Univariate Regression

(Part 2b – Estimation and Measures of Fit)

Dragos Ailoae
dailoae@gradcenter.cuny.edu

Advanced Economics and Business Statistics
ECON-4400w

Brooklyn College

The OLS Estimator, Predicted Values, and Residuals

The OLS estimators of the slope β_1 and the intercept β_0 are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{s_{XY}}{s_X^2} \tag{4.7}$$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}. \tag{4.8}$$

The OLS predicted values $\hat{Y}_i$ and residuals $\hat{u}_i$ are

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i, \quad i = 1, \ldots, n \tag{4.9}$$

$$\hat{u}_i = Y_i - \hat{Y}_i, \quad i = 1, \ldots, n. \tag{4.10}$$

The estimated intercept ($\hat{\beta}_0$), slope ($\hat{\beta}_1$), and residual ($\hat{u}_i$) are computed from a sample of n observations of X_i and Y_i, $i = 1, \ldots, n$. These are estimates of the unknown true population intercept (β_0), slope (β_1), and error term (u_i).

Example

For the regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$ find the OLS estimators $\hat{\beta}_0$, $\hat{\beta}_1$ given the following data sample:

Y_i	X_i
4	1
5	4
7	5
12	6

i	**Y**	**X**	**y**	**x**	**xy**	**x^2**	**Y_hat**	**u_hat**
1	4	1	-3	-3	9	9	2.929	1.071
2	5	4	-2	0	0	0	7.000	-2.000
3	7	5	0	1	0	1	8.357	-1.357
4	12	6	5	2	10	4	9.714	2.286
Sum	**28**	**16**	**0**	**0**	**19**	**14**	**28**	**0**
Avg (Ybar Xbar)	**7**	**4**						
	My Calcs	**Excel Output**	**Diff.**					
n	**4**	4	0					
b1 = 19/14	**1.357**	1.357	0					
b0=Ybar - b1*Xbar	**1.571**	1.571	0					

Intuition: Regression Slope Coefficient vs Correlation Coefficient

Recall that the correlation coefficient is:

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

where $\sigma_X = \sqrt{Var(X)}$ and $\sigma_Y = \sqrt{Var(Y)}$

We've established that the slope coefficient for univariate regression is:

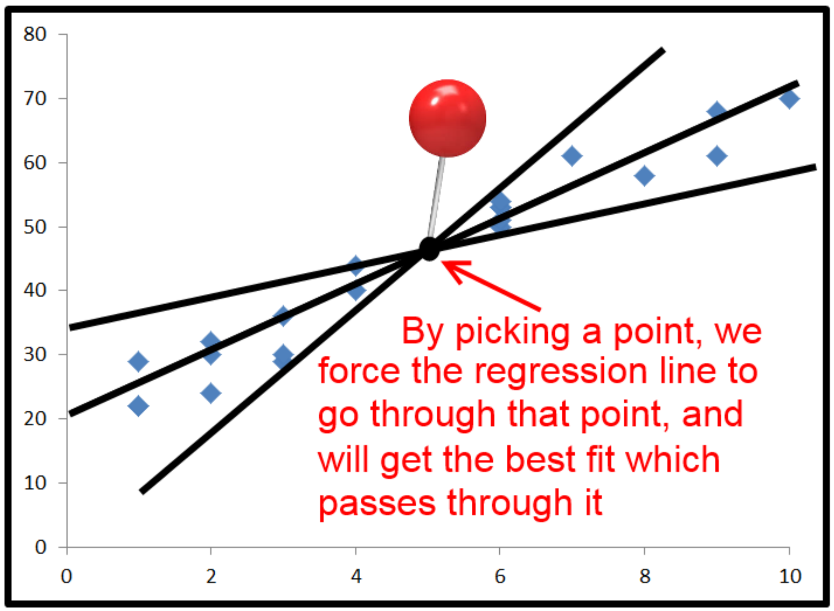
$$\hat{\beta}_1 = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} \equiv \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_X}$$

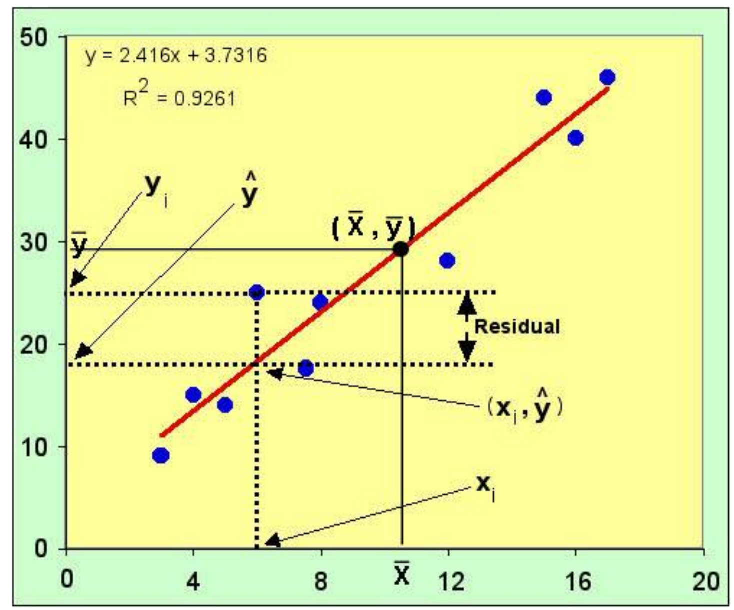
Which implies the following relationship between the regression slope coefficient and the correlation coefficient:

