Homework 5

Sandy Auttelet

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Determine whether or not the following sets are polyhedra. Use clear reasoning and appeal to the definition of a polyhedron and/or relevant geometric theorems from Chapter 2 of the textbook.

1. $\Omega_1 = \emptyset$.
2. $\Omega_2 = \{x \in \mathbb{R} | x \geq 5, x \leq 0\}$.
3. $\Omega_3 = \{x \in \mathbb{R} | x^2 - 9 \leq 0\}$.
4. $\Omega_4 = \{x \in \mathbb{R}^n | \ ||x|| \leq 1\}$.

Definition: A **polyhedron** is a set that can be described in the form $\{x \in \mathbb{R}^n | Ax \geq b\}$, where A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

Definition: A set $\Omega \subseteq \mathbb{R}^n$ is said to be **convex** if for every $x, y \in \Omega$ and all $0 \leq \lambda \leq 1, z = \lambda x + (1-\lambda)y \in \Omega$.

Answer:

1. $\Omega_1 = \emptyset$. Suppose we redefine $\Omega_1 = \{x \in \mathbb{R} | x \geq 5, x \leq 1\}$. Then there does not exist an $x \in \mathbb{R}$ where these two inequalities hold. Then Ω_1 is empty and can be expressed as a set of inequalities where for the first inequality, $A = [1]$ and $b = [5]$ and for our other inequality, $A = [-1]$ and $b = [1]$. Therefore, Ω_1 is a polyhedron.
2. $\Omega_2 = \{x \in \mathbb{R} | x \geq 5, x \leq 0\}$. Here there does not exist an $x \in \mathbb{R}$ where these two inequalities, $x \geq 5, x \leq 0$, hold. Then Ω_2 is empty and can be expressed as a set of inequalities where for the first inequality, $A = [1]$ and $b = [5]$ and for our other inequality, $A = [-1]$ and $b = [0]$. Therefore, Ω_2 is a polyhedron.

3. $\Omega_3 = \{x \in \mathbb{R} | x^2 - 9 \leq 0\}$. A depiction of this set,

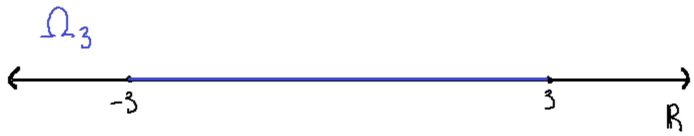

Figure 1: Ω_3

Solving for x,

$$x^2 - 9 \leq 0 \Rightarrow x^2 \leq 9 \Rightarrow x \leq \pm 3.$$

Then we can express Ω_3 as a set of two inequalities where $x \leq 3$ and $x \geq -3$. Then for the first inequality, $A = [-1]$ and $b = [3]$, and for the second inequality $A = [1]$ and $b = [-3]$. Therefore, Ω_3 is a polyhedron.

4. $\Omega_4 = \{x \in \mathbb{R}^n | \ ||x|| \leq 1\}$. A depiction of this set where $n = 2$,

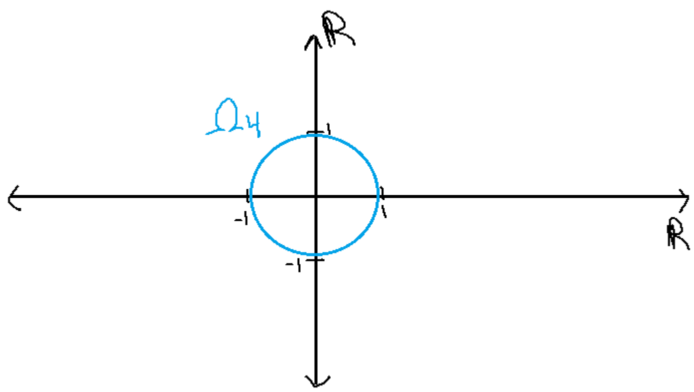

Figure 2: Ω_4 in $\mathbb{R}^2$, the unit ball.

Suppose without loss of generality that we restrict ourselves to the case where $n = 2$. Then we create a cover over Ω_4 by drawing a line tangent to each point on the boundary of Ω_4, denoted by $\partial\Omega_4$. Because $\mathbb{R}^2$ contains an uncountably infinite number of points (x, y) for $x, y \in \mathbb{R}$, $\partial\Omega_4$ also has an uncountably infinite number of points, and therefore an uncountably infinite number of tangent lines. Suppose we then take the union of all tangent lines and call this our polyhedron. However, we have reached a contradiction in the definition of a polyhedron as we cannot represent this union as a finite intersection of halfspaces since we have an infinite number of tangent lines. Therefore, Ω_4 cannot be a polyhedron. In my depiction of Ω_4 in $\mathbb{R}^2$, my drawing is not a true ball in the sense that we are limited to pixel length here, and therefore would behave discretely. If you zoom in, you can see where the pixelation takes over to represent a polyhedron. Computers cannot draw true balls because they use polyhedron inequalities to draw, like in this paint application. Even when drawing a ball on paper, you are limited to the discreteness of your pencil as the graphite atoms hit the paper.