14. 2. Limits and Continuity

2018년 9월 5일 수요일 10:25

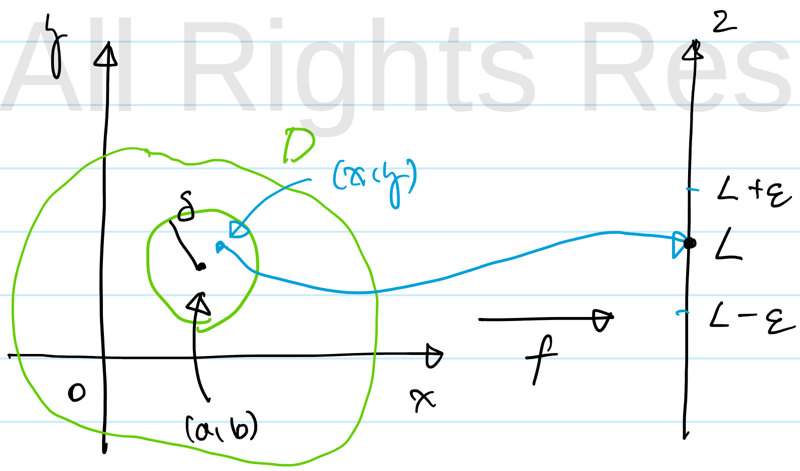
< Limits >

Def Let $f(x,y)$ be a function whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.

905



1. If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.
2. Even if $L_1 = L_2$, it still may not exist.

Ex 1 Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

$\to$ $f(x,y)\to 1$ as $(x,y)\to(0,0)$ along x-axis
$f(x,y)\to -1$ as $(x,y)\to(0,0)$ along y-axis

Since f has two different limits along two different lines, the given limit does not exist.

Ex 2. If $f(x,y)=\dfrac{xy}{x^2+y^2}$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

$\to$ If $y=0$, then $f(x,0)=0$.
$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along x-axis.

If $x=0$, then $f(0,y)=0$
$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along y-axis

But if $x=y$, then $f(x,y)=1/2$
$\therefore f(x,y)\to 1/2$ as $(x,y)\to(0,0)$ along $x=y$.

$\Rightarrow$ The given limit does not exist.

Ex 3. If $f(x,y)=\dfrac{xy^2}{x^2+y^4}$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

$\to$ Let $y=mx$ $\to$

$$f(x,y)=f(x,mx)=\frac{x(mx)^2}{x^2+(mx)^4}=\frac{m^2x}{1+m^4x^2}$$

$\therefore f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y=mx$.

But if $x=y^2 \to$

$$f(y^2,y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{1}{2}$$

$\therefore f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x=y^2$.

$\Rightarrow$ The given limit does not exist.

thm 1)

1. $\lim_{(x,y)\to(a,b)} x = a$.

2. $\lim_{(x,y)\to(a,b)} y = b$.

3. $\lim_{(x,y)\to(a,b)} c = c$.

2) $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$

3) $\vdots$

4) $g(x,y) \le f(x,y) \le h(x,y)$,

$$\lim_{(x,y)\to(a,b)} g(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y)$$

$$\Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = L.$$

<pf>.

1) $\lim_{(x,y)\to(a,b)} x = a$

$\Leftrightarrow \quad \forall \varepsilon > 0, \ \exists \delta > 0$ such that

$0 < |(x,y) - (a,b)| < \delta \Rightarrow |x - a| < \varepsilon.$

Let $\varepsilon > 0$ be fixed. show

$\exists \delta > 0$ such that

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |x-a| < \varepsilon.$

Note that $\quad |x-a| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta.$

So if we put $\delta = \varepsilon$ then

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$

$\Rightarrow |x-a| < \varepsilon.$

Ex 4. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2+y^2}$ if it exists.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

if $0 < \sqrt{x^2+y^2} < \delta$ then $\left| \dfrac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$

$\Rightarrow$ if $0 < \sqrt{x^2+y^2} < \delta$ then $\dfrac{3x^2|y|}{x^2+y^2} < \varepsilon.$

Since $\dfrac{x^2}{x^2+y^2} \leq 1,$

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}.$$

thus if we choose $\delta = \varepsilon/3$ and let

$0 < \sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{3x^2y}{x^2+y^2}-0\right| \leq 3\sqrt{x^2+y^2}$$

$$< 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$ by definition

⟨ Continuity ⟩

Def 1. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

or $\forall \varepsilon > 0,\ \exists \delta > 0$ such that

$|(x,y)-(a,b)| < \delta \Rightarrow |f(x,y)-f(a,b)| < \varepsilon.$

iff f is continuous at (a,b).

2. We say that $f(x,y)$ is continuous on D if $f(x,y)$ is continuous at every point of D.

Def 1. a polynomial function of two variables is a finite sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. i.e.

$$\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}x^iy^j = p(x,y).$$

2. a rational function is a ratio of polynomials.

Remarks. 1. $f(x,y)=x$, $g(x,y)=y$, $h(x,y)=c$

... is continuous on $\mathbb{R}^2$.

2. All polynomial functions are continuous on $\mathbb{R}^2$.

3. All rational functions are continuous on the domain.

EX5. $\displaystyle\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

$\to$ $= 1^2\cdot 2^3 - 1^3\cdot 2^2 + 3\cdot 1 + 2\cdot 2$

$= 11$ ($\because$ polynomials are continuous on $\mathbb{R}^2$).

EX6. Where is $\dfrac{x^2-y^2}{x^2+y^2}$ continuous?

$\to$ $D=\{(x,y) \mid (x,y)\neq(0,0)\}$. since it's not defined at $(0,0)$.

908

EX7. Let

$$g(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

$\to$ g is discontinuous on $(0,0)$. ($\because$ EX 1).

EX8.

$$f(x,y)=\begin{cases}\dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

→ f is continuous on $\mathbb{R}^2$. ($\because$ Ex4)

Thm. 1. If $f(x,y)$ is a continuous function and $g(t)$ is a continuous function of a single value defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is also a continuous function.

2. If $f(x,y)$ is continuous at (a,b) and $g(t)$ is continuous at $f(a,b)$ then $g \circ f = g(f(x,y))$ is also continuous at (a,b).

909 Ex9. Where is $h(x,y) = \arctan(y/x)$ continuous?

→ Let $f(x,y) = y/x$, $g(t) = \arctan(t)$.

f is continuous on $D : \{(x,y) \mid x \neq 0\}$.

g is continuous on $\mathbb{R}$.

Thus $g \circ f = \arctan(y/x)$ is continuous on $D : \{(x,y) \mid x \neq 0\}$.

Def

$$\lim_{(a,b,c) \to (x,y,z)} f(x,y,z) = L \quad \text{if}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < |\mathbf{x}-\mathbf{a}| = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta.$$

$$\Rightarrow \quad |f(x,y,z)-L| < \varepsilon.$$

14. 3. Partial Derivatives

2018년 9월 10일 월요일 10:40

Def $(a,b) \in \mathbb{R}^2$

$\rightsquigarrow N_\varepsilon(a,b) = \{(x,y) \mid |(x,y)-(a,b)| < \varepsilon\}$.

: an ε - neighborhood of (a,b).

Def $f(x,y)$: a function defined on $N_\varepsilon(a,b)$

fix y to b. $\Rightarrow g(x) := f(x,b)$.

$$f_x(a,b) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

: a partial derivative of $f(x,y)$ with respect to x at (a,b).

w.r.t.

Def $f(x,y)$: a function

$$\to f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

< Notations for Partial Derivatives >

If $z = f(x,y)$,

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_2 = D_2 f = D_y f$$

EX1. $f(x,y) = x^3 + x^2y^3 - 2y^2$,

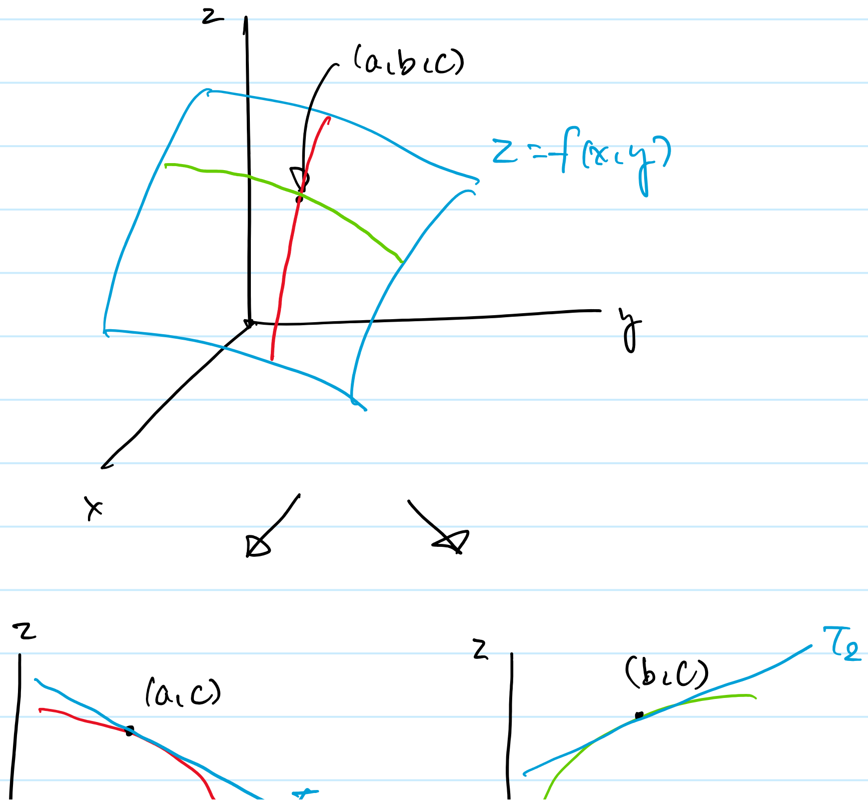
→ $f_x(2,1) = \frac{\partial}{\partial x}(x^3 + x^2y^3 - 2y^2)\Big|_{(2,1)}$

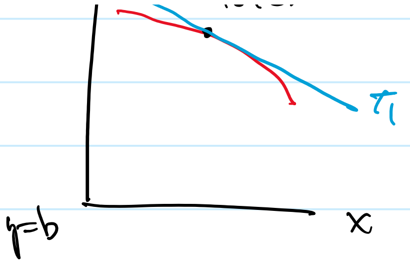
$= (3x^2 + 2xy^3)\Big|_{(2,1)} = 16.$

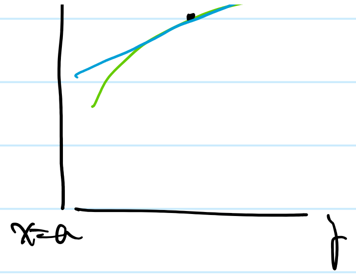
→ $f_y(2,1) = \frac{\partial}{\partial y}(x^3 + x^2y^3 - 2y^2)\Big|_{(2,1)}$

$= (3x^2y^2 - 4y)\Big|_{(2,1)} = 8.$

< Interpretations of Partial Derivatives >







$$\text{slope} = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} \qquad\qquad \text{slope} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

The plane containing T_1 and T_2 is called the tangent plane at P.

Ex 2. $f(x,y) = 4 - x^2 - 2y^2$

$$f_x(1,1) = \left.\frac{\partial}{\partial x}(4 - x^2 - 2y^2)\right|_{(1,1)}$$

$$= \left.-2x\right|_{(1,1)} = -2.$$

$$f_y(1,1) = \left.\frac{\partial}{\partial y}(4 - x^2 - 2y^2)\right|_{(1,1)}$$

$$= \left.-4y\right|_{(1,1)} = -4.$$

Ex 4. $f(x,y) = \sin\left(\frac{x}{1+y}\right).$

$$\to \frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}.$$

$$\to \frac{\partial f}{\partial y} = x\cos\left(\frac{x}{1+y}\right) \cdot \frac{-1}{(1+y)^2}$$

Ex 5. $x^3+y^3+z^3+6xyz=1.$

$\to \frac{\partial z}{\partial x}$? : $3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$

$\Rightarrow \frac{\partial z}{\partial x} = -\frac{x^2+2yz}{z^2+2xy}.$

$\to \frac{\partial z}{\partial y}$? : $3y^2+3z^2\frac{\partial z}{\partial y}+6xz+6xy\frac{\partial z}{\partial y}=0$

$\Rightarrow \frac{\partial z}{\partial y} = -\frac{y^2+2xz}{z^2+2xy}.$

Def.

- $w=f(x,y,z).$

$\to f_x(x,y,z)=\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$

$f_y(x,y,z)=\lim_{h\to 0}\frac{f(x,y+h,z)-f(x,y,z)}{h}.$

$f_z(x,y,z)=\lim_{h\to 0}\frac{f(x,y,z+h)-f(x,y,z)}{h}.$

- $\frac{\partial u}{\partial x_i}=\lim_{h\to 0}\frac{f(x_1,\cdots,x_{i-1},x_i+h,\cdots,x_n)-f(x_1,\cdots,x_n)}{h}.$

Ex 6. $f(x,y,z)=e^{xy}\ln z.$

$\to f_x=ye^{xy}\ln z,\quad f_y=xe^{xy}\ln z,$

$f_z=\frac{e^{xy}}{z}.$

< Higher Derivatives >.

$z = f(x,y)$.

$f_x(x,y)$ $\qquad\qquad$ $f_y(x,y)$

$(f_x)_x(x,y)$ $\quad$ $(f_x)_y(x,y)$ $\quad$ $(f_y)_x(x,y)$ $\quad$ $(f_y)_y(x,y)$.

$\vdots$

: $(f_x)_x = f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$ $\qquad$ $(f_y)_x = f_{yx} = \dfrac{\partial^2 f}{\partial y \partial x}$

$(f_x)_y = f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y}$ $\qquad$ $(f_y)_y = f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$

Ex7. $f(x,y) = x^3 + x^2y^3 - 2y^2$

$\rightarrow$ $f_x = 3x^2 + 2xy^3$

$f_y = 3x^2y^2 - 4y$

$\rightarrow$ $f_{xx} = 6x + 2y^3$.

$f_{xy} = 6xy^2$

$f_{yx} = 6xy^2$

$f_{yy} = 6x^2y - 4$.

