

# Review of Statistical Theory

## Part 3

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Advanced Economics and Business Statistics  
ECON-4400w

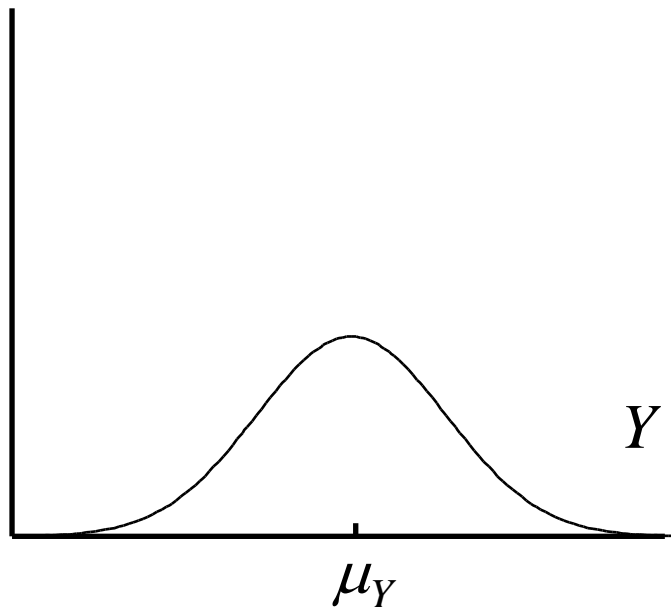
Brooklyn College

1. The probability framework for statistical inference
2. Estimation
3. **Hypothesis Testing**
4. Confidence intervals

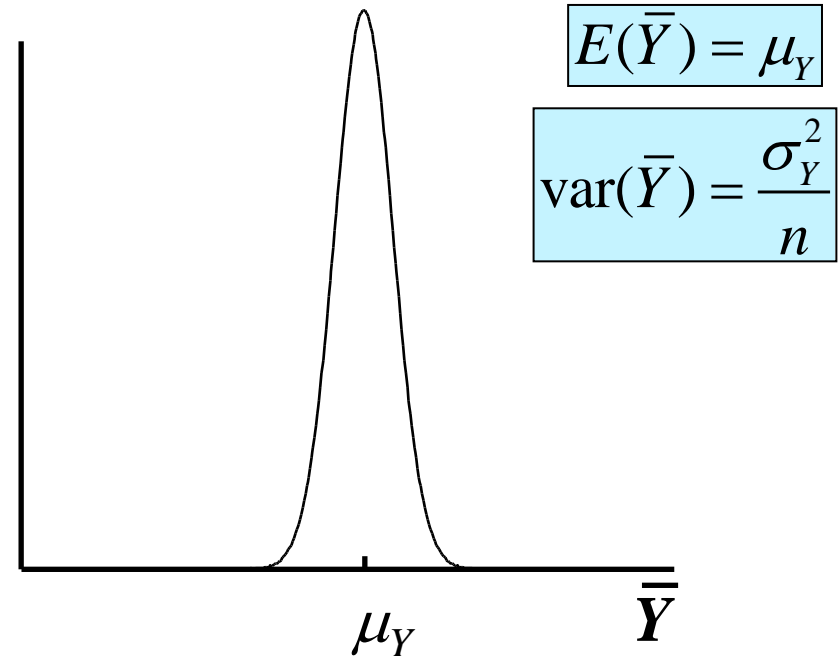
# Distribution of the sample mean

In this illustration,  $Y$  and  $\bar{Y}$  are both centered around  $\mu_Y$ , but the dispersion differs.

