Korkin-Zolotarev reduction.

INPUT:

- `delta` – (default: `0.99`) LLL parameter
- `algorithm` – (default: `"fpLLL"`) `"fpLLL"` or `"NTL"`
- `fp` – floating point number implementation
  - `None` – NTL's exact reduction or fpLLL's wrapper (default)
  - `'fp'` – double precision: NTL's FP or fpLLL's double
  - `'ld'` – long doubles (fpLLL only)
  - `'qd'` – NTL's QP
  - **`'qd1'` – quad doubles: Uses `quad_float` precision** to compute Gram-Schmidt, but uses double precision in the search phase of the block reduction algorithm. This seems adequate for most purposes, and is faster than `'qd'`, which uses quad_float precision uniformly throughout (NTL only).
  - `'xd'` – extended exponent: NTL's XD or fpLLL's dpe
  - `'rr'` – arbitrary precision: NTL'RR or fpLLL's MPFR
- `block_size` – (default: `10`) Specifies the size of the blocks in the reduction. High values yield shorter vectors, but the running time increases double exponentially with `block_size`. `block_size` should be between 2 and the number of rows of `self`.
- `proof` – (default: same as `proof.linear_algebra()`) Insist on full BKZ reduction. If disabled and fplll is called, reduction is much faster but the result is not fully BKZ reduced.

NTL SPECIFIC INPUT:

- `prune` – (default: `0`) The optional parameter `prune` can be set to any positive number to invoke the Volume Heuristic from [SH1995]. This can significantly reduce the running time, and hence allow much bigger block size, but the quality of the reduction is of course not as good in general. Higher values of `prune` mean better quality, and slower running time. When `prune` is `0`, pruning is disabled. Recommended usage: for `block_size==30`, set `10 <= prune <=15`.
- `use_givens` – Use Givens orthogonalization. Only applies to approximate reduction using NTL. This is a bit slower, but generally much more stable, and is really the preferred orthogonalization strategy. For a nice description of this, see Chapter 5 of [GL1996].

fpLLL SPECIFIC INPUT:

- `precision` – (default: `0` for automatic choice) bit precision to use if `fp='rr'` is set

- **kwds – keywords to be passed to fpylll. See fpylll.BKZ.Param for details.

Also, if the verbose level is at least 2, some output is printed during the computation.

EXAMPLES:

```
sage: A = Matrix(ZZ,3,3,range(1,10))
sage: A.BKZ()
[ 0  0  0]
[ 2  1  0]
[-1  1  3]

sage: A = Matrix(ZZ,3,3,range(1,10))
sage: A.BKZ(use_givens=True)
[ 0  0  0]
[ 2  1  0]
[-1  1  3]

sage: A = Matrix(ZZ,3,3,range(1,10))
sage: A.BKZ(fp="fp")
[ 0  0  0]
[ 2  1  0]
[-1  1  3]
```

ALGORITHM:

Calls either NTL or fpLLL.

LLL(*delta=None*, *eta=None*, *algorithm='fpLLL:wrapper'*, *fp=None*, *prec=0*, *early_red=False*, *use_givens=False*, *use_siegel=False*, *transformation=False*, ***kwds*)

Return LLL reduced or approximated LLL reduced lattice R for this matrix interpreted as a lattice.

A lattice $(b_1, b_2, ..., b_d)$ is (δ, η)-LLL-reduced if the two following conditions hold:

- For any $i > j$, we have $|\mu_{i,j}| \leq \eta$.
- For any $i < d$, we have $\delta |b_i^*|^2 \leq |b_{i+1}^* + \mu_{i+1,i} b_i^*|^2$,

where $\mu_{i,j} = \langle b_i, b_j^* \rangle / \langle b_j^*, b_j^* \rangle$ and b_i^* is the i-th vector of the Gram-Schmidt orthogonalisation of $(b_1, b_2, ..., b_d)$.

The default reduction parameters are $\delta = 0.99$ and $\eta = 0.501$. The parameters δ and η must satisfy $0.25 < \delta \leq 1.0$ and $0.5 \leq \eta < \sqrt{\delta}$. Polynomial time complexity is only guaranteed for $\delta < 1$. Not every algorithm admits the case $\delta = 1$.

The lattice is returned as a matrix. Also the rank (and the determinant) of self are cached if those are computed during the reduction. Note that in general this only happens when self.rank() == self.ncols() and the exact algorithm is used.

INPUT:

- delta – (default: 0.99) δ parameter as described above
- eta – (default: 0.501) η parameter as described above, ignored by NTL
- algorithm – string; one of the algorithms listed below (default: "fpLLL:wrapper").
- fp – floating point number implementation:
  - None – NTL's exact reduction or fpLLL's wrapper
  - 'fp' – double precision: NTL's FP or fpLLL's double

  - 'ld' – long doubles (fpLLL only)
  - 'qd' – NTL's QP
  - 'xd' – extended exponent: NTL's XD or fpLLL's dpe
  - 'rr' – arbitrary precision: NTL's RR or fpLLL's MPFR
- prec – (default: auto choose) precision, ignored by NTL
- early_red – (default: False) perform early reduction, ignored by NTL
- use_givens – (default: False) use Givens orthogonalization. Only applies to approximate reduction using NTL. This is slower but generally more stable.
- use_siegel – (default: False) use Siegel's condition instead of Lovász's condition, ignored by NTL
- **transformation – (default: False) also return transformation** matrix.
- **kwds – keywords to be passed to fpylll. See fpylll.LLL.reduction() for details.

Also, if the verbose level is at least 2, some output is printed during the computation.

AVAILABLE ALGORITHMS:

- 'NTL:LLL' - NTL's LLL + choice of fp.
- 'fpLLL:heuristic' - fpLLL's heuristic + choice of fp.
- 'fpLLL:fast' - fpLLL's fast + choice of fp.
- 'fpLLL:proved' - fpLLL's proved + choice of fp.
- 'fpLLL:wrapper' - fpLLL's automatic choice (default).

OUTPUT:

A matrix over the integers.

EXAMPLES:

```
sage: A = Matrix(ZZ,3,3,range(1,10))
sage: A.LLL()
[ 0  0  0]
[ 2  1  0]
[-1  1  3]
```

We compute the extended GCD of a list of integers using LLL, this example is from the Magma handbook:

```
sage: Q = [ 67015143, 248934363018, 109210, 25590011055, 74631449,
....:       10230248, 709487, 68965012139, 972065, 864972271 ]
sage: n = len(Q)
sage: S = 100
sage: X = Matrix(ZZ, n, n + 1)
sage: for i in range(n):
....:     X[i, i + 1] = 1
sage: for i in range(n):
....:     X[i, 0] = S * Q[i]
sage: L = X.LLL()
sage: M = L.row(n-1).list()[1:]
sage: M
[-3, -1, 13, -1, -4, 2, 3, 4, 5, -1]
```

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```
sage: add(Q[i]*M[i] for i in range(n))
-1
```

The case $\delta = 1$ is not always supported:

```
sage: L = X.LLL(delta=2)
Traceback (most recent call last):
...
TypeError: delta must be <= 1
sage: L = X.LLL(delta=1)    # not tested, will eat lots of ram
Traceback (most recent call last):
...
RuntimeError: infinite loop in LLL
sage: L = X.LLL(delta=1, algorithm='NTL:LLL')
sage: L[-1]
(-100, -3, -1, 13, -1, -4, 2, 3, 4, 5, -1)
```

We return the transformation matrix:

```
sage: A = random_matrix(ZZ, 10, 20)
sage: R, U = A.LLL(transformation=True)
sage: U*A == R
True

sage: A = random_matrix(ZZ, 10, 20)
sage: R, U = A.LLL(algorithm="NTL:LLL", transformation=True)
sage: U*A == R
True
```

Note: See `sage.libs.ntl.ntl_mat_ZZ.ntl_mat_ZZ.LLL` and `fpylll.fplll.lll` for details on the algorithms used.

Although LLL is a deterministic algorithm, the output for different implementations and CPUs (32-bit vs. 64-bit) may vary, while still being correct.

---

antitranspose()

Return the antitranspose of self, without changing self.

EXAMPLES:

```
sage: A = matrix(2,3,range(6))
sage: type(A)
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
sage: A.antitranspose()
[5 2]
[4 1]
[3 0]
sage: A
[0 1 2]
[3 4 5]

sage: A.subdivide(1,2); A
```

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```
[0 1|2]
[---+-]
[3 4|5]
sage: A.antitranspose()
[5|2]
[-+-]
[4|1]
[3|0]
```

augment(*right*, *subdivide=False*)

Return a new matrix formed by appending the matrix (or vector) `right` on the right side of `self`.

INPUT:

- `right` - a matrix, vector or free module element, whose dimensions are compatible with `self`.
- `subdivide` - default: `False` - request the resulting matrix to have a new subdivision, separating `self` from `right`.

OUTPUT:

A new matrix formed by appending `right` onto the right side of `self`. If `right` is a vector (or free module element) then in this context it is appropriate to consider it as a column vector. (The code first converts a vector to a 1-column matrix.)

EXAMPLES:

```
sage: A = matrix(ZZ, 4, 5, range(20))
sage: B = matrix(ZZ, 4, 3, range(12))
sage: A.augment(B)
[ 0  1  2  3  4  0  1  2]
[ 5  6  7  8  9  3  4  5]
[10 11 12 13 14  6  7  8]
[15 16 17 18 19  9 10 11]
```

A vector may be augmented to a matrix.

```
sage: A = matrix(ZZ, 3, 5, range(15))
sage: v = vector(ZZ, 3, range(3))
sage: A.augment(v)
[ 0  1  2  3  4  0]
[ 5  6  7  8  9  1]
[10 11 12 13 14  2]
```

The `subdivide` option will add a natural subdivision between `self` and `right`. For more details about how subdivisions are managed when augmenting, see *sage.matrix.matrix1.Matrix.augment()*.

```
sage: A = matrix(ZZ, 3, 5, range(15))
sage: B = matrix(ZZ, 3, 3, range(9))
sage: A.augment(B, subdivide=True)
[ 0  1  2  3  4| 0  1  2]
[ 5  6  7  8  9| 3  4  5]
[10 11 12 13 14| 6  7  8]
```

Errors are raised if the sizes are incompatible.

```
sage: A = matrix(ZZ, [[1, 2],[3, 4]])
sage: B = matrix(ZZ, [[10, 20], [30, 40], [50, 60]])
sage: A.augment(B)
Traceback (most recent call last):
...
TypeError: number of rows must be the same, not 2 != 3
```

charpoly(*var='x'*, *algorithm=None*)

Note: The characteristic polynomial is defined as $\det(xI - A)$.

INPUT:

- `var` - a variable name
- `algorithm` - (optional) either 'generic', 'flint' or 'linbox'. Default is set to 'linbox'.

EXAMPLES:

```
sage: A = matrix(ZZ,6, range(36))
sage: f = A.charpoly(); f
x^6 - 105*x^5 - 630*x^4
sage: f(A) == 0
True
sage: g = A.charpoly(algorithm='flint')
sage: f == g
True
sage: n=20; A = Mat(ZZ,n)(range(n^2))
sage: A.charpoly()
x^20 - 3990*x^19 - 266000*x^18
sage: A.minpoly()
x^3 - 3990*x^2 - 266000*x
```

On non square matrices, this method raises an ArithmeticError:

```
sage: matrix(ZZ, 2, 1).charpoly()
Traceback (most recent call last):
...
ArithmeticError: only valid for square matrix
```

column(*i*, *from_list=False*)

Return the i-th column of this matrix as a dense vector.

INPUT:

- `i` - integer
- `from_list` - ignored

EXAMPLES:

```
sage: m = matrix(ZZ, 3, 2, [1, -2, 3, 4, -1, 0])
sage: m.column(1)
(-2, 4, 0)
sage: m.column(1, from_list=True)
```

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```
(-2, 4, 0)
sage: m.column(-1)
(-2, 4, 0)
sage: m.column(-2)
(1, 3, -1)

sage: m.column(2)
Traceback (most recent call last):
...
IndexError: column index out of range
sage: m.column(-3)
Traceback (most recent call last):
...
IndexError: column index out of range
```

decomposition(***kwds*)

Return the decomposition of the free module on which this matrix A acts from the right (i.e., the action is x goes to x A), along with whether this matrix acts irreducibly on each factor. The factors are guaranteed to be sorted in the same way as the corresponding factors of the characteristic polynomial, and are saturated as ZZ modules.

INPUT:

- `self` - a matrix over the integers
- `**kwds` - these are passed onto to the decomposition over QQ command.

EXAMPLES:

```
sage: t = ModularSymbols(11,sign=1).hecke_matrix(2)
sage: w = t.change_ring(ZZ)
sage: w
[ 3 -2]
[ 0 -2]
sage: w.charpoly().factor()
(x - 3) * (x + 2)
sage: w.decomposition()
[
(Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[ 5 -2], True),
(Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1], True)
]
```

determinant(*algorithm='default'*, *proof=None*, *stabilize=2*)

Return the determinant of this matrix.

INPUT:

- `algorithm`
  - `'default'` – use `flint`
  - `'flint'` – let flint do the determinant

  - `'padic'` - uses a p-adic / multimodular algorithm that relies on code in IML and linbox
  - `'linbox'` - calls linbox det (you *must* set proof=False to use this!)
  - `'ntl'` - calls NTL's det function
  - `'pari'` - uses PARI

- `proof` - bool or None; if None use proof.linear_algebra(); only relevant for the padic algorithm.

  Note: It would be *VERY VERY* hard for det to fail even with proof=False.

- `stabilize` - if proof is False, require det to be the same for this many CRT primes in a row. Ignored if proof is True.

ALGORITHM: The p-adic algorithm works by first finding a random vector v, then solving $Ax = v$ and taking the denominator d. This gives a divisor of the determinant. Then we compute $\det(A)/d$ using a multimodular algorithm and the Hadamard bound, skipping primes that divide d.

EXAMPLES:

```
sage: A = matrix(ZZ,8,8,[3..66])
sage: A.determinant()
0
```

```
sage: A = random_matrix(ZZ,20,20)
sage: D1 = A.determinant()
sage: A._clear_cache()
sage: D2 = A.determinant(algorithm='ntl')
sage: D1 == D2
True
```

We have a special-case algorithm for 4 x 4 determinants:

```
sage: A = matrix(ZZ,4,[1,2,3,4,4,3,2,1,0,5,0,1,9,1,2,3])
sage: A.determinant()
270
```

Next we try the Linbox det. Note that we must have proof=False.

```
sage: A = matrix(ZZ,5,[1,2,3,4,5,4,6,3,2,1,7,9,7,5,2,1,4,6,7,8,3,2,4,6,7])
sage: A.determinant(algorithm='linbox')
Traceback (most recent call last):
...
RuntimeError: you must pass the proof=False option to the determinant command␣
↪to use LinBox's det algorithm
sage: A.determinant(algorithm='linbox', proof=False)
-21
sage: A._clear_cache()
sage: A.determinant()
-21
```

Try the other algorithms on the same example:

```
sage: A._clear_cache(); A.determinant(algorithm='padic')
-21
```

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```
sage: A._clear_cache(); A.determinant(algorithm='pari')
-21
sage: A._clear_cache(); A.determinant(algorithm='ntl')
-21
sage: A._clear_cache(); A.determinant(algorithm='padic')
-21
```

A bigger example:

```
sage: A = random_matrix(ZZ,30)
sage: d = A.determinant()
sage: A._clear_cache()
sage: A.determinant(algorithm='linbox',proof=False) == d
True
```

echelon_form(*algorithm='default'*, *proof=None*, *include_zero_rows=True*, *transformation=False*, *D=None*)

Return the echelon form of this matrix over the integers, also known as the hermite normal form (HNF).

INPUT:

- `algorithm` – String. The algorithm to use. Valid options are:
  - `'default'` – Let Sage pick an algorithm (default). Up to 75 rows or columns with no transformation matrix, use pari with flag 0; otherwise, use flint.
  - `'flint'` - use flint
  - `'ntl'` - use NTL (only works for square matrices of full rank!)
  - `'padic'` - an asymptotically fast p-adic modular algorithm, If your matrix has large coefficients and is small, you may also want to try this.
  - `'pari'` - use PARI with flag 1
  - `'pari0'` - use PARI with flag 0
  - `'pari1'` - use PARI with flag 1
  - `'pari4'` - use PARI with flag 4 (use heuristic LLL)
- `proof` - (default: True); if proof=False certain determinants are computed using a randomized hybrid p-adic multimodular strategy until it stabilizes twice (instead of up to the Hadamard bound). It is *incredibly* unlikely that one would ever get an incorrect result with proof=False.
- `include_zero_rows` - (default: True) if False, don't include zero rows
- `transformation` - if given, also compute transformation matrix; only valid for flint and padic algorithm
- `D` - (default: None) if given and the algorithm is 'ntl', then D must be a multiple of the determinant and this function will use that fact.

OUTPUT:

The Hermite normal form (=echelon form over $\mathbf{Z}$) of self as an immutable matrix.

EXAMPLES:

```
sage: A = MatrixSpace(ZZ,2)([1,2,3,4])
sage: A.echelon_form()
[1 0]
```

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```
[0 2]
sage: A = MatrixSpace(ZZ,5)(range(25))
sage: A.echelon_form()
[  5   0  -5 -10 -15]
[  0   1   2   3   4]
[  0   0   0   0   0]
[  0   0   0   0   0]
[  0   0   0   0   0]
```

Getting a transformation matrix in the nonsquare case:

```
sage: A = matrix(ZZ,5,3,[1..15])
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=False)
sage: H
[1 2 3]
[0 3 6]
sage: U
[  0   0   0   4  -3]
[  0   0   0  13 -10]
sage: U*A == H
True
```

Note: If 'ntl' is chosen for a non square matrix this function raises a ValueError.

Special cases: 0 or 1 rows:

```
sage: a = matrix(ZZ, 1,2,[0,-1])
sage: a.hermite_form()
[0 1]
sage: a.pivots()
(1,)
sage: a = matrix(ZZ, 1,2,[0,0])
sage: a.hermite_form()
[0 0]
sage: a.pivots()
()
sage: a = matrix(ZZ,1,3); a
[0 0 0]
sage: a.echelon_form(include_zero_rows=False)
[]
sage: a.echelon_form(include_zero_rows=True)
[0 0 0]
```

Illustrate using various algorithms.:

```
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='pari')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='pari0')
[1 2 3]
```

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```
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='pari4')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='padic')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='default')
[1 2 3]
[0 3 6]
[0 0 0]
```

The 'ntl' algorithm doesn't work on matrices that do not have full rank.:

```
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='ntl')
Traceback (most recent call last):
...
ValueError: ntl only computes HNF for square matrices of full rank.
sage: matrix(ZZ,3,[0] +[2..9]).hermite_form(algorithm='ntl')
[1 0 0]
[0 1 0]
[0 0 3]
```

elementary_divisors(*algorithm='pari'*)

Return the elementary divisors of self, in order.

Warning: This is MUCH faster than the *smith_form()* function.

The elementary divisors are the invariants of the finite abelian group that is the cokernel of *left* multiplication of this matrix. They are ordered in reverse by divisibility.

INPUT:

- `self` - matrix
- `algorithm` - (default: 'pari')
  - `'pari'`: works robustly, but is slower.
  - `'linbox'` - use linbox (currently off, broken)

OUTPUT: list of integers

Note: These are the invariants of the cokernel of *left* multiplication:

```
sage: M = Matrix([[3,0,1],[0,1,0]])
sage: M
[3 0 1]
[0 1 0]
sage: M.elementary_divisors()
```

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```
[1, 1]
sage: M.transpose().elementary_divisors()
[1, 1, 0]
```

EXAMPLES:

```
sage: matrix(3, range(9)).elementary_divisors()
[1, 3, 0]
sage: matrix(3, range(9)).elementary_divisors(algorithm='pari')
[1, 3, 0]
sage: C = MatrixSpace(ZZ,4)([3,4,5,6,7,3,8,10,14,5,6,7,2,2,10,9])
sage: C.elementary_divisors()
[1, 1, 1, 687]
```

```
sage: M = matrix(ZZ, 3, [1,5,7, 3,6,9, 0,1,2])
sage: M.elementary_divisors()
[1, 1, 6]
```

This returns a copy, which is safe to change:

```
sage: edivs = M.elementary_divisors()
sage: edivs.pop()
6
sage: M.elementary_divisors()
[1, 1, 6]
```

See also:

smith_form()

frobenius(*flag=0*, *var='x'*)

Return the Frobenius form (rational canonical form) of this matrix.

INPUT:

- `flag` – 0 (default), 1 or 2 as follows:
  - `0` – (default) return the Frobenius form of this matrix.
  - `1` – return only the elementary divisor polynomials, as polynomials in var.
  - `2` – return a two-components vector [F,B] where F is the Frobenius form and B is the basis change so that $M = B^{-1}FB$.
- `var` – a string (default: 'x')

ALGORITHM: uses PARI's matfrobenius()

EXAMPLES:

```
sage: A = MatrixSpace(ZZ, 3)(range(9))
sage: A.frobenius(0)
[ 0  0  0]
[ 1  0 18]
[ 0  1 12]
sage: A.frobenius(1)
```

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```
[x^3 - 12*x^2 - 18*x]
sage: A.frobenius(1, var='y')
[y^3 - 12*y^2 - 18*y]
sage: F, B = A.frobenius(2)
sage: A == B^(-1)*F*B
True
sage: a=matrix([])
sage: a.frobenius(2)
([], [])
sage: a.frobenius(0)
[]
sage: a.frobenius(1)
[]
sage: B = random_matrix(ZZ,2,3)
sage: B.frobenius()
Traceback (most recent call last):
...
ArithmeticError: frobenius matrix of non-square matrix not defined.
```

AUTHORS:

- Martin Albrect (2006-04-02)

TODO: - move this to work for more general matrices than just over Z. This will require fixing how PARI polynomials are coerced to Sage polynomials.

gcd()

Return the gcd of all entries of self; very fast.

EXAMPLES:

```
sage: a = matrix(ZZ,2, [6,15,-6,150])
sage: a.gcd()
3
```

height()

Return the height of this matrix, i.e., the max absolute value of the entries of the matrix.

OUTPUT: A nonnegative integer.

EXAMPLES:

```
sage: a = Mat(ZZ,3)(range(9))
sage: a.height()
8
sage: a = Mat(ZZ,2,3)([-17,3,-389,15,-1,0]); a
[ -17    3 -389]
[  15   -1    0]
sage: a.height()
389
```

hermite_form(*algorithm='default'*, *proof=None*, *include_zero_rows=True*, *transformation=False*, *D=None*)

Return the echelon form of this matrix over the integers, also known as the hermite normal form (HNF).

INPUT:

- `algorithm` – String. The algorithm to use. Valid options are:
  - `'default'` – Let Sage pick an algorithm (default). Up to 75 rows or columns with no transformation matrix, use pari with flag 0; otherwise, use flint.
  - `'flint'` - use flint
  - `'ntl'` - use NTL (only works for square matrices of full rank!)
  - `'padic'` - an asymptotically fast p-adic modular algorithm, If your matrix has large coefficients and is small, you may also want to try this.
  - `'pari'` - use PARI with flag 1
  - `'pari0'` - use PARI with flag 0
  - `'pari1'` - use PARI with flag 1
  - `'pari4'` - use PARI with flag 4 (use heuristic LLL)
- `proof` - (default: True); if proof=False certain determinants are computed using a randomized hybrid p-adic multimodular strategy until it stabilizes twice (instead of up to the Hadamard bound). It is *incredibly* unlikely that one would ever get an incorrect result with proof=False.
- `include_zero_rows` - (default: True) if False, don't include zero rows
- `transformation` - if given, also compute transformation matrix; only valid for flint and padic algorithm
- `D` - (default: None) if given and the algorithm is 'ntl', then D must be a multiple of the determinant and this function will use that fact.

OUTPUT:

The Hermite normal form (=echelon form over $\mathbf{Z}$) of self as an immutable matrix.

EXAMPLES:

```
sage: A = MatrixSpace(ZZ,2)([1,2,3,4])
sage: A.echelon_form()
[1 0]
[0 2]
sage: A = MatrixSpace(ZZ,5)(range(25))
sage: A.echelon_form()
[  5   0  -5 -10 -15]
[  0   1   2   3   4]
[  0   0   0   0   0]
[  0   0   0   0   0]
[  0   0   0   0   0]
```

Getting a transformation matrix in the nonsquare case:

```
sage: A = matrix(ZZ,5,3,[1..15])
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=False)
sage: H
[1 2 3]
[0 3 6]
sage: U
[  0   0   0   4  -3]
[  0   0   0  13 -10]
```

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```
sage: U*A == H
True
```

Note: If 'ntl' is chosen for a non square matrix this function raises a ValueError.

Special cases: 0 or 1 rows:

```
sage: a = matrix(ZZ, 1,2,[0,-1])
sage: a.hermite_form()
[0 1]
sage: a.pivots()
(1,)
sage: a = matrix(ZZ, 1,2,[0,0])
sage: a.hermite_form()
[0 0]
sage: a.pivots()
()
sage: a = matrix(ZZ,1,3); a
[0 0 0]
sage: a.echelon_form(include_zero_rows=False)
[]
sage: a.echelon_form(include_zero_rows=True)
[0 0 0]
```

Illustrate using various algorithms.:

```
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='pari')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='pari0')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='pari4')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='padic')
[1 2 3]
[0 3 6]
[0 0 0]
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='default')
[1 2 3]
[0 3 6]
[0 0 0]
```

The 'ntl' algorithm doesn't work on matrices that do not have full rank.:

```
sage: matrix(ZZ,3,[1..9]).hermite_form(algorithm='ntl')
Traceback (most recent call last):
```

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```
...
ValueError: ntl only computes HNF for square matrices of full rank.
sage: matrix(ZZ,3,[0] +[2..9]).hermite_form(algorithm='ntl')
[1 0 0]
[0 1 0]
[0 0 3]
```

index_in_saturation(*proof=None*)

Return the index of self in its saturation.

INPUT:

- `proof` - (default: use proof.linear_algebra()); if False, the determinant calculations are done with proof=False.

OUTPUT:

- `positive integer` - the index of the row span of this matrix in its saturation

ALGORITHM: Use Hermite normal form twice to find an invertible matrix whose inverse transforms a matrix with the same row span as self to its saturation, then compute the determinant of that matrix.

EXAMPLES:

```
sage: A = matrix(ZZ, 2,3, [1..6]); A
[1 2 3]
[4 5 6]
sage: A.index_in_saturation()
3
sage: A.saturation()
[1 2 3]
[1 1 1]
```

insert_row(*index*, *row*)

Create a new matrix from self with.

INPUT:

- `index` - integer
- `row` - a vector

EXAMPLES:

```
sage: X = matrix(ZZ,3,range(9)); X
[0 1 2]
[3 4 5]
[6 7 8]
sage: X.insert_row(1, [1,5,-10])
[  0   1   2]
[  1   5 -10]
[  3   4   5]
[  6   7   8]
sage: X.insert_row(0, [1,5,-10])
[  1   5 -10]
[  0   1   2]
[  3   4   5]
```

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```
[  6   7   8]
sage: X.insert_row(3, [1,5,-10])
[  0   1   2]
[  3   4   5]
[  6   7   8]
[  1   5 -10]
```

integer_valued_polynomials_generators()

Determine the generators of the ring of integer valued polynomials on this matrix.

OUTPUT:

A pair `(mu_B, P)` where `P` is a list of polynomials in $\mathbf{Q}[X]$ such that

$$\{f \in \mathbf{Q}[X] \mid f(B) \in M_n(\mathbf{Z})\} = \mu_B \mathbf{Q}[X] + \sum_{g \in P} g\mathbf{Z}[X]$$

where B is this matrix.

EXAMPLES:

```
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: B.integer_valued_polynomials_generators()
(x^3 + x^2 - 12*x - 20, [1, 1/4*x^2 + 3/4*x + 1/2])
```

See also:

compute_J_ideal, *integer_valued_polynomials_generators()*

inverse_of_unit()

If self is a matrix with determinant 1 or -1 return the inverse of `self` as a matrix over ZZ.

EXAMPLES:

```
sage: m = matrix(ZZ, 2, [2,1,1,1]).inverse_of_unit()
sage: m
[ 1 -1]
[-1  2]
sage: parent(m)
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring

sage: matrix(2, [2,1,0,1]).inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: non-invertible matrix
```

is_LLL_reduced(*delta=None*, *eta=None*)

Return `True` if this lattice is (δ, η)-LLL reduced. See `self.LLL` for a definition of LLL reduction.

INPUT:

- `delta` – (default: 0.99) parameter δ as described above
- `eta` – (default: 0.501) parameter η as described above

EXAMPLES:

```
sage: A = random_matrix(ZZ, 10, 10)
sage: L = A.LLL()
sage: A.is_LLL_reduced()
False
sage: L.is_LLL_reduced()
True
```

is_one()

Tests whether self is the identity matrix.

EXAMPLES:

```
sage: matrix(2, [1,0,0,1]).is_one()
True
sage: matrix(2, [1,1,0,1]).is_one()
False
sage: matrix(2, 3, [1,0,0,0,1,0]).is_one()
False
```

is_primitive()

Test whether the matrix is primitive.

An integral matrix A is primitive if all its entries are non-negative and for some positive integer n the matrix A^n has all its entries positive.

EXAMPLES:

```
sage: m = matrix(3, [1,1,0,0,0,1,1,0,0])
sage: m.is_primitive()
True
sage: m**4
[3 2 1]
[1 1 1]
[2 1 1]

sage: m = matrix(4, [[1,1,0,0],[0,0,1,0],[0,0,0,1],[1,0,0,0]])
sage: m.is_primitive()
True
sage: m**6
[4 3 2 1]
[1 1 1 1]
[2 1 1 1]
[3 2 1 1]

sage: m = matrix(4, [[0,1,0,1],[1,0,1,0],[0,1,0,1],[1,0,1,0]])
sage: m.is_primitive()
False
```

Testing extremal matrices:

```
sage: def matrix1(d):
....:     m = matrix(d)
....:     m[0,0] = 1
....:     for i in range(d-1):
```

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```
....:         m[i,i+1] = m[i+1,i] = 1
....:     return m
sage: all(matrix1(d).is_primitive() for d in range(2,20))
True

sage: def matrix2(d):
....:     m = matrix(d)
....:     for i in range(d-1):
....:         m[i,i+1] = 1
....:     m[d-1,0] = m[d-1,1] = 1
....:     return m
sage: all(matrix2(d).is_primitive() for d in range(2,20))
True
```

Non-primitive families:

```
sage: def matrix3(d):
....:     m = matrix(d)
....:     m[0,0] = 1
....:     for i in range(d-1):
....:         m[i,i+1] = 1
....:     return m
sage: any(matrix3(d).is_primitive() for d in range(2,20))
False
```

minpoly(*var='x'*, *algorithm=None*)

INPUT:

- `var` - a variable name
- `algorithm` - (optional) either 'linbox' (default) or 'generic'

EXAMPLES:

```
sage: A = matrix(ZZ, 6, range(36))
sage: A.minpoly()
x^3 - 105*x^2 - 630*x

sage: A = Mat(ZZ, 6)([k^2 for k in range(36)])
sage: A.minpoly(algorithm='linbox')
x^4 - 2695*x^3 - 257964*x^2 + 1693440*x
sage: A.minpoly(algorithm='generic')
x^4 - 2695*x^3 - 257964*x^2 + 1693440*x
```

On non square matrices, this method raises an ArithmeticError:

```
sage: matrix(ZZ, 2, 1).minpoly()
Traceback (most recent call last):
...
ArithmeticError: only valid for square matrix
```

null_ideal(*b=0*)

Return the (b)-ideal of this matrix.

Let B be a $n \times n$ matrix. The *null ideal* modulo b, or (b)-ideal, is

$$N_{(b)}(B) = \{f \in \mathbf{Z}[X] \mid f(B) \in M_n(b\mathbf{Z})\}.$$

INPUT:

- `b` – an element of $\mathbf{Z}$ (default: 0)

OUTPUT:

An ideal in $\mathbf{Z}[X]$.

EXAMPLES:

```
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: B.null_ideal()
Principal ideal (x^3 + x^2 - 12*x - 20) of
    Univariate Polynomial Ring in x over Integer Ring
sage: B.null_ideal(8)
Ideal (8, x^3 + x^2 - 12*x - 20, 2*x^2 + 6*x + 4) of
    Univariate Polynomial Ring in x over Integer Ring
sage: B.null_ideal(6)
Ideal (6, 2*x^3 + 2*x^2 - 24*x - 40, 3*x^2 + 3*x) of
    Univariate Polynomial Ring in x over Integer Ring
```

See also:

compute_J_ideal, *null_ideal()*

p_minimal_polynomials(*p*, *s_max=None*)

Compute (p^s)-minimal polynomials ν_s of this matrix.

For $s \geq 0$, a (p^s)-minimal polynomial of a matrix B is a monic polynomial $f \in \mathbf{Z}[X]$ of minimal degree such that all entries of $f(B)$ are divisible by p^s.

Compute a finite subset $\mathcal{S}$ of the positive integers and (p^s)-minimal polynomials ν_s for $s \in \mathcal{S}$.

For $0 < t \leq \max \mathcal{S}$, a (p^t)-minimal polynomial is given by ν_s where $s = \min\{r \in \mathcal{S} \mid r \geq t\}$. For $t > \max \mathcal{S}$, the minimal polynomial of B is also a (p^t)-minimal polynomial.

INPUT:

- `p` – a prime in $\mathbf{Z}$
- `s_max` – a positive integer (default: `None`); if set, only (p^s)-minimal polynomials for `s <= s_max` are computed (see below for details)

OUTPUT:

A dictionary. Keys are the finite set $\mathcal{S}$, the values are the associated (p^s)-minimal polynomials ν_s, $s \in \mathcal{S}$.

Setting `s_max` only affects the output if `s_max` is at most $\max \mathcal{S}$ where $\mathcal{S}$ denotes the full set. In that case, only those ν_s with `s <= s_max` are returned where `s_max` is always included even if it is not included in the full set $\mathcal{S}$.

EXAMPLES:

```
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: B.p_minimal_polynomials(2)
{2: x^2 + 3*x + 2}
```

See also:

compute_J_ideal, *p_minimal_polynomials()*

pivots()

Return the pivot column positions of this matrix.

OUTPUT: a tuple of Python integers: the position of the first nonzero entry in each row of the echelon form.

EXAMPLES:

```
sage: n = 3; A = matrix(ZZ,n,range(n^2)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.pivots()
(0, 1)
sage: A.echelon_form()
[ 3  0 -3]
[ 0  1  2]
[ 0  0  0]
```

prod_of_row_sums(*cols*)

Return the product of the sums of the entries in the submatrix of `self` with given columns.

INPUT:

- `cols` – a list (or set) of integers representing columns of `self`

OUTPUT: an integer

EXAMPLES:

```
sage: a = matrix(ZZ,2,3,[1..6]); a
[1 2 3]
[4 5 6]
sage: a.prod_of_row_sums([0,2])
40
sage: (1+3)*(4+6)
40
sage: a.prod_of_row_sums(set([0,2]))
40
```

randomize(*density=1*, *x=None*, *y=None*, *distribution=None*, *nonzero=False*)

Randomize `density` proportion of the entries of this matrix, leaving the rest unchanged.

The parameters are the same as the ones for the integer ring's `random_element` function.

If `x` and `y` are given, randomized entries of this matrix have to be between `x` and `y` and have density 1.

INPUT:

- `self` - a mutable matrix over ZZ
- `density` - a float between 0 and 1
- `x, y` - if not `None`, these are passed to the `ZZ.random_element` function as the upper and lower endpoints in the uniform distribution
- `distribution` - would also be passed into `ZZ.random_element` if given

- nonzero - bool (default: False); whether the new entries are guaranteed to be zero

OUTPUT:

- None, the matrix is modified in-place

EXAMPLES:

```
sage: A = matrix(ZZ, 2,3, [1..6])
sage: ranks = [True, True, True]
sage: while any(ranks):
....:     A.randomize()
....:     ranks[A.rank()] = False

sage: mini = 0
sage: maxi = 0
sage: while mini != -30 and maxi != 30:
....:     A.randomize(x=-30, y=30)
....:     mini = min(min(A.list()), mini)
....:     maxi = min(min(A.list()), maxi)
```

rank(*algorithm='modp'*)

Return the rank of this matrix.

INPUT:

- algorithm – either 'modp' (default) or 'flint' or 'linbox'

OUTPUT:

- a nonnegative integer – the rank

Note: The rank is cached.

ALGORITHM:

If set to 'modp', first check if the matrix has maximum possible rank by working modulo one random prime. If not, call LinBox's rank function.

EXAMPLES:

```
sage: a = matrix(ZZ,2,3,[1..6]); a
[1 2 3]
[4 5 6]
sage: a.rank()
2
sage: a = matrix(ZZ,3,3,[1..9]); a
[1 2 3]
[4 5 6]
[7 8 9]
sage: a.rank()
2
```

Here is a bigger example - the rank is of course still 2:

```
sage: a = matrix(ZZ,100,[1..100^2]); a.rank()
2
```

rational_reconstruction(*N*)

Use rational reconstruction to lift self to a matrix over the rational numbers (if possible), where we view self as a matrix modulo N.

INPUT:

- `N` - an integer

OUTPUT:

- `matrix` - over QQ or raise a ValueError

EXAMPLES: We create a random 4x4 matrix over ZZ.

```
sage: A = matrix(ZZ, 4, [4, -4, 7, 1, -1, 1, -1, -12, -1, -1, 1, -1, -3, 1, 5, -
↪1])
```

There isn't a unique rational reconstruction of it:

```
sage: A.rational_reconstruction(11)
Traceback (most recent call last):
...
ValueError: rational reconstruction does not exist
```

We throw in a denominator and reduce the matrix modulo 389 - it does rationally reconstruct.

```
sage: B = (A/3 % 389).change_ring(ZZ)
sage: B.rational_reconstruction(389) == A/3
True
```

row(*i*, *from_list=False*)

Return the i-th row of this matrix as a dense vector.

INPUT:

- `i` - integer
- `from_list` - ignored

EXAMPLES:

```
sage: m = matrix(ZZ, 2, [1, -2, 3, 4])
sage: m.row(0)
(1, -2)
sage: m.row(1)
(3, 4)
sage: m.row(1, from_list=True)
(3, 4)
sage: m.row(-2)
(1, -2)

sage: m.row(2)
Traceback (most recent call last):
...
IndexError: row index out of range
sage: m.row(-3)
Traceback (most recent call last):
...
IndexError: row index out of range
```

saturation(*p=0*, *proof=None*, *max_dets=5*)

Return a saturation matrix of self, which is a matrix whose rows span the saturation of the row span of self. This is not unique.

The saturation of a $\mathbf{Z}$ module M embedded in $\mathbf{Z}^n$ is a module S that contains M with finite index such that $\mathbf{Z}^n/S$ is torsion free. This function takes the row span M of self, and finds another matrix of full rank with row span the saturation of M.

INPUT:

- `p` - (default: 0); if nonzero given, saturate only at the prime p, i.e., return a matrix whose row span is a $\mathbf{Z}$-module S that contains self and such that the index of S in its saturation is coprime to p. If p is None, return full saturation of self.
- `proof` - (default: use proof.linear_algebra()); if False, the determinant calculations are done with proof=False.
- `max_dets` - (default: 5); technical parameter - max number of determinant to compute when bounding prime divisor of self in its saturation.

OUTPUT:

- `matrix` - a matrix over ZZ

Note: The result is *not* cached.

ALGORITHM: 1. Replace input by a matrix of full rank got from a subset of the rows. 2. Divide out any common factors from rows. 3. Check max_dets random dets of submatrices to see if their GCD (with p) is 1 - if so matrix is saturated and we're done. 4. Finally, use that if A is a matrix of full rank, then $hnf(transpose(A))^{-1} * A$ is a saturation of A.

EXAMPLES:

```
sage: A = matrix(ZZ, 3, 5, [-51, -1509, -71, -109, -593, -19, -341, 4, 86, 98,␣
↪0, -246, -11, 65, 217])
sage: A.echelon_form()
[      1       5    2262   20364   56576]
[      0       6   35653  320873  891313]
[      0       0   42993  386937 1074825]
sage: S = A.saturation(); S
[  -51 -1509   -71  -109  -593]
[  -19  -341     4    86    98]
[   35   994    43    51   347]
```

Notice that the saturation spans a different module than A.

```
sage: S.echelon_form()
[ 1  2  0  8 32]
[ 0  3  0 -2 -6]
[ 0  0  1  9 25]
sage: V = A.row_space(); W = S.row_space()
sage: V.is_submodule(W)
True
sage: V.index_in(W)
85986
sage: V.index_in_saturation()
85986
```

We illustrate each option:

```
sage: S = A.saturation(p=2)
sage: S = A.saturation(proof=False)
sage: S = A.saturation(max_dets=2)
```

smith_form(*transformation=True*, *integral=None*)

Return the smith normal form of this matrix, that is the diagonal matrix S with diagonal entries the ordered elementary divisors of this matrix.

INPUT:

- `transformation` – a boolean (default: `True`); whether to return the transformation matrices U and V such that $S = U \cdot self \cdot V$.
- `integral` – a subring of the base ring or `True` (default: `None`); ignored for matrices with integer entries.

Note: The *elementary_divisors()* function, which returns the diagonal entries of S, is VASTLY faster than this function.

The elementary divisors are the invariants of the finite abelian group that is the cokernel of this matrix. They are ordered in reverse by divisibility.

EXAMPLES:

```
sage: A = MatrixSpace(IntegerRing(), 3)(range(9))
sage: D, U, V = A.smith_form()
sage: D
[1 0 0]
[0 3 0]
[0 0 0]
sage: U
[ 0  2 -1]
[ 0 -1  1]
[ 1 -2  1]
sage: V
[ 0  0  1]
[-1  2 -2]
[ 1 -1  1]
sage: U*A*V
[1 0 0]
[0 3 0]
[0 0 0]
```

It also makes sense for nonsquare matrices:

```
sage: A = Matrix(ZZ,3,2,range(6))
sage: D, U, V = A.smith_form()
sage: D
[1 0]
[0 2]
[0 0]
sage: U
[ 0  2 -1]
```

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```
[ 0 -1  1]
[ 1 -2  1]
sage: V
[-1  1]
[ 1  0]
sage: U * A * V
[1 0]
[0 2]
[0 0]
```

Empty matrices are handled sensibly (see github issue #3068):

```
sage: m = MatrixSpace(ZZ, 2,0)(0); d,u,v = m.smith_form(); u*m*v == d
True
sage: m = MatrixSpace(ZZ, 0,2)(0); d,u,v = m.smith_form(); u*m*v == d
True
sage: m = MatrixSpace(ZZ, 0,0)(0); d,u,v = m.smith_form(); u*m*v == d
True
```

See also:

elementary_divisors()

symplectic_form()

Find a symplectic basis for self if self is an anti-symmetric, alternating matrix.

Return a pair (F, C) such that the rows of C form a symplectic basis for self and `F = C * self * C.transpose()`.

Raise a ValueError if self is not anti-symmetric, or self is not alternating.

Anti-symmetric means that $M = -M^t$. Alternating means that the diagonal of M is identically zero.

A symplectic basis is a basis of the form $e_1, \ldots, e_j, f_1, \ldots f_j, z_1, \ldots, z_k$ such that

- $z_i M v^t = 0$ for all vectors v
- $e_i M {e_j}^t = 0$ for all i, j
- $f_i M {f_j}^t = 0$ for all i, j
- $e_i M {f_i}^t = 1$ for all i
- $e_i M {f_j}^t = 0$ **for all** i **not equal** j.

The ordering for the factors $d_i | d_{i+1}$ and for the placement of zeroes was chosen to agree with the output of *smith_form()*.

See the example for a pictorial description of such a basis.

EXAMPLES:

```
sage: E = matrix(ZZ, 5, 5, [0, 14, 0, -8, -2, -14, 0, -3, -11, 4, 0, 3, 0, 0, 0,
↪ 8, 11, 0, 0, 8, 2, -4, 0, -8, 0]); E
[  0  14   0  -8  -2]
[-14   0  -3 -11   4]
[  0   3   0   0   0]
[  8  11   0   0   8]
```

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```
[  2  -4   0  -8   0]
sage: F, C = E.symplectic_form()
sage: F
[ 0  0  1  0  0]
[ 0  0  0  2  0]
[-1  0  0  0  0]
[ 0 -2  0  0  0]
[ 0  0  0  0  0]
sage: F == C * E * C.transpose()
True
sage: E.smith_form()[0]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 2 0 0]
[0 0 0 2 0]
[0 0 0 0 0]
```

transpose()

Return the transpose of self, without changing self.

EXAMPLES:

We create a matrix, compute its transpose, and note that the original matrix is not changed.

```
sage: A = matrix(ZZ, 2, 3, range(6))
sage: type(A)
<class 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
sage: B = A.transpose()
sage: print(B)
[0 3]
[1 4]
[2 5]
sage: print(A)
[0 1 2]
[3 4 5]
```

`.T` is a convenient shortcut for the transpose:

```
sage: A.T
[0 3]
[1 4]
[2 5]
```

```
sage: A.subdivide(None, 1); A
[0|1 2]
[3|4 5]
sage: A.transpose()
[0 3]
[---]
[1 4]
[2 5]
```

CHAPTER

SIXTEEN

SPARSE INTEGER MATRICES

AUTHORS:

- William Stein (2007-02-21)
- Soroosh Yazdani (2007-02-21)

class sage.matrix.matrix_integer_sparse.**Matrix_integer_sparse**

Bases: *Matrix_sparse*

Create a sparse matrix over the integers.

INPUT:

- `parent` – a matrix space over `ZZ`
- `entries` – see *matrix()*
- `copy` – ignored (for backwards compatibility)
- `coerce` – if False, assume without checking that the entries are of type `Integer`.

charpoly(*var='x'*, *algorithm=None*)

Return the characteristic polynomial of this matrix.

INPUT:

- `var` – (optional, default `'x'`) the name of the variable of the polynomial
- `algorithm` – (optional, default `None`) one of `None`, `'linbox'`, or an algorithm accepted by *sage.matrix.matrix_sparse.Matrix_sparse.charpoly()*

EXAMPLES:

```
sage: M = MatrixSpace(ZZ, 4, sparse=True)
sage: m = M()
sage: m[0,0] = m[1,2] = m[2,3] = m[3,3] = 1
sage: m[0,2] = m[1,3] = m[2,0] = m[3,0] = -3
sage: m[1,1] = 2
sage: m
[ 1  0 -3  0]
[ 0  2  1 -3]
[-3  0  0  1]
[-3  0  0  1]
sage: m.charpoly()
x^4 - 4*x^3 - 4*x^2 + 16*x
```

elementary_divisors(*algorithm='pari'*)

Return the elementary divisors of self, in order.

The elementary divisors are the invariants of the finite abelian group that is the cokernel of *left* multiplication by this matrix. They are ordered in reverse by divisibility.

INPUT:

- self – matrix
- algorithm – (default: 'pari')
  - 'pari': works robustly, but is slower.
  - 'linbox' – use linbox (currently off, broken)

OUTPUT:

list of integers

EXAMPLES:

```
sage: matrix(3, range(9),sparse=True).elementary_divisors()
[1, 3, 0]
sage: M = matrix(ZZ, 3, [1,5,7, 3,6,9, 0,1,2], sparse=True)
sage: M.elementary_divisors()
[1, 1, 6]
```

This returns a copy, which is safe to change:

```
sage: edivs = M.elementary_divisors()
sage: edivs.pop()
6
sage: M.elementary_divisors()
[1, 1, 6]
```

See also:

smith_form()

hermite_form(*algorithm='default'*, *cutoff=0*, ***kwds*)

Return the echelon form of `self`.

Note: This row reduction does not use division if the matrix is not over a field (e.g., if the matrix is over the integers). If you want to calculate the echelon form using division, then use `rref()`, which assumes that the matrix entries are in a field (specifically, the field of fractions of the base ring of the matrix).

INPUT:

- `algorithm` – string. Which algorithm to use. Choices are
  - `'default'`: Let Sage choose an algorithm (default).
  - `'classical'`: Gauss elimination.
  - `'partial_pivoting'`: Gauss elimination, using partial pivoting (if base ring has absolute value)
  - `'scaled_partial_pivoting'`: Gauss elimination, using scaled partial pivoting (if base ring has absolute value)

- `'scaled_partial_pivoting_valuation'`: Gauss elimination, using scaled partial pivoting (if base ring has valuation)
- `'strassen'`: use a Strassen divide and conquer algorithm (if available)

- `cutoff` – integer. Only used if the Strassen algorithm is selected.
- `transformation` – boolean. Whether to also return the transformation matrix. Some matrix backends do not provide this information, in which case this option is ignored.

OUTPUT:

The reduced row echelon form of `self`, as an immutable matrix. Note that self is *not* changed by this command. Use `echelonize()` to change `self` in place.

If the optional parameter `transformation=True` is specified, the output consists of a pair (E, T) of matrices where E is the echelon form of `self` and T is the transformation matrix.

EXAMPLES:

```
sage: MS = MatrixSpace(GF(19),2,3)
sage: C = MS.matrix([1,2,3,4,5,6])
sage: C.rank()
2
sage: C.nullity()
0
sage: C.echelon_form()
[ 1  0 18]
[ 0  1  2]
```

The matrix library used for $\mathbf{Z}/p$-matrices does not return the transformation matrix, so the `transformation` option is ignored:

```
sage: C.echelon_form(transformation=True)
[ 1  0 18]
[ 0  1  2]

sage: D = matrix(ZZ, 2, 3, [1,2,3,4,5,6])
sage: D.echelon_form(transformation=True)
(
[1 2 3]  [ 1  0]
[0 3 6], [ 4 -1]
)
sage: E, T = D.echelon_form(transformation=True)
sage: T*D == E
True
```

minpoly(*var='x'*, *algorithm=None*)

Return the minimal polynomial of this matrix.

INPUT:

- `var` – (optional, default `'x'`) the name of the variable of the polynomial
- `algorithm` – (optional, default `None`) one of `None`, `'linbox'`, or an algorithm accepted by `sage.matrix.matrix_sparse.Matrix_sparse.minpoly()`

EXAMPLES:

```
sage: M = MatrixSpace(ZZ, 4, sparse=True)
sage: m = M({(0, 0):2, (1, 1):1, (2, 1):-8, (2, 2):2, (2, 3):-1, (3, 3):1})
sage: m
[ 2  0  0  0]
[ 0  1  0  0]
[ 0 -8  2 -1]
[ 0  0  0  1]
sage: m.minpoly()
x^2 - 3*x + 2
```

rank(*algorithm=None*)

Compute the rank of this matrix.

INPUT:

- `algorithm` – (optional) one of `None`, `'linbox'` or `'generic'`

EXAMPLES:

```
sage: M = MatrixSpace(ZZ, 3, 2, sparse=True)
sage: m = M([1, 0, 2, 3, -1, 0])
sage: m.rank()
2
```

rational_reconstruction(*N*)

Use rational reconstruction to lift self to a matrix over the rational numbers (if possible), where we view self as a matrix modulo N.

EXAMPLES:

```
sage: A = matrix(ZZ, 3, 4, [(1/3)%500, 2, 3, (-4)%500, 7, 2, 2, 3, 4, 3, 4, (5/
↪7)%500], sparse=True)
sage: A.rational_reconstruction(500)
[1/3   2   3  -4]
[  7   2   2   3]
[  4   3   4 5/7]
```

smith_form(*transformation=True*, *integral=None*)

Return the smith normal form of this matrix, that is the diagonal matrix S with diagonal entries the ordered elementary divisors of this matrix.

INPUT:

- `transformation` – a boolean (default: `True`); whether to return the transformation matrices U and V such that $S = U \cdot self \cdot V$.
- `integral` – a subring of the base ring or `True` (default: `None`); ignored for matrices with integer entries.

This version is for sparse matrices and simply makes the matrix dense and calls the version for dense integer matrices.

Note: The *elementary_divisors()* function, which returns the diagonal entries of S, is VASTLY faster than this function.

The elementary divisors are the invariants of the finite abelian group that is the cokernel of this matrix. They are ordered in reverse by divisibility.

EXAMPLES:

```
sage: A = MatrixSpace(IntegerRing(), 3, sparse=True)(range(9))
sage: D, U, V = A.smith_form()
sage: D
[1 0 0]
[0 3 0]
[0 0 0]
sage: U
[ 0  2 -1]
[ 0 -1  1]
[ 1 -2  1]
sage: V
[ 0  0  1]
[-1  2 -2]
[ 1 -1  1]
sage: U*A*V
[1 0 0]
[0 3 0]
[0 0 0]
```

It also makes sense for nonsquare matrices:

```
sage: A = Matrix(ZZ,3,2,range(6), sparse=True)
sage: D, U, V = A.smith_form()
sage: D
[1 0]
[0 2]
[0 0]
sage: U
[ 0  2 -1]
[ 0 -1  1]
[ 1 -2  1]
sage: V
[-1  1]
[ 1  0]
sage: U * A * V
[1 0]
[0 2]
[0 0]
```

The examples above show that github issue #10626 has been implemented.

See also:

elementary_divisors()

CHAPTER

SEVENTEEN

MODULAR ALGORITHM TO COMPUTE HERMITE NORMAL FORMS OF INTEGER MATRICES

AUTHORS:

- Clement Pernet and William Stein (2008-02-07): initial version

sage.matrix.matrix_integer_dense_hnf.**add_column**(*B*, *H_B*, *a*, *proof*)

The add column procedure.

INPUT:

- B – a square matrix (may be singular)
- H_B – the Hermite normal form of B
- a – an n x 1 matrix, where B has n rows
- proof – bool; whether to prove result correct, in case we use fallback method.

OUTPUT:

- x – a vector such that H' = H_B.augment(x) is the HNF of A = B.augment(a).

EXAMPLES:

```
sage: B = matrix(ZZ, 3, 3, [1,2,5, 0,-5,3, 1,1,2])
sage: H_B = B.echelon_form()
sage: a = matrix(ZZ, 3, 1, [1,8,-2])
sage: import sage.matrix.matrix_integer_dense_hnf as hnf
sage: x = hnf.add_column(B, H_B, a, True); x
[18]
[ 3]
[23]
sage: H_B.augment(x)
[ 1  0 17 18]
[ 0  1  3  3]
[ 0  0 18 23]
sage: B.augment(a).echelon_form()
[ 1  0 17 18]
[ 0  1  3  3]
[ 0  0 18 23]
```

sage.matrix.matrix_integer_dense_hnf.**add_column_fallback**(*B*, *a*, *proof*)

Simplistic version of add_column, in case the powerful clever one fails (e.g., B is singular).

INPUT:

B – a square matrix (may be singular) a – an n x 1 matrix, where B has n rows proof – bool; whether to prove result correct

OUTPUT:

x – a vector such that H' = H_B.augment(x) is the HNF of A = B.augment(a).

EXAMPLES:

```
sage: B = matrix(ZZ,3, [-1, -1, 1, -3, 8, -2, -1, -1, -1])
sage: a = matrix(ZZ,3,1, [1,2,3])
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.add_column_fallback(B, a, True)
[-3]
[-7]
[-2]
sage: matrix_integer_dense_hnf.add_column_fallback(B, a, False)
[-3]
[-7]
[-2]
sage: B.augment(a).hermite_form()
[ 1  1  1 -3]
[ 0 11  1 -7]
[ 0  0  2 -2]
```

sage.matrix.matrix_integer_dense_hnf.**add_row**(*A*, *b*, *pivots*, *include_zero_rows*)

The add row procedure.

INPUT:

- A – a matrix in Hermite normal form with n column
- b – an n x 1 row matrix
- pivots – sorted list of integers; the pivot positions of A.

OUTPUT:

- H – the Hermite normal form of A.stack(b).
- new_pivots – the pivot columns of H.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as hnf
sage: A = matrix(ZZ, 2, 3, [-21, -7, 5, 1,20,-7])
sage: b = matrix(ZZ, 1,3, [-1,1,-1])
sage: hnf.add_row(A, b, A.pivots(), True)
(
[ 1  6 29]
[ 0  7 28]
[ 0  0 46], [0, 1, 2]
)
sage: A.stack(b).echelon_form()
[ 1  6 29]
[ 0  7 28]
[ 0  0 46]
```

sage.matrix.matrix_integer_dense_hnf.**benchmark_hnf**(*nrange*, *bits=4*)

Run benchmark program.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as hnf
sage: hnf.benchmark_hnf([10,25],32)
('sage', 10, 32, ...),
('sage', 25, 32, ...),
```

sage.matrix.matrix_integer_dense_hnf.**benchmark_magma_hnf**(*nrange*, *bits=4*)

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as hnf
sage: hnf.benchmark_magma_hnf([50,100],32)     # optional - magma
('magma', 50, 32, ...),
('magma', 100, 32, ...),
```

sage.matrix.matrix_integer_dense_hnf.**det_from_modp_and_divisor**(*A*, *d*, *p*, *z_mod*, *moduli*, *z_so_far=1*, *N_so_far=1*)

This is used for internal purposes for computing determinants quickly (with the hybrid p-adic / multimodular algorithm).

INPUT:

- A – a square matrix
- d – a divisor of the determinant of A
- p – a prime
- z_mod – values of det/d (mod ...)
- moduli – the moduli so far
- z_so_far – for a modulus p in the list moduli, (z_so_far mod p) is the determinant of A modulo p.
- N_so_far – N_so_far is the product over the primes in the list moduli.

OUTPUT:

- A triple (det bound, new z_so_far, new N_so_far).

EXAMPLES:

```
sage: a = matrix(ZZ, 3, [6, 1, 2, -56, -2, -1, -11, 2, -3])
sage: factor(a.det())
-1 * 13 * 29
sage: d = 13
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.det_from_modp_and_divisor(a, d, 97, [], [])
(-377, -29, 97)
sage: a.det()
-377
```

sage.matrix.matrix_integer_dense_hnf.**det_given_divisor**(*A*, *d*, *proof=True*, *stabilize=2*)

Given a divisor d of the determinant of A, compute the determinant of A.

INPUT:

- A – a square integer matrix

- d – a nonzero integer that is assumed to divide the determinant of A
- proof – bool (default: True) compute det modulo enough primes so that the determinant is computed provably correctly (via the Hadamard bound). It would be VERY hard for det() to fail even with proof=False.
- stabilize – int (default: 2) if proof = False, then compute the determinant modulo p until stabilize successive modulo determinant computations stabilize.

OUTPUT:

integer – determinant

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: a = matrix(ZZ,3,[-1, -1, -1, -20, 4, 1, -1, 1, 2])
sage: matrix_integer_dense_hnf.det_given_divisor(a, 3)
-30
sage: matrix_integer_dense_hnf.det_given_divisor(a, 3, proof=False)
-30
sage: matrix_integer_dense_hnf.det_given_divisor(a, 3, proof=False, stabilize=1)
-30
sage: a.det()
-30
```

Here we illustrate proof=False giving a wrong answer:

```
sage: p = matrix_integer_dense_hnf.max_det_prime(2)
sage: q = previous_prime(p)
sage: a = matrix(ZZ, 2, [p, 0, 0, q])
sage: p * q
70368442188091
sage: matrix_integer_dense_hnf.det_given_divisor(a, 1, proof=False, stabilize=2)
0
```

This still works, because we do not work modulo primes that divide the determinant bound, which is found using a p-adic algorithm:

```
sage: a.det(proof=False, stabilize=2)
70368442188091
```

3 primes is enough:

```
sage: matrix_integer_dense_hnf.det_given_divisor(a, 1, proof=False, stabilize=3)
70368442188091
sage: matrix_integer_dense_hnf.det_given_divisor(a, 1, proof=False, stabilize=5)
70368442188091
sage: matrix_integer_dense_hnf.det_given_divisor(a, 1, proof=True)
70368442188091
```

sage.matrix.matrix_integer_dense_hnf.**det_padic**(*A*, *proof=True*, *stabilize=2*)

Return the determinant of A, computed using a p-adic/multimodular algorithm.

INPUT:

- A – a square matrix
- proof – boolean

- `stabilize` (default: 2) – if proof False, number of successive primes so that CRT det must stabilize.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as h
sage: a = matrix(ZZ, 3, [1..9])
sage: h.det_padic(a)
0
sage: a = matrix(ZZ, 3, [1,2,5,-7,8,10,192,5,18])
sage: h.det_padic(a)
-3669
sage: a.determinant(algorithm='ntl')
-3669
```

sage.matrix.matrix_integer_dense_hnf.**double_det**(*A*, *b*, *c*, *proof*)

Compute the determinants of the stacked integer matrices A.stack(b) and A.stack(c).

INPUT:

- A – an (n-1) x n matrix
- b – an 1 x n matrix
- c – an 1 x n matrix
- proof – whether or not to compute the det modulo enough times to provably compute the determinant.

OUTPUT:

- a pair of two integers.

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_hnf import double_det
sage: A = matrix(ZZ, 2, 3, [1,2,3, 4,-2,5])
sage: b = matrix(ZZ, 1, 3, [1,-2,5])
sage: c = matrix(ZZ, 1, 3, [8,2,10])
sage: A.stack(b).det()
-48
sage: A.stack(c).det()
42
sage: double_det(A, b, c, False)
(-48, 42)
```

sage.matrix.matrix_integer_dense_hnf.**extract_ones_data**(*H*, *pivots*)

Compute ones data and corresponding submatrices of H.