919 **thm** < Clairaut's Theorem >

Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

EX8. $f(x,y,z) = \sin(3x+yz) \rightarrow f_{xxyz} = ?$

$\rightarrow$

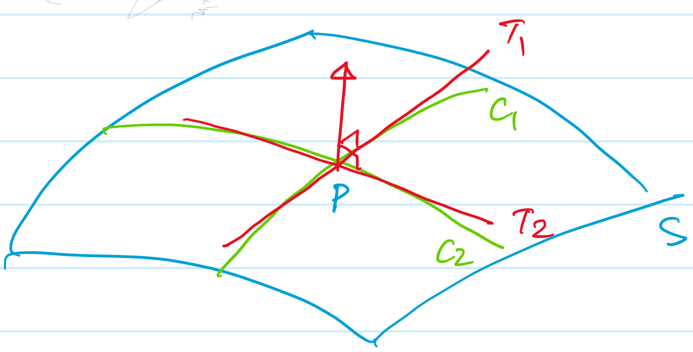
$$f_x = 3\cos(3x+yz)$$
$$f_{xx} = -9\sin(3x+yz)$$
$$f_{xxy} = -9z\cos(3x+yz)$$
$$f_{xxyz} = -9\cos(3x+yz) + 9yz\sin(3x+yz).$$

14. 4. Tangent Plane and Linear Approximation

2018년 9월 12일 수요일 10:33

928

< Tangent Planes >.



The plane containing T_1 and T_2 is called the **tangent plane** of S at P.

Thm Suppose $z = f(x,y)$ has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Ex1. $z = 2x^2 + y^2$, $(1, 1, 3) \to$?.

$$f_x(1,1) = (4x)\big|_{(1,1)} = 4$$
$$f_y(1,1) = (2y)\big|_{(1,1)} = 2$$

$$\therefore\ z - 3 = 4(x-1) + 2(y-1)$$
$$\Leftrightarrow\ z = 4x + 2y - 3.$$

$$f(x,y) \sim L(x,y) = 4x+2y-3$$

: linearization of f at (a,b)

linear approximation

or tangent plane approximation

Def $L(x,y)$

$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Review $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists

$\Rightarrow$ f is differentiable at $x=a$.

$$f(x)-f(a) = f'(a)(x-a) + \varepsilon(\Delta x)(x-a)$$

$$\Rightarrow \lim_{x\to a} \varepsilon(\Delta x) = 0.$$

Def If $z=f(x,y)$, then f is said to be differentiable at (a,b) if

$$\Delta z = f(x-a, y-b) - f(a,b).$$

can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1(\Delta x, \Delta y)\Delta x + \varepsilon_2(\Delta x, \Delta y)\Delta y$$

where $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$.

Thm If f_x and f_y exists near (a,b) and are continuous at (a,b), then $f(x,y)$ is differentiable at (a,b).

Ex2. $f(x,y) = xe^{xy}$. ... $(1,0)$

$\rightarrow$ $f_x(x,y) = e^{xy} + xy\,e^{xy}$

$f_y(x,y) = x^2 e^{xy}$

both f_x and f_y are continuous functions, so f is differentiable. The linearization is

$$L(x,y) = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0)$$
$$= x-1+y+1$$
$$= x+y.$$

$\rightarrow$ $L(1.1, -0.1) = 1$

$f(1.1, -0.1) \approx 0.98542$.

932 < Differentials >.

Def $z = f(x,y)$.

$(a,b) \rightarrow (x,y)$

- $dx = \Delta x$ $\rightarrow$ differential of x.
- $dy = \Delta y$ $\rightarrow$ differential of y.

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$
$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\rightarrow$ the total differential of $z = f(x,y)$.

$\ast$ $\Delta z = f(\underset{dx}{a+\Delta x}, \underset{dy}{b+\Delta y}) - f(a,b)$.

$$f(x,y) = f(a,b) + \Delta z.$$

$$\Delta z \approx dz. \Rightarrow f(x,y) \approx f(a,b) + dz$$
$$= L(x,y).$$

Ex 4. Total differential of

(a) $z = f(x,y) = x^2 + 3xy - y^2$

$$\Rightarrow dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$
$$= (2x+3y)dx + (3x-2y)dy.$$

(b) $x : 2 \to 2.05$
$y : 3 \to 2.96.$ $\quad \Delta z \; ? \; dz \, ?$

$$\Delta z = f(2.05, 2.96) - f(2,3)$$
$$= \underline{0.6449}$$

$$dz = (4+9) \cdot 0.05 + (6-6)(-0.04)$$
$$= 13 \times 0.05$$
$$= \underline{0.65}.$$

Def $w = f(x,y,z)$

$\to L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz.$$

Def $u = f(x_1, x_2, \cdots, x_n).$

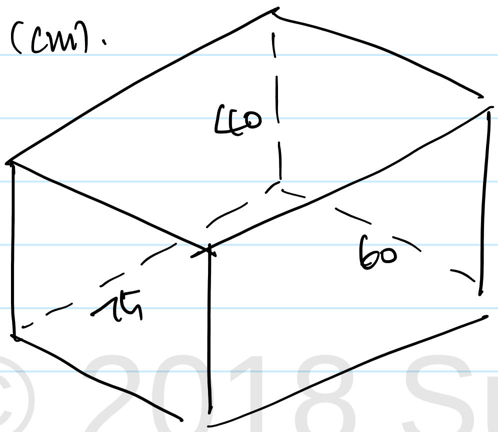
$\to L(x_1, x_2, \cdots, x_n) = f(a_1, a_2, \cdots, a_n)$

$$+\sum_{i=1}^{n} f_i(a_1,a_2,\cdots,a_n)(x_i-a_i).$$

$$du=\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}dx_i.$$

EX8.

(cm).



$|\Delta x|\le 0.2$

$|\Delta y|\le 0.2$

$|\Delta z|\le 0.2$

$V=xyz.$

$$dV=\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz$$

$$=yzdx+xzdy+xydz.$$

$a=75\quad b=60\quad c=40.$

$dx=dy=dz=0.2$

max $dV=0.2(xy+xz+yz)$

max $\Delta V=0.2(75\cdot 60+75\cdot 40+60\cdot 40)$

$=0.2(9900)$

$=1980\ (cm^3).$

14. 5. The Chain Rule

2018년 9월 17일 월요일 10:35

Thm ⟨The Chain Rule⟩ Case 1:

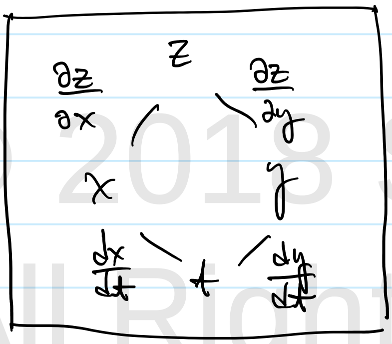
$z = f(x, y)$: differentiable

$x = g(t)$

$y = h(t)$.

$$\Rightarrow \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$



Ex 1. If $z = x^2y + 3xy^4$, where $x = \sin 2t$, $y = \cos t$, $dz/dt = ?$

$$\Rightarrow \frac{dz}{dt}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(x^2y + 3xy^4)\cdot 2\cos 2t +$$

$$\frac{\partial}{\partial y}(x^2y + 3xy^4)\cdot(-\sin t).$$

$$= 2(2xy + 3y^4)\cos 2t - (x^2 + 12xy^3)\sin t.$$

Thm <The Chain Rule> Case 2.

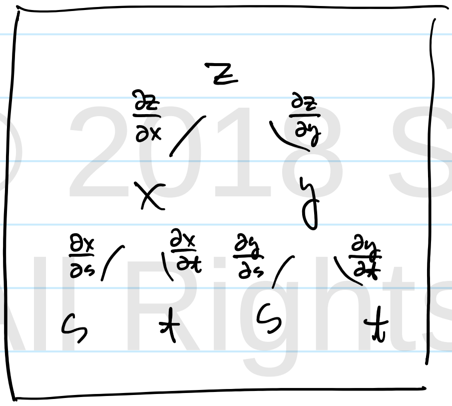
$z = f(x, y)$: differentiable.

$x = g(s, t)$

$y = h(s, t)$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}.$$



EX3. $z = e^x \sin y$

$x = st^2 \qquad y = s^2 t$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= e^x \cdot t^2 + e^x \cos y \cdot 2st$$

$$= te^x\,(t + 2s\cos y).$$

$$= te^{st^2}\,(t + 2s\cos(s^2 t)).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$= e^x \cdot 2st + e^x \cos y \cdot s^2$

$= se^x (2t + s\cos y)$

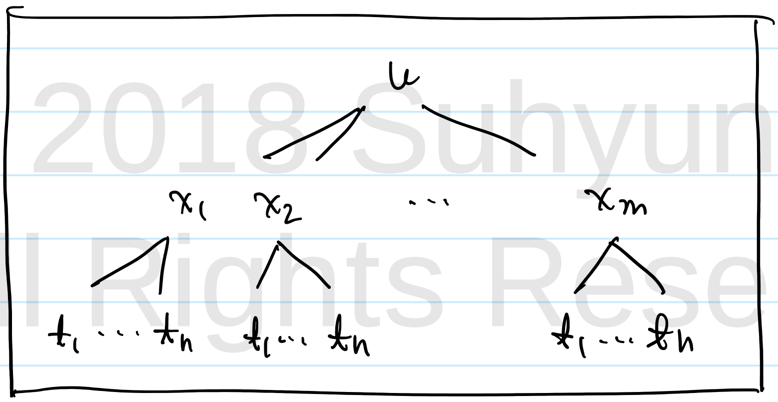
$= se^{st^2} (t + s\cos(s^2 t))$.

Thm < The Chain Rule > : General Version.

$u(x_1, x_2, \cdots, x_n)$: differentiable.

each x_i is a differentiable function of m variables $t_1, \cdots, t_m$. then

$$\frac{\partial u}{\partial t_i} = \sum_{k=1}^{m} \frac{\partial u}{\partial x_k} \cdot \frac{\partial x_k}{\partial t_i}.$$



Ex 4. $w = f(x, y, z, t)$,

$x = x(u,v)$, $y = y(u,v)$,

$z = z(u,v)$, $t = t(u,v)$

$$\rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial u}.$$

Ex 6. $g(s,t) = f(s^2 - t^2, t^2 - s^2)$

$\Rightarrow\ t\dfrac{\partial g}{\partial s}+s\dfrac{\partial g}{\partial t}$?

$\Rightarrow$ Let $x=s^2-t^2$, $y=t^2-s^2$

$\Rightarrow g(s,t)=f(x,y)$.

$$\frac{\partial g}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

$$=\frac{\partial f}{\partial x}(2s)-\frac{\partial f}{\partial y}(2s).$$

$$\frac{\partial g}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}-\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=\frac{\partial f}{\partial x}(-2t)+\frac{\partial f}{\partial y}(2t).$$

$\therefore\ t\dfrac{\partial g}{\partial s}+s\dfrac{\partial g}{\partial t}$

$$=2st\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)-2st\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)=0.$$

Ex 7.

$z=f(x,y)$. $x=r^2+s^2$, $y=2rs$

(a) $$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$=\frac{\partial z}{\partial x}(2r)+\frac{\partial z}{\partial y}(2s).$$

(b) $$\frac{\partial^2 z}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}(2r)+\frac{\partial z}{\partial y}(2s)\right)$$

$$=2\frac{\partial z}{\partial x}+2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)+2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right).$$

(Product Rule)

$$= 2\frac{\partial z}{\partial x}$$

$$+2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}\right)$$

$$+2s\left(2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

(Chain Rule)

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}.$$

< Implicit Differentiation >

1) $F(x,y)=0 \Rightarrow \frac{dy}{dx} = ?$ $\quad (y = f(x))$.

$$\to \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \quad \text{if } \frac{\partial F}{\partial y} \neq 0.$$

2) $F(x,y,z)=0 \Rightarrow \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = ?$ $\quad (z = f(x,y))$

$$\to \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial x} = -\frac{\quad}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\quad}{\frac{\partial F}{\partial z}}.$$

Ex 8. $x^3 + y^3 = 6xy \rightarrow y'$?

$$F(x,y) = x^3 + y^3 - 6xy = 0$$

$$\Downarrow$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Ex 9. $x^3 + y^3 + z^3 + 6xyz = 1$. $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$?

$\rightarrow F(x,y,z) := x^3 + y^3 + z^3 + 6xyz - 1$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

14. 6. Directional Derivatives and the Gradient Vector

2018년 9월 17일 월요일 11:38

$*$ $z = f(x, y)$

$\Rightarrow f_x(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

$f_y(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0, y_0+h) - f(x_0, y_0)}{h}.$

Def The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}.$$

Thm If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b.$$

<pf> $g(h) = f(x_0 + ha, y_0 + hb)$

$\Rightarrow g'(0) = \lim_{h \to 0} \dfrac{g(h) - g(0)}{h}$

$= \lim_{h \to 0} \dfrac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$

$= D_{\mathbf{u}} f(x_0, y_0).$

also, $g'(h) = \dfrac{\partial f}{\partial x} \dfrac{dx}{dh} + \dfrac{\partial f}{\partial y} \dfrac{dy}{dh}$

$$\therefore\ g'(0) = f_x(x,y)\,a + f_y(x,y)\,b.$$

$$\Rightarrow D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$
$$= \langle f_x(x,y),\ f_y(x,y)\rangle \cdot \mathbf{u}$$

EX 2.

$$f(x,y) = x^3 - 3xy + 4y^2.$$
$$\theta = \frac{\pi}{6} \rightarrow D_u f(1,2)?$$

$$\rightarrow D_{\mathbf{u}} f(1,2) = \cos\frac{\pi}{6} f_x(1,2) + \sin\frac{\pi}{6} f_y(1,2)$$
$$= \frac{\sqrt{3}}{2}(3x^2-3y)\Big|_{(1,2)} + \frac{1}{2}(-3x+8y)\Big|_{(1,2)}$$
$$= \frac{13-3\sqrt{3}}{2}.$$

Def

$$\overrightarrow{\text{grad}}\, f \overset{\text{del}}{=} \nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y)\rangle.$$

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y)\cdot \mathbf{u}$$

EX 3. $f(x,y) = \sin x + e^{xy}$
$$\rightarrow \nabla f(x,y) = \langle \cos x + ye^{xy},\ xe^{xy}\rangle.$$

EX 4. $f(x,y) = x^2y^3 - 4y.$ $\rightarrow$

Def.