probability density  
function of  $Y$

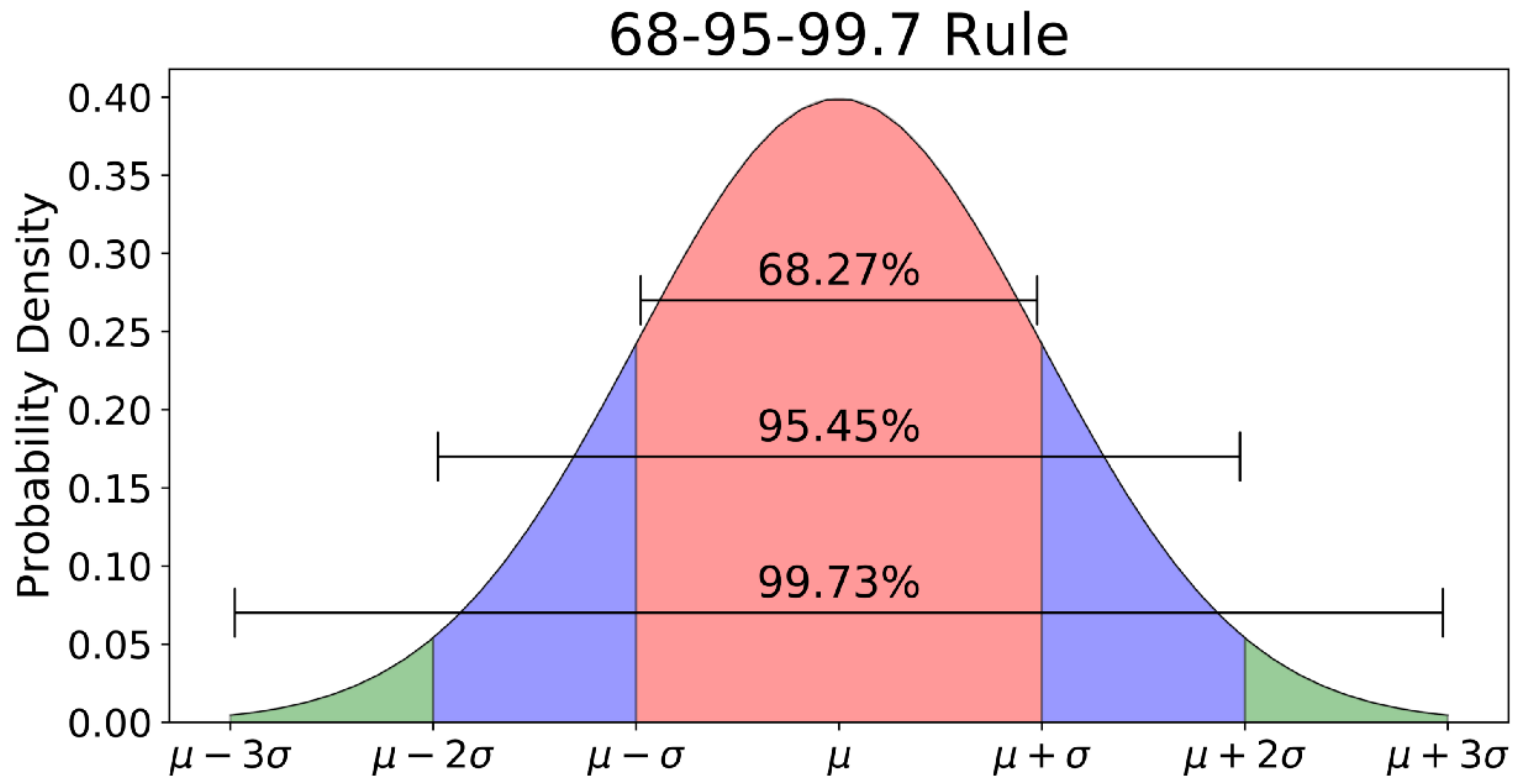


probability density  
function of  $\bar{Y}$

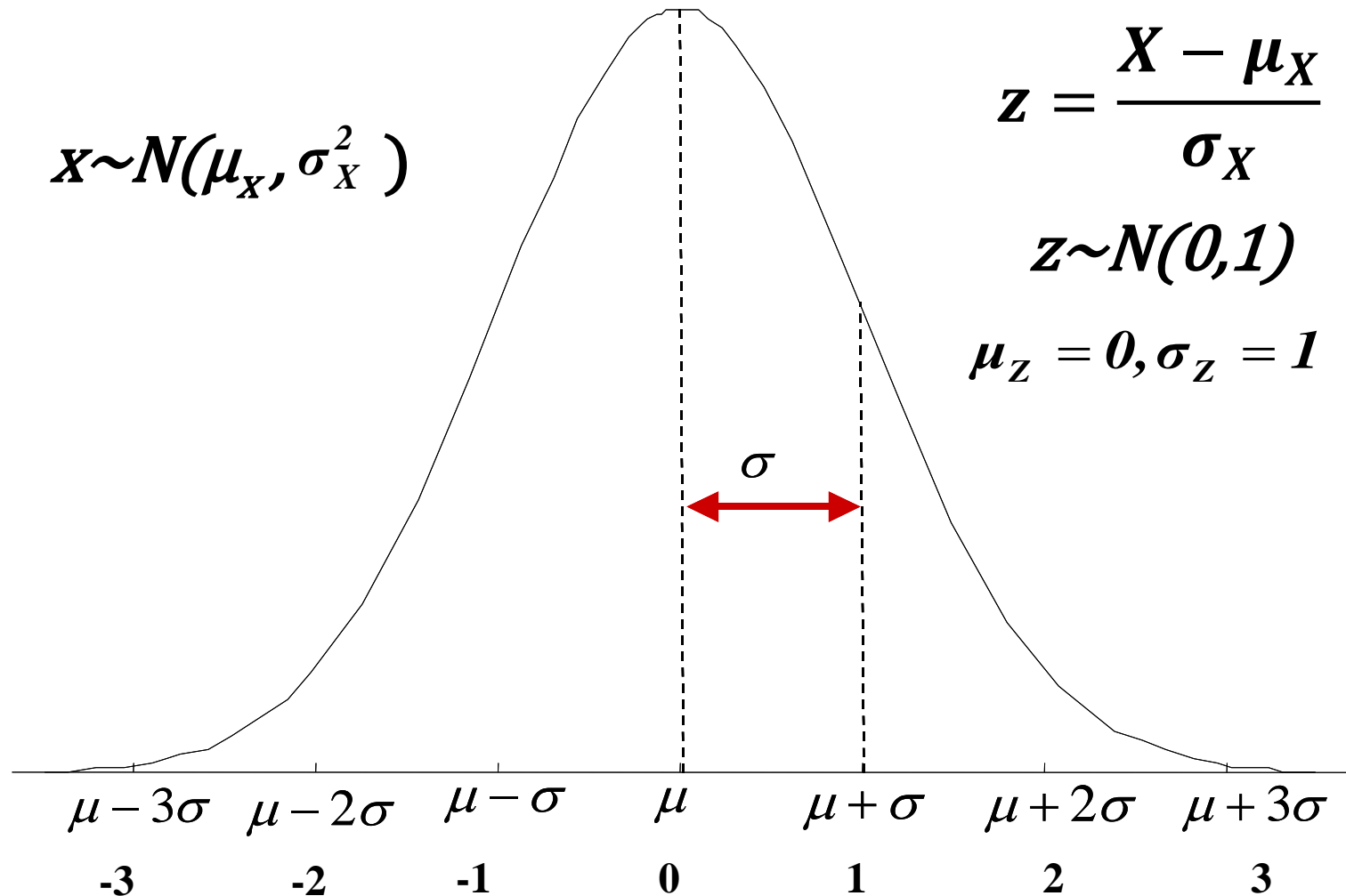


Note that  $\bar{Y}$  is normally distributed even when the underlying random variable  $Y$  is not! Remember in our CLT simulation  $Y$  was uniformly distributed.

