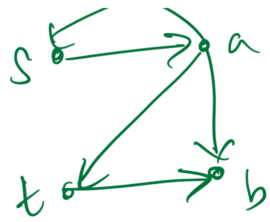
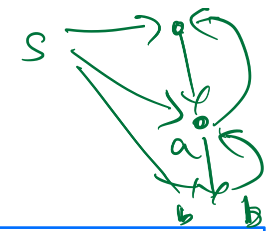
Hamiltonian Path





Given a directed graph G, and designated vertices s, t of G, is there a path from s to t that goes through each vertex of G exactly once?

HAM-PATH = { <G, s, t> | G is a directed graph, G has a Hamiltonian path from s to t }

Theorem: HAM-PATH is NP-complete.

Proof: (1) HAM-PATH ∈ NP. Easy

→ Guess $n-2$ vertices other than s, t:
Say $v_1, v_2, \dots v_{n-2}$.

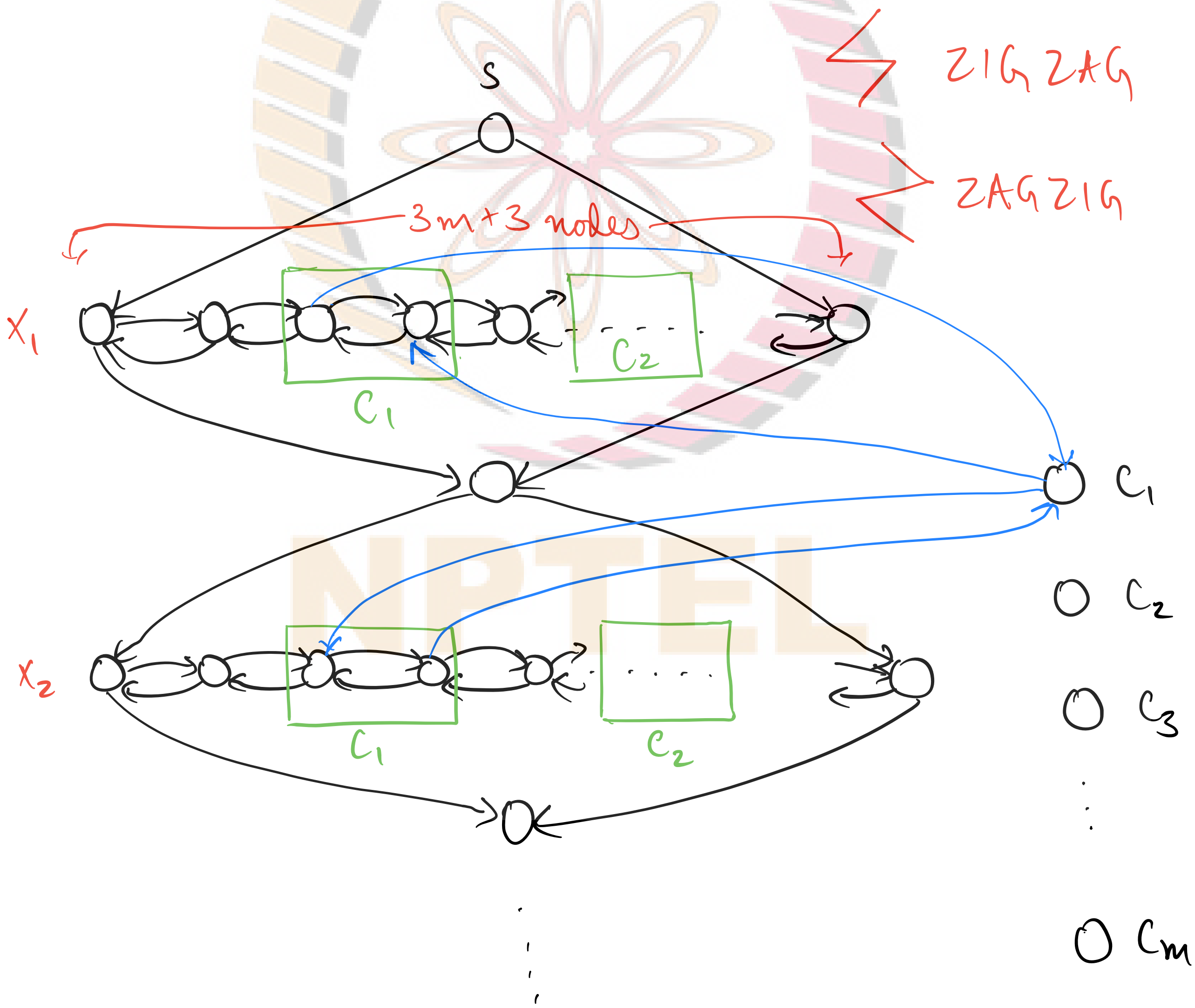
→ Check that $s\, v_1\, v_2 \dots v_{n-2}\, t$ is a path and that all the above are distinct.