$w = f(x,y,z)$ $\quad u = \langle a, b, c \rangle.$

$\rightarrow D_u f(x,y,z)$

$$= \lim_{h\to 0} \frac{f(x_0+ah,\, y_0+bh,\, z_0+ch) - f(x_0,y_0,z_0)}{h}.$$

thm

1) $D_u f(x_0) = \lim_{h\to 0} \frac{f(x_0+uh) - f(x_0)}{h}$

2) $D_u f(x_0,y_0,z_0) = f_x a + f_y b + f_z c.$

3) $\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

EX 5. $f(x,y,z) = x \sin yz.$

(a) $\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$= \langle \sin yz,\ xz\cos yz,\ xy\cos yz \rangle.$

(b) $\langle 1,2,-1\rangle \rightarrow \frac{\langle 1,2,-1\rangle}{\|\langle 1,2,-1\rangle\|} = \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle$

$\therefore D_u f(1,3,0) = \nabla f(1,3,0)\cdot u$

$= \langle 0,0,3\rangle \cdot \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle.$

$= 3\cdot\frac{1}{\sqrt{6}}\cdot(-1) = -\sqrt{\frac{3}{2}}.$

Question In what of these directions does f change fastest and what is the maximum rate of change?

thm Suppose f is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

→ pf $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$

$= |\nabla f(\mathbf{x})| |\mathbf{u}| \cos\theta$.

where θ is the angle between $\nabla f(\mathbf{x})$ and $\mathbf{u}$.
The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$.
$\therefore$ the maximum value of $D_{\mathbf{u}}$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$.

Ex 6.

(a) $f(x,y) = xe^{y}$

→ rate of change of f at $P(2,0)$ in the direction of P to $Q(\frac{1}{2}, 2)$.

→ $\mathbf{u} = \dfrac{\langle \frac{1}{2}, 2\rangle - \langle 2, 0\rangle}{|\langle \frac{1}{2}, 2\rangle - \langle 2,0\rangle|} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$.

$\nabla f(x,y) = \langle e^{y}, xe^{y}\rangle$.

$\therefore D_{\mathbf{u}} f(2,0) = \nabla f(2,0) \cdot \mathbf{u}$

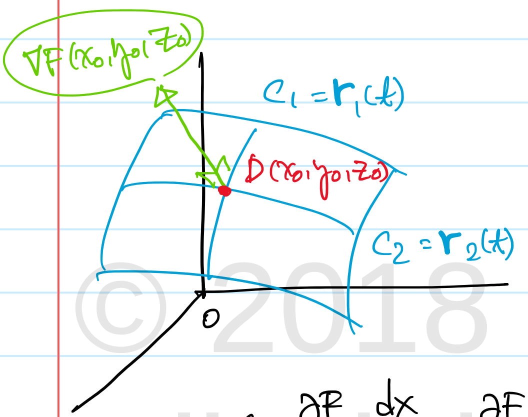
$$= \langle 1, 2\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$

$$= 1.$$

(b) $|\nabla f(2,0)| = \sqrt{5}$.

※ $w = F(x,y,z)$

$F(x,y,z) = k$: level surface of $f(x,y,z)$.



C: $\mathbf{r} = \mathbf{r}(t)$.

$t = t_0 \Rightarrow \mathbf{r}(t) = \langle x_0, y_0, z_0\rangle$

so $F(x(t), y(t), z(t)) = k$.

$\Rightarrow F(\mathbf{r}(t)) = k$.

$$\therefore \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

$$\Rightarrow \nabla F \cdot \mathbf{r}'(t) = 0$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0.$$

∴ The gradient at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes P.

I. If $F(x,y,z) = k \rightarrow$

① The plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$ is the tangent plane to the level surface $F(x,y,z) = k$ at $P(x_0, y_0, z_0)$.

② The equation of the tangent plane is

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0.$$

③ The normal line to S at P is the line passing through P and perpendicular to the tangent plane.

④ The equation of the normal line is

$$\frac{x-x_0}{F_x} = \frac{y-y_0}{F_y} = \frac{z-z_0}{F_z}.$$

II. If $z=f(x,y) \rightarrow F(x,y,z) = f(x,y)-z = 0.$

② $f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = 0.$

④ $$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{-1}.$$

955

EX 8

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \quad \cdots \quad (-2, 1, -3) \ ?$$

Let $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$

$\rightarrow F_x = \frac{x}{2}. \quad F_y = 2y. \quad F_z = \frac{2}{9}z.$

$\therefore F_x(-2,1,-3) = -1$

$F_y(-2,1,-3) = 2$

$F_z(-2,1,-3) = -2/3.$

→ tangent plane :

$-(x+2) + 2(y-1) - 2/3(z+3) = 0.$

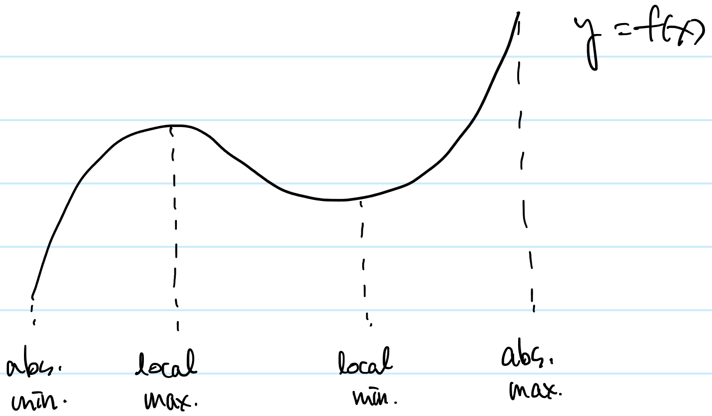
$\Leftrightarrow 3x - 6y + 2z + 18 = 0.$

→ normal line :

$$\frac{x+2}{-1} = \frac{y-1}{2} = -\frac{3(z+3)}{2}.$$

14. 7. Maximum and Minimum Values

2018년 10월 1일 월요일 10:55



Def 1. $f(x,y)$ has a local maximum if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b), or $\exists\, \delta > 0$ such that

$$f(x,y) \le f(a,b) \quad \forall (x,y) \in N_\delta(a,b)$$

2. f has a local minimum if $\exists\, \delta > 0$ such that

$$f(x,y) \ge f(a,b) \quad \forall (x,y) \in N_\delta(a,b).$$

3. f has an absolute maximum if

$$f(x,y) \le f(a,b) \quad \forall (x,y) \in D_f.$$

4. f has an absolute minimum if

$$f(x,y) \ge f(a,b) \quad \forall (x,y) \in D_f.$$

thm If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

<pf>. $g(x) := f(x,b)$, $h(y) := f(a,y)$ $\Rightarrow$ g, h has a local max. (or min.)

at (a, b), $\therefore g'(a) = 0$, $h'(b) = 0$. by Fermat's Theorem.

Def A point (a, b) is called a critical point of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or if one of these partial derivatives does not exist.

EX1. $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

$\rightarrow$ $f_x(x,y) = 2x - 2$ $f_x = 0 \rightarrow x = 1$

$f_y(x,y) = 2y - 6$. $f_y = 0 \rightarrow y = 3$.

$\therefore$ critical point $\rightsquigarrow$ $(1, 3)$.

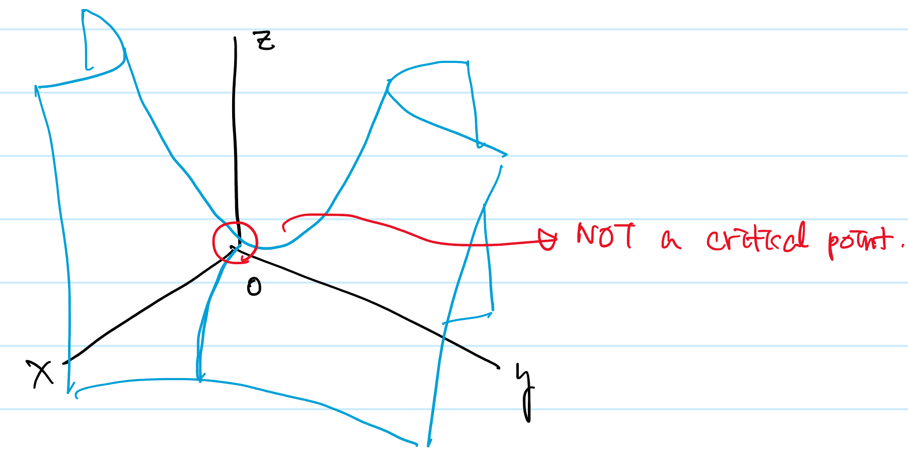
$f(x, y) = (x-1)^2 + (y-3)^2 + 4$

$\therefore$ local minimum at $(1, 3)$.

EX2. Extreme values of $f(x,y) = y^2 - x^2$?

$\rightarrow$ $f_x(x,y) = -2x$ $f_x = 0 \rightarrow x = 0$

$f_y(x,y) = 2y$ $f_y = 0 \rightarrow y = 0$.



961

Thm < Second Derivatives Test >.

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$. i.e., (a,b) is a critical point of f. Let

$$D = D(a,b) := f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2.$$

(a) $D>0$ and $f_{xx}(a,b)>0$ $\rightarrow$ local min.

(b) $D>0$ and $f_{xx}(a,b)<0$ $\rightarrow$ local max.

(c) $D<0$ $\rightarrow$ $\times$. (saddle point)

$$* \quad D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

If $D=0$, the test gives no information.

EX 3. $f(x,y) = x^4 + y^4 - 4xy + 1$.

$$\rightarrow \begin{cases} f_x(x,y) = 4x^3 - 4y \\ f_y(x,y) = 4y^3 - 4x \end{cases}$$

$$\rightarrow \begin{cases} x^3 = y \\ y^3 = x \end{cases}$$ $\therefore$ critical points $(0,0)$, $(1,1)$, $(-1,-1)$

$$\begin{cases} f_{xx}(x,y) = 12x^2 \\ f_{xy}(x,y) = -4 \\ f_{yy}(x,y) = 12y^2 \end{cases} \rightarrow D = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16.$$

$(0,0) \rightarrow D = -16$. $\therefore$ saddle point

$(1,1) \rightarrow D = 128 > 0$.

$f_{xx}(1,1) = 12 > 0$ $\therefore$ local minimum.

$(-1,-1) \rightarrow D = 128 > 0$.

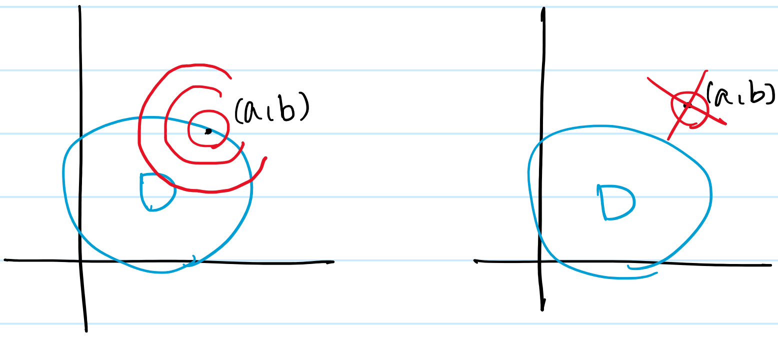
$f_{xx}(-1,-1) = 12 > 0$. $\therefore$ local minimum.

Def $D \subseteq \mathbb{R}^2$

① If every disk with center (a,b) contains points in D and also not in D, i.e.

$$\forall \varepsilon > 0, \quad N_\varepsilon(a,b) \cap D \neq \emptyset \quad \text{and}$$
$$N_\varepsilon(a,b) \cap D^c \neq \emptyset,$$

then (a,b) is called a boundary point of D.



② The set of all boundary points of D is called the boundary of D.

Def

① A set in $\mathbb{R}^2$ that contains all its boundary points is called a closed set.

② A bounded set in $\mathbb{R}^2$ is a set that is contained within some disk. i.e.

$$\exists \varepsilon > 0 \quad \text{such that} \quad D \subseteq N_\varepsilon(0,0)$$

Thm < Extreme Value Theorem for Functions of Two Variables >.

If $f(x,y)$ is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in D. i.e.

$$\exists (x_1,y_1), (x_2,y_2) \in D \quad \text{such that}$$

$$f(x_2, y_2) \le f(x, y) \le f(x_1, y_1),$$
$$\forall (x, y) \in D.$$

Finding Extreme Values

To find the absolute maximum and minimum values of some continuous function f on a closed and bounded set D,

1. Find the values of f at the critical point of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from step 1 and 2 is the absolute maximum value ; the smallest of these is the absolute minimum.

EX7. Find the absolute maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

→ ① $f_x = 2x - 2y$
$f_y = 2 - 2x$
$\therefore$ the only critical point is $(1, 1)$.

② $y = 0 \rightarrow f(x, 0) = x^2 \quad 0 \le x \le 3$
$x = 3 \rightarrow f(3, y) = 9 - 4y \quad 0 \le y \le 2$
$y = 2 \rightarrow f(x, 2) = x^2 - 4x + 4 \quad 0 \le x \le 3$
$x = 0 \rightarrow f(0, y) = 2y \quad 0 \le y \le 2.$

$\therefore$ on the boundary, minimum is 0 and maximum is 9.

③ $f(1,1) = 1.$

So the absolute maximum value is 9, while the minimum is 0.

14. 8. Lagrange Multipliers

2018년 10월 8일 월요일 11:09

암기 < Method of Lagrange Multipliers >

To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x,y,z)=k$,

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $g(x,y,z) = k$

(b) Evaluate f at all the points (x,y,z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

EX 1. $12\ m^2$ of cardboard → max. volume?

→ $V = xyz$

$g(x,y,z) = xy + 2(yz + zx)$.

→ $V_x = yz$ $\quad$ $g_x = y + 2z$

$V_y = xz$ $\quad$ $g_y = x + 2z$

$V_z = xy$ $\quad$ $g_z = 2x + 2y$.

→ $\begin{cases} V_x = \lambda g_x \\ V_y = \lambda g_y \\ V_z = \lambda g_z \end{cases}$; $\begin{cases} yz = \lambda(y+2z) \\ xz = \lambda(x+2z) \\ xy = \lambda(2x+2y) \\ 12 = xy + 2(yz+zx). \end{cases}$

$$\rightarrow \begin{cases} xyz = \lambda(xy+2xz) \\ xyz = \lambda(xy+2yz) \\ xyz = \lambda(2xz+2yz). \end{cases} \rightarrow x=y$$

$$\rightarrow \begin{cases} x^2z = \lambda(x^2+2xz) \\ x^2z = \lambda\cdot 4xz \\ 12 = x^2+4xz. \end{cases} \rightarrow \begin{array}{l} x^2=2xz \\ x=2z. \end{array}$$

$$\rightarrow 12 = (2z)^2+4(2z)(z)$$

$$\therefore x=2,\ y=2,\ z=1.$$

Ex2. Find the extreme values of $f(x,y) = x^2+2y^2$ on $x^2+y^2=1$.