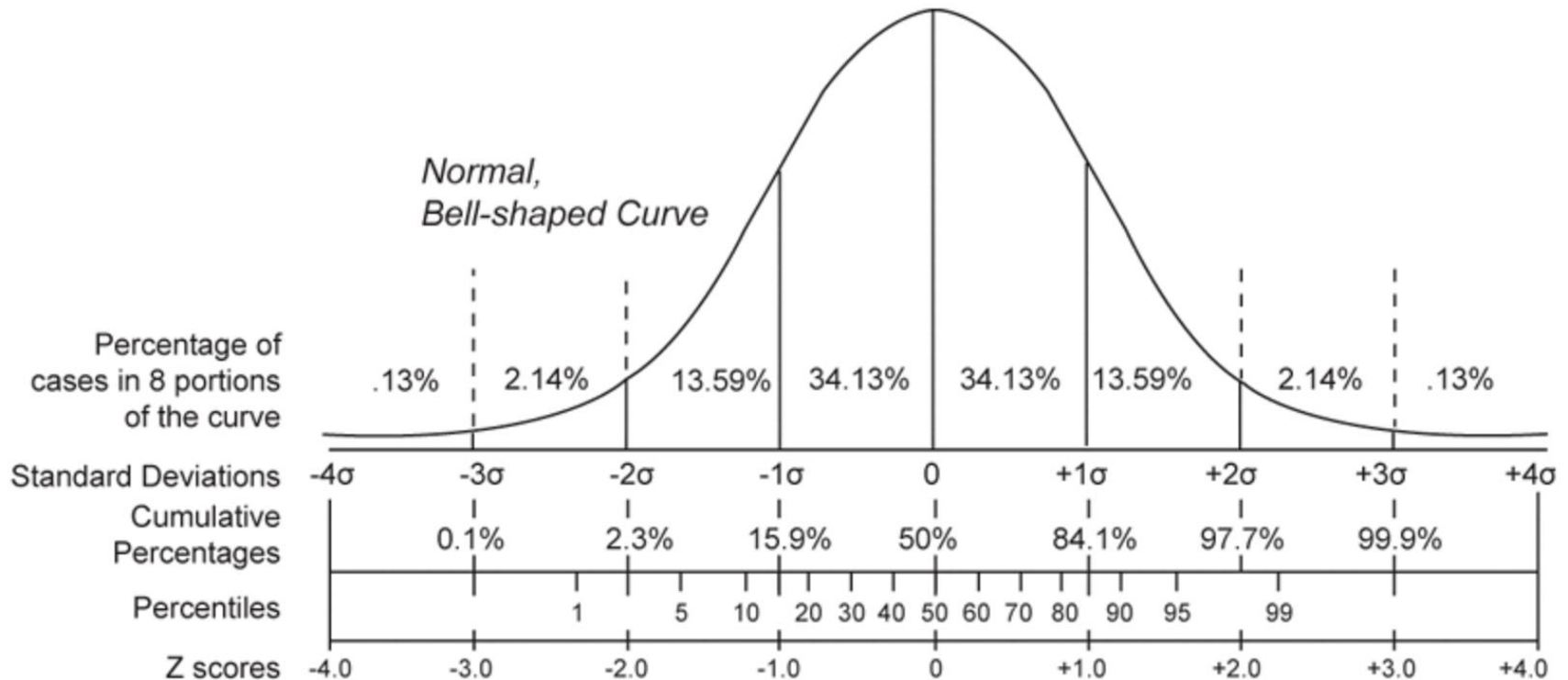
# The normal distribution



# All normal distributions can be translated into standard normal



# The standard normal profile



- For other values see statistical tables like the one in textbook Appendix or here: <https://wwwedu.github.io/BC4400S23/Admin/StatsTables.pdf>
- <https://demonstrations.wolfram.com/AreaOfANormalDistribution/>

# Hypothesis Testing

The *hypothesis testing* problem (for the mean): make a provisional decision based on the evidence at hand whether a null hypothesis is true, or instead that some alternative hypothesis is true. That is, test