$$\hat{\beta}_1 = \rho_{X,Y} \frac{\sigma_Y}{\sigma_X}$$

Intuition: Regression line must pass through point $(\bar{X}, \bar{Y})$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X}.$$





Measures of Fit (SW Section 4.3)

Two regression statistics provide complementary measures of how well the regression line “fits” or explains the data:

- The ***regression R^2*** (aka "coefficient of determination") measures the fraction of the variance of *Y* that is explained by *X*; it is unitless and ranges between zero (no fit) and one (perfect fit)
- The ***standard error of the regression* (*SER*)** measures the magnitude of a typical regression residual in the units of *Y*.

The *regression* R^2 is the fraction of the sample variance of Y_i "explained" by the regression.

$Y_i = \hat{Y}_i + \hat{u}_i =$ OLS prediction + OLS residual
$\rightarrow$ sample var (Y) = sample var$(\hat{Y}_i)$ + sample var$(\hat{u}_i)$
$\rightarrow$ total sum of squares = "explained" SS + "residual" SS

$$\text{TSS} \qquad = \qquad \text{ESS} \qquad + \qquad \text{SSR}$$

$$Definition\ of\ R^2: \qquad R^2 = \frac{ESS}{TSS} = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

$$= 1 - \frac{SSR}{TSS} = 1 - \frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

- $R^2 = 0$ means $ESS = 0$
- $R^2 = 1$ means $ESS = TSS$
- $0 \le R^2 \le 1$, higher R^2 means better fit
- For regression with a single X, R^2 = the square of the correlation coefficient between X and Y

The Standard Error of the Regression (SER)

The *SER* measures the spread of the distribution of *u*. The *SER* is (almost) the sample standard deviation of the OLS residuals:

$$SER = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(\hat{u}_i - \bar{\hat{u}})^2}$$
$$= \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2}$$

The second equality holds because $\bar{\hat{u}} = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i = 0$.

$$SER = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2}$$

The *SER*:

has the units of *u*, which are the units of *Y*

measures the average "size" of the OLS residual (the average "mistake" made by the OLS regression line)

The ***root mean squared error*** (*RMSE*) is closely related to the *SER*:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2}$$

This measures the same thing as the *SER* – the minor difference is division by 1/*n* instead of 1/(*n* – 2).

Example (continued)

For the regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$, find the coefficient of determination (R^2) given the following data sample:

Y_i	X_i
4	1
5	4
7	5
12	6

i	Y	X	y	x	xy	x^2	Y_hat	u_hat	y_hat	y_hat^2	y^2	u_hat^2
1	4	1	-3	-3	9	9	2.929	1.071	-4.071	16.577	9	1.148
2	5	4	-2	0	0	0	7.000	-2.000	0.000	0.000	4	4.000
3	7	5	0	1	0	1	8.357	-1.357	1.357	1.842	0	1.842
4	12	6	5	2	10	4	9.714	2.286	2.714	7.367	25	5.224
Sum	**28**	**16**	**0**	**0**	**19**	**14**	**28**	**0**	**0**	**25.7857**	**38**	**12.214**
Avg (Ybar Xbar)	**7**	**4**								**ESS**	**TSS**	**RSS**
	My Calcs	**Excel Output**	**Diff.**									
n	**4**	4	0									
b1 = 19/14	**1.357**	1.357	0							**TSS Check (ESS+RSS)**		Diff.
b0=Ybar - b1*Xbar	**1.571**	1.571	0								**38.000**	0.000
R2 = ESS/TSS	**0.679**	0.679	0									
R2 = 1 - RSS/TSS	**0.679**	0.679	0									
SER = sqrt(RSS/(n-2))	**2.471**	2.471	0									
RMSR = sqrt(RSS/n)	**1.747**											

Next week – finish chapter 4

1. Probability framework for linear regression
2. The ordinary least squares (OLS) estimator and the sample regression line
3. Measures of fit of the sample regression
4. **The least squares model assumptions**
5. **The sampling distribution of the OLS estimator**