Hypothesis Tests and Confidence Intervals in Multiple Regression (SW Ch. 7)
Part 1

Dragos Ailoae
dailoae@gradcenter.cuny.edu

Advanced Economics and Business Statistics
ECON-4400w - Fall 2022

Brooklyn College
Nov 14, 2022

Understanding regression output tables

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.996
R Square	0.992
Adjusted R Square	0.992
Standard Error	0.990
Observations	100

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	11763.151	5881.576	5994.941	0.000
Residual	97	95.166	0.981		
Total	99	11858.317			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	9.990	0.099	100.829	0.000	9.793	10.186
X	1.986	0.061	32.457	0.000	1.865	2.107
Z	3.019	0.047	64.493	0.000	2.926	3.112

Outline

1. **Hypothesis tests and confidence intervals for one coefficient**
2. **Joint hypothesis tests on multiple coefficients**
3. **Other types of hypotheses involving multiple coefficients**
4. Model specification: how to decide which variables to include in a regression model

Hypothesis Tests and Confidence Intervals for a Single Coefficient (SW Section 7.1)

- Hypothesis tests and confidence intervals for a single coefficient in multiple regression follow the same logic and recipe as for the slope coefficient in a single-regressor model.

- $\dfrac{\hat{\beta}_1 - E(\hat{\beta}_1)}{\sqrt{\text{var}(\hat{\beta}_1)}}$ is approximately distributed N(0,1) (CLT).

- Thus hypotheses on β_1 can be tested using the usual t-statistic, and confidence intervals are constructed as $\{\hat{\beta}_1 \pm 1.96 \times \text{SE}\hat{\beta}_1)\}$.

- So too for $\beta_2,\ldots, \beta_k$.

Example: The California class size data

1. $\widehat{TestScore} = 698.9 - 2.28 \times STR$
   $\qquad (10.4) \quad (0.52)$

2. $\widehat{TestScore} = 686.0 - 1.10 \times STR - 0.650PctEL$
   $\qquad (8.7) \quad (0.43) \qquad\qquad (0.031)$

- The coefficient on *STR* in (2) is the effect on *TestScores* of a unit change in *STR*, holding constant the percentage of English Learners in the district
- The coefficient on *STR* falls by one-half
- The 95% confidence interval for coefficient on *STR* in (2) is $\{-1.10 \pm 1.96 \times 0.43\} = (-1.95, -0.26)$
- The t-statistic testing $\beta_{STR} = 0$ is $t = -1.10/0.43 = -2.54$, so we reject the hypothesis at the 5% significance level

Standard errors in multiple regression in STATA

```
reg testscr str pctel, robust;
Regression with robust standard errors                 Number of obs =      420
                                                       F(  2,   417) =   223.82
                                                       Prob > F      =   0.0000
                                                       R-squared     =   0.4264
                                                       Root MSE      =   14.464

------------------------------------------------------------------------------
             |               Robust
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |  -1.101296   .4328472    -2.54   0.011     -1.95213   -.2504616
       pctel |  -.6497768   .0310318   -20.94   0.000     -.710775   -.5887786
       _cons |   686.0322   8.728224    78.60   0.000     668.8754     703.189
------------------------------------------------------------------------------
```

$$\widehat{TestScore} = \underset{(8.7)}{686.0} - \underset{(0.43)}{1.10} \times STR - \underset{(0.031)}{0.650} PctEL$$

We use heteroskedasticity-robust standard errors – for exactly the same reason as in the case of a single regressor.

