BP equations

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BP

At fixed f, v, a

$$\mu^l_{fv}(a) \propto \sum_{\mathrm{Conf}_{(v,f)}(a)} \prod_{v' \in \mathcal{M}(f) \setminus \{v\}} \mu^l_{v'f}(c_{v'}) \tag{1}$$

The set $\mathrm{Conf}_{(v,f)}(a)$, is given by

$$\left\{ \mathbf{c}_{v'} : \sum_{v' \in M(f) \setminus v} h_{fv'} c_{v'} + h_{fv} a = 0 \right\} = \left\{ \mathbf{c}_{v'} : \sum_{v' \in M(f) \setminus v} h_{fv'} c_{v'} = -h_{fv} a \right\} \tag{2}$$

$$= \left\{ \mathbf{c}_{v'} : \sum_{v' \in M(f) \setminus v} h_{fv'} c_{v'} = h_{fv} a \right\} \tag{3}$$

$$= \left\{ \mathbf{c}_{v'} : \sum_{v' \in M(f) \setminus v} \frac{h_{fv'}}{h_{fv}} c_{v'} = a \right\} \tag{4}$$

where the summation is intended on $\mathrm{GF}(q)$.
Defining the weights $w_{v'} := \frac{h_{fv'}}{h_{fv}} = h_{fv'} \cdot \mathrm{inv}(h_{fv})$ (where '$\cdot$' indicates multiplication on $\mathrm{GF}(q)$), and functions $g_{v'}(c_{v'}) := \mu_{v'f}(c_{v'})$ for a convolution, we get

$$\mu^l_{fv}(a) \propto \sum_{\left\{ \mathbf{c}_{v'} : \sum_{v' \in M(f) \setminus v} w_{v'} c_{v'} = a \right\}} \prod_{v' \in \mathcal{M}(f) \setminus \{v\}} g_{v'}(c_{v'}) \tag{5}$$

To get rid of the weights, define $\tilde{c}_{v'} := w_{v'} c_{v'}$ and $\tilde{g}_{v'}(\tilde{c}_{v'}) := g_{v'}(\tilde{c}_{v'} \cdot \mathrm{inv}(w_{v'}))$, where $\mathrm{inv}(w_{v'})$ is the inverse of $w_{v'}$ in $\mathrm{GF}(q)$. The equation becomes

$$\mu^l_{fv}(a) \propto \sum_{\left\{ \tilde{\mathbf{c}}_{v'} : \sum_{v' \in M(f) \setminus v} \tilde{c}_{v'} = a \right\}} \prod_{v' \in \mathcal{M}(f) \setminus \{v\}} \tilde{g}_{v'}(\tilde{c}_{v'}) \tag{6}$$

which is a convolution in a more familiar form.
Using the property (I hope this holds!) that $\text{inv}(x \cdot \text{inv}(y)) = y \cdot \text{inv}(x)$, one can further re-write

$$\tilde{g}_{v'}(\tilde{c}_{v'}) = g_{v'}\big(\tilde{c}_{v'} \cdot \text{inv}(w_{v'})\big) \tag{7}$$
$$= g_{v'}\big(\tilde{c}_{v'} \cdot \text{inv}(h_{fv'} \cdot \text{inv}(h_{fv})\big) \tag{8}$$
$$= g_{v'}\big(\tilde{c}_{v'} \cdot h_{fv} \cdot \text{inv}(h_{fv'})\big) \tag{9}$$
$$= \mu^l_{v'f}\big(\tilde{c}_{v'} \cdot h_{fv} \cdot \text{inv}(h_{fv'})\big) \tag{10}$$

Finally

$$\mu^l_{fv}(a) \propto \sum_{\left\{\tilde{\mathbf{c}}_{v'}:\ \sum_{v' \in M(f)\setminus v} \tilde{c}_{v'}=a\right\}} \prod_{v' \in \mathcal{M}(f)\setminus\{v\}} \mu^l_{v'f}\big(\tilde{c}_{v'} \cdot h_{fv} \cdot \text{inv}(h_{fv'})\big) \tag{11}$$

Messages from variable to factor are simpler

$$\mu^{l+1}_{vf}(a) \propto \mu^1_v(a) \prod_{f' \in \mathcal{N}(v)\setminus\{f\}} \mu^l_{f'v}(a) \tag{12}$$

On a fixed point, beliefs are given by

$$\mathbf{g}^{l+1}_v(a) \propto \mu^1_v(a) \prod_{f \in \mathcal{N}(v)} \mu^l_{fv}(a) \tag{13}$$

MS

Define

$$\nu^l_{vf}(a) \coloneqq \frac{1}{\beta} \log \mu^l_{vf}(a) \tag{14}$$

Now (1) becomes

$$\mu^l_{fv}(a) \propto \sum_{\text{Conf}_{(v,f)}(a)} \prod_{v' \in \mathcal{M}(f)\setminus\{v\}} e^{\beta \nu^l_{v'f}(c_{v'})} \tag{15}$$
$$= \sum_{\text{Conf}_{(v,f)}(a)} \exp\left(\beta \sum_{v' \in \mathcal{M}(f)\setminus\{v\}} \nu^l_{v'f}(c_{v'})\right) \tag{16}$$
$$= \max_{\text{Conf}_{(v,f)}(a)} \exp\left(\beta \sum_{v' \in \mathcal{M}(f)\setminus\{v\}} \nu^l_{v'f}(c_{v'})\right) \tag{17}$$
$$= \exp\left(\beta \max_{\text{Conf}_{(v,f)}(a)} \sum_{v' \in \mathcal{M}(f)\setminus\{v\}} \nu^l_{v'f}(c_{v'})\right) \tag{18}$$

Substitute back (14)

$$\nu_{fv}^{l}(a) = \frac{1}{\beta} \log \exp \left(\beta \max_{\mathrm{Conf}_{(v,f)}(a)} \sum_{v' \in \mathcal{M}(f) \setminus \{v\}} \nu_{v'f}^{l}(c_{v'}) \right) + \mathrm{const} \tag{19}$$

$$= \max_{\mathrm{Conf}_{(v,f)}(a)} \sum_{v' \in \mathcal{M}(f) \setminus \{v\}} \nu_{v'f}^{l}(c_{v'}) + \mathrm{const} \tag{20}$$

solve using max-sum convolution.

Messages from variable to factor

$$\nu_{vf}^{l+1}(a) = \frac{1}{\beta} \log \mu_{v}^{1}(a) + \sum_{f' \in \mathcal{N}(v) \setminus \{f\}} \nu_{f'v}^{l}(a) + \mathrm{const} \tag{21}$$

since $\mu_{v}^{1}(a) = \exp\left[-\beta L d_H(y_v, a)\right]$,

$$\nu_{vf}^{l+1}(a) = -L d_H(y_v, a) + \sum_{f' \in \mathcal{N}(v) \setminus \{f\}} \nu_{f'v}^{l}(a) + \mathrm{const} \tag{22}$$

Beliefs

$$\nu_{vf}^{l+1}(a) = -L d_H(y_v, a) + \sum_{f' \in \mathcal{N}(v)} \nu_{f'v}^{l}(a) + \mathrm{const} \tag{23}$$