INTEGER LINEAR PROGRAM (ILP)

We will first see what is a Linear Program.

Linear Programming

Find $x_1, x_2, \dots x_n$ that

maximizes $\sum_{i=1}^{n} c_i x_i$

subject to $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$

$Ax \leq b$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$

$x_i \geq 0 \quad \forall i$

Here a_{ij}, b_j, c_i are given as part of the LP instance.

→ Formalized during second world war, to plan expenditures of the army.

→ Many applications – food, transportation, scheduling etc.

→ LP can be solved efficiently. The decision version is in P.

Dantzig, Khachiyan, Karmakar are some who have proposed algorithms.

An integer linear program (ILP) is a linear program with the additional constraint that the x_i's should be integers.

This "minor" change makes the problem difficult!

We will consider a special case of ILP called the 0/1 ILP, where each x_i has to be from $\{0, 1\}$.

The decision version of 0/1 ILP is the following.

Does there exist $x_1, x_2, \ldots x_n \in \{0,1\}$ such that

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$
$$x_i \in \{0,1\} \quad \forall i$$

Theorem: The decision version of 0/1 ILP is NP-complete.

Proof: (1) 0/1 ILP $\in$ NP : Guess and verify.

(2) SUBSET-SUM $\leq_p$ 0/1 ILP.

Given a SUBSET-SUM instance $S = \{s_1, s_2 \ldots s_k\}$ and target sum t, we can encode it into a 0/1 ILP as follows.

Does there exist $x_1, x_2, \ldots x_k$ such that

$$s_1 x_1 + s_2 x_2 + \ldots + s_k x_k \leq t \qquad \text{--- } (1)$$

$$-s_1 x_1 + (-s_2) x_2 + \ldots + (-s_k) x_k \leq -t \qquad \text{--- } (2)$$

$$x_i \in \{0, 1\} \quad \forall i.$$

It is easy to see that the inequality (2) is equivalent to the following

$$s_1 x_1 + s_2 x + \ldots + s_k x_k \geq t \qquad \text{--- } (3)$$

(1) & (3) $\Rightarrow \sum_{i \in 1}^{k} s_i x_i = t$

When $x_i \in \{0, 1\}$, this is the same as asking if there is a subset of S that sums to t, which is the SUBSET-SUM problem.

Thus 0/1 ILP is NP-complete.

This completes time complexity.

Next week – space complexity.