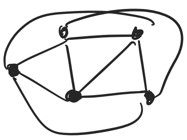
Other NP-complete Problems

We have already seen the following NP-complete problems: SAT, CNF-SAT, and 3-SAT. Let us see some other languages.



CLIQUE $= \{\langle G, k\rangle \mid G$ is an undirected graph that has a clique of size $\geq k\}$

Theorem: CLIQUE is NP-complete.

Proof: We use the following observation from lecture 49.

If B is NP-complete, $C \in NP$ and $B \leq_p C$, then C is NP-complete.

Setting 3-SAT $= B$ and CLIQUE $= C$, it is enough to show the following:

(1) CLIQUE $\in$ NP, and

(2) 3-SAT $\leq_p$ CLIQUE.

In lecture 48, Theorem 7.32, we have already seen that 3-SAT $\leq_p$ CLIQUE. All that remains is to show CLIQUE $\in$ NP. We have already seen the below decider in lecture 30.

NTM decider for CLIQUE.

For $i = 1$ to n

Nondeterministically select / omit vertex i.

Each selected vertex is written on the tape.

$O(n)$

$O(k^2)$

If the number of selected vertices $\neq k$, reject.

Else, for each selected pair (i, j), check if $A_{ij} = 1$

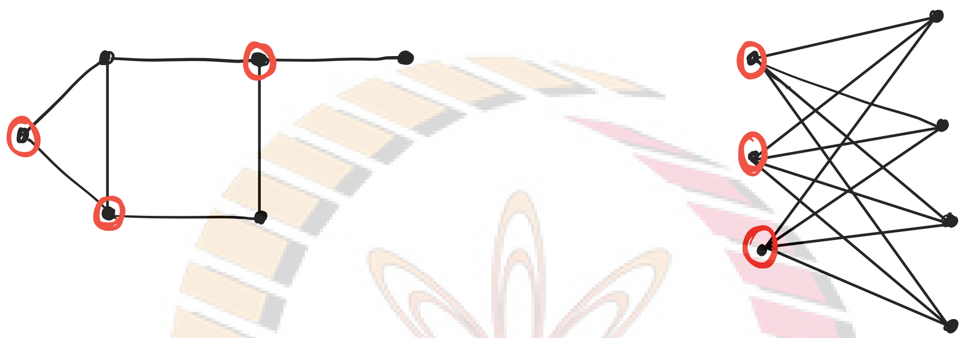
If $A_{ij} = 0$ for any such pair, reject.

A = adj. matrix

Accept if not rejected.

In any computation path, this takes at most $O(nk) \leq O(n^2)$ time. Hence CLIQUE $\in$ NP.

Vertex Cover: Given a graph $G = (V, E)$, a subset $U \subseteq V$ is a vertex cover if $\forall e \in E$, there exists an end point of e in U.



VERTEX-COVER = $\{\langle G, k\rangle \mid G$ is an undirected graph that has a vertex cover of size $k\}$

Theorem 7.44: VERTEX-COVER is NP-complete.

Proof: As per above approach that we used for CLIQUE, we need to show the following two:

(1) VERTEX-COVER $\in$ NP

(2) 3-SAT $\leq_p$ VERTEX-COVER.

For (1), we can use a "guess & verify" approach. We guess k vertices and check if all the edges are "covered" by these vertices.