$\rightarrow f_x = \lambda g_x,\quad f_y = \lambda g_y,\quad g(x,y) = x^2+y^2=1.$

$$\rightarrow \begin{cases} 2x = \lambda\cdot 2x \\ 4y = \lambda\cdot 2y \\ x^2+y^2=1 \end{cases} \rightarrow \begin{array}{lll} x=0 & \text{or} & \lambda=1. \\ \lambda=0 & & y=0 \\ y=\pm1 & & x=\pm1 \end{array}$$

$f(0,1) = 2 \qquad f(1,0)=1$

$f(0,-1) = 2 \qquad f(-1,0) = 1$

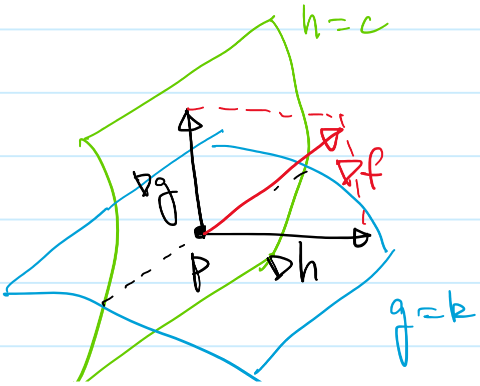
$\therefore$ maximum : $f(0,\pm1)=2$

minimum : $f(\pm1,0)=1.$

< Two Constraints >.

Find the maximum and minimum values of $f(x,y,z)$. subject to two constraints of $g(x,y,z)=k$, and $h(x,y,z)=c$.

$$\begin{cases} \nabla f = \lambda\nabla g + \mu\nabla h \\ g(x,y,z)=k \\ h(x,y,z)=c \end{cases}$$



EX 5

$f(x,y,z) = x+2y+3z$ on the curve of intersection of $x-y+z=1$ and the cylinder $x^2+y^2=1$.

$\rightarrow$ $g(x,y,z) = x-y+z=1$

$h(x,y,z) = x^2+y^2=1.$

$$\therefore \begin{cases} 1 = \lambda+2x\mu. \\ 2 = -\lambda+2y\mu. \\ 3 = \lambda. \\ x-y+z=1 \\ x^2+y^2=1. \end{cases} \rightarrow \begin{matrix} -2 = 2x\mu \\ 5 = 2y\mu. \end{matrix}$$

$\therefore x = -\frac{1}{\mu}. \quad y = \frac{5}{2\mu} \rightarrow \frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$

$\rightarrow \mu = \pm\frac{\sqrt{29}}{2}, \quad x = \mp\frac{2}{\sqrt{29}}, \quad y = \pm\frac{5}{\sqrt{29}}$

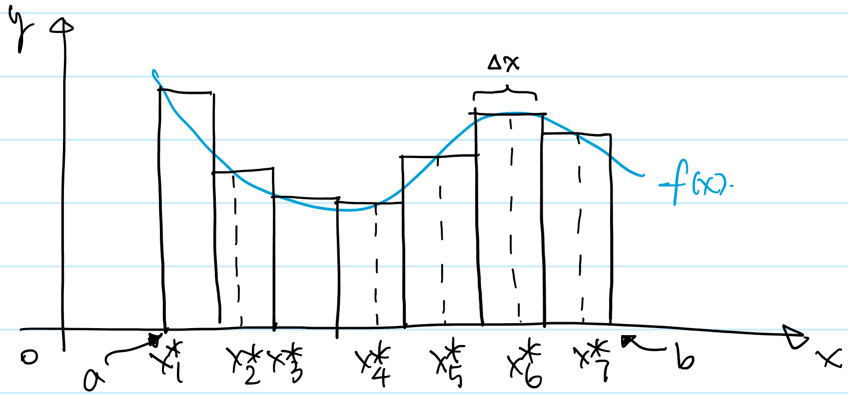
$\therefore$ maximum : $\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1\pm\frac{7}{\sqrt{29}}\right)$

$= 3\pm\sqrt{29} \rightarrow 3+\sqrt{29}.$

15. 1. Double Integrals over Rectangles

2018년 10월 10일 수요일 10:33

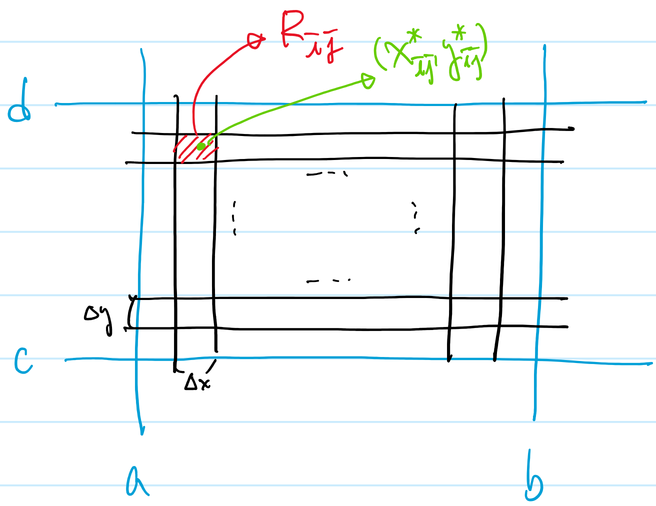
Review $f: [a,b] \to \mathbb{R}$: a bounded function.



$$\to \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x.$$

Def $f: [a,b]\times[c,d] \to \mathbb{R}$

· a bounded function with $f(x,y) \geq 0$,
$\forall (x,y) \in \mathbb{R}^2$.



$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$;
$A(R_{ij}) = \Delta A = \Delta x \Delta y$.

Choose a sample point (x_{ij}^*, y_{ij}^*) in R_{ij} $\forall i, \forall j$.

$$\rightarrow V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

(for sufficiently large i and j)

The double integral of f over the rectangle R is

$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

EX 1. Estimate: $z = 16 - x^2 - 2y^2$, by 4 sample points.

$$\rightarrow V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \Delta A$$

$$= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$$

$$= 34.$$

EX 2. $\iint_R \sqrt{1-x^2}\, dA$

$$\rightarrow V = \frac{1}{2}\pi(1)^2 \cdot 4 = 2\pi.$$

Thm < Midpoint Rule for Double Integrals >.

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j) \Delta A.$$

where $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$, $\bar{y}_i = \dfrac{y_{i-1} + y_i}{2}$.

EX 3. Use the Midpoint Rule with $m = n = 2$ to estimate $\iint_R (x - 3y^2)\, dA$. $R = [0,2] \times [1,2]$.

$$\rightarrow \iint_R (x - 3y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j) \Delta A$$

$= -11.875.$

< Iterated Integrals >

Suppose f is a function of two values, that is integrable on $R = [a,b] \times [c,d]$.

- $A(x) = \int_c^d f(x,y)\,dy.$

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right]dx$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

993

EX 4.

(a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$

$$= \int_0^3 \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2} dx = \int_0^3 \frac{3}{2}x^2\,dx$$

$$= \frac{1}{2}x^3 \Big|_0^3 = \frac{27}{2}.$$

(b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

$$= \int_1^2 \left[\frac{x^3}{3} y\right]_{x=0}^{x=3} dy = \int_1^2 9y\,dy$$

$$= \frac{9}{2}y^2 \Big|_1^2 = \frac{27}{2}.$$

Thm <Fubini's Theorem>.

If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the integrated integrals exist.

Ex 5. $R = [0,2] \times [1,2]$ →

$$\iint_R (x - 3y^2)\, dA$$

$$= \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx$$

$$= \int_0^2 (xy - y^3)\Big|_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx$$

$$= \left(\frac{1}{2}x^2 - 7x\right)\Big|_0^2 = -12.$$

Ex 6. $R = [1,2] \times [0,\pi]$.

$$\iint_R y \sin xy\, dA$$

$$= \int_0^\pi \int_1^2 y \sin xy\, dx\, dy.$$

$$= \int_0^\pi [-\cos xy]_{x=1}^{x=2}\, dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\, dy$$

$$= \left[-\frac{1}{2}\sin 2y + \sin y\right]_0^\pi = 0.$$

Ex 7. $x^2 + 2y^2 + z = 16$, $x = 2$, $y = 2$, $x = 0$, $y = 0$, $z = 0$

$\Rightarrow V = -\iint_R (x^2+2y^2-16)\, dA$

$= -\int_0^2 \int_0^2 (x^2+2y^2-16)\, dx\,dy$

$= -\int_0^2 \left[-16x+\frac{1}{3}x^3+2y^2x\right]_{y=0}^{y=2} dy$

$= \int_0^2 \left(\frac{88}{3}-4y^2\right) dy = 48.$

$\ast$ $\iint_R g(x)h(y)\, dA = \int_a^b g(x)dx \int_c^d h(y)dy$

where $R=[a,b]\times[c,d]$.

Ex8. $R=[0,\pi/2]\times[0,\pi/2]$

$\Rightarrow \iint_R \sin x \cos y\, dA$

$= \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy$

$= 1\times 1 = 1.$

< Average Value >

1) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$. is the average value of f defined on $[a,b]$.

2) $f_{ave} = \frac{1}{A(R)}\iint_R f(x,y)\,dA$ is the average value of f defined on R.

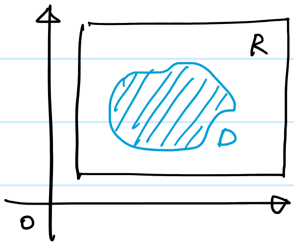
15. 2. Double Integrals over General Regions

2018년 10월 15일 월요일 10:57

Def $D \subset \mathbb{R}^2$: a bounded region in $\mathbb{R}^2$.

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$$

$$\to \iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.$$



Def 1 A plane region D is said to be type 1 if it lies between the graphs of two continuous functions of x. that is:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$.

2 $\{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$ is said to be type 2 where h_1 and h_2 are continuous on $[c,d]$.

Ex 1. $D = \{(x,y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$:

$$\iint_D (x+2y)\,dA$$

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[xy+y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx = \frac{32}{15}.$$

Ex 2. $D : \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$, x^2+y^2

$\Rightarrow V = \iint_D (x^2+y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$

$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx.$

$= \frac{216}{35}.$

Ex 3. $D: \{(x,y) \mid -2 \le y \le 4,\ \frac{1}{2}y^2 - 3 \le x \le y+1\}.$

$\Rightarrow \iint_D xy\,dA$

$= \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy$

$= \frac{1}{2} \int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy.$

$= 36.$

Ex 5.

$\int_0^1 \int_x^1 \sin y^2\,dy\,dx \quad \Rightarrow ??$

$D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$

$= \iint_D \sin y^2\,dA$

$= \int_0^1 \int_0^y \sin y^2\,dx\,dy = \int_0^1 x \sin y^2 \Big|_{x=0}^{x=y} dy$

$= \int_0^1 y \sin y^2\,dy = \frac{1}{2}(1-\cos 1).$

thm

- $\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA.$

- $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$.
- If $f \geq g \ \forall x,y$ then $\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$.
- If $\begin{cases} D = D_1 \cup D_2 \\ \emptyset = D_1 \cap D_2 \end{cases}$ then $\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$.
- $\iint_D 1\,dA = A(D)$.
- $m\,A(D) \leq \iint_D f(x,y)\,dA \leq M\,A(D)$

Ex 6. D :

$$\iint_D e^{\sin x \cos y}\,dA \ ?$$

$$\Downarrow$$

$$m\,A(D) \leq \iint_D e^{\sin x \cos y}\,dA \leq M\,A(D)$$

$$\therefore \quad \frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y}\,dA \leq 4\pi e$$

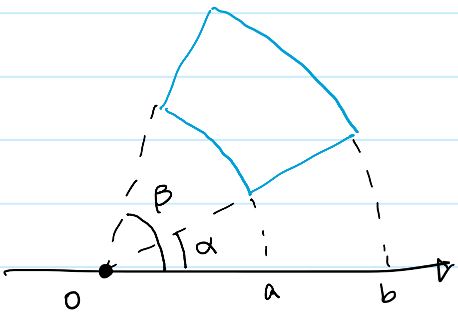
$$(\because \ e^{-1} \leq e^{\sin x \cos y} \leq e)$$

15. 3. Double Integrals in Polar Coordinates

2018년 10월 17일 수요일 10:43

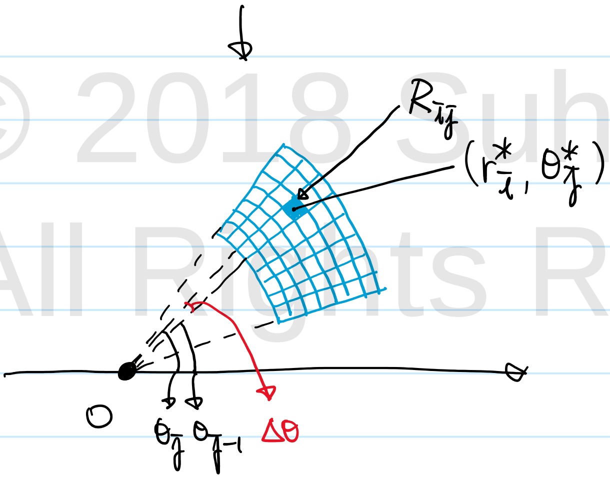
1010

Def



$R = \{ (r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta \}$.

