KNAPSACK

There are n objects indexed $1, 2, \ldots n$.

They have integer weight $w_1, w_2, \ldots w_n$.

– integer value $v_1, v_2, \ldots v_n$.

There is a sack (bag) with weight capacity W.

There is a value goal : G.

Question : Does there exist a subset of indices $I \subseteq \{1, 2, \ldots n\}$ such that

$$\text{Sum of weights} = \sum_{i \in I} w_i \leq W$$

$$\text{Sum of value} = \sum_{i \in I} v_i \geq G$$

Theorem : KNAPSACK is NP. Complete.

Proof : (1) KNAPSACK $\in$ NP.

We can "guess" a subset I and check if I meets the constraints.

(2) SUBSET-SUM $\leq_p$ KNAPSACK.

Given a set $S = \{s_1, s_2, \dots s_k\}$ and a target sum t, we will construct a KNAPSACK instance as follows.

KNAPSACK : k items with

Weights : $s_1, s_2, \dots s_k$

Values : $s_1, s_2, \dots s_k$

Weight Capacity $= t$

Value Goal $= t$.

This KNAPSACK instance is a YES instance if $\exists\, I \subseteq \{1, 2, \dots k\}$ such that

$$\sum_{i \in I} w_i = \sum_{i \in I} s_i \leq t \quad \text{(Weight capacity)}$$

$$\sum_{i \in I} v_i = \sum_{i \in I} s_i \geq t \quad \text{(Value goal)}$$

Together this implies $\sum_{i \in I} s_i = t$.

So the instance is a YES instance of SUBSET-SUM.

The other direction is immediate.

Thus KNAPSACK is NP-complete.

To show NP-completeness of C, we need to show

(1) $C \in NP$

$\boxed{(2)\ \forall A \in NP,\ A \leq_p C.}$ → NP-hard.

$\Downarrow$

can be replaced by

$B \leq_p C$

where B is an NP-complete