Post Correspondence Problem (PCP)

This is a simple undecidable problem.

Given dominos $\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots \left[\frac{t_k}{b_k}\right] \right\}$

can we arrange them (repeats allowed) so that top string = bottom string?

Example: $\left\{ \left[\frac{ab}{aba}\right], \left[\frac{ba}{abb}\right], \left[\frac{b}{ab}\right], \left[\frac{abb}{b}\right], \left[\frac{a}{bab}\right] \right\}$

$\left[\frac{ab}{aba}\right], \left[\frac{a}{bab}\right], \left[\frac{ba}{abb}\right], \left[\frac{b}{ab}\right], \left[\frac{abb}{b}\right], \left[\frac{abb}{b}\right], \left[\frac{b}{ab}\right], \left[\frac{abb}{b}\right]$

This is a match.

Example 2: $\left\{ \left[\frac{ab}{aba}\right], \left[\frac{ba}{abb}\right], \left[\frac{b}{ab}\right] \right\}$

Does this instance have a match?

Answer: NO! In all the dominos, the bottom string is longer.

To show that PCP is undecidable, we define Modified PCP (MPCP).

Modified PCP (MPCP)

It is a PCP instance, but with the extra condition that the match must start from the first domino.

We will show that MPCP $\leq_m$ PCP and then $A_{TM} \leq_m$ MPCP. Together, this will imply that PCP is undecidable.

MPCP $\leq_m$ PCP

Given MPCP instance $\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots \left[\frac{t_k}{b_k}\right] \right\}$

we will create equivalent PCP instance.

Let string $u = u_1 u_2 \cdots u_n$. $u = 011$

We define $*u$, $u*$ and $*u*$ as follows:

$*u = *u_1 * u_2 * \ldots * u_n$. $*u = *0*1*1$

$u* = u_1 * u_2 * \ldots * u_n *$. $u* = 0*1*1*$

$*u* = *u_1 * u_2 * \ldots * u_n *$. $*u* = *0*1*1*$

The MPCP $\to$ PCP reduction is as follows.

$$\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots \left[\frac{t_k}{b_k}\right] \right\}$$ (MPCP instance)

$$\to \left\{ \left[\frac{*t_1}{*b_1*}\right], \left[\frac{*t_1}{b_1*}\right], \left[\frac{*t_2}{b_2*}\right], \dots \left[\frac{*t_k}{b_k*}\right], \left[\frac{*\Diamond}{\Diamond}\right] \right\}$$

(PCP instance)

It can be seen that any match of the PCP instance has to start with $\left[\frac{*t_1}{*b_1*}\right]$. By removing the $*$'s and $\Diamond$, the match in the PCP instance corresponds to a match in the MPCP instance that starts with $\left[\frac{t_1}{b_1}\right]$.

Thus $\quad MPCP \leq_m PCP$

$A_{TM} \leq_m MPCP$

We will now use computation histories to show that $A_{TM} \leq_m MPCP$. Given $\langle M, w \rangle$, we will construct an MPCP instance.

Let $M = (Q, \Sigma, \Gamma, \delta, q_s, q_a, q_r)$

$w = w_1 w_2 \ldots\ldots w_n$.

The dominoes are the following.

1. $\left[\dfrac{\#}{\underbrace{\#q_s w_1 w_2 \cdots w_n}_{\text{Starting config of } M \text{ on } w}\#}\right]$ → As this is an MPCP inst, this must be the starting domino.

Idea: by the time the top "catches up", bottom would have moved one step.

2. For all $a, b \in \Gamma$,
for all $q, r \in Q$,
where $q \neq q_{rej}$
if $\delta(q,a) = (r, b, R)$

$\left[\dfrac{q\ a}{b\ r}\right]$

$0\ 0\ 1\ \boxed{q\ a}\ 2$

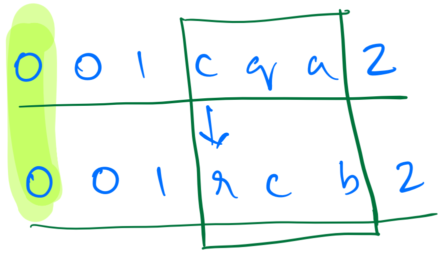
$0\ 0\ 1\ \boxed{b\ r}\ 2$

$0\ 0\ 1\ b\ 2$
$\uparrow$
r

3. For all $a, b, c \in \Gamma$
and every $q, r \in Q$
where $q \neq q_{rej}$
if $\delta(q,a) = (r, b, L)$

$\left[\dfrac{c\ q\ a}{r\ c\ b}\right]$

$0\ 0\ 1\ c\ b\ 2$
$\uparrow$
r



4. For all $a \in \Gamma$, add $\left[\frac{a}{a}\right]$

5. $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$

6. For all $a \in \Gamma$, add $\left[\frac{a\, q_{acc}}{q_{acc}}\right]$, $\left[\frac{q_{acc}\, a}{q_{acc}}\right]$

7. $\left[\frac{q_{acc}\#\#}{\#}\right]$

$\langle M, w\rangle \in A_{TM} \iff$ M accepts w

$\iff$ There is an accepting CH of M on w

$\iff$ The above MPCP instance has a match.

$$\frac{\#\, q_0\, 0\, 1\, 0\, 0\, \#\, 2\, q_3\, 1\, 0\, 0\, \#\, 2\, 5\, q_7\, 0\, 0\, \#}{\#\, q_0\, 0\, 1\, 0\, 0\, \#\, 2\, q_3\, 1\, 0\, 0\, \#\, 2\, 5\, q_7\, 0\, 0\, \#\, 2\, 5\, 1\, q_a\, 0\, \#}$$

$\delta(q_0, 0) = (q_3, 2, R)$ $\qquad$ $\delta(q_3, 1) = (q_7, 5, R)$

$$\frac{q_0\, 0}{2\, q_3} \qquad \frac{q_3\, 1}{5\, q_7}$$

$\delta(q_7, 0) = (q_a, 1, R)$

$$\frac{q_7\, 0}{1\, q_a} \qquad \frac{q_a\, 0}{q_a} \qquad \frac{1\, q_a}{q_a}$$

$$\frac{2\, 5\, 1\, q_a\, 0\, \#\, 2\, 5\, 1\, q_a\, \#\, 2\, 5\, q_a\, \#}{2\, 5\, 1\, q_a\, 0\, \#\, 2\, 5\, 1\, q_a\, \#\, 2\, 5\, q_a\, \#\, 2\, q_a\, \#}$$

$$\frac{2\, q_a\, \#\, q_a\, \#\, \#}{2\, q_a\, \#\, q_a\, \#\, \#}$$

This completes the match!

So $A_{TM} \le_m MPCP$

Rules

1. Empty match not allowed
2. Repetitions allowed
3. Need not have all the dominoes in the match.