Tests of Joint Hypotheses (SW Section 7.2)

Let *Expn* = expenditures per pupil and consider the population regression model:

$$TestScore_i = \beta_0 + \beta_1 STR_i + \beta_2 Expn_i + \beta_3 PctEL_i + u_i$$

The null hypothesis that "school resources don't matter," and the alternative that they do, corresponds to:

$$H_0\text{: } \beta_1 = 0 \textbf{\textit{ and }} \beta_2 = 0$$

$$\text{vs. } H_1\text{: } \textbf{\textit{either }} \beta_1 \neq 0 \textbf{\textit{ or }} \beta_2 \neq 0 \textbf{\textit{ or both}}$$

$$TestScore_i = \beta_0 + \beta_1 STR_i + \beta_2 Expn_i + \beta_3 PctEL_i + u_i$$

Tests of Joint Hypotheses (SW Section 7.2)

- H_0: $\beta_1 = 0$ ***and*** $\beta_2 = 0$
- vs. H_1: ***either*** $\beta_1 \neq 0$ ***or*** $\beta_2 \neq 0$ ***or both***
- A ***joint hypothesis*** specifies a value for two or more coefficients, that is, it imposes a restriction on two or more coefficients.
- In general, a joint hypothesis will involve q restrictions. In the example above, $q = 2$, and the two restrictions are $\beta_1 = 0$ and $\beta_2 = 0$.
- A "common sense" idea is to reject if either of the individual t-statistics exceeds 1.96 in absolute value.
- But this "one at a time" test isn't valid: the resulting test rejects too often under the null hypothesis (more than 5%)!

Why can't we just test the coefficients one at a time?

Because the rejection rate under the null isn't 5%. We'll calculate the probability of incorrectly rejecting the null using the "common sense" test based on the two individual *t*-statistics. To simplify the calculation, suppose that and are independently distributed (this isn't true in general – just in this example). Let t_1 and t_2 be the *t*-statistics:

$$t_1 = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} \quad \text{and } t_2 = \frac{\hat{\beta}_2 - 0}{SE(\hat{\beta}_2)}$$

The "one at time" test is:

reject H_0: $\beta_1 = \beta_2 = 0$ if $|t_1| > 1.96$ and/or $|t_2| > 1.96$

What is the probability that this "one at a time" test rejects H_0, when H_0 is actually true? (It *should* be 5%.)

Suppose t_1 and t_2 are independent (for this example).

The probability of incorrectly rejecting the null hypothesis using the "one at a time" test

$$= \Pr_{H_0}[|t_1| > 1.96 \text{ and/or } |t_2| > 1.96]$$

$$= 1 - \Pr_{H_0}[|t_1| \leq 1.96 \text{ and } |t_2| \leq 1.96]$$

$$= 1 - \Pr_{H_0}[|t_1| \leq 1.96] \times \Pr_{H_0}[|t_2| \leq 1.96]$$

(because t_1 and t_2 are independent by assumption)

$= 1 – (.95)^2$

$= .0975 = 9.75\%$ – which is ***not*** the desired 5%!!

The *size* of a test is the actual rejection rate under the null hypothesis.

- The size of the "common sense" test isn't 5%!
- In fact, its size depends on the correlation between t_1 and t_2 (and thus on the correlation between $\hat{\beta}_1$ and $\hat{\beta}_2$).

Two Solutions:

- Use a different critical value in this procedure – not 1.96 (this is the "Bonferroni method – see SW App. 7.1) (this method is rarely used in practice however)
- Use a different test statistic designed to test *both* β_1 and β_2 at once: the *F*-statistic (this is common practice)

More on *F*-statistics.

There is a simple formula for the F-statistic that holds only under homoskedasticity (so it isn't very useful) but which nevertheless might help you understand what the F-statistic is doing.

The homoskedasticity-only *F*-statistic

When the errors are homoskedastic, there is a simple formula for computing the "homoskedasticity-only" *F*-statistic:

- Run two regressions, one under the null hypothesis (the "restricted" regression) and one under the alternative hypothesis (the "unrestricted" regression).
- Compare the fits of the regressions – the R^2s – if the "unrestricted" model fits sufficiently better, reject the null

The "restricted" and "unrestricted" regressions

Example: are the coefficients on *STR* and *Expn* zero?