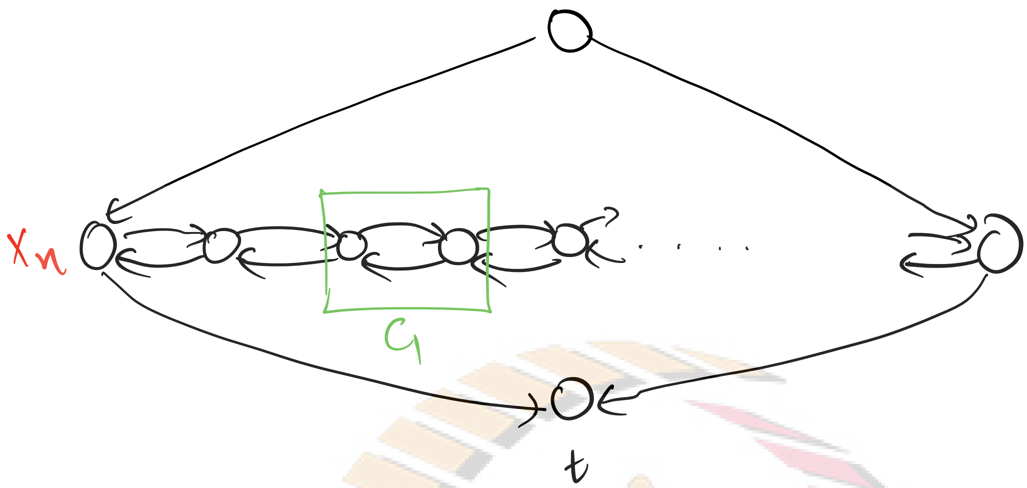
(2) 3-SAT $\leq_p$ HAM-PATH.

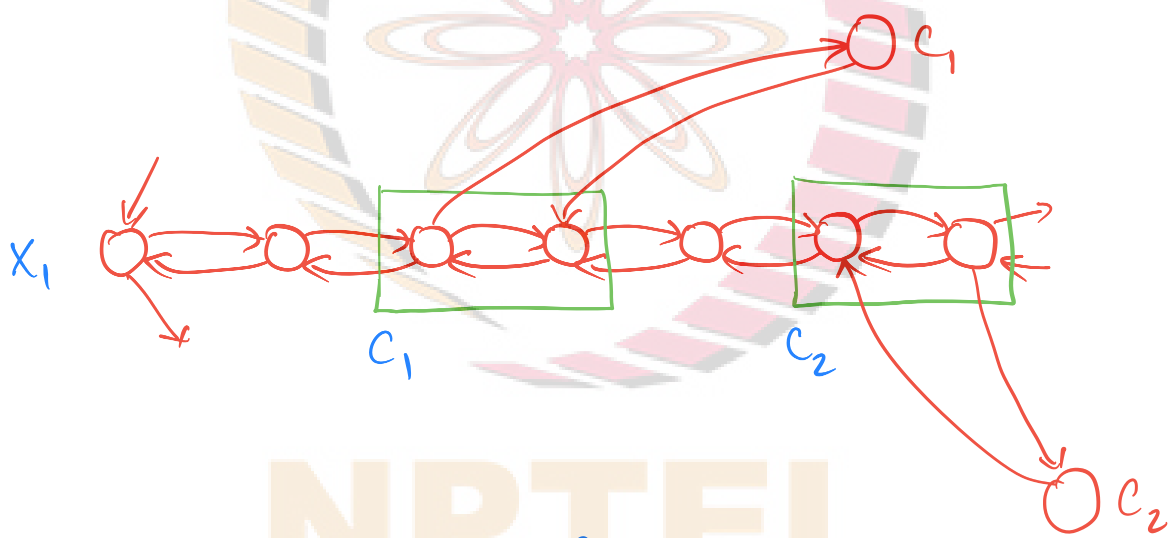
Given ϕ, we need to construct $\langle G, s, t\rangle$ such that

