Sphere-Sphere Collision Detection

Wagner L. Truppel

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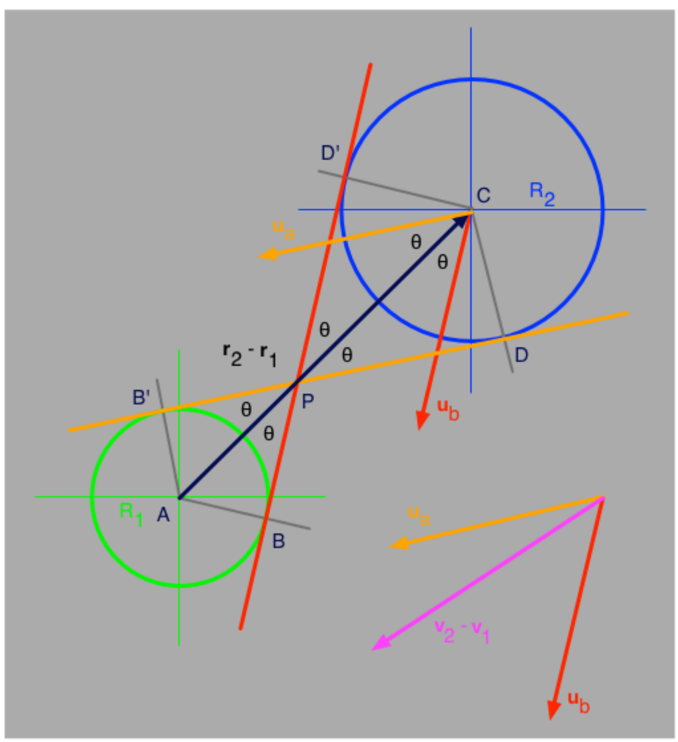
Suppose two spheres (of radii R_1 and R_2, respectively) are currently located at position vectors $\vec{r}_1$ and $\vec{r}_2$, moving with velocity vectors $\vec{v}_1$ and $\vec{v}_2$, respectively. Will they collide? If so, when and where?

It's easier to analyze this problem in the reference frame of one of the spheres, say, the one at position vector $\vec{r}_1$ (green in the picture below). Then, the position vector of sphere 2's center, relative to sphere 1's center, is $\vec{r}_2 - \vec{r}_1$ and the relative velocity of sphere 2 with respect to sphere 1 is $\vec{v}_2 - \vec{v}_1$.



Now, imagine constructing the two families of planes that are instantaneously tangent to both spheres, in a criss-crossed fashion. They form a pair of cones with a common vertex, the point P in the figure. Those two cones intersect the plane of the figure on the red and orange lines. It's easy to see that the two spheres will only collide when their relative velocity $\vec{v}_2 - \vec{v}_1$ points towards sphere 1 and is contained inside another cone, whose intersections with the plane of the figure are the two vectors labeled $\hat{u}_a$ and $\hat{u}_b$, as depicted in the side-picture of the main picture.

Now, both triangles $\widehat{ABP}$ and $\widehat{CDP}$ are rectangular, so

$$\begin{aligned} d_1 &\equiv \overline{AP} = \frac{R_1}{\sin\theta} \\ d_2 &\equiv \overline{CP} = \frac{R_2}{\sin\theta} \end{aligned} \qquad \Rightarrow \qquad \frac{1}{\sin\theta} = \frac{d_1}{R_1} = \frac{d_2}{R_2}\,.$$

We also have

$$\overline{AC} = |\vec{r}_2 - \vec{r}_1| = d_1 + d_2\,.$$

We can then solve for d_1 and d_2

$$\boxed{d_1 = \overline{AP} = R_1\,\frac{|\vec{r}_2 - \vec{r}_1|}{R_1 + R_2} \quad \text{and} \quad d_2 = \overline{CP} = R_2\,\frac{|\vec{r}_2 - \vec{r}_1|}{R_1 + R_2}}$$

from which it follows

$$\boxed{\sin\theta = \frac{R_1}{d_1} = \frac{R_2}{d_2} = \frac{R_2 + R_1}{|\vec{r}_2 - \vec{r}_1|} = \frac{R_1 + R_2}{d_1 + d_2} \quad \text{and} \quad \cos\theta = \sqrt{d_1^2 - R_1^2} = \sqrt{d_2^2 - R_2^2}\,.}$$

Now, to obtain the unit vectors $\hat{u}_a$ and $\hat{u}_b$, we can just rotate the unit vector along $\vec{r}_2 - \vec{r}_1$ by an angle $\pm\theta$ around any axis perpendicular to $\vec{r}_2 - \vec{r}_1$, and then flip the sign:

$$\boxed{\{\hat{u}_a, \hat{u}_b\} = -\,R_{\pm\theta}\,\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}\,.}$$

The next task is to determine if the relative velocity $\vec{V} \equiv \vec{v}_2 - \vec{v}_1$ lies inside the cone limited by these two unit vectors. The figure below shows the signs of the cross-products between the two unit vectors above and $\vec{V}$, in that order:

Thus, there will be a collision if and only if

$$\hat{u}_a \times \vec{V} \geq 0 \quad \text{and} \quad \hat{u}_b \times \vec{V} \leq 0\,.$$

The equalities happen when the collision is just grazing. Note that that's not the same as testing for $(\hat{u}_a \times \vec{V})\,(\hat{u}_b \times \vec{V}) \leq 0$.

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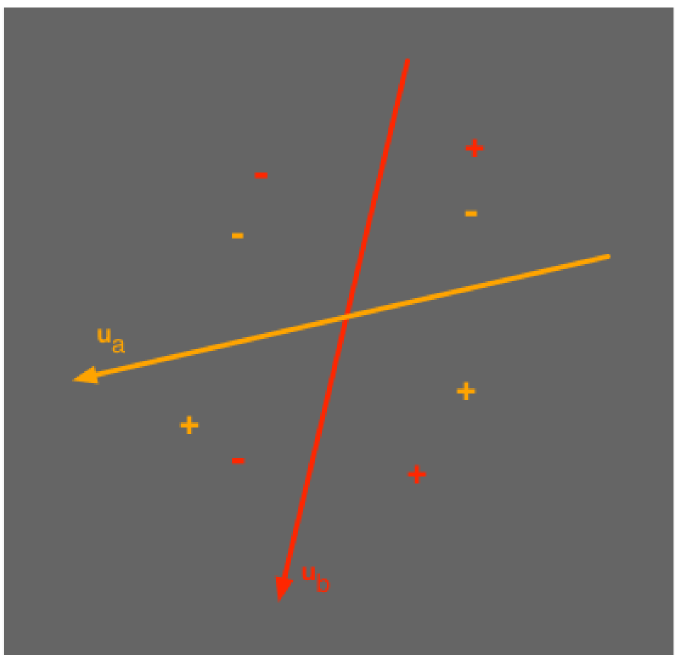
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