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Key Points:
- We derive a method to analytically determine the correct initial conditions for an aquifer flow with constant boundary conditions
- An analytical series solution is derived that is valid for both discharging and recharging aquifers, generalizing previous results
- We compare the initial conditions derived and the series solutions computed with numerical results for some illustrative cases

Supporting Information:
- Supporting Information S1
- Python Script S1

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A Generalized Series Solution for the Boussinesq Equation With Constant Boundary Conditions

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Abstract In this work we consider the Boussinesq equation applied to a one-dimensional homogeneous aquifer with constant boundary conditions. Such a problem is generally solved (both analytically and numerically) by transforming it from a boundary value problem into an equivalent initial value problem that has to be solved by means of a trial-and-error approach. We devise a method that eliminates the need for trial and error that is purely analytical, which can benefit both numerical and analytical solutions. Furthermore, we present a general series solution that solves this equation analytically with arbitrary precision. The series solution generalizes two particular cases that already exist in the literature. With the new proposed series, the need for any numerical integration of differential equations is removed, and fully analytical benchmark solutions are obtained.

1. Introduction

We study the problem of groundwater flow through an unconfined porous medium with constant porosity n_e and saturated hydraulic conductivity k_0. We consider a semi-infinite aquifer that is initially full (with a given uniform water head height H throughout the domain) and which gradually interacts with an adjacent stream which has a different fixed water level $H_0 \geq 0$. Such a setup is depicted in Figure 1 and can be modeled by the Boussinesq equation (Brutsaert, 2005)

$$n_e \frac{\partial h}{\partial t} = k_0 \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right), \tag{1}$$

where $h(x,t)$ is the water table height, t is time, and x is the horizontal coordinate, under the initial and boundary conditions

$$h(x,0) = H, \qquad h(\infty,t) = H, \qquad h(0,t) = H_0. \tag{2}$$

Note that equations (1) and (2) model the water table height h in a semi-infinite aquifer that spans from $x = 0$ to $x \to \infty$. Such a semi-infinite domain limits the applicability of the solution for finite-domain cases, serving instead as an approximation for their early-time behavior (Basha & Maalouf, 2005; Troch et al., 2013). The important question of the duration of the validity of the semi-infinite problem (1) and (2) in a *finite* domain is discussed in Text S1 of the supporting information (Bogaart et al., 2013; Boussinesq, 1904; Brutsaert & Nieber, 1977; Chor & Dias, 2015).

Because of the symmetry of the boundary conditions, the Boltzmann similarity transformation is applicable, and the problem reduces to (Dias et al., 2014)

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\phi \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) + 2\xi \frac{\mathrm{d}\phi}{\mathrm{d}\xi} = 0, \tag{3}$$

with the equivalent boundary conditions

$$\phi(0) = \frac{H_0}{H} = \phi_0, \qquad \phi(\infty) = 1, \qquad \phi_0 \geq 0. \tag{4}$$

Here $\phi = h/H$ is the normalized water head height, $\xi = \frac{x}{\sqrt{4Dt}}$ is the similarity variable, and $D = Hk_0/n_e$. Note that in the present setup, the aquifer can be recharging ($\phi_0 > 1$), discharging ($0 \leq \phi_0 < 1$), or in

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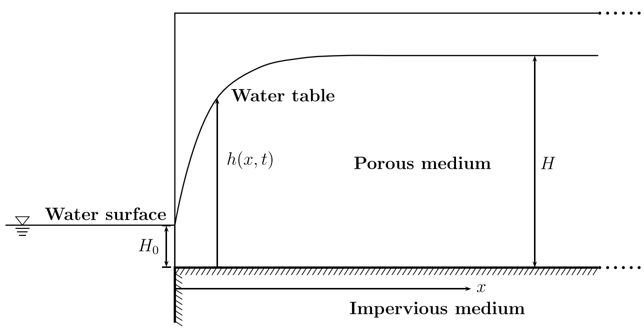

Figure 1. Representation of a semi-infinite horizontal homogeneous aquifer for which the Boussinesq equation is applicable. Here $H_0 \geq 0$ can be larger or smaller than H, corresponding to a recharging and discharging aquifer, respectively (figure shows the discharging case).

equilibrium ($\phi_0 = 1$). We focus the present effort on the case of discharging aquifers, noting differences with the recharging case in the text.

