Pumping lemma for Regular languages - Examples

→ Pumping lemma is a necessary condition for a regular language.

→ We can use Pumping lemma to show that languages are not regular.

(Note: We cannot use Pumping lemma to show that languages are regular)

Theorem 1.70 (Pumping lemma): If A is a regular language, then there exists a number p (pumping length) such that, if s is any string in A of length at least p ($\geq p$), then s can be divided into three pieces $s = xyz$, such that

(1) for each $i \geq 0$, $xy^i z \in A$ ($xz, xyz, xyyz,$ $xyyyz, \ldots$)

(2) $|y| > 0$ $(y \neq \varepsilon)$

(3) $|xy| \leq p$.

We use the following steps to argue that a language is not regular.

→ Assume that the language is regular.

→ By Pumping lemma, there exists a pumping length p.

→ Choose a string $s \in$ language s.t. $|s| \geq p$.
(Usually the creative step)

→ Show that this string s cannot be split as $s = xyz$, satisfying the three conditions.

→ Contradiction!

Examples

$B \subseteq 0^*1^*$

1. $B = \{0^n 1^n \mid n \geq 0\} = \{\varepsilon, 01, 0011, 000111, \ldots\}$

Assume B is regular. Then there exists a pumping length p, by pumping lemma.

Now consider $s = 0^p 1^p$. This string $s \in B$ and $|s| \geq p$. By pumping lemma, it follows that s can be written as $s = xyz$, satisfying the three conditions. We will show that this is not possible.

$$s = \underbrace{000\ldots0}_{p}\underbrace{111\ldots1}_{p}$$

with y marked over the last 0 and the first 1.

There are three possibilities.

(1) y has only 0's. $\Rightarrow$ $s' = xy^2z = xyyz \notin B$. has more 0's than 1's.

(2) y has only 1's $\Rightarrow$ $s' = xyyz$ has more 1's than 0's. $s' \notin B$.

(3) y has 0's and 1's $\Rightarrow$ $y = 00..011..1$

$$s' = xyyz = \underbrace{00..0}_{x}\underbrace{0..01..1}_{y}\underbrace{00..01\ldots1}_{y}\underbrace{111..1}_{z}$$

This is not of the form 0^*1^* and hence $s' \notin B$. Hence $s = 0^p1^p$ cannot be "pumped". Hence B is not regular.

Note: We did not use condition (3) of the lemma.

(3) says that $|xy| \leq p$. This implies that y is entirely comprising of 0's. Hence $xyyz$ must have more 0's than 1's.

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(2) $C = \{ w \in \{0,1\}^* \mid w$ has an equal number of 0's and 1's $\}$ $\{01, 10, 1010, 101100 \ldots\}$

If C is regular, there is a pumping length p.

Consider $s = 0^p 1^p \in C$. Also $|s| \geq p$. By pumping lemma, s can be "pumped". Suppose $s = xyz$.

The condition (3) states that $|xy| \leq p$. This implies that $xy = 00\ldots0$. Hence $y = 0^l$ for some $l \geq 1$.

$$\underbrace{000\ldots0}_{p}\underbrace{11\ldots1}_{p}$$

So $xyyz$ contains more 0's than 1's.

Hence $xyyz \notin C$. Contradiction. Hence C is not reg.

Note: The choice of s is critical here. There may be other strings that can be "pumped". Say we chose $s = (01)^p = 0101\ldots01$. This can be written as $x = 01$, $y = 01$, $z = (01)^{p-2}$.

This split satisfies all the conditions!

Note: Another way to show that C is not regular is to appeal to the closure properties. If C is regular, then $C \cap 0^*1^*$ is also regular. This is because regular languages are closed under intersection.

But $C \cap 0^*1^* = B$, which we have shown to be not regular. Hence C is not regular.

Example: $F = \{ww \mid w \in \{0,1\}^*\}$

$101011 \notin F$
$101101 \in F$

$\{\varepsilon, 1, 1111, 111111111, \ldots\}$

(3) Unary language: $D = \{1^{n^2} \mid n \geq 0\}$

The length of all the strings in D are perfect squares.

D regular $\Rightarrow$ pumping length p. Consider $s = 1^{p^2}$.

If $s = xyz$, we have $|xy| \le p \Rightarrow |y| \le p$.

$$\underbrace{|xyz|}_{= p^2} < |xy^2z| = |xyz| + |y|$$

Since $y \neq \varepsilon$

$$= p^2 + |y|$$

$$\le p^2 + p < (p+1)^2 = p^2 + 2p + 1$$

Thus $\quad p^2 < |xyyz| < (p+1)^2$ not a perfect square

Hence $xyyz \notin D$. So D is not regular.

<u>Example</u>: $E = \{0^i 1^j \mid i > j\}$

Here, "<u>pumping down</u>" is necessary.

Instead of $xyyz$, we will show that xz is not in the lang.

<u>Summary</u>: If A is regular, $\forall s \in A$ such that $|s| \ge p$,

$\exists$ a split $s = xyz$, such that $|xy| \leq p$, $|y| > 0$,
and $\forall i \geq 0$, $xy^i z \in A$.