: a polar rectangle.



$$r_i^* = \frac{1}{2}(r_{i-1} + r_i)$$
$$\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

$$\rightarrow \Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta$$
$$= \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$
$$= \underline{\frac{1}{2}(r_i + r_{i-1})}\ \underline{(r_i - r_{i-1})}\ \Delta\theta$$
$$= r_i^* \Delta r \Delta\theta.$$

$$\therefore \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\ \Delta A_i$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\, r_i^*\, \Delta r \Delta\theta.$$

$$\downarrow\ g(r,\theta) \overset{\text{def}}{=} r f(r\cos\theta, r\sin\theta)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*) \Delta r \Delta\theta.$$

$$\rightsquigarrow \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*) \Delta r \Delta\theta$$

$$= \int_\alpha^\beta \int_a^b g(r,\theta)\, dr\, d\theta.$$

$$\therefore \iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, \underline{r}\, dr\, d\theta.$$

Thm If f is continuous on R given by
$0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where
$0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

1012 **EX 1.** $R : \{ (x,y) \mid y \ge 0,\ 1 \le x^2+y^2 \le 4 \}.$

$$\to \iint_R (3x + 4y^2)\, dA$$

$$= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta$$

$$= \frac{15}{2}\pi.$$

EX 2. region bounded by $z=0$ and $z = 1 - x^2 - y^2$.

$\to D = \{ (r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi \}.$

$$V = \iint_D (1 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 (r - r^3)\, dr = \frac{\pi}{2}.$$

Thm If f is continuous on a polar region of the form

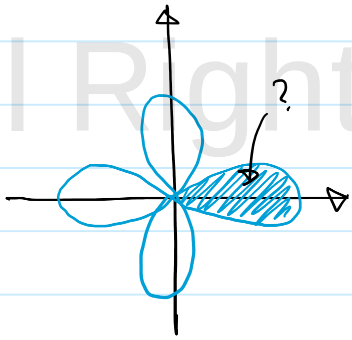
$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then $\displaystyle\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$

or when $D = \{(r, \theta) \mid a \le r \le b,\ g_1(r) \le \theta \le g_2(r)\}$

then $\displaystyle\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Ex 3 $r = \cos 2\theta$



$$D = \{(r, \theta) \mid -\tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{4},\ 0 \le r \le \cos 2\theta\}$$

$$\to A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta = \frac{\pi}{8}.$$

Ex 4 $S: z = x^2 + y^2$, above the xy-plane

$$\to D = \{(r, \theta) \mid -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le 2\cos\theta\}$$

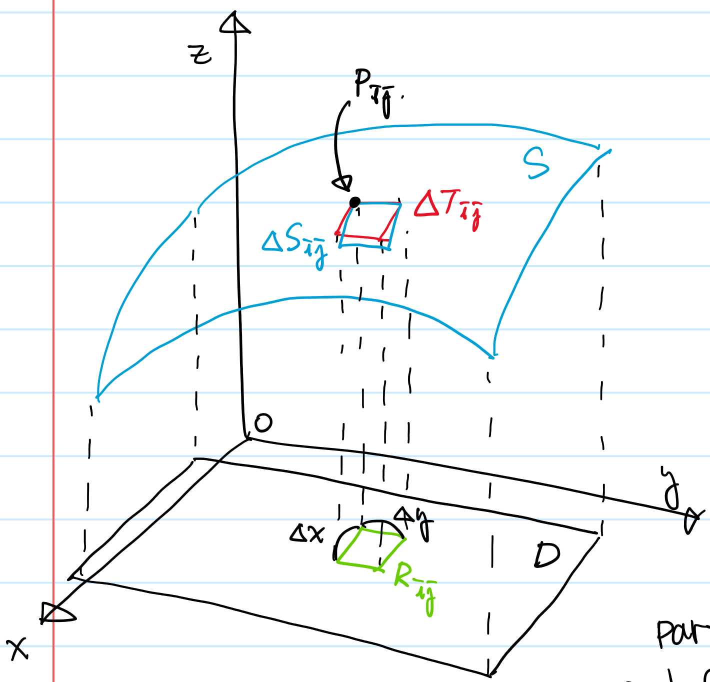
$$\therefore V = \iint_D (x^2+y^2)\, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r \, dr\, d\theta$$

$$= 8 \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta = \frac{3\pi}{2}$$

15. 5. Surface Area

2018년 10월 17일 수요일 11:27



Let S be a surface with equation $z = f(x,y)$ where f has continuous partial derivatives and $f(x,y) \geq 0$ on D.

then the surface area $A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$.

Let $\Delta T_{ij} = |\mathbf{a}\times\mathbf{b}|$. then

$$\mathbf{a} = \Delta x\,\mathbf{i} + f_x(x_i, y_j)\,\Delta x\,\mathbf{k}$$

$$\mathbf{b} = \Delta y\,\mathbf{j} + f_y(x_i, y_j)\,\Delta y\,\mathbf{k}$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix}$$

$$= [-f_x(x_i, y_j)\,\mathbf{i} - f_y(x_i, y_j)\,\mathbf{j} + \mathbf{k}]\,\Delta A$$

$$\therefore \Delta T_{ij} = |\mathbf{a}\times\mathbf{b}| = \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\cdot\Delta A$$

$$\to A(S) = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\; dA$$

Ex 1. $S:\ z = x^2 + 2y$.

$T:\ (0,0),\ (1,0),\ (1,1)$.

$$\Rightarrow A(S) = \iint_T \sqrt{(2x)^2 + 2^2 + 1}\, dA$$

$$= \iint_T \sqrt{4x^2+5}\, dA$$

$$= \int_0^1 \int_0^x \sqrt{4x^2+5}\, dy\, dx$$

$$= \int_0^1 x\sqrt{4x^2+5}\, dx = \frac{27-5\sqrt{5}}{12}.$$

EX 2. $z = x^2+y^2$ under $z=9$

$$\Rightarrow A = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA$$

$$= \iint_D \sqrt{1+4(x^2+y^2)}\, dA$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= \frac{\pi}{6}(37\sqrt{37}-1).$$

15. 6. Triple Integrals

2018년 10월 29일 월요일 10:52

Def $B = [a,b]\times[c,d]\times[r,s]$

: $B = \{(x,y,z) \mid a\le x\le b,\ c\le y\le d,\ r\le z\le s\}$.

Choose a sample point $P^*_{ijk}(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each B_{ijk}. Then we form the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}).$$

The triple integral of f over B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$$

Thm Fubini's Theorem for Triple Integrals.

If f is continuous on the rectangular box B,

then

$$\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Ex 1.

$$\iiint_B xyz^2\,dV \quad ; \quad B=[0,1]\times[-1,2]\times[0,3]$$

$$= \int_0^3\int_{-3}^2\int_0^1 xyz^2\,dx\,dy\,dz$$

$$= \int_0^3\int_{-3}^2 \frac{yz^2}{2}\,dy\,dz = \int_0^3 \frac{3z^2}{4}\,dz = \frac{27}{4}.$$

※ Let E be a bounded region in $\mathbb{R}^3$. Choose a box containing E:

1) Type 1 :

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dx\,dA$$

2) Type 2.

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dA.$$

3) Type 3.

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dx\,dA.$$

Ex 3.

$$\iiint_E \sqrt{x^2+z^2}\,dV \quad \text{where } E:\ x^2+z^2 \le y \le 4.$$

$$\to \iiint_E \sqrt{x^2+z^2}\,dV$$

$$= \iint_{D_1} (4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)r\cdot r\,dr\,d\theta = \frac{128\pi}{15}.$$

Ex 4.

$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$$

$$\Downarrow\ E=\{(x,y,z)\mid 0\le x\le 1,\ 0\le y\le x^2,\ 0\le z\le y\}$$

$$\iiint_E f(x,y,z)\,dV.$$

$$= \int_0^1\int_0^y\int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy.$$

< Applications of Triple Integrals >.

* Let E be a solid in $\mathbb{R}^3$. then

$$V(E) = \iiint_E dV.$$

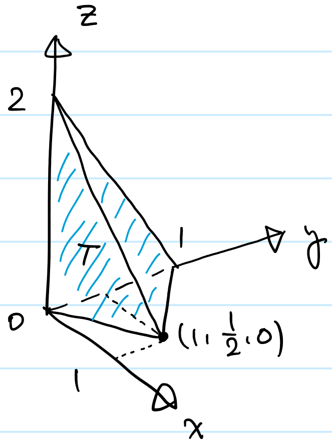
Ex 5. T: bounded by

$x + 2y + z = 2$

$x = 2y$

$x = 0$

$z = 0$.



→ T:

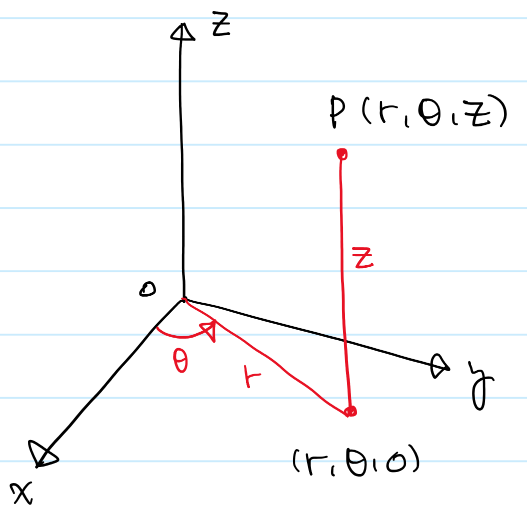
$$V(T) = \iiint_T dV$$

$$= \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} dz\,dy\,dx = \frac{1}{3}.$$

15. 7. Triple Integrals in Cylindrical Coordinates

2018년 10월 31일 수요일 10:57

<Cylindrical Coordinates>



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z. \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x. \end{cases}$$

EX1.

(a) cylindrical : $\left(2, \frac{2}{3}\pi, 1\right)$

→ cartesian : $(-1, \sqrt{3}, 1)$

(b) cartesian : $(2, -2, -7)$

→ cylindrical : $\left(3\sqrt{2}, -\frac{\pi}{4}, -7\right)$.

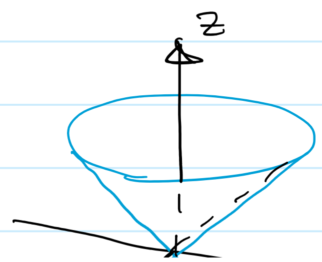
EX2.

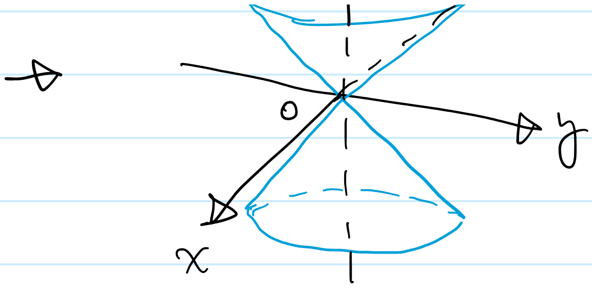
$z = r$

→ $z^2 = r^2 = x^2 + y^2$.

→ $z = \pm\sqrt{x^2+y^2}$

→





< Evaluating Triple Integrals with Cylindrical Coordinates >

$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

$$\iiint_E f(x,y,z)\,dV$$

$$= \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA.$$

$$= \iint_D \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta,\, r\sin\theta,\, z)\,dz\,dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} r f(r\cos\theta,\, r\sin\theta,\, z)\,dz\,dr\,d\theta.$$

EX3'. E : below $z=4$, above $z=1-y^2-x^2$, lies within $x^2+y^2=1$

$$V = \iiint_E 1\,dV$$

$$= \iint_D \int_0^{1-r^2} 1\,dz\,dA$$

$$= \int_0^{2\pi} \int_0^1 r \cdot \int_{1-r^2}^{4} dz \cdot dr\,d\theta.$$

$$= \int_0^{2\pi} \int_0^1 r\cdot(3+r^2)\,dr\,d\theta.$$

$$= 2\pi \left(\frac{3}{2} r^2 + \frac{1}{4} r^4 \right) \Big|_0^1 = \frac{7}{2}\pi.$$

Ex 4

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\,dy\,dx.$$

$$= \iiint_E (x^2+y^2)\, dV.$$

$$: E = \{ (x,y,z) \mid 2 \le x \le -2,$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2},$$
$$\sqrt{x^2+y^2} \le z \le 2 \}$$

$$\Leftrightarrow E = \{ (r,\theta,z) \mid -2 \le r \le 2,\ r \le z \le 2 \}.$$

$$= \iint_D \int_r^2 r^2\, dz\, dA$$

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, dz \cdot r \cdot dr\, d\theta$$

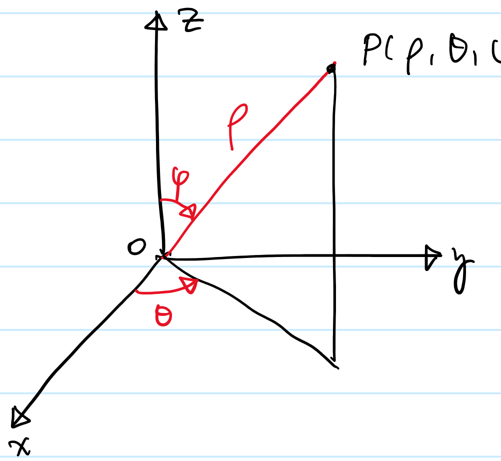
$$= 2\pi \int_0^2 (2-r) r^2 \cdot r\, dr.$$

$$= 2\pi \left(\frac{1}{2} r^4 - \frac{1}{5} r^5 \right) \Big|_0^2 = \frac{16}{5}\pi.$$

15. 8. Triple Integrals in Spherical Coordinates

2018년 11월 5일 월요일 10:33

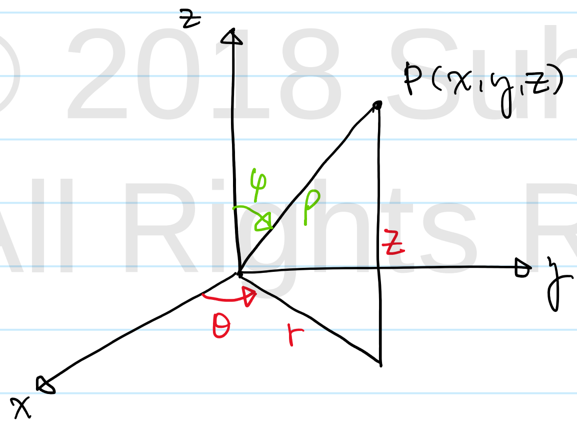
< Spherical Coordinates >



$P(\rho, \theta, \varphi)$: the spherical coords. of P.

: $\rho \geq 0$
$0 \leq \varphi \leq \pi$.

*



$$\begin{cases} z = \rho\cos\varphi \\ r = \rho\sin\varphi \end{cases}$$

$$\begin{cases} x = r\cos\theta = \rho\cos\theta\sin\varphi. \\ y = r\sin\theta = \rho\sin\theta\sin\varphi. \\ z \qquad\quad = \rho\cos\varphi. \end{cases}$$

$\Rightarrow x^2 + y^2 + z^2 = \rho^2$.

