Proving Undecidability using Reductions

If $A \leq_m B$ and A is undecidable, then B is undecidable.

$A_{TM} = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

Real halting problem

$HALT_{TM} = \{\langle M, w\rangle \mid M$ is a TM and M halts on $w\}$

Theorem 5.1: $HALT_{TM}$ is undecidable.

Proof: Suppose $HALT_{TM}$ is decidable. Let R be the TM that decides $HALT_{TM}$. We will use R to build a decider for A_{TM}.

A_{TM} decider

Given $\langle M, w\rangle$ — A_{TM} instance

Run $\langle M, w\rangle$ in R. If R rejects, then reject

M does not halt on w

If R accepts, then simulate M on w.

- Accept if M accepts
- Reject if M rejects

- M accepts w : R accepts $\langle M, w \rangle$.
- M rejects w : R accepts $\langle M, w \rangle$.
- M loops on w : R rejects $\langle M, w \rangle$. : Reject

Then we run M on w.

Since we cannot get a decider for A_{TM}, this implies that we cannot have a decider for $HALT_{TM}$.

In the above proof, we used hypothetical $HALT_{TM}$ decider to build an A_{TM} decider. We can also prove it by constructing a reduction, and then using Cor 5.23.

Given an A_{TM} instance $\langle M, w \rangle$, what are the possibilities?

A_{TM} YES inst : M accepts w : $HALT_{TM}$ YES inst.
A_{TM} NO inst : M rejects w : $HALT_{TM}$ YES inst.
A_{TM} NO inst : M loops on w : $HALT_{TM}$ NO inst.

This case needs to be handled.

q_{rej} in M
&
q_{loop} in M'

Given $\langle M, w \rangle$, construct M' in the following manner.

M' : When M rejects an input, modify it so that M' enters into an infinite loop.

Reduction machine will output $\langle M', w \rangle$.

$\langle M, w \rangle \in A_{TM} \iff \langle M', w \rangle \in HALT_{TM}$.

The existence of $\langle M', w \rangle$ is not enough. It should be computable.

$A_{TM} \leq_m HALT_{TM}$

Theorem 5.2: E_{TM} is undecidable.

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM, and } L(M) = \phi \}$

Proof: Can we reduce $A_{TM} \rightarrow E_{TM}$?

Given $\langle M, w \rangle$, we construct $\langle M_w \rangle$

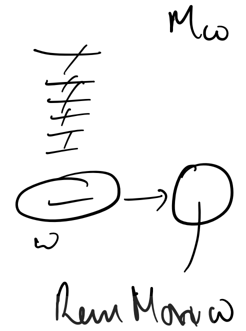
A_{TM} instance

E_{TM} instance

accepts only w

M_w: On input x

1. If $x \neq w$, reject.
2. If $x = w$, run M on input w. Accept iff M accepts.



M accepts $w \iff L(M_w)$ is not empty

$$\langle M, w\rangle \in A_{TM} \iff \langle M_w\rangle \in \overline{E_{TM}}$$

So $A_{TM} \leq_m \overline{E_{TM}}$

Thus $\overline{E_{TM}}$ is undecidable. This implies that E_{TM} is undecidable.

Theorem 5.4: EQ_{TM} is undecidable.

Proof idea: $E_{TM} \leq_m EQ_{TM}$

Construct a TM that rejects every string.

EXERCISE: Read the details from the book.

Theorem 5.3: $REGULAR_{TM}$ is undecidable.

$REGULAR_{TM} = \{\langle M\rangle \mid M$ is a TM, and $L(M)$ is regular$\}$

Proof: $A_{TM} \leq_m REGULAR_{TM}$.

Below we describe how to construct the reduction.

Given $\langle M, w\rangle$, A_{TM} instance, as input.
We construct M' as follows.

M': On input x.

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If x has the form $0^n 1^n$, accept.
If x does not have this form, then run M on w, and accept x if M accepts w.

M accepts $w \Rightarrow L(M') = \Sigma^* \rightarrow$ regular

M does not accept $w \Rightarrow L(M') = \underbrace{\{0^n 1^n \mid n \geq 0\}}_{\text{non regular.}}$.

So $\langle M, w\rangle \in A_{TM} \iff L(M')$ is regular

$\iff \langle M'\rangle \in REGULAR_{TM}$.

So $REGULAR_{TM}$ is undecidable.

In the next lecture, we will see Rice's theorem.

$PROPERTY_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ has some property $\}$

Rice's theorem: $PROPERTY_{TM}$ is undecidable.