- $H_0: E(Y) = \mu_{Y,0}$  vs.  $H_1: E(Y) > \mu_{Y,0}$  (1-sided,  $>$ )
- $H_0: E(Y) = \mu_{Y,0}$  vs.  $H_1: E(Y) < \mu_{Y,0}$  (1-sided,  $<$ )
- $H_0: E(Y) = \mu_{Y,0}$  vs.  $H_1: E(Y) \neq \mu_{Y,0}$  (2-sided)

# *Some terminology for testing statistical hypotheses* (1 of 2)

***p-value*** = probability of drawing a statistic (e.g.  $\bar{Y}$ ) at least as adverse to the null as the value actually computed with your data, assuming that the null hypothesis is true.

The ***significance level*** of a test is a pre-specified probability of incorrectly rejecting the null, when the null is true.

***Calculating the p-value*** based on  $\bar{Y}$ :

$$p - \text{value} = \Pr[|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|]$$

Where  $\bar{Y}^{act}$  is the value of  $\bar{Y}$  actually observed (nonrandom)



# *Some terminology for testing statistical hypotheses* (2 of 2)

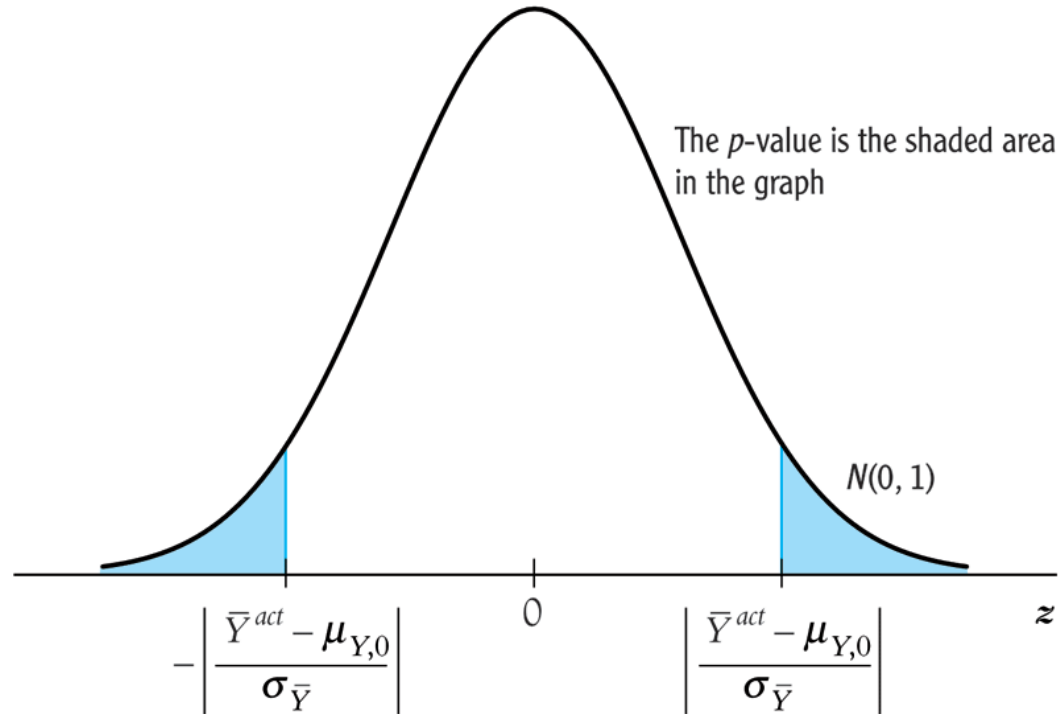
- To compute the  $p$ -value, you need to know the sampling distribution of  $\bar{Y}$ , which is complicated if  $n$  is small.
- If  $n$  is large, you can use the normal approximation (CLT):

$$\begin{aligned} p\text{-value} &= \Pr_{H_0} [|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|], \\ &= \Pr_{H_0} \left[ \left| \frac{\bar{Y} - \mu_{Y,0}}{\sigma_Y / \sqrt{n}} \right| > \left| \frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_Y / \sqrt{n}} \right| \right] \\ &= \Pr_{H_0} \left[ \left| \frac{\bar{Y} - \mu_{Y,0}}{\sigma_{\bar{Y}}} \right| > \left| \frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_{\bar{Y}}} \right| \right] \end{aligned}$$

$\cong$  probability under left + right  $N(0,1)$  tails

where  $\sigma_{\bar{Y}} = \text{std. dev. of the distribution of } \bar{Y} = \sigma_Y / \sqrt{n}$ .

