Problem 5.21 : It is undecidable to check if a given CFG is ambiguous.

$$AMBIG_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG, and is ambiguous}\}$$

Theorem: $AMBIG_{CFG}$ is undecidable.

Proof: $PCP \leq_m AMBIG_{CFG}$.

Given PCP instance, we will construct a CFG which is ambiguous if and only if the PCP has a match.

let $P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots \left[\frac{t_k}{b_k}\right] \right\}$

We construct the CFG G as follows:

$$
\begin{aligned}
G: \quad & S \to T \mid B \\
& T \to t_1 T a_1 \mid \dots \mid t_k T a_k \mid t_1 a_1 \mid \dots \mid t_k a_k \\
& B \to b_1 B a_1 \mid \dots \mid b_k B a_k \mid b_1 a_1 \mid \dots \mid b_k a_k
\end{aligned}
$$

Here t_i, b_i are the strings in the dominoes in P.
a_i are new distinct symbols (not in t_i, b_i)

($\Longrightarrow$) Suppose P has a match. Let the match be

$$t_{i_1} t_{i_2} \cdots t_{i_m} = b_{i_1} b_{i_2} \cdots b_{i_m}$$

We have two derivations of the same string.

$$S \to T \to t_{i_1} T a_{i_1} \to t_{i_1} (t_{i_2} T a_{i_2}) a_{i_1} \to$$
$$\to t_{i_1} t_{i_2} \cdots t_{i_m} a_{i_m} \cdots a_{i_2} a_{i_1}$$

$$S \to B \to b_{i_1} B a_{i_1} \to b_{i_1} (b_{i_2} B a_{i_2}) a_{i_1} \to$$
$$\to b_{i_1} b_{i_2} \cdots b_{i_m} a_{i_m} \cdots a_{i_2} a_{i_1}$$

So G is ambiguous

($\Longleftarrow$) Suppose G is ambiguous. Let the string w have two derivations.

Because of the structure of G, we can infer that $w = w' a_{j_m} a_{j_{m-1}} \cdots a_{j_1}$, where w' does not consist of any a_j's.

This means that the two derivations are necessarily the following:

$S \to T \to t_{j_1} T a_{j_1} \to t_{j_1}(t_{j_2} T a_{j_2}) a_{j_1}$
$\to t_{j_1} \cdots t_{j_m} a_{j_m} \cdots a_{j_1}$

$S \to B \to b_{j_1} B a_{j_1} \to b_{j_1}(b_{j_2} B a_{j_2}) a_{j_1}$
$\to b_{j_1} \cdots b_{j_m} a_{j_m} \cdots a_{j_1}$

This implies that $t_{j_1} \cdots t_{j_m} = b_{j_1} \cdots b_{j_m}$.

This is a match for P.

Thus the reduction is complete.

$P \in PCP \iff G \in AMBIG_{CFG}$.

Hence $PCP \leq_m AMBIG_{CFG}$.

Thus $AMBIG_{CFG}$ is undecidable.

This also marks the end of computability theory.