Such a problem is well-known in the literature (Rupp & Selker, 2006), and equations (3) and (4) are usually solved using numerical methods (Chor et al., 2013; Yeh, 1970), series solutions (Furtak-Cole et al., 2018; Gravanis & Akylas, 2017; Polubarinova-Kochina, 1962; Song et al., 2007; Telyakovskiy et al., 2010), or semianalytical approaches such as the Adomian Decomposition Method (ADM; Adomian & Adomian, 1984; Serrano & Adomian, 1996; Serrano & Workman, 1998), which can be very efficient numerically. In the two former cases, this boundary value problem is usually transformed into an initial value problem (IVP) that satisfies the original boundary conditions. This transformation is done in virtually every diffusion equation with a concentration-dependent coefficient and other nonlinear equations such as the Blasius equation for boundary layer flows (Allan, 1997). For the Boussinesq equation, this is done by replacing the conditions given in equation (4) by

$$\phi(0) = \phi_0, \qquad \psi(0) = \psi_0, \qquad \phi_0 \geq 0, \tag{5}$$

where ψ is related to the water flow across a section of the aquifer and is defined as

$$\psi \equiv \phi \frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \tag{6}$$

with ψ_0 being the limit of ψ as ξ tends to zero. Thus, the boundary condition for ϕ at $\xi \to \infty$ is replaced with a condition for ψ at $\xi = 0$. Such a transformation assumes an a priori knowledge of ψ_0, which is generally not the case: Usually, the value of ψ_0 is known a posteriori, after a trial-and-error approach. Dias et al. (2014) overcome this issue by using a shooting method (Na T., 1979) to calibrate an empirical expression for ψ_0, which may be enough for some cases but falls short in the sense that they arrive at an arbitrary expression with fitted coefficients and no particular physical meaning. In this work we introduce a method to obtain ψ_0 analytically for any value of $\phi_0 \in [0, 1]$ in which no approximations or calibrations are necessary. This means we are able to transform the original boundary value problem into an equivalent IVP with exactly the same solution, which should benefit both numerical and series solutions from this point forward.

Furthermore, the ability to analytically obtain ψ_0 for any value of ϕ_0 allows us to derive a generalized series solution for equations (3) and (5). Although Chor et al. (2013) and Dias et al. (2014) have presented solutions for this problem for the specific cases of $\phi_0 = 0$ and $\phi_0 > 0$, respectively, their solutions are not able to analytically reproduce $\phi(\xi)$ throughout the complete semi-infinite domain due to finite radii of convergence. The solution presented in this work not only generalizes the solutions by Chor et al. (2013) and Dias et al. (2014) into one general series, but it can also cover the full domain of the problem (ξ in the interval $[0, \infty)$) by means of analytic continuation (Henrici, 1974). We finish by applying the derived approaches to some cases of equations (3) and (5) in order to assure the boundary conditions are indeed satisfied by the IVP and that the analytical series indeed reproduces the correct solution in the full semi-infinite domain. Finally, comparisons with numerically-obtained solutions (whose initial conditions are obtained analytically) confirm their validity.

2. Determination of the Correct Initial Conditions for the IVP

The correct determination of the initial conditions for equations (3) and (5) means determining ψ_0 for any given ϕ_0 such that $\phi(\infty) = 1$. In order to accomplish this, we need a differential equation that relates ψ and ϕ without any dependence on ξ. We can obtain such an equation from equation (3) by applying the chain rule, using the definition of ψ and differentiating with respect to ϕ. The detailed steps are given in Appendix A, and the final result is the following differential equation:

$$\psi \frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} = -2\phi, \tag{7}$$