This is used to optimized the *add_row()* function.

INPUT:

- H – a matrix in HNF
- pivots – list of all pivot column positions of H

OUTPUT:

C, D, E, onecol, onerow, non_onecol, non_onerow where onecol, onerow, non_onecol, non_onerow are as for the ones function, and C, D, E are matrices:

- C – submatrix of all non-onecol columns and onecol rows

- D – all non-onecol columns and other rows
- E – inverse of D

If D is not invertible or there are 0 or more than 2 non onecols, then C, D, and E are set to None.

EXAMPLES:

```
sage: H = matrix(ZZ, 3, 4, [1, 0, 0, 7, 0, 1, 5, 2, 0, 0, 6, 6])
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.extract_ones_data(H, [0,1,2])
(
[0]
[5], [6], [1/6], [0, 1], [0, 1], [2], [2]
)
```

Here we get None's since the (2,2) position submatrix is not invertible.
sage: H = matrix(ZZ, 3, 5, [1, 0, 0, 45, -36, 0, 1, 0, 131, -107, 0, 0, 0, 178, -145]); H [1 0 0 45 -36] [0 1 0 131 -107] [0 0 0 178 -145] sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf sage: matrix_integer_dense_hnf.extract_ones_data(H, [0,1,3]) (None, None, None, [0, 1], [0, 1], [2], [2])

sage.matrix.matrix_integer_dense_hnf.**hnf**(*A*, *include_zero_rows=True*, *proof=True*)

Return the Hermite Normal Form of a general integer matrix A, along with the pivot columns.

INPUT:

- A – an n x m matrix A over the integers.
- include_zero_rows – bool (default: True) whether or not to include zero rows in the output matrix
- proof – whether or not to prove the result correct.

OUTPUT:

- matrix – the Hermite normal form of A
- pivots – the pivot column positions of A

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: a = matrix(ZZ,3,5,[-2, -6, -3, -17, -1, 2, -1, -1, -2, -1, -2, -2, -6, 9, 2])
sage: matrix_integer_dense_hnf.hnf(a)
(
[   2    0   26  -75  -10]
[   0    1   27  -73   -9]
[   0    0   37 -106  -13], [0, 1, 2]
)
sage: matrix_integer_dense_hnf.hnf(a.transpose())
(
[1 0 0]
[0 1 0]
[0 0 1]
[0 0 0]
[0 0 0], [0, 1, 2]
)
sage: matrix_integer_dense_hnf.hnf(a.transpose(), include_zero_rows=False)
(
```

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```
[1 0 0]
[0 1 0]
[0 0 1], [0, 1, 2]
)
```

sage.matrix.matrix_integer_dense_hnf.**hnf_square**(*A*, *proof*)

INPUT:

- a nonsingular n x n matrix A over the integers.

OUTPUT:

- the Hermite normal form of A.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as hnf
sage: A = matrix(ZZ, 3, [-21, -7, 5, 1,20,-7, -1,1,-1])
sage: hnf.hnf_square(A, False)
[ 1  6 29]
[ 0  7 28]
[ 0  0 46]
sage: A.echelon_form()
[ 1  6 29]
[ 0  7 28]
[ 0  0 46]
```

sage.matrix.matrix_integer_dense_hnf.**hnf_with_transformation**(*A*, *proof=True*)

Compute the HNF H of A along with a transformation matrix U such that U*A = H.

INPUT:

- A – an n x m matrix A over the integers.
- proof – whether or not to prove the result correct.

OUTPUT:

- matrix – the Hermite normal form H of A
- U – a unimodular matrix such that U * A = H

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: A = matrix(ZZ, 2, [1, -5, -10, 1, 3, 197]); A
[  1  -5 -10]
[  1   3 197]
sage: H, U = matrix_integer_dense_hnf.hnf_with_transformation(A)
sage: H
[  1   3 197]
[  0   8 207]
sage: U
[ 0  1]
[-1  1]
sage: U*A
[  1   3 197]
[  0   8 207]
```

sage.matrix.matrix_integer_dense_hnf.**hnf_with_transformation_tests**(*n=10*, *m=5*, *trials=10*)

Use this to randomly test that hnf with transformation matrix is working.

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_hnf import hnf_with_transformation_tests
sage: hnf_with_transformation_tests(n=15, m=10, trials=10)
0 1 2 3 4 5 6 7 8 9
```

sage.matrix.matrix_integer_dense_hnf.**interleave_matrices**(*A*, *B*, *cols1*, *cols2*)

INPUT:

- A, B – matrices with the same number of rows
- cols1, cols2 – disjoint lists of integers

OUTPUT:

construct a new matrix C by sticking the columns of A at the positions specified by cols1 and the columns of B at the positions specified by cols2.

EXAMPLES:

```
sage: A = matrix(ZZ, 2, [1,2,3,4]); B = matrix(ZZ, 2, [-1,5,2,3])
sage: A
[1 2]
[3 4]
sage: B
[-1  5]
[ 2  3]
sage: import sage.matrix.matrix_integer_dense_hnf as hnf
sage: hnf.interleave_matrices(A, B, [1,3], [0,2])
[-1  1  5  2]
[ 2  3  3  4]
```

sage.matrix.matrix_integer_dense_hnf.**is_in_hnf_form**(*H*, *pivots*)

Return whether the matrix `H` is in Hermite normal form with given pivot columns.

INPUT:

- `H` – matrix
- `pivots` – sorted list of integers

OUTPUT:

boolean

EXAMPLES:

```
sage: a = matrix(ZZ,3,5,[-2, -6, -3, -17, -1, 2, -1, -1, -2, -1, -2, -2, -6, 9, 2])
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.is_in_hnf_form(a,range(3))
False
sage: e = a.hermite_form(); p = a.pivots()
sage: matrix_integer_dense_hnf.is_in_hnf_form(e, p)
True
```

sage.matrix.matrix_integer_dense_hnf.**max_det_prime**(*n*)

Return the largest prime so that it is reasonably efficient to compute modulo that prime with n x n matrices in LinBox.

INPUT:

- n – a positive integer

OUTPUT:

a prime number

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_hnf import max_det_prime
sage: max_det_prime(10000)
8388593
sage: max_det_prime(1000)
8388593
sage: max_det_prime(10)
8388593
```

sage.matrix.matrix_integer_dense_hnf.**ones**(*H*, *pivots*)

Find all 1 pivot columns of the matrix H in Hermite form, along with the corresponding rows, and also the non 1 pivot columns and non-pivot rows. Here a 1 pivot column is a pivot column so that the leading bottom entry is 1.

INPUT:

- H – matrix in Hermite form
- pivots – list of integers (all pivot positions of H).

OUTPUT:

4-tuple of integer lists: onecol, onerow, non_oneol, non_onerow

EXAMPLES:

```
sage: H = matrix(ZZ, 3, 5, [1, 0, 0, 45, -36, 0, 1, 0, 131, -107, 0, 0, 0, 178, -
↪145]); H
[   1    0    0   45  -36]
[   0    1    0  131 -107]
[   0    0    0  178 -145]
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.ones(H, [0,1,3])
([0, 1], [0, 1], [2], [2])
```

sage.matrix.matrix_integer_dense_hnf.**pad_zeros**(*A*, *nrows*)

Add zeros to the bottom of A so that the resulting matrix has nrows.

INPUT:

- A – a matrix
- nrows – an integer that is at least as big as the number of rows of A.

OUTPUT:

a matrix with nrows rows.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: a = matrix(ZZ, 2, 4, [1, 0, 0, 7, 0, 1, 5, 2])
sage: matrix_integer_dense_hnf.pad_zeros(a, 4)
[1 0 0 7]
[0 1 5 2]
[0 0 0 0]
[0 0 0 0]
sage: matrix_integer_dense_hnf.pad_zeros(a, 2)
[1 0 0 7]
[0 1 5 2]
```

sage.matrix.matrix_integer_dense_hnf.**pivots_of_hnf_matrix**(*H*)

Return the pivot columns of a matrix H assumed to be in HNF.

INPUT:

- H – a matrix that must be HNF

OUTPUT:

- list – list of pivots

EXAMPLES:

```
sage: H = matrix(ZZ, 3, 5, [1, 0, 0, 45, -36, 0, 1, 0, 131, -107, 0, 0, 0, 178, -
↪145]); H
[   1    0    0   45  -36]
[   0    1    0  131 -107]
[   0    0    0  178 -145]
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.pivots_of_hnf_matrix(H)
[0, 1, 3]
```

sage.matrix.matrix_integer_dense_hnf.**probable_hnf**(*A*, *include_zero_rows*, *proof*)

Return the HNF of A or raise an exception if something involving the randomized nature of the algorithm goes wrong along the way.

Calling this function again a few times should result it in it working, at least if proof=True.

INPUT:

- A – a matrix
- include_zero_rows – bool
- proof – bool

OUTPUT:

the Hermite normal form of A. cols – pivot columns

EXAMPLES:

```
sage: a = matrix(ZZ,4,3,[-1, -1, -1, -20, 4, 1, -1, 1, 2,1,2,3])
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.probable_hnf(a, True, True)
(
[1 0 0]
[0 1 0]
```

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```
[0 0 1]
[0 0 0], [0, 1, 2]
)
sage: matrix_integer_dense_hnf.probable_hnf(a, False, True)
(
[1 0 0]
[0 1 0]
[0 0 1], [0, 1, 2]
)
sage: matrix_integer_dense_hnf.probable_hnf(a, False, False)
(
[1 0 0]
[0 1 0]
[0 0 1], [0, 1, 2]
)
```

sage.matrix.matrix_integer_dense_hnf.**probable_pivot_columns**(*A*)

INPUT:

- A – a matrix

OUTPUT:

a tuple of integers

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: a = matrix(ZZ,3,[0, -1, -1, 0, -20, 1, 0, 1, 2])
sage: a
[  0  -1  -1]
[  0 -20   1]
[  0   1   2]
sage: matrix_integer_dense_hnf.probable_pivot_columns(a)
(1, 2)
```

sage.matrix.matrix_integer_dense_hnf.**probable_pivot_rows**(*A*)

Return rows of A that are very likely to be pivots.

This really finds the pivots of A modulo a random prime.

INPUT:

- A – a matrix

OUTPUT:

a tuple of integers

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: a = matrix(ZZ,3,[0, -1, -1, 0, -20, 1, 0, 1, 2])
sage: a
[  0  -1  -1]
[  0 -20   1]
[  0   1   2]
```

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```
sage: matrix_integer_dense_hnf.probable_pivot_rows(a)
(0, 1)
```

sage.matrix.matrix_integer_dense_hnf.**sanity_checks**(*times=50*, *n=8*, *m=5*, *proof=True*, *stabilize=2*, *check_using_magma=True*)

Run random sanity checks on the modular p-adic HNF with tall and wide matrices both dense and sparse.

INPUT:

- times – number of times to randomly try matrices with each shape
- n – number of rows
- m – number of columns
- proof – test with proof true
- stabilize – parameter to pass to hnf algorithm when proof is False
- check_using_magma – if True use Magma instead of PARI to check correctness of computed HNF's. Since PARI's HNF is buggy and slow (as of 2008-02-16 non-pivot entries sometimes are not normalized to be nonnegative) the default is Magma.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_hnf as matrix_integer_dense_hnf
sage: matrix_integer_dense_hnf.sanity_checks(times=5, check_using_magma=False)
small 8 x 5
0 1 2 3 4  (done)
big 8 x 5
0 1 2 3 4  (done)
small 5 x 8
0 1 2 3 4  (done)
big 5 x 8
0 1 2 3 4  (done)
sparse 8 x 5
0 1 2 3 4  (done)
sparse 5 x 8
0 1 2 3 4  (done)
ill conditioned -- 1000*A -- 8 x 5
0 1 2 3 4  (done)
ill conditioned -- 1000*A but one row -- 8 x 5
0 1 2 3 4  (done)
```

sage.matrix.matrix_integer_dense_hnf.**solve_system_with_difficult_last_row**(*B*, *a*)

Solve B*x = a when the last row of B contains huge entries using a clever trick that reduces the problem to solve C*x = a where C is B but with the last row replaced by something small, along with one easy null space computation. The latter are both solved p-adically.

INPUT:

- B – a square n x n nonsingular matrix with painful big bottom row.
- a – an n x 1 column matrix

OUTPUT:

- the unique solution to B*x = a.

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_hnf import solve_system_with_difficult_
↪last_row
sage: B = matrix(ZZ, 3, [1,2,4, 3,-4,7, 939082,2930982,132902384098234])
sage: a = matrix(ZZ,3,1, [1,2,5])
sage: z = solve_system_with_difficult_last_row(B, a)
sage: z
[ 106321906985474/132902379815497]
[132902385037291/1329023798154970]
[        -5221794/664511899077485]
sage: B*z
[1]
[2]
[5]
```

CHAPTER

EIGHTEEN

SATURATION OVER ZZ

sage.matrix.matrix_integer_dense_saturation.**index_in_saturation**(*A*, *proof=True*)

The index of A in its saturation.

INPUT:

- A – matrix over **Z**
- proof – boolean (True or False)

OUTPUT:

An integer

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_saturation import index_in_saturation
sage: A = matrix(ZZ, 2, 2, [3,2,3,4]); B = matrix(ZZ, 2,3,[1,2,3,4,5,6]); C = A*B; C
[11 16 21]
[19 26 33]
sage: index_in_saturation(C)
18
sage: W = C.row_space()
sage: S = W.saturation()
sage: W.index_in(S)
18
```

For any zero matrix the index in its saturation is 1 (see github issue #13034):

```
sage: m = matrix(ZZ, 3)
sage: m
[0 0 0]
[0 0 0]
[0 0 0]
sage: m.index_in_saturation()
1
sage: m = matrix(ZZ, 2, 3)
sage: m
[0 0 0]
[0 0 0]
sage: m.index_in_saturation()
1
```

sage.matrix.matrix_integer_dense_saturation.**p_saturation**(*A*, *p*, *proof=True*)

INPUT:

- A – a matrix over ZZ
- p – a prime
- proof – bool (default: True)

OUTPUT:

The p-saturation of the matrix A, i.e., a new matrix in Hermite form whose row span a ZZ-module that is p-saturated.

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_saturation import p_saturation
sage: A = matrix(ZZ, 2, 2, [3,2,3,4]); B = matrix(ZZ, 2,3,[1,2,3,4,5,6])
sage: A.det()
6
sage: C = A*B; C
[11 16 21]
[19 26 33]
sage: C2 = p_saturation(C, 2); C2
[ 1  8 15]
[ 0  9 18]
sage: C2.index_in_saturation()
9
sage: C3 = p_saturation(C, 3); C3
[ 1  0 -1]
[ 0  2  4]
sage: C3.index_in_saturation()
2
```

sage.matrix.matrix_integer_dense_saturation.**random_sublist_of_size**(*k*, *n*)

INPUT:

- k – an integer
- n – an integer

OUTPUT:

a randomly chosen sublist of range(k) of size n.

EXAMPLES:

```
sage: import sage.matrix.matrix_integer_dense_saturation as s
sage: l = s.random_sublist_of_size(10, 3)
sage: len(l)
3
sage: l_check = [-1] + l + [10]
sage: all(l_check[i] < l_check[i+1] for i in range(4))
True
sage: l = s.random_sublist_of_size(10, 7)
sage: len(l)
7
sage: l_check = [-1] + l + [10]
sage: all(l_check[i] < l_check[i+1] for i in range(8))
True
```

sage.matrix.matrix_integer_dense_saturation.**saturation**(*A*, *proof=True*, *p=0*, *max_dets=5*)

Compute a saturation matrix of A.

INPUT:

- A – a matrix over ZZ
- proof – bool (default: True)
- p – int (default: 0); if not 0 only guarantees that output is p-saturated
- max_dets – int (default: 4) max number of dets of submatrices to compute.

OUTPUT:

matrix – saturation of the matrix A.

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_saturation import saturation
sage: A = matrix(ZZ, 2, 2, [3,2,3,4]); B = matrix(ZZ, 2,3,[1,2,3,4,5,6]); C = A*B
sage: C
[11 16 21]
[19 26 33]
sage: C.index_in_saturation()
18
sage: S = saturation(C); S
[11 16 21]
[-2 -3 -4]
sage: S.index_in_saturation()
1
sage: saturation(C, proof=False)
[11 16 21]
[-2 -3 -4]
sage: saturation(C, p=2)
[11 16 21]
[-2 -3 -4]
sage: saturation(C, p=2, max_dets=1)
[11 16 21]
[-2 -3 -4]
```

sage.matrix.matrix_integer_dense_saturation.**solve_system_with_difficult_last_row**(*B*, *A*)

Solve the matrix equation B*Z = A when the last row of B contains huge entries.

INPUT:

- B – a square n x n nonsingular matrix with painful big bottom row.
- A – an n x k matrix.

OUTPUT:

the unique solution to B*Z = A.

EXAMPLES:

```
sage: from sage.matrix.matrix_integer_dense_saturation import solve_system_with_
↪difficult_last_row
sage: B = matrix(ZZ, 3, [1,2,3, 3,-1,2,939239082,39202803080,2939028038402834]); A␣
↪= matrix(ZZ,3,2,[1,2,4,3,-1,0])
```

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```
sage: X = solve_system_with_difficult_last_row(B, A); X
[  290668794698843/226075992027744          468068726971/409557956572]
[-226078357385539/1582531944194208       1228691305937/2866905696004]
[      2365357795/1582531944194208           -17436221/2866905696004]
sage: B*X == A
True
```

CHAPTER

NINETEEN

DENSE MATRICES OVER THE RATIONAL FIELD

EXAMPLES:

We create a 3x3 matrix with rational entries and do some operations with it.

```
sage: a = matrix(QQ, 3,3, [1,2/3, -4/5, 1,1,1, 8,2, -3/19]); a
[    1   2/3  -4/5]
[    1     1     1]
[    8     2 -3/19]
sage: a.det()
2303/285
sage: a.charpoly()
x^3 - 35/19*x^2 + 1259/285*x - 2303/285
sage: b = a^(-1); b
[ -615/2303  -426/2303   418/2303]
[ 2325/2303  1779/2303  -513/2303]
[-1710/2303   950/2303    95/2303]
sage: b.det()
285/2303
sage: a == b
False
sage: a < b
False
sage: b < a
True
sage: a > b
True
sage: a*b
[1 0 0]
[0 1 0]
[0 0 1]
```

class sage.matrix.matrix_rational_dense.**MatrixWindow**

Bases: object

class sage.matrix.matrix_rational_dense.**Matrix_rational_dense**

Bases: *Matrix_dense*

INPUT:

- parent – a matrix space over QQ
- entries – see *matrix()*
- copy – ignored (for backwards compatibility)

- coerce – if False, assume without checking that the entries are of type Rational.

LLL(**args*, ***kwargs*)

Return an LLL reduced or approximated LLL reduced lattice for self interpreted as a lattice.

For details on input parameters, see *sage.matrix.matrix_integer_dense.Matrix_integer_dense.LLL()*.

EXAMPLES:

```
sage: A = Matrix(QQ, 3, 3, [1/n for n in range(1, 10)])
sage: A.LLL()
[ 1/28 -1/40 -1/18]
[ 1/28 -1/40  1/18]
[    0 -3/40     0]
```

add_to_entry(*i*, *j*, *elt*)

Add elt to the entry at position (i,j)

EXAMPLES:

```
sage: m = matrix(QQ, 2, 2)
sage: m.add_to_entry(0, 0, -1/3)
sage: m
[-1/3    0]
[   0    0]
```

antitranspose()

Returns the antitranspose of self, without changing self.

EXAMPLES:

```
sage: A = matrix(QQ,2,3,range(6))
sage: type(A)
<class 'sage.matrix.matrix_rational_dense.Matrix_rational_dense'>
sage: A.antitranspose()
[5 2]
[4 1]
[3 0]
sage: A
[0 1 2]
[3 4 5]

sage: A.subdivide(1,2); A
[0 1|2]
[---+-]
[3 4|5]
sage: A.antitranspose()
[5|2]
[-+-]
[4|1]
[3|0]
```

change_ring(*R*)

Create the matrix over R with entries the entries of self coerced into R.

EXAMPLES:

```
sage: a = matrix(QQ,2,[1/2,-1,2,3])
sage: a.change_ring(GF(3))
[2 2]
[2 0]
sage: a.change_ring(ZZ)
Traceback (most recent call last):
...
TypeError: matrix has denominators so can...t change to ZZ
sage: b = a.change_ring(QQ['x']); b
[1/2  -1]
[  2   3]
sage: b.parent()
Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x␣
↪over Rational Field
```

charpoly(*var='x'*, *algorithm=None*)

Return the characteristic polynomial of this matrix.

Note: The characteristic polynomial is defined as $\det(xI - A)$.

INPUT:

- `var` – (optional) name of the variable as a string
- `algorithm` – an optional specification of an algorithm. It can be one of:
  - `None`: (default) will use flint for small dimensions and linbox otherwise
  - `'flint'`: uses flint library
  - `'linbox'`: uses linbox library
  - `'generic'`: uses Sage generic implementation

OUTPUT: a polynomial over the rational numbers.

EXAMPLES:

```
sage: a = matrix(QQ, 3, [4/3, 2/5, 1/5, 4, -3/2, 0, 0, -2/3, 3/4])
sage: f = a.charpoly(); f
x^3 - 7/12*x^2 - 149/40*x + 97/30
sage: f(a)
[0 0 0]
[0 0 0]
[0 0 0]
```

column(*i*, *from_list=False*)

Return the i-th column of this matrix as a dense vector.

INPUT:

- `i` - integer
- `from_list` - ignored

EXAMPLES:

```
sage: m = matrix(QQ, 3, 2, [1/5,-2/3,3/4,4/9,-1,0])
sage: m.column(1)
(-2/3, 4/9, 0)
sage: m.column(1,from_list=True)
(-2/3, 4/9, 0)
sage: m.column(-1)
(-2/3, 4/9, 0)
sage: m.column(-2)
(1/5, 3/4, -1)

sage: m.column(2)
Traceback (most recent call last):
...
IndexError: column index out of range
sage: m.column(-3)
Traceback (most recent call last):
...
IndexError: column index out of range
```

decomposition(*is_diagonalizable=False*, *dual=False*, *algorithm=None*, *height_guess=None*, *proof=None*)

Returns the decomposition of the free module on which this matrix A acts from the right (i.e., the action is x goes to x A), along with whether this matrix acts irreducibly on each factor. The factors are guaranteed to be sorted in the same way as the corresponding factors of the characteristic polynomial.

Let A be the matrix acting from the on the vector space V of column vectors. Assume that A is square. This function computes maximal subspaces W_1, ..., W_n corresponding to Galois conjugacy classes of eigenvalues of A. More precisely, let f(X) be the characteristic polynomial of A. This function computes the subspace $W_i = ker(g_(A)^n)$, where g_i(X) is an irreducible factor of f(X) and g_i(X) exactly divides f(X). If the optional parameter is_diagonalizable is True, then we let W_i = ker(g(A)), since then we know that ker(g(A)) = $ker(g(A)^n)$.

If dual is True, also returns the corresponding decomposition of V under the action of the transpose of A. The factors are guaranteed to correspond.

INPUT:

- `is_diagonalizable` - ignored
- `dual` - whether to also return decompositions for the dual
- `algorithm` - an optional specification of an algorithm
  - `None` - (default) use default algorithm for computing Echelon forms
  - 'multimodular': much better if the answers factors have small height
- `height_guess` - positive integer; only used by the multimodular algorithm
- `proof` - bool or None (default: None, see proof.linear_algebra or sage.structure.proof); only used by the multimodular algorithm. Note that the Sage global default is proof=True.

Note: IMPORTANT: If you expect that the subspaces in the answer are spanned by vectors with small height coordinates, use algorithm='multimodular' and height_guess=1; this is potentially much faster than the default. If you know for a fact the answer will be very small, use algorithm='multimodular', height_guess=bound on height, proof=False.

You can get very very fast decomposition with proof=False.

EXAMPLES:

```
sage: a = matrix(QQ,3,[1..9])
sage: a.decomposition()
[
(Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -2  1], True),
(Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2], True)
]
```

denominator()

Return the denominator of this matrix.

OUTPUT: a Sage Integer

EXAMPLES:

```
sage: b = matrix(QQ,2,range(6)); b[0,0]=-5007/293; b
[-5007/293         1         2]
[        3         4         5]
sage: b.denominator()
293

sage: matrix(QQ, 2, [1/2, 1/3, 1/4, 1/5]).denominator()
60
```

determinant(*algorithm=None*, *proof=None*)

Return the determinant of this matrix.

INPUT:

- `algorithm` – an optional specification of an algorithm. It can be one of
  - `None`: (default) uses flint
  - `'flint'`: uses flint library
  - `'pari'`: uses PARI library
  - **`'integer'`: removes denominator and call determinant on the corresponding** integer matrix
  - `'generic'`: calls the generic Sage implementation
- `proof` - bool or None; if None use proof.linear_algebra(); only relevant for the padic algorithm.

Note: It would be *VERY VERY* hard for det to fail even with proof=False.

EXAMPLES:

```
sage: m = matrix(QQ,3,[1,2/3,4/5, 2,2,2, 5,3,2/5])
sage: m.determinant()
-34/15
```

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```
sage: m.charpoly()
x^3 - 17/5*x^2 - 122/15*x + 34/15

sage: m = matrix(QQ, 3, [(1/i)**j for i in range(2,5) for j in range(3)])
sage: m.determinant(algorithm="flint")
-1/288

sage: m = matrix(QQ, 4, [(-1)**n/n for n in range(1,17)])
sage: m.determinant(algorithm="pari")
2/70945875

sage: m = matrix(QQ, 5, [1/(i+j+1) for i in range(5) for j in range(5)])
sage: m.determinant(algorithm="integer")
1/266716800000
```

On non-square matrices, the method raises a `ValueError`:

```
sage: matrix(QQ, 2, 3).determinant(algorithm='flint')
Traceback (most recent call last):
...
ValueError: non square matrix
sage: matrix(QQ, 2, 3).determinant(algorithm='pari')
Traceback (most recent call last):
...
ValueError: non square matrix
sage: matrix(QQ, 2, 3).determinant(algorithm='integer')
Traceback (most recent call last):
...
ValueError: non square matrix
sage: matrix(QQ, 2, 3).determinant(algorithm='generic')
Traceback (most recent call last):
...
ValueError: non square matrix
```

echelon_form(*algorithm=None*, *height_guess=None*, *proof=None*, ***kwds*)

Return the echelon form of this matrix.

The (row) echelon form of a matrix, see Wikipedia article Row_echelon_form, is the matrix obtained by performing Gauss elimination on the rows of the matrix.

INPUT: See *echelonize()* for the options.

EXAMPLES:

```
sage: a = matrix(QQ, 4, range(16)); a[0,0] = 1/19; a[0,1] = 1/5; a
[1/19  1/5    2    3]
[   4    5    6    7]
[   8    9   10   11]
[  12   13   14   15]
sage: a.echelon_form()
[      1       0       0 -76/157]
[      0       1       0  -5/157]
[      0       0       1 238/157]
[      0       0       0       0]
```

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```
sage: a.echelon_form(algorithm='multimodular')
[      1       0       0 -76/157]
[      0       1       0  -5/157]
[      0       0       1 238/157]
[      0       0       0       0]
```

The result is an immutable matrix, so if you want to modify the result then you need to make a copy. This checks that github issue #10543 is fixed.:

```
sage: A = matrix(QQ, 2, range(6))
sage: E = A.echelon_form()
sage: E.is_mutable()
False
sage: F = copy(E)
sage: F[0,0] = 50
sage: F
[50  0 -1]
[ 0  1  2]
```

echelonize(*algorithm=None*, *height_guess=None*, *proof=None*, ***kwds*)

Transform the matrix `self` into reduced row echelon form in place.

INPUT:

- `algorithm` – an optional specification of an algorithm. One of
- `None`: (default) uses flint for small dimension and multimodular otherwise
- `'flint'`: use the flint library,
- `'padic'`: an algorithm based on the IML p-adic solver,
- `'multimodular'`: uses a multimodular algorithm the uses linbox modulo many primes (likely to be faster when coefficients are huge),
- `'classical'`: just clear each column using Gauss elimination.
- `height_guess`, `**kwds` - all passed to the multimodular algorithm; ignored by other algorithms.
- `proof` - bool or None (default: None, see proof.linear_algebra or sage.structure.proof). Passed to the multimodular algorithm. Note that the Sage global default is `proof=True`.

EXAMPLES:

```
sage: a = matrix(QQ, 4, range(16)); a[0,0] = 1/19; a[0,1] = 1/5; a
[1/19  1/5    2    3]
[   4    5    6    7]
[   8    9   10   11]
[  12   13   14   15]
sage: a.echelonize()
sage: a
[      1       0       0 -76/157]
[      0       1       0  -5/157]
[      0       0       1 238/157]
[      0       0       0       0]
```

```
sage: a = matrix(QQ, 4, range(16)); a[0,0] = 1/19; a[0,1] = 1/5
sage: a.echelonize(algorithm='multimodular')
sage: a
[      1       0       0 -76/157]
[      0       1       0  -5/157]
[      0       0       1 238/157]
[      0       0       0       0]
```

height()

Return the height of this matrix, which is the maximum of the absolute values of all numerators and denominators of entries in this matrix.

OUTPUT: an Integer

EXAMPLES:

```
sage: b = matrix(QQ,2,range(6)); b[0,0]=-5007/293; b
[-5007/293          1          2]
[         3          4          5]
sage: b.height()
5007
```

inverse(*algorithm=None*, *check_invertible=True*)

Return the inverse of this matrix

INPUT:

- `algorithm` – an optional specification of an algorithm. It can be one of
  - `None`: (default) uses flint
  - `'flint'`: uses flint library
  - `'pari'`: uses PARI library
  - `'iml'`: uses IML library
- `check_invertible` - only used when `algorithm=iml`. Whether to check that matrix is invertible

EXAMPLES:

```
sage: a = matrix(QQ,3,[1,2,5,3,2,1,1,1,1,])
sage: a.inverse()
[1/2 3/2  -4]
[ -1  -2   7]
[1/2 1/2  -2]

sage: a = matrix(QQ, 2, [1, 5, 17, 3])
sage: a.inverse(algorithm="flint")
[-3/82  5/82]
[17/82 -1/82]
sage: a.inverse(algorithm="flint")  * a
[1 0]
[0 1]

sage: a = matrix(QQ, 2, [-1, 5, 12, -3])
sage: a.inverse(algorithm="iml")
```

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```
[1/19 5/57]
[4/19 1/57]
sage: a.inverse(algorithm="iml") * a
[1 0]
[0 1]

sage: a = matrix(QQ, 4, primes_first_n(16))
sage: a.inverse(algorithm="pari")
[   3/11  -12/55    -1/5    2/11]
[  -5/11   -2/55    3/10   -3/22]
[ -13/22 307/440   -1/10   -9/88]
[  15/22  -37/88       0    7/88]
```

On singular matrices this method raises a `ZeroDivisionError`:

```
sage: a = matrix(QQ, 2)
sage: a.inverse(algorithm="flint")
Traceback (most recent call last):
...
ZeroDivisionError: input matrix must be nonsingular
sage: a.inverse(algorithm="iml")
Traceback (most recent call last):
...
ZeroDivisionError: input matrix must be nonsingular
sage: a.inverse(algorithm="pari")
Traceback (most recent call last):
...
ZeroDivisionError: input matrix must be nonsingular
```

matrix_from_columns(*columns*)

Return the matrix constructed from self using columns with indices in the columns list.

EXAMPLES:

```
sage: A = matrix(QQ, 3, range(9))
sage: A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.matrix_from_columns([2,1])
[2 1]
[5 4]
[8 7]
sage: A.matrix_from_columns((2,1,0,2))
[2 1 0 2]
[5 4 3 5]
[8 7 6 8]
```

minpoly(*var='x'*, *algorithm=None*)

Return the minimal polynomial of this matrix

INPUT:

- `var` - (optional) the variable name as a string (default is 'x')

- `algorithm` - an optional specification of an algorithm. It can be one of
  - `None`: (default) will use linbox
  - `'linbox'`: uses the linbox library
  - `'generic'`: uses the generic Sage implementation

OUTPUT: a polynomial over the rationals

EXAMPLES:

```
sage: a = matrix(QQ, 3, [4/3, 2/5, 1/5, 4, -3/2, 0, 0, -2/3, 3/4])
sage: f = a.minpoly(); f
x^3 - 7/12*x^2 - 149/40*x + 97/30
sage: a = Mat(ZZ,4)(range(16))
sage: f = a.minpoly(); f.factor()
x * (x^2 - 30*x - 80)
sage: f(a) == 0
True
```

```
sage: a = matrix(QQ, 4, [1..4^2])
sage: factor(a.minpoly())
x * (x^2 - 34*x - 80)
sage: factor(a.minpoly('y'))
y * (y^2 - 34*y - 80)
sage: factor(a.charpoly())
x^2 * (x^2 - 34*x - 80)
sage: b = matrix(QQ, 4, [-1, 2, 2, 0, 0, 4, 2, 2, 0, 0, -1, -2, 0, -4, 0, 4])
sage: a = matrix(QQ, 4, [1, 1, 0,0, 0,1,0,0, 0,0,5,0, 0,0,0,5])
sage: c = b^(-1)*a*b
sage: factor(c.minpoly())
(x - 5) * (x - 1)^2
sage: factor(c.charpoly())
(x - 5)^2 * (x - 1)^2
```

Check consistency:

```
sage: for _ in range(100):
....:     dim = randint(0, 10)
....:     m = random_matrix(QQ, dim, num_bound=8, den_bound=8)
....:     p_linbox = m.charpoly(algorithm='linbox'); m._clear_cache()
....:     p_generic = m.charpoly(algorithm='generic')
....:     assert p_linbox == p_generic
```

prod_of_row_sums(*cols*)

randomize(*density=1*, *num_bound=2*, *den_bound=2*, *distribution=None*, *nonzero=False*)

Randomize `density` proportion of the entries of this matrix, leaving the rest unchanged.

If `x` and `y` are given, randomized entries of this matrix have numerators and denominators bounded by `x` and `y` and have density 1.

INPUT:

- `density` - number between 0 and 1 (default: 1)
- `num_bound` - numerator bound (default: 2)

- den_bound - denominator bound (default: 2)
- distribution - None or '1/n' (default: None); if '1/n' then num_bound, den_bound are ignored and numbers are chosen using the GMP function mpq_randomize_entry_recip_uniform

OUTPUT:

- None, the matrix is modified in-space

EXAMPLES:

The default distribution:

```
sage: from collections import defaultdict
sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: def add_samples(distribution=None):
....:     global dic, total_count
....:     for _ in range(100):
....:         A = Matrix(QQ, 2, 4, 0)
....:         A.randomize(distribution=distribution)
....:         for a in A.list():
....:             dic[a] += 1
....:             total_count += 1.0

sage: expected = {-2: 1/9, -1: 3/18, -1/2: 1/18, 0: 3/9,
....:             1/2: 1/18, 1: 3/18, 2: 1/9}
sage: add_samples()
sage: while not all(abs(dic[a]/total_count - expected[a]) < 0.001 for a in dic):
....:     add_samples()
```

The distribution '1/n':

```
sage: def mpq_randomize_entry_recip_uniform():
....:     r = 2*random() - 1
....:     if r == 0: r = 1
....:     num = int(4/(5*r))
....:     r = random()
....:     if r == 0: r = 1
....:     den = int(1/random())
....:     return Integer(num)/Integer(den)

sage: total_count = 0
sage: dic = defaultdict(Integer)
sage: dic2 = defaultdict(Integer)
sage: add_samples('1/n')
sage: for _ in range(8):
....:     dic2[mpq_randomize_entry_recip_uniform()] += 1
sage: while not all(abs(dic[a] - dic2[a])/total_count < 0.005 for a in dic):
....:     add_samples('1/n')
....:     for _ in range(800):
....:         dic2[mpq_randomize_entry_recip_uniform()] += 1
```

The default can be used to obtain matrices of different rank:

```
sage: ranks = [False]*11
sage: while not all(ranks):
....:     for dens in (0.05, 0.1, 0.2, 0.5):
....:         A = Matrix(QQ, 10, 10, 0)
....:         A.randomize(dens)
....:         ranks[A.rank()] = True
```

The default density is $6/9$:

```
sage: def add_sample(density, num_rows, num_cols):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     A = Matrix(QQ, num_rows, num_cols, 0)
....:     A.randomize(density)
....:     density_sum += float(A.density())

sage: density_sum = 0.0
sage: total_count = 0.0
sage: expected_density = 6/9
sage: add_sample(1.0, 100, 100)
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(1.0, 100, 100)
```

The modified density depends on the number of columns:

```
sage: density_sum = 0.0
sage: total_count = 0.0
sage: expected_density = 6/9*0.5
sage: add_sample(0.5, 100, 2)
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.5, 100, 2)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: expected_density = 6/9*(1.0 - (99/100)^50)
sage: expected_density
0.263...

sage: add_sample(0.5, 100, 100)
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.5, 100, 100)
```

Modifying the bounds for numerator and denominator:

```
sage: num_dic = defaultdict(Integer)
sage: den_dic = defaultdict(Integer)
sage: while not (all(num_dic[i] for i in range(-200, 201))
....:            and all(den_dic[i] for i in range(1, 101))):
....:     a = matrix(QQ, 2, 4)
....:     a.randomize(num_bound=200, den_bound=100)
....:     for q in a.list():
....:         num_dic[q.numerator()] += 1
....:         den_dic[q.denominator()] += 1
```

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```
sage: len(num_dic)
401
sage: len(den_dic)
100
```

rank(*algorithm=None*)

Return the rank of this matrix.

INPUT:

- `algorithm` - an optional specification of an algorithm. One of
  - `None`: (default) will use flint
  - `'flint'`: uses the flint library
  - `'pari'`: uses the PARI library
  - `'integer'`: eliminate denominators and calls the rank function on the corresponding integer matrix

EXAMPLES:

```
sage: matrix(QQ,3,[1..9]).rank()
2
sage: matrix(QQ,100,[1..100^2]).rank()
2
```

row(*i*, *from_list=False*)

Return the i-th row of this matrix as a dense vector.

INPUT:

- `i` - integer
- `from_list` - ignored

EXAMPLES:

```
sage: m = matrix(QQ, 2, [1/5, -2/3, 3/4, 4/9])
sage: m.row(0)
(1/5, -2/3)
sage: m.row(1)
(3/4, 4/9)
sage: m.row(1, from_list=True)
(3/4, 4/9)
sage: m.row(-2)
(1/5, -2/3)

sage: m.row(2)
Traceback (most recent call last):
...
IndexError: row index out of range
sage: m.row(-3)
Traceback (most recent call last):
...
IndexError: row index out of range
```

set_row_to_multiple_of_row(*i*, *j*, *s*)

Set row i equal to s times row j.

EXAMPLES:

```
sage: a = matrix(QQ,2,3,range(6)); a
[0 1 2]
[3 4 5]
sage: a.set_row_to_multiple_of_row(1,0,-3)
sage: a
[ 0  1  2]
[ 0 -3 -6]
```

transpose()

Returns the transpose of self, without changing self.

EXAMPLES:

We create a matrix, compute its transpose, and note that the original matrix is not changed.

```
sage: A = matrix(QQ, 2, 3, range(6))
sage: type(A)
<class 'sage.matrix.matrix_rational_dense.Matrix_rational_dense'>
sage: B = A.transpose()
sage: print(B)
[0 3]
[1 4]
[2 5]
sage: print(A)
[0 1 2]
[3 4 5]
```

`.T` is a convenient shortcut for the transpose:

```
sage: print(A.T)
[0 3]
[1 4]
[2 5]
```

```
sage: A.subdivide(None, 1); A
[0|1 2]
[3|4 5]
sage: A.transpose()
[0 3]
[---]
[1 4]
[2 5]
```

CHAPTER

TWENTY

SPARSE RATIONAL MATRICES

AUTHORS:

- William Stein (2007-02-21)
- Soroosh Yazdani (2007-02-21)

class sage.matrix.matrix_rational_sparse.**Matrix_rational_sparse**

Bases: *Matrix_sparse*

Create a sparse matrix over the rational numbers.

INPUT:

- `parent` – a matrix space over `QQ`
- `entries` – see *matrix()*
- `copy` – ignored (for backwards compatibility)
- `coerce` – if False, assume without checking that the entries are of type `Rational`.

add_to_entry(*i*, *j*, *elt*)

Add `elt` to the entry at position `(i, j)`.

EXAMPLES:

```
sage: m = matrix(QQ, 2, 2, sparse=True)
sage: m.add_to_entry(0, 0, -1/3)
sage: m
[-1/3    0]
[   0    0]
sage: m.add_to_entry(0, 0, 1/3)
sage: m
[0 0]
[0 0]
sage: m.nonzero_positions()
[]
```

denominator()

Return the denominator of this matrix.

OUTPUT:

– Sage Integer

EXAMPLES:

```
sage: b = matrix(QQ,2,range(6)); b[0,0]=-5007/293; b
[-5007/293         1         2]
[        3         4         5]
sage: b.denominator()
293
```

dense_matrix()

Return dense version of this matrix.

EXAMPLES:

```
sage: a = matrix(QQ,2,[1..4],sparse=True); type(a)
<class 'sage.matrix.matrix_rational_sparse.Matrix_rational_sparse'>
sage: type(a.dense_matrix())
<class 'sage.matrix.matrix_rational_dense.Matrix_rational_dense'>
sage: a.dense_matrix()
[1 2]
[3 4]
```

Check that subdivisions are preserved when converting between dense and sparse matrices:

```
sage: a.subdivide([1,1], [2])
sage: b = a.dense_matrix().sparse_matrix().dense_matrix()
sage: b.subdivisions() == a.subdivisions()
True
```

echelon_form(*algorithm='default'*, *height_guess=None*, *proof=True*, ***kwds*)

INPUT:

height_guess, proof, **kwds – all passed to the multimodular algorithm; ignored by the p-adic algorithm.

OUTPUT:

self is no in reduced row echelon form.

EXAMPLES:

```
sage: a = matrix(QQ, 4, range(16), sparse=True); a[0,0] = 1/19; a[0,1] = 1/5; a
[1/19  1/5    2    3]
[   4    5    6    7]
[   8    9   10   11]
[  12   13   14   15]
sage: a.echelon_form()
[      1       0       0 -76/157]
[      0       1       0  -5/157]
[      0       0       1 238/157]
[      0       0       0       0]
```

echelonize(*height_guess=None*, *proof=True*, ***kwds*)

Transform the matrix self into reduced row echelon form in place.

INPUT:

height_guess, proof, **kwds – all passed to the multimodular algorithm; ignored by the p-adic algorithm.

OUTPUT:

Nothing. The matrix `self` is transformed into reduced row echelon form in place.

ALGORITHM: a multimodular algorithm.

EXAMPLES:

```
sage: a = matrix(QQ, 4, range(16), sparse=True); a[0,0] = 1/19; a[0,1] = 1/5; a
[1/19  1/5    2    3]
[   4    5    6    7]
[   8    9   10   11]
[  12   13   14   15]
sage: a.echelonize(); a
[      1       0       0 -76/157]
[      0       1       0  -5/157]
[      0       0       1 238/157]
[      0       0       0       0]
```

github issue #10319 has been fixed:

```
sage: m = Matrix(QQ, [1], sparse=True); m.echelonize()
sage: m = Matrix(QQ, [1], sparse=True); m.echelonize(); m
[1]
```

height()

Return the height of this matrix, which is the least common multiple of all numerators and denominators of elements of this matrix.

OUTPUT:

– Integer

EXAMPLES:

```
sage: b = matrix(QQ,2,range(6), sparse=True); b[0,0]=-5007/293; b
[-5007/293         1         2]
[        3         4         5]
sage: b.height()
5007
```

set_row_to_multiple_of_row(*i*, *j*, *s*)

Set row i equal to s times row j.

EXAMPLES:

```
sage: a = matrix(QQ,2,3,range(6), sparse=True); a
[0 1 2]
[3 4 5]
sage: a.set_row_to_multiple_of_row(1,0,-3)
sage: a
[ 0  1  2]
[ 0 -3 -6]
```

CHAPTER

TWENTYONE

DENSE MATRICES USING A NUMPY BACKEND

This serves as a base class for dense matrices over Real Double Field and Complex Double Field.

AUTHORS:

- Jason Grout, Sep 2008: switch to NumPy backend, factored out the Matrix_double_dense class
- Josh Kantor
- William Stein: many bug fixes and touch ups.

EXAMPLES:

```
sage: b = Mat(RDF,2,3).basis()
sage: b[0,0]
[1.0 0.0 0.0]
[0.0 0.0 0.0]
```

We deal with the case of zero rows or zero columns:

```
sage: m = MatrixSpace(RDF,0,3)
sage: m.zero_matrix()
[]
```

class sage.matrix.matrix_double_dense.**Matrix_double_dense**

Bases: Matrix_numpy_dense

Base class for matrices over the Real Double Field and the Complex Double Field. These are supposed to be fast matrix operations using C doubles. Most operations are implemented using numpy which will call the underlying BLAS on the system.

This class cannot be instantiated on its own. The numpy matrix creation depends on several variables that are set in the subclasses.

EXAMPLES:

```
sage: m = Matrix(RDF, [[1,2],[3,4]])
sage: m**2
[ 7.0 10.0]
[15.0 22.0]
sage: m^(-1)         # rel tol 1e-15
[-1.9999999999999996  0.9999999999999998]
[ 1.4999999999999998 -0.4999999999999999]
```

LU()

Return a decomposition of the (row-permuted) matrix as a product of a lower-triangular matrix ("L") and an upper-triangular matrix ("U").

OUTPUT:

For an $m \times n$ matrix A this method returns a triple of immutable matrices P, L, U such that

- A = P*L*U
- P is a square permutation matrix, of size $m \times m$, so is all zeroes, but with exactly a single one in each row and each column.
- L is lower-triangular, square of size $m \times m$, with every diagonal entry equal to one.
- U is upper-triangular with size $m \times n$, i.e. entries below the "diagonal" are all zero.

The computed decomposition is cached and returned on subsequent calls, thus requiring the results to be immutable.

Effectively, P permutes the rows of A. Then L can be viewed as a sequence of row operations on this matrix, where each operation is adding a multiple of a row to a subsequent row. There is no scaling (thus 1's on the diagonal of L) and no row-swapping (P does that). As a result U is close to being the result of Gaussian-elimination. However, round-off errors can make it hard to determine the zero entries of U.

Note: The behaviour of LU() has changed in Sage version 9.1. Earlier, LU() returned P,L,U such that P*A=L*U, where P represents the permutation and is the matrix inverse of the P returned by this method. The computation of this matrix inverse can be accomplished quickly with just a transpose as the matrix is orthogonal/unitary.

For details see github issue #18365.

EXAMPLES:

```
sage: m = matrix(RDF,4,range(16))
sage: P,L,U = m.LU()
sage: P*L*U # rel tol 2e-16
[ 0.0  1.0  2.0  3.0]
[ 4.0  5.0  6.0  7.0]
[ 8.0  9.0 10.0 11.0]
[12.0 13.0 14.0 15.0]
```

Below example illustrates the change in behaviour of LU().

```
sage: (m - P*L*U).norm() < 1e-14
True
sage: (P*m - L*U).norm() < 1e-14
False
```

github issue #10839 made this routine available for rectangular matrices.

```
sage: A = matrix(RDF, 5, 6, range(30)); A
[ 0.0  1.0  2.0  3.0  4.0  5.0]
[ 6.0  7.0  8.0  9.0 10.0 11.0]
[12.0 13.0 14.0 15.0 16.0 17.0]
[18.0 19.0 20.0 21.0 22.0 23.0]
[24.0 25.0 26.0 27.0 28.0 29.0]
```

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```
sage: P, L, U = A.LU()
sage: P
[0.0 1.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 1.0]
[0.0 0.0 1.0 0.0 0.0]
[0.0 0.0 0.0 1.0 0.0]
[1.0 0.0 0.0 0.0 0.0]
sage: L.zero_at(0)   # Use zero_at(0) to get rid of signed zeros
[ 1.0  0.0  0.0  0.0  0.0]
[ 0.0  1.0  0.0  0.0  0.0]
[ 0.5  0.5  1.0  0.0  0.0]
[0.75 0.25  0.0  1.0  0.0]
[0.25 0.75  0.0  0.0  1.0]
sage: U.zero_at(0)   # Use zero_at(0) to get rid of signed zeros
[24.0 25.0 26.0 27.0 28.0 29.0]
[ 0.0  1.0  2.0  3.0  4.0  5.0]
[ 0.0  0.0  0.0  0.0  0.0  0.0]
[ 0.0  0.0  0.0  0.0  0.0  0.0]
[ 0.0  0.0  0.0  0.0  0.0  0.0]
sage: P.transpose()*A-L*U
[0.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 0.0]
sage: P*L*U
[ 0.0  1.0  2.0  3.0  4.0  5.0]
[ 6.0  7.0  8.0  9.0 10.0 11.0]
[12.0 13.0 14.0 15.0 16.0 17.0]
[18.0 19.0 20.0 21.0 22.0 23.0]
[24.0 25.0 26.0 27.0 28.0 29.0]
```

Trivial cases return matrices of the right size and characteristics.

```
sage: A = matrix(RDF, 5, 0)
sage: P, L, U = A.LU()
sage: P.parent()
Full MatrixSpace of 5 by 5 dense matrices over Real Double Field
sage: L.parent()
Full MatrixSpace of 5 by 5 dense matrices over Real Double Field
sage: U.parent()
Full MatrixSpace of 5 by 0 dense matrices over Real Double Field
sage: A-P*L*U
[]
```

The results are immutable since they are cached.

```
sage: P, L, U = matrix(RDF, 2, 2, range(4)).LU()
sage: L[0,0] = 0
Traceback (most recent call last):
    ...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
```

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```
↪copy(M) to change a copy of M).
sage: P[0,0] = 0
Traceback (most recent call last):
    ...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
↪copy(M) to change a copy of M).
sage: U[0,0] = 0
Traceback (most recent call last):
    ...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
↪copy(M) to change a copy of M).
```

LU_valid()

Return True if the LU form of this matrix has already been computed.

EXAMPLES:

```
sage: A = random_matrix(RDF,3) ; A.LU_valid()
False
sage: P, L, U = A.LU()
sage: A.LU_valid()
True
```

QR()

Return a factorization into a unitary matrix and an upper-triangular matrix.

INPUT:

Any matrix over RDF or CDF.

OUTPUT:

Q, R – a pair of matrices such that if A is the original matrix, then

$$A = QR, \quad Q^*Q = I$$

where R is upper-triangular. Q^* is the conjugate-transpose in the complex case, and just the transpose in the real case. So Q is a unitary matrix (or rather, orthogonal, in the real case), or equivalently Q has orthogonal columns. For a matrix of full rank this factorization is unique up to adjustments via multiples of rows and columns by multiples with scalars having modulus 1. So in the full-rank case, R is unique if the diagonal entries are required to be positive real numbers.

The resulting decomposition is cached.

ALGORITHM:

Calls "linalg.qr" from SciPy, which is in turn an interface to LAPACK routines.

EXAMPLES:

Over the reals, the inverse of Q is its transpose, since including a conjugate has no effect. In the real case, we say Q is orthogonal.

```
sage: A = matrix(RDF, [[-2, 0, -4, -1, -1],
....:                  [-2, 1, -6, -3, -1],
....:                  [1, 1, 7, 4, 5],
....:                  [3, 0, 8, 3, 3],
```

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```
....:                  [-1, 1, -6, -6, 5]])
sage: Q, R = A.QR()
```

At this point, Q is only well-defined up to the signs of its columns, and similarly for R and its rows, so we normalize them:

```
sage: Qnorm = Q._normalize_columns()
sage: Rnorm = R._normalize_rows()
sage: Qnorm.round(6).zero_at(10^-6)
[ 0.458831  0.126051  0.381212  0.394574   0.68744]
[ 0.458831  -0.47269 -0.051983 -0.717294  0.220963]
[-0.229416 -0.661766  0.661923  0.180872 -0.196411]
[-0.688247 -0.189076 -0.204468  -0.09663  0.662889]
[ 0.229416 -0.535715 -0.609939  0.536422 -0.024551]
sage: Rnorm.round(6).zero_at(10^-6)
[ 4.358899 -0.458831 13.076697  6.194225  2.982405]
[      0.0  1.670172  0.598741  -1.29202  6.207997]
[      0.0       0.0  5.444402  5.468661 -0.682716]
[      0.0       0.0       0.0  1.027626   -3.6193]
[      0.0       0.0       0.0       0.0  0.024551]
sage: (Q*Q.transpose())  # tol 1e-14
[0.9999999999999994                0.0                0.0                0.0 ␣
↪            0.0]
[               0.0                1.0                0.0                0.0 ␣
↪            0.0]
[               0.0                0.0 0.9999999999999999                0.0 ␣
↪            0.0]
[               0.0                0.0                0.0 0.9999999999999998 ␣
↪            0.0]
[               0.0                0.0                0.0                0.0 1.
↪0000000000000002]
sage: (Q*R - A).zero_at(10^-14)
[0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0]
```

Now over the complex numbers, demonstrating that the SciPy libraries are (properly) using the Hermitian inner product, so that Q is a unitary matrix (its inverse is the conjugate-transpose).

```
sage: A = matrix(CDF, [[-8, 4*I + 1, -I + 2, 2*I + 1],
....:                  [1, -2*I - 1, -I + 3, -I + 1],
....:                  [I + 7, 2*I + 1, -2*I + 7, -I + 1],
....:                  [I + 2, 0, I + 12, -1]])
sage: Q, R = A.QR()
sage: Q._normalize_columns()  # tol 1e-6
[                         0.7302967433402214    0.20705664550556482 + 0.
↪5383472783144685*I   0.24630498099986423 - 0.07644563587232917*I   0.
↪23816176831943323 - 0.10365960327796941*I]
[                        -0.09128709291752768  -0.20705664550556482 - 0.
↪37787837804765584*I   0.37865595338630315 - 0.19522214955246678*I    0.
```

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```
↪7012444502144682 - 0.36437116509865947*I]
[  -0.6390096504226938 - 0.09128709291752768*I    0.17082173254209104 + 0.
↪6677576817554466*I -0.03411475806452064 + 0.04090198741767l426*I   0.
↪31401710855067644 - 0.08251917187054114*I]
[ -0.18257418583505536 - 0.09128709291752768*I  -0.03623491296347384 + 0.
↪07246982592694771*I    0.8632284069415112 + 0.06322839976356195*I  -0.
↪44996948676115206 - 0.01161191812089182*I]
sage: R._normalize_rows().zero_at(1e-15)  # tol 1e-6
[                       10.954451150103322                       -1.
↪9170289512680814*I   5.385938482134133 - 2.1908902300206643*I -0.
↪2738612787525829 - 2.1908902300206643*I]
[                                      0.0                              4.
↪8295962564173  -0.8696379111233719 - 5.864879483945123*I  0.993871898426711 -␣
↪0.30540855212070794*I]
[                                      0.0                                  ␣
↪   0.0                            12.001607609358l4 -0.2709533402297273 + 0.
↪4420629644486325*I]
[                                      0.0                                  ␣
↪   0.0                                      0.0                          1.
↪9429639442589917]
sage: (Q.conjugate().transpose()*Q).zero_at(1e-15)  # tol 1e-15
[               1.0                0.0                0.0                0.0]
[               0.0 0.9999999999999994                0.0                0.0]
[               0.0                0.0 1.0000000000000002                0.0]
[               0.0                0.0                0.0 1.0000000000000004]
sage: (Q*R - A).zero_at(10^-14)
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
```

An example of a rectangular matrix that is also rank-deficient. If you run this example yourself, you may see a very small, nonzero entries in the third row, in the third column, even though the exact version of the matrix has rank 2. The final two columns of `Q` span the left kernel of `A` (as evidenced by the two zero rows of `R`). Different platforms will compute different bases for this left kernel, so we do not exhibit the actual matrix.

```
sage: Arat = matrix(QQ, [[2, -3, 3],
....:                    [-1, 1, -1],
....:                    [-1, 3, -3],
....:                    [-5, 1, -1]])
sage: Arat.rank()
2
sage: A = Arat.change_ring(CDF)
sage: Q, R = A.QR()
sage: R._normalize_rows()  # abs tol 1e-14
[     5.567764362830022    -2.6940795304016243     2.6940795304016243]
[                   0.0     3.5695847775155825    -3.5695847775155825]
[                   0.0                    0.0 2.4444034681064287e-16]
[                   0.0                    0.0                    0.0]
sage: (Q.conjugate_transpose()*Q)  # abs tol 1e-14
[     1.0000000000000002  -5.185196889911925e-17 -4.1457180570414476e-17  -2.
```

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```
↪909388767229071e-17]
[ -5.185196889911925e-17      1.0000000000000002  -9.286869233696149e-17 -1.
↪1035822863186828e-16]
[-4.1457180570414476e-17  -9.286869233696149e-17                     1.0  4.
↪4159215672155694e-17]
[ -2.909388767229071e-17 -1.1035822863186828e-16  4.4159215672155694e-17       ␣
↪             1.0]
```

Results are cached, meaning they are immutable matrices. Make a copy if you need to manipulate a result.

```
sage: A = random_matrix(CDF, 2, 2)
sage: Q, R = A.QR()
sage: Q.is_mutable()
False
sage: R.is_mutable()
False
sage: Q[0,0] = 0
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
↪copy(M) to change a copy of M).
sage: Qcopy = copy(Q)
sage: Qcopy[0,0] = 679
sage: Qcopy[0,0]
679.0
```

SVD()

Return the singular value decomposition of this matrix.

The U and V matrices are not unique and may be returned with different values in the future or on different systems. The S matrix is unique and contains the singular values in descending order.

The computed decomposition is cached and returned on subsequent calls.

INPUT:

- A – a matrix

OUTPUT:

- U, S, V – immutable matrices such that $A = U * S * V.conj().transpose()$ where U and V are orthogonal and S is zero off of the diagonal.

Note that if self is m-by-n, then the dimensions of the matrices that this returns are (m,m), (m,n), and (n, n).

Note: If all you need is the singular values of the matrix, see the more convenient *singular_values()*.