Ex 1 $(2, \pi/4, \pi/3)$

$$\Rightarrow \begin{cases} x = 2\cos\pi/4 \sin\pi/3 = \sqrt{3/2} \\ y = 2\sin\pi/4\sin\pi/3 = \sqrt{3/2} \\ z = 2\cos\pi/3 = 1 \end{cases}$$

$\rightarrow \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right).$

EX2. $(0, 2\sqrt{3}, -2)$

$\rightarrow \rho^2 = 0^2 + (2\sqrt{3})^2 + (-2)^2 = 4^2.$

$\therefore \rho = 4.$

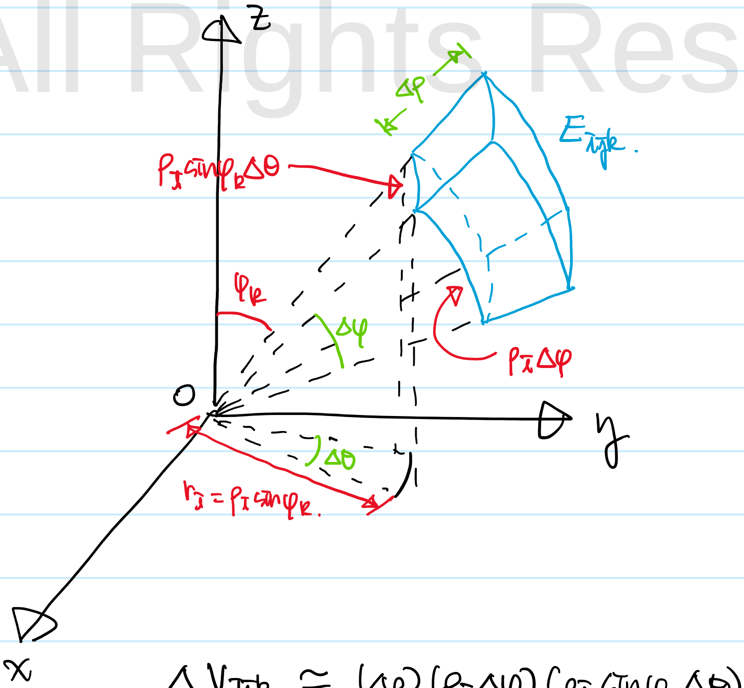
$\cos\varphi = -\frac{1}{2} \rightarrow \varphi = \frac{2}{3}\pi$

$\cos\theta = 0 \rightarrow \theta = \frac{1}{2}\pi$

$\rightarrow \left(4, \frac{\pi}{2}, \frac{2}{3}\pi\right).$

<Evaluating Triple Integrals with Spherical Coordinates>

$E = \{(\rho, \theta, \varphi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \varphi \le d\}$



$$\Delta V_{ijk} \simeq (\Delta\rho)(\rho_i \Delta\varphi)(\rho_i \sin\varphi_k \Delta\theta)$$
$$= \rho_i^2 \sin\varphi_k \Delta\rho \Delta\theta \Delta\varphi$$

$$\rightarrow \Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\varphi}_k \Delta\rho\Delta\theta\Delta\varphi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\varphi}_k)$ is some point in E_{ijk}. then

If $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\varphi}_k) = (x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$:

$$\iiint_E f(x,y,z)\,dV$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V_{ijk}.$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \Big[$$

$$f(\tilde{\rho}_i \sin\tilde{\varphi}_k \cos\tilde{\theta}_j,\ \tilde{\rho}_i \sin\tilde{\varphi}_k \sin\tilde{\theta}_j,\ \tilde{\rho}_i \cos\tilde{\varphi}_k)$$

$$\times\ \tilde{\rho}_i^2 \sin\tilde{\varphi}_k\,\Delta\rho\Delta\theta\Delta\varphi\ \Big].$$

$\therefore\ F(\rho, \theta, \varphi) =$

$$f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\ \rho^2\sin\varphi.$$

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\ \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

EX3. $B : \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

$$\iiint_B \exp\left[(x^2+y^2+z^2)^{3/2}\right] dV$$

$\downarrow$ $B : \{(r,\theta,\varphi) \mid r \le 1\}$.

$$= \int_0^\pi\int_0^{2\pi}\int_0^1 \rho^2\exp(\rho^3)\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

$$= \int_0^\pi \sin\varphi\,d\varphi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2 e^{\rho^3}\,d\rho.$$

$$= \frac{4}{3}\pi(e-1).$$

Ex 4. Above $z=\sqrt{x^2+y^2}$, below $x^2+y^2+z^2=z$?

Sphere : $\rho^2=\rho\cos\varphi$. $\rightarrow$ $\rho=\cos\varphi$.

Cone : $\rho\cos\varphi=\sqrt{\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta}=\rho\sin\varphi$

$\rightarrow$ $\cos\varphi=\sin\varphi$, $\varphi=\pi/4$.

$\therefore$ $E:\{(\rho,\theta,\varphi)\mid 0\le\theta\le2\pi,\ 0\le\varphi\le\pi/4,\ 0\le\rho\le\cos\varphi\}$.

$\therefore$ $V(E)=\iiint_E dV$

$$=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\varphi\left(\frac{\rho^3}{3}\right)\Bigg|_0^{\cos\varphi}d\varphi$$

$$=\frac{\pi}{8}.$$

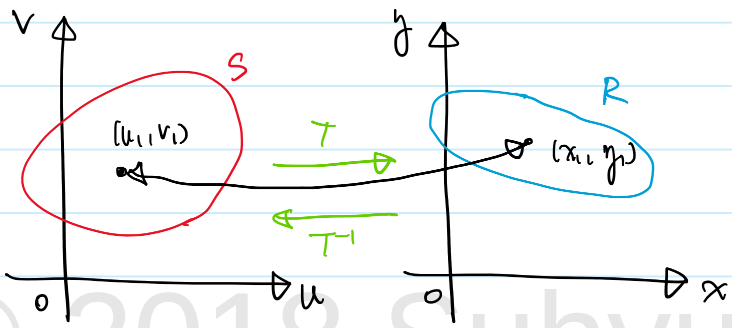
15. 9. Change of Variables in Multiple Integrals

2018년 11월 5일 월요일 11:28

$$* \int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$.

Def $S, R \in \mathbb{R}^2$.



- A transformation T is a function whose domain and range are both subsets of $\mathbb{R}^2$.
- $T(u_1, v_1) = R(x_1, y_1)$

  $\Rightarrow$ (x_1, y_1) : the image of (u_1, v_1).
- If no two points have the same image, T is called one-to-one.
- $S^* \subseteq S \rightarrow T(S^*) = \{T(u,v) \mid (u,v) \in S^*\}$.

  : the image of S^* under T.
- **Thm** $S, R \subseteq \mathbb{R}^2$

  $\rightarrow$ $T : S \rightarrow R$ such that T is one-to-one

  $T(S) = R$

  then $\exists\, T^{-1} : R \rightarrow S$: the inverse of T.

Def The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

$$\to \quad \Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v.$$

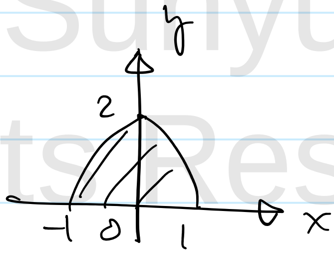
Change of Variables in a Double Integral

$$T : \underset{uv}{S} \longrightarrow \underset{xy}{R}$$

$$\to \iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Ex 2.

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

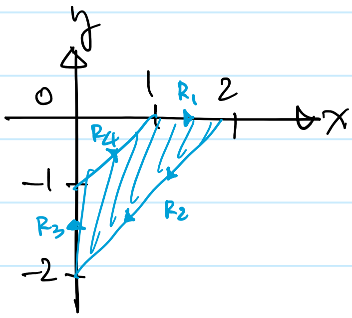


$$\to \iint_R y\,dA = \iint_S 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

$$= \int_0^1\int_0^1 2uv \cdot 4(u^2+v^2)\,du\,dv = 2.$$

Ex3.

R:

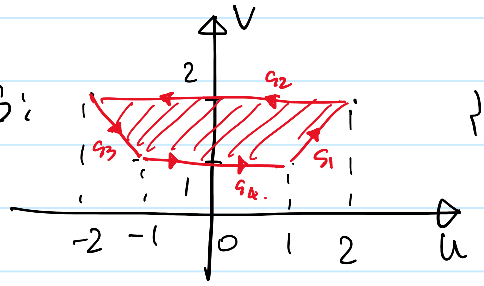


$$\iint_R e^{\frac{x+y}{x-y}}\,dA\ ?$$

T^{-1} : $u = x+y$, $v = x-y$.

T $\downarrow$ $\therefore x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v).$

S:



$\{(u,v) \mid 1 \le v \le 2,\ -v \le u \le v\}.$

$$\therefore \iint_R e^{\frac{x+y}{x-y}}\, dA$$

$$= \iint_S e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

$$= \int_1^2 \int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}}\, du\,dv$$

$$= \frac{1}{2}\int_1^2 v e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 \left(e - \frac{1}{e}\right) v\, dv$$

$$= \frac{3}{4}\left(e - \frac{1}{e}\right).$$

<Triple Integrals>.

$T: S \to R,\quad S, R \in \mathbb{R}^3.$

$$(u,v,w) \to (x,y,z) = (g(u,v,w),\ h(u,v,w),\ k(u,v,w))$$

→ $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

→ $$\iiint_R f(x,y,z)\,dV$$

$$= \iiint_S f(x(u,v,w),\ y(u,v,w),\ z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

EX 4. $(x,y,z) \to (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)$

→ $$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\theta & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= -\rho^2\sin\theta.$$

$$\therefore \iiint_R f(x,y,z)\,dV$$

$$= \iiint_S f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

16. 1. Vector Fields

2018년 11월 12일 월요일 10:51

Def 1) $D \subseteq \mathbb{R}^2$

$\mathbb{F} : D \to \mathbb{R}^2$ or V_2

$(x, y) \mapsto \mathbb{F}(x, y)$.

$= (P(x,y), Q(x,y))$.

$= P\mathbf{i} + Q\mathbf{j}$

: a vector field on $\mathbb{R}^2$.

2) $D \subseteq \mathbb{R}^3$

$\mathbb{F} : D \to \mathbb{R}^3$ or V_3.

$(x, y, z) \to \mathbb{F}(x, y, z)$

$= (P(x,y,z), Q(x,y,z), R(x,y,z))$

$= P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

: a vector field on $\mathbb{R}^3$.

< Gradient Fields >.

$\nabla f(x, y) = f_x\mathbf{i} + f_y\mathbf{j}$

$\nabla f(x, y, z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.

Def A vector field $\mathbb{F}$ is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbb{F} = \nabla f$. In this situation, f is called a potential function for $\mathbb{F}$.

16. 2. Line Integrals

2018년 11월 12일 월요일 11:10

Def $C : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ or $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

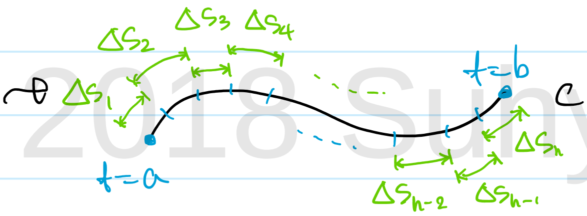
If $\mathbf{r}'$ is continuous and $\mathbf{r}' \neq \mathbf{0}$, then the curve C is called a smooth curve.

Def $C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

: a smooth curve.

f: a function defined on C.

$a = t_0 < t_1 < t_2 < \cdots < t_n = b$



Choose any point $P_i^*(x_i^*, y_i^*)$ in the i^{th} subarc.

then the line integral of f along C is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i.$$

if this limit exists.

Thms

1) $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$

2) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

3) $\int_C f(x,y)\,ds$

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

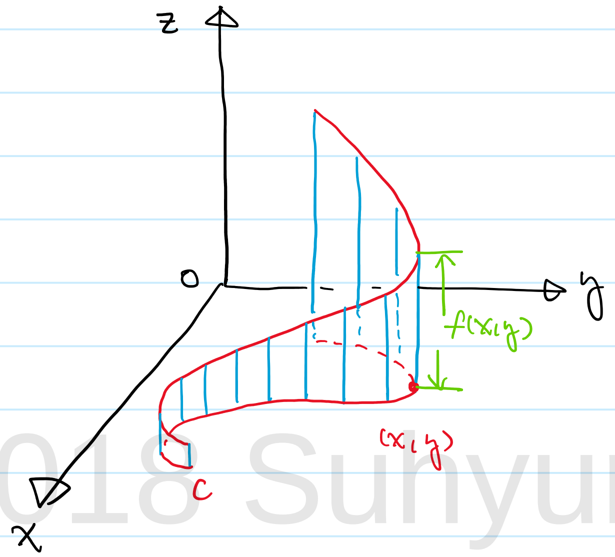
a) C: the line segment that joins $(a, 0)$ to $(b, 0)$.

C; $x = t$, $y = 0$ $\quad a \le t \le b$.

$$\to \int_C f(x,y)\,ds = \int_a^b f(t,0)\sqrt{1^2+0^2}\,dt$$

$$= \int_a^b f(t,0)\,dt.$$

b) If $f(x,y) \geq 0$, then



$\int_C f(x,y)\,ds$ represents the area of one side of the "fence" or "curtain" whose base is C and whose height above the point (x,y) is $f(x,y)$.

EX 1. $\int_C (2+x^2y)\,ds$ where C: $y = \sqrt{1-x^2}$?

$$\to \int_C (2+x^2y)\,ds$$

$\downarrow$ $x = \cos t$, $y = \sin t$.

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\,dt = \underline{2\pi + \frac{2}{3}}.$$

Def If a curve C is a union of finite number of smooth curves $C_1, C_2, \cdots, C_n$, where the initial point of C_{n+1} is the terminal point of C_n. then C is called a piecewise-smooth curve.

Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

EX2. $C_1 : y = x^2 \quad (0,0) \to (1,1)$

$C_2 : x = 1 \quad (1,1) \to (1,2)$

$$\int_C 2x\,ds$$

$$= \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx + \int_1^2 2\cdot\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy$$

$$= \frac{5\sqrt{5}+11}{6}.$$

Def $C : x = x(t),\ y = y(t),\quad a \le t \le b$

1) $$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$

$$= \int_a^b f(x(t), y(t))\, x'(t)\,dt$$

: the line integral of $f(x,y)$ along C with respect to x.

