Nonlinear Functions of a Single Independent Variable
(SW 8.2)

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Logarithms refresher

Number	How Many 10s	Base-10 Logarithm	
.. etc..			
1000	$1 \times 10 \times 10 \times 10$	$\log_{10}(1000)$	$= 3$
100	$1 \times 10 \times 10$	$\log_{10}(100)$	$= 2$
10	1×10	$\log_{10}(10)$	$= 1$
1	1	$\log_{10}(1)$	$= 0$
0.1	$1 \div 10$	$\log_{10}(0.1)$	$= -1$
0.01	$1 \div 10 \div 10$	$\log_{10}(0.01)$	$= -2$
0.001	$1 \div 10 \div 10 \div 10$	$\log_{10}(0.001)$	$= -3$
.. etc..			

Converting between log bases

$$\log_5(12) = \frac{\log_{10}(12)}{\log_{10}(5)}$$

$$= \frac{\log(12)}{\log(5)}$$

$$= \frac{1.079181246...}{0.6989700043...}$$

$$\log_5(12) \approx 1.544$$

Natural logs (ln)

What the heck is a log? If e (a constant equal to 2.71828) to the "*b*th power" produces x, then b is the log of x:

$$\text{b is the log of x to the base e if:} \quad e^b = x$$

Thus, a **log** (or logarithm) is the exponent to which a given base must be taken in order to produce a specific number. While logs come in more than one variety, we'll use only **natural logs** (logs to the base e) in this text. The symbol for a natural log is "ln," so $\ln(x) = b$ means that $(2.71828)^b = x$ or, more simply,

$$\ln(x) = b \qquad \text{means that} \qquad e^b = x$$

For example, since $e^2 = (2.71828)^2 = 7.389$, we can state that:

$$\ln(7.389) = 2$$

Thus, the natural log of 7.389 is 2! Two is the power of e that produces 7.389. Let's look at some other natural log calculations:

$$\begin{aligned} \ln(100) &= 4.605 \\ \ln(1000) &= 6.908 \end{aligned}$$

Logarithmic functions of *Y* and/or *X*

- ln(*X*) = the natural logarithm of *X*
- Logarithmic transforms permit modeling relations in "percentage" terms (like elasticities), rather than linearly.

$$\textit{Here's why}:\quad \ln(x+\Delta x)-\ln(x)=\ln\left(1+\frac{\Delta x}{x}\right)\cong\frac{\Delta x}{x}$$

$$(\text{calculus}: \frac{d\ln(x)}{dx}=\frac{1}{x})$$

$$\textit{Numerically}:$$

$$\ln(1.01)=.00995\cong .01;$$

$$\ln(1.10)=.0953\cong .10\ (\text{sort of})$$

Functional form

The best way to choose a functional form for a regression model is to select the specification that best matches the underlying theory of the equation. In a majority of cases, the linear form will be adequate, and for most of the rest, common sense will point out a fairly easy choice from the following alternatives:

Functional Form	Equation (one X only)	The Change in Y when X Changes
Linear	$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$	If X increases by one unit, Y will change by β_1 units.
Double-log	$\ln Y_i = \beta_0 + \beta_1 \ln X_i + \epsilon_i$	If X increases by one percent, Y will change by β_1 percent. (Thus β_1 is the elasticity of Y with respect to X.)
Semilog (lnX)	$Y_i = \beta_0 + \beta_1 \ln X_i + \epsilon_i$	If X increases by one percent, Y will change by $\beta_1/100$ units.
Semilog (lnY)	$\ln Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$	If X increases by one unit, Y will change by roughly $100\beta_1$ percent.
Polynomial	$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \epsilon_i$	If X increases by one unit, Y will change by $(\beta_1 + 2\beta_2 X)$ units.

Linear-log population regression function (1 of 2)

Compute Y "before" and "after" changing X:

$$Y = \beta_0 + \beta_1 \ln(X) \qquad \text{("before")}$$

Now change X: $$Y + \Delta Y = \beta_0 + \beta_1 \ln(X + \Delta X) \qquad \text{("after")}$$

Subtract ("after") – ("before"): $\Delta Y = \beta_1[\ln(X + \Delta X) - \ln(X)]$

now $$\ln(X + \Delta X) - \ln(X) \cong \frac{\Delta X}{X}$$

so $$\Delta Y \cong \beta_1 \frac{\Delta X}{X}$$

or $$\beta_1 \cong \frac{\Delta Y}{\Delta X / X} \text{ (small } \Delta X\text{)}$$

Linear-log population regression function (2 of 2)

$$Y_i = \beta_0 + \beta_1 \ln(X_i) + u_i$$

for small ΔX,

$$\beta_1 \cong \frac{\Delta Y}{\Delta X / X}$$

Now $100 \times \dfrac{\Delta X}{X}$ = percentage change in X, so ***a 1% increase in X (multiplying X by 1.01) is associated with a .01β_1 change in Y.***

(1% increase in $X \rightarrow .01$ increase in $\ln(X) \rightarrow .01\beta_1$ increase in Y)

Example: TestScore vs. ln(Income)

