Closure Properties of CFL's.

We saw in Lecture 16 that CFL's are closed under union.

Theorem 1: CFL's are closed under concatenation.

Proof: Suppose L_1 is generated by CFG G_1 and L_2 is generated by CFG G_2. Let S_1 and S_2 be the respective start variables of G_1 and G_2.

$$L_1 \circ L_2 = \{ xy \mid x \in L_1, y \in L_2 \}$$

We can create a new CFG G as follows, with new start variable S.

$$\boxed{\begin{array}{l} S \rightarrow \underline{S_1 S_2} \\ \text{Rules of } G_1 \\ \text{Rules of } G_2 \end{array}}$$

$\rightarrow$ CFG G that generates $L_1 \circ L_2$.

Theorem 2: CFL's are closed under Kleene star

Proof: Suppose L_1 is generated by CFG G_1 with start variable S_1.

$$L_1^* = \{ x_1 x_2 \ldots x_k \mid x_i \in L_1 \text{ for each } x_i, k \geq 0 \}$$

We create a new CFG G for L_1^*, with a new start variable S.

$$\boxed{\begin{array}{l} S \to S\,S_1 \\ S \to \varepsilon \\ \text{Rules of } G_1 \end{array}}$$

A CFG for the language L_1^*

$S \to S S_1$
$\to S S_1 S_1$
$\to S S_1 S_1 S_1$
$\to S_1 S_1 S_1$

$S \to \varepsilon$

Exercise: Verify that the above CFG's indeed generate $L_1 \cdot L_2$ and L_1^* respectively.

What about intersection? Complement?

Consider $L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$

$L_2 = \{a^n b^m c^m \mid n, m \geq 0\}$

Exercise: Show that L_1, L_2 are context-free.

$a^* b^* c^*$

$$L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$$

We will see that $L_1 \cap L_2$ is not context-free.
Hence it follows that CFL's are not closed under intersection.

This also implies that CFL's are not closed under complement.

De Morgan's law: $(L_1^c \cup L_2^c)^c = L_1 \cap L_2$

If CFL's were closed under complement, by De Morgan's law, it would imply that CFL's are closed under intersection $\rightarrow$ we know this is not the case.

Hence it follows that CFL's are not closed under complement.