## Calculating the $p$ -value with $\sigma_Y$ known:



- For large  $n$ ,  $p$ -value = the probability that a  $N(0,1)$  random variable falls outside  $|(\bar{Y}^{act} - \mu_{Y,0})/\sigma_{\bar{Y}}|$
- In practice,  $\sigma_{\bar{Y}}$  is unknown – it must be estimated

## *Estimator of the variance of $Y$ :*

$$s_Y^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \text{“sample variance of } Y\text{”}$$

Fact:

If  $(Y_1, \dots, Y_n)$  are i.i.d. and  $E(Y^4) < \infty$ , then

$$s_Y^2 \xrightarrow{p} \sigma_Y^2$$

### The Standard Error of $\bar{Y}$

KEY CONCEPT

3.4

The standard error of  $\bar{Y}$  is an estimator of the standard deviation of  $\bar{Y}$ . The standard error of  $\bar{Y}$  is denoted  $SE(\bar{Y})$  or  $\hat{\sigma}_{\bar{Y}}$ . When  $Y_1, \dots, Y_n$  are i.i.d.,

$$SE(\bar{Y}) = \hat{\sigma}_{\bar{Y}} = s_Y / \sqrt{n}. \quad (3.8)$$

## *Computing the $p$ -value with $\sigma_Y^2$ estimated:*

$$\begin{aligned} p\text{-value} &= \Pr_{H_0} [|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|], \\ &= \Pr_{H_0} \left[ \left| \frac{\bar{Y} - \mu_{Y,0}}{\sigma_Y / \sqrt{n}} \right| > \left| \frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_Y / \sqrt{n}} \right| \right] \\ &\cong \Pr_{H_0} \left[ \left| \frac{\bar{Y} - \mu_{Y,0}}{s_Y / \sqrt{n}} \right| > \left| \frac{\bar{Y}^{act} - \mu_{Y,0}}{s_Y / \sqrt{n}} \right| \right] \quad (\text{large } n) \end{aligned}$$