2) $$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\, y'(t)\,dt$$

3) $$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

$$= \int_C P(x,y)\,dx + Q(x,y)\,dy.$$

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$$

: a vector equation of a line segment.

Ex 4.

(a) C_1 : $(-5,-3) \cdots (0,2)$

$\Rightarrow$ $x = 5t-5$, $y = 5t-3$.

$$\int_{C_1} y^2\,dx + x\,dy$$

$$= \int_0^1 (5t-3)^2 (5\,dt) + (5t-5)(5\,dt)$$

$$= -\frac{5}{6}.$$

(b) C_2 : $(-5,-3) \cdots (0,2)$ $\quad x = 4-y^2$.

$\Rightarrow$ $x = 4-y^2$, $y = y$.

$$\int_{C_2} y^2\,dx + x\,dy$$

$$= \int_{-3}^{2} y^2(-2y)\,dy + (4-y^2)\,dy$$

$$= \frac{245}{6}.$$

In general, a parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an orientation of a curve C, with the positive direction corresponding to increasing values of the parameter t.

If $-C$ denotes the curve consisting of the same points of C but with opposite orientation, then we have

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$$

$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy.$$

But if we integrate with respect to arc length, the value of the line integral does not change when we reverse the orientation of the curve :

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds.$$

This is because Δs is always positive.

< Line Integrals in Space > .

$C:\ x = x(t),\ y = y(t),\ z = z(t), \quad a \le t \le b.$

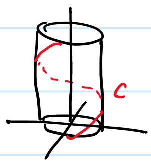
then the line integral of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

$$= \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

EX 5. $C:\ x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le 2\pi.$



$$\int_C y \sin z\,ds$$

$$= \int_0^{2\pi} \sin^2 t \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sin^2 t \cdot \sqrt{2}\,dt = \sqrt{2}\pi.$$

EX 6. $C_1 : (2,0,0) \to (3,4,5)$

$C_2 : (3,4,5) \to (3,4,0)$

$$\int_C y\,dx + z\,dy + x\,dz$$

$C_1 : \mathbf{r}(t) = (1-t)(2,0,0) + t(3,4,5)$
$= (2+t,\ 4t,\ 5t)$

$C_2 : \mathbf{r}(t) = (1-t)(3,4,5) + t(3,4,0)$
$= (3,4,5-5t).$

$$= \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 \left(4t + 5t\cdot 4 + (2+t)\cdot 5\right) dt$$

$$+ \int_0^1 \left(4\cdot 0 + (5-5t)\cdot 0 + 3\cdot(-5)\right) dt.$$

$$= \frac{49}{2} - 15 = \frac{19}{2}.$$

☆ $C_2 : x=3,\ y=4,\ z=t,\quad \underline{t : 5 \to 0}\ \to ?$

< Line Integrals of Vector Fields >

Def Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$.

Then the line integral of f along C is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$$

$$= \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

$$= \int_a^b \left[P x'(t) + Q y'(t) + R z'(t)\right] dt$$

Ex 7. $\mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j}$

$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \qquad 0 \le t \le \pi/2.$

$$\rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} (-2\cos^2 t \sin t)\,dt.$$

$$= 2\left.\frac{\cos^3 t}{3}\right|_0^{\frac{\pi}{2}} = -\frac{2}{3}.$$

Ex8. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$

$C:\ x=t,\ y=t^2,\ z=t^3 \quad 0\le t\le 1.$

$$\rightarrow \mathbf{r}(t) = (t,\ t^2,\ t^3)$$

$$\mathbf{r}'(t) = (1,\ 2t,\ 3t^2).$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$= \int_0^1 (t^3+5t^6)\,dt = \frac{27}{28}.$$

$*\ \int_C \mathbf{F}\,d\mathbf{r}$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t))\,\mathbf{r}'(t)\,dt$$

$$= \int_a^b \left[P x'(t) + Q y'(t) + R z'(t)\right] dt$$

16. 3. The Fundamental Theorem for Line Integrals

2018년 11월 19일 월요일 10:38

Thm Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of 2~3 variables, whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

<pf>

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t))\mathbf{r}'(t)\, dt$$

$$= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

by the Fundamental Theorem of Calculus.

EX 1. $(3, 4, 12) \to (2, 2, 0)$

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x} \quad ?$$

$$\to W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(2, 2, 0) - f(3, 4, 12)$$

$$= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right).$$

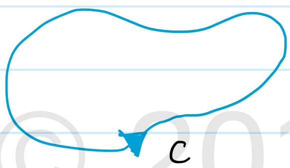
⟨ Independence of Path ⟩

Def A piecewise - smooth curve is called a path.

Thm If ∇f is continuous on D, then

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$
$$= f(\mathbf{r}(a)) - f(\mathbf{r}(b)).$$

Def If $\mathbf{F}$ is a continuous vector field with domain D, we say that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path if $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points.



Def A curve C is called closed if the terminal point coincides with the initial point. i.e. $\mathbf{r}(b) = \mathbf{r}(a)$.

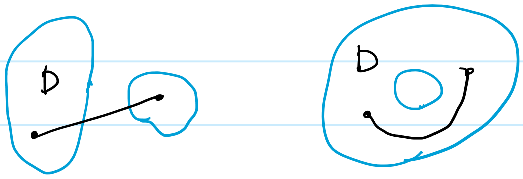
Thm pf at p.1089

$\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path D iff

$\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

Def 1). If $\forall p \in D$, there is a disk with center P that lies entirely in D, i.e. $\forall p \in D$, $\exists\, \varepsilon > 0$ such that $N_\varepsilon(p) \subset D$, then D is said to be open.

2)



If any two points in D can be joined by a path that lies in D, then D is said to be connected.

Thm Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D. i.e,

$\exists f(x,y)$ such that $\nabla f = \mathbf{F}$.

Thm If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

⟨pf⟩ Since $\mathbf{F}$ is conservative, $\exists f$ such that

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = P\mathbf{i} + Q\mathbf{j}.$$

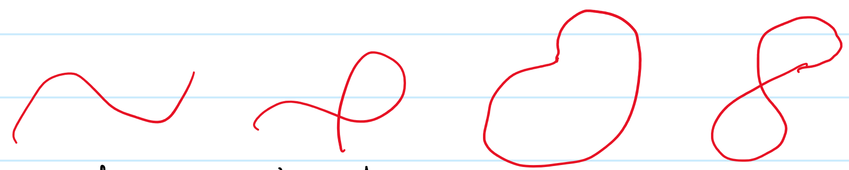
By Clairaut's theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}. \text{ on } D.$$

Def 1) A curve that doesn't intersect itself anywhere between its endpoints is called a simple curve.

2) A simple curve with $\mathbf{r}(a) = \mathbf{r}(b)$ is called a simple closed curve.



simple, not closed — not simple, not closed — simple, closed — not simple, closed.

3) A connected region D such that every simple closed curves in D encloses only points that are in D is called a simply-connected region in the plane.

Thm Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D, Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Then $\mathbf{F}$ is conservative. i.e. $\exists f$ such that

$$\nabla f = \mathbf{F}.$$

<pf> Suppose that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

Show : $\exists f(x,y)$ such that

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = P\mathbf{i} + Q\mathbf{j}$$

i.e. $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$.

Let $f(x,y) = \int P(x,y)\,dx - g(y)$.

Since $\frac{\partial f}{\partial y} = Q$, $\frac{\partial f}{\partial y} = Q = \frac{\partial}{\partial y}\int P\,dx - g'(y)$.

$\therefore g'(y) = Q - \frac{\partial}{\partial y}\int P\,dx.$

Since $\frac{\partial}{\partial x}\left(Q - \frac{\partial}{\partial y}\int P\,dx\right)$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0,$$

$\therefore g(y) = \int\left[Q - \frac{\partial}{\partial y}\int P\,dx\right]dy + C.$

Thus

$$f(x,y) = \int P\,dx + \int \left[Q - \frac{\partial}{\partial y}\int P\,dx \right] dy + C.$$

Note that

$$f_x = P \ ; \quad f_y = \frac{\partial}{\partial y}\int P\,dx + Q - \frac{\partial}{\partial y}\int P\,dx$$
$$= Q.$$

Thus $\nabla f = \mathbf{F}$.

Ex 3. $\mathbf{F}(x,y) = (3+2xy)\,\mathbf{i} + (x^2-3y^2)\,\mathbf{j}$

$\Rightarrow \dfrac{\partial P}{\partial y} = 2x = \dfrac{\partial Q}{\partial x}$, $\therefore$ conservative.

Ex 5. $\mathbf{F}(x,y,z) = y^2\,\mathbf{i} + (2xy+e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$.

$\Rightarrow \nabla f = \mathbf{F}$?

$$\Rightarrow \begin{cases} f_x(x,y,z) = y^2 \\ f_y(x,y,z) = 2xy + e^{3z} \\ f_z(x,y,z) = 3ye^{3z}. \end{cases}$$

$f(x,y,z) = xy^2 + g(x,y)$

$f_y(x,y,z) = 2xy + g_y(x,y) = 2xy + e^{3z}$

$\Rightarrow g_y(x,y) = e^{3z}$

$g(y,z) = ye^{3z} + h(z)$.

$f(x,y,z) = xy^2 + ye^{3z} + h(z)$.

$f_z(x,y,z) = 3ye^{3z} + h'(z)$.

$\Rightarrow h'(z) = 0$,

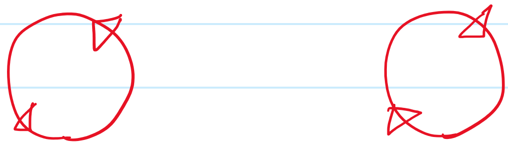
$\therefore f(x,y,z) = xy^2 + ye^{3z} + C$.

$\Rightarrow \nabla f = \mathbf{F}$.

16. 4. Green's Theorem

2018년 11월 21일 수요일 11:26

Def The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.



Def The region which is type I and type II is called a simple region.

Thm ⟨Green's Theorem⟩

Let C be a positively oriented, piecewise-smooth, simple closed curve and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

(proof at p. 1097)

Note

1) $\oint_C P\,dx + Q\,dy$ or $\oint P\,dx + Q\,dy$

is sometimes used to indicate that the line integral is calculated using the positive orientation.

2) $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$

EX 1. C: $(0,0) \to (1,0)$, $(1,0) \to (0,1)$,

$(0,1) \to (0,0)$

$$\int_C x^4\,dx + xy\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \int_0^{1-x} (y-0)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{6}.$$

EX2. $C: x^2+y^2=9.$

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy.$$

$$= \iint_D \left[\frac{\partial}{\partial x}(7x+\sqrt{y^4+1}) + \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi}\int_0^3 (7-3)\,r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi.$$

< Application >

D: a region in $\mathbb{R}^2$.

$$A = A(D) = \oint_{\partial D} x\,dy = \iint_D 1\,dA$$

$$= \oint_{\partial D} -y\,dx = \iint_D 1\,dA$$

EX3. $\frac{1}{a^2} + \frac{1}{b^2}$

$$\Rightarrow A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt$$

$$-\frac{1}{2}\int_0^{2\pi} (b\sin t)(-a\sin t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = ab\pi.$$

Ex 4. C: $D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$$\oint_C y^2\,dx + 3xy\,dy.$$

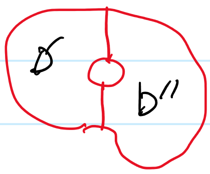
$$= \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D (3y - 2y)\,dy.$$

$$= \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = \frac{14}{3}.$$

$$F(x,y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2+y^2)$$

$$\rightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dt.$$



$$= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx$$

$$- \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) P$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

16. 5. Curl and Divergence

2018년 11월 26일 월요일 11:38

*

1) $\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$

$$= \left(\mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} \right)$$

2) $\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$

: the vector differential operator.

Def If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} ;$$

also

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix},$$

Ex 1. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\mathbf{k}$.

$$\rightarrow \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= -y(2+x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}.$$

Thm If $f(x,y,z)$ is a function that have continuous 2^{nd} order partial derivatives then

$$\text{curl}(\nabla f) = \mathbf{0}.$$

i.e. If $\mathbf{F}$ is conservative, then $\text{curl}\,\mathbf{F} = \mathbf{0}$.

EX 2 Show that the vector field $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

$$\rightarrow \text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ xyz & -y^2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xz & -y^2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xz & xyz \end{vmatrix}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -(2y + xy)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k} \neq \mathbf{0}$$

Thm If $\mathbf{F}$ is a vector field on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl}\,\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

EX 3. $\mathbf{F}(x,y,z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$

$\rightarrow P, Q, R \in C^1(\mathbb{R}^3)$. $\leftarrow$ continuous derivatives

(a)

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2xyz^3 & 3xy^2z^2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ y^2z^3 & 3xy^2z^2 \end{vmatrix}$$

$$+ \mathbf{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ y^2z^3 & 2xyz^3 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\mathbf{i} - (3y^2z^2 - 3y^2z^2)\mathbf{j}$$
$$+ (2yz^3 - 2yz^3)\mathbf{k} = \mathbf{0}$$

thus $\mathbf{F}$ is conservative.