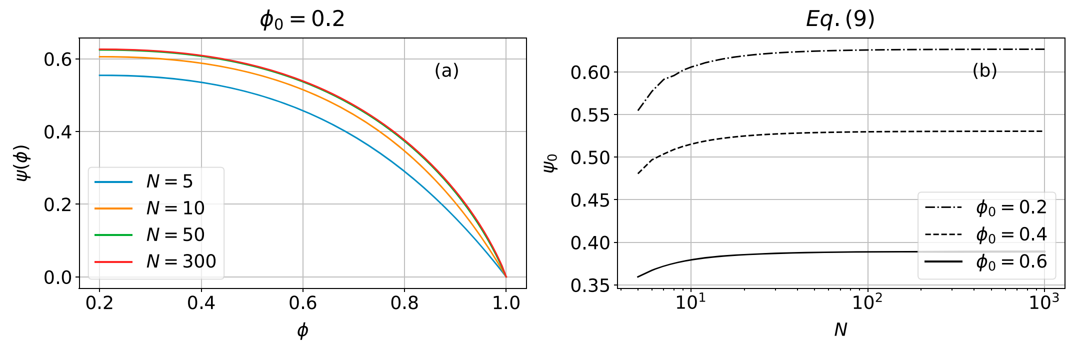

Figure 2. (a) $\psi(\phi)$ calculated with the series expansion of equation (8) for $\phi_0 = 0.2$ and ψ_0 calculated with equation (11) with $N = 5, 10, 50$, and 300. (b) Convergence rate of our approach by showing calculations of ψ_0 with equation (11) for different values of ϕ_0 as a function of the number of N.

which is singular for $\phi = 1$ but has the advantage of eliminating ξ and treating ϕ as the independent variable, thus reducing the problem from a semi-infinite domain to a finite one (from ϕ_0 to $\phi = 1$). From conditions at $\xi = 0$ and $\xi \to \infty$, the inversion of variables produces

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi}(\phi_0) = 0, \qquad \psi(1) = 0,$$

respectively.

A solution of equation (7) in the form of a power series can be sought by assuming that $\psi(\phi)$ can be expressed as

$$\psi = \sum_{n=0}^{\infty} b_n (\phi - \phi_0)^n. \tag{8}$$

Substituting equation (8) into equation (7) along with the boundary conditions, one can derive

$$b_0 = \psi_0, \quad b_1 = 0, \quad b_2 = -\frac{\phi_0}{\psi_0}, \quad b_3 = -\frac{1}{3\psi_0}. \tag{9}$$

Furthermore, it is possible to write the recurrence relation for b_n as

$$b_{n+2} = -\frac{1}{\psi_0(n+2)(n+1)} \sum_{k=1}^{n} (n-k+2)(n-k+1) b_k b_{n-k+2}, \qquad n > 1. \tag{10}$$

We show results for equations (8)–(10) for $\phi_0 = 0.2$ with (8) summed up to increasing values of N in Figure 2a to illustrate the convergence of the series solution. Note that equations (8)–(10) reproduce $\psi(\phi)$ over the whole interval from $\phi = 0$ to $\phi = 1$; however, in order to obtain the correct value for ψ_0 (which is used to calculate the terms b_n and which is the focus of this section), one only has to consider the series at the point $\phi = 1$, thus enforcing the condition $\psi(\phi = 1) = 0$. Since the series given by equations (8)–(10) converges at $\phi = 1$ for $0 \le \phi_0 < 1$ (see theorem 4 of Callegari & Friedman, 1968, for a proof for the $\phi_0 = 0$ case), it can be used to apply such condition successfully, and we estimate ψ_0 by solving the algebraic equation

$$\sum_{n=0}^{N} b_n (1 - \phi_0)^n = 0 \tag{11}$$

for a sufficiently large N. We show results for ψ_0 calculated with equation (11) with different values of N and ϕ_0 in Figure 2b. Equation (11) is the main result of this work, since it facilitates both numerical and series solutions of the problem at hand.

Although equation (11) is solved iteratively, as opposed to explicitly, this is much faster and simpler than the previously available approaches of numerically integrating equation (3) progressively, with different values of ψ_0, up to a large value of ξ (for which one can consider $\xi = \infty$) in a trial-and-error search for the value of ψ_0 that matches the boundary condition $\phi(\infty) = 1$ closely. Determination of ψ_0 via equation (11) is, as far as we know, the first method capable of computing the value of ψ_0 with arbitrary precision analytically for every value of $0 \le \phi_0 < 1$. Roughly, computation with around 300 terms gives a precision of at least 5×10^{-4} when obtaining ψ_0. We provide a Python script that performs this computation as Supporting Information S1 along with this paper.