EXAMPLES:

```
sage: m = matrix(RDF,4,range(1,17))
sage: U,S,V = m.SVD()
sage: U*S*V.transpose()  # tol 1e-14
[0.9999999999999993 1.9999999999999987  3.000000000000001  4.000000000000002]
[ 4.999999999999998  5.999999999999998  6.999999999999998                 8.0]
```

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```
[ 8.999999999999998  9.999999999999996 10.999999999999998                12.0]
[12.999999999999998                14.0                15.0                16.0]
```

A non-square example:

```
sage: m = matrix(RDF, 2, range(1,7)); m
[1.0 2.0 3.0]
[4.0 5.0 6.0]
sage: U, S, V = m.SVD()
sage: U*S*V.transpose()  # tol 1e-14
[0.9999999999999994 1.9999999999999998  2.999999999999999]
[ 4.000000000000001  5.000000000000002  6.000000000000001]
```

S contains the singular values:

```
sage: S.round(4)
[ 9.508    0.0    0.0]
[   0.0 0.7729    0.0]
sage: [N(sqrt(abs(x)), digits=4) for x in (S*S.transpose()).eigenvalues()]
[9.508, 0.7729]
```

U and V are orthogonal matrices:

```
sage: U # random, SVD is not unique
[-0.386317703119 -0.922365780077]
[-0.922365780077  0.386317703119]
[-0.274721127897 -0.961523947641]
[-0.961523947641  0.274721127897]
sage: (U*U.transpose())  # tol 1e-15
[               1.0                0.0]
[               0.0 1.0000000000000004]
sage: V # random, SVD is not unique
[-0.428667133549  0.805963908589  0.408248290464]
[-0.566306918848  0.112382414097 -0.816496580928]
[-0.703946704147 -0.581199080396  0.408248290464]
sage: (V*V.transpose())  # tol 1e-15
[0.9999999999999999                0.0                0.0]
[               0.0                1.0                0.0]
[               0.0                0.0 0.9999999999999999]
```

cholesky()

Return the Cholesky factorization of a matrix that is real symmetric, or complex Hermitian.

INPUT:

Any square matrix with entries from RDF that is symmetric, or with entries from CDF that is Hermitian. The matrix must be positive definite for the Cholesky decomposition to exist.

OUTPUT:

For a matrix A the routine returns a lower triangular matrix L such that,

$$A = LL^*$$

where L^* is the conjugate-transpose in the complex case, and just the transpose in the real case. If the

matrix fails to be positive definite (perhaps because it is not symmetric or Hermitian), then this function raises a `ValueError`.

IMPLEMENTATION:

The existence of a Cholesky decomposition and the positive definite property are equivalent. So this method and the *is_positive_definite()* method compute and cache both the Cholesky decomposition and the positive-definiteness. So the *is_positive_definite()* method or catching a `ValueError` from the *cholesky()* method are equally expensive computationally and if the decomposition exists, it is cached as a side-effect of either routine.

EXAMPLES:

A real matrix that is symmetric, Hermitian, and positive definite:

```
sage: M = matrix(RDF,[[ 1,  1,    1,     1,     1],
....:                 [ 1,  5,   31,   121,   341],
....:                 [ 1, 31,  341,  1555,  4681],
....:                 [ 1,121, 1555,  7381, 22621],
....:                 [ 1,341, 4681, 22621, 69905]])
sage: M.is_symmetric()
True
sage: M.is_hermitian()
True
sage: L = M.cholesky()
sage: L.round(6).zero_at(10^-10)
[   1.0    0.0         0.0        0.0     0.0]
[   1.0    2.0         0.0        0.0     0.0]
[   1.0   15.0   10.723805        0.0     0.0]
[   1.0   60.0   60.985814   7.792973     0.0]
[   1.0  170.0  198.623524  39.366567  1.7231]
sage: (L*L.transpose()).round(6).zero_at(10^-10)
[ 1.0     1.0     1.0     1.0     1.0]
[ 1.0     5.0    31.0   121.0   341.0]
[ 1.0    31.0   341.0  1555.0  4681.0]
[ 1.0   121.0  1555.0  7381.0 22621.0]
[ 1.0   341.0  4681.0 22621.0 69905.0]
```

A complex matrix that is Hermitian and positive definite.

```
sage: A = matrix(CDF, [[        23,  17*I + 3,  24*I + 25,     21*I],
....:                  [ -17*I + 3,         38, -69*I + 89, 7*I + 15],
....:                  [-24*I + 25, 69*I + 89,        976, 24*I + 6],
....:                  [     -21*I, -7*I + 15,  -24*I + 6,       28]])
sage: A.is_hermitian()
True
sage: L = A.cholesky()
sage: L.round(6).zero_at(10^-10)
[               4.795832                      0.0                     0.0       0.
↪0]
[  0.625543 - 3.544745*I                 5.004346                     0.0       0.
↪0]
[   5.21286 - 5.004346*I 13.588189 + 10.721116*I              24.984023       0.
↪0]
[            -4.378803*I  -0.104257 - 0.851434*I  -0.21486 + 0.371348*I  2.
↪811799]
```

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```
sage: (L*L.conjugate_transpose()).round(6).zero_at(10^-10)
[         23.0  3.0 + 17.0*I 25.0 + 24.0*I         21.0*I]
[ 3.0 - 17.0*I          38.0 89.0 - 69.0*I  15.0 + 7.0*I]
[25.0 - 24.0*I 89.0 + 69.0*I          976.0  6.0 + 24.0*I]
[      -21.0*I  15.0 - 7.0*I  6.0 - 24.0*I           28.0]
```

This routine will recognize when the input matrix is not positive definite. The negative eigenvalues are an equivalent indicator. (Eigenvalues of a Hermitian matrix must be real, so there is no loss in ignoring the imprecise imaginary parts).

```
sage: A = matrix(RDF, [[ 3,  -6,   9,   6,  -9],
....:                  [-6,  11, -16, -11,  17],
....:                  [ 9, -16,  28,  16, -40],
....:                  [ 6, -11,  16,   9, -19],
....:                  [-9,  17, -40, -19,  68]])
sage: A.is_symmetric()
True
sage: A.eigenvalues()
[108.07..., 13.02..., -0.02..., -0.70..., -1.37...]
sage: A.cholesky()
Traceback (most recent call last):
...
ValueError: matrix is not positive definite

sage: B = matrix(CDF, [[      2, 4 - 2*I, 2 + 2*I],
....:                  [4 + 2*I,       8,    10*I],
....:                  [2 - 2*I,   -10*I,      -3]])
sage: B.is_hermitian()
True
sage: [ev.real() for ev in B.eigenvalues()]
[15.88..., 0.08..., -8.97...]
sage: B.cholesky()
Traceback (most recent call last):
...
ValueError: matrix is not positive definite
```

condition(*p='frob'*)

Return the condition number of a square nonsingular matrix.

Roughly speaking, this is a measure of how sensitive the matrix is to round-off errors in numerical computations. The minimum possible value is 1.0, and larger numbers indicate greater sensitivity.

INPUT:

- p - default: 'frob' - controls which norm is used to compute the condition number, allowable values are 'frob' (for the Frobenius norm), integers -2, -1, 1, 2, positive and negative infinity. See output discussion for specifics.

OUTPUT:

The condition number of a matrix is the product of a norm of the matrix times the norm of the inverse of the matrix. This requires that the matrix be square and invertible (nonsingular, full rank).

Returned value is a double precision floating point value in RDF, or Infinity. Row and column sums described below are sums of the absolute values of the entries, where the absolute value of the complex

number $a + bi$ is $\sqrt{a^2 + b^2}$. Singular values are the "diagonal" entries of the "S" matrix in the singular value decomposition.

- `p = 'frob'`: the default norm employed in computing the condition number, the Frobenius norm, which for a matrix $A = (a_{ij})$ computes

$$\left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2}$$

- `p = 'sv'`: the quotient of the maximal and minimal singular value.
- `p = Infinity` or `p = oo`: the maximum row sum.
- `p = -Infinity` or `p = -oo`: the minimum column sum.
- `p = 1`: the maximum column sum.
- `p = -1`: the minimum column sum.
- `p = 2`: the 2-norm, equal to the maximum singular value.
- `p = -2`: the minimum singular value.

ALGORITHM:

Computation is performed by the `cond()` function of the SciPy/NumPy library.

EXAMPLES:

First over the reals.

```
sage: A = matrix(RDF, 4, [(1/4)*x^3 for x in range(16)]); A
[   0.0   0.25    2.0   6.75]
[  16.0  31.25   54.0  85.75]
[ 128.0 182.25  250.0 332.75]
[ 432.0 549.25  686.0 843.75]
sage: A.condition()
9923.88955...
sage: A.condition(p='frob')
9923.88955...
sage: A.condition(p=Infinity)  # tol 3e-14
22738.50000000045
sage: A.condition(p=-Infinity)  # tol 2e-14
17.50000000000028
sage: A.condition(p=1)
12139.21...
sage: A.condition(p=-1)  # tol 2e-14
550.0000000000093
sage: A.condition(p=2)
9897.8088...
sage: A.condition(p=-2)
0.000101032462...
```

And over the complex numbers.

```
sage: B = matrix(CDF, 3, [x + x^2*I for x in range(9)]); B
[          0.0   1.0 + 1.0*I   2.0 + 4.0*I]
[ 3.0 + 9.0*I 4.0 + 16.0*I 5.0 + 25.0*I]
```

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```
[6.0 + 36.0*I 7.0 + 49.0*I 8.0 + 64.0*I]
sage: B.condition()
203.851798...
sage: B.condition(p='frob')
203.851798...
sage: B.condition(p=Infinity)
369.55630...
sage: B.condition(p=-Infinity)
5.46112969...
sage: B.condition(p=1)
289.251481...
sage: B.condition(p=-1)
20.4566639...
sage: B.condition(p=2)
202.653543...
sage: B.condition(p=-2)
0.00493453005...
```

Hilbert matrices are famously ill-conditioned, while an identity matrix can hit the minimum with the right norm.

```
sage: A = matrix(RDF, 10, [1/(i+j+1) for i in range(10) for j in range(10)])
sage: A.condition()  # tol 2e-4
16332197709146.014
sage: id = identity_matrix(CDF, 10)
sage: id.condition(p=1)
1.0
```

Return values are in RDF.

```
sage: A = matrix(CDF, 2, range(1,5))
sage: A.condition() in RDF
True
```

Rectangular and singular matrices raise errors if p is not 'sv'.

```
sage: A = matrix(RDF, 2, 3, range(6))
sage: A.condition()
Traceback (most recent call last):
...
TypeError: matrix must be square if p is not 'sv', not 2 x 3

sage: A.condition('sv')
7.34...

sage: A = matrix(QQ, 5, range(25))
sage: A.is_singular()
True
sage: B = A.change_ring(CDF)
sage: B.condition()
+Infinity
```

Improper values of p are caught.

```
sage: A = matrix(CDF, 2, range(1,5))
sage: A.condition(p='bogus')
Traceback (most recent call last):
...
ValueError: condition number 'p' must be +/- infinity, 'frob', 'sv' or an␣
↪integer, not bogus
sage: A.condition(p=632)
Traceback (most recent call last):
...
ValueError: condition number integer values of 'p' must be -2, -1, 1 or 2, not␣
↪632
```

conjugate()

Return the conjugate of this matrix, i.e. the matrix whose entries are the conjugates of the entries of self.

EXAMPLES:

```
sage: A = matrix(CDF, [[1+I, 3-I], [0, 2*I]])
sage: A.conjugate()
[1.0 - 1.0*I 3.0 + 1.0*I]
[        0.0      -2.0*I]
```

There is a shorthand notation:

```
sage: A.conjugate() == A.C
True
```

Conjugates work (trivially) for real matrices:

```
sage: B = matrix.random(RDF, 3)
sage: B == B.conjugate()
True
```

determinant()

Return the determinant of self.

ALGORITHM:

Use numpy

EXAMPLES:

```
sage: m = matrix(RDF,2,range(4)); m.det()
-2.0
sage: m = matrix(RDF,0,[]); m.det()
1.0
sage: m = matrix(RDF, 2, range(6)); m.det()
Traceback (most recent call last):
...
ValueError: self must be a square matrix
```

eigenvalues(*other=None*, *algorithm='default'*, *tol=None*, *homogeneous=False*)

Return a list of ordinary or generalized eigenvalues.

INPUT:

- `self` - a square matrix

- other – a square matrix B (default: None) in a generalized eigenvalue problem; if None, an ordinary eigenvalue problem is solved; if algorithm is 'symmetric' or 'hermitian', B must be real symmetric or hermitian positive definite, respectively
- algorithm - default: 'default'
  - 'default' - applicable to any matrix with double-precision floating point entries. Uses the eigvals() method from SciPy.
  - 'symmetric' - converts the matrix into a real matrix (i.e. with entries from RDF), then applies the algorithm for Hermitian matrices. This algorithm can be significantly faster than the 'default' algorithm.
  - 'hermitian' - uses the eigh() method from SciPy, which applies only to real symmetric or complex Hermitian matrices. Since Hermitian is defined as a matrix equaling its conjugate-transpose, for a matrix with real entries this property is equivalent to being symmetric. This algorithm can be significantly faster than the 'default' algorithm.
- 'tol' – (default: None); if set to a value other than None, this is interpreted as a small real number used to aid in grouping eigenvalues that are numerically similar, but is ignored when homogeneous is set. See the output description for more information.
- homogeneous – boolean (default: False); if True, use homogeneous coordinates for the output (see *eigenvectors_right()* for details)

Warning: When using the 'symmetric' or 'hermitian' algorithms, no check is made on the input matrix, and only the entries below, and on, the main diagonal are employed in the computation.

Methods such as is_symmetric() and *is_hermitian()* could be used to verify this beforehand.

OUTPUT:

Default output for a square matrix of size n is a list of n eigenvalues from the complex double field, CDF. If the 'symmetric' or 'hermitian' algorithms are chosen, the returned eigenvalues are from the real double field, RDF.

If a tolerance is specified, an attempt is made to group eigenvalues that are numerically similar. The return is then a list of pairs, where each pair is an eigenvalue followed by its multiplicity. The eigenvalue reported is the mean of the eigenvalues computed, and these eigenvalues are contained in an interval (or disk) whose radius is less than 5*tol for $n < 10,000$ in the worst case.

More precisely, for an $n \times n$ matrix, the diameter of the interval containing similar eigenvalues could be as large as sum of the reciprocals of the first n integers times tol.

Warning: Use caution when using the tol parameter to group eigenvalues. See the examples below to see how this can go wrong.

EXAMPLES:

```
sage: m = matrix(RDF, 2, 2, [1,2,3,4])
sage: ev = m.eigenvalues(); ev
[-0.372281323..., 5.37228132...]
sage: ev[0].parent()
Complex Double Field

sage: m = matrix(RDF, 2, 2, [0,1,-1,0])
```

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```
sage: m.eigenvalues(algorithm='default')
[1.0*I, -1.0*I]

sage: m = matrix(CDF, 2, 2, [I,1,-I,0])
sage: m.eigenvalues()
[-0.624810533... + 1.30024259...*I, 0.624810533... - 0.30024259...*I]
```

The adjacency matrix of a graph will be symmetric, and the eigenvalues will be real.

```
sage: A = graphs.PetersenGraph().adjacency_matrix()
sage: A = A.change_ring(RDF)
sage: ev = A.eigenvalues(algorithm='symmetric'); ev  # tol 1e-14
[-2.0, -2.0, -2.0, -2.0, 1.0, 1.0, 1.0, 1.0, 1.0, 3.0]
sage: ev[0].parent()
Real Double Field
```

The matrix `A` is "random", but the construction of `C` provides a positive-definite Hermitian matrix. Note that the eigenvalues of a Hermitian matrix are real, and the eigenvalues of a positive-definite matrix will be positive.

```
sage: A = matrix([[ 4*I + 5,  8*I + 1,  7*I + 5, 3*I + 5],
....:             [ 7*I - 2, -4*I + 7, -2*I + 4, 8*I + 8],
....:             [-2*I + 1,  6*I + 6,  5*I + 5,  -I - 4],
....:             [ 5*I + 1,  6*I + 2,    I - 4, -I + 3]])
sage: C = (A*A.conjugate_transpose()).change_ring(CDF)
sage: ev = C.eigenvalues(algorithm='hermitian'); ev
[2.68144025..., 49.5167998..., 274.086188..., 390.71557...]
sage: ev[0].parent()
Real Double Field
```

A tolerance can be given to aid in grouping eigenvalues that are similar numerically. However, if the parameter is too small it might split too finely. Too large, and it can go wrong very badly. Use with care.

```
sage: G = graphs.PetersenGraph()
sage: G.spectrum()
[3, 1, 1, 1, 1, 1, -2, -2, -2, -2]

sage: A = G.adjacency_matrix().change_ring(RDF)
sage: A.eigenvalues(algorithm='symmetric', tol=1.0e-5)  # tol 1e-15
[(-2.0, 4), (1.0, 5), (3.0, 1)]

sage: A.eigenvalues(algorithm='symmetric', tol=2.5)  # tol 1e-15
[(-2.0, 4), (1.3333333333333333, 6)]
```

An (extreme) example of properly grouping similar eigenvalues.

```
sage: G = graphs.HigmanSimsGraph()
sage: A = G.adjacency_matrix().change_ring(RDF)
sage: A.eigenvalues(algorithm='symmetric', tol=1.0e-5)  # tol 2e-15
[(-8.0, 22), (2.0, 77), (22.0, 1)]
```

In this generalized eigenvalue problem, the homogeneous coordinates explain the output obtained for the eigenvalues:

```
sage: A = matrix.identity(RDF, 2)
sage: B = matrix(RDF, [[3, 5], [6, 10]])
sage: A.eigenvalues(B)  # tol 1e-14
[0.0769230769230769, +infinity]
sage: E = A.eigenvalues(B, homogeneous=True); E  # random
[(0.9999999999999999, 13.000000000000002), (0.9999999999999999, 0.0)]
sage: [alpha/beta for alpha, beta in E]  # tol 1e-14
[0.0769230769230769, NaN + NaN*I]
```

See also:

eigenvectors_left(), *eigenvectors_right()*, *Matrix.eigenmatrix_left()*, *Matrix.eigenmatrix_right()*.

eigenvectors_left(*other=None*, *homogeneous=False*)

Compute the ordinary or generalized left eigenvectors of a matrix of double precision real or complex numbers (i.e. RDF or CDF).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved
- `homogeneous` – boolean (default: `False`); if `True`, use homogeneous coordinates for the eigenvalues in the output

OUTPUT:

A list of triples, each of the form `(e,[v],1)`, where `e` is the eigenvalue, and `v` is an associated left eigenvector such that

$$vA = ev.$$

If the matrix A is of size n, then there are n triples.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$vA = evB.$$

If `homogeneous` is set, each eigenvalue is returned as a tuple (α, β) of homogeneous coordinates such that

$$\beta vA = \alpha vB.$$

The format of the output is designed to match the format for exact results. However, since matrices here have numerical entries, the resulting eigenvalues will also be numerical. No attempt is made to determine if two eigenvalues are equal, or if eigenvalues might actually be zero. So the algebraic multiplicity of each eigenvalue is reported as 1. Decisions about equal eigenvalues or zero eigenvalues should be addressed in the calling routine.

The SciPy routines used for these computations produce eigenvectors normalized to have length 1, but on different hardware they may vary by a complex sign. So for doctests we have normalized output by forcing their eigenvectors to have their first non-zero entry equal to one.

ALGORITHM:

Values are computed with the SciPy library using `scipy:scipy.linalg.eig()`.

EXAMPLES:

```
sage: m = matrix(RDF, [[-5, 3, 2, 8],[10, 2, 4, -2],[-1, -10, -10, -17],[-2, 7,␣
↪6, 13]])
sage: m
[ -5.0   3.0   2.0   8.0]
[ 10.0   2.0   4.0  -2.0]
[ -1.0 -10.0 -10.0 -17.0]
[ -2.0   7.0   6.0  13.0]
sage: spectrum = m.left_eigenvectors()
sage: for i in range(len(spectrum)):
....:     spectrum[i][1][0] = matrix(RDF, spectrum[i][1]).echelon_form()[0]
sage: spectrum[0]  # tol 1e-13
(2.0, [(1.0, 1.0, 1.0, 1.0)], 1)
sage: spectrum[1]  # tol 1e-13
(1.0, [(1.0, 0.8, 0.8, 0.6)], 1)
sage: spectrum[2]  # tol 1e-13
(-2.0, [(1.0, 0.4, 0.6, 0.2)], 1)
sage: spectrum[3]  # tol 1e-13
(-1.0, [(1.0, 1.0, 2.0, 2.0)], 1)
```

A generalized eigenvalue problem:

```
sage: A = matrix(CDF, [[1+I, -2], [3, 4]])
sage: B = matrix(CDF, [[0, 7-I], [2, -3]])
sage: E = A.eigenvectors_left(B)
sage: all((v * A - e * v * B).norm() < 1e-14 for e, [v], _ in E)
True
```

In a generalized eigenvalue problem with a singular matrix B, we can check the eigenvector property using homogeneous coordinates, even though the quotient α/β is not always defined:

```
sage: A = matrix.identity(CDF, 2)
sage: B = matrix(CDF, [[2, 1+I], [4, 2+2*I]])
sage: E = A.eigenvectors_left(B, homogeneous=True)
sage: all((beta * v * A - alpha * v * B).norm() < 1e-14
....:     for (alpha, beta), [v], _ in E)
True
```

See also:

eigenvalues(), *eigenvectors_right()*, *Matrix.eigenmatrix_left()*.

eigenvectors_right(*other=None*, *homogeneous=False*)

Compute the ordinary or generalized right eigenvectors of a matrix of double precision real or complex numbers (i.e. RDF or CDF).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved
- `homogeneous` – boolean (default: `False`); if `True`, use homogeneous coordinates for the eigenvalues in the output

OUTPUT:

A list of triples, each of the form `(e,[v],1)`, where `e` is the eigenvalue, and `v` is an associated right

eigenvector such that

$$Av = ev.$$

If the matrix A is of size n, then there are n triples.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$Av = eBv.$$

If `homogeneous` is set, each eigenvalue is returned as a tuple (α, β) of homogeneous coordinates such that

$$\beta Av = \alpha Bv.$$

The format of the output is designed to match the format for exact results. However, since matrices here have numerical entries, the resulting eigenvalues will also be numerical. No attempt is made to determine if two eigenvalues are equal, or if eigenvalues might actually be zero. So the algebraic multiplicity of each eigenvalue is reported as 1. Decisions about equal eigenvalues or zero eigenvalues should be addressed in the calling routine.

The SciPy routines used for these computations produce eigenvectors normalized to have length 1, but on different hardware they may vary by a complex sign. So for doctests we have normalized output by forcing their eigenvectors to have their first non-zero entry equal to one.

ALGORITHM:

Values are computed with the SciPy library using `scipy:scipy.linalg.eig()`.

EXAMPLES:

```
sage: m = matrix(RDF, [[-9, -14, 19, -74],[-1, 2, 4, -11],[-4, -12, 6, -32],[0,␣
↪-2, -1, 1]])
sage: m
[ -9.0 -14.0  19.0 -74.0]
[ -1.0   2.0   4.0 -11.0]
[ -4.0 -12.0   6.0 -32.0]
[  0.0  -2.0  -1.0   1.0]
sage: spectrum = m.right_eigenvectors()
sage: for i in range(len(spectrum)):
....:   spectrum[i][1][0] = matrix(RDF, spectrum[i][1]).echelon_form()[0]
sage: spectrum[0]  # tol 1e-13
(2.0, [(1.0, -2.0, 3.0, 1.0)], 1)
sage: spectrum[1]  # tol 1e-13
(1.0, [(1.0, -0.666666666666633, 1.333333333333286, 0.3333333333331555)], 1)
sage: spectrum[2]  # tol 1e-13
(-2.0, [(1.0, -0.2, 1.0, 0.2)], 1)
sage: spectrum[3]  # tol 1e-13
(-1.0, [(1.0, -0.5, 2.0, 0.5)], 1)
```

A generalized eigenvalue problem:

```
sage: A = matrix(CDF, [[1+I, -2], [3, 4]])
sage: B = matrix(CDF, [[0, 7-I], [2, -3]])
sage: E = A.eigenvectors_right(B)
sage: all((A * v - e * B * v).norm() < 1e-14 for e, [v], _ in E)
True
```

In a generalized eigenvalue problem with a singular matrix B, we can check the eigenvector property using homogeneous coordinates, even though the quotient α/β is not always defined:

```
sage: A = matrix.identity(RDF, 2)
sage: B = matrix(RDF, [[3, 5], [6, 10]])
sage: E = A.eigenvectors_right(B, homogeneous=True)
sage: all((beta * A * v - alpha * B * v).norm() < 1e-14
....:     for (alpha, beta), [v], _ in E)
True
```

See also:

eigenvalues(), *eigenvectors_left()*, *Matrix.eigenmatrix_right()*.

exp()

Calculate the exponential of this matrix X, which is the matrix

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

EXAMPLES:

```
sage: A = matrix(RDF, 2, [1,2,3,4]); A
[1.0 2.0]
[3.0 4.0]
sage: A.exp()  # tol 1e-14
[51.968956198705044  74.73656456700327]
[112.10484685050491 164.07380304920997]
sage: A = matrix(CDF, 2, [1,2+I,3*I,4]); A
[        1.0 2.0 + 1.0*I]
[      3.0*I         4.0]
sage: A.exp()  # tol 1.1e-14
[-19.614602953804912 + 12.517743846762578*I   3.7949636449582176 + 28.
↪883799306580 99*I]
[ -32.383580980922254 + 21.88423595789845*I   2.269633004093535 + 44.
↪901324827684824*I]
```

is_hermitian(*tol=1e-12*, *algorithm='naive'*)

Return `True` if the matrix is equal to its conjugate-transpose.

INPUT:

- `tol` - default: `1e-12` - the largest value of the absolute value of the difference between two matrix entries for which they will still be considered equal.
- `algorithm` – string (default: "naive"); either "naive" or "orthonormal"

OUTPUT:

`True` if the matrix is square and equal to the transpose with every entry conjugated, and `False` otherwise.

Note that if conjugation has no effect on elements of the base ring (such as for integers), then the `is_symmetric()` method is equivalent and faster.

The tolerance parameter is used to allow for numerical values to be equal if there is a slight difference due to round-off and other imprecisions.

The result is cached, on a per-tolerance and per-algorithm basis.

ALGORITHMS:

The naive algorithm simply compares corresponding entries on either side of the diagonal (and on the diagonal itself) to see if they are conjugates, with equality controlled by the tolerance parameter.

The orthonormal algorithm first computes a Schur decomposition (via the *schur()* method) and checks that the result is a diagonal matrix with real entries.

So the naive algorithm can finish quickly for a matrix that is not Hermitian, while the orthonormal algorithm will always compute a Schur decomposition before going through a similar check of the matrix entry-by-entry.

EXAMPLES:

```
sage: A = matrix(CDF, [[ 1 + I,  1 - 6*I, -1 - I],
....:                  [-3 - I,     -4*I,     -2],
....:                  [-1 + I, -2 - 8*I,  2 + I]])
sage: A.is_hermitian(algorithm='orthonormal')
False
sage: A.is_hermitian(algorithm='naive')
False
sage: B = A*A.conjugate_transpose()
sage: B.is_hermitian(algorithm='orthonormal')
True
sage: B.is_hermitian(algorithm='naive')
True
```

A matrix that is nearly Hermitian, but for one non-real diagonal entry.

```
sage: A = matrix(CDF, [[    2,   2-I, 1+4*I],
....:                  [  2+I,   3+I, 2-6*I],
....:                  [1-4*I, 2+6*I,     5]])
sage: A.is_hermitian(algorithm='orthonormal')
False
sage: A[1,1] = 132
sage: A.is_hermitian(algorithm='orthonormal')
True
```

We get a unitary matrix from the SVD routine and use this numerical matrix to create a matrix that should be Hermitian (indeed it should be the identity matrix), but with some imprecision. We use this to illustrate that if the tolerance is set too small, then we can be too strict about the equality of entries and may achieve the wrong result (depending on the system):

```
sage: A = matrix(CDF, [[ 1 + I,  1 - 6*I, -1 - I],
....:                  [-3 - I,     -4*I,     -2],
....:                  [-1 + I, -2 - 8*I,  2 + I]])
sage: U, _, _ = A.SVD()
sage: B=U*U.conjugate_transpose()
sage: B.is_hermitian(algorithm='naive')
True
sage: B.is_hermitian(algorithm='naive', tol=1.0e-17)  # random
False
sage: B.is_hermitian(algorithm='naive', tol=1.0e-15)
True
```

A square, empty matrix is trivially Hermitian.

```
sage: A = matrix(RDF, 0, 0)
sage: A.is_hermitian()
True
```

Rectangular matrices are never Hermitian, no matter which algorithm is requested.

```
sage: A = matrix(CDF, 3, 4)
sage: A.is_hermitian()
False
```

AUTHOR:

- Rob Beezer (2011-03-30)

is_normal(*tol=1e-12*, *algorithm='orthonormal'*)

Return `True` if the matrix commutes with its conjugate-transpose.

INPUT:

- `tol` - default: `1e-12` - the largest value of the absolute value of the difference between two matrix entries for which they will still be considered equal.
- `algorithm` - default: 'orthonormal' - set to 'orthonormal' for a stable procedure and set to 'naive' for a fast procedure.

OUTPUT:

`True` if the matrix is square and commutes with its conjugate-transpose, and `False` otherwise.

Normal matrices are precisely those that can be diagonalized by a unitary matrix.

The tolerance parameter is used to allow for numerical values to be equal if there is a slight difference due to round-off and other imprecisions.

The result is cached, on a per-tolerance and per-algorithm basis.

ALGORITHMS:

The naive algorithm simply compares entries of the two possible products of the matrix with its conjugate-transpose, with equality controlled by the tolerance parameter.

The orthonormal algorithm first computes a Schur decomposition (via the *schur()* method) and checks that the result is a diagonal matrix. An orthonormal diagonalization is equivalent to being normal.

So the naive algorithm can finish fairly quickly for a matrix that is not normal, once the products have been computed. However, the orthonormal algorithm will compute a Schur decomposition before going through a similar check of a matrix entry-by-entry.

EXAMPLES:

First over the complexes. `B` is Hermitian, hence normal.

```
sage: A = matrix(CDF, [[ 1 + I,  1 - 6*I, -1 - I],
....:                  [-3 - I,     -4*I,     -2],
....:                  [-1 + I, -2 - 8*I,  2 + I]])
sage: B = A*A.conjugate_transpose()
sage: B.is_hermitian()
True
sage: B.is_normal(algorithm='orthonormal')
True
sage: B.is_normal(algorithm='naive')
```

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```
True
sage: B[0,0] = I
sage: B.is_normal(algorithm='orthonormal')
False
sage: B.is_normal(algorithm='naive')
False
```

Now over the reals. Circulant matrices are normal.

```
sage: G = graphs.CirculantGraph(20, [3, 7])
sage: D = digraphs.Circuit(20)
sage: A = 3*D.adjacency_matrix() - 5*G.adjacency_matrix()
sage: A = A.change_ring(RDF)
sage: A.is_normal()
True
sage: A.is_normal(algorithm = 'naive')
True
sage: A[19,0] = 4.0
sage: A.is_normal()
False
sage: A.is_normal(algorithm = 'naive')
False
```

Skew-Hermitian matrices are normal.

```
sage: A = matrix(CDF, [[ 1 + I,  1 - 6*I, -1 - I],
....:                  [-3 - I,     -4*I,     -2],
....:                  [-1 + I, -2 - 8*I,  2 + I]])
sage: B = A - A.conjugate_transpose()
sage: B.is_hermitian()
False
sage: B.is_normal()
True
sage: B.is_normal(algorithm='naive')
True
```

A small matrix that does not fit into any of the usual categories of normal matrices.

```
sage: A = matrix(RDF, [[1, -1],
....:                  [1,  1]])
sage: A.is_normal()
True
sage: not A.is_hermitian() and not A.is_skew_symmetric()
True
```

Sage has several fields besides the entire complex numbers where conjugation is non-trivial.

```
sage: F.<b> = QuadraticField(-7)
sage: C = matrix(F, [[-2*b - 3,  7*b - 6, -b + 3],
....:                [-2*b - 3, -3*b + 2,   -2*b],
....:                [   b + 1,        0,     -2]])
sage: C = C*C.conjugate_transpose()
```

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```
sage: C.is_normal()
True
```

A square, empty matrix is trivially normal.

```
sage: A = matrix(CDF, 0, 0)
sage: A.is_normal()
True
```

Rectangular matrices are never normal, no matter which algorithm is requested.

```
sage: A = matrix(CDF, 3, 4)
sage: A.is_normal()
False
```

AUTHOR:

- Rob Beezer (2011-03-31)

is_positive_definite()

Determines if a matrix is positive definite.

A matrix A is positive definite if it is square, is Hermitian (which reduces to symmetric in the real case), and for every nonzero vector $\vec{x}$,

$$\vec{x}^* A \vec{x} > 0$$

where $\vec{x}^*$ is the conjugate-transpose in the complex case and just the transpose in the real case. Equivalently, a positive definite matrix has only positive eigenvalues and only positive determinants of leading principal submatrices.

INPUT:

Any matrix over RDF or CDF.

OUTPUT:

True if and only if the matrix is square, Hermitian, and meets the condition above on the quadratic form. The result is cached.

IMPLEMENTATION:

The existence of a Cholesky decomposition and the positive definite property are equivalent. So this method and the *cholesky()* method compute and cache both the Cholesky decomposition and the positive-definiteness. So the *is_positive_definite()* method or catching a ValueError from the *cholesky()* method are equally expensive computationally and if the decomposition exists, it is cached as a side-effect of either routine.

EXAMPLES:

A matrix over RDF that is positive definite.

```
sage: M = matrix(RDF,[[ 1,  1,    1,     1,     1],
....:                 [ 1,  5,   31,   121,   341],
....:                 [ 1, 31,  341,  1555,  4681],
....:                 [ 1,121, 1555,  7381, 22621],
....:                 [ 1,341, 4681, 22621, 69905]])
sage: M.is_symmetric()
```

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```
True
sage: M.eigenvalues()
[77547.66..., 82.44..., 2.41..., 0.46..., 0.011...]
sage: [round(M[:i,:i].determinant()) for i in range(1, M.nrows()+1)]
[1, 4, 460, 27936, 82944]
sage: M.is_positive_definite()
True
```

A matrix over CDF that is positive definite.

```
sage: C = matrix(CDF, [[        23,  17*I + 3,  24*I + 25,     21*I],
....:                  [ -17*I + 3,         38, -69*I + 89, 7*I + 15],
....:                  [-24*I + 25, 69*I + 89,        976, 24*I + 6],
....:                  [     -21*I, -7*I + 15,  -24*I + 6,       28]])
sage: C.is_hermitian()
True
sage: [x.real() for x in C.eigenvalues()]
[991.46..., 55.96..., 3.69..., 13.87...]
sage: [round(C[:i,:i].determinant().real()) for i in range(1, C.nrows()+1)]
[23, 576, 359540, 2842600]
sage: C.is_positive_definite()
True
```

A matrix over RDF that is not positive definite.

```
sage: A = matrix(RDF, [[ 3,  -6,   9,   6,  -9],
....:                  [-6,  11, -16, -11,  17],
....:                  [ 9, -16,  28,  16, -40],
....:                  [ 6, -11,  16,   9, -19],
....:                  [-9,  17, -40, -19,  68]])
sage: A.is_symmetric()
True
sage: A.eigenvalues()
[108.07..., 13.02..., -0.02..., -0.70..., -1.37...]
sage: [round(A[:i,:i].determinant()) for i in range(1, A.nrows()+1)]
[3, -3, -15, 30, -30]
sage: A.is_positive_definite()
False
```

A matrix over CDF that is not positive definite.

```
sage: B = matrix(CDF, [[      2, 4 - 2*I, 2 + 2*I],
....:                  [4 + 2*I,       8,    10*I],
....:                  [2 - 2*I,   -10*I,      -3]])
sage: B.is_hermitian()
True
sage: [ev.real() for ev in B.eigenvalues()]
[15.88..., 0.08..., -8.97...]
sage: [round(B[:i,:i].determinant().real()) for i in range(1, B.nrows()+1)]
[2, -4, -12]
sage: B.is_positive_definite()
False
```

A large random matrix that is guaranteed by theory to be positive definite.

```
sage: R = random_matrix(CDF, 200)
sage: H = R.conjugate_transpose()*R
sage: H.is_positive_definite()
True
```

AUTHOR:

- Rob Beezer (2012-05-28)

is_skew_hermitian(*tol=1e-12*, *algorithm='orthonormal'*)

Return `True` if the matrix is equal to the negative of its conjugate transpose.

INPUT:

- `tol` - default: `1e-12` - the largest value of the absolute value of the difference between two matrix entries for which they will still be considered equal.
- `algorithm` - default: 'orthonormal' - set to 'orthonormal' for a stable procedure and set to 'naive' for a fast procedure.

OUTPUT:

`True` if the matrix is square and equal to the negative of its conjugate transpose, and `False` otherwise.

Note that if conjugation has no effect on elements of the base ring (such as for integers), then the `is_skew_symmetric()` method is equivalent and faster.

The tolerance parameter is used to allow for numerical values to be equal if there is a slight difference due to round-off and other imprecisions.

The result is cached, on a per-tolerance and per-algorithm basis.

ALGORITHMS:

The naive algorithm simply compares corresponding entries on either side of the diagonal (and on the diagonal itself) to see if they are conjugates, with equality controlled by the tolerance parameter.

The orthonormal algorithm first computes a Schur decomposition (via the *schur()* method) and checks that the result is a diagonal matrix with real entries.

So the naive algorithm can finish quickly for a matrix that is not Hermitian, while the orthonormal algorithm will always compute a Schur decomposition before going through a similar check of the matrix entry-by-entry.

EXAMPLES:

```
sage: A = matrix(CDF, [[0, -1],
....:                  [1,  0]])
sage: A.is_skew_hermitian(algorithm='orthonormal')
True
sage: A.is_skew_hermitian(algorithm='naive')
True
```

A matrix that is nearly skew-Hermitian, but for a non-real diagonal entry.

```
sage: A = matrix(CDF, [[  -I, -1, 1-I],
....:                  [   1,  1,  -1],
....:                  [-1-I,  1,  -I]])
sage: A.is_skew_hermitian()
```

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```
False
sage: A[1,1] = -I
sage: A.is_skew_hermitian()
True
```

We get a unitary matrix from the SVD routine and use this numerical matrix to create a matrix that should be skew-Hermitian (indeed it should be the identity matrix multiplied by I), but with some imprecision. We use this to illustrate that if the tolerance is set too small, then we can be too strict about the equality of entries and may achieve the wrong result (depending on the system):

```
sage: A = matrix(CDF, [[ 1 + I,  1 - 6*I, -1 - I],
....:                  [-3 - I,     -4*I,     -2],
....:                  [-1 + I, -2 - 8*I,  2 + I]])
sage: U, _, _ = A.SVD()
sage: B=1j*U*U.conjugate_transpose()
sage: B.is_skew_hermitian(algorithm='naive')
True
sage: B.is_skew_hermitian(algorithm='naive', tol=1.0e-17)  # random
False
sage: B.is_skew_hermitian(algorithm='naive', tol=1.0e-15)
True
```

A square, empty matrix is trivially Hermitian.

```
sage: A = matrix(RDF, 0, 0)
sage: A.is_skew_hermitian()
True
```

Rectangular matrices are never Hermitian, no matter which algorithm is requested.

```
sage: A = matrix(CDF, 3, 4)
sage: A.is_skew_hermitian()
False
```

AUTHOR:

- Rob Beezer (2011-03-30)

is_unitary(*tol=1e-12*, *algorithm='orthonormal'*)

Return `True` if the columns of the matrix are an orthonormal basis.

For a matrix with real entries this determines if a matrix is "orthogonal" and for a matrix with complex entries this determines if the matrix is "unitary."

INPUT:

- `tol` - default: `1e-12` - the largest value of the absolute value of the difference between two matrix entries for which they will still be considered equal.
- `algorithm` - default: 'orthonormal' - set to 'orthonormal' for a stable procedure and set to 'naive' for a fast procedure.

OUTPUT:

`True` if the matrix is square and its conjugate-transpose is its inverse, and `False` otherwise. In other words, a matrix is orthogonal or unitary if the product of its conjugate-transpose times the matrix is the identity matrix.

The tolerance parameter is used to allow for numerical values to be equal if there is a slight difference due to round-off and other imprecisions.

The result is cached, on a per-tolerance and per-algorithm basis.

ALGORITHMS:

The naive algorithm simply computes the product of the conjugate-transpose with the matrix and compares the entries to the identity matrix, with equality controlled by the tolerance parameter.

The orthonormal algorithm first computes a Schur decomposition (via the *schur()* method) and checks that the result is a diagonal matrix with entries of modulus 1, which is equivalent to being unitary.

So the naive algorithm might finish fairly quickly for a matrix that is not unitary, once the product has been computed. However, the orthonormal algorithm will compute a Schur decomposition before going through a similar check of a matrix entry-by-entry.

EXAMPLES:

A matrix that is far from unitary.

```
sage: A = matrix(RDF, 4, range(16))
sage: A.conjugate().transpose()*A
[224.0 248.0 272.0 296.0]
[248.0 276.0 304.0 332.0]
[272.0 304.0 336.0 368.0]
[296.0 332.0 368.0 404.0]
sage: A.is_unitary()
False
sage: A.is_unitary(algorithm='naive')
False
sage: A.is_unitary(algorithm='orthonormal')
False
```

The QR decomposition will produce a unitary matrix as Q and the SVD decomposition will create two unitary matrices, U and V.

```
sage: A = matrix(CDF, [[   1 - I,   -3*I,  -2 + I,         1, -2 + 3*I],
....:                  [   1 - I, -2 + I, 1 + 4*I,         0,    2 + I],
....:                  [      -1, -5 + I,  -2 + I,     1 + I, -5 - 4*I],
....:                  [-2 + 4*I,  2 - I, 8 - 4*I,  1 - 8*I,  3 - 2*I]])
sage: Q, R = A.QR()
sage: Q.is_unitary()
True
sage: U, S, V = A.SVD()
sage: U.is_unitary(algorithm='naive')
True
sage: U.is_unitary(algorithm='orthonormal')
True
sage: V.is_unitary(algorithm='naive')
True
```

If we make the tolerance too strict we can get misleading results.

```
sage: A = matrix(RDF, 10, 10, [1/(i+j+1) for i in range(10) for j in range(10)])
sage: Q, R = A.QR()
sage: Q.is_unitary(algorithm='naive', tol=1e-16)
```

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```
False
sage: Q.is_unitary(algorithm='orthonormal', tol=1e-17)
False
```

Rectangular matrices are not unitary/orthogonal, even if their columns form an orthonormal set.

```
sage: A = matrix(CDF, [[1,0], [0,0], [0,1]])
sage: A.is_unitary()
False
```

The smallest cases:

```
sage: P = matrix(CDF, 0, 0)
sage: P.is_unitary(algorithm='naive')
True

sage: P = matrix(CDF, 1, 1, [1])
sage: P.is_unitary(algorithm='orthonormal')
True

sage: P = matrix(CDF, 0, 0,)
sage: P.is_unitary(algorithm='orthonormal')
True
```

AUTHOR:

- Rob Beezer (2011-05-04)

left_eigenvectors(*other=None*, *homogeneous=False*)

Compute the ordinary or generalized left eigenvectors of a matrix of double precision real or complex numbers (i.e. RDF or CDF).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved
- `homogeneous` – boolean (default: `False`); if `True`, use homogeneous coordinates for the eigenvalues in the output

OUTPUT:

A list of triples, each of the form `(e,[v],1)`, where `e` is the eigenvalue, and `v` is an associated left eigenvector such that

$$vA = ev.$$

If the matrix A is of size n, then there are n triples.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$vA = evB.$$

If `homogeneous` is set, each eigenvalue is returned as a tuple (α, β) of homogeneous coordinates such that

$$\beta vA = \alpha vB.$$

The format of the output is designed to match the format for exact results. However, since matrices here have numerical entries, the resulting eigenvalues will also be numerical. No attempt is made to determine if two eigenvalues are equal, or if eigenvalues might actually be zero. So the algebraic multiplicity of each eigenvalue is reported as 1. Decisions about equal eigenvalues or zero eigenvalues should be addressed in the calling routine.

The SciPy routines used for these computations produce eigenvectors normalized to have length 1, but on different hardware they may vary by a complex sign. So for doctests we have normalized output by forcing their eigenvectors to have their first non-zero entry equal to one.

ALGORITHM:

Values are computed with the SciPy library using `scipy:scipy.linalg.eig()`.

EXAMPLES:

```
sage: m = matrix(RDF, [[-5, 3, 2, 8],[10, 2, 4, -2],[-1, -10, -10, -17],[-2, 7,␣
↪6, 13]])
sage: m
[ -5.0   3.0   2.0   8.0]
[ 10.0   2.0   4.0  -2.0]
[ -1.0 -10.0 -10.0 -17.0]
[ -2.0   7.0   6.0  13.0]
sage: spectrum = m.left_eigenvectors()
sage: for i in range(len(spectrum)):
....:     spectrum[i][1][0] = matrix(RDF, spectrum[i][1]).echelon_form()[0]
sage: spectrum[0]  # tol 1e-13
(2.0, [(1.0, 1.0, 1.0, 1.0)], 1)
sage: spectrum[1]  # tol 1e-13
(1.0, [(1.0, 0.8, 0.8, 0.6)], 1)
sage: spectrum[2]  # tol 1e-13
(-2.0, [(1.0, 0.4, 0.6, 0.2)], 1)
sage: spectrum[3]  # tol 1e-13
(-1.0, [(1.0, 1.0, 2.0, 2.0)], 1)
```

A generalized eigenvalue problem:

```
sage: A = matrix(CDF, [[1+I, -2], [3, 4]])
sage: B = matrix(CDF, [[0, 7-I], [2, -3]])
sage: E = A.eigenvectors_left(B)
sage: all((v * A - e * v * B).norm() < 1e-14 for e, [v], _ in E)
True
```

In a generalized eigenvalue problem with a singular matrix B, we can check the eigenvector property using homogeneous coordinates, even though the quotient α/β is not always defined:

```
sage: A = matrix.identity(CDF, 2)
sage: B = matrix(CDF, [[2, 1+I], [4, 2+2*I]])
sage: E = A.eigenvectors_left(B, homogeneous=True)
sage: all((beta * v * A - alpha * v * B).norm() < 1e-14
....:     for (alpha, beta), [v], _ in E)
True
```

See also:

eigenvalues(), *eigenvectors_right()*, *Matrix.eigenmatrix_left()*.

log_determinant()

Compute the log of the absolute value of the determinant using LU decomposition.

Note: This is useful if the usual determinant overflows.

EXAMPLES:

```
sage: m = matrix(RDF,2,2,range(4)); m
[0.0 1.0]
[2.0 3.0]
sage: RDF(log(abs(m.determinant())))
0.6931471805599453
sage: m.log_determinant()
0.6931471805599453
sage: m = matrix(RDF,0,0,[]); m
[]
sage: m.log_determinant()
0.0
sage: m = matrix(CDF,2,2,range(4)); m
[0.0 1.0]
[2.0 3.0]
sage: RDF(log(abs(m.determinant())))
0.6931471805599453
sage: m.log_determinant()
0.6931471805599453
sage: m = matrix(CDF,0,0,[]); m
[]
sage: m.log_determinant()
0.0
```

norm(*p=2*)

Return the norm of the matrix.

INPUT:

- p - default: 2 - controls which norm is computed, allowable values are 'frob' (for the Frobenius norm), integers -2, -1, 1, 2, positive and negative infinity. See output discussion for specifics.

OUTPUT:

Returned value is a double precision floating point value in RDF. Row and column sums described below are sums of the absolute values of the entries, where the absolute value of the complex number $a + bi$ is $\sqrt{a^2 + b^2}$. Singular values are the "diagonal" entries of the "S" matrix in the singular value decomposition.

- p = 'frob': the Frobenius norm, which for a matrix $A = (a_{ij})$ computes

$$\left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2}$$

- p = Infinity or p = oo: the maximum row sum.
- p = -Infinity or p = -oo: the minimum column sum.
- p = 1: the maximum column sum.
- p = -1: the minimum column sum.

- `p = 2`: the induced 2-norm, equal to the maximum singular value.
- `p = -2`: the minimum singular value.

ALGORITHM:

Computation is performed by the `norm()` function of the SciPy/NumPy library.

EXAMPLES:

First over the reals.

```
sage: A = matrix(RDF, 3, range(-3, 6)); A
[-3.0 -2.0 -1.0]
[ 0.0  1.0  2.0]
[ 3.0  4.0  5.0]
sage: A.norm()
7.99575670...
sage: A.norm(p='frob')
8.30662386...
sage: A.norm(p=Infinity)
12.0
sage: A.norm(p=-Infinity)
3.0
sage: A.norm(p=1)
8.0
sage: A.norm(p=-1)
6.0
sage: A.norm(p=2)
7.99575670...
sage: A.norm(p=-2) < 10^-15
True
```

And over the complex numbers.

```
sage: B = matrix(CDF, 2, [[1+I, 2+3*I],[3+4*I,3*I]]); B
[1.0 + 1.0*I 2.0 + 3.0*I]
[3.0 + 4.0*I       3.0*I]
sage: B.norm()
6.66189877...
sage: B.norm(p='frob')
7.0
sage: B.norm(p=Infinity)
8.0
sage: B.norm(p=-Infinity)
5.01976483...
sage: B.norm(p=1)
6.60555127...
sage: B.norm(p=-1)
6.41421356...
sage: B.norm(p=2)
6.66189877...
sage: B.norm(p=-2)
2.14921023...
```

Since it is invariant under unitary multiplication, the Frobenius norm is equal to the square root of the sum of squares of the singular values.

```
sage: A = matrix(RDF, 5, range(1,26))
sage: f = A.norm(p='frob')
sage: U, S, V = A.SVD()
sage: s = sqrt(sum([S[i,i]^2 for i in range(5)]))
sage: abs(f-s) < 1.0e-12
True
```

Return values are in RDF.

```
sage: A = matrix(CDF, 2, range(4))
sage: A.norm() in RDF
True
```

Improper values of p are caught.

```
sage: A.norm(p='bogus')
Traceback (most recent call last):
...
ValueError: matrix norm 'p' must be +/- infinity, 'frob' or an integer, not␣
↪bogus
sage: A.norm(p=632)
Traceback (most recent call last):
...
ValueError: matrix norm integer values of 'p' must be -2, -1, 1 or 2, not 632
```

right_eigenvectors(*other=None*, *homogeneous=False*)

Compute the ordinary or generalized right eigenvectors of a matrix of double precision real or complex numbers (i.e. RDF or CDF).

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved
- `homogeneous` – boolean (default: `False`); if `True`, use homogeneous coordinates for the eigenvalues in the output

OUTPUT:

A list of triples, each of the form `(e,[v],1)`, where `e` is the eigenvalue, and `v` is an associated right eigenvector such that

$$Av = ev.$$

If the matrix A is of size n, then there are n triples.

If a matrix B is passed as optional argument, the output is a solution to the generalized eigenvalue problem such that

$$Av = eBv.$$

If `homogeneous` is set, each eigenvalue is returned as a tuple (α, β) of homogeneous coordinates such that

$$\beta Av = \alpha Bv.$$

The format of the output is designed to match the format for exact results. However, since matrices here have numerical entries, the resulting eigenvalues will also be numerical. No attempt is made to determine

if two eigenvalues are equal, or if eigenvalues might actually be zero. So the algebraic multiplicity of each eigenvalue is reported as 1. Decisions about equal eigenvalues or zero eigenvalues should be addressed in the calling routine.

The SciPy routines used for these computations produce eigenvectors normalized to have length 1, but on different hardware they may vary by a complex sign. So for doctests we have normalized output by forcing their eigenvectors to have their first non-zero entry equal to one.

ALGORITHM:

Values are computed with the SciPy library using `scipy:scipy.linalg.eig()`.

EXAMPLES:

```
sage: m = matrix(RDF, [[-9, -14, 19, -74],[-1, 2, 4, -11],[-4, -12, 6, -32],[0,␣
↪-2, -1, 1]])
sage: m
[ -9.0 -14.0  19.0 -74.0]
[ -1.0   2.0   4.0 -11.0]
[ -4.0 -12.0   6.0 -32.0]
[  0.0  -2.0  -1.0   1.0]
sage: spectrum = m.right_eigenvectors()
sage: for i in range(len(spectrum)):
....:   spectrum[i][1][0] = matrix(RDF, spectrum[i][1]).echelon_form()[0]
sage: spectrum[0]  # tol 1e-13
(2.0, [(1.0, -2.0, 3.0, 1.0)], 1)
sage: spectrum[1]  # tol 1e-13
(1.0, [(1.0, -0.666666666666633, 1.333333333333286, 0.33333333333331555)], 1)
sage: spectrum[2]  # tol 1e-13
(-2.0, [(1.0, -0.2, 1.0, 0.2)], 1)
sage: spectrum[3]  # tol 1e-13
(-1.0, [(1.0, -0.5, 2.0, 0.5)], 1)
```

A generalized eigenvalue problem:

```
sage: A = matrix(CDF, [[1+I, -2], [3, 4]])
sage: B = matrix(CDF, [[0, 7-I], [2, -3]])
sage: E = A.eigenvectors_right(B)
sage: all((A * v - e * B * v).norm() < 1e-14 for e, [v], _ in E)
True
```

In a generalized eigenvalue problem with a singular matrix B, we can check the eigenvector property using homogeneous coordinates, even though the quotient α/β is not always defined:

```
sage: A = matrix.identity(RDF, 2)
sage: B = matrix(RDF, [[3, 5], [6, 10]])
sage: E = A.eigenvectors_right(B, homogeneous=True)
sage: all((beta * A * v - alpha * B * v).norm() < 1e-14
....:     for (alpha, beta), [v], _ in E)
True
```

See also:

eigenvalues(), *eigenvectors_left()*, *Matrix.eigenmatrix_right()*.

round(*ndigits=0*)

Return a copy of the matrix where all entries have been rounded to a given precision in decimal digits (default 0 digits).

INPUT:

- `ndigits` - The precision in number of decimal digits

OUTPUT:

A modified copy of the matrix

EXAMPLES:

```
sage: M = matrix(CDF, [[10.234r + 34.2343jr, 34e10r]])
sage: M
[10.234 + 34.2343*I     340000000000.0]
sage: M.round(2)
[10.23 + 34.23*I  340000000000.0]
sage: M.round()
[ 10.0 + 34.0*I 340000000000.0]
```

schur(*base_ring=None*)

Return the Schur decomposition of the matrix.

INPUT:

- `base_ring` - optional, defaults to the base ring of `self`. Use this to request the base ring of the returned matrices, which will affect the format of the results.

OUTPUT:

A pair of immutable matrices. The first is a unitary matrix Q. The second, T, is upper-triangular when returned over the complex numbers, while it is almost upper-triangular over the reals. In the latter case, there can be some 2×2 blocks on the diagonal which represent a pair of conjugate complex eigenvalues of `self`.

If `self` is the matrix A, then

$$A = QT(\overline{Q})^t$$

where the latter matrix is the conjugate-transpose of `Q`, which is also the inverse of `Q`, since `Q` is unitary.

Note that in the case of a normal matrix (Hermitian, symmetric, and others), the upper-triangular matrix is a diagonal matrix with eigenvalues of `self` on the diagonal, and the unitary matrix has columns that form an orthonormal basis composed of eigenvectors of `self`. This is known as "orthonormal diagonalization".

Warning: The Schur decomposition is not unique, as there may be numerous choices for the vectors of the orthonormal basis, and consequently different possibilities for the upper-triangular matrix. However, the diagonal of the upper-triangular matrix will always contain the eigenvalues of the matrix (in the complex version), or 2×2 block matrices in the real version representing pairs of conjugate complex eigenvalues.

In particular, results may vary across systems and processors.

EXAMPLES:

First over the complexes. The similar matrix is always upper-triangular in this case.

```
sage: A = matrix(CDF, 4, 4, range(16)) + matrix(CDF, 4, 4, [x^3*I for x in␣
↪range(0, 16)])
sage: Q, T = A.schur()
```

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```
sage: (Q*Q.conjugate().transpose()).zero_at(1e-12)  # tol 1e-12
[ 0.999999999999999                0.0                 0.0                 0.0]
[               0.0 0.9999999999999996                 0.0                 0.0]
[               0.0                0.0 0.9999999999999992                 0.0]
[               0.0                0.0                 0.0 0.9999999999999999]
sage: all(T.zero_at(1.0e-12)[i,j] == 0 for i in range(4) for j in range(i))
True
sage: (Q*T*Q.conjugate().transpose()-A).zero_at(1.0e-11)
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
sage: eigenvalues = [T[i,i] for i in range(4)]; eigenvalues
[30.733... + 4648.541...*I, -0.184... - 159.057...*I, -0.523... + 11.158...*I, -
↪0.025... - 0.642...*I]
sage: A.eigenvalues()
[30.733... + 4648.541...*I, -0.184... - 159.057...*I, -0.523... + 11.158...*I, -
↪0.025... - 0.642...*I]
sage: abs(A.norm()-T.norm()) < 1e-10
True
```

We begin with a real matrix but ask for a decomposition over the complexes. The result will yield an upper-triangular matrix over the complex numbers for `T`.

```
sage: A = matrix(RDF, 4, 4, [x^3 for x in range(16)])
sage: Q, T = A.schur(base_ring=CDF)
sage: (Q*Q.conjugate().transpose()).zero_at(1e-12)  # tol 1e-12
[0.9999999999999987                0.0                 0.0                 0.0]
[               0.0 0.9999999999999999                 0.0                 0.0]
[               0.0                0.0 1.0000000000000013                 0.0]
[               0.0                0.0                 0.0 1.0000000000000007]
sage: T.parent()
Full MatrixSpace of 4 by 4 dense matrices over Complex Double Field
sage: all(T.zero_at(1.0e-12)[i,j] == 0 for i in range(4) for j in range(i))
True
sage: (Q*T*Q.conjugate().transpose()-A).zero_at(1.0e-11)
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
```

Now totally over the reals. But with complex eigenvalues, the similar matrix may not be upper-triangular. But "at worst" there may be some 2×2 blocks on the diagonal which represent a pair of conjugate complex eigenvalues. These blocks will then just interrupt the zeros below the main diagonal. This example has a pair of these of the blocks.

```
sage: A = matrix(RDF, 4, 4, [[1, 0, -3, -1],
....:                        [4, -16, -7, 0],
....:                        [1, 21, 1, -2],
....:                        [26, -1, -2, 1]])
sage: Q, T = A.schur()
sage: (Q*Q.conjugate().transpose())  # tol 1e-12
```

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```
[0.9999999999999994                0.0                0.0                0.0]
[               0.0 1.0000000000000013                0.0                0.0]
[               0.0                0.0 1.0000000000000004                0.0]
[               0.0                0.0                0.0 1.0000000000000016]
sage: all(T.zero_at(1.0e-12)[i,j] == 0 for i in range(4) for j in range(i))
False
sage: all(T.zero_at(1.0e-12)[i,j] == 0 for i in range(4) for j in range(i-1))
True
sage: (Q*T*Q.conjugate().transpose()-A).zero_at(1.0e-11)
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
sage: sorted(T[0:2,0:2].eigenvalues() + T[2:4,2:4].eigenvalues())
[-5.710... - 8.382...*I, -5.710... + 8.382...*I, -0.789... - 2.336...*I, -0.789.
↪.. + 2.336...*I]
sage: sorted(A.eigenvalues())
[-5.710... - 8.382...*I, -5.710... + 8.382...*I, -0.789... - 2.336...*I, -0.789.
↪.. + 2.336...*I]
sage: abs(A.norm()-T.norm()) < 1e-12
True
```

Starting with complex numbers and requesting a result over the reals will never happen.

```
sage: A = matrix(CDF, 2, 2, [[2+I, -1+3*I], [5-4*I, 2-7*I]])
sage: A.schur(base_ring=RDF)
Traceback (most recent call last):
...
TypeError: unable to convert input matrix over CDF to a matrix over RDF
```

If theory predicts your matrix is real, but it contains some very small imaginary parts, you can specify the cutoff for "small" imaginary parts, then request the output as real matrices, and let the routine do the rest.

```
sage: A = matrix(RDF, 2, 2, [1, 1, -1, 0]) + matrix(CDF, 2, 2, [1.0e-14*I]*4)
sage: B = A.zero_at(1.0e-12)
sage: B.parent()
Full MatrixSpace of 2 by 2 dense matrices over Complex Double Field
sage: Q, T = B.schur(RDF)
sage: Q.parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Double Field
sage: T.parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Double Field
sage: Q.round(6)
[ 0.707107  0.707107]
[-0.707107  0.707107]
sage: T.round(6)
[ 0.5  1.5]
[-0.5  0.5]
sage: (Q*T*Q.conjugate().transpose()-B).zero_at(1.0e-11)
[0.0 0.0]
[0.0 0.0]
```

A Hermitian matrix has real eigenvalues, so the similar matrix will be upper-triangular. Furthermore, a

Hermitian matrix is diagonalizable with respect to an orthonormal basis, composed of eigenvectors of the matrix. Here that basis is the set of columns of the unitary matrix.

```
sage: A = matrix(CDF, [[        52,   -9*I - 8,    6*I - 187,  -188*I + 2],
....:                  [   9*I - 8,         12,   -58*I + 59,   30*I + 42],
....:                  [-6*I - 187,  58*I + 59,         2677, 2264*I + 65],
....:                  [ 188*I + 2, -30*I + 42, -2264*I + 65,        2080]])
sage: Q, T = A.schur()
sage: T = T.zero_at(1.0e-12).change_ring(RDF)
sage: T.round(6)
[4680.13301        0.0        0.0        0.0]
[       0.0 102.715967        0.0        0.0]
[       0.0        0.0  35.039344        0.0]
[       0.0        0.0        0.0    3.11168]
sage: (Q*Q.conjugate().transpose()).zero_at(1e-12)  # tol 1e-12
[1.0000000000000004                0.0                0.0                0.0]
[               0.0 0.9999999999999989                0.0                0.0]
[               0.0                0.0 1.0000000000000002                0.0]
[               0.0                0.0                0.0 0.9999999999999992]
sage: (Q*T*Q.conjugate().transpose()-A).zero_at(1.0e-11)
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
```

Similarly, a real symmetric matrix has only real eigenvalues, and there is an orthonormal basis composed of eigenvectors of the matrix.