$\langle \phi \rangle \in$ 3-SAT $\iff \langle G, s, t \rangle \in$ HAM-PATH.

Let ϕ have n variables $x_1, x_2, \ldots x_n$ and m clauses.





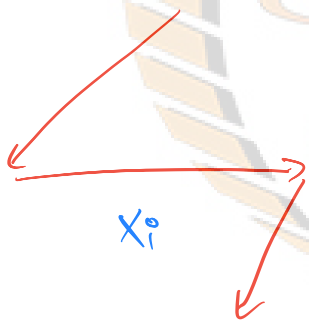


x_1 appears in C_1

$\bar{x}_1$ appears in C_2.

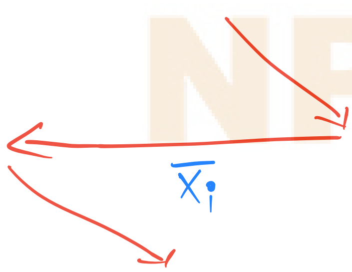
→ How can we go from s to t? At each x_i "diamond", we can go zigzag or we can zagzig

This gives us 2^n ways to go from s to t, covering all the nodes, except the clause nodes. To cover the clause nodes, we need to take detours.



ZIGZAG

We can cover clauses in which x_i is present.



ZAGZIG

We can cover clauses in which $\bar{x}_i$ is present.

This completes the construction. Given ϕ, this takes only time polynomial in n and m.

($\Longrightarrow$) Suppose $\phi \in$ 3-SAT. That is, there exists an assignment satisfying all the clauses. For each clause C_j, there is a true literal.

We start from s and move down. If x_i is set to true in the sat. assignment, we zigzag on x_i. If x_i is set to false, we zag zig.

Each clause can be covered by a detour from one of the true literals in it.