Note that although equation (7) can be applied for both discharging and recharging aquifers, a few notes are due when considering its application on recharging aquifers ($\phi_0 > 1$). The principal difference is that the infinite sum in equation (11) does not converge for $\phi_0 > 1$. This fact prevents us from obtaining ψ_0 with arbitrary precision for a recharging aquifer. However, with $\phi_0 > 1$, equation (11) presents a behavior typical of asymptotic series: It converges up to a finite N, after which the error increases (Henrici, 1977, Section 11.1). This behavior makes it possible to obtain reasonable values of ϕ_0 in such a situation (provided that the value of ϕ_0 is not too large) albeit with a limited precision. Future investigation on this topic is needed in order to compare the limited precision approximation it may provide for ψ_0.

3. Deriving a General Series Solution

In this section we derive a series solution that generalizes both series presented in Chor et al. (2013) and Dias et al. (2014). The present solution significantly improves on those previous results in that it applies for both homogeneous and nonhomogeneous boundary conditions at the origin and, more importantly, it allows the solution to be expanded about any point in the ξ domain, which makes it possible to use the principle of analytic continuation (Henrici, 1974, Section 3.2) to effectively reproduce the series at any point of the domain.

We begin by assuming that the solution ϕ of equations (3) and (4) can be expressed as

$$\phi(\xi) = \sum_{n=0}^{\infty} a_n (\xi - \xi_a)^{n/2}, \tag{12}$$

where ξ_a is the point of expansion of the series. After substitution of equation (12) into an integrated version of equation (3), namely,

$$\phi \frac{\mathrm{d}\phi}{\mathrm{d}\xi} - \psi_a + 2\left(\xi\,\phi \Big|_{\xi_a}^{\xi} - \int_{\xi_a}^{\xi} \phi\,\mathrm{d}\xi \right) = 0, \tag{13}$$

we obtain

$$\begin{aligned} a_1 a_0 = a_1 \phi_a &= 0, \\ \frac{a_1^2}{2} + a_2 a_0 &= \psi_a, \\ 3(a_1 a_2 + a_3 a_0) + 4\xi_a a_1 &= 0, \end{aligned} \tag{14}$$

where $\phi_a = \phi(\xi_a)$ and $\psi_a = \psi(\xi_a)$. The detailed steps of this derivation can be found in Appendix B.

From the same substitution, we also obtain two recurrence relations valid for $n \ge 0$. The first one should be used if $\phi_a \ne 0$:

$$a_{n+4} = -\frac{1}{(n+4)a_0} \left(\frac{n+4}{2} \sum_{k=1}^{n+3} a_k a_{n-k+4} + \frac{4n}{n+2} a_n + 4\xi_a a_{n+2} \right). \tag{15}$$

If, however, $\phi_a = 0$ (i.e., $a_0 = 0$), equation (15) clearly does not hold. Instead, we then have

$$a_{n+3} = -\frac{1}{(n+4)a_1} \left(\frac{n+4}{2} \sum_{k=2}^{n+2} a_k a_{n-k+4} + \frac{4n}{n+2} a_n + 4\xi_a a_{n+2} \right). \tag{16}$$

Note from equations (12)–(16) that if the series is being expanded about a point such that $\phi_a = 0$, then $a_1 = \sqrt{2\psi_a}$. In this case the solution has a singular behavior at ξ_a due to the square root-type terms in the series. However, if $\phi_a \ne 0$, then all the odd numbered coefficients are zero, canceling out all the fractional exponents and effectively producing a Taylor series with no singular behavior. We provide a Python script that calculates coefficients a_n as Supporting Information S1.

Figure 3 illustrates the series expansion for the homogeneous case (Figure 3a) and for $\phi_0 = 0.6$ (Figure 3b) as continuous lines. Analytic continuation was used to compute ϕ and ψ up to $\xi = 3$ in both cases (which