Unrestricted population regression (under H_1):

$$TestScore_i = \beta_0 + \beta_1 STR_i + \beta_2 Expn_i + \beta_3 PctEL_i + u_i$$

Restricted population regression (that is, under H_0):

$$TestScore_i = \beta_0 + \beta_3 PctEL_i + u_i \qquad (why?)$$

- The number of restrictions under H_0 is $q = 2$ (*why?*).
- The fit will be better (R^2 will be higher) in the unrestricted regression (*why?*)

By how much must the R^2 increase for the coefficients on *Expn* and *PctEL* to be judged statistically significant?

The homoskedasticity-only F-statistic:

$$F = \frac{\left(SSR_{\text{restricted}} - SSR_{\text{unrestricted}}\right)/q}{SSR_{\text{unrestricted}}/\left(n - k_{\text{unrestricted}} - 1\right)}$$

where:

$SSR_{\text{restricted}}$ = sum of squared residuals from the restricted regression

$SSR_{\text{unrestricted}}$ = sum of squared residuals from the unrestricted regression

q = number of restrictions under the null hypothesis

$k_{\text{unrestricted}}$ = number of regressors in the unrestricted regression

Alternate formula for the homoskedasticity-only F-statistic:

$$F = \frac{(R^2_{unrestricted} - R^2_{restricted})/q}{(1 - R^2_{unrestricted})/(n - k_{unrestricted} - 1)}$$

where:

$R^2_{restricted}$ = the R^2 for the restricted regression

$R^2_{unrestricted}$ = the R^2 for the unrestricted regression

q = the number of restrictions under the null

$k_{unrestricted}$ = the number of regressors in the unrestricted regression.

- The bigger the difference between the restricted and unrestricted R^2s – the greater the improvement in fit by adding the variables in question – the larger is the homoskedasticity-only F.

Example:

Restricted regression:

$$\widehat{TestScore} = 644.7 - 0.671PctEL, \quad R^2_{restricted} = 0.4149$$
$$\qquad (1.0) \quad (0.032)$$

Unrestricted regression:

$$\widehat{TestScore} = 649.6 - 0.29STR + 3.87Expn - 0.656PctEL$$
$$\qquad (15.5) \quad (0.48) \qquad (1.59) \qquad (0.032)$$

$$R^2_{unrestricted} = 0.4366, \;\; k_{unrestricted} = 3, \quad q = 2$$

So $$F = \frac{(R^2_{unrestricted} - R^2_{restricted})/q}{(1 - R^2_{unrestricted})/(n - k_{unrestricted} - 1)}$$

$$= \frac{(.4366 - .4149)/2}{(1 - .4366)/(420 - 3 - 1)} = 8.01$$

Note: Heteroskedasticity-robust F = **5.43**…

The homoskedasticity-only F-statistic – summary

$$F = \frac{(R^2_{unrestricted} - R^2_{restricted})/q}{(1 - R^2_{unrestricted})/(n - k_{unrestricted} - 1)}$$

- The homoskedasticity-only F-statistic rejects when adding the two variables increased the R^2 by "enough" – that is, when adding the two variables improves the fit of the regression by "enough"
- If the errors are homoskedastic, then the homoskedasticity-only F-statistic has a large-sample distribution that is χ^2_q/q.
- But if the errors are heteroskedastic, the large-sample distribution of the homoskedasticity-only F-statistic is <u>not</u> χ^2_q/q

The F distribution

Your regression printouts might refer to the "F" distribution.

If the four multiple regression LS assumptions hold ***and if***:

5. u_i is homoskedastic, that is, var($u|X_1,\ldots,X_k$) does not depend on X's

6. $u_1,\ldots,u_n$ are normally distributed then the homoskedasticity-only F-statistic has the "$F_{q,n\text{-}k-1}$" distribution, where q = the number of restrictions and k = the number of regressors under the alternative (the unrestricted model).

