Pumping lemma for CFL's – Examples

Pumping lemma (Theorem 2.34): If A is a CFL, then there is a number p (pumping length) where if s is any string in A, $|s| \geq p$, then there exists a partition $s = uvxyz$, satisfying

1. For each $i \geq 0$, $uv^i xy^i z \in A$.
2. $|vy| > 0$ (either $v \neq \varepsilon$ or $y \neq \varepsilon$)
3. $|vxy| \leq p$.

uxz
$uvxyz$
$uvvxyyz$
$uvvvxyyyz$
$\vdots$

$\{\varepsilon, abc, aabbcc, a^3b^3c^3, \ldots\}$

Example 2.36: $B = \{a^n b^n c^n \mid n \geq 0\} \subseteq a^* b^* c^*$

Show that B is not context-free.

Assume that B is context-free. Then by pumping lemma, there exists a pumping length $p \geq 0$. Let $s = a^p b^p c^p$. Suppose s can be written as $s = uvxyz$.

$aa \ldots abbb \ldots bbbcc \ldots cc$

$|\xleftarrow{\quad v \quad}|$

$aaaaabbaabbbb\ldots$

uv^2xy^2z

There are two cases.

1) Either v or y contains two types of symbols. Then uv^2xy^2z is not of the form $a^* b^* c^*$.

2) If v and y each contain only one type of symbol, then uv^2xy^2z does not contain the same no. of a's, b's and c's.

Hence B is not a CFL.

Example 2.37: $C = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\} \subseteq a^* b^* c^*$

Like before, we assume that C is context-free. Pumping lemma implies the existence of a pumping length p. Let $s = a^p b^p c^p$.

a a .... a a b b ..... b b c c c ... c c

$|\xleftarrow{\quad v \quad}|$ $\quad uv^2xy^2z = aa..abbaabb\ldots$

1. If v or y contains two different types of symbols, then uv^2xy^2z is not of the type $a^* b^* c^*$.

So v and y each contain only one type of symbol.

2. If vy avoids a, then consider uv^0xy^0z. Since $|vy| \neq 0$, then the count of b's or c's must go down.

If vy avoids b, but contains a, then uv^2xy^2z has more a's than b's.

If vy avoids b, but contains c, then uv^0xy^0z has more b's than c's.

If vy avoids c, then uv^2xy^2z contains more a's than c's or more b's than c's.

Hence C is not context-free.

Exercise: $D = \{w\#w \mid w \in \{0,1\}^*\}$.
Show that D is not context-free.

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(Similar to Example 2.38: $\{ww \mid w \in \{0,1\}^*\}$)