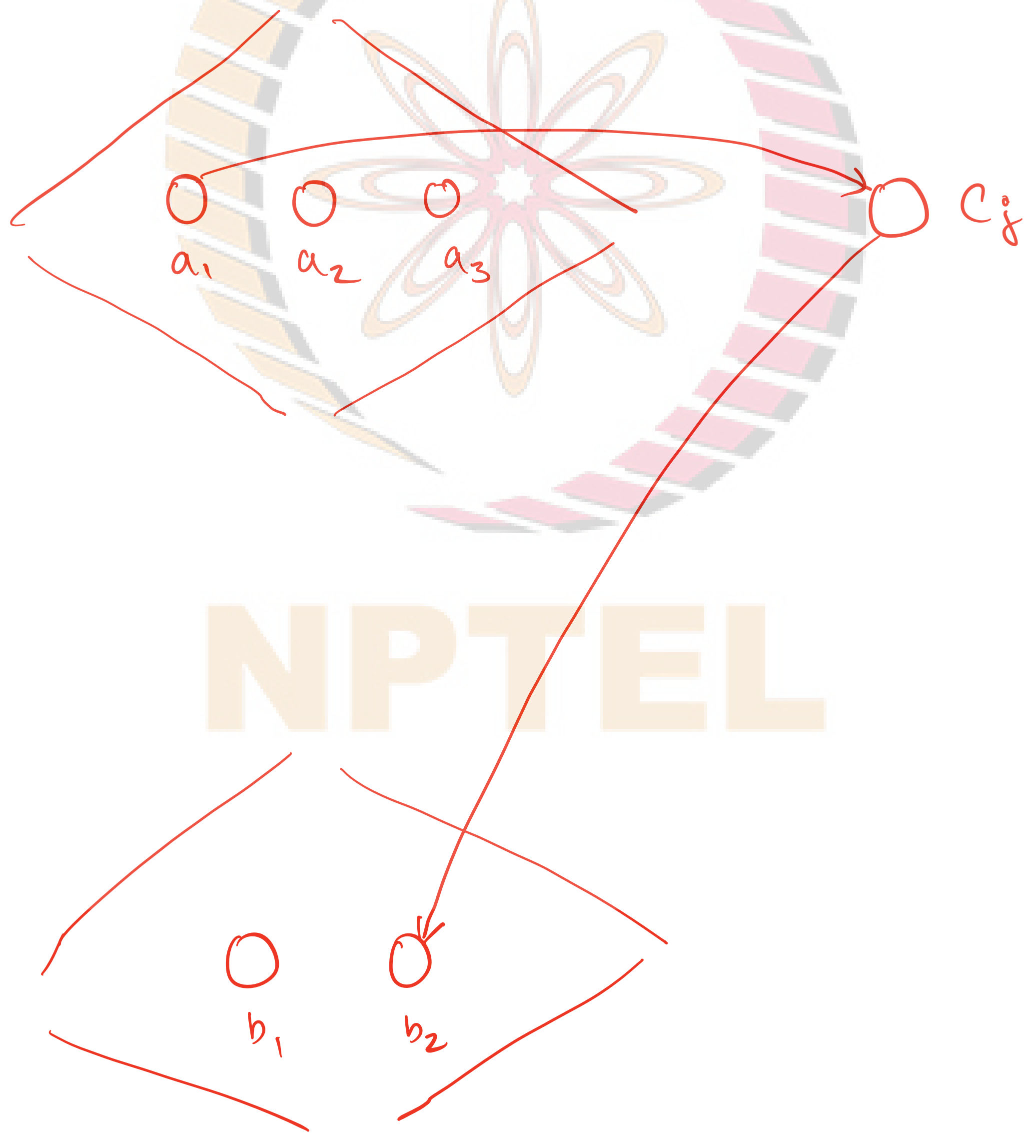
Hence G has an s-t Hamiltonian path.

($\Longleftarrow$) Suppose G has an s-t Hamiltonian path. There is a path from s to t that goes through all the vertices.

If this path is normal (goes through all the diamonds in order from top to bottom), set each variable to True/False accordingly. (as per zigzag or zagzig).

Each clause is covered and has to have at least one true literal.

How can a path not be normal? This can happen only through C_j nodes. Enter a C_j from x_5 and exit into x_{10}, say.



Either a_2 or a_3 is a separator node.

→ If a_2 is separator, then the only way to enter a_2 is through a_1 (covered) or a_3. If we enter a_2 from a_3, then no exit is possible.

→ If a_3 is separator, then the only way to enter a_2 is through a_1 (covered), c_j (also covered) or a_3. If we enter a_2 from a_3, then no exit is possible.

So the path has to be normal. We can assign True/False to each x_i as per whether we zig zag or zag-zig at the respective diamond. This gives us a satisfying assignment for ϕ.

Exercises: (1) Show that UHAMPATH is NP-complete.

HAMPATH in undirected graph.

We can reduce from HAMPATH.

(2) Show that HAM-CYCLE is NP-complete.

<u>Hamiltonian Cycle</u>: A cycle that goes through all the vertices exactly once.