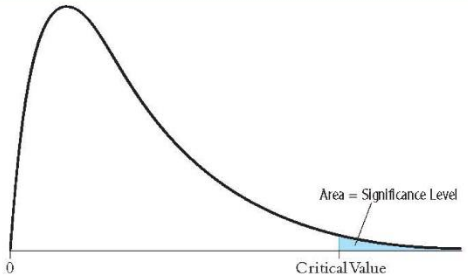
- **The F distribution is to the χ^2_q / q distribution what the t_{n-1} distribution is to the $N(0,1)$ distribution**

The $F_{q,n-k-1}$ *distribution* (1 of 2)

- The F distribution is tabulated many places
- As $n \to \infty$, the $F_{q,n\text{-}k-1}$ distribution asymptotes to the χ_q^2/q distribution:
- **The $F_{q,\infty}$ and χ_q^2/q distributions are the same.**
- For q not too big and $n \geq 100$, the $F_{q,n-k-1}$ distribution and the χ_q^2/q distribution are essentially identical.
- Many regression packages (including STATA) compute p-values of F-statistics using the F distribution
- You will encounter the F distribution in published empirical work.

The $F_{q,n-k-1}$ *distribution* (2 of 2)

TABLE 4 Critical Values for the $F_{m,\infty}$ Distribution



	Significance Level		
Degrees of Freedom	**10%**	**5%**	**1%**
1	2.71	3.84	6.63
2	2.30	3.00	4.61
3	2.08	2.60	3.78
4	1.94	2.37	3.32
5	1.85	2.21	3.02
6	1.77	2.10	2.80
7	1.72	2.01	2.64
8	1.67	1.94	2.51
9	1.63	1.88	2.41
10	1.60	1.83	2.32
11	1.57	1.79	2.25
12	1.55	1.75	2.18
13	1.52	1.72	2.13
14	1.50	1.69	2.08
15	1.49	1.67	2.04
16	1.47	1.64	2.00
17	1.46	1.62	1.97
18	1.44	1.60	1.93
19	1.43	1.59	1.90
20	1.42	1.57	1.88
21	1.41	1.56	1.85
22	1.40	1.54	1.83
23	1.39	1.53	1.81
24	1.38	1.52	1.79
25	1.38	1.51	1.77
26	1.37	1.50	1.76
27	1.36	1.49	1.74
28	1.35	1.48	1.72
29	1.35	1.47	1.71
30	1.34	1.46	1.70

This table contains the 90th, 95th, and 99th percentiles of the $F_{m,\infty}$ distribution. These serve as critical values for tests with significance levels of 10%, 5%, and 1%.

Testing Single Restrictions on Multiple Coefficients (SW Section 7.3) (1 of 2)

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i,\ \ i = 1,\dots,n$$

Consider the null and alternative hypothesis,

$$H_0: \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \beta_1 \neq \beta_2$$

This null imposes a *single* restriction ($q = 1$) on *multiple* coefficients – it is not a joint hypothesis with multiple restrictions (compare with $\beta_1 = 0$ and $\beta_2 = 0$).

Rearrange ("transform") the regression (1 of 2)

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

$$H_0\text{: } \beta_1 = \beta_2 \quad \text{vs.} \quad H_1\text{: } \beta_1 \neq \beta_2$$

Add and subtract $\beta_2 X_{1i}$:

$$Y_i = \beta_0 + (\beta_1 - \beta_2)\, X_{1i} + \beta_2 (X_{1i} + X_{2i}) + u_i$$

or

$$Y_i = \beta_0 + \gamma_1 X_{1i} + \beta_2 W_i + u_i$$

Where

$$\gamma_1 = \beta_1 - \beta_2$$

$$W_i = X_{1i} + X_{2i}$$

Rearrange (“transform”) the regression (2 of 2)

(a) Original equation:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

$$H_0: \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \beta_1 \neq \beta_2$$

(b) Rearranged (“transformed”) equation:

$$Y_i = \beta_0 + \gamma_1 X_{1i} + \beta_2 W_i + u_i$$

where $\gamma_1 = \beta_1 - \beta_2$ and $W_i = X_{1i} + X_{2i}$

So $\quad H_0: \gamma_1 = 0 \;$ vs. $\; H_1: \gamma_1 \neq 0$

- These two regressions ((a) and (b)) have the same R^2, the same predicted values, and the same residuals.
- The testing problem is now a simple one: test whether $\gamma_1 = 0$ in regression (b).