```
sage: A = matrix(RDF, [[ 1, -2, 5, -3],
....:                  [-2,  9, 1,  5],
....:                  [ 5,  1, 3 , 7],
....:                  [-3,  5, 7, -8]])
sage: Q, T = A.schur()
sage: Q.round(4)
[-0.3027  -0.751   0.576 -0.1121]
[  0.139 -0.3892 -0.2648  0.8713]
[ 0.4361   0.359  0.7599  0.3217]
[ -0.836  0.3945  0.1438  0.3533]
sage: T = T.zero_at(10^-12)
sage: all(abs(e) < 10^-4 for e in (T - diagonal_matrix(RDF, [-13.5698, -0.8508,␣
↪7.7664, 11.6542])).list())
True
sage: (Q*Q.transpose())  # tol 1e-12
[0.9999999999999998                0.0                0.0                0.0]
[               0.0                1.0                0.0                0.0]
[               0.0                0.0 0.9999999999999998                0.0]
[               0.0                0.0                0.0 0.9999999999999996]
sage: (Q*T*Q.transpose()-A).zero_at(1.0e-11)
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0]
```

The results are cached, both as a real factorization and also as a complex factorization. This means the

returned matrices are immutable.

```
sage: A = matrix(RDF, 2, 2, [[0, -1], [1, 0]])
sage: Qr, Tr = A.schur(base_ring=RDF)
sage: Qc, Tc = A.schur(base_ring=CDF)
sage: all(M.is_immutable() for M in [Qr, Tr, Qc, Tc])
True
sage: Tr.round(6) != Tc.round(6)
True
```

AUTHOR:

- Rob Beezer (2011-03-31)

singular_values(*eps=None*)

Return a sorted list of the singular values of the matrix.

INPUT:

- `eps` - default: `None` - the largest number which will be considered to be zero. May also be set to the string 'auto'. See the discussion below.

OUTPUT:

A sorted list of the singular values of the matrix, which are the diagonal entries of the "S" matrix in the SVD decomposition. As such, the values are real and are returned as elements of `RDF`. The list is sorted with larger values first, and since theory predicts these values are always positive, for a rank-deficient matrix the list should end in zeros (but in practice may not). The length of the list is the minimum of the row count and column count for the matrix.

The number of non-zero singular values will be the rank of the matrix. However, as a numerical matrix, it is impossible to control the difference between zero entries and very small non-zero entries. As an informed consumer it is up to you to use the output responsibly. We will do our best, and give you the tools to work with the output, but we cannot give you a guarantee.

With `eps` set to `None` you will get the raw singular values and can manage them as you see fit. You may also set `eps` to any positive floating point value you wish. If you set `eps` to 'auto' this routine will compute a reasonable cutoff value, based on the size of the matrix, the largest singular value and the smallest nonzero value representable by the 53-bit precision values used. See the discussion at page 268 of [Wat2010].

See the examples for a way to use the "verbose" facility to easily watch the zero cutoffs in action.

ALGORITHM:

The singular values come from the SVD decomposition computed by SciPy/NumPy.

EXAMPLES:

Singular values close to zero have trailing digits that may vary on different hardware. For exact matrices, the number of non-zero singular values will equal the rank of the matrix. So for some of the doctests we round the small singular values that ideally would be zero, to control the variability across hardware.

This matrix has a determinant of one. A chain of two or three theorems implies the product of the singular values must also be one.

```
sage: A = matrix(QQ, [[ 1,  0,  0,  0,  0,  1,  3],
....:                 [-2,  1,  1, -2,  0, -4,  0],
....:                 [ 1,  0,  1, -4, -6, -3,  7],
....:                 [-2,  2,  1,  1,  7,  1, -1],
....:                 [-1,  0, -1,  5,  8,  4, -6],
```

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```
....:                 [ 4, -2, -2,  1, -3,  0,  8],
....:                 [-2,  1,  0,  2,  7,  3, -4]])
sage: A.determinant()
1
sage: B = A.change_ring(RDF)
sage: sv = B.singular_values(); sv  # tol 1e-12
[20.523980658874265, 8.486837028536643, 5.86168134845073, 2.4429165899286978, 0.
↪5831970144724045, 0.26933287286576313, 0.0025524488076110402]
sage: prod(sv)  # tol 1e-12
0.9999999999999525
```

An exact matrix that is obviously not of full rank, and then a computation of the singular values after conversion to an approximate matrix.

```
sage: A = matrix(QQ, [[1/3, 2/3, 11/3],
....:                 [2/3, 1/3,  7/3],
....:                 [2/3, 5/3, 27/3]])
sage: A.rank()
2
sage: B = A.change_ring(CDF)
sage: sv = B.singular_values()
sage: sv[0:2]
[10.1973039..., 0.487045871...]
sage: sv[2] < 1e-14
True
```

A matrix of rank 3 over the complex numbers.

```
sage: A = matrix(CDF, [[46*I - 28, -47*I - 50, 21*I + 51, -62*I - 782, 13*I +␣
↪22],
....:                  [35*I - 20, -32*I - 46, 18*I + 43, -57*I - 670, 7*I + 3],
....:                  [22*I - 13, -23*I - 23, 9*I + 24, -26*I - 347, 7*I + 13],
....:                  [-44*I + 23, 41*I + 57, -19*I - 54, 60*I + 757, -11*I -␣
↪9],
....:                  [30*I - 18, -30*I - 34, 14*I + 34, -42*I - 522, 8*I +␣
↪12]])
sage: sv = A.singular_values()
sage: sv[0:3]  # tol 1e-14
[1440.7336659952966, 18.4044403413369227, 6.839707797136151]
sage: (sv[3] < 10^-13) or sv[3]
True
sage: (sv[4] < 10^-14) or sv[4]
True
```

A full-rank matrix that is ill-conditioned. We use this to illustrate ways of using the various possibilities for `eps`, including one that is ill-advised. Notice that the automatically computed cutoff gets this (difficult) example slightly wrong. This illustrates the impossibility of any automated process always getting this right. Use with caution and judgement.

```
sage: entries = [1/(i+j+1) for i in range(12) for j in range(12)]
sage: B = matrix(QQ, 12, 12, entries)
sage: B.rank()
12
```

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```
sage: A = B.change_ring(RDF)
sage: A.condition() > 1.59e16 or A.condition()
True

sage: A.singular_values(eps=None)  # abs tol 7e-16
[1.7953720595619975, 0.38027524595503703, 0.04473854875218107, 0.
↪0037223122378911614, 0.0002330890890217751, 1.116335748323284e-05, 4.
↪82376110397296e-07, 1.1228610675717613e-08, 2.2519645713496478e-10, 3.
↪1113486853814003e-12, 2.6500422260778388e-14, 9.87312834948426e-17]
sage: A.singular_values(eps='auto')  # abs tol 7e-16
[1.7953720595619975, 0.38027524595503703, 0.04473854875218107, 0.
↪0037223122378911614, 0.0002330890890217751, 1.116335748323284e-05, 4.
↪82376110397296e-07, 1.1228610675717613e-08, 2.2519645713496478e-10, 3.
↪1113486853814003e-12, 2.6500422260778388e-14, 0.0]
sage: A.singular_values(eps=1e-4)  # abs tol 7e-16
[1.7953720595619975, 0.38027524595503703, 0.04473854875218107, 0.
↪0037223122378911614, 0.0002330890890217751, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]
```

With Sage's "verbose" facility, you can compactly see the cutoff at work. In any application of this routine, or those that build upon it, it would be a good idea to conduct this exercise on samples. We also test here that all the values are returned in RDF since singular values are always real.

```
sage: A = matrix(CDF, 4, range(16))
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(1)
sage: sv = A.singular_values(eps='auto'); sv
verbose 1 (<module>) singular values,
smallest-non-zero:cutoff:largest-zero,
2.2766...:6.2421...e-14:...
[35.13996365902..., 2.27661020871472..., 0.0, 0.0]
sage: set_verbose(0)

sage: all(s in RDF for s in sv)
True
```

AUTHOR:

- Rob Beezer - (2011-02-18)

zero_at(*eps*)

Return a copy of the matrix where elements smaller than or equal to `eps` are replaced with zeroes. For complex matrices, the real and imaginary parts are considered individually.

This is useful for modifying output from algorithms which have large relative errors when producing zero elements, e.g. to create reliable doctests.

INPUT:

- `eps` - Cutoff value

OUTPUT:

A modified copy of the matrix.

EXAMPLES:

```
sage: a = matrix(CDF, [[1, 1e-4r, 1+1e-100jr], [1e-8+3j, 0, 1e-58r]])
sage: a
[           1.0         0.0001 1.0 + 1e-100*I]
[ 1e-08 + 3.0*I            0.0           1e-58]
sage: a.zero_at(1e-50)
[          1.0        0.0001           1.0]
[1e-08 + 3.0*I           0.0           0.0]
sage: a.zero_at(1e-4)
[  1.0   0.0   1.0]
[3.0*I   0.0   0.0]
```

CHAPTER

TWENTYTWO

DENSE MATRICES OVER THE REAL DOUBLE FIELD USING NUMPY

EXAMPLES:

```
sage: b = Mat(RDF,2,3).basis()
sage: b[0,0]
[1.0 0.0 0.0]
[0.0 0.0 0.0]
```

We deal with the case of zero rows or zero columns:

```
sage: m = MatrixSpace(RDF,0,3)
sage: m.zero_matrix()
[]
```

AUTHORS:

- Jason Grout (2008-09): switch to NumPy backend, factored out the Matrix_double_dense class
- Josh Kantor
- William Stein: many bug fixes and touch ups.

class sage.matrix.matrix_real_double_dense.**Matrix_real_double_dense**

Bases: *Matrix_double_dense*

Class that implements matrices over the real double field. These are supposed to be fast matrix operations using C doubles. Most operations are implemented using numpy which will call the underlying BLAS on the system.

EXAMPLES:

```
sage: m = Matrix(RDF, [[1,2],[3,4]])
sage: m**2
[ 7.0 10.0]
[15.0 22.0]
sage: n = m^(-1); n     # rel tol 1e-15
[-1.9999999999999996  0.9999999999999998]
[ 1.4999999999999998 -0.4999999999999999]
```

To compute eigenvalues, use the method *left_eigenvectors()* or *right_eigenvectors()*.

```
sage: p,e = m.right_eigenvectors()
```

The result is a pair `(p,e)`, where `p` is a diagonal matrix of eigenvalues and `e` is a matrix whose columns are the eigenvectors.

To solve a linear system $Ax = b$ where `A = [[1,2],[3,4]]` and $b = [5, 6]$:

```
sage: b = vector(RDF,[5,6])
sage: m.solve_right(b)  # rel tol 1e-15
(-3.9999999999999987, 4.499999999999999)
```

See the methods *QR()*, *LU()*, and *SVD()* for QR, LU, and singular value decomposition.

CHAPTER

TWENTYTHREE

DENSE MATRICES OVER GF(2) USING THE M4RI LIBRARY

AUTHOR: Martin Albrecht <malb@informatik.uni-bremen.de>

EXAMPLES:

```
sage: a = matrix(GF(2),3,range(9),sparse=False); a
[0 1 0]
[1 0 1]
[0 1 0]
sage: a.rank()
2
sage: type(a)
<class 'sage.matrix.matrix_mod2_dense.Matrix_mod2_dense'>
sage: a[0,0] = 1
sage: a.rank()
3
sage: parent(a)
Full MatrixSpace of 3 by 3 dense matrices over Finite Field of size 2

sage: a^2
[0 1 1]
[1 0 0]
[1 0 1]
sage: a+a
[0 0 0]
[0 0 0]
[0 0 0]

sage: b = a.new_matrix(2,3,range(6)); b
[0 1 0]
[1 0 1]

sage: a*b
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'Full MatrixSpace of 3 by 3 dense␣
↪matrices over Finite Field of size 2' and 'Full MatrixSpace of 2 by 3 dense matrices␣
↪over Finite Field of size 2'
sage: b*a
[1 0 1]
[1 0 0]
```

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```
sage: TestSuite(a).run()
sage: TestSuite(b).run()

sage: a.echelonize(); a
[1 0 0]
[0 1 0]
[0 0 1]
sage: b.echelonize(); b
[1 0 1]
[0 1 0]
```

Todo:

- make LinBox frontend and use it
  - charpoly ?
  - minpoly ?
- make Matrix_modn_frontend and use it (?)

class sage.matrix.matrix_mod2_dense.**Matrix_mod2_dense**

Bases: *Matrix_dense*

Dense matrix over GF(2).

augment(*right*, *subdivide=False*)

Augments `self` with `right`.

EXAMPLES:

```
sage: MS = MatrixSpace(GF(2),3,3)
sage: A = MS([0, 1, 0, 1, 1, 0, 1, 1, 1]); A
[0 1 0]
[1 1 0]
[1 1 1]
sage: B = A.augment(MS(1)); B
[0 1 0 1 0 0]
[1 1 0 0 1 0]
[1 1 1 0 0 1]
sage: B.echelonize(); B
[1 0 0 1 1 0]
[0 1 0 1 0 0]
[0 0 1 0 1 1]
sage: C = B.matrix_from_columns([3,4,5]); C
[1 1 0]
[1 0 0]
[0 1 1]
sage: C == ~A
True
sage: C*A == MS(1)
True
```

A vector may be augmented to a matrix.

```
sage: A = matrix(GF(2), 3, 4, range(12))
sage: v = vector(GF(2), 3, range(3))
sage: A.augment(v)
[0 1 0 1 0]
[0 1 0 1 1]
[0 1 0 1 0]
```

The `subdivide` option will add a natural subdivision between `self` and `right`. For more details about how subdivisions are managed when augmenting, see *sage.matrix.matrix1.Matrix.augment()*.

```
sage: A = matrix(GF(2), 3, 5, range(15))
sage: B = matrix(GF(2), 3, 3, range(9))
sage: A.augment(B, subdivide=True)
[0 1 0 1 0|0 1 0]
[1 0 1 0 1|1 0 1]
[0 1 0 1 0|0 1 0]
```

density(*approx=False*)

Return the density of this matrix.

By density we understand the ratio of the number of nonzero positions and the self.nrows() * self.ncols(), i.e. the number of possible nonzero positions.

INPUT:

- approx – return floating point approximation (default: False)

EXAMPLES:

```
sage: A = random_matrix(GF(2), 1000, 1000)
sage: d = A.density()
sage: float(d) == A.density(approx=True)
True
sage: len(A.nonzero_positions())/1000^2 == d
True

sage: total = 1.0
sage: density_sum = A.density()
sage: while abs(density_sum/total - 0.5) > 0.001:
....:     A = random_matrix(GF(2), 1000, 1000)
....:     total += 1
....:     density_sum += A.density()
```

determinant()

Return the determinant of this matrix over GF(2).

EXAMPLES:

```
sage: matrix(GF(2),2,[1,1,0,1]).determinant()
1
sage: matrix(GF(2),2,[1,1,1,1]).determinant()
0
```

echelonize(*algorithm='heuristic'*, *cutoff=0*, *reduced=True*, ***kwds*)

Puts self in (reduced) row echelon form.

INPUT:

- self – a mutable matrix
- algorithm
  - 'heuristic' – uses M4RI and PLUQ (default)
  - 'm4ri' – uses M4RI
  - 'pluq' – uses PLUQ factorization
  - 'classical' – uses classical Gaussian elimination
- k – the parameter 'k' of the M4RI algorithm. It MUST be between 1 and 16 (inclusive). If it is not specified it will be calculated as 3/4 * log_2(min(nrows, ncols)) as suggested in the M4RI paper.
- reduced – return reduced row echelon form (default:True)

EXAMPLES:

```
sage: A = random_matrix(GF(2), 10, 10)
sage: B = A.__copy__(); B.echelonize() # fastest
sage: C = A.__copy__(); C.echelonize(k=2) # force k
sage: E = A.__copy__(); E.echelonize(algorithm='classical') # force Gaussian␣
↪elimination
sage: B == C == E
True
```

ALGORITHM:

Uses M4RI library

REFERENCES:

- [Bar2006]

randomize(*density=1*, *nonzero=False*)

Randomize `density` proportion of the entries of this matrix, leaving the rest unchanged.

INPUT:

- `density` - float; proportion (roughly) to be considered for changes
- `nonzero` - Bool (default: `False`); whether the new entries are forced to be non-zero

OUTPUT:

- None, the matrix is modified in-space

EXAMPLES:

```
sage: A = matrix(GF(2), 5, 5, 0)
sage: A.randomize(0.5)
sage: A.density() < 0.5
True
sage: expected = 0.5
sage: A = matrix(GF(2), 5, 5, 0)
sage: A.randomize()
sage: density_sum = float(A.density())
sage: total = 1
sage: while abs(density_sum/total - expected) > 0.001:
....:     A = matrix(GF(2), 5, 5, 0)
....:     A.randomize()
```

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```
....:     density_sum += float(A.density())
....:     total += 1
```

rank(*algorithm='ple'*)

Return the rank of this matrix.

On average 'ple' should be faster than 'm4ri' and hence it is the default choice. However, for small - i.e. quite few thousand rows & columns - and sparse matrices 'm4ri' might be a better choice.

INPUT:

- `algorithm` - either "ple" or "m4ri"

EXAMPLES:

```
sage: while random_matrix(GF(2), 1000, 1000).rank() != 999:
....:     pass

sage: A = matrix(GF(2),10, 0)
sage: A.rank()
0
```

row(*i*, *from_list=False*)

Return the `i`'th row of this matrix as a vector.

This row is a dense vector if and only if the matrix is a dense matrix.

INPUT:

- `i` - integer
- `from_list` - bool (default: `False`); if `True`, returns the `i`'th element of `self.rows()` (see `rows()`), which may be faster, but requires building a list of all rows the first time it is called after an entry of the matrix is changed.

EXAMPLES:

```
sage: l = [GF(2).random_element() for _ in range(100)]
sage: A = matrix(GF(2), 10, 10 , l)
sage: list(A.row(0)) == l[:10]
True
sage: list(A.row(-1)) == l[-10:]
True

sage: list(A.row(2, from_list=True)) == l[20:30]
True

sage: A = Matrix(GF(2),1,0)
sage: A.row(0)
()
```

str(*rep_mapping=None*, *zero=None*, *plus_one=None*, *minus_one=None*, *unicode=False*, *shape=None*, *character_art=False*)

Return a nice string representation of the matrix.

INPUT:

- `rep_mapping` – a dictionary or callable used to override the usual representation of elements. For a dictionary, keys should be elements of the base ring and values the desired string representation.
- `zero` – string (default: `None`); if not `None` use the value of `zero` as the representation of the zero element.
- `plus_one` – string (default: `None`); if not `None` use the value of `plus_one` as the representation of the one element.
- `minus_one` – Ignored. Only for compatibility with generic matrices.
- `unicode` – boolean (default: `False`). Whether to use Unicode symbols instead of ASCII symbols for brackets and subdivision lines.
- `shape` – one of `"square"` or `"round"` (default: `None`). Switches between round and square brackets. The default depends on the setting of the `unicode` keyword argument. For Unicode symbols, the default is round brackets in accordance with the TeX rendering, while the ASCII rendering defaults to square brackets.
- `character_art` – boolean (default: `False`); if `True`, the result will be of type `AsciiArt` or `UnicodeArt` which support line breaking of wide matrices that exceed the window width

EXAMPLES:

```
sage: B = matrix(GF(2), 3, 3, [0, 1, 0, 0, 1, 1, 0, 0, 0])
sage: B  # indirect doctest
[0 1 0]
[0 1 1]
[0 0 0]
sage: block_matrix([[B, 1], [0, B]])
[0 1 0|1 0 0]
[0 1 1|0 1 0]
[0 0 0|0 0 1]
[-----+-----]
[0 0 0|0 1 0]
[0 0 0|0 1 1]
[0 0 0|0 0 0]
sage: B.str(zero='.')
'[. 1 .]\n[. 1 1]\n[. . .]'

sage: M = matrix.identity(GF(2), 3)
sage: M.subdivide(None, 2)
sage: print(M.str(unicode=True, shape='square'))
⎡1 0|0⎤
⎢     ⎥
⎢0 1|0⎥
⎢     ⎥
⎣0 0|1⎦
sage: print(unicode_art(M))  # indirect doctest
⎛1 0|0⎞
⎜     ⎟
⎜0 1|0⎟
⎜     ⎟
⎝0 0|1⎠
```

submatrix(*row=0*, *col=0*, *nrows=-1*, *ncols=-1*)

Return submatrix from the index row, col (inclusive) with dimension nrows x ncols.

INPUT:

- row – index of start row
- col – index of start column
- nrows – number of rows of submatrix
- ncols – number of columns of submatrix

EXAMPLES:

```
sage: A = random_matrix(GF(2),200,200)
sage: A[0:2,0:2] == A.submatrix(0,0,2,2)
True
sage: A[0:100,0:100] == A.submatrix(0,0,100,100)
True
sage: A == A.submatrix(0,0,200,200)
True

sage: A[1:3,1:3] == A.submatrix(1,1,2,2)
True
sage: A[1:100,1:100] == A.submatrix(1,1,99,99)
True
sage: A[1:200,1:200] == A.submatrix(1,1,199,199)
True
```

TESTS for handling of default arguments (github issue #18761):

```
sage: A.submatrix(17,15) == A.submatrix(17,15,183,185)
True
sage: A.submatrix(row=100,col=37,nrows=1,ncols=3) == A.submatrix(100,37,1,3)
True
```

transpose()

Returns transpose of self and leaves self untouched.

EXAMPLES:

```
sage: A = Matrix(GF(2),3,5,[1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0])
sage: A
[1 0 1 0 0]
[0 1 1 0 0]
[1 1 0 1 0]
sage: B = A.transpose(); B
[1 0 1]
[0 1 1]
[1 1 0]
[0 0 1]
[0 0 0]
sage: B.transpose() == A
True
```

.T is a convenient shortcut for the transpose:

```
sage: A.T
[1 0 1]
[0 1 1]
```

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```
[1 1 0]
[0 0 1]
[0 0 0]
```

sage.matrix.matrix_mod2_dense.**from_png**(*filename*)

Returns a dense matrix over GF(2) from a 1-bit PNG image read from `filename`. No attempt is made to verify that the filename string actually points to a PNG image.

INPUT:

- filename – a string

EXAMPLES:

```
sage: from sage.matrix.matrix_mod2_dense import from_png, to_png
sage: A = random_matrix(GF(2),10,10)
sage: fn = tmp_filename()
sage: to_png(A, fn)
sage: B = from_png(fn)
sage: A == B
True
```

sage.matrix.matrix_mod2_dense.**parity**(*a*)

Return the parity of the number of bits in a.

EXAMPLES:

```
sage: from sage.matrix.matrix_mod2_dense import parity
sage: parity(1)
1
sage: parity(3)
0
sage: parity(0x10000101011)
1
```

sage.matrix.matrix_mod2_dense.**ple**(*A*, *algorithm='standard'*, *param=0*)

Return PLE factorization of A.

INPUT:

- A – matrix
- algorithm
  - 'standard' asymptotically fast (default)
  - 'russian' M4RI inspired
  - 'naive' naive cubic
- param – either k for 'mmpf' is chosen or matrix multiplication cutoff for 'standard' (default: 0)

EXAMPLES:

```
sage: from sage.matrix.matrix_mod2_dense import ple

sage: A = matrix(GF(2), 4, 4, [0, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 1, 0])
sage: A
```

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```
[0 1 0 1]
[0 1 1 1]
[0 0 0 1]
[0 1 1 0]

sage: LU, P, Q = ple(A)
sage: LU
[1 0 0 1]
[1 1 0 0]
[0 0 1 0]
[1 1 1 0]

sage: P
[0, 1, 2, 3]

sage: Q
[1, 2, 3, 3]

sage: A = random_matrix(GF(2),1000,1000)
sage: ple(A) == ple(A,'russian') == ple(A,'naive')
True
```

sage.matrix.matrix_mod2_dense.**pluq**(*A*, *algorithm='standard'*, *param=0*)

Return PLUQ factorization of A.

INPUT:

- A – matrix
- algorithm
  - 'standard' asymptotically fast (default)
  - 'mmpf' M4RI inspired
  - 'naive' naive cubic
- param – either k for 'mmpf' is chosen or matrix multiplication cutoff for 'standard' (default: 0)

EXAMPLES:

```
sage: from sage.matrix.matrix_mod2_dense import pluq
sage: A = matrix(GF(2), 4, 4, [0, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 1, 0])
sage: A
[0 1 0 1]
[0 1 1 1]
[0 0 0 1]
[0 1 1 0]
sage: LU, P, Q = pluq(A)
sage: LU
[1 0 1 0]
[1 1 0 0]
[0 0 1 0]
[1 1 1 0]

sage: P
```

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```
[0, 1, 2, 3]

sage: Q
[1, 2, 3, 3]
```

sage.matrix.matrix_mod2_dense.**to_png**(*A*, *filename*)

Saves the matrix A to filename as a 1-bit PNG image.

INPUT:

- A - a matrix over GF(2)
- filename - a string for a file in a writable position

EXAMPLES:

```
sage: from sage.matrix.matrix_mod2_dense import from_png, to_png
sage: A = random_matrix(GF(2),10,10)
sage: fn = tmp_filename()
sage: to_png(A, fn)
sage: B = from_png(fn)
sage: A == B
True
```

sage.matrix.matrix_mod2_dense.**unpickle_matrix_mod2_dense_v2**(*r*, *c*, *data*, *size*, *immutable=False*)

Deserialize a matrix encoded in the string s.

INPUT:

- r – number of rows of matrix
- c – number of columns of matrix
- s – a string
- size – length of the string s
- immutable – (default: False) whether the matrix is immutable or not

EXAMPLES:

```
sage: A = random_matrix(GF(2),100,101)
sage: _, (r,c,s,s2,i) = A.__reduce__()
sage: from sage.matrix.matrix_mod2_dense import unpickle_matrix_mod2_dense_v2
sage: unpickle_matrix_mod2_dense_v2(r,c,s,s2,i) == A
True
sage: loads(dumps(A)) == A
True
```

CHAPTER

TWENTYFOUR

DENSE MATRICES OVER $\mathbf{F}_{2^E}$ FOR $2 \leq E \leq 16$ USING THE M4RIE LIBRARY

The M4RIE library offers two matrix representations:

1) `mzed_t`

   m x n matrices over $\mathbf{F}_{2^e}$ are internally represented roughly as m x (en) matrices over $\mathbf{F}_2$. Several elements are packed into words such that each element is filled with zeroes until the next power of two. Thus, for example, elements of $\mathbf{F}_{2^3}$ are represented as `[0xxx|0xxx|0xxx|0xxx|...]`. This representation is wrapped as *Matrix_gf2e_dense* in Sage.

   Multiplication and elimination both use "Newton-John" tables. These tables are simply all possible multiples of a given row in a matrix such that a scale+add operation is reduced to a table lookup + add. On top of Newton-John multiplication M4RIE implements asymptotically fast Strassen-Winograd multiplication. Elimination uses simple Gaussian elimination which requires $O(n^3)$ additions but only $O(n^2 * 2^e)$ multiplications.

2) `mzd_slice_t`

   m x n matrices over $\mathbf{F}_{2^e}$ are internally represented as slices of m x n matrices over $\mathbf{F}_2$. This representation allows for very fast matrix times matrix products using Karatsuba's polynomial multiplication for polynomials over matrices. However, it is not feature complete yet and hence not wrapped in Sage for now.

See http://m4ri.sagemath.org for more details on the M4RIE library.

EXAMPLES:

```
sage: K.<a> = GF(2^8)
sage: A = random_matrix(K, 3,4)
sage: E = A.echelon_form()
sage: A.row_space() == E.row_space()
True
sage: all(r[r.nonzero_positions()[0]] == 1 for r in E.rows() if r)
True
```

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Todo: Wrap `mzd_slice_t`.

REFERENCES:

- [BB2009]

class sage.matrix.matrix_gf2e_dense.**M4RIE_finite_field**

Bases: object

A thin wrapper around the M4RIE finite field class such that we can put it in a hash table. This class is not meant for public consumption.

class sage.matrix.matrix_gf2e_dense.**Matrix_gf2e_dense**

Bases: *Matrix_dense*

Create new matrix over $GF(2^e)$ for $2 \leq e \leq 16$.

INPUT:

- parent – a matrix space over GF(2^e)
- entries – see *matrix()*
- copy – ignored (for backwards compatibility)
- coerce – if False, assume without checking that the entries lie in the base ring

EXAMPLES:

```
sage: K.<a> = GF(2^4)
sage: l = [K.random_element() for _ in range(3*4)]

sage: A = Matrix(K, 3, 4, l)
sage: l == A.list()
True

sage: l[0] == A[0,0]
True

sage: A = Matrix(K, 3, 3, a); A
[a 0 0]
[0 a 0]
[0 0 a]
```

augment(*right*)

Augments self with right.

INPUT:

- right - a matrix

EXAMPLES:

```
sage: K.<a> = GF(2^4)
sage: MS = MatrixSpace(K,3,3)
sage: A = random_matrix(K,3,3)
sage: B = A.augment(MS(1))
sage: B.echelonize()
sage: C = B.matrix_from_columns([3,4,5])
sage: A.rank() < 3 or C == ~A
True
sage: A.rank() < 3 or C*A == MS(1)
True
```

cling(**C*)

Pack the matrices over $\mathbf{F}_2$ into this matrix over $\mathbf{F}_{2^e}$.

Elements in $\mathbf{F}_{2^e}$ can be represented as $\sum c_i a^i$ where a is a root the minimal polynomial. If this matrix is A then this function writes $c_i a^i$ to the entry $A[x, y]$ where c_i is the entry $C_i[x, y]$.

INPUT:

- `C` - a list of matrices over GF(2)

EXAMPLES:

```
sage: K.<a> = GF(2^2)
sage: A = matrix(K, 5, 5)
sage: A0 = random_matrix(GF(2), 5, 5)
sage: A1 = random_matrix(GF(2), 5, 5)
sage: A.cling(A0, A1)
sage: all(A.list()[i] == A0.list()[i] + a*A1.list()[i] for i in range(25))
True
```

Slicing and clinging are inverse operations:

```
sage: B0, B1 = A.slice()
sage: B0 == A0 and B1 == A1
True
```

echelonize(*algorithm='heuristic'*, *reduced=True*, ***kwds*)

Compute the row echelon form of `self` in place.

INPUT:

- `algorithm` - one of the following - `heuristic` - let M4RIE decide (default) - `newton_john` - use newton_john table based algorithm - `ple` - use PLE decomposition - `naive` - use naive cubic Gaussian elimination (M4RIE implementation) - `builtin` - use naive cubic Gaussian elimination (Sage implementation)
- `reduced` - if `True` return reduced echelon form. No guarantee is given that the matrix is *not* reduced if `False` (default: `True`)

EXAMPLES:

```
sage: K.<a> = GF(2^4)
sage: m,n  = 3, 5
sage: A = random_matrix(K, 3, 5)
sage: R = A.row_space()
sage: A.echelonize()
sage: all(r[r.nonzero_positions()[0]] == 1 for r in A.rows() if r)
True
sage: A.row_space() == R
True

sage: K.<a> = GF(2^3)
sage: m,n  = 3, 5
sage: MS = MatrixSpace(K,m,n)
sage: A = random_matrix(K, 3, 5)
sage: B = copy(A).echelon_form('newton_john')
sage: C = copy(A).echelon_form('naive')
```

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```
sage: D = copy(A).echelon_form('builtin')
sage: B == C == D
True
sage: all(r[r.nonzero_positions()[0]] == 1 for r in B.rows() if r)
True
```

randomize(*density=1*, *nonzero=False*, **args*, ***kwds*)

Randomize `density` proportion of the entries of this matrix, leaving the rest unchanged.

INPUT:

- `density` - float; proportion (roughly) to be considered for changes
- `nonzero` - Bool (default: `False`); whether the new entries are forced to be non-zero

OUTPUT:

- None, the matrix is modified in-place

EXAMPLES:

```
sage: K.<a> = GF(2^4)
sage: total_count = 0
sage: from collections import defaultdict
sage: dic = defaultdict(Integer)
sage: def add_samples():
....:     global dic, total_count
....:     for _ in range(100):
....:         A = Matrix(K,3,3)
....:         A.randomize()
....:         for a in A.list():
....:             dic[a] += 1
....:             total_count += 1.0
sage: add_samples()
sage: while not all(abs(dic[a]/total_count - 1/16) < 0.01 for a in dic):
....:     add_samples()

sage: def add_sample(density):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     density_sum += random_matrix(K, 1000, 1000, density=density).density()

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.1)
sage: while abs(density_sum/total_count - 0.1) > 0.001:
....:     add_sample(0.1)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(1.0)
sage: while abs(density_sum/total_count - 1.0) > 0.001:
....:     add_sample(1.0)

sage: density_sum = 0.0
```

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```
sage: total_count = 0.0
sage: add_sample(0.5)
sage: while abs(density_sum/total_count - 0.5) > 0.001:
....:     add_sample(0.5)
```

Note, that the matrix is updated and not zero-ed out before being randomized:

```
sage: def add_sample(density, nonzero):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     A = matrix(K, 1000, 1000)
....:     A.randomize(nonzero=nonzero, density=density)
....:     A.randomize(nonzero=nonzero, density=density)
....:     density_sum += A.density()

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.1, True)
sage: while abs(density_sum/total_count - (1 - 0.9^2)) > 0.001:
....:     add_sample(0.1, True)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.1, False)
sage: while abs(density_sum/total_count - (1 - 0.9^2)*15/16) > 0.001:
....:     add_sample(0.1, False)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.05, True)
sage: while abs(density_sum/total_count - (1 - 0.95^2)) > 0.001:
....:     add_sample(0.05, True)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.5, True)
sage: while abs(density_sum/total_count - (1 - 0.5^2)) > 0.001:
....:     add_sample(0.5, True)
```

rank()

Return the rank of this matrix (cached).

EXAMPLES:

```
sage: K.<a> = GF(2^4)
sage: A = random_matrix(K, 10, 10, algorithm="unimodular")
sage: A.rank()
10
sage: A = matrix(K, 10, 0)
sage: A.rank()
0
```

slice()

Unpack this matrix into matrices over $\mathbf{F}_2$.

Elements in $\mathbf{F}_{2^e}$ can be represented as $\sum c_i a^i$ where a is a root the minimal polynomial. This function returns a tuple of matrices C whose entry $C_i[x, y]$ is the coefficient of c_i in $A[x, y]$ if this matrix is A.

EXAMPLES:

```
sage: K.<a> = GF(2^2)
sage: A = random_matrix(K, 5, 5)
sage: A0, A1 = A.slice()
sage: all(A.list()[i] == A0.list()[i] + a*A1.list()[i] for i in range(25))
True

sage: K.<a> = GF(2^3)
sage: A = random_matrix(K, 5, 5)
sage: A0, A1, A2 = A.slice()
sage: all(A.list()[i] == A0.list()[i] + a*A1.list()[i] + a^2*A2.list()[i] for i␣
↪in range(25))
True
```

Slicing and clinging are inverse operations:

```
sage: B = matrix(K, 5, 5)
sage: B.cling(A0, A1, A2)
sage: B == A
True
```

submatrix(*row=0*, *col=0*, *nrows=-1*, *ncols=-1*)

Return submatrix from the index `row,col` (inclusive) with dimension `nrows x ncols`.

INPUT:

- `row` – index of start row
- `col` – index of start column
- `nrows` – number of rows of submatrix
- `ncols` – number of columns of submatrix

EXAMPLES:

```
sage: K.<a> = GF(2^10)
sage: A = random_matrix(K,200,200)
sage: A[0:2,0:2] == A.submatrix(0,0,2,2)
True
sage: A[0:100,0:100] == A.submatrix(0,0,100,100)
True
sage: A == A.submatrix(0,0,200,200)
True

sage: A[1:3,1:3] == A.submatrix(1,1,2,2)
True
sage: A[1:100,1:100] == A.submatrix(1,1,99,99)
True
sage: A[1:200,1:200] == A.submatrix(1,1,199,199)
True
```

TESTS for handling of default arguments (github issue #18761):

```
sage: A.submatrix(17,15) == A.submatrix(17,15,183,185)
True
sage: A.submatrix(row=100,col=37,nrows=1,ncols=3) == A.submatrix(100,37,1,3)
True
```

sage.matrix.matrix_gf2e_dense.**unpickle_matrix_gf2e_dense_v0**(*a*, *base_ring*, *nrows*, *ncols*)

EXAMPLES:

```
sage: K.<a> = GF(2^2)
sage: A = random_matrix(K,10,10)
sage: f, s= A.__reduce__()
sage: from sage.matrix.matrix_gf2e_dense import unpickle_matrix_gf2e_dense_v0
sage: f == unpickle_matrix_gf2e_dense_v0
True
sage: f(*s) == A
True
```

We can still unpickle pickles from before github issue #19240:

```
sage: old_pickle = b'x\x9c\x85RKo\xd3@\x10\xae\xdd$$\xdb&\xe5U\x1e-\x8f\xc2\xc9\
↪x12RD#$\xce\xa0\xb4\x80\x07\xa2\xca\xc2\x07\x0e\xd5\xe2:\x1b\xdb\x8acg\x1c\xa7J\
↪x85*!\xa4\x90\xe6\x07p\xe0\xc4\x01q\xe5\xc4\x19\xf5\xd0?\xc1\x81\xdf\x80\xb8q\x0b\
↪xb3\x8eMS\xa1\x82V;;\xb3\xdf\xce\xf7\xcd\x8e\xe6\xb5j\xf7,GT;V\x1cy\x83\xf4\xe0\
↪x9d\xb0Y\x13\xbc)\x82\x9e`\xfd\xa0\xeb\xd9m_\xf0\xbf1\xbe{\x97\xa1\xa2\x9d\xc6\
↪xf0\x0f\x82,\x7f\x9d\xa1\xaa\x81\n\xb9m\x9c\xd7\xf4\xf1d2\x81-h\xc0#(\x03\x83\x15\
↪xdas\xc9*\xc3\x13x\x0cu0\xd28\x97\x9e*(0\x9f\xfa\x1b\xd0\xd2\x7fH\x82\xb5\xf4\
↪xa2@TO\xe19\x01I\xac\x136\x991\x9f\xa4\xf9&\xcd\x07i\xbe\xcb\xd4ib\t\xba\xa4\xf6\
↪x02zIT\xd1\x8f2(u\x15\xfd\x9d<\xee@\x05V\xd3\x94E*\xb0\x0e\x0fH\xad\xa8\xbf\x97\
↪xa0\r\x03\xfd\xf0\xb8\x1aU\xff\x92\x90\xe8?\xa5\xd6\x814_\xa5\xf9(\xcd\xafc\xe99\
↪xe2\xd9\xa0\x06\xd4\xf5\xcf\xf2\xf2!\xbc\xd4\xdf\x90#\xc0\x8f\r\xccM\x1b\xdd\x8b\
↪xa3\xbe\x1d\xf7#QmYv\x1cF{\xcc\x11\x81\x88<\x9b\xa71\xcf:\xce0\xaf\x9d\x96\xe3\
↪x87a\xbb\xdf\xe5\x8e\x1f\xeeX>\xc3\x82\xb9\xb0\xe9\x05^,6=\xe17\xf1\xcc\xd0\xc0"u\
↪xb0d\xe6wDl\xdd\x1fa)e\x8a\xbc\xc0\xe9U\xbd \x16\x8e\x88X\xc7j\x0b\x9e\x05\xc8L\
↪xe5\x1e%.\x98\x8a5\xc4\xc5\xd9\xf7\xdd\xd0\xdf\x0b\xc2\x8eg\xf93.wZ\xb5\xc1\x94B\
↪xf8\xa2#\x82\x98a\xf9\xffY\x12\xe3v\x18L\xff\x14Fl\xeb\x0ff\x10\xc4\xb0\xa2\xb9y\
↪xcd-\xba%\xcd\xa5\x8ajT\xd1\x92\xa9\x0c\x86x\xb6a\xe6h\xf8\x02<g\xaa\xaf\xf6\xdd%\
↪x89\xae\x13z\xfe \xc6\x0b\xfb1^4p\x99\x1e6\xc6\xd4\xebK\xdbx\xf9\xc4\x8f[Iw\xf8\
↪x89\xef\xcbQf\xcfh\xe3\x95\x8c\xebj&\xb9\xe2.\x8f\x0c\\ui\x89\xf1x\xf4\xd6\xc0kf\
↪xc1\xf1v\xad(\xc4\xeb\x89~\xfa\xf0\x06\xa8\xa4\x7f\x93\xf4\xd7\x0c\xbcE#\xad\x92\
↪xfc\xed\xeao\xefX\\\x03'
sage: loads(old_pickle)
[    0     a]
[a + 1     1]
```

CHAPTER

TWENTYFIVE

DENSE MATRICES OVER $\mathbb{Z}/N\mathbb{Z}$ FOR $N < 2^{23}$ USING LINBOX'S MODULAR<DOUBLE>

AUTHORS:

- Burcin Erocal
- Martin Albrecht

class sage.matrix.matrix_modn_dense_double.**Matrix_modn_dense_double**

Bases: *Matrix_modn_dense_template*

Dense matrices over $\mathbf{Z}/n\mathbf{Z}$ for $n < 2^{23}$ using LinBox's Modular<double>

These are matrices with integer entries mod n represented as floating-point numbers in a 64-bit word for use with LinBox routines. This allows for n up to 2^{23}. The analogous Matrix_modn_dense_float class is used for smaller moduli.

Routines here are for the most basic access, see the matrix_modn_dense_template.pxi file for higher-level routines.

class sage.matrix.matrix_modn_dense_double.**Matrix_modn_dense_template**

Bases: *Matrix_dense*

Create a new matrix.

INPUT:

- parent – a matrix space
- entries – see *matrix()*
- copy – ignored (for backwards compatibility)
- coerce - perform modular reduction first?

EXAMPLES:

```
sage: A = random_matrix(GF(3),1000,1000)
sage: type(A)
<class 'sage.matrix.matrix_modn_dense_float.Matrix_modn_dense_float'>
sage: A = random_matrix(Integers(10),1000,1000)
sage: type(A)
<class 'sage.matrix.matrix_modn_dense_float.Matrix_modn_dense_float'>
sage: A = random_matrix(Integers(2^16),1000,1000)
sage: type(A)
<class 'sage.matrix.matrix_modn_dense_double.Matrix_modn_dense_double'>
```

charpoly(*var='x'*, *algorithm='linbox'*)

Return the characteristic polynomial of `self`.

INPUT:

- `var` - a variable name
- `algorithm` - 'generic', 'linbox' or 'all' (default: linbox)

EXAMPLES:

```
sage: A = random_matrix(GF(19), 10, 10)
sage: B = copy(A)
sage: char_p = A.characteristic_polynomial()
sage: char_p(A) == 0
True
sage: B == A              # A is not modified
True

sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p.divides(char_p)
True
```

```
sage: A = random_matrix(GF(2916337), 7, 7)
sage: B = copy(A)
sage: char_p = A.characteristic_polynomial()
sage: char_p(A) == 0
True
sage: B == A               # A is not modified
True

sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p.divides(char_p)
True

sage: A = Mat(Integers(6),3,3)(range(9))
sage: A.charpoly()
x^3
```

ALGORITHM: Uses LinBox if `self.base_ring()` is a field, otherwise use Hessenberg form algorithm.

determinant()

Return the determinant of this matrix.

EXAMPLES:

```
sage: s = set()
sage: while s != set(GF(7)):
....:     A = random_matrix(GF(7), 10, 10)
....:     s.add(A.determinant())
```

```
sage: A = random_matrix(GF(7), 100, 100)
sage: A.determinant() == A.transpose().determinant()
True

sage: B = random_matrix(GF(7), 100, 100)
```

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```
sage: (A*B).determinant() == A.determinant() * B.determinant()
True
```

```
sage: A = random_matrix(GF(16007), 10, 10)
sage: A.determinant().parent() is GF(16007)
True
```

```
sage: A = random_matrix(GF(16007), 100, 100)
sage: A.determinant().parent() is GF(16007)
True


sage: A.determinant() == A.transpose().determinant()
True

sage: B = random_matrix(GF(16007), 100, 100)
sage: (A*B).determinant() == A.determinant() * B.determinant()
True
```

Parallel computation:

```
sage: A = random_matrix(GF(65521),200)
sage: B = copy(A)
sage: Parallelism().set('linbox', nproc=2)
sage: d = A.determinant()
sage: Parallelism().set('linbox', nproc=1) # switch off parallelization
sage: e = B.determinant()
sage: d==e
True
```

echelonize(*algorithm='linbox_noefd'*, ***kwds*)

Put `self` in reduced row echelon form.

INPUT:

- `self` - a mutable matrix
- `algorithm`
  - `linbox` - uses the LinBox library (wrapping fflas-ffpack)
  - `linbox_noefd` - uses the FFPACK directly, less memory and faster (default)
  - `gauss` - uses a custom slower $O(n^3)$ Gauss elimination implemented in Sage.
  - `all` - compute using both algorithms and verify that the results are the same.
- `**kwds` - these are all ignored

OUTPUT:

- `self` is put in reduced row echelon form.
- the rank of self is computed and cached
- the pivot columns of self are computed and cached.
- the fact that self is now in echelon form is recorded and cached so future calls to echelonize return immediately.

EXAMPLES:

```
sage: A = random_matrix(GF(7), 10, 20)
sage: E = A.echelon_form()
sage: A.row_space() == E.row_space()
True
sage: all(r[r.nonzero_positions()[0]] == 1 for r in E.rows() if r)
True
```

```
sage: A = random_matrix(GF(13), 10, 10)
sage: while A.rank() != 10:
....:     A = random_matrix(GF(13), 10, 10)
sage: MS = parent(A)
sage: B = A.augment(MS(1))
sage: B.echelonize()
sage: A.rank()
10
sage: C = B.submatrix(0,10,10,10)
sage: ~A == C
True
```

```
sage: A = random_matrix(Integers(10), 10, 20)
sage: A.echelon_form()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪10'.
```

```
sage: A = random_matrix(GF(16007), 10, 20)
sage: E = A.echelon_form()
sage: A.row_space() == E.row_space()
True
sage: all(r[r.nonzero_positions()[0]] == 1 for r in E.rows() if r)
True
```

```
sage: A = random_matrix(Integers(10000), 10, 20)
sage: A.echelon_form()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪10000'.
```

Parallel computation:

```
sage: A = random_matrix(GF(65521),100,200)
sage: Parallelism().set('linbox', nproc=2)
sage: E = A.echelon_form()
sage: Parallelism().set('linbox', nproc=1) # switch off parallelization
sage: F = A.echelon_form()
sage: E==F
True
```

hessenbergize()

Transforms self in place to its Hessenberg form.

EXAMPLES:

```
sage: A = random_matrix(GF(17), 10, 10, density=0.1)
sage: B = copy(A)
sage: A.hessenbergize()
sage: all(A[i,j] == 0 for j in range(10) for i in range(j+2, 10))
True
sage: A.charpoly() == B.charpoly()
True
```

lift()

Return the lift of this matrix to the integers.

EXAMPLES:

```
sage: A = matrix(GF(7),2,3,[1..6])
sage: A.lift()
[1 2 3]
[4 5 6]
sage: A.lift().parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring

sage: A = matrix(GF(16007),2,3,[1..6])
sage: A.lift()
[1 2 3]
[4 5 6]
sage: A.lift().parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

Subdivisions are preserved when lifting:

```
sage: A.subdivide([], [1,1]); A
[1|2 3]
[4|5 6]
sage: A.lift()
[1|2 3]
[4|5 6]
```

minpoly(*var='x'*, *algorithm='linbox'*, *proof=None*)

Returns the minimal polynomial of`` self``.

INPUT:

- `var` - a variable name
- `algorithm` - `generic` or `linbox` (default: `linbox`)
- `proof` – (default: `True`); whether to provably return the true minimal polynomial; if `False`, we only guarantee to return a divisor of the minimal polynomial. There are also certainly cases where the computed results is frequently not exactly equal to the minimal polynomial (but is instead merely a divisor of it).

Warning: If `proof=True`, minpoly is insanely slow compared to `proof=False`. This matters since proof=True is the default, unless you first type `proof.linear_algebra(False)`.

EXAMPLES:

```
sage: A = random_matrix(GF(17), 10, 10)
sage: B = copy(A)
sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p(A) == 0
True
sage: B == A
True

sage: char_p = A.characteristic_polynomial()
sage: min_p.divides(char_p)
True
```

```
sage: A = random_matrix(GF(1214471), 10, 10)
sage: B = copy(A)
sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p(A) == 0
True
sage: B == A
True

sage: char_p = A.characteristic_polynomial()
sage: min_p.divides(char_p)
True
```

EXAMPLES:

```
sage: R.<x>=GF(3)[]
sage: A = matrix(GF(3),2,[0,0,1,2])
sage: A.minpoly()
x^2 + x

sage: A.minpoly(proof=False) in [x, x+1, x^2+x]
True
```

randomize(*density=1*, *nonzero=False*)

Randomize `density` proportion of the entries of this matrix, leaving the rest unchanged.

INPUT:

- **density - Integer; proportion (roughly) to be considered** for changes
- **nonzero - Bool (default: `False`); whether the new** entries are forced to be non-zero

OUTPUT:

- None, the matrix is modified in-space

EXAMPLES:

```
sage: A = matrix(GF(5), 5, 5, 0)
sage: total_count = 0
sage: from collections import defaultdict
sage: dic = defaultdict(Integer)
sage: def add_samples(density):
....:     global dic, total_count
....:     for _ in range(100):
....:         A = Matrix(GF(5), 5, 5, 0)
....:         A.randomize(density)
....:         for a in A.list():
....:             dic[a] += 1
....:             total_count += 1.0

sage: add_samples(1.0)
sage: while not all(abs(dic[a]/total_count - 1/5) < 0.01 for a in dic):
....:     add_samples(1.0)

sage: def add_sample(density):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     density_sum += random_matrix(GF(5), 1000, 1000, density=density).
↪density()

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.5)
sage: expected_density = 1.0 - (999/1000)^500
sage: expected_density
0.3936...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.5)
```

The matrix is updated instead of overwritten:

```
sage: def add_sample(density):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     A = random_matrix(GF(5), 1000, 1000, density=density)
....:     A.randomize(density=density, nonzero=True)
....:     density_sum += A.density()

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.5)
sage: expected_density = 1.0 - (999/1000)^1000
sage: expected_density
0.6323...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.5)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.1)
```

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```
sage: expected_density = 1.0 - (999/1000)^200
sage: expected_density
0.1813...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.1)
```

rank()

Return the rank of this matrix.

EXAMPLES:

```
sage: A = random_matrix(GF(3), 100, 100)
sage: B = copy(A)
sage: _ = A.rank()
sage: B == A
True

sage: A = random_matrix(GF(3), 100, 100, density=0.01)
sage: A.transpose().rank() == A.rank()
True

sage: A = matrix(GF(3), 100, 100)
sage: A.rank()
0
```

Rank is not implemented over the integers modulo a composite yet.:

```
sage: M = matrix(Integers(4), 2, [2,2,2,2])
sage: M.rank()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪4'.
```

```
sage: A = random_matrix(GF(16007), 100, 100)
sage: B = copy(A)
sage: A.rank()
100
sage: B == A
True
sage: MS = A.parent()
sage: MS(1) == ~A*A
True
```

right_kernel_matrix(*algorithm='linbox'*, *basis='echelon'*)

Returns a matrix whose rows form a basis for the right kernel of `self`.

If the base ring is the ring of integers modulo a composite, the keyword arguments are ignored and the computation is delegated to `Matrix_dense.right_kernel_matrix()`.

INPUT:

- `algorithm` – (default: `'linbox'`) a parameter that is passed on to `self.echelon_form`, if computation of an echelon form is required; see that routine for allowable values

- `basis` – (default: `'echelon'`) a keyword that describes the format of the basis returned, allowable values are:
  - `'echelon'`: the basis matrix is in echelon form
  - **`'pivot'`: the basis matrix is such that the submatrix obtained** by taking the columns that in `self` contain no pivots, is the identity matrix
  - **`'computed'`: no work is done to transform the basis; in** the current implementation the result is the negative of that returned by `'pivot'`

OUTPUT:

A matrix `X` whose rows are a basis for the right kernel of `self`. This means that `self * X.transpose()` is a zero matrix.

The result is not cached, but the routine benefits when `self` is known to be in echelon form already.

EXAMPLES:

```
sage: M = matrix(GF(5),6,6,range(36))
sage: M.right_kernel_matrix(basis='computed')
[4 2 4 0 0 0]
[3 3 0 4 0 0]
[2 4 0 0 4 0]
[1 0 0 0 0 4]
sage: M.right_kernel_matrix(basis='pivot')
[1 3 1 0 0 0]
[2 2 0 1 0 0]
[3 1 0 0 1 0]
[4 0 0 0 0 1]
sage: M.right_kernel_matrix()
[1 0 0 0 0 4]
[0 1 0 0 1 3]
[0 0 1 0 2 2]
[0 0 0 1 3 1]
sage: M * M.right_kernel_matrix().transpose()
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

`submatrix`(*row=0*, *col=0*, *nrows=-1*, *ncols=-1*)

Return the matrix constructed from self using the specified range of rows and columns.

INPUT:

- `row`, `col` – index of the starting row and column. Indices start at zero
- `nrows`, `ncols` – (optional) number of rows and columns to take. If not provided, take all rows below and all columns to the right of the starting entry

See also:

The functions `matrix_from_rows()`, `matrix_from_columns()`, and `matrix_from_rows_and_columns()` allow one to select arbitrary subsets of rows and/or columns.

EXAMPLES:

Take the 3×3 submatrix starting from entry $(1, 1)$ in a 4×4 matrix:

```
sage: m = matrix(GF(17),4, [1..16])
sage: m.submatrix(1, 1)
[ 6  7  8]
[10 11 12]
[14 15 16]
```

Same thing, except take only two rows:

```
sage: m.submatrix(1, 1, 2)
[ 6  7  8]
[10 11 12]
```

And now take only one column:

```
sage: m.submatrix(1, 1, 2, 1)
[ 6]
[10]
```

You can take zero rows or columns if you want:

```
sage: m.submatrix(0, 0, 0)
[]
sage: parent(m.submatrix(0, 0, 0))
Full MatrixSpace of 0 by 4 dense matrices over Finite Field of size 17
```

transpose()

Return the transpose of `self`, without changing `self`.