so

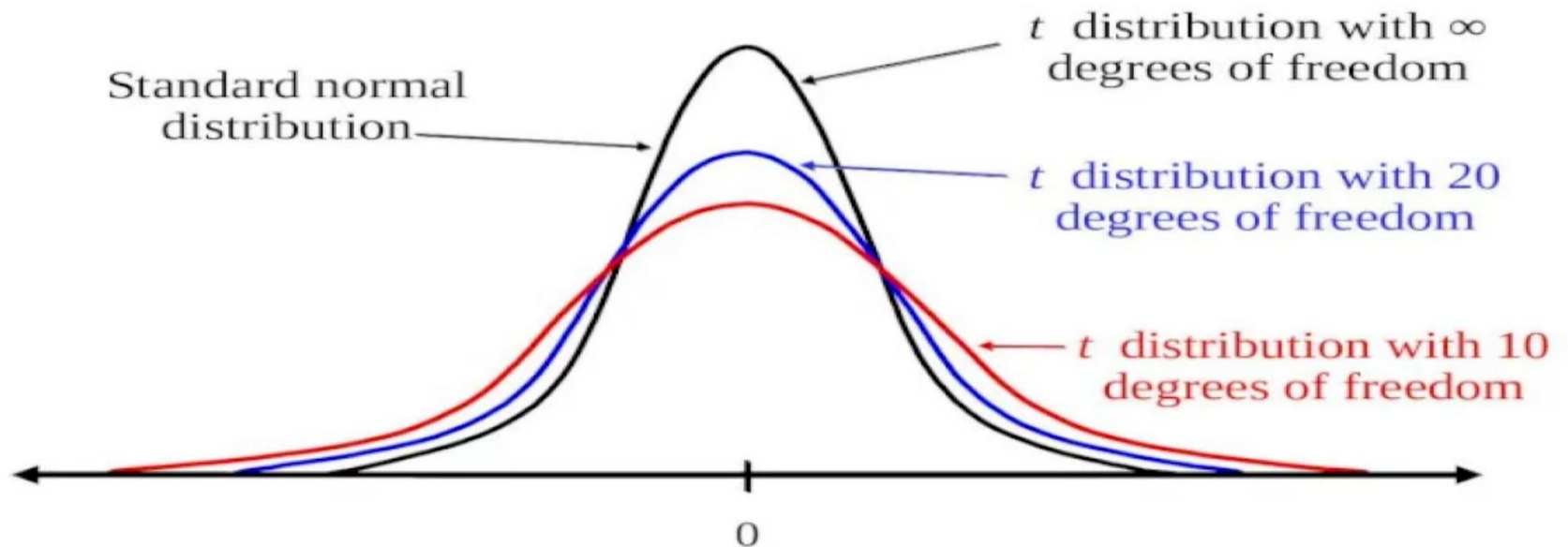
$$p\text{-value} = \Pr_{H_0} [|t| > |t^{act}|] \quad (\sigma_Y^2 \text{ estimated})$$

$\cong$  probability under normal tails outside  $|t^{act}|$

where  $t = \frac{\bar{Y} - \mu_{Y,0}}{s_Y / \sqrt{n}}$  (the usual  $t$ -statistic)

# *The Student-t distribution*

The t-distribution is used when  $n$  is **small** and  $\sigma$  is **unknown**.



# *The Student- $t$ distribution*

- For  $n > 30$ , the  $t$ -distribution and  $N(0,1)$  are very close (as  $n$  grows large, the  $t_{n-1}$  distribution converges to  $N(0,1)$ )
- The  $t$ -distribution is an artifact from days when sample sizes were small and “computers” were people
- For historical reasons, statistical software typically uses the  $t$ -distribution to compute  $p$ -values – but this is irrelevant when the sample size is moderate or large.
- For these reasons, in this class we will focus on the large- $n$  approximation given by the CLT

# Hypothesis test example 1

The average adult male height in a certain country is 170 cm. We suspect that the men in a certain city in that country might have a different average height due to some environmental factors. We pick a random sample of size 9 from the adult males in the city and obtain the following values for their heights (in cm ):

176.2   157.9   160.1   180.9   165.1   167.2   162.9   155.7   166.2

Assume that the height distribution in this population is normally distributed. Here, we need to decide between

$$H_0: \mu = 170$$

$$H_1: \mu \neq 170$$

Based on the observed data, is there enough evidence to reject  $H_0$  at significance level  $\alpha = 0.05$  ?

# Hypothesis test example 1

## **Solution:**

Let's first compute the sample mean and the sample standard deviation. The sample mean is

$$\begin{aligned}\bar{X} &= \frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9}{9} \\ &= 165.8\end{aligned}$$

The sample variance is given by

$$S^2 = \frac{1}{9-1} \sum_{k=1}^9 (X_k - \bar{X})^2 = 68.01$$

The sample standard deviation is given by  $S = \sqrt{S^2} = 8.25$



# Hypothesis test example 1

Now, our test statistic is

$$W(X_1, X_2, \dots, X_9) = \frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}} = \frac{165.8 - 170}{\frac{8.25}{3}} = -1.52$$

Thus,  $|W| = 1.52$ . Also, we have

$$t_{\frac{\alpha}{2}, n-1} = t_{0.025, 8} \approx 2.31$$

Thus, we conclude

$$|W| \leq t_{\frac{\alpha}{2}, n-1}$$

Therefore, we cannot reject  $H_0$ . In other words, we do not have enough evidence to conclude that the average height in the city is different from the average height in the country.

