Savitch's Theorem

(Walter Savitch - 1970)

Theorem: For $f: \mathbb{N} \to \mathbb{R}^+$, $f(n) \geq \log n$,

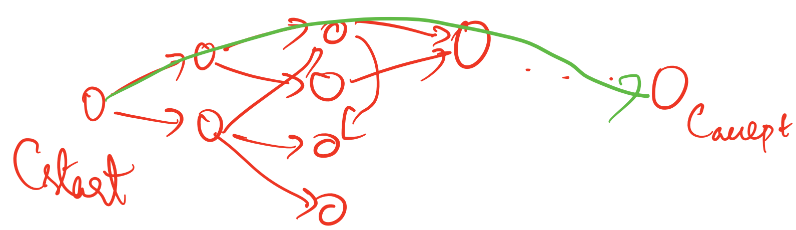
$$NSPACE(f(n)) \subseteq DSPACE((f(n))^2)$$

* If we can square the space usage, then we can give up on nondeterminism.

* As seen in previous lectures, simulating a space bounded NTM can be accomplished by searching for a path from C_{start} to C_{accept} in the config graph.

Assume unique C_{accept}.

Let $A \in NSPACE(f(n))$. Let N be the $O(f(n))$ space NTM that decides A. There are $2^{O(f(n))}$ $= 2^{d f(n)}$ configurations for N. We need to decide if there is a path from C_{start} to C_{accept} that uses at most $2^{d f(n)}$ vertices.



PATH(v_1, v_2, t)

If $t=1$, ACCEPT if $v_1 = v_2$ or if $v_1 \rightarrow v_2$ is an edge.

Else for all vertices w,

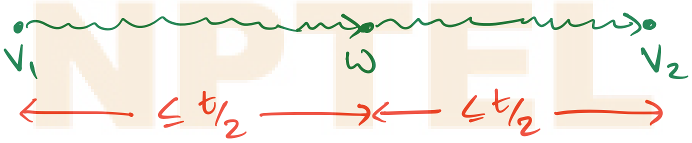
Run PATH$(v_1, w, t/2)$

Run PATH$(w, v_2, t/2)$

ACCEPT if both ACCEPT.

REJECT if not accepted yet.

<u>Basic idea</u>: There is a path from v_1 to v_2 using at most t vertices if there is a vertex w such that



The correctness is evident from the above.

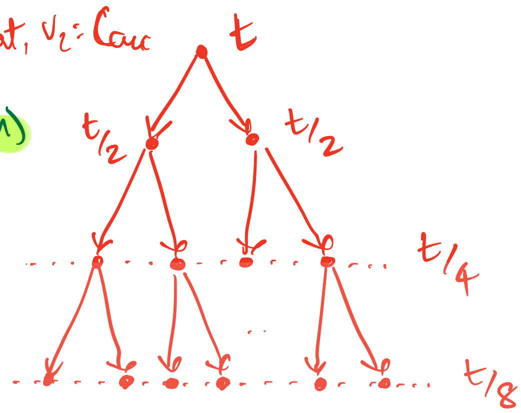
What is the space usage?

We need $\log t$ levels.

$v_1 = C_{start}, v_2 = C_{acc}$

The initial call has $t = 2^{d f(n)}$.

For each recursive call, we need to store v_1, v_2, t, w in the stack.



* Each of v_1, v_2, t, w requires $\log 2^{d f(n)} = d f(n)$ bits. So space usage is $\underline{4 \cdot d f(n)}$ per recursive call. No. of levels $\underline{\le d f(n)}$. $(= \log t)$

* So total space $\le 4 \cdot (d f(n))^2$

$$= \underline{\underline{O(f(n))^2}}.$$

Finally, the simulating machine can do this without knowing $f(n)$ as well. The simulating machine can try out $f(n) = 1, 2, 3, \ldots$ till it finds a decision. If accept is reachable, we accept. If no config of bigger length is reachable, we reject. Else we move to the next value of $f(n)$.

<u>Consequences</u>

1. $NL \subseteq DSPACE(\log^2 n)$
2. $NSPACE(n) \subseteq DSPACE(n^2)$
3. Non deterministic polynomial space, NPSPACE.

$$\begin{aligned} NPSPACE &= \bigcup_{k=1}^{\infty} NSPACE(n^k) \\ &\subseteq \bigcup_{k=1}^{\infty} DSPACE(n^{2k}) \quad \downarrow \text{Savitch} \\ &\subseteq \bigcup_{k=1}^{\infty} DSPACE(n^k) \\ &= PSPACE. \end{aligned}$$

<u>Theorem</u>: NPSPACE = PSPACE.