EXAMPLES:

We create a matrix, compute its transpose, and note that the original matrix is not changed.

```
sage: M = MatrixSpace(GF(41),  2)
sage: A = M([1,2,3,4])
sage: B = A.transpose()
sage: B
[1 3]
[2 4]
sage: A
[1 2]
[3 4]
```

`.T` is a convenient shortcut for the transpose:

```
sage: A.T
[1 3]
[2 4]
```

```
sage: A.subdivide(None, 1); A
[1|2]
[3|4]
sage: A.transpose()
[1 3]
```

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```
[---]
[2 4]
```

CHAPTER

TWENTYSIX

DENSE MATRICES OVER $\mathbb{Z}/N\mathbb{Z}$ FOR $N < 2^{11}$ USING LINBOX'S MODULAR<FLOAT>

AUTHORS: - Burcin Erocal - Martin Albrecht

class sage.matrix.matrix_modn_dense_float.**Matrix_modn_dense_float**

Bases: *Matrix_modn_dense_template*

Dense matrices over $\mathbf{Z}/n\mathbf{Z}$ for $n < 2^{11}$ using LinBox's `Modular<float>`

These are matrices with integer entries mod `n` represented as floating-point numbers in a 32-bit word for use with LinBox routines. This allows for `n` up to 2^{11}. The `Matrix_modn_dense_double` class is used for larger moduli.

Routines here are for the most basic access, see the `matrix_modn_dense_template.pxi` file for higher-level routines.

class sage.matrix.matrix_modn_dense_float.**Matrix_modn_dense_template**

Bases: *Matrix_dense*

Create a new matrix.

INPUT:

- `parent` – a matrix space
- `entries` – see *matrix()*
- `copy` – ignored (for backwards compatibility)
- `coerce` - perform modular reduction first?

EXAMPLES:

```
sage: A = random_matrix(GF(3),1000,1000)
sage: type(A)
<class 'sage.matrix.matrix_modn_dense_float.Matrix_modn_dense_float'>
sage: A = random_matrix(Integers(10),1000,1000)
sage: type(A)
<class 'sage.matrix.matrix_modn_dense_float.Matrix_modn_dense_float'>
sage: A = random_matrix(Integers(2^16),1000,1000)
sage: type(A)
<class 'sage.matrix.matrix_modn_dense_double.Matrix_modn_dense_double'>
```

charpoly(*var='x'*, *algorithm='linbox'*)

Return the characteristic polynomial of `self`.

INPUT:

- `var` - a variable name
- `algorithm` - 'generic', 'linbox' or 'all' (default: linbox)

EXAMPLES:

```
sage: A = random_matrix(GF(19), 10, 10)
sage: B = copy(A)
sage: char_p = A.characteristic_polynomial()
sage: char_p(A) == 0
True
sage: B == A              # A is not modified
True

sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p.divides(char_p)
True
```

```
sage: A = random_matrix(GF(2916337), 7, 7)
sage: B = copy(A)
sage: char_p = A.characteristic_polynomial()
sage: char_p(A) == 0
True
sage: B == A               # A is not modified
True

sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p.divides(char_p)
True

sage: A = Mat(Integers(6),3,3)(range(9))
sage: A.charpoly()
x^3
```

ALGORITHM: Uses LinBox if `self.base_ring()` is a field, otherwise use Hessenberg form algorithm.

determinant()

Return the determinant of this matrix.

EXAMPLES:

```
sage: s = set()
sage: while s != set(GF(7)):
....:     A = random_matrix(GF(7), 10, 10)
....:     s.add(A.determinant())
```

```
sage: A = random_matrix(GF(7), 100, 100)
sage: A.determinant() == A.transpose().determinant()
True

sage: B = random_matrix(GF(7), 100, 100)
sage: (A*B).determinant() == A.determinant() * B.determinant()
True
```

```
sage: A = random_matrix(GF(16007), 10, 10)
sage: A.determinant().parent() is GF(16007)
True
```

```
sage: A = random_matrix(GF(16007), 100, 100)
sage: A.determinant().parent() is GF(16007)
True


sage: A.determinant() == A.transpose().determinant()
True

sage: B = random_matrix(GF(16007), 100, 100)
sage: (A*B).determinant() == A.determinant() * B.determinant()
True
```

Parallel computation:

```
sage: A = random_matrix(GF(65521),200)
sage: B = copy(A)
sage: Parallelism().set('linbox', nproc=2)
sage: d = A.determinant()
sage: Parallelism().set('linbox', nproc=1) # switch off parallelization
sage: e = B.determinant()
sage: d==e
True
```

echelonize(*algorithm='linbox_noefd'*, ***kwds*)

Put `self` in reduced row echelon form.

INPUT:

- `self` - a mutable matrix
- `algorithm`
  - `linbox` - uses the LinBox library (wrapping fflas-ffpack)
  - `linbox_noefd` - uses the FFPACK directly, less memory and faster (default)
  - `gauss` - uses a custom slower $O(n^3)$ Gauss elimination implemented in Sage.
  - `all` - compute using both algorithms and verify that the results are the same.
- `**kwds` - these are all ignored

OUTPUT:

- `self` is put in reduced row echelon form.
- the rank of self is computed and cached
- the pivot columns of self are computed and cached.
- the fact that self is now in echelon form is recorded and cached so future calls to echelonize return immediately.

EXAMPLES:

```
sage: A = random_matrix(GF(7), 10, 20)
sage: E = A.echelon_form()
sage: A.row_space() == E.row_space()
True
sage: all(r[r.nonzero_positions()[0]] == 1 for r in E.rows() if r)
True
```

```
sage: A = random_matrix(GF(13), 10, 10)
sage: while A.rank() != 10:
....:     A = random_matrix(GF(13), 10, 10)
sage: MS = parent(A)
sage: B = A.augment(MS(1))
sage: B.echelonize()
sage: A.rank()
10
sage: C = B.submatrix(0,10,10,10)
sage: ~A == C
True
```

```
sage: A = random_matrix(Integers(10), 10, 20)
sage: A.echelon_form()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪10'.
```

```
sage: A = random_matrix(GF(16007), 10, 20)
sage: E = A.echelon_form()
sage: A.row_space() == E.row_space()
True
sage: all(r[r.nonzero_positions()[0]] == 1 for r in E.rows() if r)
True
```

```
sage: A = random_matrix(Integers(10000), 10, 20)
sage: A.echelon_form()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪10000'.
```

Parallel computation:

```
sage: A = random_matrix(GF(65521),100,200)
sage: Parallelism().set('linbox', nproc=2)
sage: E = A.echelon_form()
sage: Parallelism().set('linbox', nproc=1) # switch off parallelization
sage: F = A.echelon_form()
sage: E==F
True
```

hessenbergize()

Transforms self in place to its Hessenberg form.

EXAMPLES:

```
sage: A = random_matrix(GF(17), 10, 10, density=0.1)
sage: B = copy(A)
sage: A.hessenbergize()
sage: all(A[i,j] == 0 for j in range(10) for i in range(j+2, 10))
True
sage: A.charpoly() == B.charpoly()
True
```

lift()

Return the lift of this matrix to the integers.

EXAMPLES:

```
sage: A = matrix(GF(7),2,3,[1..6])
sage: A.lift()
[1 2 3]
[4 5 6]
sage: A.lift().parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring

sage: A = matrix(GF(16007),2,3,[1..6])
sage: A.lift()
[1 2 3]
[4 5 6]
sage: A.lift().parent()
Full MatrixSpace of 2 by 3 dense matrices over Integer Ring
```

Subdivisions are preserved when lifting:

```
sage: A.subdivide([], [1,1]); A
[1|2 3]
[4|5 6]
sage: A.lift()
[1|2 3]
[4|5 6]
```

minpoly(*var='x'*, *algorithm='linbox'*, *proof=None*)

Returns the minimal polynomial of`` self``.

INPUT:

- `var` - a variable name
- `algorithm` - `generic` or `linbox` (default: `linbox`)
- `proof` – (default: `True`); whether to provably return the true minimal polynomial; if `False`, we only guarantee to return a divisor of the minimal polynomial. There are also certainly cases where the computed results is frequently not exactly equal to the minimal polynomial (but is instead merely a divisor of it).

> **Warning:** If `proof=True`, minpoly is insanely slow compared to `proof=False`. This matters since proof=True is the default, unless you first type `proof.linear_algebra(False)`.

EXAMPLES:

```
sage: A = random_matrix(GF(17), 10, 10)
sage: B = copy(A)
sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p(A) == 0
True
sage: B == A
True

sage: char_p = A.characteristic_polynomial()
sage: min_p.divides(char_p)
True
```

```
sage: A = random_matrix(GF(1214471), 10, 10)
sage: B = copy(A)
sage: min_p = A.minimal_polynomial(proof=True)
sage: min_p(A) == 0
True
sage: B == A
True

sage: char_p = A.characteristic_polynomial()
sage: min_p.divides(char_p)
True
```

EXAMPLES:

```
sage: R.<x>=GF(3)[]
sage: A = matrix(GF(3),2,[0,0,1,2])
sage: A.minpoly()
x^2 + x

sage: A.minpoly(proof=False) in [x, x+1, x^2+x]
True
```

randomize(*density=1*, *nonzero=False*)

Randomize `density` proportion of the entries of this matrix, leaving the rest unchanged.

INPUT:

- **`density` - Integer; proportion (roughly) to be considered**
  for changes
- **`nonzero` - Bool (default: `False`); whether the new**
  entries are forced to be non-zero

OUTPUT:

- None, the matrix is modified in-space

EXAMPLES:

```
sage: A = matrix(GF(5), 5, 5, 0)
sage: total_count = 0
sage: from collections import defaultdict
```

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```
sage: dic = defaultdict(Integer)
sage: def add_samples(density):
....:     global dic, total_count
....:     for _ in range(100):
....:         A = Matrix(GF(5), 5, 5, 0)
....:         A.randomize(density)
....:         for a in A.list():
....:             dic[a] += 1
....:             total_count += 1.0

sage: add_samples(1.0)
sage: while not all(abs(dic[a]/total_count - 1/5) < 0.01 for a in dic):
....:     add_samples(1.0)

sage: def add_sample(density):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     density_sum += random_matrix(GF(5), 1000, 1000, density=density).
↪density()

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.5)
sage: expected_density = 1.0 - (999/1000)^500
sage: expected_density
0.3936...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.5)
```

The matrix is updated instead of overwritten:

```
sage: def add_sample(density):
....:     global density_sum, total_count
....:     total_count += 1.0
....:     A = random_matrix(GF(5), 1000, 1000, density=density)
....:     A.randomize(density=density, nonzero=True)
....:     density_sum += A.density()

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.5)
sage: expected_density = 1.0 - (999/1000)^1000
sage: expected_density
0.6323...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.5)

sage: density_sum = 0.0
sage: total_count = 0.0
sage: add_sample(0.1)
sage: expected_density = 1.0 - (999/1000)^200
sage: expected_density
```

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```
0.1813...
sage: while abs(density_sum/total_count - expected_density) > 0.001:
....:     add_sample(0.1)
```

rank()

Return the rank of this matrix.

EXAMPLES:

```
sage: A = random_matrix(GF(3), 100, 100)
sage: B = copy(A)
sage: _ = A.rank()
sage: B == A
True

sage: A = random_matrix(GF(3), 100, 100, density=0.01)
sage: A.transpose().rank() == A.rank()
True

sage: A = matrix(GF(3), 100, 100)
sage: A.rank()
0
```

Rank is not implemented over the integers modulo a composite yet.:

```
sage: M = matrix(Integers(4), 2, [2,2,2,2])
sage: M.rank()
Traceback (most recent call last):
...
NotImplementedError: Echelon form not implemented over 'Ring of integers modulo␣
↪4'.
```

```
sage: A = random_matrix(GF(16007), 100, 100)
sage: B = copy(A)
sage: A.rank()
100
sage: B == A
True
sage: MS = A.parent()
sage: MS(1) == ~A*A
True
```

right_kernel_matrix(*algorithm='linbox'*, *basis='echelon'*)

Returns a matrix whose rows form a basis for the right kernel of `self`.

If the base ring is the ring of integers modulo a composite, the keyword arguments are ignored and the computation is delegated to `Matrix_dense.right_kernel_matrix()`.

INPUT:

- `algorithm` – (default: `'linbox'`) a parameter that is passed on to `self.echelon_form`, if computation of an echelon form is required; see that routine for allowable values
- `basis` – (default: `'echelon'`) a keyword that describes the format of the basis returned, allowable values are:

- `'echelon'`: the basis matrix is in echelon form
- **`'pivot'`: the basis matrix is such that the submatrix obtained** by taking the columns that in `self` contain no pivots, is the identity matrix
- **`'computed'`: no work is done to transform the basis; in** the current implementation the result is the negative of that returned by `'pivot'`

OUTPUT:

A matrix `X` whose rows are a basis for the right kernel of `self`. This means that `self * X.transpose()` is a zero matrix.

The result is not cached, but the routine benefits when `self` is known to be in echelon form already.

EXAMPLES:

```
sage: M = matrix(GF(5),6,6,range(36))
sage: M.right_kernel_matrix(basis='computed')
[4 2 4 0 0 0]
[3 3 0 4 0 0]
[2 4 0 0 4 0]
[1 0 0 0 0 4]
sage: M.right_kernel_matrix(basis='pivot')
[1 3 1 0 0 0]
[2 2 0 1 0 0]
[3 1 0 0 1 0]
[4 0 0 0 0 1]
sage: M.right_kernel_matrix()
[1 0 0 0 0 4]
[0 1 0 0 1 3]
[0 0 1 0 2 2]
[0 0 0 1 3 1]
sage: M * M.right_kernel_matrix().transpose()
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

`submatrix`(*row=0*, *col=0*, *nrows=-1*, *ncols=-1*)

Return the matrix constructed from self using the specified range of rows and columns.

INPUT:

- `row`, `col` – index of the starting row and column. Indices start at zero
- `nrows`, `ncols` – (optional) number of rows and columns to take. If not provided, take all rows below and all columns to the right of the starting entry

See also:

The functions `matrix_from_rows()`, `matrix_from_columns()`, and `matrix_from_rows_and_columns()` allow one to select arbitrary subsets of rows and/or columns.

EXAMPLES:

Take the 3×3 submatrix starting from entry $(1, 1)$ in a 4×4 matrix:

```
sage: m = matrix(GF(17),4, [1..16])
sage: m.submatrix(1, 1)
[ 6  7  8]
[10 11 12]
[14 15 16]
```

Same thing, except take only two rows:

```
sage: m.submatrix(1, 1, 2)
[ 6  7  8]
[10 11 12]
```

And now take only one column:

```
sage: m.submatrix(1, 1, 2, 1)
[ 6]
[10]
```

You can take zero rows or columns if you want:

```
sage: m.submatrix(0, 0, 0)
[]
sage: parent(m.submatrix(0, 0, 0))
Full MatrixSpace of 0 by 4 dense matrices over Finite Field of size 17
```

transpose()

Return the transpose of `self`, without changing `self`.

EXAMPLES:

We create a matrix, compute its transpose, and note that the original matrix is not changed.

```
sage: M = MatrixSpace(GF(41),  2)
sage: A = M([1,2,3,4])
sage: B = A.transpose()
sage: B
[1 3]
[2 4]
sage: A
[1 2]
[3 4]
```

`.T` is a convenient shortcut for the transpose:

```
sage: A.T
[1 3]
[2 4]
```

```
sage: A.subdivide(None, 1); A
[1|2]
[3|4]
sage: A.transpose()
[1 3]
[---]
[2 4]
```

CHAPTER

TWENTYSEVEN

SPARSE MATRICES OVER $\mathbb{Z}/N\mathbb{Z}$ FOR N SMALL

This is a compiled implementation of sparse matrices over $\mathbf{Z}/n\mathbf{Z}$ for n small.

Todo: move vectors into a Cython vector class - add _add_ and _mul_ methods.

EXAMPLES:

```
sage: a = matrix(Integers(37),3,3,range(9),sparse=True); a
[0 1 2]
[3 4 5]
[6 7 8]
sage: type(a)
<class 'sage.matrix.matrix_modn_sparse.Matrix_modn_sparse'>
sage: parent(a)
Full MatrixSpace of 3 by 3 sparse matrices over Ring of integers modulo 37
sage: a^2
[15 18 21]
[ 5 17 29]
[32 16  0]
sage: a+a
[ 0  2  4]
[ 6  8 10]
[12 14 16]
sage: b = a.new_matrix(2,3,range(6)); b
[0 1 2]
[3 4 5]
sage: a*b
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'Full MatrixSpace of 3 by 3 sparse␣
↪matrices over Ring of integers modulo 37' and 'Full MatrixSpace of 2 by 3 sparse␣
↪matrices over Ring of integers modulo 37'
sage: b*a
[15 18 21]
[ 5 17 29]
```

```
sage: TestSuite(a).run()
sage: TestSuite(b).run()
```

```
sage: a.echelonize(); a
[ 1  0 36]
[ 0  1  2]
[ 0  0  0]
sage: b.echelonize(); b
[ 1  0 36]
[ 0  1  2]
sage: a.pivots()
(0, 1)
sage: b.pivots()
(0, 1)
sage: a.rank()
2
sage: b.rank()
2
sage: a[2,2] = 5
sage: a.rank()
3
```

class sage.matrix.matrix_modn_sparse.**Matrix_modn_sparse**

Bases: *Matrix_sparse*

Create a sparse matrix over the integers modulo n.

INPUT:

- parent – a matrix space over the integers modulo n
- entries – see *matrix()*
- copy – ignored (for backwards compatibility)
- coerce – if False, assume without checking that the entries lie in the base ring

density()

Return the density of self, i.e., the ratio of the number of nonzero entries of self to the total size of self.

EXAMPLES:

```
sage: A = matrix(QQ,3,3,[0,1,2,3,0,0,6,7,8],sparse=True)
sage: A.density()
2/3
```

Notice that the density parameter does not ensure the density of a matrix; it is only an upper bound.

```
sage: A = random_matrix(GF(127), 200, 200, density=0.3, sparse=True)
sage: density_sum = float(A.density())
sage: total = 1
sage: expected_density = 1.0 - (199/200)^60
sage: expected_density
0.2597...
sage: while abs(density_sum/total - expected_density) > 0.001:
....:     A = random_matrix(GF(127), 200, 200, density=0.3, sparse=True)
....:     density_sum += float(A.density())
....:     total += 1
```

determinant(*algorithm=None*)

Return the determinant of this matrix.

INPUT:

- `algorithm` - either `"linbox"` (default) or `"generic"`.

EXAMPLES:

```
sage: A = matrix(GF(3), 4, range(16), sparse=True)
sage: B = identity_matrix(GF(3), 4, sparse=True)
sage: (A + B).det()
2
sage: (A + B).det(algorithm="linbox")
2
sage: (A + B).det(algorithm="generic")
2
sage: (A + B).det(algorithm="hey")
Traceback (most recent call last):
...
ValueError: no algorithm 'hey'

sage: matrix(GF(11), 1, 2, sparse=True).det()
Traceback (most recent call last):
...
ValueError: self must be a square matrix
```

matrix_from_columns(*cols*)

Return the matrix constructed from self using columns with indices in the columns list.

EXAMPLES:

```
sage: M = MatrixSpace(GF(127),3,3,sparse=True)
sage: A = M(range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.matrix_from_columns([2,1])
[2 1]
[5 4]
[8 7]
```

matrix_from_rows(*rows*)

Return the matrix constructed from self using rows with indices in the rows list.

INPUT:

- `rows` - list or tuple of row indices

EXAMPLES:

```
sage: M = MatrixSpace(GF(127),3,3,sparse=True)
sage: A = M(range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: A.matrix_from_rows([2,1])
```

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```
[6 7 8]
[3 4 5]
```

p

rank(*algorithm=None*)

Return the rank of this matrix.

INPUT:

- `algorithm` - either `"linbox"` (only available for matrices over prime fields) or `"generic"`

EXAMPLES:

```
sage: A = matrix(GF(127), 2, 2, sparse=True)
sage: A[0,0] = 34
sage: A[0,1] = 102
sage: A[1,0] = 55
sage: A[1,1] = 74
sage: A.rank()
2

sage: A._clear_cache()
sage: A.rank(algorithm="generic")
2
sage: A._clear_cache()
sage: A.rank(algorithm="hey")
Traceback (most recent call last):
...
ValueError: no algorithm 'hey'
```

REFERENCES:

- Jean-Guillaume Dumas and Gilles Villars. 'Computing the Rank of Large Sparse Matrices over Finite Fields'. Proc. CASC'2002, The Fifth International Workshop on Computer Algebra in Scientific Computing, Big Yalta, Crimea, Ukraine, 22-27 sept. 2002, Springer-Verlag, http://perso.ens-lyon.fr/gilles.villard/BIBLIOGRAPHIE/POSTSCRIPT/rankjgd.ps

---

Note: For very sparse matrices Gaussian elimination is faster because it barely has anything to do. If the fill in needs to be considered, 'Symbolic Reordering' is usually much faster.

---

swap_rows(*r1*, *r2*)

transpose()

Return the transpose of self.

EXAMPLES:

```
sage: A = matrix(GF(127),3,3,[0,1,0,2,0,0,3,0,0],sparse=True)
sage: A
[0 1 0]
[2 0 0]
[3 0 0]
sage: A.transpose()
```

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```
[0 2 3]
[1 0 0]
[0 0 0]
```

`.T` is a convenient shortcut for the transpose:

```
sage: A.T
[0 2 3]
[1 0 0]
[0 0 0]
```

CHAPTER

TWENTYEIGHT

SYMBOLIC MATRICES

EXAMPLES:

```
sage: matrix(SR, 2, 2, range(4))
[0 1]
[2 3]
sage: matrix(SR, 2, 2, var('t'))
[t 0]
[0 t]
```

Arithmetic:

```
sage: -matrix(SR, 2, range(4))
[ 0 -1]
[-2 -3]
sage: m = matrix(SR, 2, [1..4]); sqrt(2)*m
[  sqrt(2) 2*sqrt(2)]
[3*sqrt(2) 4*sqrt(2)]
sage: m = matrix(SR, 4, [1..4^2])
sage: m * m
[ 90 100 110 120]
[202 228 254 280]
[314 356 398 440]
[426 484 542 600]

sage: m = matrix(SR, 3, [1, 2, 3]); m
[1]
[2]
[3]
sage: m.transpose() * m
[14]
```

Computing inverses:

```
sage: M = matrix(SR, 2, var('a,b,c,d'))
sage: ~M
[1/a - b*c/(a^2*(b*c/a - d))           b/(a*(b*c/a - d))]
[          c/(a*(b*c/a - d))              -1/(b*c/a - d)]
sage: (~M*M).simplify_rational()
[1 0]
[0 1]
sage: M = matrix(SR, 3, 3, range(9)) - var('t')
```

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```
sage: (~M * M).simplify_rational()
[1 0 0]
[0 1 0]
[0 0 1]

sage: matrix(SR, 1, 1, 1).inverse()
[1]
sage: matrix(SR, 0, 0).inverse()
[]
sage: matrix(SR, 3, 0).inverse()
Traceback (most recent call last):
...
ArithmeticError: self must be a square matrix
```

Transposition:

```
sage: m = matrix(SR, 2, [sqrt(2), -1, pi, e^2])
sage: m.transpose()
[sqrt(2)      pi]
[     -1     e^2]
```

`.T` is a convenient shortcut for the transpose:

```
sage: m.T
[sqrt(2)      pi]
[     -1     e^2]
```

Test pickling:

```
sage: m = matrix(SR, 2, [sqrt(2), 3, pi, e]); m
[sqrt(2)       3]
[     pi       e]
sage: TestSuite(m).run()
```

Comparison:

```
sage: m = matrix(SR, 2, [sqrt(2), 3, pi, e])
sage: m == m
True
sage: m != 3
True
sage: m = matrix(SR,2,[1..4]); n = m^2
sage: (exp(m+n) - exp(m)*exp(n)).simplify_rational() == 0       # indirect test
True
```

Determinant:

```
sage: M = matrix(SR, 2, 2, [x,2,3,4])
sage: M.determinant()
4*x - 6
sage: M = matrix(SR, 3,3,range(9))
sage: M.det()
0
```

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```
sage: t = var('t')
sage: M = matrix(SR, 2, 2, [cos(t), sin(t), -sin(t), cos(t)])
sage: M.det()
cos(t)^2 + sin(t)^2
sage: M = matrix([[sqrt(x),0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1]])
sage: det(M)
sqrt(x)
```

Permanents:

```
sage: M = matrix(SR, 2, 2, [x,2,3,4])
sage: M.permanent()
4*x + 6
```

Rank:

```
sage: M = matrix(SR, 5, 5, range(25))
sage: M.rank()
2
sage: M = matrix(SR, 5, 5, range(25)) - var('t')
sage: M.rank()
5

.. warning::

    :meth:`rank` may return the wrong answer if it cannot determine that a
    matrix element that is equivalent to zero is indeed so.
```

Copying symbolic matrices:

```
sage: m = matrix(SR, 2, [sqrt(2), 3, pi, e])
sage: n = copy(m)
sage: n[0,0] = sin(1)
sage: m
[sqrt(2)       3]
[     pi       e]
sage: n
[sin(1)      3]
[    pi      e]
```

Conversion to Maxima:

```
sage: m = matrix(SR, 2, [sqrt(2), 3, pi, e])
sage: m._maxima_()
matrix([sqrt(2),3],[%pi,%e])
```

class sage.matrix.matrix_symbolic_dense.**Matrix_symbolic_dense**

Bases: *Matrix_generic_dense*

arguments()

Return a tuple of the arguments that `self` can take.

EXAMPLES:

```
sage: var('x,y,z')
(x, y, z)
sage: M = MatrixSpace(SR,2,2)
sage: M(x).arguments()
(x,)
sage: M(x+sin(x)).arguments()
(x,)
```

canonicalize_radical()

Choose a canonical branch of each entry of `self` by calling `Expression.canonicalize_radical()` componentwise.

EXAMPLES:

```
sage: var('x','y')
(x, y)
sage: l1 = [sqrt(2)*sqrt(3)*sqrt(6) , log(x*y)]
sage: l2 = [sin(x/(x^2 + x)) , 1]
sage: m = matrix([l1, l2])
sage: m
[sqrt(6)*sqrt(3)*sqrt(2)                log(x*y)]
[       sin(x/(x^2 + x))                       1]
sage: m.canonicalize_radical()
[              6 log(x) + log(y)]
[ sin(1/(x + 1))               1]
```

charpoly(*var='x'*, *algorithm=None*)

Compute the characteristic polynomial of `self`, using maxima.

Note: The characteristic polynomial is defined as $\det(xI - A)$.

INPUT:

- `var` – (default: 'x') name of variable of charpoly

EXAMPLES:

```
sage: M = matrix(SR, 2, 2, var('a,b,c,d'))
sage: M.charpoly('t')
t^2 + (-a - d)*t - b*c + a*d
sage: matrix(SR, 5, [1..5^2]).charpoly()
x^5 - 65*x^4 - 250*x^3
```

echelonize(***kwds*)

Echelonize using the classical algorithm.

eigenvalues(*extend=True*)

Compute the eigenvalues by solving the characteristic polynomial in maxima.

The argument `extend` is ignored but kept for compatibility with other matrix classes.

EXAMPLES:

```
sage: a=matrix(SR,[[1,2],[3,4]])
sage: a.eigenvalues()
[-1/2*sqrt(33) + 5/2, 1/2*sqrt(33) + 5/2]
```

eigenvectors_left(*other=None*)

Compute the left eigenvectors of a matrix.

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved (currently supported only if the base ring of `self` is `RDF` or `CDF`)

OUTPUT:

For each distinct eigenvalue, returns a list of the form (e,V,n) where e is the eigenvalue, V is a list of eigenvectors forming a basis for the corresponding left eigenspace, and n is the algebraic multiplicity of the eigenvalue.

EXAMPLES:

```
sage: A = matrix(SR,3,3,range(9)); A
[0 1 2]
[3 4 5]
[6 7 8]
sage: es = A.eigenvectors_left(); es
[(-3*sqrt(6) + 6, [(1, -1/5*sqrt(6) + 4/5, -2/5*sqrt(6) + 3/5)], 1),
 (3*sqrt(6) + 6, [(1, 1/5*sqrt(6) + 4/5, 2/5*sqrt(6) + 3/5)], 1),
 (0, [(1, -2, 1)], 1)]
sage: eval, [evec], mult = es[0]
sage: delta = eval*evec - evec*A
sage: abs(abs(delta)) < 1e-10
3/5*sqrt(((2*sqrt(6) - 3)*(sqrt(6) - 2) + 7*sqrt(6) - 18)^2 + ((sqrt(6) -␣
↪2)*(sqrt(6) - 4) + 6*sqrt(6) - 14)^2) < (1.00000000000000e-10)
sage: abs(abs(delta)).n() < 1e-10
True
```

```
sage: A = matrix(SR, 2, 2, var('a,b,c,d'))
sage: A.eigenvectors_left()
[(1/2*a + 1/2*d - 1/2*sqrt(a^2 + 4*b*c - 2*a*d + d^2), [(1, -1/2*(a - d +␣
↪sqrt(a^2 + 4*b*c - 2*a*d + d^2))/c)], 1), (1/2*a + 1/2*d + 1/2*sqrt(a^2 +␣
↪4*b*c - 2*a*d + d^2), [(1, -1/2*(a - d - sqrt(a^2 + 4*b*c - 2*a*d + d^2))/c)],
↪ 1)]
sage: es = A.eigenvectors_left(); es
[(1/2*a + 1/2*d - 1/2*sqrt(a^2 + 4*b*c - 2*a*d + d^2), [(1, -1/2*(a - d +␣
↪sqrt(a^2 + 4*b*c - 2*a*d + d^2))/c)], 1), (1/2*a + 1/2*d + 1/2*sqrt(a^2 +␣
↪4*b*c - 2*a*d + d^2), [(1, -1/2*(a - d - sqrt(a^2 + 4*b*c - 2*a*d + d^2))/c)],
↪ 1)]
sage: eval, [evec], mult = es[0]
sage: delta = eval*evec - evec*A
sage: delta.apply_map(lambda x: x.full_simplify())
(0, 0)
```

This routine calls Maxima and can struggle with even small matrices with a few variables, such as a 3×3 matrix with three variables. However, if the entries are integers or rationals it can produce exact values in a reasonable time. These examples create 0-1 matrices from the adjacency matrices of graphs and illustrate

how the format and type of the results differ when the base ring changes. First for matrices over the rational numbers, then the same matrix but viewed as a symbolic matrix.

```
sage: G=graphs.CycleGraph(5)
sage: am = G.adjacency_matrix()
sage: spectrum = am.eigenvectors_left()
sage: qqbar_evalue = spectrum[2][0]
sage: type(qqbar_evalue)
<class 'sage.rings.qqbar.AlgebraicNumber'>
sage: qqbar_evalue
0.618033988749895?

sage: am = G.adjacency_matrix().change_ring(SR)
sage: spectrum = am.eigenvectors_left()
sage: symbolic_evalue = spectrum[2][0]
sage: type(symbolic_evalue)
<class 'sage.symbolic.expression.Expression'>
sage: symbolic_evalue
1/2*sqrt(5) - 1/2

sage: bool(qqbar_evalue == symbolic_evalue)
True
```

A slightly larger matrix with a "nice" spectrum.

```
sage: G = graphs.CycleGraph(6)
sage: am = G.adjacency_matrix().change_ring(SR)
sage: am.eigenvectors_left()
[(-1, [(1, 0, -1, 1, 0, -1), (0, 1, -1, 0, 1, -1)], 2), (1, [(1, 0, -1, -1, 0,␣
↪1), (0, 1, 1, 0, -1, -1)], 2), (-2, [(1, -1, 1, -1, 1, -1)], 1), (2, [(1, 1,␣
↪1, 1, 1, 1)], 1)]
```

eigenvectors_right(*other=None*)

Compute the right eigenvectors of a matrix.

INPUT:

- `other` – a square matrix B (default: `None`) in a generalized eigenvalue problem; if `None`, an ordinary eigenvalue problem is solved (currently supported only if the base ring of `self` is `RDF` or `CDF`)

OUTPUT:

For each distinct eigenvalue, returns a list of the form (e,V,n) where e is the eigenvalue, V is a list of eigenvectors forming a basis for the corresponding right eigenspace, and n is the algebraic multiplicity of the eigenvalue.

EXAMPLES:

```
sage: A = matrix(SR,2,2,range(4)); A
[0 1]
[2 3]
sage: right = A.eigenvectors_right(); right
[(-1/2*sqrt(17) + 3/2, [(1, -1/2*sqrt(17) + 3/2)], 1), (1/2*sqrt(17) + 3/2, [(1,
↪ 1/2*sqrt(17) + 3/2)], 1)]
```

The right eigenvectors are nothing but the left eigenvectors of the transpose matrix:

```
sage: left  = A.transpose().eigenvectors_left(); left
[(-1/2*sqrt(17) + 3/2, [(1, -1/2*sqrt(17) + 3/2)], 1), (1/2*sqrt(17) + 3/2, [(1,
↪ 1/2*sqrt(17) + 3/2)], 1)]
sage: right[0][1] == left[0][1]
True
```

exp()

Return the matrix exponential of this matrix X, which is the matrix

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

This function depends on maxima's matrix exponentiation function, which does not deal well with floating point numbers. If the matrix has floating point numbers, they will be rounded automatically to rational numbers during the computation.

EXAMPLES:

```
sage: m = matrix(SR,2, [0,x,x,0]); m
[0 x]
[x 0]
sage: m.exp()
[1/2*(e^(2*x) + 1)*e^(-x) 1/2*(e^(2*x) - 1)*e^(-x)]
[1/2*(e^(2*x) - 1)*e^(-x) 1/2*(e^(2*x) + 1)*e^(-x)]
sage: exp(m)
[1/2*(e^(2*x) + 1)*e^(-x) 1/2*(e^(2*x) - 1)*e^(-x)]
[1/2*(e^(2*x) - 1)*e^(-x) 1/2*(e^(2*x) + 1)*e^(-x)]
```

Exp works on 0x0 and 1x1 matrices:

```
sage: m = matrix(SR,0,[]); m
[]
sage: m.exp()
[]
sage: m = matrix(SR,1,[2]); m
[2]
sage: m.exp()
[e^2]
```

Commuting matrices m, n have the property that $e^{m+n} = e^m e^n$ (but non-commuting matrices need not):

```
sage: m = matrix(SR,2,[1..4]); n = m^2
sage: m*n
[ 37  54]
[ 81 118]
sage: n*m
[ 37  54]
[ 81 118]

sage: a = exp(m+n) - exp(m)*exp(n)
sage: a.simplify_rational() == 0
True
```

The input matrix must be square:

```
sage: m = matrix(SR,2,3,[1..6]); exp(m)
Traceback (most recent call last):
...
ValueError: exp only defined on square matrices
```

In this example we take the symbolic answer and make it numerical at the end:

```
sage: exp(matrix(SR, [[1.2, 5.6], [3,4]])).change_ring(RDF)  # rel tol 1e-15
[ 346.5574872980695  661.7345909344504]
[354.50067371488416  677.4247827652946]
```

Another example involving the reversed identity matrix, which we clumsily create:

```
sage: m = identity_matrix(SR,4); m = matrix(list(reversed(m.rows()))) * x
sage: exp(m)
[1/2*(e^(2*x) + 1)*e^(-x)                         0                         0 1/
↪2*(e^(2*x) - 1)*e^(-x)]
[                         0 1/2*(e^(2*x) + 1)*e^(-x) 1/2*(e^(2*x) - 1)*e^(-x)
↪                   0]
[                         0 1/2*(e^(2*x) - 1)*e^(-x) 1/2*(e^(2*x) + 1)*e^(-x)
↪                   0]
[1/2*(e^(2*x) - 1)*e^(-x)                         0                         0 1/
↪2*(e^(2*x) + 1)*e^(-x)]
```

expand()

Operate point-wise on each element.

EXAMPLES:

```
sage: M = matrix(2, 2, range(4)) - var('x')
sage: M*M
[      x^2 + 2      -2*x + 3]
[     -4*x + 6 (x - 3)^2 + 2]
sage: (M*M).expand()
[       x^2 + 2       -2*x + 3]
[      -4*x + 6 x^2 - 6*x + 11]
```

factor()

Operate point-wise on each element.

EXAMPLES:

```
sage: M = matrix(SR, 2, 2, x^2 - 2*x + 1); M
[x^2 - 2*x + 1             0]
[            0 x^2 - 2*x + 1]
sage: M.factor()
[(x - 1)^2         0]
[        0 (x - 1)^2]
```

fcp(*var='x'*)

Return the factorization of the characteristic polynomial of `self`.

INPUT:

- `var` – (default: 'x') name of variable of charpoly

EXAMPLES:

```
sage: a = matrix(SR,[[1,2],[3,4]])
sage: a.fcp()
x^2 - 5*x - 2
sage: [i for i in a.fcp()]
[(x^2 - 5*x - 2, 1)]
sage: a = matrix(SR,[[1,0],[0,2]])
sage: a.fcp()
(x - 2) * (x - 1)
sage: [i for i in a.fcp()]
[(x - 2, 1), (x - 1, 1)]
sage: a = matrix(SR, 5, [1..5^2])
sage: a.fcp()
(x^2 - 65*x - 250) * x^3
sage: list(a.fcp())
[(x^2 - 65*x - 250, 1), (x, 3)]
```

function(**args*)

Return a matrix over a callable symbolic expression ring.

EXAMPLES:

```
sage: x, y = var('x,y')
sage: v = matrix([[x,y],[x*sin(y), 0]])
sage: w = v.function([x,y]); w
[       (x, y) |--> x         (x, y) |--> y]
[(x, y) |--> x*sin(y)         (x, y) |--> 0]
sage: w.parent()
Full MatrixSpace of 2 by 2 dense matrices over Callable function ring with␣
↪arguments (x, y)
```

jordan_form(*subdivide=True*, *transformation=False*)

Return a Jordan normal form of `self`.

INPUT:

- `self` – a square matrix
- `subdivide` – boolean (default: `True`)
- `transformation` – boolean (default: `False`)

OUTPUT:

If `transformation` is `False`, only a Jordan normal form (unique up to the ordering of the Jordan blocks) is returned. Otherwise, a pair `(J, P)` is returned, where `J` is a Jordan normal form and `P` is an invertible matrix such that `self` equals `P * J * P^(-1)`.

If `subdivide` is `True`, the Jordan blocks in the returned matrix `J` are indicated by a subdivision in the sense of `subdivide()`.

EXAMPLES:

We start with some examples of diagonalisable matrices:

```
sage: a,b,c,d = var('a,b,c,d')
sage: matrix([a]).jordan_form()
```

(continues on next page)

(continued from previous page)

```
[a]
sage: matrix([[a, 0], [1, d]]).jordan_form(subdivide=True)
[d|0]
[-+-]
[0|a]
sage: matrix([[a, 0], [1, d]]).jordan_form(subdivide=False)
[d 0]
[0 a]
sage: matrix([[a, x, x], [0, b, x], [0, 0, c]]).jordan_form()
[c|0|0]
[-+-+-]
[0|b|0]
[-+-+-]
[0|0|a]
```

In the following examples, we compute Jordan forms of some non-diagonalisable matrices:

```
sage: matrix([[a, a], [0, a]]).jordan_form()
[a 1]
[0 a]
sage: matrix([[a, 0, b], [0, c, 0], [0, 0, a]]).jordan_form()
[c|0 0]
[-+---]
[0|a 1]
[0|0 a]
```

The following examples illustrate the `transformation` flag. Note that symbolic expressions may need to be simplified to make consistency checks succeed:

```
sage: A = matrix([[x - a*c, a^2], [-c^2, x + a*c]])
sage: J, P = A.jordan_form(transformation=True)
sage: J, P
(
[x 1]  [-a*c    1]
[0 x], [-c^2    0]
)
sage: A1 = P * J * ~P; A1
[             -a*c + x (a*c - x)*a/c + a*x/c]
[                -c^2               a*c + x]
sage: A1.simplify_rational() == A
True

sage: B = matrix([[a, b, c], [0, a, d], [0, 0, a]])
sage: J, T = B.jordan_form(transformation=True)
sage: J, T
(
[a 1 0]  [b*d   c   0]
[0 a 1]  [  0   d   0]
[0 0 a], [  0   0   1]
)
sage: (B * T).simplify_rational() == T * J
True
```

Finally, some examples involving square roots:

```
sage: matrix([[a, -b], [b, a]]).jordan_form()
[a - I*b|      0]
[-------+-------]
[      0|a + I*b]
sage: matrix([[a, b], [c, d]]).jordan_form(subdivide=False)
[1/2*a + 1/2*d - 1/2*sqrt(a^2 + 4*b*c - 2*a*d + d^2)                                 ␣
↪                       0]
[                                                 0 1/2*a + 1/2*d + 1/2*sqrt(a^
↪2 + 4*b*c - 2*a*d + d^2)]
```

minpoly(*var='x'*)

Return the minimal polynomial of `self`.

EXAMPLES:

```
sage: M = Matrix.identity(SR, 2)
sage: M.minpoly()
x - 1

sage: t = var('t')
sage: m = matrix(2, [1, 2, 4, t])
sage: m.minimal_polynomial()
x^2 + (-t - 1)*x + t - 8
```

number_of_arguments()

Return the number of arguments that `self` can take.

EXAMPLES:

```
sage: var('a,b,c,x,y')
(a, b, c, x, y)
sage: m = matrix([[a, (x+y)/(x+y)], [x^2, y^2+2]]); m
[      a       1]
[    x^2 y^2 + 2]
sage: m.number_of_arguments()
3
```

simplify()

Simplify `self`.

EXAMPLES:

```
sage: var('x,y,z')
(x, y, z)
sage: m = matrix([[z, (x+y)/(x+y)], [x^2, y^2+2]]); m
[      z       1]
[    x^2 y^2 + 2]
sage: m.simplify()
[      z       1]
[    x^2 y^2 + 2]
```

simplify_full()

Simplify a symbolic matrix by calling `Expression.simplify_full()` componentwise.

INPUT:

- `self` – the matrix whose entries we should simplify.

OUTPUT:

A copy of `self` with all of its entries simplified.

EXAMPLES:

Symbolic matrices will have their entries simplified:

```
sage: a,n,k = SR.var('a,n,k')
sage: f1 = sin(x)^2 + cos(x)^2
sage: f2 = sin(x/(x^2 + x))
sage: f3 = binomial(n,k)*factorial(k)*factorial(n-k)
sage: f4 = x*sin(2)/(x^a)
sage: A = matrix(SR, [[f1,f2],[f3,f4]])
sage: A.simplify_full()
[                1    sin(1/(x + 1))]
[     factorial(n) x^(-a + 1)*sin(2)]
```

simplify_rational()

EXAMPLES:

```
sage: M = matrix(SR, 3, 3, range(9)) - var('t')
sage: (~M*M)[0,0]
t*(3*(2/t + (6/t + 7)/((t - 3/t - 4)*t))*(2/t + (6/t + 5)/((t - 3/t
- 4)*t))/(t - (6/t + 7)*(6/t + 5)/(t - 3/t - 4) - 12/t - 8) + 1/t +
3/((t - 3/t - 4)*t^2)) - 6*(2/t + (6/t + 5)/((t - 3/t - 4)*t))/(t -
(6/t + 7)*(6/t + 5)/(t - 3/t - 4) - 12/t - 8) - 3*(6/t + 7)*(2/t +
(6/t + 5)/((t - 3/t - 4)*t))/((t - (6/t + 7)*(6/t + 5)/(t - 3/t -
4) - 12/t - 8)*(t - 3/t - 4)) - 3/((t - 3/t - 4)*t)
sage: expand((~M*M)[0,0])
1
sage: (~M * M).simplify_rational()
[1 0 0]
[0 1 0]
[0 0 1]
```

simplify_trig()

EXAMPLES:

```
sage: theta = var('theta')
sage: M = matrix(SR, 2, 2, [cos(theta), sin(theta), -sin(theta), cos(theta)])
sage: ~M
[1/cos(theta) - sin(theta)^2/((sin(theta)^2/cos(theta) + cos(theta))*cos(theta)^
↪2)                   -sin(theta)/((sin(theta)^2/cos(theta) +␣
↪cos(theta))*cos(theta))]
[                   sin(theta)/((sin(theta)^2/cos(theta) +␣
↪cos(theta))*cos(theta))                                                  1/
↪(sin(theta)^2/cos(theta) + cos(theta))]
sage: (~M).simplify_trig()
[ cos(theta) -sin(theta)]
[ sin(theta)  cos(theta)]
```

variables()

Return the variables of `self`.

EXAMPLES:

```
sage: var('a,b,c,x,y')
(a, b, c, x, y)
sage: m = matrix([[x, x+2], [x^2, x^2+2]]); m
[      x   x + 2]
[    x^2 x^2 + 2]
sage: m.variables()
(x,)
sage: m = matrix([[a, b+c], [x^2, y^2+2]]); m
[      a   b + c]
[    x^2 y^2 + 2]
sage: m.variables()
(a, b, c, x, y)
```

CHAPTER

TWENTYNINE

DENSE MATRICES OVER THE COMPLEX DOUBLE FIELD USING NUMPY

EXAMPLES:

```
sage: b = Mat(CDF,2,3).basis()
sage: b[0,0]
[1.0 0.0 0.0]
[0.0 0.0 0.0]
```

We deal with the case of zero rows or zero columns:

```
sage: m = MatrixSpace(CDF,0,3)
sage: m.zero_matrix()
[]
```

AUTHORS:

- Jason Grout (2008-09): switch to NumPy backend
- Josh Kantor
- William Stein: many bug fixes and touch ups.

class sage.matrix.matrix_complex_double_dense.**Matrix_complex_double_dense**

Bases: *Matrix_double_dense*

Class that implements matrices over the real double field. These are supposed to be fast matrix operations using C doubles. Most operations are implemented using numpy which will call the underlying BLAS on the system.

EXAMPLES:

```
sage: m = Matrix(CDF, [[1,2*I],[3+I,4]])
sage: m**2
[-1.0 + 6.0*I       10.0*I]
[15.0 + 5.0*I 14.0 + 6.0*I]
sage: n= m^(-1); n  # abs tol 1e-15
[  0.333333333333333 + 0.333333333333333*I 0.16666666666666669 - 0.
↪16666666666666666*I]
[-0.16666666666666666 - 0.333333333333333*I 0.08333333333333331 + 0.
↪08333333333333333*I]
```

To compute eigenvalues, use the methods *left_eigenvectors()* or *right_eigenvectors()*:

```
sage: p,e = m.right_eigenvectors()
```

The result is a pair `(p,e)`, where `p` is a diagonal matrix of eigenvalues and `e` is a matrix whose columns are the eigenvectors.

To solve a linear system $Ax = b$ where `A = [[1,2*I],[3+I,4]]` and `b = [5,6]`:

```
sage: b = vector(CDF,[5,6])
sage: m.solve_right(b)  # abs tol 1e-14
(2.6666666666666665 + 0.6666666666666669*I, -0.3333333333333333 - 1.
↪1666666666666667*I)
```

See the methods *QR()*, *LU()*, and *SVD()* for QR, LU, and singular value decomposition.

CHAPTER

THIRTY

ARBITRARY PRECISION COMPLEX BALL MATRICES USING ARB

AUTHORS:

- Clemens Heuberger (2014-10-25): Initial version.

This is a rudimentary binding to the Arb library; it may be useful to refer to its documentation for more details.

class sage.matrix.matrix_complex_ball_dense.**Matrix_complex_ball_dense**

Bases: *Matrix_dense*

Matrix over a complex ball field. Implemented using the `acb_mat` type of the Arb library.

EXAMPLES:

```
sage: MatrixSpace(CBF, 3)(2)
[2.000000000000000                  0                  0]
[                 0 2.000000000000000                  0]
[                 0                  0 2.000000000000000]
sage: matrix(CBF, 1, 3, [1, 2, -3])
[ 1.000000000000000  2.000000000000000 -3.000000000000000]
```

charpoly(*var='x'*, *algorithm=None*)

Compute the characteristic polynomial of this matrix.

EXAMPLES:

```
sage: from sage.matrix.benchmark import hilbert_matrix
sage: mat = hilbert_matrix(5).change_ring(ComplexBallField(10))
sage: mat.charpoly()
x^5 + ([-1.8 +/- 0.0258])*x^4 + ([0.3 +/- 0.05...)*x^3 +
([+/- 0.0...])*x^2 + ([+/- 0.0...])*x + [+/- 0.0...]
```

contains(*other*)

Test if the set of complex matrices represented by `self` is contained in that represented by `other`.

EXAMPLES:

```
sage: b = CBF(0, RBF(0, rad=.1r)); b
[+/- 0.101]*I
sage: matrix(CBF, [0, b]).contains(matrix(CBF, [0, 0]))
True
sage: matrix(CBF, [0, b]).contains(matrix(CBF, [b, 0]))
False
sage: matrix(CBF, [b, b]).contains(matrix(CBF, [b, 0]))
True
```

determinant()

Compute the determinant of this matrix.

EXAMPLES:

```
sage: matrix(CBF, [[1/2, 1/3], [1, 1]]).determinant()
[0.1666666666666667 +/- ...e-17]
sage: matrix(CBF, [[1/2, 1/3], [1, 1]]).det()
[0.1666666666666667 +/- ...e-17]
sage: matrix(CBF, [[1/2, 1/3]]).determinant()
Traceback (most recent call last):
...
ValueError: self must be a square matrix
```

eigenvalues(*other=None*, *extend=None*)

(Experimental.) Compute rigorous enclosures of the eigenvalues of this matrix.

INPUT:

- `self` – an $n \times n$ matrix
- `other` – unsupported (generalized eigenvalue problem), should be `None`
- `extend` – ignored

OUTPUT:

A `Sequence` of complex balls of length equal to the size of the matrix.

Each element represents one eigenvalue with the correct multiplicities in case of overlap. The output intervals are either disjoint or identical, and identical intervals are guaranteed to be grouped consecutively. Each complete run of k identical balls thus represents a cluster of exactly k eigenvalues which could not be separated from each other at the current precision, but which could be isolated from the other eigenvalues.

There is currently no guarantee that the algorithm converges as the working precision is increased.

See the Arb documentation for more information.

EXAMPLES:

```
sage: from sage.matrix.benchmark import hilbert_matrix
sage: mat = hilbert_matrix(5).change_ring(CBF)
sage: mat.eigenvalues()
doctest:...: FutureWarning: This class/method/function is marked as␣
↪experimental.
...
[[1.567050691098...] + [+/- ...]*I, [0.208534218611...] + [+/- ...]*I,
[3.287928...e-6...] + [+/- ...]*I, [0.000305898040...] + [+/- ...]*I,
[0.011407491623...] + [+/- ...]*I]

sage: mat = Permutation([2, 1, 4, 5, 3]).to_matrix().dense_matrix().change_
↪ring(CBF)
sage: mat.eigenvalues()
Traceback (most recent call last):
...
ValueError: unable to certify the eigenvalues
sage: precond = matrix(ZZ, [[-1, -2, 2, 2, -2], [2, -2, -2, -2, 2],
....:     [-2, 2, -1, 2, 1], [2, 1, -1, 0, 2], [-2, 0, 1, -1, 1]])
sage: (~precond*mat*precond).eigenvalues()
```

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```
[[-0.5000000000000...] + [-0.8660254037844...]*I, [-1.000000000000...] + [+/- ..
↪.]*I,
 [-0.5000000000000...] + [0.8660254037844...]*I,
 [1.000000000000...] + [+/- ...]*I, [1.000000000000...] + [+/- ...]*I]
```

See also:

eigenvectors_right()

eigenvectors_left(*other=None*, *extend=True*)

(Experimental.) Compute rigorous enclosures of the eigenvalues and left eigenvectors of this matrix.

INPUT:

- `self` – an $n \times n$ matrix
- `other` – unsupported (generalized eigenvalue problem), should be `None`
- `extend` – ignored

OUTPUT:

A list of triples of the form `(eigenvalue, [eigenvector], 1)`.

Unlike *eigenvalues()* and *eigenvectors_left_approx()*, this method currently fails in the presence of multiple eigenvalues.

Additionally, there is currently no guarantee that the algorithm converges as the working precision is increased.

See the Arb documentation for more information.

EXAMPLES:

```
sage: mat = matrix(CBF, 3, [2, 3, 5, 7, 11, 13, 17, 19, 23])
sage: eigval, eigvec, _ = mat.eigenvectors_left()[0]
sage: eigval
[1.1052996349...] + [+/- ...]*I
sage: eigvec[0]
([0.69817246751...] + [+/- ...]*I, [-0.67419514369...] + [+/- ...]*I, [0.
↪240865343781...] + [+/- ...]*I)
sage: eigvec[0] * (mat - eigval)
([+/- ...] + [+/- ...]*I, [+/- ...] + [+/- ...]*I, [+/- ...] + [+/- ...]*I)
```

See also:

eigenvectors_right(), *eigenvalues()*, *eigenvectors_left_approx()*

eigenvectors_left_approx(*other=None*, *extend=None*)

(Experimental.) Compute *non-rigorous* approximations of the left eigenvalues and eigenvectors of this matrix.

INPUT:

- `self` – an $n \times n$ matrix
- `other` – unsupported (generalized eigenvalue problem), should be `None`
- `extend` – ignored

OUTPUT:

A list of triples of the form `(eigenvalue, [eigenvector], 1)`. The eigenvalue and the entries of the eigenvector are complex balls with zero radius.

No guarantees are made about the accuracy of the output.

See the Arb documentation for more information.

EXAMPLES:

```
sage: mat = matrix(CBF, 3, [2, 3, 5, 7, 11, 13, 17, 19, 23])
sage: eigval, eigvec, _ = mat.eigenvectors_left_approx()[0]
sage: eigval
[1.1052996349... +/- ...]
sage: eigvec[0]
([0.69817246751...], [-0.67419514369...], [0.240865343781...])
sage: eigvec[0] * (mat - eigval)
([+/- ...], [+/- ...], [+/- ...])
```

See also:

eigenvectors_left()

eigenvectors_right(*other=None*, *extend=None*)

(Experimental.) Compute rigorous enclosures of the eigenvalues and eigenvectors of this matrix.

INPUT:

- `self` – an $n \times n$ matrix
- `other` – unsupported (generalized eigenvalue problem), should be `None`
- `extend` – ignored

OUTPUT:

A list of triples of the form `(eigenvalue, [eigenvector], 1)`.

Unlike *eigenvalues()* and *eigenvectors_right_approx()*, this method currently fails in the presence of multiple eigenvalues.

Additionally, there is currently no guarantee that the algorithm converges as the working precision is increased.

See the Arb documentation for more information.

EXAMPLES:

```
sage: from sage.matrix.benchmark import hilbert_matrix
sage: mat = hilbert_matrix(3).change_ring(CBF)
sage: eigval, eigvec, _ = mat.eigenvectors_right()[0]
doctest:...: FutureWarning: This class/method/function is marked as␣
↪experimental.
...
sage: eigval
[1.40831892712...] + [+/- ...]*I
sage: eigvec
[([0.82704492697...] + [+/- ...]*I, [0.45986390436...] + [+/- ...]*I, [0.
↪32329843524...] + [+/- ...]*I)]
sage: (mat - eigval)*eigvec[0]
([+/- ...] + [+/- ...]*I, [+/- ...] + [+/- ...]*I, [+/- ...] + [+/- ...]*I)
```

See also:

eigenvectors_right_approx(), *eigenvalues()*

eigenvectors_right_approx(*other=None*, *extend=None*)

(Experimental.) Compute *non-rigorous* approximations of the eigenvalues and eigenvectors of this matrix.

INPUT:

- `self` – an $n \times n$ matrix
- `other` – unsupported (generalized eigenvalue problem), should be `None`
- `extend` – ignored

OUTPUT:

A list of triples of the form `(eigenvalue, [eigenvector], 1)`. The eigenvalue and the entries of the eigenvector are complex balls with zero radius.

No guarantees are made about the accuracy of the output.

See the Arb documentation for more information.

EXAMPLES:

```
sage: from sage.matrix.benchmark import hilbert_matrix
sage: mat = hilbert_matrix(3).change_ring(CBF)
sage: eigval, eigvec, _ = mat.eigenvectors_right_approx()[0]
doctest:...: FutureWarning: This class/method/function is marked as␣
↪experimental.
...
sage: eigval
[1.40831892712...]
sage: eigval.rad()
0.00000000
sage: eigvec
[([0.8270449269720...], [0.4598639043655...], [0.3232984352444...])]
sage: (mat - eigval)*eigvec[0]
([1e-15 +/- ...], [2e-15 +/- ...], [+/- ...])
```

See also:

eigenvectors_right()

exp()

Compute the exponential of this matrix.

EXAMPLES:

```
sage: matrix(CBF, [[i*pi, 1], [0, i*pi]]).exp()
[[-1.00000000000000 +/- ...e-16] + [+/- ...e-16]*I [-1.00000000000000 +/- ...e-
↪16] + [+/- ...e-16]*I]
[                                                0 [-1.00000000000000 +/- ...e-
↪16] + [+/- ...e-16]*I]
sage: matrix(CBF, [[1/2, 1/3]]).exp()
Traceback (most recent call last):
...
ValueError: self must be a square matrix
```

identical(*other*)

Test if the corresponding entries of two complex ball matrices represent the same balls.

EXAMPLES:

```
sage: a = matrix(CBF, [[1/3,2],[3,4]])
sage: b = matrix(CBF, [[1/3,2],[3,4]])
sage: a == b
False
sage: a.identical(b)
True
```

overlaps(*other*)

Test if two matrices with complex ball entries represent overlapping sets of complex matrices.

EXAMPLES:

```
sage: b = CBF(0, RBF(0, rad=0.1r)); b
[+/- 0.101]*I
sage: matrix(CBF, [0, b]).overlaps(matrix(CBF, [b, 0]))
True
sage: matrix(CBF, [1, 0]).overlaps(matrix(CBF, [b, 0]))
False
```

trace()

Compute the trace of this matrix.

EXAMPLES:

```
sage: matrix(CBF, [[1/3, 1/3], [1, 1]]).trace()
[1.333333333333333 +/- ...e-16]
sage: matrix(CBF, [[1/2, 1/3]]).trace()
Traceback (most recent call last):
...
ValueError: self must be a square matrix
```

transpose()

Return the transpose of `self`.

EXAMPLES:

```
sage: m = matrix(CBF, 2, 3, [1, 2, 3, 4, 5, 6])
sage: m.transpose()
[1.000000000000000 4.000000000000000]
[2.000000000000000 5.000000000000000]
[3.000000000000000 6.000000000000000]
sage: m.transpose().parent()
Full MatrixSpace of 3 by 2 dense matrices over Complex ball field with 53 bits␣
↪of precision
```

CHAPTER

THIRTYONE

DENSE MATRICES OVER UNIVARIATE POLYNOMIALS OVER FIELDS

The implementation inherits from Matrix_generic_dense but some algorithms are optimized for polynomial matrices.

AUTHORS:

- Kwankyu Lee (2016-12-15): initial version with code moved from other files.
- Johan Rosenkilde (2017-02-07): added weak_popov_form()
- Vincent Neiger (2018-06-13): added basic functions (row/column degrees, leading positions, leading matrix, testing reduced and canonical forms)
- Vincent Neiger (2018-09-29): added functions for computing and for verifying minimal approximant bases
- Vincent Neiger (2020-04-01): added functions for computing and for verifying minimal kernel bases
- Vincent Neiger (2021-03-11): added matrix-wise basic functions for univariate polynomials (shifts, reverse, truncate, get coefficient of specified degree)
- Vincent Neiger (2021-07-29): added popov_form(). Added more options to weak_popov_form() (column-wise, ordered, zero rows).
- Vincent Neiger (2021-08-07): added inverse_series_trunc(), solve_{left/right}_series_trunc(), {left/right}_quo_rem(), reduce().

class sage.matrix.matrix_polynomial_dense.**Matrix_polynomial_dense**

Bases: *Matrix_generic_dense*

Dense matrix over a univariate polynomial ring over a field.

For a field $\mathbf{K}$, we consider matrices over the univariate polynomial ring $\mathbf{K}[x]$.

They are often used to represent bases of some $\mathbf{K}[x]$-modules. In this context, there are two possible representations which are both commonly used in the literature.

- Working column-wise: each column of the matrix is a vector in the basis; then, a $\mathbf{K}[x]$-submodule of $\mathbf{K}[x]^m$ of rank n is represented by an $m \times n$ matrix, whose columns span the module (via $\mathbf{K}[x]$-linear combinations). This matrix has full rank, and $n \leq m$.
- Working row-wise: each row of the matrix is a vector in the basis; then, a $\mathbf{K}[x]$-submodule of $\mathbf{K}[x]^n$ of rank m is represented by an $m \times n$ matrix, whose rows span the module (via $\mathbf{K}[x]$-linear combinations). This matrix has full rank, and $m \leq n$.

For the rest of this class description, we assume that one is working row-wise. For a given such module, all its bases are equivalent under left-multiplication by a unimodular matrix, that is, a square matrix which has determinant in $\mathbf{K} \setminus \{0\}$.

There are bases which are called reduced or minimal: their rows have the minimal degree possible among all bases of this module; here the degree of a row is the maximum of the degrees of the entries of the row. An equivalent condition is that the leading matrix of this basis has full rank (see *leading_matrix()*, *reduced_form()*,

is_reduced()). There is a unique minimal basis, called the Popov basis of the module, which satisfies some additional normalization condition (see *popov_form()*, *is_popov()*).

These notions can be extended via a more general degree measure, involving a tuple of integers which is called shift and acts as column degree shifts in the definition of row degree. Precisely, for given $s_1, \ldots, s_n \in \mathbf{Z}$ and a row vector $[p_1 \ \cdots \ p_n] \in \mathbf{K}[x]^{1\times n}$, its shifted row degree is the maximum of $\deg(p_j) + s_j$ for $1 \leq j \leq n$ (see *row_degrees()*). Then, reduced bases and Popov bases are defined similarly, with respect to this notion of degree.

Another important canonical basis is the Hermite basis, which is an upper triangular matrix satisfying a normalization condition similar to that for the Popov basis. In fact, if d is the largest degree appearing in the Hermite basis, then the Hermite basis coincide with the shifted Popov basis with the shifts $((n-1)d, \ldots, 2d, d, 0)$.

coefficient_matrix(*d*, *row_wise=True*)

Return the constant matrix which is obtained from this matrix by taking the coefficient of its entries with degree specified by d.

- if d is an integer, this selects the coefficient of d for all entries;
- if d is a list $(d_1, \ldots, d_m)$ and `row_wise` is `True`, this selects the coefficient of degree d_i for all entries of the $i`throwforeach`i$;
- if d is a list $(d_1, \ldots, d_n)$ and `row_wise` is `False`, this selects the coefficient of degree d_i for all entries of the $j`thcolumnforeach`j$.

INPUT:

- `d` – a list of integers, or an integer,
- `row_wise` – (optional, default: `True`) boolean, if `True` (resp. `False`) then d should be a list of length equal to the row (resp. column) dimension of this matrix.

OUTPUT: a matrix over the base field.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: M = Matrix([
....:     [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:     [       6*x^2+3*x+1,        1,            2,          0],
....:     [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:      ])
sage: M.coefficient_matrix(2)
[5 0 0 0]
[6 0 0 0]
[4 0 2 1]
sage: M.coefficient_matrix(0) == M.constant_matrix()
True
```

Row-wise and column-wise coefficient extraction are available:

```
sage: M.coefficient_matrix([3,2,1])
[1 0 0 0]
[6 0 0 0]
[6 5 5 5]

sage: M.coefficient_matrix([2,0,1,3], row_wise=False)
[5 5 6 0]
```

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```
[6 1 0 0]
[4 1 5 0]
```

Negative degrees give zero coefficients:

```
sage: M.coefficient_matrix([-1,0,1,3], row_wise=False)
[0 5 6 0]
[0 1 0 0]
[0 1 5 0]
```

Length of list of degrees is checked:

```
sage: M.coefficient_matrix([2,1,1,2])
Traceback (most recent call last):
...
ValueError: length of input degree list should be the row
dimension of the input matrix

sage: M.coefficient_matrix([3,2,1], row_wise=False)
Traceback (most recent call last):
...
ValueError: length of input degree list should be the column
dimension of the input matrix
```

column_degrees(*shifts=None*)

Return the (shifted) column degrees of this matrix.

For a given polynomial matrix $M = (M_{i,j})_{i,j}$ with m rows and n columns, its column degrees is the tuple $(d_1, \ldots, d_n)$ where $d_j = \max_i(\deg(M_{i,j}))$ for $1 \leq j \leq n$. Thus, $d_j = -1$ if the j-th column of M is zero, and $d_j \geq 0$ otherwise.

For given shifts $s_1, \ldots, s_m \in \mathbf{Z}$, the shifted column degrees of M is $(d_1, \ldots, d_n)$ where $d_j = \max_i(\deg(M_{i,j}) + s_i)$. Here, if the j-th column of M is zero then $d_j = \min(s_1, \ldots, s_m) - 1$; otherwise d_j is larger than this value.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.

OUTPUT: a list of integers.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [3*x+1, 0, 1], [x^3+3, 0, 0] ])
sage: M.column_degrees()
[3, -1, 0]

sage: M.column_degrees(shifts=[0,2])
[5, -1, 0]
```

A zero column in a polynomial matrix can be identified in the (shifted) column degrees as the entries equal to `min(shifts)-1`:

```
sage: M.column_degrees(shifts=[-2,1])
[4, -3, -2]
```

The column degrees of an empty matrix ($0 \times n$ or $m \times 0$) is not defined:

```
sage: M = Matrix( pR, 0, 3 )
sage: M.column_degrees()
Traceback (most recent call last):
...
ValueError: empty matrix does not have column degrees

sage: M = Matrix( pR, 3, 0 )
sage: M.column_degrees()
Traceback (most recent call last):
...
ValueError: empty matrix does not have column degrees
```

See also:

The documentation of *row_degrees()*.

constant_matrix()

Return the constant coefficient of this matrix seen as a polynomial with matrix coefficients; this is also this matrix evaluated at zero.

OUTPUT: a matrix over the base field.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: M = Matrix([
....:     [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:     [       6*x^2+3*x+1,        1,            2,          0],
....:     [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:      ])
sage: M.constant_matrix()
[1 5 4 0]
[1 1 2 0]
[4 1 5 6]
```

degree()

Return the degree of this matrix.

For a given polynomial matrix, its degree is the maximum of the degrees of all its entries. If the matrix is nonzero, this is a nonnegative integer; here, the degree of the zero matrix is -1.

OUTPUT: an integer.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix( pR, [[3*x+1, 0, 1], [x^3+3, 0, 0]])
sage: M.degree()
3
```

The zero matrix has degree -1:

```
sage: M = Matrix( pR, 2, 3 )
sage: M.degree()
-1
```

For an empty matrix, the degree is not defined:

```
sage: M = Matrix( pR, 3, 0 )
sage: M.degree()
Traceback (most recent call last):
...
ValueError: empty matrix does not have a degree
```

degree_matrix(*shifts=None*, *row_wise=True*)

Return the matrix of the (shifted) degrees in this matrix.