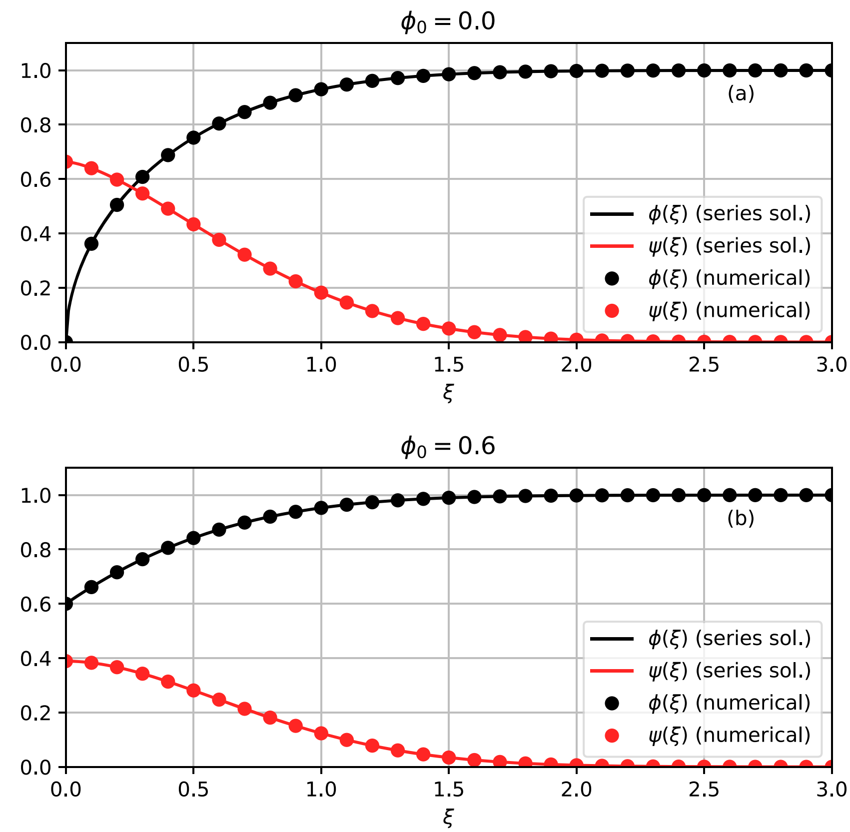

Figure 3. Functions ϕ and ψ for the homogeneous case $\phi_0 = 0$ (a) and for the case $\phi_0 = 0.6$ (b). The initial condition ψ_0 was obtained by solving equation (11) iteratively. All solid lines were obtained with equation (12), along with its recurrence relation (15) and (16). The points correspond to numerical solutions obtained according to Chor et al. (2013) with the initial condition ψ_0 also given by equation (11). All calculations were done using $N = 150$. It is straightforward to assess that the boundary conditions at infinity are satisfied.

is straightforward here because of the general expression given by equation (12)). It is worth noting that to analytically continue the series, the value for $\psi_a = \psi(\xi_a)$ also has to be computed for $\xi_a > 0$, which can be done using the definition of ψ and the series solution for $\phi(\xi)$. We focused on the discharging aquifers case for application of the series solution presented here, even though it equally applies to recharging aquifers ($\phi_0 > 1$) with no need for any adaptation.

Numerical solutions shown in Figure 3 were obtained with classical fourth-order Runge-Kutta method (Press et al., 1993), using a step of 1×10^{-5}. Both analytical and series solutions were compared with a more accurate numerical solution from an adaptive Runge-Kutta 4–5 solver (Dormand & Prince, 1980). For the homogeneous case, both series and numerical solutions have similar errors (the errors integrated over the domain were $\approx 1 \times10^{-8}$) and similar wall-clock times. The errors for numerical solutions decrease as ϕ_0 increases, as expected for nonsingular problems $\phi_0 > 0$. It is worth noting that we calculated the series solution at 301 points for the comparison. However, one advantage of the series solution is that if we are interested in its value at some specific point, for example, the value $\phi(\xi = 3)$, only the points used to analytically continue the series are needed (4 points in this case), while the number of points to obtain the same value via numerical solution is significantly larger. Moreover, since computational efficiency is not the main goal of a series solution, the computational code used was not extensively optimized in either case. The main advantage of the series solution is that it constitutes an exact result. This is not the case for numerical solutions, which always solve an *approximation* of the governing equations. For these reasons, exact series solutions are important benchmarks for numerical solutions.

4. Conclusions

In this work we have considered the problem of groundwater flow through a porous medium with the Boussinesq equation for constant boundary conditions in a semi-infinite domain, given by equations (3) and (4). Under these circumstances, we have eliminated the need of any numerical integrations of differential equations to find the solution $\phi(\xi)$, which can now be represented in a purely analytical manner.

First, we devised a method to analytically transform the boundary condition at $\xi \to \infty$ into an initial condition at the origin. Its advantage lies in the fact that it solves a single algebraic equation, whereas the shooting method consists in a trial-and-error search that involves solving several IVPs until the correct ψ_0 is obtained. The correct analytical determination of the proper initial condition for this problem benefits both series solutions and numerical solutions of the Boussinesq equation, since the usual approaches involve solving the equivalent IVP.

