Rice's Theorem

(Proved by Henry Gordon Rice, 1951)

In the last lecture we saw that E_{TM} and $REGULAR_{TM}$ are undecidable.

P_5 : $E_{TM} = \{ \langle M \rangle \mid M$ is a TM, and $L(M) = \phi \}$

$REGULAR_{TM} = \{ \langle M \rangle \mid M$ is a TM, and $L(M)$ is regular $\}$

In both these cases, we need to decide if the language of the given TM satisfies a certain "property". We can ask other similar questions.

P_2 → Does the language recognized by the given TM contain only strings of odd length?

P_3 → Does the given TM accept at least one palindrome?

→ Is the language recognized by the given TM a finite language?

$P_4 \rightarrow$ Does the given TM accept the string 1011?

Rice's theorem says that all these are undecidable.
(Features as Problem 5.28)

Def 1: A property P is a subset of all Turing recognizable languages.

We say that a language L has property P if $L \in P$. We will also say that the TM M that recognizes L has property P.

Examples: (1) P_1 = Set of all regular languages.

(2) P_2 = Set of all languages that contain only odd length strings.

(3) P_3 = Set of all languages that contain at least one string that is a palindrome.

(4) P_4 = Set of all languages that contain 1011.

(5) $P_5 = \{\phi\}$ = The set containing the empty language.

(6) $P_6 = \phi$ = The empty property
(no language is in this property)

This is not undecidable!
This is a trivial property!

Def 2: Property P is said to be non-trivial if $P \neq \phi$ and $P \neq$ the set of all Turing-recognizable languages.

Example: All the above except $P_6 = \phi$ are non-trivial properties.

For example: P_4. There are Turing-recognizable languages that contain 1011 ($\in P_4$) as well as Turing-recognizable languages that don't contain 1011 ($\notin P_4$).

P_5: The empty language is Turing recognizable (so $\phi \in P_5$) and there are other Turing recognizable languages that are not in P_5. (say $\{0,1\} \notin P_5$).

Rice's Theorem: Let P be a non-trivial property of Turing-recognizable languages.

Let $P_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \in P\}$

Then P_{TM} is undecidable. It is not possible to decide if TM M has property P.

Proof: By reduction from A_{TM}.

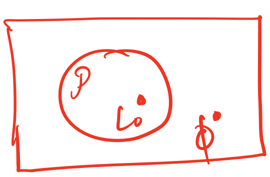
$$A_{TM} = \{\langle M, w\rangle \mid M \text{ accepts } w\}$$

We will show that $A_{TM} \leq_m P_{TM}$.

Empty language $\notin P$

P is non-trivial. We assume that $\phi \notin P$. If $\phi \in P$, we consider $\overline{P}$ instead of P for the rest of the proof.

$$\overline{P} = \left\{\text{Set of all Turing-recognizable languages}\right\} - P$$



→ set of all T. recog. languages.

Since $\overline{P}_{TM}$ is the complement of P_{TM} (kind of), if $\overline{P}_{TM}$ is undecidable, it follows that P_{TM} is also undecidable. So WLOG, we assume $\phi \notin P$.

Since P is non trivial, $\exists\, L_0 \in P$ such that L_0 is Turing recognizable. That is, $\exists$ TM M_0 such that $L(M_0) = L_0$.

Now we are ready to construct the reduction.

Given $\underbrace{\langle M, w\rangle}_{A_{TM}\text{ instance}}$, we create $\underbrace{\langle M'\rangle}_{P_{TM}\text{ instance}}$

Reduction:

M': Given input x.
Simulate M on w.
If M accepts w, simulate M_0 on x.
and accept/reject as per M_0
If M rejects w, reject x.

Let us see how this constitutes a reduction.

$\langle M, w \rangle \in A_{TM} \implies M$ accepts w

$\implies M'$ simulate M_0 on x.

$\implies L(M') = L(M_0) = L_0$

$\implies L(M') \in P$

$\implies \langle M' \rangle \in P_{TM}$.

$\langle M, w \rangle \notin A_{TM} \implies M$ does not accept w.

$\implies L(M') = \phi$

$\implies L(M') \notin P$

$\implies \langle M' \rangle \notin P_{TM}$

So $\langle M, w \rangle \in A_{TM} \iff \langle M' \rangle \in P_{TM}$.

This completes the proof.