For a given polynomial matrix $M = (M_{i,j})_{i,j}$, its degree matrix is the matrix $(\deg(M_{i,j}))_{i,j}$ formed by the degrees of its entries. Here, the degree of the zero polynomial is -1.

For given shifts $s_1, \ldots, s_m \in \mathbf{Z}$, the shifted degree matrix of M is either $(\deg(M_{i,j}) + s_j)_{i,j}$ if working row-wise, or $(\deg(M_{i,j}) + s_i)_{i,j}$ if working column-wise. In the former case, m has to be the number of columns of M; in the latter case, the number of its rows. Here, if $M_{i,j} = 0$ then the corresponding entry in the shifted degree matrix is $\min(s_1, \ldots, s_m) - 1$. For more on shifts and working row-wise versus column-wise, see the class documentation.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, if `True` then shifts apply to the columns of the matrix and otherwise to its rows (see the class description for more details).

OUTPUT: an integer matrix.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix( pR, [[3*x+1, 0, 1], [x^3+3, 0, 0]])
sage: M.degree_matrix()
[ 1 -1  0]
[ 3 -1 -1]

sage: M.degree_matrix(shifts=[0,1,2])
[ 1 -1  2]
[ 3 -1 -1]
```

The zero entries in the polynomial matrix can be identified in the (shifted) degree matrix as the entries equal to `min(shifts)-1`:

```
sage: M.degree_matrix(shifts=[-2,1,2])
[-1 -3  2]
[ 1 -3 -3]
```

Using `row_wise=False`, the function supports shifts applied to the rows of the matrix (which, in terms of modules, means that we are working column-wise, see the class documentation):

```
sage: M.degree_matrix(shifts=[-1,2], row_wise=False)
[ 0 -2 -1]
[ 5 -2 -2]
```

hermite_form(*include_zero_rows=True*, *transformation=False*)

Return the Hermite form of this matrix.

See *is_hermite()* for a definition of Hermite forms. If the input is a matrix A, then its Hermite form is the unique matrix H in Hermite form such that $UA = H$ for some unimodular matrix U.

INPUT:

- `include_zero_rows` – boolean (default: `True`); if `False`, the zero rows in the output matrix are deleted.
- `transformation` – boolean (default: `False`); if `True`, return the transformation matrix.

OUTPUT:

- the Hermite normal form H of this matrix A .
- (optional) transformation matrix U such that $UA = H$.

EXAMPLES:

```
sage: M.<x> = GF(7)[]
sage: A = matrix(M, 2, 3, [x, 1, 2*x, x, 1+x, 2])
sage: A.hermite_form()
[      x       1     2*x]
[      0       x 5*x + 2]
sage: A.hermite_form(transformation=True)
(
[      x       1     2*x]  [1 0]
[      0       x 5*x + 2], [6 1]
)
sage: A = matrix(M, 2, 3, [x, 1, 2*x, 2*x, 2, 4*x])
sage: A.hermite_form(transformation=True, include_zero_rows=False)
([  x   1 2*x], [0 4])
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=True); H, U
(
[  x   1 2*x]  [0 4]
[  0   0   0], [5 1]
)
sage: U * A == H
True
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=False)
sage: U * A
[  x   1 2*x]
sage: U * A == H
True
```

See also:

is_hermite() , *popov_form()* .

inverse_series_trunc(*d*)

Return a matrix polynomial approximation of precision `d` of the inverse series of this matrix polynomial.

Here matrix polynomial means that `self` is seen as a univariate polynomial with matrix coefficients, meaning that this method has the same output as if one: 1) converts this matrix to a univariate polynomial with matrix coefficients, 2) calls

`sage.rings.polynomial.polynomial_element.Polynomial.inverse_series_trunc()`

on that univariate polynomial, and 3) converts back to a matrix of polynomials.

Raises a `ZeroDivisionError` if the constant matrix of `self` is not invertible (i.e. has zero determinant); raises an `ArithmeticError` if `self` is nonsquare; and raises a `ValueError` if the precision `d` is not positive.

INPUT: a positive integer d .

OUTPUT: the unique polynomial matrix B of degree less than d such that AB and BA are the identity matrix modulo x^d, where A is `self`.

ALGORITHM: This uses Newton iteration, performing about $\log(d)$ polynomial matrix multiplications in size $m \times m$ and in degree less than $2d$, where m is the row dimension of `self`.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: A = Matrix(pR, 3, 3,                                \
    [[4*x+5,           5*x^2 + x + 1, 4*x^2 + 4],      \
     [6*x^2 + 6*x + 6, 4*x^2 + 5*x,   4*x^2 + x + 3], \
     [3*x^2 + 2,       4*x + 1,       x^2 + 3*x]])
sage: B = A.inverse_series_trunc(4); B
[    x^3 + 5*x^2 + x + 4   x^3 + 5*x^2 + 6*x + 4         6*x^2 + 5*x + 3]
[        4*x^2 + 5*x + 6     6*x^3 + x^2 + x + 6       3*x^3 + 2*x^2 + 2]
[5*x^3 + 5*x^2 + 6*x + 6 4*x^3 + 2*x^2 + 6*x + 4   6*x^3 + x^2 + 6*x + 1]
sage: (B*A).truncate(4) == 1
True

sage: A.inverse_series_trunc(0)
Traceback (most recent call last):
...
ValueError: the precision must be positive

sage: A[:2,:].inverse_series_trunc(4)
Traceback (most recent call last):
...
ArithmeticError: the input matrix must be square

sage: A[0,:] = A[0,:] - A[0,:](0) + A[1,:](0) + A[2,:](0)
sage: A.inverse_series_trunc(4)
Traceback (most recent call last):
...
ZeroDivisionError: the constant matrix term self(0) must be invertible
```

See also:

`sage.rings.polynomial.polynomial_element.Polynomial.inverse_series_trunc()` .

Todo: in the current state of polynomial matrix multiplication (July 2021), it would be highly beneficial to use conversions and rely on polynomials with matrix coefficients when the matrix size is "large" and the degree "small", see github issue #31472#comment:5.

`is_constant()`

Return `True` if and only if this polynomial matrix is constant, that is, all its entries are constant.

OUTPUT: a boolean.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: M = Matrix([
....:    [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:    [       6*x^2+3*x+1,        1,            2,          0],
....:    [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:     ])
sage: M.is_constant()
False
sage: M = Matrix(pR,[[1,5,2],[3,1,5]]); M.is_constant()
True
sage: M = Matrix.zero(pR,3,5); M.is_constant()
True
```

See also:

sage.rings.polynomial.polynomial_element.Polynomial.is_constant()

is_hermite(*row_wise=True*, *lower_echelon=False*, *include_zero_vectors=True*)

Return a boolean indicating whether this matrix is in Hermite form.

If working row-wise, a polynomial matrix is said to be in Hermite form if it is in row echelon form with all pivot entries monic and such that all entries above a pivot have degree less than this pivot. Being in row echelon form means that all zero rows are gathered at the bottom of the matrix, and in each nonzero row the pivot (leftmost nonzero entry) is strictly to the right of the pivot of the row just above this row.

Note that, for any integer d strictly greater than all degrees appearing in the Hermite form, then the Hermite form coincides with the shifted Popov form with the shifts $((n-1)d, \ldots, 2d, d, 0)$, where n is the column dimension.

If working column-wise, a polynomial matrix is said to be in Hermite form if it is in column echelon form with all pivot entries monic and such that all entries to the left of a pivot have degree less than this pivot. Being in column echelon form means that all zero columns are gathered at the right-hand side of the matrix, and in each nonzero column the pivot (topmost nonzero entry) is strictly below the pivot of the column just to the left of this row.

Optional arguments provide support of alternative definitions, concerning the choice of upper or lower echelon forms and concerning whether zero rows (resp. columns) are allowed.

INPUT:

- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `lower_echelon` – (optional, default: `False`) boolean, `False` if working with upper triangular Hermite forms, `True` if working with lower triangular Hermite forms.
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if one does not allow zero rows (resp. zero columns) in Hermite forms.

OUTPUT: a boolean.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [x^4+6*x^3+4*x+4, 3*x+6,      3  ],
....:                  [0,               x^2+5*x+5, 2  ],
....:                  [0,               0,         x+5] ])
```

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```
sage: M.is_hermite()
True
sage: M.is_hermite(row_wise=False)
True
sage: M.is_hermite(row_wise=False, lower_echelon=True)
False

sage: N = Matrix(pR, [ [x+5, 0,               0        ],
....:                  [2,   x^4+6*x^3+4*x+4, 0        ],
....:                  [3,   3*x^3+6,         x^2+5*x+5] ])
sage: N.is_hermite()
False
sage: N.is_hermite(lower_echelon=True)
True
sage: N.is_hermite(row_wise=False)
False
sage: N.is_hermite(row_wise=False, lower_echelon=True)
False
```

Rectangular matrices with zero rows are supported. Zero rows (resp. columns) can be forbidden, and otherwise they should be at the bottom (resp. the right-hand side) of the matrix:

```
sage: N[:,1:].is_hermite(lower_echelon=True)
False
sage: N[[1,2,0],1:].is_hermite(lower_echelon=True)
True
sage: N[:2,:].is_hermite(row_wise=False, lower_echelon=True)
True
sage: N[:2,:].is_hermite(row_wise=False,
....:                    lower_echelon=True,
....:                    include_zero_vectors=False)
False
```

See also:

hermite_form() .

is_minimal_approximant_basis(*pmat*, *order*, *shifts=None*, *row_wise=True*, *normal_form=False*)

Return `True` if and only if this matrix is an approximant basis in `shifts`-ordered weak Popov form for the polynomial matrix `pmat` at order `order`.

If `normal_form` is `True`, then the polynomial matrix must furthermore be in `shifts`-Popov form. An error is raised if the input dimensions are not sound. If a single integer is provided for `order`, then it is interpreted as a list of repeated integers with this value. (See *minimal_approximant_basis()* for definitions and more details.)

INPUT:

- `pmat` – a polynomial matrix.
- `order` – a list of positive integers, or a positive integer.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, if `True` then the basis considered row-wise and operates on the left of `pmat`; otherwise it is column-wise and operates on the right of `pmat`.

- `normal_form` – (optional, default: `False`) boolean, if `True` then checks for a basis in `shifts`-Popov form.

OUTPUT: a boolean.

ALGORITHM:

Verification that the matrix is formed by approximants is done via a truncated matrix product; verification that the matrix is square, nonsingular and in shifted weak Popov form is done via *is_weak_popov()*; verification that the matrix generates the module of approximants is done via the characterization in Theorem 2.1 of [GN2018] .

EXAMPLES:

```
sage: pR.<x> = GF(97)[]
```

We consider the following example from [Arne Storjohann, Notes on computing minimal approximant bases, 2006]:

```
sage: order = 8; shifts = [1,1,0,0,0]
sage: pmat = Matrix(pR, 5, 1, [ \
        pR([35,  0, 41, 87,  3, 42, 22, 90]), \
        pR([80, 15, 62, 87, 14, 93, 24,  0]), \
        pR([42, 57, 90, 87, 22, 80, 71, 53]), \
        pR([37, 72, 74,  6,  5, 75, 23, 47]), \
        pR([36, 10, 74,  1, 29, 44, 87, 74]) ])
sage: appbas = Matrix(pR, [ \
       [x+47,   57, 58*x+44,     9*x+23,      93*x+76], \
       [  15, x+18, 52*x+23,     15*x+58,     93*x+88], \
       [  17,   86, x^2+77*x+16, 76*x+29,     90*x+78], \
       [  44,   36, 3*x+42,      x^2+50*x+26, 85*x+44], \
       [   2,   22, 54*x+94,     73*x+24,     x^2+2*x+25] ])
sage: appbas.is_minimal_approximant_basis(pmat,\
        order, shifts, row_wise=True, normal_form=True)
True
```

The matrix $x^8\mathrm{Id}_5$ is square, nonsingular, in Popov form, and its rows are approximants for `pmat` at order 8. However, it is not an approximant basis since its rows generate a module strictly contained in the set of approximants for `pmat` at order 8:

```
sage: (x^8*Matrix.identity(pR, 5)).is_minimal_approximant_basis(\
                                                        pmat, 8)
False
```

Since `pmat` is a single column, with nonzero constant coefficient, its column-wise approximant bases at order 8 are all 1×1 matrices $[cx^8]$ for some nonzero field element c:

```
sage: Matrix(pR, [x^8]).is_minimal_approximant_basis(pmat, \
        8, row_wise=False, normal_form=True)
True
```

Exceptions are raised if input dimensions are not sound:

```
sage: appbas.is_minimal_approximant_basis(pmat, [8,8], shifts)
Traceback (most recent call last):
...
```

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```
ValueError: order length should be the column dimension
            of the input matrix

sage: appbas.is_minimal_approximant_basis(pmat, \
        order, shifts, row_wise=False)
Traceback (most recent call last):
...
ValueError: shifts length should be the column dimension
            of the input matrix

sage: Matrix(pR, [x^8]).is_minimal_approximant_basis(pmat, 8)
Traceback (most recent call last):
...
ValueError: column dimension should be the row dimension of the
input matrix
```

See also:

minimal_approximant_basis() .

is_minimal_kernel_basis(*pmat*, *shifts=None*, *row_wise=True*, *normal_form=False*)

Return `True` if and only if this matrix is a left kernel basis in `shifts`-ordered weak Popov form for the polynomial matrix `pmat`.

If `normal_form` is `True`, then the kernel basis must furthermore be in `shifts`-Popov form. An error is raised if the input dimensions are not sound.

INPUT:

- `pmat` – a polynomial matrix.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, if `True` then the basis is considered row-wise and operates on the left of `pmat`; otherwise it is column-wise and operates on the right of `pmat`.
- `normal_form` – (optional, default: `False`) boolean, if `True` then checks for a basis in `shifts`-Popov form.

OUTPUT: a boolean.

ALGORITHM:

Verification that the matrix has full rank and is in shifted weak Popov form is done via *is_weak_popov()*; verification that the matrix is a left kernel basis is done by checking that the rank is correct, that the product is indeed zero, and that the matrix is saturated, i.e. it has unimodular column bases (see Lemma 6.10 of https://arxiv.org/pdf/1807.01272.pdf for details).

EXAMPLES:

```
sage: pR.<x> = GF(97)[]
sage: pmat = Matrix(pR, [[1],[x],[x**2]])

sage: kerbas = Matrix(pR, [[x,-1,0],[0,x,-1]])
sage: kerbas.is_minimal_kernel_basis(pmat)
True
```

A matrix in Popov form which has the right rank, all rows in the kernel, but does not generate the kernel:

```
sage: kerbas = Matrix(pR, [[x**2,0,-1],[0,x,-1]])
sage: kerbas.is_minimal_kernel_basis(pmat)
False
```

Shifts and right kernel bases are supported (with `row_wise`), and one can test whether the kernel basis is normalized in shifted-Popov form (with `normal_form`):

```
sage: kerbas = Matrix(pR, [[-x,-x**2],[1,0],[0,1]])
sage: kerbas.is_minimal_kernel_basis(pmat.transpose(),row_wise=False,normal_
↪form=True,shifts=[0,1,2])
True
```

is_popov(*shifts=None*, *row_wise=True*, *up_to_permutation=False*, *include_zero_vectors=True*)

Return a boolean indicating whether this matrix is in (shifted) Popov form.

If working row-wise (resp. column-wise), a polynomial matrix is said to be in Popov form if it has no zero row above a nonzero row (resp. no zero column to the left of a nonzero column), the leading positions of its nonzero rows (resp. columns) are strictly increasing, and for each row (resp. column) the pivot entry is monic and has degree strictly larger than the other entries in its column (resp. row).

Since other conventions concerning the ordering of the rows (resp. columns) are sometimes useful, an optional argument allows one to test whether the matrix is in Popov form up to row (resp. column) permutation. For example, there is an alternative definition which replaces "leading positions strictly increasing" by "row (resp. column) degree nondecreasing, and for rows (resp. columns) of same degree, leading positions increasing".

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `up_to_permutation` – (option, default: `False`) boolean, `True` if testing Popov form up to row permutation (if working row-wise).
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if one does not allow zero rows (resp. zero columns) in Popov forms.

OUTPUT: a boolean.

REFERENCES:

For the square case, without shifts: [Pop1972] and [Kai1980] (Section 6.7.2). For the general case: [BLV2006] .

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [x^4+6*x^3+4*x+4, 3*x+6,     3  ],
....:                  [x^2+6*x+6,       x^2+5*x+5, 2  ],
....:                  [3*x,             6*x+5,     x+5] ])
sage: M.is_popov()
True

sage: M.is_popov(shifts=[0,1,2])
True

sage: M[:,:2].is_popov()
```

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```
False

sage: M[:2,:].is_popov(shifts=[0,1,2])
True

sage: M = Matrix(pR, [ [x^4+3*x^3+x^2+2*x+6, x^3+5*x^2+5*x+1],
....:                  [6*x+1,               x^2+4*x+1      ],
....:                  [6,                   6              ] ])
sage: M.is_popov(row_wise=False)
False

sage: M.is_popov(shifts=[0,2,3], row_wise=False)
True
```

One can forbid zero rows (or columns if not working row-wise):

```
sage: N = Matrix(pR, [ [x^4+3*x^3+x^2+2*x+6, 6*x+1     ],
....:                  [5*x^2+5*x+1,         x^2+4*x+1 ],
....:                  [0,                   0         ] ])

sage: N.is_popov()
True

sage: N.is_popov(include_zero_vectors=False)
False
```

One can verify Popov form up to row permutation (or column permutation if not working row-wise):

```
sage: M.swap_columns(0,1)
sage: M.is_popov(shifts=[0,2,3], row_wise=False)
False

sage: M.is_popov(shifts=[0,2,3], row_wise=False,
....:   up_to_permutation=True)
True

sage: N.swap_rows(0,2)

sage: N.is_popov()
False

sage: N.is_popov(up_to_permutation=True)
True
```

is_reduced(*shifts=None*, *row_wise=True*, *include_zero_vectors=True*)

Return a boolean indicating whether this matrix is in (shifted) reduced form.

An $m \times n$ univariate polynomial matrix M is said to be in shifted row reduced form if it has k nonzero rows with $k \leq n$ and its shifted leading matrix has rank k. Equivalently, when considering all the matrices obtained by left-multiplying M by a unimodular matrix, then the shifted row degrees of M – once sorted in nondecreasing order – is lexicographically minimal.

Similarly, M is said to be in shifted column reduced form if it has k nonzero columns with $k \leq m$ and its shifted leading matrix has rank k.

Sometimes, one forbids M to have zero rows (resp. columns) in the above definitions; an optional parameter allows one to adopt this more restrictive setting.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if one does not allow zero rows in row reduced forms (resp. zero columns in column reduced forms).

OUTPUT: a boolean value.

REFERENCES:

[Wol1974] (Section 2.5, without shifts) and [VBB1992] (Section 3).

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [3*x+1, 0, 1], [x^3+3, 0, 0] ])
sage: M.is_reduced()
False

sage: M.is_reduced(shifts=[0,1,2])
True

sage: M.is_reduced(shifts=[2,0], row_wise=False)
True

sage: M.is_reduced(shifts=[2,0], row_wise=False,
....:                            include_zero_vectors=False)
False

sage: M = Matrix(pR, [ [3*x+1, 0, 1], [x^3+3, 0, 0], [0, 1, 0] ])
sage: M.is_reduced(shifts=[2,0,0], row_wise=False)
True
```

See also:

leading_matrix() , *reduced_form()* .

is_weak_popov(*shifts=None*, *row_wise=True*, *ordered=False*, *include_zero_vectors=True*)

Return a boolean indicating whether this matrix is in (shifted) (ordered) weak Popov form.

If working row-wise (resp. column-wise), a polynomial matrix is said to be in weak Popov form if the leading positions of its nonzero rows (resp. columns) are pairwise distinct. For the ordered weak Popov form, these positions must be strictly increasing, except for the possibly repeated -1 entries which are at the end. For the shifted variants, see the class description for an introduction to shifts.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `ordered` – (optional, default: `False`) boolean, `True` if checking for an ordered weak Popov form.
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if one does not allow zero rows (resp. zero columns) in (ordered) weak Popov forms.

OUTPUT: a boolean.

REFERENCES:

[Kai1980] (Section 6.7.2, square case without shifts), [MS2003] (without shifts), [BLV1999] .

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix([ [x^3+3*x^2+6*x+6, 3*x^2+3*x+6, 4*x^2+x+3],
....:              [5,               1,           0        ],
....:              [2*x^2+2,         2*x+5,       x^2+4*x+6] ])
sage: M.is_weak_popov()
True
```

One can check whether the leading positions, in addition to being pairwise distinct, are actually in increasing order:

```
sage: M.is_weak_popov(ordered=True)
True

sage: N = M.with_swapped_rows(1,2)
sage: N.is_weak_popov()
True
sage: N.is_weak_popov(ordered=True)
False
```

Shifts and orientation (row-wise or column-wise) are supported:

```
sage: M.is_weak_popov(shifts=[2,3,1])
False

sage: M.is_weak_popov(shifts=[0,2,0],row_wise=False,ordered=True)
True
```

Rectangular matrices are supported:

```
sage: M = Matrix([
....:    [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:    [      6*x^2+3*x+1,        1,            2,          0],
....:    [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:     ])
sage: M.is_weak_popov(shifts=[0,2,1,3])
True

sage: M.is_weak_popov(shifts=[0,2,1,3],ordered=True)
True
```

Zero rows (resp. columns) can be forbidden:

```
sage: M = Matrix([
....:   [      6*x+4,        0,              5*x+1, 0],
....:   [          2, 5*x + 1,        6*x^2+3*x+1, 0],
....:   [2*x^2+5*x+5,        1, 2*x^3+4*x^2+6*x+4, 0]
....:   ])
sage: M.is_weak_popov(shifts=[2,1,0], row_wise=False, ordered=True)
```

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```
True

sage: M.is_weak_popov(shifts=[2,1,0], row_wise=False,
....:     include_zero_vectors=False)
False
```

See also:

weak_popov_form() .

leading_matrix(*shifts=None*, *row_wise=True*)

Return the (shifted) leading matrix of this matrix.

Let M be a univariate polynomial matrix in $\mathbf{K}[x]^{m \times n}$. Working row-wise and without shifts, its leading matrix is the matrix in $\mathbf{K}^{m \times n}$ formed by the leading coefficients of the entries of M which reach the degree of the corresponding row.

More precisely, if working row-wise, let $s_1, \ldots, s_n \in \mathbf{Z}$ be a shift, and let $(d_1, \ldots, d_m)$ denote the shifted row degrees of M. Then, the shifted leading matrix of M is the matrix in $\mathbf{K}^{m \times n}$ whose entry i, j is the coefficient of degree $d_i - s_j$ of the entry i, j of M.

If working column-wise, let $s_1, \ldots, s_m \in \mathbf{Z}$ be a shift, and let $(d_1, \ldots, d_n)$ denote the shifted column degrees of M. Then, the shifted leading matrix of M is the matrix in $\mathbf{K}^{m \times n}$ whose entry i, j is the coefficient of degree $d_j - s_i$ of the entry i, j of M.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).

OUTPUT: a matrix over the base field.

REFERENCES:

[Wol1974] (Section 2.5, without shifts) and [VBB1992] (Section 3).

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [3*x+1, 0, 1], [x^3+3, 0, 0] ])
sage: M.leading_matrix()
[3 0 0]
[1 0 0]

sage: M.leading_matrix().base_ring()
Finite Field of size 7

sage: M.leading_matrix(shifts=[0,1,2])
[0 0 1]
[1 0 0]

sage: M.leading_matrix(row_wise=False)
[0 0 1]
[1 0 0]

sage: M.leading_matrix(shifts=[-2,1], row_wise=False)
[0 0 1]
```

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```
[1 0 0]

sage: M.leading_matrix(shifts=[2,0], row_wise=False)
[3 0 1]
[1 0 0]
```

leading_positions(*shifts=None*, *row_wise=True*, *return_degree=False*)

Return the (shifted) leading positions (also known as the pivot indices), and optionally the (shifted) pivot degrees of this matrix.

If working row-wise, for a given shift $s_1, \ldots, s_n \in \mathbf{Z}$, taken as $(0, \ldots, 0)$ by default, and a row vector of univariate polynomials $[p_1, \ldots, p_n]$, the leading position of this vector is the index j of the rightmost nonzero entry p_j such that $\deg(p_j) + s_j$ is equal to the shifted row degree of the vector. Then the pivot degree of the vector is the degree $\deg(p_j)$.

For the zero row, both the leading positions and degree are -1. For a $m \times n$ polynomial matrix, the leading positions and pivot degrees are the two lists containing the leading positions and the pivot degrees of its rows.

The definition is similar if working column-wise (instead of rightmost nonzero entry, we choose the bottommost nonzero entry).

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `return_degree` – (optional, default: `False`) boolean, `True` implies that the pivot degrees are returned.

OUTPUT: a list of integers if `return_degree=False`; a pair of lists of integers otherwise.

REFERENCES:

[Kai1980] (Section 6.7.2, without shifts).

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [3*x+1, 0, 1], [x^3+3, 0, 0] ])
sage: M.leading_positions()
[0, 0]

sage: M.leading_positions(return_degree=True)
([0, 0], [1, 3])

sage: M.leading_positions(shifts=[0,5,2], return_degree=True)
([2, 0], [0, 3])

sage: M.leading_positions(row_wise=False, return_degree=True)
([1, -1, 0], [3, -1, 0])

sage: M.leading_positions(shifts=[1,2], row_wise=False,
....:   return_degree=True)
([1, -1, 0], [3, -1, 0])
```

In case several entries in the row (resp. column) reach the shifted row (resp. column) degree, the leading position is chosen as the rightmost (resp. bottommost) such entry:

```
sage: M.leading_positions(shifts=[0,5,1],return_degree=True)
([2, 0], [0, 3])

sage: M.leading_positions(shifts=[2,0], row_wise=False,return_degree=True)
([1, -1, 0], [3, -1, 0])
```

The leading positions and pivot degrees of an empty matrix ($0 \times n$ or $m \times 0$) is not defined:

```
sage: M = Matrix( pR, 0, 3 )
sage: M.leading_positions()
Traceback (most recent call last):
...
ValueError: empty matrix does not have leading positions

sage: M.leading_positions(row_wise=False)
Traceback (most recent call last):
...
ValueError: empty matrix does not have leading positions

sage: M = Matrix( pR, 3, 0 )
sage: M.leading_positions(row_wise=False)
Traceback (most recent call last):
...
ValueError: empty matrix does not have leading positions
```

left_quo_rem(B)

Return, if it exists, the quotient and remainder (Q, R) such that `self` is $BQ + R$, where R has row degrees less than those of B entrywise.

This method directly calls *right_quo_rem()* on transposed matrices, and transposes the result. See *right_quo_rem()* for a complete documentation and more examples.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: A = Matrix(pR, 3, 2,                                \
            [[      3*x^3 + 3*x,          2*x^3 + 4], \
             [  3*x^3 + 6*x + 5, 6*x^3 + 5*x^2 + 1], \
             [  2*x^3 + 2*x + 6,   3*x^2 + 2*x + 2]])
sage: B = Matrix(pR, 3, 3,                                        \
            [[              3,       x + 3,               6], \
             [3*x^3 + 3*x + 1, 4*x^2 + 3*x,   6*x^3 + x + 4], \
             [  4*x^2 + x + 4, 3*x^2 + 4*x, 3*x^2 + 3*x + 2]])
sage: Q,R = A.left_quo_rem(B); (Q,R)
(
[2*x^2 + 4*x + 6 6*x^2 + 4*x + 1]  [              3               1]
[    3*x^2 + 5*x   2*x^2 + x + 5]  [              6 5*x^2 + 2*x + 3]
[    6*x^2 + 3*x 4*x^2 + 6*x + 1], [        2*x + 3         6*x + 3]
)
sage: rdegR = R.row_degrees(); rdegB = B.row_degrees()
sage: A == B*Q+R and all([rdegR[i] < rdegB[i] for i in range(3)])
True
```

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```
sage: A[:2,:].left_quo_rem(B)
Traceback (most recent call last):
...
ValueError: row dimension of self should be the row dimension of
the input matrix
```

Rectangular or rank-deficient matrices are supported but there may be no quotient and remainder (unless the matrix has full row rank, see *right_quo_rem()*):

```
sage: Q,R = A[:2,:].left_quo_rem(B[:2,:]); (Q,R)
(
[      3*x + 3       2*x + 1]
[  3*x^2 + 5*x 2*x^2 + x + 5]  [            5             0]
[            0             0], [4*x^2 + x + 2     4*x^2 + x]
)
sage: rdegR = R.row_degrees(); rdegB = B[:2,:].row_degrees()
sage: A[:2,:] == B[:2,:]*Q+R
True
sage: all([rdegR[i] < rdegB[i] for i in range(len(rdegR))])
True

sage: A.left_quo_rem(B[:,:2])
Traceback (most recent call last):
...
ValueError: division of these matrices does not admit a remainder
with the required degree property
```

See also:

right_quo_rem() , *reduce()* .

minimal_approximant_basis(*order*, *shifts=None*, *row_wise=True*, *normal_form=False*)

Return an approximant basis in `shifts`-ordered weak Popov form for this polynomial matrix at order `order`.

Assuming we work row-wise, if F is an $m \times n$ polynomial matrix and $(d_0, \ldots, d_{n-1})$ are positive integers, then an approximant basis for F at order $(d_0, \ldots, d_{n-1})$ is a polynomial matrix whose rows form a basis of the module of approximants for F at order $(d_0, \ldots, d_{n-1})$. The latter approximants are the polynomial vectors p of size m such that the column j of pF has valuation at least d_j, for all $0 \leq j \leq n-1$.

If `normal_form` is `True`, then the output basis P is furthermore in `shifts`-Popov form. By default, P is considered row-wise, that is, its rows are left-approximants for `self`; if `row_wise` is `False` then its columns are right-approximants for `self`. It is guaranteed that the degree of the output basis is at most the maximum of the entries of `order`, independently of `shifts`.

An error is raised if the input dimensions are not sound: if working row-wise (resp. column-wise), the length of `order` must be the number of columns (resp. rows) of `self`, while the length of `shifts` must be the number of rows (resp. columns) of `self`.

If a single integer is provided for `order`, then it is converted into a list of repeated integers with this value.

INPUT:

- `order` – a list of positive integers, or a positive integer.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.

- `row_wise` – (optional, default: `True`) boolean, if `True` then the output basis is considered row-wise and operates on the left of `self`; otherwise it is column-wise and operates on the right of `self`.
- `normal_form` – (optional, default: `False`) boolean, if `True` then the output basis is in `shifts`-Popov form.

OUTPUT: a polynomial matrix.

ALGORITHM:

The implementation is inspired from the iterative algorithms described in [VBB1992] and [BL1994] ; for obtaining the normal form, it relies directly on Lemmas 3.3 and 4.1 in [JNSV2016] .

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: order = [4, 3]; shifts = [-1, 2, 0]
sage: F = Matrix(pR, [[5*x^3 + 4*x^2 + 4*x + 6, 5*x^2 + 4*x + 1], \
                      [        2*x^2 + 2*x + 3, 6*x^2 + 6*x + 3], \
                      [4*x^3         +   x + 1, 4*x^2 + 2*x + 3] ])
sage: P = F.minimal_approximant_basis(order, shifts)
sage: P.is_minimal_approximant_basis(F, order, shifts)
True
```

By default, the computed basis is not required to be in normal form (and will not be except in rare special cases):

```
sage: P.is_minimal_approximant_basis(F, order, shifts, \
                                     normal_form=True)
False
sage: P = F.minimal_approximant_basis(order, shifts, normal_form=True)
sage: P.is_minimal_approximant_basis(F, order, shifts, \
                                     normal_form=True)
True
```

If shifts are not specified, they are chosen as uniform $[0, \ldots, 0]$ by default. Besides, if the orders are all the same, one can rather give a single integer:

```
sage: F.minimal_approximant_basis(3) == \
       F.minimal_approximant_basis([3,3], shifts=None)
True
```

One can work column-wise by specifying `row_wise=False`:

```
sage: P = F.minimal_approximant_basis([5,2,2], [0,1], row_wise=False)
sage: P.is_minimal_approximant_basis(F, [5,2,2], \
                   shifts=[0,1], row_wise=False)
True
sage: F.minimal_approximant_basis(3, row_wise=True) == \
    F.transpose().minimal_approximant_basis(3, row_wise=False).transpose()
True
```

Errors are raised if the input dimensions are not sound:

```
sage: P = F.minimal_approximant_basis([4], shifts)
Traceback (most recent call last):
```

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```
...
ValueError: order length should be the column dimension

sage: P = F.minimal_approximant_basis(order, [0,0,0,0])
Traceback (most recent call last):
...
ValueError: shifts length should be the row dimension
```

An error is raised if order does not contain only positive integers:

```
sage: P = F.minimal_approximant_basis([1,0], shifts)
Traceback (most recent call last):
...
ValueError: order should consist of positive integers
```

minimal_kernel_basis(*shifts=None*, *row_wise=True*, *normal_form=False*)

Return a left kernel basis in `shifts`-ordered weak Popov form for this polynomial matrix.

Assuming we work row-wise, if F is an $m \times n$ polynomial matrix, then a left kernel basis for F is a polynomial matrix whose rows form a basis of the left kernel of F, which is the module of polynomial vectors p of size m such that pF is zero.

If `normal_form` is `True`, then the output basis P is furthermore in `shifts`-Popov form. By default, P is considered row-wise, that is, its rows are left kernel vectors for `self`; if `row_wise` is `False` then its columns are right kernel vectors for `self`.

An error is raised if the input dimensions are not sound: if working row-wise (resp. column-wise), the length of `shifts` must be the number of rows (resp. columns) of `self`.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, if `True` then the output basis considered row-wise and operates on the left of `self`; otherwise it is column-wise and operates on the right of `self`.
- `normal_form` – (optional, default: `False`) boolean, if `True` then the output basis is in `shifts`-Popov form.

OUTPUT: a polynomial matrix.

ALGORITHM: uses minimal approximant basis computation at a sufficiently large order so that the approximant basis contains a kernel basis as a submatrix.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: pmat = Matrix([[(x+1)*(x+3)],[(x+1)*(x+3)+1]])
sage: pmat.minimal_kernel_basis()
[6*x^2 + 3*x + 3   x^2 + 4*x + 3]

sage: pmat = Matrix([[(x+1)*(x+3)],[(x+1)*(x+4)]])
sage: pmat.minimal_kernel_basis()
[6*x + 3   x + 3]

sage: pmat.minimal_kernel_basis(row_wise=False)
[]
```

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```
sage: pmat = Matrix(pR, [[1,x,x**2]])
sage: pmat.minimal_kernel_basis(row_wise=False,normal_form=True)
[x 0]
[6 x]
[0 6]

sage: pmat.minimal_kernel_basis(row_wise=False,normal_form=True,shifts=[0,1,2])
[  6*x 6*x^2]
[    1     0]
[    0     1]
```

Some particular cases (matrix is zero, dimension is zero, column is zero):

```
sage: Matrix(pR, 2, 1).minimal_kernel_basis()
[1 0]
[0 1]

sage: Matrix(pR, 2, 0).minimal_kernel_basis()
[1 0]
[0 1]

sage: Matrix(pR, 0, 2).minimal_kernel_basis()
[]

sage: Matrix(pR, 3, 2, [[1,0],[1,0],[1,0]]).minimal_kernel_basis()
[6 1 0]
[6 0 1]

sage: Matrix(pR, 3, 2, [[x,0],[1,0],[x+1,0]]).minimal_kernel_basis()
[6 x 0]
[6 6 1]
```

popov_form(*transformation=False*, *shifts=None*, *row_wise=True*, *include_zero_vectors=True*)

Return the (shifted) Popov form of this matrix.

See *is_popov()* for a definition of Popov forms. If the input matrix is A, the (shifted) Popov form of A is the unique matrix P in (shifted) Popov form and such that $UA = P$ for some unimodular matrix U. The latter matrix is called the transformation, and the first optional argument allows one to specify whether to return this transformation. We refer to the description of *weak_popov_form()* for an explanation of the option `include_zero_vectors` .

INPUT:

- `transformation` – (optional, default: `False`). If this is `True`, the transformation matrix U will be returned as well.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if zero rows (resp. zero columns) should be discarded from the Popov forms.

OUTPUT:

- A polynomial matrix which is the Popov form of `self` if `transformation` is `False`; otherwise two polynomial matrices which are the Popov form of `self` and the corresponding unimodular transformation.

ALGORITHM:

This method implements the Mulders-Storjohann algorithm of [MS2003] for transforming a weak Popov form into Popov form, straightforwardly extended to the case of shifted forms.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [                                  \
    [      6*x+4,       5*x^3+5*x,       6*x^2+2*x+2], \
    [4*x^2+5*x+2, x^4+5*x^2+2*x+4, 4*x^3+6*x^2+6*x+5]])

sage: P,U = M.popov_form(transformation=True)
sage: P
[            4 x^2 + 4*x + 1             3]
[            0       4*x + 1 x^2 + 6*x + 1]
sage: U
[            x             2]
[5*x^2 + x + 6       3*x + 2]
sage: P.is_popov() and U.is_invertible() and U*M==P
True
```

Demonstrating shifts and specific case of Hermite form:

```
sage: P = M.popov_form(shifts=[0,2,4]); P
[              4*x^2 + 3*x + 4 x^4 + 3*x^3 + 5*x^2 + 5*x + 5                ␣
↪          0]
[                            6               5*x^2 + 6*x + 5                ␣
↪          1]
sage: P.is_popov(shifts=[0,2,4])
True
sage: P==M.popov_form(shifts=[-6,-4,-2])
True
sage: dd=sum(M.row_degrees())+1
sage: M.popov_form(shifts=[2*dd,dd,0]) == M.hermite_form()
True
```

Column-wise form is the row-wise form of the transpose:

```
sage: M.popov_form() == M.T.popov_form(row_wise=False).T
True
```

Zero vectors can be discarded:

```
sage: M.popov_form(row_wise=False)
[x + 2     6     0]
[    0     1     0]

sage: P,U = M.popov_form(transformation=True,      \
                         row_wise=False,          \
                         include_zero_vectors=False)
```

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```
sage: P
[x + 2     6]
[    0     1]
sage: U
[        3*x^2 + 6*x + 3         5*x^2 + 4*x + 4 3*x^3 + 3*x^2 + 2*x + 4]
[                      3                       1                 2*x + 1]
[                5*x + 2                       2                       6]
sage: M*U[:,:2] == P and (M*U[:,2]).is_zero()
True
```

See also:

is_popov() , *reduced_form()* , *weak_popov_form()* , *hermite_form()* .

reduce(*B*, *shifts=None*, *row_wise=True*, *return_quotient=False*)

Reduce `self`, i.e. compute its normal form, modulo the row space of B with respect to `shifts`.

If `self` is a $k \times n$ polynomial matrix (written A below), and the input B is an $m \times n$ polynomial matrix, this computes the normal form R of A with respect the row space of B and the monomial order defined by `shifts` (written s below). This means that the i th row of R is equal to the i th row of A up to addition of an element in the row space of B, and if $J = (j_1, \ldots, j_r)$ are the s-leading positions of the s-Popov form P of A, then the submatrix $R_{*,J}$ (submatrix of R formed by its columns in J) has column degrees smaller entrywise than the column degrees of $P_{*,J}$.

If the option `row_wise` is set to `False`, the same operation is performed, but with everything considered column-wise: column space of B, i th column of R and A, column-wise s-leading positions and s-Popov form, and submatrices $R_{J,*}$ and $P_{J,*}$.

The operation above can be seen as a matrix generalization of division with remainder for univariate polynomials. If the option `return_quotient` is set to `True`, this method returns both the normal form R and a quotient matrix Q such that $A = QB + R$ (or $A = BQ + R$ if `row_wise` is `False`). Whereas the remainder is unique, this quotient is not unique unless B has a trivial left kernel i.e. has full row rank (or right kernel, full column rank if `row_wise` is `False`).

This method checks whether B is in s-Popov form, and if not, computes the s-Popov form P of B, which can take some time. Therefore, if P is already known or is to be re-used, this method should be called directly with P, yielding the same normal form R since P and B have the same row space (or column space, if `row_wise` is `False`).

A `ValueError` is raised if the dimensions of the shifts and/or of the matrices are not conformal.

INPUT:

- `B` – polynomial matrix.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `return_quotient` – (optional, default: `False`). If this is `True`, the quotient will be returned as well.

OUTPUT: a polynomial matrix if `return_quotient=False`, two polynomial matrices otherwise.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: B = Matrix(pR, [                                  \
    [      6*x+4,       5*x^3+5*x,       6*x^2+2*x+2], \
    [4*x^2+5*x+2, x^4+5*x^2+2*x+4, 4*x^3+6*x^2+6*x+5]])
```

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```
sage: A = Matrix(pR, 1, 3, [ \
    [3*x^4+3*x^3+4*x^2+5*x+1, x^4+x^3+5*x^2+4*x+4, 4*x^4+2*x^3+x]])

sage: (Q,R) = A.reduce(B,return_quotient=True); R
[3*x^4 + 3*x^3 + 4*x + 3                 2*x + 2                 2*x + 6]
sage: A == Q*B + R
True
sage: P = B.popov_form(); P.leading_positions(return_degree=True)
([1, 2], [2, 2])
sage: R.degree_matrix()
[4 1 1]
sage: A.reduce(P) == R
True
sage: A.reduce(P[:,:2])
Traceback (most recent call last):
...
ValueError: column dimension of self should be the column
dimension of the input matrix
```

Demonstrating shifts:

```
sage: (Qs,Rs) = A.reduce(B,shifts=[0,2,4],return_quotient=True); Rs
[3*x^4 + 3*x^3 + 6*x + 2             4*x^3 + 5*x                         0]
sage: A == Qs*B + Rs
True
sage: Ps = B.popov_form(shifts=[0,2,4])
sage: Ps.leading_positions(shifts=[0,2,4],return_degree=True)
([1, 2], [4, 0])
sage: Rs.degree_matrix()
[ 4  3 -1]
sage: A.reduce(Ps, shifts=[0,2,4]) == Rs
True
```

If `return_quotient` is `False`, only the normal form is returned:

```
sage: R == A.reduce(B) and Rs == A.reduce(B,shifts=[0,2,4])
True
```

Demonstrating column-wise normal forms, with a matrix A which has several columns, and a matrix B which does not have full column rank (its column-wise Popov form has a zero column):

```
sage: A = Matrix(pR, 2, 2,                                  \
    [[5*x^3 + 2*x^2 + 4*x + 1,           x^3 + 4*x + 4], \
     [2*x^3 + 5*x^2 + 2*x + 4,         2*x^3 + 3*x + 2]])
sage: (Q,R) = A.reduce(B,row_wise=False,return_quotient=True); R
[0 3]
[0 0]
sage: A == B*Q + R
True
sage: P = B.popov_form(row_wise=False); P
[x + 2     6     0]
[    0     1     0]
sage: P.leading_positions(row_wise=False, return_degree=True)
```

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```
([0, 1, -1], [1, 0, -1])
sage: R.degree_matrix()
[-1  0]
[-1 -1]
```

See also:

left_quo_rem() , *right_quo_rem()* .

reduced_form(*transformation=None*, *shifts=None*, *row_wise=True*, *include_zero_vectors=True*)

Return a row reduced form of this matrix (resp. a column reduced form if the optional parameter `row_wise` is set to `False`).

An $m \times n$ univariate polynomial matrix M is said to be in (shifted) row reduced form if it has k nonzero rows with $k \leq n$ and its (shifted) leading matrix has rank k. See *is_reduced()* for more information.

Currently, the implementation of this method is a direct call to *weak_popov_form()*.

INPUT:

- `transformation` – (optional, default: `False`). If this is `True`, the transformation matrix U will be returned as well: this is a unimodular matrix over $\mathbf{K}[x]$ such that `self` equals UR, where R is the output matrix.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if one does not allow zero rows in row reduced forms (resp. zero columns in column reduced forms).

OUTPUT:

- A polynomial matrix R which is a reduced form of `self` if `transformation=False`; otherwise two polynomial matrices R, U such that $UA = R$ and R is reduced and U is unimodular where A is `self`.

EXAMPLES:

```
sage: pR.<x> = GF(3)[]
sage: A = matrix(pR,3,[x,   x^2, x^3,
....:                  x^2, x^1, 0,
....:                  x^3, x^3, x^3])
sage: R = A.reduced_form(); R
[        x           x^2       x^3]
[      x^2             x         0]
[  x^3 + 2*x x^3 + 2*x^2         0]
sage: R.is_reduced()
True
sage: R2 = A.reduced_form(shifts=[6,3,0]); R2
[                x               x^2                 x^3]
[                0         2*x^2 + x       2*x^4 + x^3]
[                0                 0 2*x^5 + x^4 + x^3]
sage: R2.is_reduced(shifts=[6,3,0])
True
sage: R2.is_reduced()
False
```

If the matrix is an $n \times 1$ matrix with at least one non-zero entry, R has a single non-zero entry and that entry is a scalar multiple of the greatest-common-divisor of the entries of the matrix:

```
sage: A = matrix([[x*(x-1)*(x+1)],[x*(x-2)*(x+2)],[x]])
sage: R = A.reduced_form()
sage: R
[x]
[0]
[0]
```

A zero matrix is already reduced:

```
sage: A = matrix(pR, 2, [0,0,0,0])
sage: A.reduced_form()
[0 0]
[0 0]
```

In the following example, the original matrix is already reduced, but the output is a different matrix: currently this method is an alias for *weak_popov_form()*, which is a stronger reduced form:

```
sage: R.<x> = QQ['x']
sage: A = matrix([[x,x,x],[0,0,x]]); A
[x x x]
[0 0 x]
sage: A.is_reduced()
True
sage: W = A.reduced_form(); W
[ x  x  x]
[-x -x  0]
sage: W.is_weak_popov()
True
```

The last example shows the usage of the transformation parameter:

```
sage: Fq.<a> = GF(2^3)
sage: pR.<x> = Fq[]
sage: A = matrix(pR, [[x^2+a,  x^4+a],
....:                 [  x^3,  a*x^4]])
sage: W,U = A.reduced_form(transformation=True)
sage: W,U
(
[          x^2 + a           x^4 + a]  [1 0]
[x^3 + a*x^2 + a^2               a^2], [a 1]
)
sage: W.is_reduced()
True
sage: U*W == A
True
sage: U.is_invertible()
True
```

See also:

is_reduced() , *weak_popov_form()* , *popov_form()* , *hermite_form()* .

reverse(*degree=None*, *row_wise=True*, *entry_wise=False*)

Return the matrix which is obtained from this matrix after reversing all its entries with respect to the degree as specified by `degree`.

Reversing a polynomial with respect to an integer d follows the convention for univariate polynomials, in particular it uses truncation or zero-padding as necessary if d differs from the degree of this polynomial.

If `entry_wise` is `True`: the input `degree` and `row_wise` are ignored, and all entries of the matrix are reversed with respect to their respective degrees.

If `entry_wise` is `False` (the default):

- if `degree` is an integer, all entries are reversed with respect to it;
- if `degree` is not provided, then all entries are reversed with respect to the degree of the whole matrix;
- if `degree` is a list $(d_1, \ldots, d_m)$ and `row_wise` is `True`, all entries of the $i`throware reversed with respect to` d_i$ for each i;
- if `degree` is a list $(d_1, \ldots, d_n)$ and `row_wise` is `False`, all entries of the $j`th column are reversed with respect to` d_j$ for each j.

INPUT:

- `degree` – (optional, default: `None`) a list of nonnegative integers, or a nonnegative integer,
- `row_wise` – (optional, default: `True`) boolean, if `True` (resp. `False`) then `degree` should be a list of length equal to the row (resp. column) dimension of this matrix.
- `entry_wise` – (optional, default: `False`) boolean, if `True` then the input `degree` and `row_wise` are ignored.

OUTPUT: a polynomial matrix.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: M = Matrix([
....:     [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:     [       6*x^2+3*x+1,        1,            2,          0],
....:     [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:      ])
sage: M.reverse()
[  x^3 + 5*x^2 + 5*x + 1                    5*x^3           4*x^3 +
6*x^2                          0]
[      x^3 + 3*x^2 + 6*x                      x^3
2*x^3                          0]
[4*x^3 + 6*x^2 + 4*x + 2              x^3 + 5*x^2     5*x^3 + 5*x^2
+ 2*x       6*x^3 + 5*x^2 + x]

sage: M.reverse(1)
[  x + 5     5*x 4*x + 6       0]
[  x + 3       x     2*x       0]
[4*x + 6   x + 5 5*x + 5 6*x + 5]

sage: M.reverse(0) == M.constant_matrix()
True
```

Entry-wise reversing with respect to each entry's degree:

```
sage: M.reverse(entry_wise=True)
[  x^3 + 5*x^2 + 5*x + 1                       5
```

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```
4*x + 6                         0]
[          x^2 + 3*x + 6                      1
2                         0]
[4*x^3 + 6*x^2 + 4*x + 2                   x + 5         5*x^2 +
5*x + 2          6*x^2 + 5*x + 1]
```

Row-wise and column-wise degree reversing are available:

```
sage: M.reverse([2,3,1])
[    x^2 + 5*x + 5             5*x^2       4*x^2 + 6*x
0]
[x^3 + 3*x^2 + 6*x               x^3             2*x^3
0]
[          4*x + 6             x + 5           5*x + 5
6*x + 5]

sage: M.reverse(M.column_degrees(),row_wise=False)
[  x^3 + 5*x^2 + 5*x + 1                      5*x             4*x^2
+ 6*x                         0]
[      x^3 + 3*x^2 + 6*x                        x
2*x^2                         0]
[4*x^3 + 6*x^2 + 4*x + 2                   x + 5         5*x^2 +
5*x + 2          6*x^2 + 5*x + 1]
```

Wrong length or negativity of input degree raise errors:

sage: M.reverse([1,3,1,4]) Traceback (most recent call last): … ValueError: length of input degree list should be the row dimension of the input matrix

sage: M.reverse([5,2,1], row_wise=False) Traceback (most recent call last): … ValueError: length of input degree list should be the column dimension of the input matrix

sage: M.reverse([2,3,-1]) Traceback (most recent call last): … OverflowError: can't convert negative value to unsigned long

See also:

`sage.rings.polynomial.polynomial_element.Polynomial.reverse()` .

right_quo_rem(B)

Return, if it exists, the quotient and remainder (Q, R) such that `self` is $QB + R$, where R has column degrees less than those of B entrywise.

If `self` is a $k \times m$ polynomial matrix (written A below), and the input B is an $m \times m$ polynomial matrix in column reduced form, then (Q, R) is unique. Both Q and R have dimensions $k \times m$. In this case, this method implements Newton iteration of a reversed polynomial matrix B, generalizing to matrices the fast division of univariate polynomials.

If A is a $k \times n$ polynomial matrix, and the input B is an $m \times n$ polynomial matrix such that B has full column rank, or more generally such that the matrix equation $A = XB$ has a rational solution, then there exists such a (Q, R) but it may not be unique; the algorithm returns one such quotient and remainder. Here Q is $k \times m$ and R is $k \times n$. In this case, this method follows the folklore approach based on solving the matrix equation $A = XB$ and splitting X into its polynomial part and proper fraction part.

Finally, if the matrix equation $A = XB$ has no rational solution, this method computes the normal form R and quotient Q of the rows of A with respect to the row space of B (see *reduce()*). Doing this for a well-chosen shift ensures that either R does not have column degrees less than those of B, and then there

is no valid quotient and remainder, or it does satisfy this degree constraint, and then this R can be returned as a remainder along with the quotient Q.

A `ValueError` is raised if the dimensions of `self` and B are not conformal, or if there exists no quotient and remainder.

EXAMPLES:

Case where B is a square, column reduced matrix:

```
sage: pR.<x> = GF(7)[]
sage: A = Matrix(pR, 2, 3,                                  \
    [[3*x^3 + 3*x, 3*x^3 + 6*x + 5,   2*x^3 + 2*x + 6], \
     [2*x^3 + 4,   6*x^3 + 5*x^2 + 1, 3*x^2 + 2*x + 2]])

sage: B = Matrix(pR, 3, 3,                                    \
    [[4*x^2 + 3*x + 3, 3*x^2 + 3*x + 1,   4*x^2 + x + 4], \
     [6*x^2 + 2*x + 3,     4*x^2 + 3*x,     3*x^2 + 4*x], \
     [5*x^2 + 3*x + 6,   6*x^2 + x + 4, 3*x^2 + 3*x + 2]])
sage: B.is_reduced(row_wise=False)
True
sage: Q,R = A.right_quo_rem(B); (Q,R)
(
[    4*x   x + 2 6*x + 1]  [  x + 2 6*x + 1 5*x + 4]
[4*x + 3   x + 6 3*x + 4], [4*x + 2 2*x + 3 4*x + 3]
)
sage: A == Q*B+R and R.degree() < 2
True
sage: A[:,:2].right_quo_rem(B)
Traceback (most recent call last):
...
ValueError: column dimension of self should be the column dimension
of the input matrix

sage: B = Matrix(pR, 3, 3,                                  \
    [[3,     3*x^3 + 3*x + 1, 4*x^2 + x + 4], \
     [x + 3, 4*x^2 + 3*x,     3*x^2 + 4*x],   \
     [6,     6*x^3 + x + 4,   3*x^2 + 3*x + 2]])
sage: B.is_reduced(row_wise=False)
True
sage: Q,R = A.right_quo_rem(B); (Q,R)
(
[2*x^2 + 4*x + 6     3*x^2 + 5*x     6*x^2 + 3*x]
[6*x^2 + 4*x + 1   2*x^2 + x + 5 4*x^2 + 6*x + 1],

[              3               6         2*x + 3]
[              1 5*x^2 + 2*x + 3         6*x + 3]
)
sage: cdegR = R.column_degrees(); cdegB = B.column_degrees()
sage: A == Q*B+R and all([cdegR[i] < cdegB[i] for i in range(3)])
True
```

With a nonsingular but also non-reduced matrix, there exists a solution, but it might not be unique:

```
sage: B = Matrix(pR, 3, 3,                                  \
```

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```
        [[              5,               0, 2*x + 6], \
         [           4*x, 3*x^2 + 4*x + 5,   x + 1], \
         [3*x^2 + 5*x + 2, 6*x^3 + 4*x + 6,       3]])
sage: B.det() != 0 and (not B.is_reduced(row_wise=False))
True
sage: Q,R = A.right_quo_rem(B); (Q,R)
(
[    6*x^2 + 3*x 4*x^2 + 3*x + 1         5*x + 1]
[  x^2 + 5*x + 5 5*x^2 + 3*x + 5           x + 2],

[      4*x + 5 x^2 + 2*x + 1             2]
[      6*x + 3     5*x^2 + 6             3]
)
sage: cdegR = R.column_degrees(); cdegB = B.column_degrees()
sage: A == Q*B+R and all([cdegR[i] < cdegB[i] for i in range(3)])
True

sage: Q2 = Matrix(pR, 2, 3,                              \
        [[6*x^2 + 3*x + 1, 4*x^2 + 3*x + 6, 5*x + 1], \
         [  x^2 + 5*x + 3, 5*x^2 + 3*x + 2,   x + 2]])
sage: R2 = Matrix(pR, 2, 3,     \
        [[    5*x, 3*x + 4, 5], \
         [4*x + 6,     5*x, 4]])
sage: A == Q2*B + R2
True
```

The same remark holds more generally for full column rank matrices: there exists a solution, but it might not be unique. However, for all other cases (rank-deficient matrix B or matrix B having strictly fewer rows than columns) there may be no solution:

```
sage: C = B.stack(B[1,:] + B[2,:]) # matrix 4 x 3, full column rank
sage: Q,R = A.right_quo_rem(C); (Q,R)
(
[    6*x^2 + 3*x 4*x^2 + 3*x + 1         5*x + 1               0]
[  x^2 + 5*x + 5 5*x^2 + 3*x + 5           x + 2               0],

[      4*x + 5 x^2 + 2*x + 1             2]
[      6*x + 3     5*x^2 + 6             3]
)

sage: A.right_quo_rem(B[:2,:]) # matrix 2 x 3, full row rank
Traceback (most recent call last):
...
ValueError: division of these matrices does not admit a remainder
with the required degree property
sage: D = copy(B); D[2,:] = B[0,:]+B[1,:] # square, singular
sage: A.right_quo_rem(D)
Traceback (most recent call last):
...
ValueError: division of these matrices does not admit a remainder
with the required degree property
```

In the latter case (rank-deficient or strictly fewer rows than columns, with no solution to $A = XB$), there

might stil be a quotient and remainder, in which case this method will find it via normal form computation:

```
sage: B = Matrix(pR, 1, 2, [[x, x]])
sage: A = Matrix(pR, 1, 2, [[x, x+2]])
sage: A.right_quo_rem(B)
([1], [0 2])
sage: A == 1*B + Matrix([[0,2]])
True
```

See also:

left_quo_rem() , *reduce()* .

row_degrees(*shifts=None*)

Return the (shifted) row degrees of this matrix.

For a given polynomial matrix $M = (M_{i,j})_{i,j}$ with m rows and n columns, its row degrees is the tuple $(d_1, \ldots, d_m)$ where $d_i = \max_j(\deg(M_{i,j}))$ for $1 \leq i \leq m$. Thus, $d_i = -1$ if the i-th row of M is zero, and $d_i \geq 0$ otherwise.

For given shifts $s_1, \ldots, s_n \in \mathbf{Z}$, the shifted row degrees of M is $(d_1, \ldots, d_m)$ where $d_i = \max_j(\deg(M_{i,j}) + s_j)$. Here, if the i-th row of M is zero then $d_i = \min(s_1, \ldots, s_n) - 1$; otherwise, d_i is larger than this value.

INPUT:

- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.

OUTPUT: a list of integers.

REFERENCES:

- [Wol1974] (Section 2.5, without shifts), and [VBB1992] (Section 3).
- Up to changes of signs, shifted row degrees coincide with the notion of *defect* commonly used in the rational approximation literature (see for example [Bec1992]).

