Undecidable languages

Undecidable languages : there are no TM's that can decide these languages. How can we show that no TM can decide a language? We will first show that the below language is undecidable.

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM}, M \text{ accepts } w\}$$

Q: Is A_{TM} Turing recognizable? Why / Why not?

Yes! Run machine M on w.

This is not a decider since it may loop if M loops on w.

In order to show undecidability, we need to set up more theory.

Comparison of infinite sets

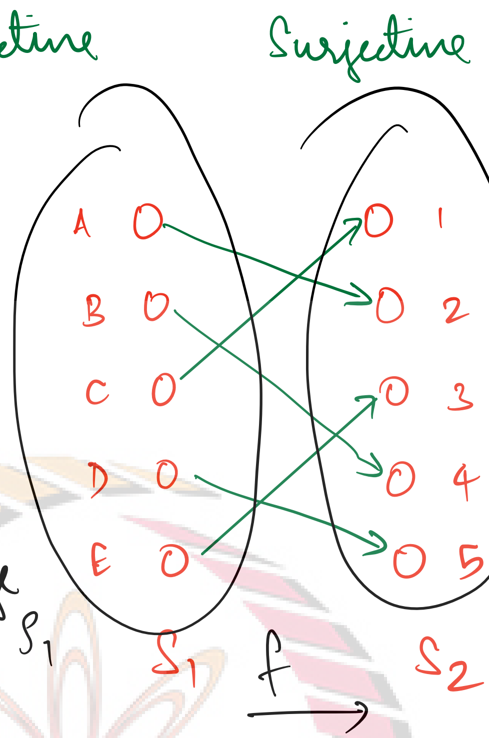
Mapping $f: S_1 \to S_2$

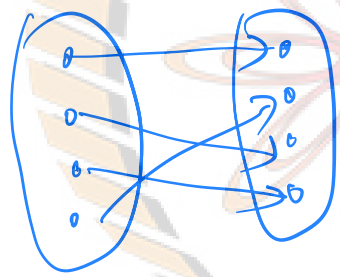
Bijection : One-to-one and onto

Called Correspondence by Sipser

One-to one: $f(x) \neq f(y)$ if $x \neq y$.

Onto: All the elements in S_2 are an image of some element in S_1





There is a bijection from set S to T iff $|S| = |T|$.

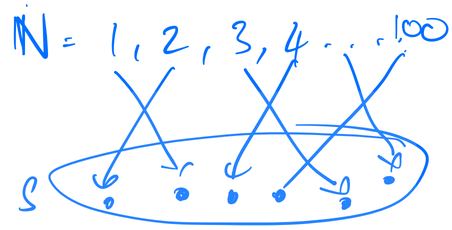
We have 5 apples. The set of 5 apples has a bijection with the set $\{1, 2, 3, 4, 5\}$.

What about infinite sets?

Def 4.14: A set A is countable if it is finite or if it has a bijection with $\mathbb{N}$.

$\mathbb{N} = \{1, 2, 3, 4 \ldots \}$

the set of natural numbers.



Countable = Bijection with $\mathbb{N}$

= Same "size" as $\mathbb{N}$

= Can list all the elements without missing out on any element

"Count - able"

<u>Examples</u>: $\mathbb{N}$, Even numbers in $\mathbb{N}$, $\mathbb{N}^2$, $\mathbb{N}^3$, $\mathbb{Z}$, rational numbers $\mathbb{Q}$

$\{1 \quad 2 \quad 3 \quad 4 \ldots \}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\mathbb{N} = \{1 \quad 2 \quad 3 \quad 4 \ldots \}$

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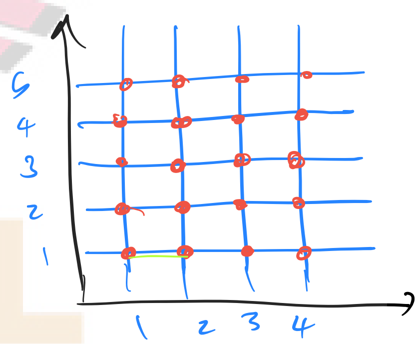
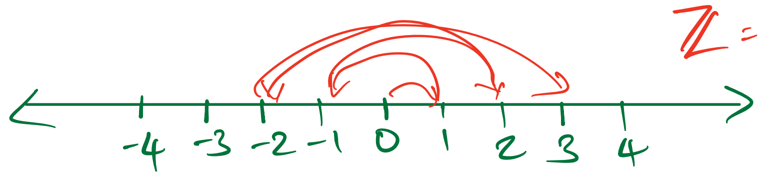
Even nos in $\mathbb{N}$ = $\{2 \quad 4 \quad 6 \quad 8 \ldots \} = E$



$\mathbb{N}^2 = \{(x, y) \mid x, y \in \mathbb{N}\}$

With infinite sets, even sets that are "seemingly" bigger, can be the same size.

$\mathbb{Z}$ = Set of all integers



Theorem 4.17: $\mathbb{R}$ is uncountable. ← The set of real numbers.