(b)

$$\nabla f = \mathbf{F}$$

$$\begin{cases} f_x = y^2z^3 \\ f_y = 2xyz^3 \\ f_z = 3xy^2z^2 \end{cases}$$

$$f(x,y,z) = \int f_x \, dx = xy^2z^3 + g(y,z).$$

$$\downarrow \partial/\partial y \qquad \downarrow \partial/\partial z$$

$$f_y(x,y,z) = \frac{\partial}{\partial y}\left(xy^2z^3 + g(y,z)\right)$$

$$= 2xyz^3 + g_y(y,z) = 2xyz^3$$

$$\therefore g_y(y,z) = 0.$$

$$g(y,z) = h(z)$$

$$f_z(x,y,z) = \frac{\partial}{\partial z}\left(xy^2z^3 + h(z)\right)$$

$$= 3xy^2z^2 + h'(z) = 3xy^2z^2 \quad \therefore h'(z) = 0$$

$$\therefore f(x,y,z) = xy^2z^3 + K.$$

Def If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Ex4 If $\mathbf{F}(x,y,z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$,

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z} y^2$$

$$= z + xz.$$

Thm If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous 2^{nd}-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0.$$

* $$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is called the Laplace operator because of its relation to Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Also,

$$\nabla^2 \mathbf{F} = \nabla^2 P\mathbf{i} + \nabla^2 Q\mathbf{j} + \nabla^2 R\mathbf{k}.$$

< Vector Forms of Green's Theorem >

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$$

$\rightarrow$ $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$

$\rightarrow$ $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

$\therefore$ $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA$,

$\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div } \mathbf{F}(x,y) \, dA.$

16. 6. Parametric Surfaces and Their Areas

2018년 11월 28일 수요일 11:30

< Parametric Surfaces >

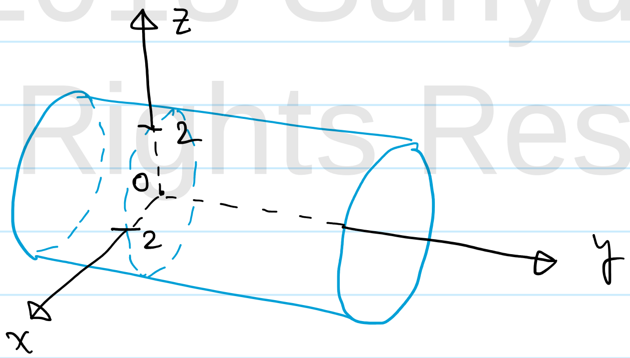
Def The set of all points (x,y,z) in $\mathbb{R}^3$ such that

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$$

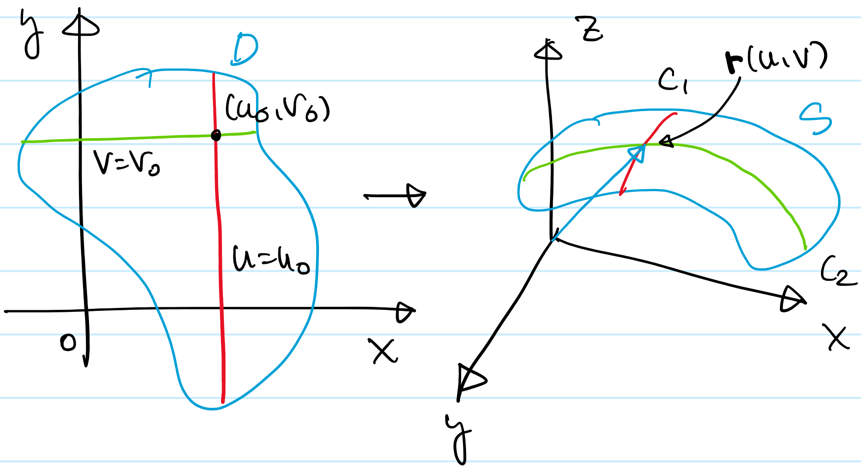
and (u,v) varies throughout D, is called a parametric surface S and the equations above are called parametric equations of S.

Ex 1. $\mathbf{r}(u,v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$

→



Def Let a parametric surface be given by $\mathbf{r}(u,v)$.



C_1, C_2 : grid curves.

EX 3. Plane passes P_0 and contains $\mathbf{a}$ and $\mathbf{b}$?

$$\to \mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

$$\therefore \mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

$$\begin{aligned} \mathbf{r} &= (x, y, z) \\ \mathbf{r}_0 &= (x_0, y_0, z_0) \\ \mathbf{a} &= (a_1, a_2, a_3) \\ \mathbf{b} &= (b_1, b_2, b_3) \end{aligned} \quad \to \quad \begin{cases} x = x_0 + ua_1 + vb_1 \\ y = y_0 + ua_2 + vb_2 \\ z = z_0 + ua_3 + vb_3. \end{cases}$$

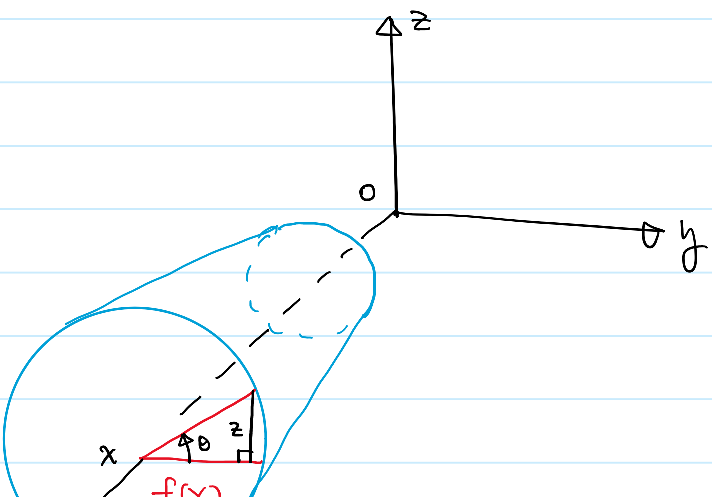
EX 4. $x^2 + y^2 + z^2 = a^2$

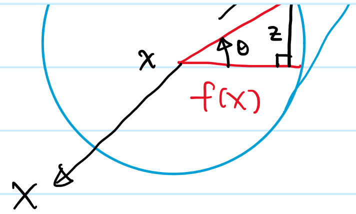
$$\to \begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi. \end{cases}$$

$$\mathbf{r}(\varphi, \theta) = a\sin\varphi\cos\theta\,\mathbf{i} + a\sin\varphi\sin\theta\,\mathbf{j} + a\cos\varphi\,\mathbf{k}$$

< Surfaces of Revolution >

Let S be the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \geq 0$.





$$\begin{cases} x = x \\ y = f(x)\cos\theta \\ z = f(x)\sin\theta. \end{cases}$$

$a \le x \le b,\ 0 \le \theta \le 2\pi.$

Ex 8. $y = \sin x,\ 0 \le x \le 2\pi$, rotate about the x-axis.

$$\to \begin{cases} x = x \\ y = \sin x \cos\theta \\ z = \sin x \sin\theta. \end{cases}$$

< Tangent Planes >

S: $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$

: a parametric surface, $(u,v) \in D$.

The tangent vector to C_1 at P_0 is

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\,\mathbf{k}$$

Def

1) $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0} \to S$ is called "smooth"

2) For a smooth surface, the tangent plane is the plane that contains the tangent vectors $\mathbf{r}_u, \mathbf{r}_v$. and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Ex 9.
$$\begin{cases} x = u^2 \\ y = v^2 \\ z = u + 2v \end{cases}$$

$$\to \mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k}$$

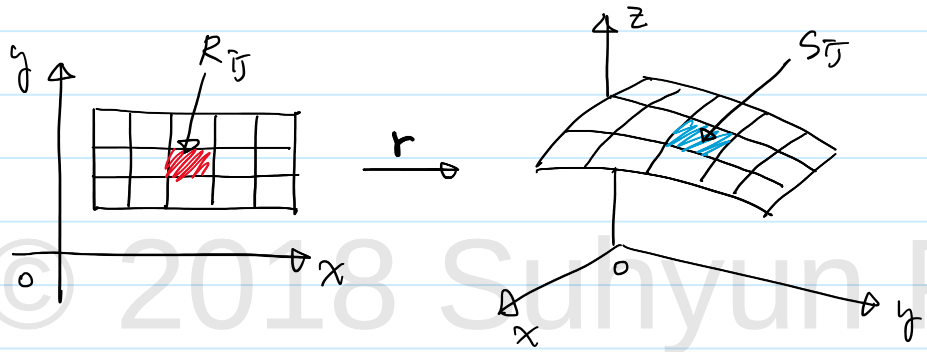
$\downarrow$ $(1,1,3) \rightarrow u=1,\ v=1$

normal vector : $(-2, -4, 4)$

equation : $x + 2y - 2z + 3 = 0.$

< Surface Area >

$S : \mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$



$$\Delta S_{ij} \simeq |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

$$\rightarrow A(S) \simeq \sum_{ij} \Delta(S_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

$$\therefore A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

$$= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$: the surface area.

Ex 10. $\rho = a$

$$\rightarrow \begin{cases} x = a \sin\varphi \cos\theta \\ y = a \sin\varphi \sin\theta \\ z = a \cos\varphi \end{cases}$$

$$\to \mathbf{r}_\varphi = \frac{\partial x}{\partial \varphi}\mathbf{i} + \frac{\partial y}{\partial \varphi}\mathbf{j} + \frac{\partial z}{\partial \varphi}\mathbf{k}$$

$$= a\cos\varphi\cos\theta\,\mathbf{i} + a\cos\varphi\sin\theta\,\mathbf{j} - a\sin\varphi\,\mathbf{k}$$

$$\mathbf{r}_\theta = \frac{\partial x}{\partial \theta}\mathbf{i} + \frac{\partial y}{\partial \theta}\mathbf{j} + \frac{\partial z}{\partial \theta}\mathbf{k}$$

$$= -a\sin\varphi\sin\theta\,\mathbf{i} + a\sin\varphi\cos\theta\,\mathbf{j}$$

$\therefore\ \mathbf{r}_\varphi \times \mathbf{r}_\theta$

$$= a^2\sin^2\varphi\cos\theta\,\mathbf{i} + a^2\sin^2\varphi\sin\theta\,\mathbf{j} + a^2\sin\varphi\cos\theta\,\mathbf{k},$$

$|\mathbf{r}_\varphi \times \mathbf{r}_\theta|$

$$= \sqrt{a^4\sin^4\varphi\cos^2\theta + a^4\sin^4\varphi\sin^2\theta + a^4\sin^2\varphi\cos^2\theta}$$

$$= a^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = a^2\sin\varphi.$$

$$\therefore\ A = \iint_D |\mathbf{r}_\varphi \times \mathbf{r}_\theta|\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi\,d\varphi\,d\theta$$

$$= 4\pi a^2.$$

⟨Surface Area of the Graph of a Function⟩

$S:\ z = f(x,y),\ (x,y) \in D.$

$f \in C^1(D).$

$$\to S: \begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases},\quad \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

$$\therefore\ A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA.$$

⟨Surface Area of Surface of Revolution⟩.

$$S = \begin{cases} x = x \\ y = f(x)\cos\theta \\ z = f(x)\sin\theta \end{cases},$$

$$\mathbf{r}(x,\theta) = x\mathbf{i} + f(x)\cos\theta\,\mathbf{j} + f(x)\sin\theta\,\mathbf{k}$$

$$\mathbf{r}_x = \mathbf{i} + f'(x)\cos\theta\,\mathbf{j} + f'(x)\sin\theta\,\mathbf{k}$$

$$\mathbf{r}_\theta = -f(x)\sin\theta\,\mathbf{j} + f(x)\cos\theta\,\mathbf{k}$$

$\therefore\ \mathbf{r}_x \times \mathbf{r}_\theta = f(x)f'(x)\,\mathbf{i} + f(x)\cos\theta\,\mathbf{j} - f(x)\sin\theta\,\mathbf{k}$

$|\mathbf{r}_x \times \mathbf{r}_\theta| = f(x)\sqrt{1+f'(x)^2}$.

$$\therefore\ A = \iint_D f(x)\sqrt{1+f'(x)^2}\,dA$$

$$= \int_0^{2\pi}\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx\,d\theta$$

$$= 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx .$$

16. 7. Surface Integrals

2018년 12월 3일 월요일 11:41

$S: \mathbf{r}(u,v)$. $\Delta S_{ij} = |\mathbf{r}_u \times \mathbf{r}_v| \, du dv$.

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

$$* \quad A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

Ex 1. $S: x^2+y^2+z^2=1$

$\iint_S x^2 \, dS$?

$$S \to \begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases} \qquad \begin{matrix} \varphi & 0 \cdots \pi \\ \theta & 0 \cdots 2\pi \end{matrix}$$

$|\mathbf{r}_\varphi \times \mathbf{r}_\theta| = \sin\varphi$.

$\therefore \iint_S x^2 \, dS$

$$= \iint_D (\sin\varphi\cos\theta)^2 \cdot \sin\varphi \, dA$$

$$= \iint_D \sin^3\varphi\cos^2\theta \, dA$$

$$= \int_0^\pi \sin^3\varphi \, d\varphi \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{4\pi}{3}.$$

⟨ Graphs of Functions ⟩.

$S: z = g(x,y)$

$$\begin{cases} x = x \\ y = y \end{cases} \quad : \quad \begin{cases} x = u \\ y = v \end{cases}$$

$$\begin{cases} y = y \\ z = g(x,y) \end{cases} \quad : \quad \begin{cases} y = v \\ z = g(u,v). \end{cases}$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

$$\therefore A = \iint_S f(x,y,z)\, dS$$

$$= \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\,dy$$

EX 2. $S: z = x + y^2$.

$$\iint_S y\, dS$$

$$= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_0^1 dx \int_0^2 \sqrt{2}\sqrt{1+2y^3}\, dy = \frac{13\sqrt{2}}{3}.$$

* If S is a piecewise - smooth surface, i.e.

$S = S_1 \cup S_2 \cup \cdots \cup S_n$,

then

$$\iint_S f(x,y,z)\, dS = \sum_{i=1}^{n} \iint_{S_i} f(x,y,z)\, dS.$$

EX 3. p. 1125

⟨ Oriented Surfaces ⟩.

* 1) $S: z = g(x,y)$.

$$\mathbf{r}_x \times \mathbf{r}_y = -\left(\frac{\partial g}{\partial x}\right)\mathbf{i} - \left(\frac{\partial g}{\partial y}\right)\mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} : \text{the normal vector.}$$

2) $S: \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}.$$

3) $d\mathbf{S} = \mathbf{n}\, dS.$

Def For a closed surface, i.e. a surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E, and inward-pointing normals gives the negative number.

Def If $\mathbf{F}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

This integral is also called the flux of $\mathbf{F}$ across S.

In the case of a surface S given by a graph $z = g(x,y)$, we can think of x and y as parameters and write

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA.$$

EX 5. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$,

$S: z = 1 - x^2 - y^2$, $z = 0$

$$\to \iint_{S} \mathbf{F} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$$

$$\to \iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$$

$$= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2) r\, dr\, d\theta$$

$$= \frac{\pi}{2}.$$

$\sim$

16. 8. Stokes' Theorem

2018년 12월 10일 월요일 11:10

Thm <Stokes' Theorem>

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

(pf. @ p. 1135)

16. 9. The Divergence Theorem

2018년 12월 12일 수요일 10:48

Thm

Let E be simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

(pf. @ p. 1141)