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [ [3*x+1, 0, 1], [x^3+3, 0, 0] ])
sage: M.row_degrees()
[1, 3]

sage: M.row_degrees(shifts=[0,1,2])
[2, 3]
```

A zero row in a polynomial matrix can be identified in the (shifted) row degrees as the entries equal to `min(shifts)-1`:

```
sage: M = Matrix(pR, [[3*x+1, 0, 1], [x^3+3, 0, 0], [0, 0, 0]])
sage: M.row_degrees()
[1, 3, -1]

sage: M.row_degrees(shifts=[-2,1,2])
[2, 1, -3]
```

The row degrees of an empty matrix ($0 \times n$ or $m \times 0$) is not defined:

```
sage: M = Matrix( pR, 0, 3 )
sage: M.row_degrees()
Traceback (most recent call last):
...
ValueError: empty matrix does not have row degrees

sage: M = Matrix( pR, 3, 0 )
sage: M.row_degrees()
Traceback (most recent call last):
...
ValueError: empty matrix does not have row degrees
```

shift(*d*, *row_wise=True*)

Return the matrix which is obtained from this matrix after shifting all its entries as specified by d.

- if d is an integer, the shift is by d for all entries;
- if d is a list $(d_1, \ldots, d_m)$ and `row_wise` is `True`, all entries of the $i`throwareshiftedby`d_i$ for each i;
- if d is a list $(d_1, \ldots, d_n)$ and `row_wise` is `False`, all entries of the $j`thcolumnareshiftedby`d_j$ for each j.

Shifting by d means multiplying by the variable to the power d; if d is negative then terms of negative degree after shifting are discarded.

INPUT:

- `d` – a list of integers, or an integer,
- `row_wise` – (optional, default: `True`) boolean, if `True` (resp. `False`) then d should be a list of length equal to the row (resp. column) dimension of this matrix.

OUTPUT: a polynomial matrix.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: M = Matrix([
....:     [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:     [       6*x^2+3*x+1,        1,            2,          0],
....:     [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:      ])
sage: M.shift(-2)
[  x + 5       0       0       0]
[      6       0       0       0]
[2*x + 4       0       2       1]
```

Row-wise and column-wise shifting are available:

```
sage: M.shift([-1,2,-2])
[      x^2 + 5*x + 5                   0                   6                ␣
↪0]
[6*x^4 + 3*x^3 + x^2                 x^2               2*x^2                ␣
↪0]
[            2*x + 4                   0                   2                ␣
```

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```
↪1]

sage: M.shift([-1,1,0,0], row_wise=False)
[  x^2 + 5*x + 5             5*x         6*x + 4               0]
[        6*x + 3               x               2               0]
[2*x^2 + 4*x + 6       5*x^2 + x 2*x^2 + 5*x + 5   x^2 + 5*x + 6]

sage: M.shift([-d for d in M.row_degrees()]) == M.leading_matrix()
True
```

Length of input shift degree list is checked:

```
sage: M.shift([1,3,1,4])
Traceback (most recent call last):
...
ValueError: length of input shift list should be the row
dimension of the input matrix

sage: M.shift([5,2,-1], row_wise=False)
Traceback (most recent call last):
...
ValueError: length of input shift list should be the column
dimension of the input matrix
```

See also:

sage.rings.polynomial.polynomial_element.Polynomial.shift() .

solve_left_series_trunc(*B*, *d*)

Try to find a solution X to the equation $XA = B$, at precision `d`.

If `self` is a matrix A, then this function returns a vector or matrix X such that $XA = B \bmod x^d$. If B is a vector then X is a vector, and if B is a matrix then X is a matrix.

Raises `ValueError` if `d` is not strictly positive, or if there is a dimension mismatch between A and B, or if there is no solution to the given matrix equation at the specified precision.

INPUT:

- `B` – a polynomial matrix or polynomial vector.
- `d` – a positive integer.

OUTPUT:

A solution to the matrix equation, returned as polynomial matrix of degree less than `d` if `B` is a polynomial matrix; a polynomial vector of degree less than `d` if B is a polynomial vector.

ALGORITHM:

If A is square with invertible constant term, then the unique solution is found by calling *inverse_series_trunc()* and using the Dixon & Moenck-Carter iteration. Otherwise, a left minimal approximant basis of a matrix formed by A and B is computed, for an appropriate shift which ensures that this basis reveals either a solution X or the fact that no such solution exists.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: A = Matrix(pR, 3, 3,                                \
    [[4*x+5,           5*x^2 + x + 1, 4*x^2 + 4],     \
     [6*x^2 + 6*x + 6, 4*x^2 + 5*x,   4*x^2 + x + 3], \
     [3*x^2 + 2,       4*x + 1,       x^2 + 3*x]])
sage: A.is_square() and A.constant_matrix().is_invertible()
True
sage: B = vector([2*x^2 + 6*x + 6, 0, x + 6])
sage: X = A.solve_left_series_trunc(B,4); X
(3*x^3 + 3*x^2 + 2*x + 4, 4*x^3 + x^2 + 2*x + 6, 6*x^3 + x + 3)
sage: B == X*A % x**4
True

sage: B = Matrix(pR, 2, 3,                    \
        [[3*x, x^2 + x + 2, x^2 + 2*x + 3], \
        [  0,   6*x^2 + 1,             1]])
sage: A.solve_left_series_trunc(B,3)
[6*x^2 + 2*x + 2         4*x + 3     2*x^2 + 3*x]
[3*x^2 + 4*x + 5       4*x^2 + 3   x^2 + 6*x + 3]
sage: X = A.solve_left_series_trunc(B,37); B == X*A % x**37
True
```

Dimensions of input are checked:

```
sage: A.solve_left_series_trunc(B[:,:2],3)
Traceback (most recent call last):
...
ValueError: number of columns of self must equal number of columns of right-
↪hand side
```

Raises an exception when no solution:

```
sage: A[2:,:].solve_left_series_trunc(B,4)
Traceback (most recent call last):
...
ValueError: matrix equation has no solutions

sage: Ax = x*A; C = vector(pR, [1,1,1])
sage: Ax.solve_left_series_trunc(C,5)
Traceback (most recent call last):
...
ValueError: matrix equation has no solutions
```

Supports rectangular and rank-deficient cases:

```
sage: A[:,:2].solve_left_series_trunc(B[:,:2],4)
[5*x^2 + 2*x + 5         5*x + 5         2*x + 4]
[5*x^3 + 2*x + 1 2*x^2 + 2*x + 5           4*x^2]

sage: V = Matrix([[3*x^2 + 4*x + 1, 4*x]])
sage: A[:2,:].solve_left_series_trunc(V*A[:2,:], 4) == V
True
```

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```
sage: A[1,:] = (x+1) * A[0,:]; A[2,:] = (x+5) * A[0,:]
sage: B = (3*x^3+x^2+2)*A[0,:]
sage: A.solve_left_series_trunc(B, 6)
[4*x^2 + 6*x + 2       3*x^2 + x               0]
```

See also:

solve_right_series_trunc() .

solve_right_series_trunc(B, d)

Try to find a solution X to the equation $AX = B$, at precision d.

If self is a matrix A, then this function returns a vector or matrix X such that $AX = B \bmod x^d$. If B is a vector then X is a vector, and if B is a matrix then X is a matrix.

Raises ValueError if d is not strictly positive, or if there is a dimension mismatch between A and B, or if there is no solution to the given matrix equation at the specified precision.

INPUT:

- B – a polynomial matrix or polynomial vector.
- d – a positive integer.

OUTPUT:

A solution to the matrix equation, returned as polynomial matrix of degree less than d if B is a polynomial matrix; a polynomial vector of degree less than d if B is a polynomial vector.

ALGORITHM:

If A is square with invertible constant term, then the unique solution is found by calling *inverse_series_trunc()* and using the Dixon & Moenck-Carter iteration. Otherwise, a right minimal approximant basis of a matrix formed by A and B is computed, for an appropriate shift which ensures that this basis reveals either a solution X or the fact that no such solution exists.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: A = Matrix(pR, 3, 3,                                \
    [[4*x+5,           5*x^2 + x + 1, 4*x^2 + 4],      \
     [6*x^2 + 6*x + 6, 4*x^2 + 5*x,   4*x^2 + x + 3], \
     [3*x^2 + 2,       4*x + 1,       x^2 + 3*x]])
sage: A.is_square() and A.constant_matrix().is_invertible()
True
sage: B = vector([2*x^2 + 6*x + 6, 0, x + 6])
sage: X = A.solve_right_series_trunc(B,4); X
(2*x^3 + x^2, 5*x^3 + x^2 + 5*x + 6, 4*x^3 + 6*x^2 + 4*x)
sage: B == A*X % x**4
True

sage: B = Matrix(pR, 3, 2,                          \
            [[5*x^2 + 6*x + 3, 4*x^2 + 6*x + 4], \
             [  x^2 + 4*x + 2,         5*x + 2], \
             [        5*x + 3,               0]])
sage: A.solve_right_series_trunc(B,3)
[  3*x^2 + x + 1 5*x^2 + 4*x + 3]
[6*x^2 + 3*x + 1         4*x + 1]
```

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```
[      6*x^2 + 1   2*x^2 + x + 4]
sage: X = A.solve_right_series_trunc(B,37); B == A*X % x**37
True
```

Dimensions of input are checked:

```
sage: A.solve_right_series_trunc(B[:2,:],3)
Traceback (most recent call last):
...
ValueError: number of rows of self must equal number of rows of right-hand side
```

Raises an exception when no solution:

```
sage: A[:,2:].solve_right_series_trunc(B,4)
Traceback (most recent call last):
...
ValueError: matrix equation has no solutions

sage: Ax = x*A; C = vector(pR, [1,1,1])
sage: Ax.solve_right_series_trunc(C,5)
Traceback (most recent call last):
...
ValueError: matrix equation has no solutions
```

Supports rectangular and rank-deficient cases:

```
sage: A[:2,:].solve_right_series_trunc(B[:2,:],4)
[    5*x^2 + 4*x           x + 4]
[  x^2 + 3*x + 5 3*x^2 + 4*x + 4]
[        5*x + 3         3*x + 2]

sage: V = Matrix([[2*x^2 + 5*x + 1], [3*x^2+4]])
sage: A[:,:2].solve_right_series_trunc(A[:,:2]*V, 4) == V
True

sage: A[:,1] = (x+1) * A[:,0]; A[:,2] = (x+5) * A[:,0]
sage: B = (3*x^3+x^2+2)*A[:,0]
sage: A.solve_right_series_trunc(B, 6)
[4*x^2 + 6*x + 2]
[      3*x^2 + x]
[              0]
```

See also:

solve_left_series_trunc() .

truncate(*d*, *row_wise=True*)

Return the matrix which is obtained from this matrix after truncating all its entries according to precisions specified by d.

- if d is an integer, the truncation is at precision d for all entries;
- if d is a list $(d_1, \ldots, d_m)$ and `row_wise` is `True`, all entries of the i‘$throwaretruncatedatprecision$‘$d_i$ for each i;

- if d is a list $(d_1, \ldots, d_n)$ and `row_wise` is `False`, all entries of the $j`thcolumnaretruncatedatprecision`d_j$ for each j.

Here the convention for univariate polynomials is to take zero for the truncation for a negative d.

INPUT:

- `d` – a list of integers, or an integer,
- `row_wise` – (optional, default: `True`) boolean, if `True` (resp. `False`) then d should be a list of length equal to the row (resp. column) dimension of this matrix.

OUTPUT: a polynomial matrix.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]

sage: M = Matrix([
....:     [  x^3+5*x^2+5*x+1,        5,        6*x+4,          0],
....:     [       6*x^2+3*x+1,        1,            2,          0],
....:     [2*x^3+4*x^2+6*x+4, 5*x + 1, 2*x^2+5*x+5, x^2+5*x+6]
....:      ])
sage: M.truncate(2)
[5*x + 1       5 6*x + 4       0]
[3*x + 1       1       2       0]
[6*x + 4 5*x + 1 5*x + 5 5*x + 6]
sage: M.truncate(1) == M.constant_matrix()
True
```

Row-wise and column-wise truncation are available:

```
sage: M.truncate([3,2,1])
[5*x^2 + 5*x + 1               5         6*x + 4               0]
[         3*x + 1               1               2               0]
[               4               1               5               6]

sage: M.truncate([2,1,1,2], row_wise=False)
[5*x + 1       5       4       0]
[3*x + 1       1       2       0]
[6*x + 4       1       5 5*x + 6]
```

Length of list of truncation orders is checked:

```
sage: M.truncate([2,1,1,2])
Traceback (most recent call last):
...
ValueError: length of input precision list should be the row
dimension of the input matrix

sage: M.truncate([3,2,1], row_wise=False)
Traceback (most recent call last):
...
ValueError: length of input precision list should be the column
dimension of the input matrix
```

See also:

sage.rings.polynomial.polynomial_element.Polynomial.truncate() .

weak_popov_form(*transformation=False*, *shifts=None*, *row_wise=True*, *ordered=False*, *include_zero_vectors=True*)

Return a (shifted) (ordered) weak Popov form of this matrix.

See *is_weak_popov()* for a definition of weak Popov forms. If the input matrix is A, a weak Popov form of A is any matrix P in weak Popov form and such that $UA = P$ for some unimodular matrix U. The latter matrix is called the transformation, and the first optional argument allows one to specify whether to return this transformation.

Sometimes, one forbids weak Popov forms to have zero rows (resp. columns) in the above definitions; an optional parameter allows one to adopt this more restrictive setting. If zero rows (resp. columns) are allowed, the convention here is to place them as the bottommost rows (resp. the rightmost columns) of the output weak Popov form.

Note that, if asking for the transformation and discarding zero vectors (i.e. `transformation=True` and `include_zero_vectors=False`), then the returned transformation is still the complete unimodular matrix, including its bottommost rows (resp. rightmost columns) which correspond to zero rows (resp. columns) of the complete weak Popov form. In fact, this bottom part of the transformation yields a basis of the left (resp. right) kernel of the input matrix.

INPUT:

- `transformation` – (optional, default: `False`). If this is `True`, the transformation matrix U will be returned as well.
- `shifts` – (optional, default: `None`) list of integers; `None` is interpreted as `shifts=[0,...,0]`.
- `row_wise` – (optional, default: `True`) boolean, `True` if working row-wise (see the class description).
- `ordered` – (optional, default: `False`) boolean, `True` if seeking an ordered weak Popov form.
- `include_zero_vectors` – (optional, default: `True`) boolean, `False` if zero rows (resp. zero columns) should be discarded from the (ordered) weak Popov forms.

OUTPUT:

- A polynomial matrix which is a weak Popov form of `self` if `transformation` is `False`; otherwise two polynomial matrices which are a weak Popov form of `self` and the corresponding unimodular transformation.

ALGORITHM:

This method implements the Mulders-Storjohann algorithm of [MS2003], straightforwardly extended to the case of shifted forms.

EXAMPLES:

```
sage: pR.<x> = GF(7)[]
sage: M = Matrix(pR, [                                \
    [      6*x+4,       5*x^3+5*x,       6*x^2+2*x+2], \
    [4*x^2+5*x+2, x^4+5*x^2+2*x+4, 4*x^3+6*x^2+6*x+5]])

sage: P,U = M.weak_popov_form(transformation=True)
sage: P
[              4             x^2   6*x^2 + x + 2]
[              2 4*x^2 + 2*x + 4               5]
sage: U
```

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```
[2*x^2 + 1       4*x]
[      4*x         1]
sage: P.is_weak_popov() and U.is_invertible() and U*M==P
True
```

Demonstrating the `ordered` option:

```
sage: P.leading_positions()
[2, 1]
sage: PP = M.weak_popov_form(ordered=True); PP
[              2 4*x^2 + 2*x + 4               5]
[              4             x^2   6*x^2 + x + 2]
sage: PP.leading_positions()
[1, 2]
```

Demonstrating shifts:

```
sage: P = M.weak_popov_form(shifts=[0,2,4]); P
[            6*x^2 + 6*x + 4 5*x^4 + 4*x^3 + 5*x^2 + 5*x                    ␣
↪2*x + 2]
[                          2             4*x^2 + 2*x + 4                     ␣
↪    5]
sage: P==M.weak_popov_form(shifts=[-10,-8,-6])
True
```

Column-wise form is the row-wise form of the transpose:

```
sage: M.weak_popov_form() == M.T.weak_popov_form(row_wise=False).T
True
```

Zero vectors can be discarded:

```
sage: M.weak_popov_form(row_wise=False)
[x + 4     6     0]
[    5     1     0]

sage: P,U = M.weak_popov_form(transformation=True,      \
                              row_wise=False,           \
                              include_zero_vectors=False)
sage: P
[x + 4     6]
[    5     1]
sage: U
[                5*x + 2         5*x^2 + 4*x + 4 3*x^3 + 3*x^2 + 2*x + 4]
[                      1                       1                 2*x + 1]
[                5*x + 5                       2                       6]
sage: M*U[:,:2] == P and (M*U[:,2]).is_zero()
True
```

See also:

is_weak_popov() , *reduced_form()* , *popov_form()* , *hermite_form()* .

CHAPTER

THIRTYTWO

DENSE MATRICES OVER MULTIVARIATE POLYNOMIALS OVER FIELDS

This implementation inherits from Matrix_generic_dense, i.e. it is not optimized for speed only some methods were added.

AUTHOR:

- Martin Albrecht <malb@informatik.uni-bremen.de>

class sage.matrix.matrix_mpolynomial_dense.**Matrix_mpolynomial_dense**

Bases: *Matrix_generic_dense*

Dense matrix over a multivariate polynomial ring over a field.

determinant(*algorithm=None*)

Return the determinant of this matrix

INPUT:

- `algorithm` – ignored

EXAMPLES:

We compute the determinant of the arbitrary $3x3$ matrix:

```
sage: R = PolynomialRing(QQ, 9, 'x')
sage: A = matrix(R, 3, R.gens())
sage: A
[x0 x1 x2]
[x3 x4 x5]
[x6 x7 x8]
sage: A.determinant()
-x2*x4*x6 + x1*x5*x6 + x2*x3*x7 - x0*x5*x7 - x1*x3*x8 + x0*x4*x8
```

We check if two implementations agree on the result:

```
sage: R.<x,y> = QQ[]
sage: C = matrix(R, [[-6/5*x*y - y^2, -6*y^2 - 1/4*y],
....:                [  -1/3*x*y - 3, x*y - x]])
sage: C.determinant()
-6/5*x^2*y^2 - 3*x*y^3 + 6/5*x^2*y + 11/12*x*y^2 - 18*y^2 - 3/4*y

sage: C.change_ring(R.change_ring(QQbar)).det()
(-6/5)*x^2*y^2 + (-3)*x*y^3 + 6/5*x^2*y + 11/12*x*y^2 + (-18)*y^2 + (-3/4)*y
```

Finally, we check whether the Singular interface is working:

```
sage: R.<x,y> = RR[]
sage: C = matrix(R, [[0.368965517352886*y^2 + 0.425700773972636*x, -0.
↪800362171389760*y^2 - 0.807635502485287],
....:                [0.706173539423122*y^2 - 0.915986060298440, 0.
↪897165181570476*y + 0.107903328188376]])
sage: C.determinant()
0.565194587390682*y^4 + 0.33102301536914...*y^3 + 0.381923912175852*x*y - 0.
↪122977163520282*y^2 + 0.0459345303240150*x - 0.739782862078649
```

ALGORITHM: Calls Singular, libSingular or native implementation.

echelon_form(*algorithm='row_reduction'*, ***kwds*)

Return an echelon form of `self` using chosen algorithm.

By default only a usual row reduction with no divisions or column swaps is returned.

If Gauss-Bareiss algorithm is chosen, column swaps are recorded and can be retrieved via *swapped_columns()*.

INPUT:

- `algorithm` – string, which algorithm to use (default: 'row_reduction'). Valid options are:
  - `'row_reduction'` (default) – reduce as far as possible, only divide by constant entries
  - `'frac'` – reduced echelon form over fraction field
  - `'bareiss'` – fraction free Gauss-Bareiss algorithm with column swaps

OUTPUT:

The row echelon form of A depending on the chosen algorithm, as an immutable matrix. Note that `self` is *not* changed by this command. Use `A.echelonize()` to change A in place.

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(GF(127), 2)
sage: A = matrix(P, 2, 2, [1, x, 1, y])
sage: A
[1 x]
[1 y]
sage: A.echelon_form()
[     1      x]
[     0 -x + y]
```

The reduced row echelon form over the fraction field is as follows:

```
sage: A.echelon_form('frac') # over fraction field
[1 0]
[0 1]
```

Alternatively, the Gauss-Bareiss algorithm may be chosen:

```
sage: E = A.echelon_form('bareiss'); E
[    1     y]
[    0 x - y]
```

After the application of the Gauss-Bareiss algorithm the swapped columns may inspected:

```
sage: E.swapped_columns(), E.pivots()
((0, 1), (0, 1))
sage: A.swapped_columns(), A.pivots()
(None, (0, 1))
```

Another approach is to row reduce as far as possible:

```
sage: A.echelon_form('row_reduction')
[     1      x]
[     0 -x + y]
```

echelonize(*algorithm='row_reduction'*, ***kwds*)

Transform self into a matrix in echelon form over the same base ring as `self`.

If Gauss-Bareiss algorithm is chosen, column swaps are recorded and can be retrieved via *swapped_columns()*.

INPUT:

- `algorithm` – string, which algorithm to use. Valid options are:
  - `'row_reduction'` – reduce as far as possible, only divide by constant entries
  - `'bareiss'` – fraction free Gauss-Bareiss algorithm with column swaps

EXAMPLES:

```
sage: P.<x,y> = PolynomialRing(QQ, 2)
sage: A = matrix(P, 2, 2, [1/2, x, 1, 3/4*y+1])
sage: A
[      1/2          x]
[        1 3/4*y + 1]

sage: B = copy(A)
sage: B.echelonize('bareiss'); B
[              1       3/4*y + 1]
[              0 x - 3/8*y - 1/2]

sage: B = copy(A)
sage: B.echelonize('row_reduction'); B
[               1              2*x]
[               0 -2*x + 3/4*y + 1]

sage: P.<x,y> = PolynomialRing(QQ, 2)
sage: A = matrix(P,2,3,[2,x,0,3,y,1]); A
[2 x 0]
[3 y 1]

sage: E = A.echelon_form('bareiss'); E
[1 3 y]
[0 2 x]
sage: E.swapped_columns()
(2, 0, 1)
sage: A.pivots()
(0, 1, 2)
```

pivots()

Return the pivot column positions of this matrix as a list of integers.

This returns a list, of the position of the first nonzero entry in each row of the echelon form.

OUTPUT:

A list of Python ints.

EXAMPLES:

```
sage: matrix([PolynomialRing(GF(2), 2, 'x').gen()]).pivots()
(0,)
sage: K = QQ['x,y']
sage: x, y = K.gens()
sage: m = matrix(K, [(-x, 1, y, x - y), (-x*y, y, y^2 - 1, x*y - y^2 + x), (-
↪x*y + x, y - 1, y^2 - y - 2, x*y - y^2 + x + y)])
sage: m.pivots()
(0, 2)
sage: m.rank()
2
```

swapped_columns()

Return which columns were swapped during the Gauss-Bareiss reduction

OUTPUT:

Return a tuple representing the column swaps during the last application of the Gauss-Bareiss algorithm (see *echelon_form()* for details).

The tuple as length equal to the rank of self and the value at the i-th position indicates the source column which was put as the i-th column.

If no Gauss-Bareiss reduction was performed yet, None is returned.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: C = random_matrix(R, 2, 2, terms=2)
sage: while C.rank() != 2:
....:     C = random_matrix(R, 2, 2, terms=2)
sage: C.swapped_columns()
sage: E = C.echelon_form('bareiss')
sage: sorted(E.swapped_columns())
[0, 1]
```

CHAPTER

THIRTYTHREE

MATRICES OVER CYCLOTOMIC FIELDS

The underlying matrix for a Matrix_cyclo_dense object is stored as follows: given an n x m matrix over a cyclotomic field of degree d, we store a d x (nm) matrix over QQ, each column of which corresponds to an element of the original matrix. This can be retrieved via the _rational_matrix method. Here is an example illustrating this:

EXAMPLES:

```
sage: F.<zeta> = CyclotomicField(5)
sage: M = Matrix(F, 2, 3, [zeta, 3, zeta**4+5, (zeta+1)**4, 0, 1])
sage: M
[                       zeta                                3  -zeta^3 - zeta^2 - zeta + 4]
[3*zeta^3 + 5*zeta^2 + 3*zeta                                0                             1]

sage: M._rational_matrix()
[ 0  3  4  0  0  1]
[ 1  0 -1  3  0  0]
[ 0  0 -1  5  0  0]
[ 0  0 -1  3  0  0]
```

AUTHORS:

- William Stein
- Craig Citro

class sage.matrix.matrix_cyclo_dense.**Matrix_cyclo_dense**

Bases: *Matrix_dense*

Initialize a newly created cyclotomic matrix.

INPUT:

- `parent` – a matrix space over a cyclotomic number field
- `entries` – see *matrix()*
- `copy` – ignored (for backwards compatibility)
- `coerce` – if False, assume without checking that the entries lie in the base ring

EXAMPLES:

This function is called implicitly when you create new cyclotomic dense matrices:

```
sage: W.<a> = CyclotomicField(100)
sage: A = matrix(2, 3, [1, 1/a, 1-a,a, -2/3*a, a^19])
sage: A
```

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```
[                        1 -a^39 + a^29 - a^19 + a^9                      -a + 1]
[                        a                    -2/3*a                        a^19]
sage: TestSuite(A).run()
```

charpoly(*var='x'*, *algorithm='multimodular'*, *proof=None*)

Return the characteristic polynomial of self, as a polynomial over the base ring.

INPUT:

- algorithm
  - 'multimodular' (default): reduce modulo primes, compute charpoly mod p, and lift (very fast)
  - 'pari': use pari (quite slow; comparable to Magma v2.14 though)
  - 'hessenberg': put matrix in Hessenberg form (double dog slow)
- proof – bool (default: None) proof flag determined by global linalg proof.

OUTPUT:

polynomial

EXAMPLES:

```
sage: K.<z> = CyclotomicField(5)
sage: a = matrix(K, 3, [1,z,1+z^2, z/3,1,2,3,z^2,1-z])
sage: f = a.charpoly(); f
x^3 + (z - 3)*x^2 + (-16/3*z^2 - 2*z)*x - 2/3*z^3 + 16/3*z^2 - 5*z + 5/3
sage: f(a)
[0 0 0]
[0 0 0]
[0 0 0]
sage: a.charpoly(algorithm='pari')
x^3 + (z - 3)*x^2 + (-16/3*z^2 - 2*z)*x - 2/3*z^3 + 16/3*z^2 - 5*z + 5/3
sage: a.charpoly(algorithm='hessenberg')
x^3 + (z - 3)*x^2 + (-16/3*z^2 - 2*z)*x - 2/3*z^3 + 16/3*z^2 - 5*z + 5/3

sage: Matrix(K, 1, [0]).charpoly()
x
sage: Matrix(K, 1, [5]).charpoly(var='y')
y - 5

sage: Matrix(CyclotomicField(13),3).charpoly()
x^3
sage: Matrix(CyclotomicField(13),3).charpoly()[2].parent()
Cyclotomic Field of order 13 and degree 12
```

coefficient_bound()

Return an upper bound for the (complex) absolute values of all entries of self with respect to all embeddings.

Use `self.height()` for a sharper bound.

This is computed using just the Cauchy-Schwarz inequality, i.e., we use the fact that

```
\left| \sum_i a_i\zeta^i \right| \leq \sum_i |a_i|,
```

as $|\zeta| = 1$.

EXAMPLES:

```
sage: W.<z> = CyclotomicField(5)
sage: A = matrix(W, 2, 2, [1+z, 0, 9*z+7, -3 + 4*z]); A
[  z + 1       0]
[9*z + 7 4*z - 3]
sage: A.coefficient_bound()
16
```

The above bound is just $9 + 7$, coming from the lower left entry. A better bound would be the following:

```
sage: (A[1,0]).abs()
12.997543663...
```

denominator()

Return the denominator of the entries of this matrix.

OUTPUT:

integer – the smallest integer d so that d * self has
entries in the ring of integers

EXAMPLES:

```
sage: W.<z> = CyclotomicField(5)
sage: A = matrix(W, 2, 2, [-2/7,2/3*z+z^2,-z,1+z/19]); A
[       -2/7 z^2 + 2/3*z]
[         -z  1/19*z + 1]
sage: d = A.denominator(); d
399
```

echelon_form(*algorithm='multimodular'*, *height_guess=None*)

Find the echelon form of self, using the specified algorithm.

The result is cached for each algorithm separately.

EXAMPLES:

```
sage: W.<z> = CyclotomicField(3)
sage: A = matrix(W, 2, 3, [1+z, 2/3, 9*z+7, -3 + 4*z, z, -7*z]); A
[  z + 1     2/3 9*z + 7]
[4*z - 3       z    -7*z]
sage: A.echelon_form()
[                  1                   0  -192/97*z - 361/97]
[                  0                   1 1851/97*z + 1272/97]
sage: A.echelon_form(algorithm='classical')
[                  1                   0  -192/97*z - 361/97]
[                  0                   1 1851/97*z + 1272/97]
```

We verify that the result is cached and that the caches are separate:

```
sage: A.echelon_form() is A.echelon_form()
True
sage: A.echelon_form() is A.echelon_form(algorithm='classical')
False
```

height()

Return the height of self.

If we let a_{ij} be the i, j entry of self, then we define the height of self to be

$$\max_v \max_{i,j} |a_{ij}|_v,$$

where v runs over all complex embeddings of `self.base_ring()`.

EXAMPLES:

```
sage: W.<z> = CyclotomicField(5)
sage: A = matrix(W, 2, 2, [1+z, 0, 9*z+7, -3 + 4*z]); A
[  z + 1       0]
[9*z + 7 4*z - 3]
sage: A.height()
12.997543663...
sage: (A[1,0]).abs()
12.997543663...
```

randomize(*density=1*, *num_bound=2*, *den_bound=2*, *distribution=None*, *nonzero=False*, **args*, ***kwds*)

Randomize the entries of `self`.

Choose rational numbers according to `distribution`, whose numerators are bounded by num_bound and whose denominators are bounded by den_bound.

EXAMPLES:

```
sage: A = Matrix(CyclotomicField(5),2,2,range(4)) ; A
[0 1]
[2 3]
sage: A.randomize()
sage: A   # random output
[       1/2*zeta5^2 + zeta5                          1/2]
[        -zeta5^2 + 2*zeta5 -2*zeta5^3 + 2*zeta5^2 + 2]
```

set_immutable()

Change this matrix so that it is immutable.

EXAMPLES:

```
sage: W.<z> = CyclotomicField(5)
sage: A = matrix(W, 2, 2, [1,2/3*z+z^2,-z,1+z/2])
sage: A[0,0] = 10
sage: A.set_immutable()
sage: A[0,0] = 20
Traceback (most recent call last):
...
ValueError: matrix is immutable; please change a copy instead (i.e., use␣
↪copy(M) to change a copy of M).
```

Note that there is no function to set a matrix to be mutable again, since such a function would violate the whole point. Instead make a copy, which is always mutable by default.:

```
sage: A.set_mutable()
Traceback (most recent call last):
...
```

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```
AttributeError: 'sage.matrix.matrix_cyclo_dense.Matrix_cyclo_dense' object has␣
↪no attribute 'set_mutable'
sage: B = A.__copy__()
sage: B[0,0] = 20
sage: B[0,0]
20
```

tensor_product(*A*, *subdivide=True*)

Return the tensor product of two matrices.

INPUT:

- `A` – a matrix
- `subdivide` – (default: `True`) whether or not to return natural subdivisions with the matrix

OUTPUT:

Replace each element of `self` by a copy of `A`, but first create a scalar multiple of `A` by the element it replaces. So if `self` is an $m \times n$ matrix and `A` is a $p \times q$ matrix, then the tensor product is an $mp \times nq$ matrix. By default, the matrix will be subdivided into submatrices of size $p \times q$.

EXAMPLES:

```
sage: C = CyclotomicField(12)
sage: M = matrix.random(C, 3, 3)
sage: N = matrix.random(C, 50, 50)
sage: M.tensor_product(M) == super(type(M), M).tensor_product(M)
True
sage: N = matrix.random(C, 15, 20)
sage: M.tensor_product(N) == super(type(M), M).tensor_product(N)
True
```

CHAPTER

THIRTYFOUR

OPERATION TABLES

This module implements general operation tables, which are very matrix-like.

class sage.matrix.operation_table.**OperationTable**(*S*, *operation*, *names='letters'*, *elements=None*)

Bases: SageObject

An object that represents a binary operation as a table.

Primarily this object is used to provide a multiplication_table() for objects in the category of magmas (monoids, groups, ...) and addition_table() for objects in the category of commutative additive magmas (additive monoids, groups, ...).

INPUT:

- S - a finite algebraic structure (or finite iterable)
- **operation - a function of two variables that accepts pairs**
  of elements from S. A natural source of such functions is the Python operator module, and in particular operator.add() and operator.mul(). This may also be a function defined with lambda or def.
- names - (default: 'letters') The type of names used, values are:
  - 'letters' - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by *column_keys()*, padded to a common width with leading 'a's.
  - 'digits' - base 10 representation of the elements' positions in the list given by *column_keys()*, padded to a common width with leading zeros.
  - 'elements' - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- elements - (default: None) A list of elements of S, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements of S', perhaps when this is used in the context of a particular structure. The default is to use whatever ordering the S.list() method returns. $elements$' can also be a subset which is closed under the operation, useful perhaps when the set is infinite.

OUTPUT: An object with methods that abstracts multiplication tables, addition tables, Cayley tables, etc. It should be general enough to be useful for any finite algebraic structure whose elements can be combined with a binary operation. This is not necessarily meant be constructed directly, but instead should be useful for constructing operation tables of various algebraic structures that have binary operations.

EXAMPLES:

In its most basic use, the table needs a structure and an operation:

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=SymmetricGroup(3)
sage: OperationTable(G, operation=operator.mul)
*  a b c d e f
 +------------
a| a b c d e f
b| b a d c f e
c| c e a f b d
d| d f b e a c
e| e c f a d b
f| f d e b c a
```

With two operations present, we can specify which operation we want:

```
sage: from sage.matrix.operation_table import OperationTable
sage: R=Integers(6)
sage: OperationTable(R, operation=operator.add)
+  a b c d e f
 +------------
a| a b c d e f
b| b c d e f a
c| c d e f a b
d| d e f a b c
e| e f a b c d
f| f a b c d e
```

The default symbol set for elements is lowercase ASCII letters, which take on a base 26 flavor for structures with more than 26 elements.

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=DihedralGroup(14)
sage: OperationTable(G, operator.mul, names='letters')
 *  aa ab ac ad ae af ag ah ai aj ak al am an ao ap aq ar as at au av aw ax ay az␣
↪ba bb
  +------------------------------------------------------------------------------
↪---
aa| aa ab ac ad ae af ag ah ai aj ak al am an ao ap aq ar as at au av aw ax ay az␣
↪ba bb
ab| ab aa ad ac af ae ah ag aj ai al ak an am ap ao ar aq at as av au ax aw az ay␣
↪bb ba
ac| ac ba aa ae ad ag af ai ah ak aj am al ao an aq ap as ar au at aw av ay ax bb␣
↪ab az
ad| ad bb ab af ac ah ae aj ag al ai an ak ap am ar ao at aq av as ax au az aw ba␣
↪aa ay
ae| ae az ba ag aa ai ad ak af am ah ao aj aq al as an au ap aw ar ay at bb av ab␣
↪ac ax
af| af ay bb ah ab aj ac al ae an ag ap ai ar ak at am av ao ax aq az as ba au aa␣
↪ad aw
ag| ag ax az ai ba ak aa am ad ao af aq ah as aj au al aw an ay ap bb ar ab at ac␣
↪ae av
ah| ah aw ay aj bb al ab an ac ap ae ar ag at ai av ak ax am az ao ba aq aa as ad␣
↪af au
ai| ai av ax ak az am ba ao aa aq ad as af au ah aw aj ay al bb an ab ap ac ar ae␣
```

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```
 ↪ag at
aj| aj au aw al ay an bb ap ab ar ac at ae av ag ax ai az ak ba am aa ao ad aq af␣
 ↪ah as
ak| ak at av am ax ao az aq ba as aa au ad aw af ay ah bb aj ab al ac an ae ap ag␣
 ↪ai ar
al| al as au an aw ap ay ar bb at ab av ac ax ae az ag ba ai aa ak ad am af ao ah␣
 ↪aj aq
am| am ar at ao av aq ax as az au ba aw aa ay ad bb af ab ah ac aj ae al ag an ai␣
 ↪ak ap
an| an aq as ap au ar aw at ay av bb ax ab az ac ba ae aa ag ad ai af ak ah am aj␣
 ↪al ao
ao| ao ap ar aq at as av au ax aw az ay ba bb aa ab ad ac af ae ah ag aj ai al ak␣
 ↪am an
ap| ap ao aq ar as at au av aw ax ay az bb ba ab aa ac ad ae af ag ah ai aj ak al␣
 ↪an am
aq| aq an ap as ar au at aw av ay ax bb az ab ba ac aa ae ad ag af ai ah ak aj am␣
 ↪ao al
ar| ar am ao at aq av as ax au az aw ba ay aa bb ad ab af ac ah ae aj ag al ai an␣
 ↪ap ak
as| as al an au ap aw ar ay at bb av ab ax ac az ae ba ag aa ai ad ak af am ah ao␣
 ↪aq aj
at| at ak am av ao ax aq az as ba au aa aw ad ay af bb ah ab aj ac al ae an ag ap␣
 ↪ar ai
au| au aj al aw an ay ap bb ar ab at ac av ae ax ag az ai ba ak aa am ad ao af aq␣
 ↪as ah
av| av ai ak ax am az ao ba aq aa as ad au af aw ah ay aj bb al ab an ac ap ae ar␣
 ↪at ag
aw| aw ah aj ay al bb an ab ap ac ar ae at ag av ai ax ak az am ba ao aa aq ad as␣
 ↪au af
ax| ax ag ai az ak ba am aa ao ad aq af as ah au aj aw al ay an bb ap ab ar ac at␣
 ↪av ae
ay| ay af ah bb aj ab al ac an ae ap ag ar ai at ak av am ax ao az aq ba as aa au␣
 ↪aw ad
az| az ae ag ba ai aa ak ad am af ao ah aq aj as al au an aw ap ay ar bb at ab av␣
 ↪ax ac
ba| ba ac ae aa ag ad ai af ak ah am aj ao al aq an as ap au ar aw at ay av bb ax␣
 ↪az ab
bb| bb ad af ab ah ac aj ae al ag an ai ap ak ar am at ao av aq ax as az au ba aw␣
 ↪ay aa
```

Another symbol set is base 10 digits, padded with leading zeros to make a common width.

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=AlternatingGroup(4)
sage: OperationTable(G, operator.mul, names='digits')
 *  00 01 02 03 04 05 06 07 08 09 10 11
  +-----------------------------------
00| 00 01 02 03 04 05 06 07 08 09 10 11
01| 01 02 00 05 03 04 07 08 06 11 09 10
02| 02 00 01 04 05 03 08 06 07 10 11 09
03| 03 06 09 00 07 10 01 04 11 02 05 08
04| 04 08 10 02 06 11 00 05 09 01 03 07
```

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```
05| 05 07 11 01 08 09 02 03 10 00 04 06
06| 06 09 03 10 00 07 04 11 01 08 02 05
07| 07 11 05 09 01 08 03 10 02 06 00 04
08| 08 10 04 11 02 06 05 09 00 07 01 03
09| 09 03 06 07 10 00 11 01 04 05 08 02
10| 10 04 08 06 11 02 09 00 05 03 07 01
11| 11 05 07 08 09 01 10 02 03 04 06 00
```

If the group's elements are not too cumbersome, or the group is small, then the string representation of the elements can be used.

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=AlternatingGroup(3)
sage: OperationTable(G, operator.mul, names='elements')
      *       () (1,2,3) (1,3,2)
       +------------------------
     ()|       () (1,2,3) (1,3,2)
(1,2,3)| (1,2,3) (1,3,2)       ()
(1,3,2)| (1,3,2)       () (1,2,3)
```

You can give the elements any names you like, but they need to be ordered in the same order as returned by the *column_keys()* method.

```
sage: from sage.matrix.operation_table import OperationTable
sage: G = QuaternionGroup()
sage: T = OperationTable(G, operator.mul)
sage: T.column_keys()
((), (1,2,3,4)(5,6,7,8), ..., (1,8,3,6)(2,7,4,5))
sage: names=['1', 'I', '-1', '-I', 'J', '-K', '-J', 'K']
sage: T.change_names(names=names)
sage: sorted(T.translation().items())
[('-1', (1,3)(2,4)(5,7)(6,8)), ..., ('K', (1,8,3,6)(2,7,4,5))]
sage: T
 *   1  I -1 -I  J -K -J  K
  +------------------------
 1|  1  I -1 -I  J -K -J  K
 I|  I -1 -I  1  K  J -K -J
-1| -1 -I  1  I -J  K  J -K
-I| -I  1  I -1 -K -J  K  J
 J|  J -K -J  K -1 -I  1  I
-K| -K -J  K  J  I -1 -I  1
-J| -J  K  J -K  1  I -1 -I
 K|  K  J -K -J -I  1  I -1
```

With the right functions and a list comprehension, custom names can be easier. A multiplication table for hex digits (without carries):

```
sage: from sage.matrix.operation_table import OperationTable
sage: R = Integers(16)
sage: names=['{:x}'.format(Integer(a)) for a in R]
sage: OperationTable(R, operation=operator.mul, names=names)
*  0 1 2 3 4 5 6 7 8 9 a b c d e f
 +--------------------------------
```

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```
0| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1| 0 1 2 3 4 5 6 7 8 9 a b c d e f
2| 0 2 4 6 8 a c e 0 2 4 6 8 a c e
3| 0 3 6 9 c f 2 5 8 b e 1 4 7 a d
4| 0 4 8 c 0 4 8 c 0 4 8 c 0 4 8 c
5| 0 5 a f 4 9 e 3 8 d 2 7 c 1 6 b
6| 0 6 c 2 8 e 4 a 0 6 c 2 8 e 4 a
7| 0 7 e 5 c 3 a 1 8 f 6 d 4 b 2 9
8| 0 8 0 8 0 8 0 8 0 8 0 8 0 8 0 8
9| 0 9 2 b 4 d 6 f 8 1 a 3 c 5 e 7
a| 0 a 4 e 8 2 c 6 0 a 4 e 8 2 c 6
b| 0 b 6 1 c 7 2 d 8 3 e 9 4 f a 5
c| 0 c 8 4 0 c 8 4 0 c 8 4 0 c 8 4
d| 0 d a 7 4 1 e b 8 5 2 f c 9 6 3
e| 0 e c a 8 6 4 2 0 e c a 8 6 4 2
f| 0 f e d c b a 9 8 7 6 5 4 3 2 1
```

This should be flexible enough to create a variety of such tables.

```
sage: from sage.matrix.operation_table import OperationTable
sage: from operator import xor
sage: T=OperationTable(ZZ, xor, elements=range(8))
sage: T
.  a b c d e f g h
 +---------------
a| a b c d e f g h
b| b a d c f e h g
c| c d a b g h e f
d| d c b a h g f e
e| e f g h a b c d
f| f e h g b a d c
g| g h e f c d a b
h| h g f e d c b a
sage: names=['000', '001','010','011','100','101','110','111']
sage: T.change_names(names)
sage: T.set_print_symbols('^', '\\land')
sage: T
  ^  000 001 010 011 100 101 110 111
   +--------------------------------
000| 000 001 010 011 100 101 110 111
001| 001 000 011 010 101 100 111 110
010| 010 011 000 001 110 111 100 101
011| 011 010 001 000 111 110 101 100
100| 100 101 110 111 000 001 010 011
101| 101 100 111 110 001 000 011 010
110| 110 111 100 101 010 011 000 001
111| 111 110 101 100 011 010 001 000

sage: T = OperationTable([False, True], operator.or_, names = 'elements')
sage: T
    .  False  True
     +------------
```

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```
False| False  True
 True|  True  True
```

AUTHORS:

- Rob Beezer (2010-03-15)
- Bruno Edwards (2022-10-31)

change_names(*names*)

For an existing operation table, change the names used for the elements.

INPUT:

- `names` - the type of names used, values are:
  - `'letters'` - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by `list()`, padded to a common width with leading 'a's.
  - `'digits'` - base 10 representation of the elements' positions in the list given by `list()`, padded to a common width with leading zeros.
  - `'elements'` - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.

OUTPUT: `None`. This method changes the table "in-place", so any printed version will change and the output of the `dict()` will also change. So any items of interest about a particular table need to be copied/saved prior to calling this method.

EXAMPLES:

More examples can be found in the documentation for *OperationTable* since creating a new operation table uses the same routine.

```
sage: from sage.matrix.operation_table import OperationTable
sage: D=DihedralGroup(2)
sage: T=OperationTable(D, operator.mul)
sage: T
*  a b c d
 +--------
a| a b c d
b| b a d c
c| c d a b
d| d c b a
sage: T.translation()['c']
(1,2)
sage: T.change_names('digits')
sage: T
*  0 1 2 3
 +--------
0| 0 1 2 3
1| 1 0 3 2
2| 2 3 0 1
3| 3 2 1 0
sage: T.translation()['2']
(1,2)
sage: T.change_names('elements')
```

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```
sage: T
        *          ()      (3,4)      (1,2) (1,2)(3,4)
         +-----------------------------------------
       ()|          ()      (3,4)      (1,2) (1,2)(3,4)
    (3,4)|      (3,4)         () (1,2)(3,4)      (1,2)
    (1,2)|      (1,2) (1,2)(3,4)         ()      (3,4)
(1,2)(3,4)| (1,2)(3,4)      (1,2)      (3,4)         ()
sage: T.translation()['(1,2)']
(1,2)
sage: T.change_names(['w', 'x', 'y', 'z'])
sage: T
*  w x y z
 +--------
w| w x y z
x| x w z y
y| y z w x
z| z y x w
sage: T.translation()['y']
(1,2)
```

color_table(*element_names=True*, *cmap=None*, ***options*)

Return a graphic image as a square grid where entries are color coded.

INPUT:

- `element_names` - (default : `True`) Whether to display text with element names on the image
- `cmap` – (default: `matplotlib.cm.gist_rainbow`) color map for plot, see `matplotlib.cm`
- `**options` – passed on to `matrix_plot()`

EXAMPLES:

```
sage: from sage.matrix.operation_table import OperationTable
sage: OTa = OperationTable(SymmetricGroup(3), operation=operator.mul)       #␣
↪optional - sage.plot, sage.groups
sage: OTa.color_table()                                                     #␣
↪optional - sage.plot, sage.groups
Graphics object consisting of 37 graphics primitives
```

column_keys()

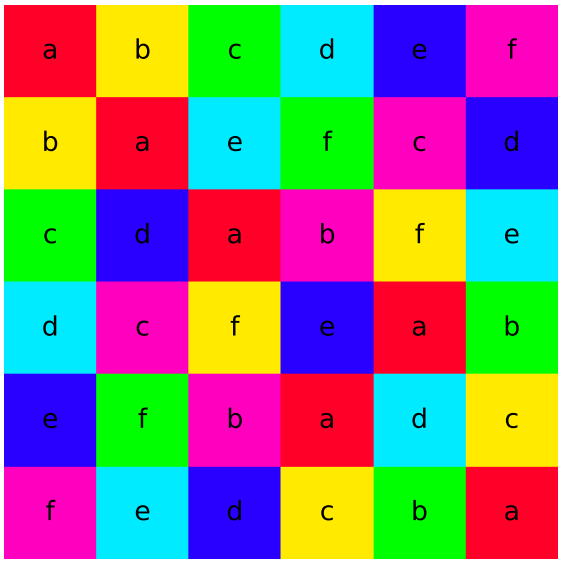
Returns a tuple of the elements used to build the table.

Note: `column_keys` and `row_keys` are identical. Both list the elements in the order used to label the table.

OUTPUT:

The elements of the algebraic structure used to build the table, as a list. But most importantly, elements are present in the list in the order which they appear in the table's column headings.

EXAMPLES:




```
sage: from sage.matrix.operation_table import OperationTable
sage: G=AlternatingGroup(3)
sage: T=OperationTable(G, operator.mul)
sage: T.column_keys()
((), (1,2,3), (1,3,2))
```

gray_table(***options*)

Return a graphic image as a square grid where entries are displayed in grayscale.

INPUT:

- `element_names` – (default: `True`) whether to display text with element names on the image
- `**options` – passed on to `matrix_plot()`

EXAMPLES:

```
sage: from sage.matrix.operation_table import OperationTable
sage: OTa = OperationTable(SymmetricGroup(3), operation=operator.mul)       #␣
↪optional - sage.plot, sage.groups
sage: OTa.gray_table()                                                      #␣
↪optional - sage.plot, sage.groups
Graphics object consisting of 37 graphics primitives
```

matrix_of_variables()

This method provides some backward compatibility for Cayley tables of groups, whose output was restricted to this single format.

EXAMPLES:

The output here is from the doctests for the old `cayley_table()` method for permutation groups.

```
sage: from sage.matrix.operation_table import OperationTable
sage: G = PermutationGroup(['(1,2,3)', '(2,3)'])
sage: T = OperationTable(G, operator.mul)
sage: T.matrix_of_variables()
```

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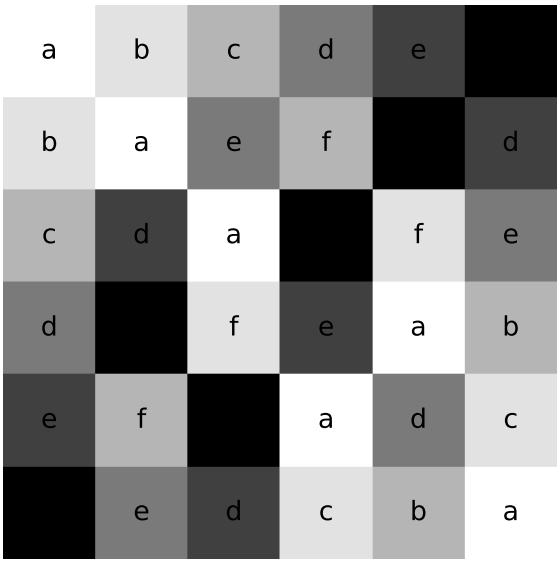

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```
[x0 x1 x2 x3 x4 x5]
[x1 x0 x3 x2 x5 x4]
[x2 x4 x0 x5 x1 x3]
[x3 x5 x1 x4 x0 x2]
[x4 x2 x5 x0 x3 x1]
[x5 x3 x4 x1 x2 x0]
sage: T.column_keys()[2]*T.column_keys()[2] == T.column_keys()[0]
True
```

row_keys()

Returns a tuple of the elements used to build the table.

Note: `column_keys` and `row_keys` are identical. Both list the elements in the order used to label the table.

OUTPUT:

The elements of the algebraic structure used to build the table, as a list. But most importantly, elements are present in the list in the order which they appear in the table's column headings.

EXAMPLES:

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=AlternatingGroup(3)
sage: T=OperationTable(G, operator.mul)
sage: T.column_keys()
((), (1,2,3), (1,3,2))
```

set_print_symbols(*ascii*, *latex*)

Set the symbols used for text and LaTeX printing of operation tables.

INPUT:

- `ascii` - a single character for text table

- `latex` - a string to represent an operation in LaTeX math mode. Note the need for double-backslashes to escape properly.

EXAMPLES:

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=AlternatingGroup(3)
sage: T=OperationTable(G, operator.mul)
sage: T.set_print_symbols('@', '\\times')
sage: T
@  a b c
 +------
a| a b c
b| b c a
c| c a b
sage: T._latex_()
'{\\setlength{\\arraycolsep}{2ex}\n\\begin{array}{r|*{3}{r}}\n\\multicolumn{1}
↪{c|}{\\times}&a&b&c\\\\\\hline\n{}a&a&b&c\\\\\n{}b&b&c&a\\\\\n{}c&c&a&b\\\\\n\
↪\end{array}}'
```

table()

Returns the table as a list of lists, using integers to reference the elements.

OUTPUT: The rows of the table, as a list of rows, each row being a list of integer entries. The integers correspond to the order of the elements in the headings of the table and the order of the output of the `list()` method.

EXAMPLES:

```
sage: from sage.matrix.operation_table import OperationTable
sage: C=CyclicPermutationGroup(3)
sage: T=OperationTable(C, operator.mul)
sage: T.table()
[[0, 1, 2], [1, 2, 0], [2, 0, 1]]
```

translation()

Returns a dictionary associating names with elements.

OUTPUT: A dictionary whose keys are strings used as names for entries of the table and values that are the actual elements of the algebraic structure.

EXAMPLES:

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=AlternatingGroup(3)
sage: T=OperationTable(G, operator.mul, names=['p','q','r'])
sage: T.translation()
{'p': (), 'q': (1,2,3), 'r': (1,3,2)}
```

CHAPTER

THIRTYFIVE

ACTIONS USED BY THE COERCION MODEL FOR MATRIX AND VECTOR MULTIPLICATIONS

Warning: The class *MatrixMulAction* and its descendants extends the class `Action`. As a consequence objects from these classes only keep weak references to the underlying sets which are acted upon. This decision was made in github issue #715 in order to allow garbage collection within the coercion framework, where actions are mainly used, and avoid memory leaks.

To ensure that the underlying set of such an object does not get garbage collected, it is sufficient to explicitly create a strong reference to it before creating the action.

```
sage: MSQ = MatrixSpace(QQ, 2)
sage: MSZ = MatrixSpace(ZZ['x'], 2)
sage: A = MSQ.get_action(MSZ)
sage: A
Left action by Full MatrixSpace of 2 by 2 dense matrices over Rational Field on Full␣
↪MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x over␣
↪Integer Ring
sage: import gc
sage: _ = gc.collect()
sage: A
Left action by Full MatrixSpace of 2 by 2 dense matrices over Rational Field on Full␣
↪MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x over␣
↪Integer Ring
```

Note: The *MatrixSpace()* function caches the objects it creates. Therefore, the underlying set `MSZ` in the above example will not be garbage collected, even if it is not strongly ref'ed. Nonetheless, there is no guarantee that the set that is acted upon will always be cached in such a way, so that following the above example is good practice.

EXAMPLES:

An action requires a common parent for the base rings, so the following doesn't work (see github issue #17859):

```
sage: vector(QQ, [1]) * matrix(Zmod(2), [[1]])
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'Vector space of
dimension 1 over Rational Field' and 'Full MatrixSpace of 1 by 1
dense matrices over Ring of integers modulo 2'
```

AUTHOR:

- Robert Bradshaw (2007-09): Initial version.

class sage.matrix.action.**MatrixMatrixAction**

Bases: *MatrixMulAction*

Action of a matrix on another matrix.

This is always implemented as a left action.

EXAMPLES:

By github issue #715, there only is a weak reference on the underlying set, so that it can be garbage collected if only the action itself is explicitly referred to. Hence, we first assign the involved matrix spaces to a variable:

```
sage: R.<x> = ZZ[]
sage: MSR = MatrixSpace(R, 3, 3)
sage: MSQ = MatrixSpace(QQ, 3, 2)
sage: from sage.matrix.action import MatrixMatrixAction
sage: A = MatrixMatrixAction(MSR, MSQ); A
Left action by Full MatrixSpace of 3 by 3 dense matrices over Univariate Polynomial␣
↪Ring in x over Integer Ring on Full MatrixSpace of 3 by 2 dense matrices over␣
↪Rational Field
sage: A.codomain()
Full MatrixSpace of 3 by 2 dense matrices over Univariate Polynomial Ring in x over␣
↪Rational Field
sage: A(matrix(R, 3, 3, x), matrix(QQ, 3, 2, range(6)))
[  0   x]
[2*x 3*x]
[4*x 5*x]
```

Note: The *MatrixSpace()* function caches the object it creates. Therefore, the underlying set MSZ in the above example will not be garbage collected, even if it is not strongly ref'ed. Nonetheless, there is no guarantee that the set that is acted upon will always be cached in such a way, so that following the above example is good practice.

class sage.matrix.action.**MatrixMulAction**

Bases: Action

Abstract base class for a matrix space acting on something.

EXAMPLES:

```
sage: MSQ = MatrixSpace(QQ, 2)
sage: MSZ = MatrixSpace(ZZ['x'], 2)
sage: A = MSQ.get_action(MSZ); A
Left action by Full MatrixSpace of 2 by 2 dense matrices over Rational Field on␣
↪Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x␣
↪over Integer Ring
sage: A.actor()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: A.domain()
Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x over␣
↪Integer Ring
sage: A.codomain()
```

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```
Full MatrixSpace of 2 by 2 dense matrices over Univariate Polynomial Ring in x over␣
↪Rational Field
```

codomain()

class sage.matrix.action.**MatrixPolymapAction**

Bases: *MatrixMulAction*

Left action of a matrix on a scheme polynomial morphism

class sage.matrix.action.**MatrixSchemePointAction**

Bases: *MatrixMulAction*

Action class for left multiplication of schemes points by matrices.

class sage.matrix.action.**MatrixVectorAction**

Bases: *MatrixMulAction*

Left action of a matrix on a vector

class sage.matrix.action.**PolymapMatrixAction**

Bases: *MatrixMulAction*

Right action of a matrix on a scheme polynomial morphism

class sage.matrix.action.**VectorMatrixAction**

Bases: *MatrixMulAction*

Right action of a matrix on a vector

CHAPTER

THIRTYSIX

FUNCTIONS FOR CHANGING THE BASE RING OF MATRICES QUICKLY

sage.matrix.change_ring.**integer_to_real_double_dense**(*A*)

Fast conversion of a matrix over the integers to a matrix with real double entries.

INPUT:

A – a dense matrix over the integers

OUTPUT:

– a dense real double matrix

EXAMPLES:

```
sage: a = matrix(ZZ,2,3,[-2,-5,3,4,8,10303398304893499 08])
sage: a.change_ring(RDF)
[                  -2.0                  -5.0                   3.0]
[                   4.0                   8.0 1.0303398304893499e+18]
sage: import sage.matrix.change_ring
sage: sage.matrix.change_ring.integer_to_real_double_dense(a)
[                  -2.0                  -5.0                   3.0]
[                   4.0                   8.0 1.0303398304893499e+18]
```

CHAPTER

THIRTYSEVEN

ECHELON MATRICES OVER FINITE FIELDS.

sage.matrix.echelon_matrix.**reduced_echelon_matrix_iterator**(*K*, *k*, *n*, *sparse=False*, *copy=True*, *set_immutable=False*)

An iterator over (k, n) reduced echelon matrices over the finite field K.

INPUT:

- `K` – a finite field
- `k` – number of rows (or the size of the subspace)
- `n` – number of columns (or the dimension of the ambient space)
- `sparse` – boolean (default is `False`)
- `copy` – boolean. If set to `False` then iterator yields the same matrix over and over (but with different entries). Default is `True` which is safer but might be slower.
- `set_immutable` – boolean. If set to `True` then the output matrices are immutable. This option automatically turns `copy` into `True`.

Note: We ensure that the iteration order is so that all matrices with given pivot columns are generated consecutively. Furthermore, the order in which the pivot columns appear is lexicographic.

It would be faster to generate the pivots columns following a Gray code. There would be only one pivot changing at a time, avoiding the possibly expensive `m0.__copy__()`. However that would modify the generation order some functions depend upon.

EXAMPLES:

```
sage: from sage.matrix.echelon_matrix import reduced_echelon_matrix_iterator
sage: it = reduced_echelon_matrix_iterator(GF(2),2,3)
sage: for m in it:
....:     print(m)
....:     print(m.pivots())
....:     print("********")
[1 0 0]
[0 1 0]
(0, 1)
********
[1 0 0]
[0 1 1]
(0, 1)
********
```

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```
[1 0 1]
[0 1 0]
(0, 1)
*******
[1 0 1]
[0 1 1]
(0, 1)
*******
[1 0 0]
[0 0 1]
(0, 2)
*******
[1 1 0]
[0 0 1]
(0, 2)
*******
[0 1 0]
[0 0 1]
(1, 2)
*******
```

CHAPTER

THIRTYEIGHT

MISCELLANEOUS MATRIX FUNCTIONS

sage.matrix.matrix_misc.**permanental_minor_polynomial**(*A*, *permanent_only=False*, *var='t'*, *prec=None*)

Return the polynomial of the sums of permanental minors of `A`.

INPUT:

- A – a matrix
- $permanent_only$ – if True, return only the permanent of A
- var – name of the polynomial variable
- $prec$ – if prec is not None, truncate the polynomial at precision $prec$

The polynomial of the sums of permanental minors is

$$\sum_{i=0}^{min(nrows,ncols)} p_i(A)x^i$$

where $p_i(A)$ is the i-th permanental minor of A (that can also be obtained through the method *permanental_minor()* via `A.permanental_minor(i)`).

The algorithm implemented by that function has been developed by P. Butera and M. Pernici, see [BP2015]. Its complexity is $O(2^n m^2 n)$ where m and n are the number of rows and columns of A. Moreover, if A is a banded matrix with width w, that is $A_{ij} = 0$ for $|i - j| > w$ and $w < n/2$, then the complexity of the algorithm is $O(4^w(w + 1)n^2)$.

INPUT:

- `A` – matrix
- `permanent_only` – optional boolean. If `True`, only the permanent is computed (might be faster).
- `var` – a variable name

EXAMPLES:

```
sage: from sage.matrix.matrix_misc import permanental_minor_polynomial
sage: m = matrix([[1,1],[1,2]])
sage: permanental_minor_polynomial(m)
3*t^2 + 5*t + 1
sage: permanental_minor_polynomial(m, permanent_only=True)
3
sage: permanental_minor_polynomial(m, prec=2)
5*t + 1
```

```
sage: M = MatrixSpace(ZZ,4,4)
sage: A = M([1,0,1,0,1,0,1,0,1,0,10,10,1,0,1,1])
sage: permanental_minor_polynomial(A)
84*t^3 + 114*t^2 + 28*t + 1
sage: [A.permanental_minor(i) for i in range(5)]
[1, 28, 114, 84, 0]
```

An example over $\mathbf{Q}$:

```
sage: M = MatrixSpace(QQ,2,2)
sage: A = M([1/5,2/7,3/2,4/5])
sage: permanental_minor_polynomial(A, True)
103/175
```

An example with polynomial coefficients:

```
sage: R.<a> = PolynomialRing(ZZ)
sage: A = MatrixSpace(R,2)([[a,1], [a,a+1]])
sage: permanental_minor_polynomial(A, True)
a^2 + 2*a
```

A usage of the `var` argument:

```
sage: m = matrix(ZZ,4,[0,1,2,3,1,2,3,0,2,3,0,1,3,0,1,2])
sage: permanental_minor_polynomial(m, var='x')
164*x^4 + 384*x^3 + 172*x^2 + 24*x + 1
```

ALGORITHM:

The permanent $perm(A)$ of a $n \times n$ matrix A is the coefficient of the $x_1 x_2 \dots x_n$ monomial in

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij} x_j \right)$$

Evaluating this product one can neglect x_i^2, that is x_i can be considered to be nilpotent of order 2.