It is worth noting that there is a relation between ψ_0 and the Blasius constant κ for the standard stationary plate and the moving plate problems (given that the velocity of the plate coincides with nonnegative values of ϕ_0; Allan, 1997). Thus, we can retrieve the initial conditions for the boundary layer problem with $\kappa = \psi_0/2$.

Furthermore, we have developed a generalized series for the solution of the Boussinesq equation that allows for expansions about any point in the domain, thus making it possible to apply the principle of analytic continuation to represent the solution analytically in the complete semi-infinite domain. This result is given by equations (12) to (16) and can be used as a benchmark for future advances for the porous aquifer problem. Finally, we compared applications of the present solution against numerical ones which confirmed their accuracy and validity.

A brief note on the ADM is due at this point. Comparisons with our work are difficult, given the very different nature of both approaches, but we note that one advantage of our method is the simple algebra which produces a recurrence relation that makes it easy to calculate as many terms as needed for the desired accuracy. In the case of the ADM, calculating new terms is not as easy, which may be a problem when the hydraulic conductivity is too low (which decreases the rate of convergence of the method; Serrano & Workman, 1998). However, the ADM is more flexible since it can solve problems with time-varying boundary conditions and finite aquifers, which may be an important characteristic in some cases. A more rigorous comparison of the two methods is not straightforward and is left for future work.

Appendix A: Details on the Derivation of Equation (7)

We begin by rewriting equation (3) by applying the chain rule under the assumption that $\mathrm{d}\phi/\mathrm{d}\xi$ is different from zero; eventually, this will lead to the elimination of ξ from the equation. Proceeding

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\phi\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + 2\xi\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = 0; \tag{A1}$$

after using equation (6) and simplifying $\mathrm{d}\phi/\mathrm{d}\xi$, we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi} + 2\xi = 0. \tag{A2}$$

Differentiating equation (A2), we are left with a second-order equation in ψ:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} = -2\frac{\mathrm{d}\xi}{\mathrm{d}\phi},$$

and with the help of equation (6), we can substitute the term in the right-hand side to produce

$$\psi\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} = -2\phi. \tag{A3}$$

From conditions at $\xi = 0$ and $\xi \to \infty$, the inversion of variables produces

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi}(\phi_0) = 0, \qquad \psi(1) = 0,$$

respectively.

Appendix B: Derivation of the Series Solution From Section 3

This appendix presents the derivation of the series solution described in section 3 in detail, which solves equation (3) with (5). To begin, we integrate equation (3) from ξ_a to ξ:

$$\underbrace{\phi\frac{\mathrm{d}\phi}{\mathrm{d}\xi}}_{\mathrm{I}} - \psi_a + 2\left(\underbrace{\xi\ \phi\Big|_{\xi_a}^{\xi}}_{\mathrm{II}} \underbrace{- \int_{\xi_a}^{\xi}\phi\,\mathrm{d}\xi}_{\mathrm{III}}\right) = 0, \tag{B1}$$

where we use the notation ξ_a for the point of expansion of the series, $\phi_a = \phi(\xi_a)$ and $\psi_a = \psi(\xi_a)$.

Let us now suppose that the Boussinesq function ϕ can be expressed as

$$\phi(\xi) = \sum_{n=0}^{\infty} a_n(\xi - \xi_a)^{n/2}. \tag{B2}$$

This postulate is a natural generalization of the particular case $\xi_a = 0, \phi_a = 0$, where the same choice of exponents successfully worked in previous studies (Heaslet & Alksne, 1961; Chor et al., 2013). Equation (B2) comprises also the case with solutions that admit Taylor expansions, already considered in Dias et al. (2014) for $\xi_a = 0, \phi_a > 0$. Moreover, note that if the exponents of $(\xi - \xi_a)$ are not of the form $n/2$, the procedure outlined below would not be possible.