## Hypothesis test example 2

Achievement test scores of all high school seniors in a state have mean 60 and variance 64. A random sample of  $n=100$  students from one large high school had a mean score of 58. Is there evidence to suggest that this high school is inferior?

**Hint:** calculate the probability that the sample mean is at most 58 when  $n=100$ .

## Hypothesis test example 2

Let  $\bar{X}$  denote the mean of a random sample of  $n = 100$  scores from a population with  $\mu = 60$  and  $\sigma^2 = 64$ . We want to approximate  $P(\bar{X} \leq 58)$ . We know from the Central Limit Theorem that  $(\bar{X} - \mu)/(\sigma/\sqrt{n})$  has a distribution that can be approximated by a standard normal distribution. Using the standard normal table we have:

$$P(\bar{X} \leq 58) = P\left(\frac{\bar{X} - 60}{8/\sqrt{100}} \leq \frac{58 - 60}{.8}\right) \approx P(Z \leq -2.5) = .0062$$

Because this probability is so small, it is unlikely that the sample from the school of interest can be regarded as a random sample from a population with  $\mu = 60$  and  $\sigma^2 = 64$ . The evidence suggests that the average score for this high school is lower than the overall average of  $\mu = 60$ .

1. The probability framework for statistical inference
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3. Testing
4. **Confidence intervals**

## Confidence Intervals

- A 95% *confidence interval* for  $\mu_Y$  is an interval that contains the true value of  $\mu_Y$  in 95% of repeated samples.
- *Digression*: What is random here? The values of  $Y_1, \dots, Y_n$  and thus any functions of them – including the confidence interval. The confidence interval will differ from one sample to the next. The population parameter,  $\mu_Y$ , is not random; we just don't know it.

# Confidence Intervals

A 95% confidence interval can always be constructed as the set of values of  $\mu_Y$  not rejected by a hypothesis test with a 5% significance level.

$$\begin{aligned}\{\mu_Y: \left| \frac{\bar{Y} - \mu_Y}{s_Y / \sqrt{n}} \right| \leq 1.96\} &= \{\mu_Y: -1.96 \leq \frac{\bar{Y} - \mu_Y}{s_Y / \sqrt{n}} \leq 1.96\} \\ &= \{\mu_Y: -1.96 \frac{s_Y}{\sqrt{n}} \leq -\mu_Y \leq 1.96 \frac{s_Y}{\sqrt{n}}\} \\ &= \{\mu_Y \in (\bar{Y} - 1.96 \frac{s_Y}{\sqrt{n}}, \bar{Y} + 1.96 \frac{s_Y}{\sqrt{n}})\}\end{aligned}$$

*This confidence interval relies on the large- $n$  results that  $\bar{Y}$  is approximately normally distributed and  $s_Y^2 \xrightarrow{P} \sigma_Y^2$ .*

# Confidence interval example

In a sample of 25,  $\bar{x} = 1.63$  and  $s = 0.51$ . Construct a 95 percent confidence interval for  $\mu$ .

**Solution:**

2.064 is the 95% critical value from a  $t$  distribution with 24 degrees of freedom.

Thus, the confidence interval is  $1.63 \pm [2.064(0.51)/5]$  or  $[1.4195, 1.8405]$ .

# Summary:

From the two assumptions of:

1. simple random sampling of a population, that is,  $\{Y_i, i = 1, \dots, n\}$  are i.i.d.
2.  $0 < E(Y^4) < \infty$

we developed, for large samples (large  $n$ ):

- Theory of estimation (sampling distribution of  $\bar{Y}$ )
- Theory of hypothesis testing (large- $n$  distribution of  $t$ -statistic and computation of the  $p$ -value)
- Theory of confidence intervals