Singular Value Decomposition

SVD methods are based on the following theorem of linear algebra:

Any *MXN* matrix **A** can be written as the product of an *MXN* column-**orthogonal** matrix **U,** an *NXN* **diagonal** matrix **W** with positive or zero elements (the *singular values),* and the **transpose** of an *NXN* **orthogonal** matrix **V.** Therefore:

$$A = U \cdot W \cdot V^T \qquad \text{(Eq. 1)}$$

If $M > N$, we have the *overdetermined* situation, i.e. more equations than unknowns.

If $M < N$, we have the *underdetermined* situation of fewer equations than unknowns.

The matrix **V** is orthogonal in the sense that its columns are orthonormal,

$$\sum_{j=0}^{N-1} V_{jk} V_{jn} = \delta_{kn}, 0 \leq k \leq N-1, 0 \leq n \leq N-1 \qquad \text{(Eq. 2)}$$

that is, $V^T \cdot V = 1$. Since **V** is square, it is also row-orthonormal, $V \cdot V^T = 1$.

When $M \geq N$, the matrix **U** is also column-orthogonal,

$$\sum_{i=0}^{M-1} U_{ik} U_{\text{¿}} = \delta_{kn}, 0 \leq k \leq N-1, 0 \leq n \leq N-1 \qquad \text{(Eq. 3)}$$

that is, $U^T \cdot U = 1$. In the case M < N (i.e. underdetermined, fewer equations than unknowns), however, two things happen:

- The singular values w_j for $j = M, \ldots, N-1$ are all zero.
- The corresponding columns of **U** are also zero. Equation (Eq. 3)then holds only for $k, n \leq M-1$.