It follows that

$$\mathrm{I} = \phi\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \sum_{n=0}^{\infty} a_n(\xi-\xi_a)^{n/2}\sum_{k=0}^{\infty}\frac{k}{2}a_k(\xi-\xi_a)^{\frac{k-2}{2}} \tag{B3}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n+1}\frac{k}{2}a_k a_{n-k+1}\right)(\xi-\xi_a)^{\frac{n-1}{2}}. \tag{B4}$$

The second term can be written as

$$\begin{aligned}\mathrm{II} = \ \xi\phi\Big|_{\xi_a}^{\xi} &= \left[(\xi-\xi_a+\xi_a)\sum_{n=0}^{\infty}a_n(\xi-\xi_a)^{\frac{n}{2}}\right]_{\xi_a}^{\xi} \\ &= \sum_{n=0}^{\infty}a_n(\xi-\xi_a)^{\frac{n+2}{2}} + \xi_a\sum_{n=0}^{\infty}a_n(\xi-\xi_a)^{\frac{n}{2}} - \xi_a a_0.\end{aligned} \tag{B5}$$

Furthermore, we have that the third term can be written as

$$\mathrm{III} = -\int_{\xi_a}^{\xi}\phi\,\mathrm{d}\xi = -\sum_{n=0}^{\infty}\frac{2}{n+2}a_n(\xi-\xi_a)^{\frac{n+2}{2}}. \tag{B6}$$

Bringing the terms together produces

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m+1}\frac{k}{2}a_k a_{m-k+1}\right)\xi^{\frac{m-1}{2}} \\ &\quad - \psi_a + 2\left(\sum_{n=0}^{\infty}\frac{n}{n+2}a_n(\xi-\xi_a)^{\frac{n+2}{2}} + \xi_a\sum_{p=0}^{\infty}a_p(\xi-\xi_a)^{\frac{p}{2}} - \xi_a a_0\right) = 0.\end{aligned} \tag{B7}$$

In order to collect terms with the same power of $(\xi - \xi_a)$, we alter indexes

$$m = n+3 \qquad p = n+2 \tag{B8}$$

without modifying the total result, which produces

$$\begin{aligned}&\sum_{n=-3}^{\infty}\left(\sum_{k=0}^{n+4}\frac{k}{2}a_k a_{n-k+4}\right)(\xi-\xi_a)^{\frac{n+2}{2}} \\ &\quad - \psi_a + 2\left(\sum_{n=0}^{\infty}\frac{n}{n+2}a_n(\xi-\xi_a)^{\frac{n+2}{2}} + \xi_a\sum_{n=-2}^{\infty}a_{n+2}(\xi-\xi_a)^{\frac{n+2}{2}} - \xi_a a_0\right) = 0.\end{aligned} \tag{B9}$$

Analyzing the values $n = -3$, $n = -2$, and $n = -1$ separately gives, respectively,

$$a_1 a_0 = a_1 \phi_a = 0, \tag{B10}$$

$$\frac{a_1^2}{2} + a_2 a_0 = \psi_a, \tag{B11}$$

$$3(a_1 a_2 + a_3 a_0) + 4\xi_a a_1 = 0. \tag{B12}$$

We can now collect terms in the equation as

$$\begin{gathered}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n+4}\frac{k}{2}a_k a_{n-k+4} + 2\frac{n}{n+2}a_n + 2\xi_a a_{n+2}\right)(\xi-\xi_a)^{\frac{n+2}{2}} = 0 \qquad \Rightarrow \\ (n+4)a_0 a_{n+4} + \frac{n+4}{2}\sum_{k=1}^{n+3}a_k a_{n-k+4} + \frac{4n}{n+2}a_n + 4\xi_a a_{n+2} = 0.\end{gathered} \tag{B13}$$

At this point there are two possibilities depending on the conditions at the point of expansion. If $\phi_a \neq 0$, then $a_0 \neq 0$, and we have

$$a_{n+4} = -\frac{1}{(n+4)a_0}\left(\frac{n+4}{2}\sum_{k=1}^{n+3}a_k a_{n-k+4} + \frac{4n}{n+2}a_n + 4\xi_a a_{n+2}\right). \tag{B14}$$

If, however, $\phi_a = 0$, that is, $a_0 = 0$, the previous algebra does not hold. In this case equation (B13) leads to

$$a_{n+3} = -\frac{1}{(n+4)a_1}\left(\frac{n+4}{2}\sum_{k=2}^{n+2}a_k a_{n-k+4} + \frac{4n}{n+2}a_n + 4\xi_a a_{n+2}\right), \tag{B15}$$

which completes our derivation.

Acknowledgments

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