Proof: Assume the contrary. Suppose $\mathbb{R}$ was countable. Let us list down the bijection from $\mathbb{N}$ to $\mathbb{R}$.

Cantor's Diagonalization argument

$i \in \mathbb{N}$	$f(i) \in \mathbb{R}$
1	1.23423....
2	3.14159....
3	5.000000...
4	3.23232....
5	0.61000....

$$x = 0.35141\ldots\ldots$$

We choose a number x as follows: the i^{th} digit of x after the decimal point is chosen to be different from that of $f(i)$. Hence $x \neq f(i)$. Hence x is not a member in this listing.

We also need to choose the i^{th} digit of x to be different from 0 and 9 to avoid situations like 5.0000... and 4.999... being the same.

So we produced a number x, which is in $\mathbb{R}$, but not included in the above listing. Hence $\mathbb{R}$ is uncountable.

Another uncountable set is the set of all infinite binary strings.

$B = \{$ all infinite binary strings $\}$

0 0 1 1 1 0 1 0 1 ....

<u>Corollary</u>: B is uncountable.

<u>Proof</u>: We can get a proof similar to the uncountability of $\mathbb{R}$.

Diagonalization

1	0	1	0	1	0	0	0	1	0	1	.....
2	1	0	1	1	1	0	1	0	0	1	.....
3	0	1	1	0	1	0	0	1	1	0	....
4	1	1	1	0	0	1	1	0	0	1	....
5					1						

⋮

$x = 1\ 1\ 0\ 1\ 0 \ldots\ldots$

So x is chosen such that i^{th} bit of x is not equal to i^{th} bit of i^{th} string in the listing.

__Theorem__: For any finite Σ, the set of all strings Σ^* is countable.

__Example__: $\Sigma = \{a, b\}$

$$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, aba, \ldots\ bbb, aaaa, \ldots\}$$

This listing demonstrates that Σ^* is countable.

Let $\mathcal{L}$ be the set of all languages over Σ.

$$\mathcal{L} = \{A \mid A \subseteq \Sigma^*\} = \mathcal{P}(\Sigma^*) \rightarrow \text{Power set of } \Sigma^*$$

__Theorem__: For any finite Σ, the set of all languages is uncountable.

Proof: We can define a bijection between $\mathcal{L}$ and B, the set of all infinite length binary strings. Since B is uncountable, it follows that $\mathcal{L}$ is uncountable as well.

Let $\underline{\Sigma^* = \{s_1, s_2, s_3, \ldots\}}$ be the (ordered) set of strings in Σ^*.

Note: The above notation implicitly assumes an ordering of Σ^*.

We define $f : \mathcal{L} \to B$ as follows:

$$f(A) = \text{infinite } 0/1 \text{ string } b.$$
$$f(A) = \chi_A$$

i^{th} bit of $b = 1 \iff s_i \in A$

$f : \mathcal{L} \to B$ is one-to-one and onto.

$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$

$A : \{\varepsilon, \quad b, aa, \quad ba, \quad \ldots\}$

$f(A) = \chi_A = 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad \ldots$

T_s Characteristic string of A.

Hence $|\mathcal{L}| = |B|$. So $\mathcal{L}$ is also uncountable.

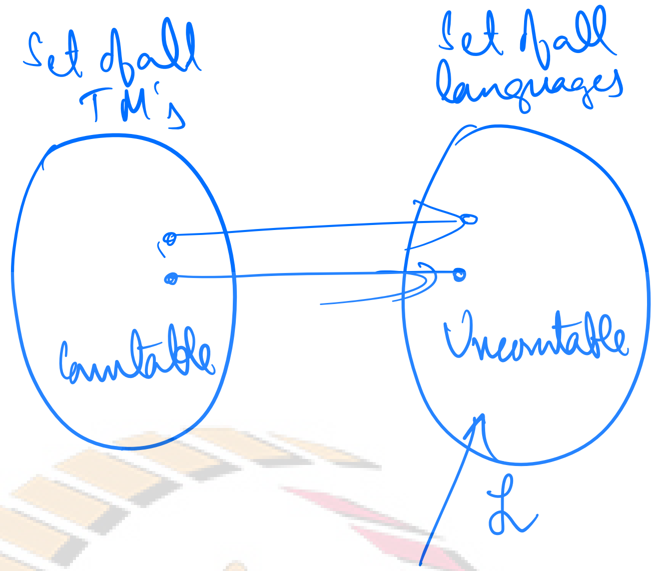
Consider the set of all Turing machines M. For any Turing machine M, the encoding $\langle M \rangle$ is finite length string. The set of all finite length strings over an alphabet Σ is countable since Σ^* is countable.

So the set of all TM's is countable.

Cor 4.18: Some languages are not Turing-recognizable.

Proof: The set of all languages $\mathcal{L}$ is uncountable. But the set of all TM's is countable. Each TM M recognizes exactly one language L(M). So there are languages $L \in \mathcal{L}$ such that $L \neq L(M)$ for any TM M.

That is, L is not Turing recognizable.



Much bigger.

L is decidable $\Rightarrow$ L is Turing recognizable

This means there are languages that are not decidable.