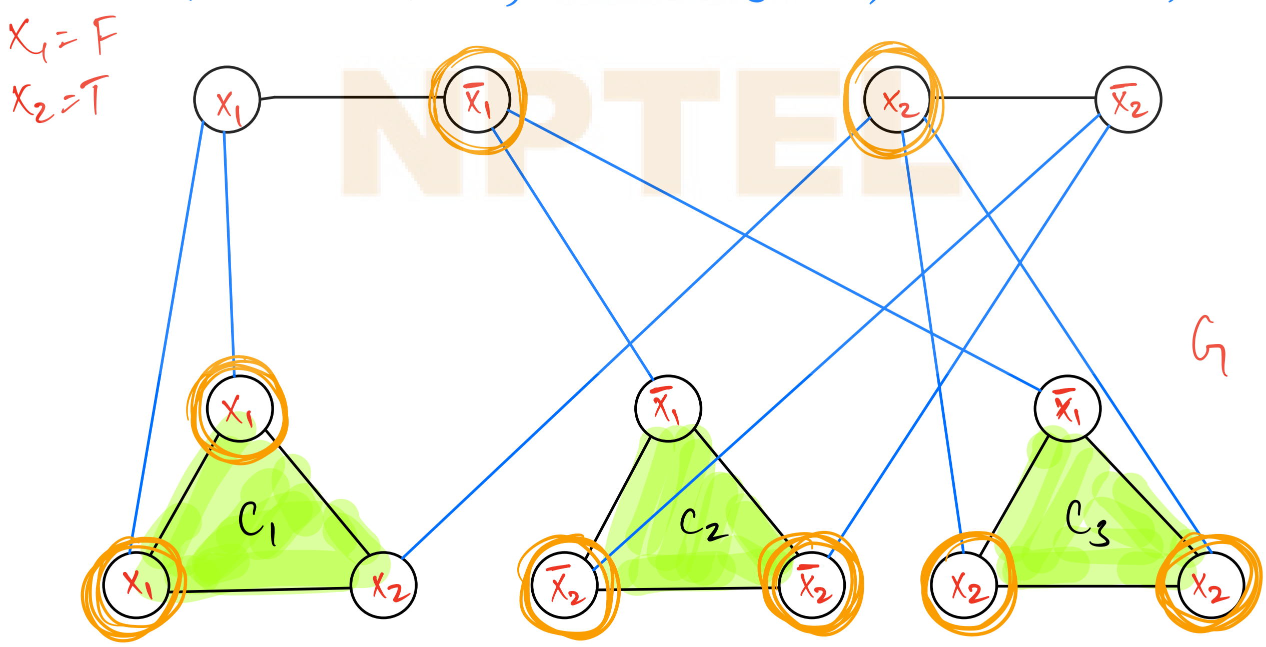
We focus on (2), for the rest of the proof.

Given a 3-CNF formula ϕ, we will produce $\langle G, k \rangle$ such that

$$\phi \in 3\text{-SAT} \iff \langle G, k \rangle \in \text{VERTEX-COVER}$$

Let ϕ have n variables and m clauses. We will demonstrate through an example.

$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$



The three triangles form the clause gadgets, while the two "dumb-bells" form variable gadgets.

Clause gadget: Three vertices that form a clique — one corresponding to each literal of that clause.

Variable gadget: Two adjacent vertices, corresponding to x_i and $\bar{x}_i$.

Edges: In addition to all the edges described above, we add edges between the vertices in the clause and variable gadget that have the same label.

$$\left.\begin{array}{c}\Phi \text{ has } n \text{ variables and} \\ m \text{ clauses}\end{array}\right\} \Rightarrow G \text{ has } 3m+2n \text{ vertices}$$

(and $6m+n$ edges)

CHECK!

We set $k = 2m + n$.

The construction takes $O(m+n)$ time. Now we need to show the following:

ϕ is satisfiable $\iff$ G has a vertex cover of size $2m+n$.

$(\Longrightarrow)$ ϕ is satisfiable $\Longrightarrow$

Select a satisfying assignment. Choose the true literal from each variable gadget, and add them to S.

This covers the edges in the variable gadget.

Choose a true literal from each clause gadget and add the other two vertices into S.

This covers the edges in the clause gadget

What remains are the edges going across variable and clause gadgets.

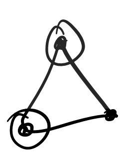
For each clause, the vertices from the clause gadgets, and the true literal from the variable gadget cover these.

$\Rightarrow$ S is a vertex cover of size $2m+n$

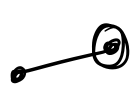
($\Leftarrow$) Suppose G has a vertex cover of size $2m+n$.

Let S be the vertex cover.

$\Rightarrow$ To cover the edges in the clause gadget, we need ≥ 2 vertices from each clause gadget.

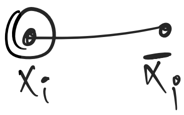


To cover the edges in the variable gadget, we need ≥ 1 vertex from each variable



gadget.

$\Rightarrow$ S contains exactly 2 from each clause gadget and 1 from each variable gadget.



Set $x_i = T$

Set $x_j = F$

$\Rightarrow$ We set the selected literals from each variable gadget to true. Since S is a vertex cover, it follows that for each clause one of the cross edges must be covered by a vertex from the variable gadget.

$\Rightarrow$ This indicates that the chosen assignment is a satisfying assignment.

$\Rightarrow$ $\phi \in$ 3-SAT.

Exercise: Instead of showing 3-SAT $\le_p$ VERTEX-COVER, it suffices to show CLIQUE $\le_p$ VER-COVER.

Show CLIQUE $\le_p$ IND-SET $\le_p$ VER-COVER.

A set of k vertices such that no two are pairwise adjacent.

$$\text{IND-SET} = \{\langle G, k\rangle \mid G \text{ has a } k\text{-sized ind. set}\}.$$