To formalize this procedure, consider the algebra $R = K[\eta_1, \eta_2, \dots, \eta_n]$ where the η_i are commuting, nilpotent of order 2 (i.e. $\eta_i^2 = 0$). Formally it is the quotient ring of the polynomial ring in $\eta_1, \eta_2, \dots, \eta_n$ quotiented by the ideal generated by the η_i^2.

We will mostly consider the ring $R[t]$ of polynomials over R. We denote a generic element of $R[t]$ by $p(\eta_1, \dots, \eta_n)$ or $p(\eta_{i_1}, \dots, \eta_{i_k})$ if we want to emphasize that some monomials in the η_i are missing.

Introduce an "integration" operation $\langle p \rangle$ over R and $R[t]$ consisting in the sum of the coefficients of the non-vanishing monomials in η_i (i.e. the result of setting all variables η_i to 1). Let us emphasize that this is *not* a morphism of algebras as $\langle \eta_1 \rangle^2 = 1$ while $\langle \eta_1^2 \rangle = 0$!

Let us consider an example of computation. Let $p_1 = 1 + t\eta_1 + t\eta_2$ and $p_2 = 1 + t\eta_1 + t\eta_3$. Then

$$p_1 p_2 = 1 + 2t\eta_1 + t(\eta_2 + \eta_3) + t^2(\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3)$$

and

$$\langle p_1 p_2 \rangle = 1 + 4t + 3t^2$$

In this formalism, the permanent is just

$$perm(A) = \langle \prod_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\eta_j \rangle$$

A useful property of $\langle . \rangle$ which makes this algorithm efficient for band matrices is the following: let $p_1(\eta_1, \ldots, \eta_n)$ and $p_2(\eta_j, \ldots, \eta_n)$ be polynomials in $R[t]$ where $j \geq 1$. Then one has

$$\langle p_1(\eta_1, \ldots, \eta_n) p_2 \rangle = \langle p_1(1, \ldots, 1, \eta_j, \ldots, \eta_n) p_2 \rangle$$

where $\eta_1, .., \eta_{j-1}$ are replaced by 1 in p_1. Informally, we can "integrate" these variables *before* performing the product. More generally, if a monomial η_i is missing in one of the terms of a product of two terms, then it can be integrated in the other term.

Now let us consider an $m \times n$ matrix with $m \leq n$. The *sum of permanental `k`-minors of `A`* is

$$perm(A, k) = \sum_{r,c} perm(A_{r,c})$$

where the sum is over the k-subsets r of rows and k-subsets c of columns and $A_{r,c}$ is the submatrix obtained from A by keeping only the rows r and columns c. Of course $perm(A, \min(m, n)) = perm(A)$ and note that $perm(A, 1)$ is just the sum of all entries of the matrix.

The generating function of these sums of permanental minors is

$$g(t) = \left\langle \prod_{i=1}^{m} \left(1 + t \sum_{j=1}^{n} A_{ij}\eta_j \right) \right\rangle$$

In fact the t^k coefficient of $g(t)$ corresponds to choosing k rows of A; η_i is associated to the i-th column; nilpotency avoids having twice the same column in a product of A's.

For more details, see the article [BP2015].

From a technical point of view, the product in $K[\eta_1, \ldots, \eta_n][t]$ is implemented as a subroutine in *prm_mul()*. The indices of the rows and columns actually start at 0, so the variables are $\eta_0, \ldots, \eta_{n-1}$. Polynomials are represented in dictionary form: to a variable η_i is associated the key 2^i (or in Python `1 << i`). The keys associated to products are obtained by considering the development in base 2: to the monomial $\eta_{i_1} \ldots \eta_{i_k}$ is associated the key $2^{i_1} + \ldots + 2^{i_k}$. So the product $\eta_1\eta_2$ corresponds to the key $6 = (110)_2$ while $\eta_0\eta_3$ has key $9 = (1001)_2$. In particular all operations on monomials are implemented via bitwise operations on the keys.

`sage.matrix.matrix_misc.`**`prm_mul`**(*p1*, *p2*, *mask_free*, *prec*)

Return the product of `p1` and `p2`, putting free variables in `mask_free` to 1.

This function is mainly use as a subroutine of *permanental_minor_polynomial()*.

INPUT:

- $p1, p2$ – polynomials as dictionaries
- `mask_free` – an integer mask that give the list of free variables (the i-th variable is free if the i-th bit of `mask_free` is 1)
- `prec` – if `prec` is not `None`, truncate the product at precision `prec`

EXAMPLES:

```
sage: from sage.matrix.matrix_misc import prm_mul
sage: t = polygen(ZZ, 't')
sage: p1 = {0: 1, 1: t, 4: t}
sage: p2 = {0: 1, 1: t, 2: t}
sage: prm_mul(p1, p2, 1, None)
{0: 2*t + 1, 2: t^2 + t, 4: t^2 + t, 6: t^2}
```

sage.matrix.matrix_misc.**row_iterator**(*A*)

CHAPTER

THIRTYNINE

MATRIX WINDOWS

class sage.matrix.matrix_window.**MatrixWindow**

Bases: object

add(*A*)

add_prod(*A*, *B*)

echelon_in_place()

Calculate the echelon form of this matrix, returning the list of pivot columns

element_is_zero(*i*, *j*)

get_unsafe(*i*, *j*)

matrix()

Returns the underlying matrix that this window is a view of.

matrix_window(*row*, *col*, *n_rows*, *n_cols*)

Returns a matrix window relative to this window of the underlying matrix.

ncols()

new_empty_window(*nrows*, *ncols*)

new_matrix_window(*matrix*, *row*, *col*, *n_rows*, *n_cols*)

This method is here only to provide a fast cdef way of constructing new matrix windows. The only implicit assumption is that self._matrix and matrix are over the same base ring (so share the zero).

nrows()

set(*src*)

set_to(*A*)

Change self, making it equal A.

set_to_diff(*A*, *B*)

set_to_prod(*A*, *B*)

set_to_sum(*A*, *B*)

set_to_zero()

set_unsafe(*i*, *j*, *x*)

subtract(*A*)

subtract_prod(*A*, *B*)

swap_rows(*a*, *b*)

to_matrix()

Returns an actual matrix object representing this view.

CHAPTER

FORTY

MISC MATRIX ALGORITHMS

Code goes here mainly when it needs access to the internal structure of several classes, and we want to avoid circular cimports.

NOTE: The whole problem of avoiding circular imports – the reason for existence of this file – is now a non-issue, since some bugs in Cython were fixed. Probably all this code should be moved into the relevant classes and this file deleted.

sage.matrix.misc.**cmp_pivots**(*x*, *y*)

Compare two sequences of pivot columns.

If x is shorter than y, return -1, i.e., x < y, "not as good". If x is longer than y, then x > y, so "better" and return +1. If the length is the same, then x is better, i.e., x > y if the entries of x are correspondingly <= those of y with one being strictly less.

INPUT:

- x, y – lists or tuples of integers

EXAMPLES:

We illustrate each of the above comparisons.

```
sage: sage.matrix.misc.cmp_pivots([1,2,3], [4,5,6,7])
-1
sage: sage.matrix.misc.cmp_pivots([1,2,3,5], [4,5,6])
1
sage: sage.matrix.misc.cmp_pivots([1,2,4], [1,2,3])
-1
sage: sage.matrix.misc.cmp_pivots([1,2,3], [1,2,3])
0
sage: sage.matrix.misc.cmp_pivots([1,2,3], [1,2,4])
1
```

sage.matrix.misc.**hadamard_row_bound_mpfr**(*A*)

Given a matrix A with entries that coerce to RR, compute the row Hadamard bound on the determinant.

INPUT:

A – a matrix over RR

OUTPUT:

integer – an integer n such that the absolute value of the determinant of this matrix is at most 10^n.

EXAMPLES:

We create a very large matrix, compute the row Hadamard bound, and also compute the row Hadamard bound of the transpose, which happens to be sharp.

```
sage: a = matrix(ZZ, 2, [2^10000,3^10000,2^50,3^19292])
sage: import sage.matrix.misc
sage: sage.matrix.misc.hadamard_row_bound_mpfr(a.change_ring(RR))
13976
sage: len(str(a.det()))
12215
sage: sage.matrix.misc.hadamard_row_bound_mpfr(a.transpose().change_ring(RR))
12215
```

Note that in the above example using RDF would overflow:

```
sage: b = a.change_ring(RDF)
sage: b._hadamard_row_bound()
Traceback (most recent call last):
...
OverflowError: cannot convert float infinity to integer
```

sage.matrix.misc.**matrix_integer_dense_rational_reconstruction**(*A*, *N*)

Given a matrix over the integers and an integer modulus, do rational reconstruction on all entries of the matrix, viewed as numbers mod N. This is done efficiently by assuming there is a large common factor dividing the denominators.

INPUT:

A – matrix N – an integer

EXAMPLES:

```
sage: B = ((matrix(ZZ, 3,4, [1,2,3,-4,7,2,18,3,4,3,4,5])/3)%500).change_ring(ZZ)
sage: sage.matrix.misc.matrix_integer_dense_rational_reconstruction(B, 500)
[ 1/3  2/3    1 -4/3]
[ 7/3  2/3    6    1]
[ 4/3    1  4/3  5/3]
```

sage.matrix.misc.**matrix_integer_sparse_rational_reconstruction**(*A*, *N*)

Given a sparse matrix over the integers and an integer modulus, do rational reconstruction on all entries of the matrix, viewed as numbers mod N.

EXAMPLES:

```
sage: A = matrix(ZZ, 3, 4, [(1/3)%500, 2, 3, (-4)%500, 7, 2, 2, 3, 4, 3, 4, (5/7)
↪%500], sparse=True)
sage: sage.matrix.misc.matrix_integer_sparse_rational_reconstruction(A, 500)
[1/3   2   3  -4]
[  7   2   2   3]
[  4   3   4 5/7]
```

sage.matrix.misc.**matrix_rational_echelon_form_multimodular**(*self*, *height_guess=None*, *proof=None*)

Returns reduced row-echelon form using a multi-modular algorithm. Does not change self.

REFERENCE: Chapter 7 of Stein's "Explicitly Computing Modular Forms".

INPUT:

- height_guess – integer or None

- proof – boolean or None (default: None, see proof.linear_algebra or sage.structure.proof). Note that the global Sage default is proof=True

OUTPUT: a pair consisting of a matrix in echelon form and a tuple of pivot positions.

ALGORITHM:

The following is a modular algorithm for computing the echelon form. Define the height of a matrix to be the max of the absolute values of the entries.

Given Matrix A with n columns (self).

0. Rescale input matrix A to have integer entries. This does not change echelon form and makes reduction modulo lots of primes significantly easier if there were denominators. Henceforth we assume A has integer entries.
1. Let c be a guess for the height of the echelon form. E.g., c=1000, e.g., if matrix is very sparse and application is to computing modular symbols.
2. Let M = n * c * H(A) + 1, where n is the number of columns of A.
3. List primes p_1, p_2, ..., such that the product of the p_i is at least M.
4. Try to compute the rational reconstruction CRT echelon form of A mod the product of the p_i. If rational reconstruction fails, compute 1 more echelon forms mod the next prime, and attempt again. Make sure to keep the result of CRT on the primes from before, so we don't have to do that computation again. Let E be this matrix.
5. Compute the denominator d of E. Attempt to prove that result is correct by checking that

   H(d*E)*ncols(A)*H(A) < (prod of reduction primes)

   where H denotes the height. If this fails, do step 4 with a few more primes.

EXAMPLES:

```
sage: A = matrix(QQ, 3, 7, [1..21])
sage: from sage.matrix.misc import matrix_rational_echelon_form_multimodular
sage: E, pivots = matrix_rational_echelon_form_multimodular(A)
sage: E
[ 1  0 -1 -2 -3 -4 -5]
[ 0  1  2  3  4  5  6]
[ 0  0  0  0  0  0  0]
sage: pivots
(0, 1)

sage: A = matrix(QQ, 3, 4, [0,0] + [1..9] + [-1/2^20])
sage: E, pivots = matrix_rational_echelon_form_multimodular(A)
sage: E
[                1                 0                 0 -10485761/1048576]
[                0                 1                 0  27262979/4194304]
[                0                 0                 1                 2]
sage: pivots
(0, 1, 2)

sage: A.echelon_form()
[                1                 0                 0 -10485761/1048576]
[                0                 1                 0  27262979/4194304]
[                0                 0                 1                 2]
```

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```
sage: A.pivots()
(0, 1, 2)
```

CHAPTER

FORTYONE

CALCULATE SYMPLECTIC BASES FOR MATRICES OVER FIELDS AND THE INTEGERS.

This module finds a symplectic basis for an anti-symmetric, alternating matrix M defined over a field or the integers.

Anti-symmetric means that $M = -M^t$, where M^t denotes the transpose of M. Alternating means that the diagonal of M is identically zero.

A symplectic basis is a basis of the form $e_1, \ldots, e_j, f_1, \ldots, f_j, z_1, \ldots, z_k$ such that

- $z_i M v^t$ = 0 for all vectors v;
- $e_i M {e_j}^t = 0$ for all i, j;
- $f_i M {f_j}^t = 0$ for all i, j;
- $e_i M {f_j}^t = 0$ for all i not equal j;

and such that the non-zero terms

- $e_i M {f_i}^t$ are "as nice as possible": 1 over fields, or integers satisfying divisibility properties otherwise.

REFERENCES:

Bourbaki gives a nice proof that can be made constructive but is not efficient (see Section 5, Number 1, Theorem 1, page 79):

Bourbaki, N. Elements of Mathematics, Algebra III, Springer Verlag 2007.

Kuperberg gives a more efficient and constructive exposition (see Theorem 18).

Kuperberg, Greg. Kasteleyn Cokernels. Electr. J. Comb. 9(1), 2002.

Todo: The routine over the integers applies over general principal ideal domains.

Warning: This code is not a good candidate for conversion to Cython. The majority of the execution time is spent adding multiples of columns and rows, which is already fast. It would be better to devise a better algorithm, perhaps modular or based on a fast `smith_form` implementation.

AUTHOR:

- Nick Alexander: initial implementation
- David Loeffler (2008-12-08): changed conventions for consistency with smith_form

sage.matrix.symplectic_basis.**symplectic_basis_over_ZZ**(M)

Find a symplectic basis for an anti-symmetric, alternating matrix M defined over the integers.

Returns a pair (F, C) such that the rows of C form a symplectic basis for M and F = C * M * C.transpose().

Anti-symmetric means that $M = -M^t$. Alternating means that the diagonal of M is identically zero.

A symplectic basis is a basis of the form $e_1, \ldots, e_j, f_1, \ldots, f_j, z_1, \ldots, z_k$ such that

- $z_i M v^t = 0$ for all vectors v;
- $e_i M {e_j}^t = 0$ for all i, j;
- $f_i M {f_j}^t = 0$ for all i, j;
- $e_i M {f_i}^t = d_i$ for all i, where d_i are positive integers such that $d_i | d_{i+1}$ for all i;
- $e_i M {f_j}^t = 0$ for all i not equal j.

The ordering for the factors $d_i | d_{i+1}$ and for the placement of zeroes was chosen to agree with the output of `smith_form`.

See the examples for a pictorial description of such a basis.

EXAMPLES:

```
sage: from sage.matrix.symplectic_basis import symplectic_basis_over_ZZ
```

An example which does not have full rank:

```
sage: E = matrix(ZZ, 4, 4, [0, 16, 0, 2, -16, 0, 0, -4, 0, 0, 0, 0, -2, 4, 0, 0]); E
[  0  16   0   2]
[-16   0   0  -4]
[  0   0   0   0]
[ -2   4   0   0]
sage: F, C = symplectic_basis_over_ZZ(E)
sage: F
[ 0  2  0  0]
[-2  0  0  0]
[ 0  0  0  0]
[ 0  0  0  0]
sage: C * E * C.transpose() == F
True
```

A larger example:

```
sage: E = matrix(ZZ, 8, 8, [0, 25, 0, 0, -37, -3, 2, -5, -25, 0, 1, -5, -54, -3, 3,␣
↪3, 0, -1, 0, 7, 0, -4, -20, 0, 0, 5, -7, 0, 0, 14, 0, -3, 37, 54, 0, 0, 0, 2, 3, -
↪12, 3, 3, 4, -14, -2, 0, -3, 2, -2, -3, 20, 0, -3, 3, 0, -2, 5, -3, 0, 3, 12, -2,␣
↪2, 0]); E
[  0  25   0   0 -37  -3   2  -5]
[-25   0   1  -5 -54  -3   3   3]
[  0  -1   0   7   0  -4 -20   0]
[  0   5  -7   0   0  14   0  -3]
[ 37  54   0   0   0   2   3 -12]
[  3   3   4 -14  -2   0  -3   2]
[ -2  -3  20   0  -3   3   0  -2]
[  5  -3   0   3  12  -2   2   0]
sage: F, C = symplectic_basis_over_ZZ(E)
```

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```
sage: F
[     0      0      0      0      1      0      0      0]
[     0      0      0      0      0      1      0      0]
[     0      0      0      0      0      0      1      0]
[     0      0      0      0      0      0      0  20191]
[    -1      0      0      0      0      0      0      0]
[     0     -1      0      0      0      0      0      0]
[     0      0     -1      0      0      0      0      0]
[     0      0      0 -20191      0      0      0      0]
sage: F == C * E * C.transpose()
True
sage: E.smith_form()[0]
[    1     0     0     0     0     0     0     0]
[    0     1     0     0     0     0     0     0]
[    0     0     1     0     0     0     0     0]
[    0     0     0     1     0     0     0     0]
[    0     0     0     0     1     0     0     0]
[    0     0     0     0     0     1     0     0]
[    0     0     0     0     0     0 20191     0]
[    0     0     0     0     0     0     0 20191]
```

An odd dimensional example:

```
sage: E = matrix(ZZ, 5, 5, [0, 14, 0, -8, -2, -14, 0, -3, -11, 4, 0, 3, 0, 0, 0, 8,␣
↪11, 0, 0, 8, 2, -4, 0, -8, 0]); E
[  0  14   0  -8  -2]
[-14   0  -3 -11   4]
[  0   3   0   0   0]
[  8  11   0   0   8]
[  2  -4   0  -8   0]
sage: F, C = symplectic_basis_over_ZZ(E)
sage: F
[ 0  0  1  0  0]
[ 0  0  0  2  0]
[-1  0  0  0  0]
[ 0 -2  0  0  0]
[ 0  0  0  0  0]
sage: F == C * E * C.transpose()
True
sage: E.smith_form()[0]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 2 0 0]
[0 0 0 2 0]
[0 0 0 0 0]

sage: F.parent()
Full MatrixSpace of 5 by 5 dense matrices over Integer Ring
sage: C.parent()
Full MatrixSpace of 5 by 5 dense matrices over Integer Ring
```

sage.matrix.symplectic_basis.**symplectic_basis_over_field**(*M*)

Find a symplectic basis for an anti-symmetric, alternating matrix M defined over a field.

Returns a pair (F, C) such that the rows of C form a symplectic basis for M and `F = C * M * C.transpose()`.

Anti-symmetric means that $M = -M^t$. Alternating means that the diagonal of M is identically zero.

A symplectic basis is a basis of the form $e_1, \ldots, e_j, f_1, \ldots f_j, z_1, \ldots, z_k$ such that

- $z_i M v^t = 0$ for all vectors v;
- $e_i M {e_j}^t = 0$ for all i, j;
- $f_i M {f_j}^t = 0$ for all i, j;
- $e_i M {f_i}^t = 1$ for all i;
- $e_i M {f_j}^t = 0$ for all i not equal j.

See the examples for a pictorial description of such a basis.

EXAMPLES:

```
sage: from sage.matrix.symplectic_basis import symplectic_basis_over_field
```

A full rank exact example:

```
sage: E = matrix(QQ, 8, 8, [0, -1/2, -2, 1/2, 2, 0, -2, 1, 1/2, 0, -1, -3, 0, 2, 5/
↪2, -3, 2, 1, 0, 3/2, -1, 0, -1, -2, -1/2, 3, -3/2, 0, 1, 3/2, -1/2, -1/2, -2, 0,␣
↪1, -1, 0, 0, 1, -1, 0, -2, 0, -3/2, 0, 0, 1/2, -2, 2, -5/2, 1, 1/2, -1, -1/2, 0, -
↪1, -1, 3, 2, 1/2, 1, 2, 1, 0]); E
[   0 -1/2   -2  1/2    2    0   -2    1]
[ 1/2    0   -1   -3    0    2  5/2   -3]
[   2    1    0  3/2   -1    0   -1   -2]
[-1/2    3 -3/2    0    1  3/2 -1/2 -1/2]
[  -2    0    1   -1    0    0    1   -1]
[   0   -2    0 -3/2    0    0  1/2   -2]
[   2 -5/2    1  1/2   -1 -1/2    0   -1]
[  -1    3    2  1/2    1    2    1    0]
sage: F, C = symplectic_basis_over_field(E); F
[ 0  0  0  0  1  0  0  0]
[ 0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  1  0]
[ 0  0  0  0  0  0  0  1]
[-1  0  0  0  0  0  0  0]
[ 0 -1  0  0  0  0  0  0]
[ 0  0 -1  0  0  0  0  0]
[ 0  0  0 -1  0  0  0  0]
sage: F == C * E * C.transpose()
True
```

An example over a finite field:

```
sage: E = matrix(GF(7), 8, 8, [0, -1/2, -2, 1/2, 2, 0, -2, 1, 1/2, 0, -1, -3, 0, 2,␣
↪5/2, -3, 2, 1, 0, 3/2, -1, 0, -1, -2, -1/2, 3, -3/2, 0, 1, 3/2, -1/2, -1/2, -2, 0,
↪ 1, -1, 0, 0, 1, -1, 0, -2, 0, -3/2, 0, 0, 1/2, -2, 2, -5/2, 1, 1/2, -1, -1/2, 0,␣
↪-1, -1, 3, 2, 1/2, 1, 2, 1, 0]); E
[0 3 5 4 2 0 5 1]
[4 0 6 4 0 2 6 4]
[2 1 0 5 6 0 6 5]
[3 3 2 0 1 5 3 3]
```

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```
[5 0 1 6 0 0 1 6]
[0 5 0 2 0 0 4 5]
[2 1 1 4 6 3 0 6]
[6 3 2 4 1 2 1 0]
sage: F, C = symplectic_basis_over_field(E); F
[0 0 0 0 1 0 0 0]
[0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 1]
[6 0 0 0 0 0 0 0]
[0 6 0 0 0 0 0 0]
[0 0 6 0 0 0 0 0]
[0 0 0 6 0 0 0 0]
sage: F == C * E * C.transpose()
True
```

The tricky case of characteristic 2:

```
sage: E = matrix(GF(2), 8, 8, [0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
↪ 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1,␣
↪0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0]); E
[0 0 1 1 0 1 0 1]
[0 0 0 0 0 0 0 0]
[1 0 0 0 0 0 1 1]
[1 0 0 0 0 0 0 1]
[0 0 0 0 0 1 1 0]
[1 0 0 0 1 0 1 1]
[0 0 1 0 1 1 0 0]
[1 0 1 1 0 1 0 0]
sage: F, C = symplectic_basis_over_field(E); F
[0 0 0 1 0 0 0 0]
[0 0 0 0 1 0 0 0]
[0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0]
sage: F == C * E * C.transpose()
True
```

An inexact example:

```
sage: E = matrix(RR, 8, 8, [0.000000000000000, 0.420674846479344, -0.
↪839702420666807, 0.658715385244413, 1.69467394825853, -1.14718543053828, 1.
↪03076138152950, -0.227739521708484, -0.420674846479344, 0.000000000000000, 0.
↪514381455379082, 0.282194064028260, -1.38977093018412, 0.278305070958958, -0.
↪0781320488361574, -0.496003664217833, 0.839702420666807, -0.514381455379082, 0.
↪000000000000000, -0.00618222322875384, -0.318386939149028, -0.0840205427053993, 1.
↪28202592892333, -0.512563654267693, -0.658715385244413, -0.282194064028260, 0.
↪00618222322875384, 0.000000000000000, 0.852525732369211, -0.356957405431611, -0.
↪699960114607661, 0.0260496330859998, -1.69467394825853, 1.38977093018412, 0.
```

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```
↪318386939149028, -0.852525732369211, 0.000000000000000, -0.836072521423577, 0.
↪450137632758469, -0.696145287292091, 1.14718543053828, -0.278305070958958, 0.
↪0840205427053993, 0.356957405431611, 0.836072521423577, 0.000000000000000, 0.
↪214878541347751, -1.20221688928379, -1.03076138152950, 0.0781320488361574, -1.
↪28202592892333, 0.699960114607661, -0.450137632758469, -0.214878541347751, 0.
↪000000000000000, 0.785074452163036, 0.227739521708484, 0.496003664217833, 0.
↪512563654267693, -0.0260496330859998, 0.696145287292091, 1.20221688928379, -0.
↪785074452163036, 0.000000000000000]); E
[   0.000000000000000    0.420674846479344   -0.839702420666807    0.
↪658715385244413     1.69467394825853    -1.14718543053828     1.03076138152950   -
↪0.227739521708484]
[  -0.420674846479344    0.000000000000000    0.514381455379082    0.
↪282194064028260    -1.38977093018412    0.278305070958958  -0.0781320488361574   -
↪0.496003664217833]
[   0.839702420666807   -0.514381455379082    0.000000000000000 -0.
↪00618222322875384   -0.318386939149028  -0.0840205427053993     1.28202592892333 ␣
↪ -0.512563654267693]
[  -0.658715385244413   -0.282194064028260  0.00618222322875384    0.
↪000000000000000    0.852525732369211   -0.356957405431611   -0.699960114607661  ␣
↪0.0260496330859998]
[   -1.69467394825853     1.38977093018412    0.318386939149028   -0.
↪852525732369211    0.000000000000000   -0.836072521423577    0.450137632758469   -
↪0.696145287292091]
[    1.14718543053828   -0.278305070958958   0.0840205427053993    0.
↪356957405431611    0.836072521423577    0.000000000000000    0.214878541347751   ␣
↪-1.20221688928379]
[   -1.03076138152950   0.0781320488361574    -1.28202592892333    0.
↪699960114607661   -0.450137632758469   -0.214878541347751    0.000000000000000   ␣
↪0.785074452163036]
[   0.227739521708484    0.496003664217833    0.512563654267693  -0.
↪0260496330859998    0.696145287292091     1.20221688928379   -0.785074452163036  ␣
↪ 0.000000000000000]
sage: F, C = symplectic_basis_over_field(E); F # random
[    0.000000000000000     0.000000000000000  2.22044604925031e-16 -2.
↪22044604925031e-16      1.00000000000000     0.000000000000000     0.
↪000000000000000 -3.33066907387547e-16]
[    0.000000000000000  8.14814392305203e-17 -1.66533453693773e-16 -1.
↪11022302462516e-16     0.000000000000000      1.00000000000000 -1.11022302462516e-
↪16     0.000000000000000]
[-5.27829526256056e-16 -2.40004077757759e-16  1.28373418199470e-16 -1.
↪11022302462516e-16     0.000000000000000 -3.15483812822081e-16      1.
↪00000000000000 -4.44089209850063e-16]
[ 1.31957381564014e-16  1.41622049084608e-16 -6.68515202578511e-17 -3.
↪95597468756028e-17 -4.85722573273506e-17 -5.32388011580111e-17 -1.31328455615552e-
↪16      1.00000000000000]
[    -1.00000000000000     0.000000000000000     0.000000000000000  4.
↪85722573273506e-17     0.000000000000000 -5.55111512312578e-17 -1.11022302462516e-
↪16  2.22044604925031e-16]
[    0.000000000000000     -1.00000000000000     0.000000000000000 -2.
↪77555756156289e-17  5.55111512312578e-17 -8.69223574327834e-17     0.
↪000000000000000 -4.44089209850063e-16]
[    0.000000000000000 -1.05042437087238e-17     -1.00000000000000  3.
```

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```
↪33066907387547e-16  1.11022302462516e-16 -1.18333563634309e-16  4.40064433050777e-
↪17  2.22044604925031e-16]
[ 5.27829526256056e-16  1.99901485752317e-16  1.65710718121313e-17     -1.
↪00000000000000 -2.22044604925031e-16  5.52150940090699e-16 -3.93560383111738e-16 ␣
↪1.01155762925061e-16]
sage: F == C * E * C.transpose()
True
sage: abs(F[0, 4] - 1) < 1e-10
True
sage: abs(F[4, 0] + 1) < 1e-10
True

sage: F.parent()
Full MatrixSpace of 8 by 8 dense matrices over Real Field with 53 bits of precision
sage: C.parent()
Full MatrixSpace of 8 by 8 dense matrices over Real Field with 53 bits of precision
```

CHAPTER

FORTYTWO

J-IDEALS OF MATRICES

Let B be an $n \times n$-matrix over a principal ideal domain D.

For an ideal J, the J-ideal of B is defined to be $N_J(B) = \{f \in D[X] \mid f(B) \in M_n(J)\}$.

For a prime element p of D and $t \geq 0$, a (p^t)-minimal polynomial of B is a monic polynomial $f \in N_{(p^t)}(B)$ of minimal degree.

This module computes these minimal polynomials.

Let p be a prime element of D. Then there is a finite set $\mathcal{S}_p$ of positive integers and monic polynomials ν_{ps} for $s \in \mathcal{S}_p$ such that for $t \geq 1$,

$$N_{(p^t)}(B) = \mu_B D[X] + p^t D[X] + \sum_{\substack{s \in \mathcal{S}_p \\ s \leq b(t)}} p^{\max\{0,t-s\}} \nu_{ps} D[X]$$

holds where $b(t) = \min\{r \in \mathcal{S}_p \mid r \geq s\}$. The degree of ν_{ps} is strictly increasing in $s \in \mathcal{S}_p$ and ν_{ps} is a (p^s)-minimal polynomial. If $t \leq \max \mathcal{S}_p$, then the summand $\mu_B D[X]$ can be omitted.

All computations are done by the class *ComputeMinimalPolynomials* where various intermediate results are cached. It provides the following methods:

- *p_minimal_polynomials()* computes $\mathcal{S}_p$ and the monic polynomials ν_{ps}.
- *null_ideal()* determines $N_{(p^t)}(B)$.
- *prime_candidates()* determines all primes p where $\mathcal{S}_p$ might be non-empty.
- *integer_valued_polynomials_generators()* determines the generators of the ring $\{f \in K[X] \mid f(B) \in M_n(D)\}$ of integer valued polynomials on B.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: C.prime_candidates()
[2, 3, 5]
sage: for t in range(4):
....:     print(C.null_ideal(2^t))
Principal ideal (1) of
    Univariate Polynomial Ring in x over Integer Ring
Ideal (2, x^2 + x) of
    Univariate Polynomial Ring in x over Integer Ring
Ideal (4, x^2 + 3*x + 2) of
    Univariate Polynomial Ring in x over Integer Ring
```

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```
Ideal (8, x^3 + x^2 - 12*x - 20, 2*x^2 + 6*x + 4) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.p_minimal_polynomials(2)
{2: x^2 + 3*x + 2}
sage: C.integer_valued_polynomials_generators()
(x^3 + x^2 - 12*x - 20, [1, 1/4*x^2 + 3/4*x + 1/2])
```

The last output means that

$$\{f \in \mathbf{Q}[X] \mid f(B) \in M_3(\mathbf{Z})\} = (x^3 + x^2 - 12x - 20)\mathbf{Q}[X] + \mathbf{Z}[X] + \frac{1}{4}(x^2 + 3x + 2)\mathbf{Z}[X].$$

Todo: Test code over PIDs other than ZZ.

This requires implementation of *frobenius()* over more general domains than ZZ.

Additionally, *lifting()* requires modification or a bug needs fixing, see AskSage Question 35555.

REFERENCES:

[Ris2016], [HR2016]

AUTHORS:

- Clemens Heuberger (2016)
- Roswitha Rissner (2016)

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42.1 Classes and Methods

class sage.matrix.compute_J_ideal.**ComputeMinimalPolynomials**(B)

Bases: SageObject

Create an object for computing (p^t)-minimal polynomials and J-ideals.

For an ideal J and a square matrix B over a principal ideal domain D, the J-ideal of B is defined to be $N_J(B) = \{f \in D[X] \mid f(B) \in M_n(J)\}$.

For a prime element p of D and $t \geq 0$, a (p^t)-minimal polynomial of B is a monic polynomial $f \in N_{(p^t)}(B)$ of minimal degree.

The characteristic polynomial of B is denoted by χ_B; n is the size of B.

INPUT:

- `B` – a square matrix over a principal ideal domain D

OUTPUT:

An object which allows to call *p_minimal_polynomials()*, *null_ideal()* and *integer_valued_polynomials_generators()*.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: C.prime_candidates()
[2, 3, 5]
sage: for t in range(4):
....:     print(C.null_ideal(2^t))
Principal ideal (1) of
    Univariate Polynomial Ring in x over Integer Ring
Ideal (2, x^2 + x) of
    Univariate Polynomial Ring in x over Integer Ring
Ideal (4, x^2 + 3*x + 2) of
    Univariate Polynomial Ring in x over Integer Ring
Ideal (8, x^3 + x^2 - 12*x - 20, 2*x^2 + 6*x + 4) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.p_minimal_polynomials(2)
{2: x^2 + 3*x + 2}
sage: C.integer_valued_polynomials_generators()
(x^3 + x^2 - 12*x - 20, [1, 1/4*x^2 + 3/4*x + 1/2])
```

current_nu(*p*, *t*, *pt_generators*, *prev_nu*)

Compute (p^t)-minimal polynomial of B.

INPUT:

- `p` – a prime element of D
- `t` – a positive integer
- `pt_generators` – a list $(g_1, \ldots, g_s)$ of polynomials in $D[X]$ such that $N_{(p^t)}(B) = (g_1, \ldots, g_s) + pN_{(p^{t-1})}(B)$
- `prev_nu` – a (p^{t-1})-minimal polynomial of B

OUTPUT:

A (p^t)-minimal polynomial of B.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: x = polygen(ZZ, 'x')
sage: nu_1 = x^2 + x
sage: generators_4 = [2*x^2 + 2*x, x^2 + 3*x + 2]
sage: C.current_nu(2, 2, generators_4, nu_1)
x^2 + 3*x + 2
```

ALGORITHM:

[HR2016], Algorithm 4.

find_monic_replacements(*p*, *t*, *pt_generators*, *prev_nu*)

Replace possibly non-monic generators of $N_{(p^t)}(B)$ by monic generators.

INPUT:

- `p` – a prime element of D

- `t` – a non-negative integer
- `pt_generators` – a list $(g_1, \ldots, g_s)$ of polynomials in $D[X]$ such that $N_{(p^t)}(B) = (g_1, \ldots, g_s) + pN_{(p^{t-1})}(B)$
- `prev_nu` – a (p^{t-1})-minimal polynomial of B

OUTPUT:

A list $(h_1, \ldots, h_r)$ of monic polynomials such that $N_{(p^t)}(B) = (h_1, \ldots, h_r) + pN_{(p^{t-1})}(B)$.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: x = polygen(ZZ, 'x')
sage: nu_1 = x^2 + x
sage: generators_4 = [2*x^2 + 2*x, x^2 + 3*x + 2]
sage: C.find_monic_replacements(2, 2, generators_4, nu_1)
[x^2 + 3*x + 2]
```

ALGORITHM:

[HR2016], Algorithms 2 and 3.

integer_valued_polynomials_generators()

Determine the generators of the ring of integer valued polynomials on B.

OUTPUT:

A pair `(mu_B, P)` where `P` is a list of polynomials in $K[X]$ such that

$$\{f \in K[X] \mid f(B) \in M_n(D)\} = \mu_B K[X] + \sum_{g \in P} gD[X]$$

where K denotes the fraction field of D.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: C.integer_valued_polynomials_generators()
(x^3 + x^2 - 12*x - 20, [1, 1/4*x^2 + 3/4*x + 1/2])
```

mccoy_column(*p*, *t*, *nu*)

Compute matrix for McCoy's criterion.

INPUT:

- `p` – a prime element in D
- `t` – a positive integer
- `nu` – a (p^t)-minimal polynomial of B

OUTPUT:

An $(n^2 + 1) \times 1$ matrix g with first entry `nu` such that $\begin{pmatrix} b & -\chi_B I \end{pmatrix} g \equiv 0 \pmod{p^t}$ where b consists of the entries of $\operatorname{adj}(X - B)$.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: x = polygen(ZZ, 'x')
sage: nu_4 = x^2 + 3*x + 2
sage: g = C.mccoy_column(2, 2, nu_4)
sage: b = matrix(9, 1, (x-B).adjugate().list())
sage: M = matrix.block([[b , -B.charpoly(x)*matrix.identity(9)]])
sage: (M*g % 4).is_zero()
True
```

ALGORITHM:

[HR2016], Algorithm 5.

null_ideal(*b=0*)

Return the (b)-ideal $N_{(b)}(B) = \{f \in D[X] \mid f(B) \in M_n(bD)\}$.

INPUT:

- b – an element of D (default: 0)

OUTPUT:

An ideal in $D[X]$.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: C.null_ideal()
Principal ideal (x^3 + x^2 - 12*x - 20) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.null_ideal(2)
Ideal (2, x^2 + x) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.null_ideal(4)
Ideal (4, x^2 + 3*x + 2) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.null_ideal(8)
Ideal (8, x^3 + x^2 - 12*x - 20, 2*x^2 + 6*x + 4) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.null_ideal(3)
Ideal (3, x^3 + x^2 - 12*x - 20) of
    Univariate Polynomial Ring in x over Integer Ring
sage: C.null_ideal(6)
Ideal (6, 2*x^3 + 2*x^2 - 24*x - 40, 3*x^2 + 3*x) of
    Univariate Polynomial Ring in x over Integer Ring
```

p_minimal_polynomials(*p*, *s_max=None*)

Compute (p^s)-minimal polynomials ν_s of B.

Compute a finite subset $\mathcal{S}$ of the positive integers and (p^s)-minimal polynomials ν_s for $s \in \mathcal{S}$.

For $0 < t \leq \max \mathcal{S}$, a (p^t)-minimal polynomial is given by ν_s where $s = \min\{r \in \mathcal{S} \mid r \geq t\}$. For $t > \max \mathcal{S}$, the minimal polynomial of B is also a (p^t)-minimal polynomial.

INPUT:

- p – a prime in D
- s_max – a positive integer (default: None); if set, only (p^s)-minimal polynomials for s <= s_max are computed (see below for details)

OUTPUT:

A dictionary. Keys are the finite set $\mathcal{S}$, the values are the associated (p^s)-minimal polynomials ν_s, $s \in \mathcal{S}$.

Setting s_max only affects the output if s_max is at most $\max \mathcal{S}$ where $\mathcal{S}$ denotes the full set. In that case, only those ν_s with s <= s_max are returned where s_max is always included even if it is not included in the full set $\mathcal{S}$.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: C.p_minimal_polynomials(2)
{2: x^2 + 3*x + 2}
sage: set_verbose(1)
sage: C = ComputeMinimalPolynomials(B)
sage: C.p_minimal_polynomials(2)
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
-----------------------------------------
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
p = 2, t = 1:
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
Result of lifting:
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
F =
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
[x^2 + x]
[      x]
[      0]
[      1]
[      1]
[  x + 1]
[      1]
[      0]
[      0]
[  x + 1]
verbose 1 (...: compute_J_ideal.py, current_nu)
-----------------------------------------
verbose 1 (...: compute_J_ideal.py, current_nu)
(x^2 + x)
verbose 1 (...: compute_J_ideal.py, current_nu)
Generators with (p^t)-generating property:
verbose 1 (...: compute_J_ideal.py, current_nu)
[x^2 + x]
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
nu = x^2 + x
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
corresponding columns for G
```

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```
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
[x^2 + x]
[  x + 2]
[      0]
[      1]
[      1]
[  x - 1]
[     -1]
[     10]
[      0]
[  x + 1]
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
----------------------------------------
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
p = 2, t = 2:
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
Result of lifting:
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
F =
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
[  2*x^2 + 2*x x^2 + 3*x + 2]
[           2*x         x + 4]
[             0             0]
[             2             1]
[             2             1]
[       2*x + 2         x + 1]
[             2            -1]
[             0            10]
[             0             0]
[       2*x + 2         x + 3]
verbose 1 (...: compute_J_ideal.py, current_nu)
----------------------------------------
verbose 1 (...: compute_J_ideal.py, current_nu)
(2*x^2 + 2*x, x^2 + 3*x + 2)
verbose 1 (...: compute_J_ideal.py, current_nu)
Generators with (p^t)-generating property:
verbose 1 (...: compute_J_ideal.py, current_nu)
[x^2 + 3*x + 2]
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
nu = x^2 + 3*x + 2
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
corresponding columns for G
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
[x^2 + 3*x + 2]
[        x + 4]
[            0]
[            1]
[            1]
[        x + 1]
[           -1]
[           10]
[            0]
```

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```
[        x + 3]
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
-----------------------------------------
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
p = 2, t = 3:
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
Result of lifting:
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
F =
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
[x^3 + 7*x^2 + 6*x x^3 + 3*x^2 + 2*x]
[        x^2 + 8*x         x^2 + 4*x]
[                0                 0]
[                x             x + 4]
[            x + 4                 x]
[    x^2 + 5*x + 4           x^2 + x]
[           -x + 4                -x]
[             10*x              10*x]
[                0                 0]
[        x^2 + 7*x     x^2 + 3*x + 4]
verbose 1 (...: compute_J_ideal.py, current_nu)
-----------------------------------------
verbose 1 (...: compute_J_ideal.py, current_nu)
(x^3 + 7*x^2 + 6*x, x^3 + 3*x^2 + 2*x)
verbose 1 (...: compute_J_ideal.py, current_nu)
Generators with (p^t)-generating property:
verbose 1 (...: compute_J_ideal.py, current_nu)
...
verbose 1 (...: compute_J_ideal.py, current_nu)
[x^3 + 3*x^2 + 2*x]
verbose 1 (...: compute_J_ideal.py, p_minimal_polynomials)
nu = x^3 + 3*x^2 + 2*x
{2: x^2 + 3*x + 2}
sage: set_verbose(0)
sage: C.p_minimal_polynomials(2, s_max=1)
{1: x^2 + x}
sage: C.p_minimal_polynomials(2, s_max=2)
{2: x^2 + 3*x + 2}
sage: C.p_minimal_polynomials(2, s_max=3)
{2: x^2 + 3*x + 2}
```

ALGORITHM:

[HR2016], Algorithm 5.

prime_candidates()

Determine those primes p where μ_B might not be a (p)-minimal polynomial.

OUTPUT:

A list of primes.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import ComputeMinimalPolynomials
sage: B = matrix(ZZ, [[1, 0, 1], [1, -2, -1], [10, 0, 0]])
sage: C = ComputeMinimalPolynomials(B)
sage: C.prime_candidates()
[2, 3, 5]
sage: C.p_minimal_polynomials(2)
{2: x^2 + 3*x + 2}
sage: C.p_minimal_polynomials(3)
{}
sage: C.p_minimal_polynomials(5)
{}
```

This means that 3 and 5 were candidates, but actually, μ_B turns out to be a (3)-minimal polynomial and a (5)-minimal polynomial.

sage.matrix.compute_J_ideal.**lifting**(p, t, A, G)

Compute generators of $\{f \in D[X]^d \mid Af \equiv 0 \pmod{p^t}\}$ given generators of $\{f \in D[X]^d \mid Af \equiv 0 \pmod{p^{t-1}}\}$.

INPUT:

- `p` – a prime element of some principal ideal domain D
- `t` – a non-negative integer
- `A` – a $c \times d$ matrix over $D[X]$
- `G` – a matrix over $D[X]$. The columns of $\begin{pmatrix} p^{t-1}I & G \end{pmatrix}$ are generators of $\{f \in D[X]^d \mid Af \equiv 0 \pmod{p^{t-1}}\}$; can be set to `None` if `t` is zero

OUTPUT:

A matrix F over $D[X]$ such that the columns of $\begin{pmatrix} p^tI & F & pG \end{pmatrix}$ are generators of $\{f \in D[X]^d \mid Af \equiv 0 \pmod{p^t}\}$.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import lifting
sage: X = polygen(ZZ, 'X')
sage: A = matrix([[1, X], [2*X, X^2]])
sage: G0 = lifting(5, 0, A, None)
sage: G1 = lifting(5, 1, A, G0); G1
[]
sage: (A*G1 % 5).is_zero()
True
sage: A = matrix([[1, X, X^2], [2*X, X^2, 3*X^3]])
sage: G0 = lifting(5, 0, A, None)
sage: G1 = lifting(5, 1, A, G0); G1
[3*X^2]
[    X]
[    1]
sage: (A*G1 % 5).is_zero()
True
sage: G2 = lifting(5, 2, A, G1); G2
[15*X^2 23*X^2]
[   5*X      X]
[     5      1]
```

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(continued from previous page)

```
sage: (A*G2 % 25).is_zero()
True
sage: lifting(5, 10, A, G1)
Traceback (most recent call last):
...
ValueError: A*G not zero mod 5^9
```

ALGORITHM:

[HR2016], Algorithm 1.

sage.matrix.compute_J_ideal.**p_part**(*f*, *p*)

Compute the p-part of a polynomial.

INPUT:

- `f` – a polynomial over D
- `p` – a prime in D

OUTPUT:

A polynomial g such that $\deg g \leq \deg f$ and all non-zero coefficients of $f - pg$ are not divisible by p.

EXAMPLES:

```
sage: from sage.matrix.compute_J_ideal import p_part
sage: X = polygen(ZZ, 'X')
sage: f = X^3 + 5*X + 25
sage: g = p_part(f, 5); g
X + 5
sage: f - 5*g
X^3
```

CHAPTER

FORTYTHREE

BENCHMARKS FOR MATRICES

This file has many functions for computing timing benchmarks of various methods for random matrices with given bounds for the entries. The systems supported are Sage and Magma.

The basic command syntax is as follows:

```
sage: import sage.matrix.benchmark as b
sage: print("starting"); import sys; sys.stdout.flush(); b.report([b.det_ZZ], 'Test',␣
↪systems=['sage'])
starting...
=============================================================================
          Test
=============================================================================
...
=============================================================================
```

sage.matrix.benchmark.**MatrixVector_QQ**(*n=1000*, *h=100*, *system='sage'*, *times=1*)

Compute product of square `n` matrix by random vector with num and denom bounded by `h` the given number of `times`.

INPUT:

- `n` - matrix dimension (default: `300`)
- `h` - numerator and denominator bound (default: `bnd`)
- `system` - either 'sage' or 'magma' (default: 'sage')
- `times` - number of experiments (default: `1`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.MatrixVector_QQ(500)
sage: tm = b.MatrixVector_QQ(500, system='magma')  # optional - magma
```

sage.matrix.benchmark.**charpoly_GF**(*n=100*, *p=16411*, *system='sage'*)

Given a n x n matrix over GF with random entries, compute the charpoly.

INPUT:

- `n` - matrix dimension (default: 100)
- `p` - prime number (default: `16411`)
- `system` - either 'magma' or 'sage' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.charpoly_GF(100)
sage: tm = b.charpoly_GF(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**charpoly_ZZ**(*n=100*, *min=0*, *max=9*, *system='sage'*)

Characteristic polynomial over ZZ: Given a n x n matrix over ZZ with random entries between min and max, compute the charpoly.

INPUT:

- `n` - matrix dimension (default: `100`)
- `min` - minimal value for entries of matrix (default: `0`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.charpoly_ZZ(100)
sage: tm = b.charpoly_ZZ(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**det_GF**(*n=400*, *p=16411*, *system='sage'*)

Dense determinant over GF(p). Given an n x n matrix A over GF with random entries compute det(A).

INPUT:

- `n` - matrix dimension (default: 300)
- `p` - prime number (default: `16411`)
- `system` - either 'magma' or 'sage' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.det_GF(1000)
sage: tm = b.det_GF(1000, system='magma')  # optional - magma
```

sage.matrix.benchmark.**det_QQ**(*n=300*, *num_bound=10*, *den_bound=10*, *system='sage'*)

Dense rational determinant over QQ. Given an n x n matrix A over QQ with random entries with numerator bound and denominator bound, compute det(A).

INPUT:

- `n` - matrix dimension (default: `200`)
- `num_bound` - numerator bound, inclusive (default: `10`)
- `den_bound` - denominator bound, inclusive (default: `10`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.det_QQ(200)
sage: ts = b.det_QQ(10, num_bound=100000, den_bound=10000)
sage: tm = b.det_QQ(200, system='magma')  # optional - magma
```

sage.matrix.benchmark.**det_ZZ**(*n=200*, *min=1*, *max=100*, *system='sage'*)

Dense integer determinant over ZZ. Given an n x n matrix A over ZZ with random entries between min and max, inclusive, compute det(A).

INPUT:

- `n` - matrix dimension (default: `200`)
- `min` - minimal value for entries of matrix (default: `1`)
- `max` - maximal value for entries of matrix (default: `100`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.det_ZZ(200)
sage: tm = b.det_ZZ(200, system='magma')  # optional - magma
```

sage.matrix.benchmark.**det_hilbert_QQ**(*n=80*, *system='sage'*)

Runs the benchmark for calculating the determinant of the hilbert matrix over rationals of dimension n.

INPUT:

- `n` - matrix dimension (default: `300`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.det_hilbert_QQ(50)
sage: tm = b.det_hilbert_QQ(50, system='magma')  # optional - magma
```

sage.matrix.benchmark.**echelon_QQ**(*n=100*, *min=0*, *max=9*, *system='sage'*)

Given a n x (2*n) matrix over QQ with random integer entries between min and max, compute the reduced row echelon form.

INPUT:

- `n` - matrix dimension (default: `300`)
- `min` - minimal value for entries of matrix (default: `-9`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.echelon_QQ(100)
sage: tm = b.echelon_QQ(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**hilbert_matrix**(*n*)

Returns the Hilbert matrix of size n over rationals.

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: b.hilbert_matrix(3)
[  1 1/2 1/3]
[1/2 1/3 1/4]
[1/3 1/4 1/5]
```

sage.matrix.benchmark.**inverse_QQ**(*n=100*, *min=0*, *max=9*, *system='sage'*)

Given a n x n matrix over QQ with random integer entries between min and max, compute the reduced row echelon form.

INPUT:

- `n` - matrix dimension (default: `300`)
- `min` - minimal value for entries of matrix (default: `-9`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.inverse_QQ(100)
sage: tm = b.inverse_QQ(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**invert_hilbert_QQ**(*n=40*, *system='sage'*)

Runs the benchmark for calculating the inverse of the hilbert matrix over rationals of dimension n.

INPUT:

- `n` - matrix dimension (default: `300`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.invert_hilbert_QQ(30)
sage: tm = b.invert_hilbert_QQ(30, system='magma')  # optional - magma
```

sage.matrix.benchmark.**matrix_add_GF**(*n=1000*, *p=16411*, *system='sage'*, *times=100*)

Given two n x n matrix over GF(p) with random entries, add them.

INPUT:

- `n` - matrix dimension (default: 300)
- `p` - prime number (default: `16411`)
- `system` - either 'magma' or 'sage' (default: 'sage')
- `times` - number of experiments (default: `100`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.matrix_add_GF(500, p=19)
sage: tm = b.matrix_add_GF(500, p=19, system='magma')  # optional - magma
```

sage.matrix.benchmark.**matrix_add_ZZ**(*n=200*, *min=-9*, *max=9*, *system='sage'*, *times=50*)

Matrix addition over ZZ Given an n x n matrix A and B over ZZ with random entries between `min` and `max`, inclusive, compute A + B `times` times.

INPUT:

- `n` - matrix dimension (default: `200`)
- `min` - minimal value for entries of matrix (default: `-9`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')
- `times` - number of experiments (default: `50`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.matrix_add_ZZ(200)
sage: tm = b.matrix_add_ZZ(200, system='magma')  # optional - magma
```

sage.matrix.benchmark.**matrix_add_ZZ_2**(*n=200*, *bits=16*, *system='sage'*, *times=50*)

Matrix addition over ZZ. Given an n x n matrix A and B over ZZ with random `bits`-bit entries, compute A + B.

INPUT:

- `n` - matrix dimension (default: `200`)
- `bits` - bitsize of entries
- `system` - either 'sage' or 'magma' (default: 'sage')
- `times` - number of experiments (default: `50`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.matrix_add_ZZ_2(200)
sage: tm = b.matrix_add_ZZ_2(200, system='magma')  # optional - magma
```

sage.matrix.benchmark.**matrix_multiply_GF**(*n=100*, *p=16411*, *system='sage'*, *times=3*)

Given an n x n matrix A over GF(p) with random entries, compute A * (A+1).

INPUT:

- `n` - matrix dimension (default: 100)
- `p` - prime number (default: `16411`)
- `system` - either 'magma' or 'sage' (default: 'sage')
- `times` - number of experiments (default: `3`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.matrix_multiply_GF(100, p=19)
sage: tm = b.matrix_multiply_GF(100, p=19, system='magma')  # optional - magma
```

sage.matrix.benchmark.**matrix_multiply_QQ**(*n=100*, *bnd=2*, *system='sage'*, *times=1*)

Given an n x n matrix A over QQ with random entries whose numerators and denominators are bounded by bnd, compute A * (A+1).

INPUT:

- `n` - matrix dimension (default: `300`)
- `bnd` - numerator and denominator bound (default: `bnd`)
- `system` - either 'sage' or 'magma' (default: 'sage')
- `times` - number of experiments (default: `1`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.matrix_multiply_QQ(100)
sage: tm = b.matrix_multiply_QQ(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**matrix_multiply_ZZ**(*n=300*, *min=-9*, *max=9*, *system='sage'*, *times=1*)

Matrix multiplication over ZZ Given an n x n matrix A over ZZ with random entries between min and max, inclusive, compute A * (A+1).

INPUT:

- `n` - matrix dimension (default: `300`)
- `min` - minimal value for entries of matrix (default: `-9`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')
- `times` - number of experiments (default: `1`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.matrix_multiply_ZZ(200)
sage: tm = b.matrix_multiply_ZZ(200, system='magma')  # optional - magma
```

sage.matrix.benchmark.**nullspace_GF**(*n=300*, *p=16411*, *system='sage'*)

Given a n+1 x n matrix over GF(p) with random entries, compute the nullspace.

INPUT:

- `n` - matrix dimension (default: 300)
- `p` - prime number (default: `16411`)
- `system` - either 'magma' or 'sage' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.nullspace_GF(300)
sage: tm = b.nullspace_GF(300, system='magma')  # optional - magma
```

sage.matrix.benchmark.**nullspace_RDF**(*n=300*, *min=0*, *max=10*, *system='sage'*)

Nullspace over RDF: Given a n+1 x n matrix over RDF with random entries between min and max, compute the nullspace.

INPUT:

- `n` - matrix dimension (default: `300`)
- `min` - minimal value for entries of matrix (default: `0`)
- `max` - maximal value for entries of matrix (default: 10‘)
- `system` - either ‘sage’ or ‘magma’ (default: ‘sage’)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.nullspace_RDF(100)  # long time
sage: tm = b.nullspace_RDF(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**nullspace_RR**(*n=300*, *min=0*, *max=10*, *system='sage'*)

Nullspace over RR: Given a n+1 x n matrix over RR with random entries between min and max, compute the nullspace.

INPUT:

- `n` - matrix dimension (default: `300`)
- `min` - minimal value for entries of matrix (default: `0`)
- `max` - maximal value for entries of matrix (default: `10`)
- `system` - either ‘sage’ or ‘magma’ (default: ‘sage’)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.nullspace_RR(100)
sage: tm = b.nullspace_RR(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**nullspace_ZZ**(*n=200*, *min=0*, *max=4294967296*, *system='sage'*)

Nullspace over ZZ: Given a n+1 x n matrix over ZZ with random entries between min and max, compute the nullspace.

INPUT:

- `n` - matrix dimension (default: `200`)
- `min` - minimal value for entries of matrix (default: `0`)
- `max` - maximal value for entries of matrix (default: `2**32`)
- `system` - either ‘sage’ or ‘magma’ (default: ‘sage’)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.nullspace_ZZ(200)
sage: tm = b.nullspace_ZZ(200, system='magma')  # optional - magma
```

sage.matrix.benchmark.**rank2_GF**(*n=500*, *p=16411*, *system='sage'*)

Rank over GF(p): Given a (n + 10) x n matrix over GF(p) with random entries, compute the rank.

INPUT:

- `n` - matrix dimension (default: 300)
- `p` - prime number (default: `16411`)

- system - either 'magma' or 'sage' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.rank2_GF(500)
sage: tm = b.rank2_GF(500, system='magma')  # optional - magma
```

sage.matrix.benchmark.**rank2_ZZ**(*n=400*, *min=0*, *max=18446744073709551616*, *system='sage'*)

Rank 2 over ZZ: Given a (n + 10) x n matrix over ZZ with random entries between min and max, compute the rank.

INPUT:

- n - matrix dimension (default: 400)
- min - minimal value for entries of matrix (default: 0)
- max - maximal value for entries of matrix (default: 2**64)
- system - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.rank2_ZZ(300)
sage: tm = b.rank2_ZZ(300, system='magma')  # optional - magma
```

sage.matrix.benchmark.**rank_GF**(*n=500*, *p=16411*, *system='sage'*)

Rank over GF(p): Given a n x (n+10) matrix over GF(p) with random entries, compute the rank.

INPUT:

- n - matrix dimension (default: 300)
- p - prime number (default: 16411)
- system - either 'magma' or 'sage' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.rank_GF(1000)
sage: tm = b.rank_GF(1000, system='magma')  # optional - magma
```

sage.matrix.benchmark.**rank_ZZ**(*n=700*, *min=0*, *max=9*, *system='sage'*)

Rank over ZZ: Given a n x (n+10) matrix over ZZ with random entries between min and max, compute the rank.

INPUT:

- n - matrix dimension (default: 700)
- min - minimal value for entries of matrix (default: 0)
- max - maximal value for entries of matrix (default: 9)
- system - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.rank_ZZ(300)
sage: tm = b.rank_ZZ(300, system='magma')  # optional - magma
```

sage.matrix.benchmark.**report**(*F*, *title*, *systems=['sage', 'magma']*, ***kwds*)

Run benchmarks with default arguments for each function in the list F.

INPUT:

- `F` - a list of callables used for benchmarking
- `title` - a string describing this report
- `systems` - a list of systems (supported entries are 'sage' and 'magma')
- `**kwds` - keyword arguments passed to all functions in `F`

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: print("starting"); import sys; sys.stdout.flush(); b.report([b.det_ZZ], 'Test
↪', systems=['sage'])
starting...
================================================================================
          Test
================================================================================
...
================================================================================
```

sage.matrix.benchmark.**report_GF**(*p=16411*, ***kwds*)

Runs all the reports for finite field matrix operations, for prime p=16411.

INPUT:

- `p` - ignored
- `**kwds` - passed through to *report()*

Note: right now, even though p is an input, it is being ignored! If you need to check the performance for other primes, you can call individual benchmark functions.

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: print("starting"); import sys; sys.stdout.flush(); b.report_GF(systems=['sage
↪'])
starting...
================================================================================
Dense benchmarks over GF with prime 16411
================================================================================
...
================================================================================
```

sage.matrix.benchmark.**report_ZZ**(***kwds*)

Reports all the benchmarks for integer matrices and few rational matrices.

INPUT:

- `**kwds` - passed through to *report()*

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: print("starting"); import sys; sys.stdout.flush(); b.report_ZZ(systems=['sage
↪'])  # long time (15s on sage.math, 2012)
starting...
====================================================================
Dense benchmarks over ZZ
====================================================================
...
====================================================================
```

sage.matrix.benchmark.**smithform_ZZ**(*n=128*, *min=0*, *max=9*, *system='sage'*)

Smith Form over ZZ: Given a n x n matrix over ZZ with random entries between min and max, compute the Smith normal form.

INPUT:

- `n` - matrix dimension (default: `128`)
- `min` - minimal value for entries of matrix (default: `0`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.smithform_ZZ(100)
sage: tm = b.smithform_ZZ(100, system='magma')  # optional - magma
```

sage.matrix.benchmark.**vecmat_ZZ**(*n=300*, *min=-9*, *max=9*, *system='sage'*, *times=200*)

Vector matrix multiplication over ZZ.

Given an n x n matrix A over ZZ with random entries between min and max, inclusive, and v the first row of A, compute the product v * A.

INPUT:

- `n` - matrix dimension (default: `300`)
- `min` - minimal value for entries of matrix (default: `-9`)
- `max` - maximal value for entries of matrix (default: `9`)
- `system` - either 'sage' or 'magma' (default: 'sage')
- `times` - number of runs (default: `200`)

EXAMPLES:

```
sage: import sage.matrix.benchmark as b
sage: ts = b.vecmat_ZZ(300)  # long time
sage: tm = b.vecmat_ZZ(300, system='magma')  # optional - magma
```

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