Homework 5

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Due: 02/13/24 at 11:59PM

Determine whether or not the following sets are polyhedra. Use clear reasoning and appeal to the definition of a polyhedron and/or relevant geometric theorems from Chapter 2 of the textbook.

1. $\Omega_1 = \emptyset$.
2. $\Omega_2 = \{x \in \mathbb{R} | x \geq 5, x \leq 0\}$.
3. $\Omega_3 = \{x \in \mathbb{R} | x^2 - 9 \leq 0\}$.
4. $\Omega_4 = \{x \in \mathbb{R}^n | \ ||x|| \leq 1\}$.

Definition: A **polyhedron** is a set that can be described in the form $\{x \in \mathbb{R}^n | Ax \geq b\}$, where A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

Definition: A set $\Omega \subseteq \mathbb{R}^n$ is said to be **convex** if for every $x, y \in \Omega$ and all $0 \leq \lambda \leq 1, z = \lambda x + (1-\lambda)y \in \Omega$.

Answer:

1. $\Omega_1 = \emptyset$. From class discussion, we stated that Ω_1 is a polyhedron. This polyhedron is obviously the trivial result of our definition.
2. $\Omega_2 = \{x \in \mathbb{R} | x \geq 5, x \leq 0\}$. A depiction of this set,

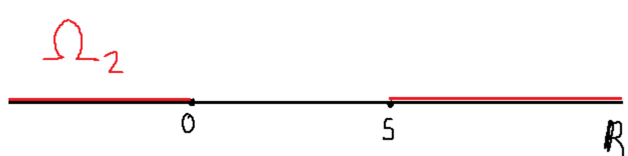

Figure 1: Ω_2

This set is not convex because it is disjoint. Suppose using our above definition, we choose $x = 0$ and $y = 5$ and we choose $\lambda = 0.5$. Then,

$$z = \lambda x + (1-\lambda)y = 0.5(0) + (1-0.5)5 = 0 + 2.5 = 2.5 \notin \Omega_2.$$

Therefore, Ω_2 is not convex, and thus not a polyhedron.

3. $\Omega_3 = \{x \in \mathbb{R} | x^2 - 9 \leq 0\}$. A depiction of this set,

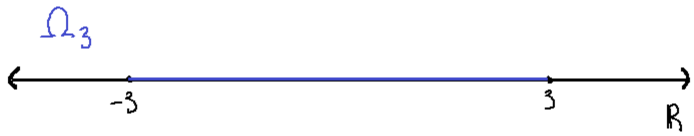

Figure 2: Ω_3

As we discussed in class, a line segment is a polyhedron because it is convex and can be discretized into an intersection of a finite number of half spaces. Therefore, Ω_3 is a polyhedron.

4. $\Omega_4 = \{x \in \mathbb{R}^n | \ ||x|| \leq 1\}$. A depiction of this set where $n = 2$,

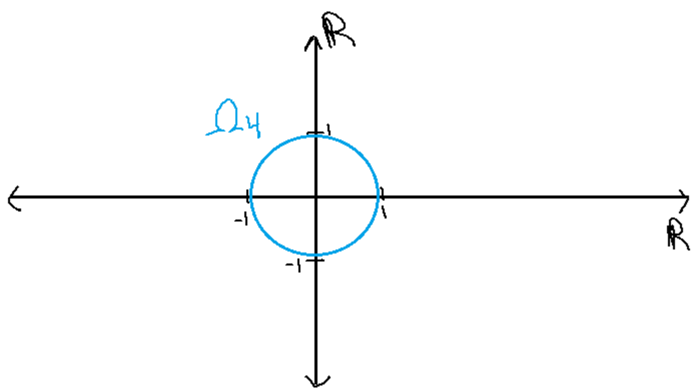

Figure 3: Ω_4 in $\mathbb{R}^2$, the unit ball.

As we discussed in class, a ball is not a polyhedron, regardless of the value here for n, and thus Ω_4 is not a polyhedron. This reason is because there is no way to discretize a true ball into a finite intersection of half spaces. However, my drawing is not a true ball in the sense that we are limited to pixel length here, and therefore would behave discretely. If you zoom in, you can see where the pixelation takes over to represent a polyhedron. Computers cannot draw true balls because they use polyhedron inequalities to draw, like in this paint application. Even when drawing a ball on paper, you are limited to the discreteness of your pencil as the graphite atoms hit the paper.