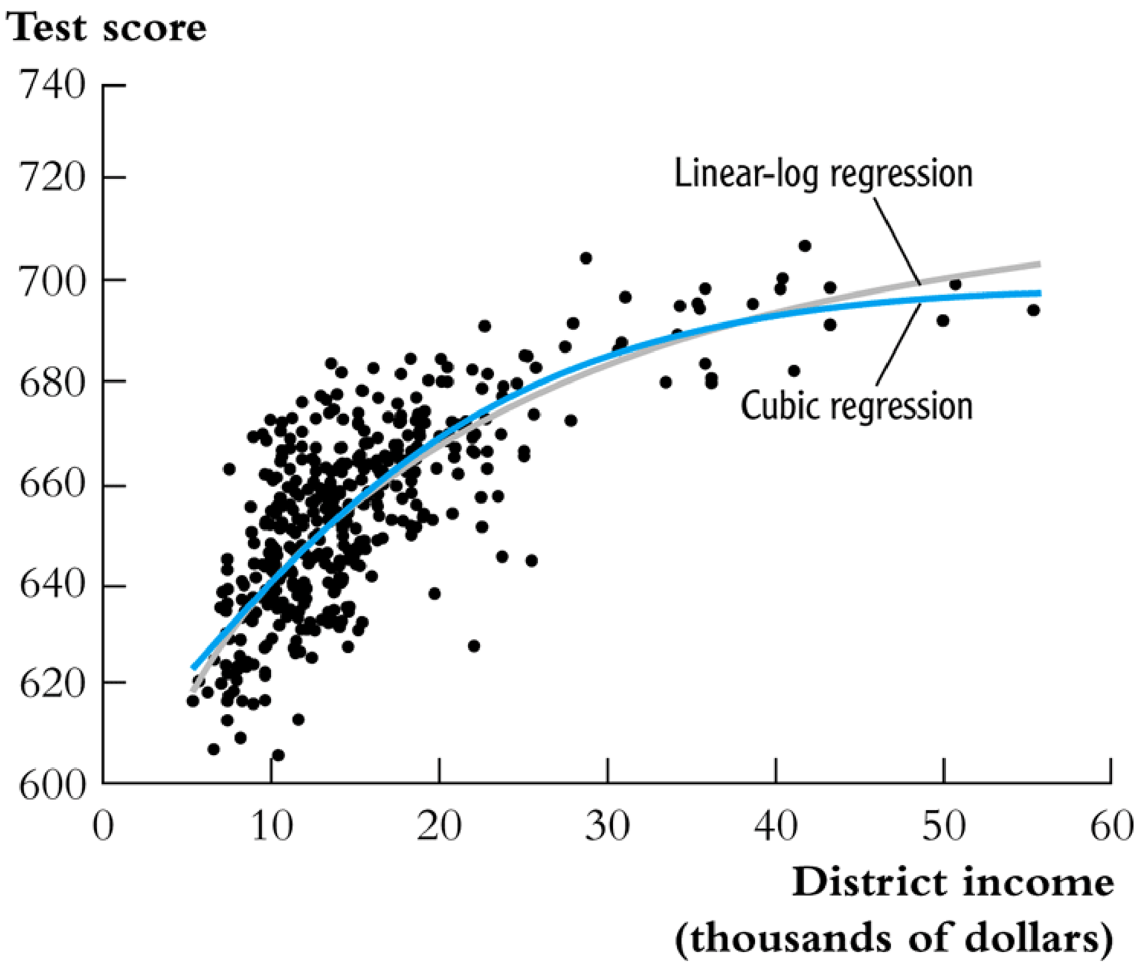
- First defining the new regressor, ln(*Income*)
- The model is now linear in ln(*Income*), so the linear-log model can be estimated by OLS:

$$\widehat{TestScore} = \underset{(3.8)}{557.8} + \underset{(1.40)}{36.42} \times \ln(Income_i)$$

so a 1% increase in *Income* is associated with an increase in *TestScore* of 0.36 points on the test.

- Standard errors, confidence intervals, R^2 – all the usual tools of regression apply here.
- How does this compare to the cubic model?

The linear-log and cubic regression functions



Log-linear population regression function (1 of 2)

$$\ln(Y) = \beta_0 + \beta_1 X \qquad \text{(b)}$$

Now change X: $$\ln(Y + \Delta Y) = \beta_0 + \beta_1(X + \Delta X) \qquad \text{(a)}$$

Subtract (a) – (b): $$\ln(Y + \Delta Y) - \ln(Y) = \beta_1 \Delta X$$

so $$\frac{\Delta Y}{Y} \cong \beta_1 \Delta X$$

or $$\beta_1 \cong \frac{\Delta Y / Y}{\Delta X} \text{ (small } \Delta X\text{)}$$

Log-linear population regression function (2 of 2)

$$\ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

for small ΔX, $\qquad \beta_1 \cong \dfrac{\Delta Y / Y}{\Delta X}$

- Now $100 \times \dfrac{\Delta Y}{Y}$ = percentage change in Y, so ***a change in X by one unit*** **($\Delta X = 1$)** ***is associated with a*** **$100\beta_1$%** ***change in Y.***
- 1 unit increase in $X \rightarrow \beta_1$ increase in $\ln(Y)$

  $\rightarrow 100\beta_1$% increase in Y
- *Note:* What are the units of u_i and the SER?
  - fractional (proportional) deviations
  - for example, SER = .2 means…

Log-log population regression function (1 of 2)

$$\ln(Y_i) = \beta_0 + \beta_1 \ln(X_i) + u_i \qquad \text{(b)}$$

Now change X: $$\ln(Y + \Delta Y) = \beta_0 + \beta_1 \ln(X + \Delta X) \qquad \text{(a)}$$

Subtract: $$\ln(Y + \Delta Y) - \ln(Y) = \beta_1[\ln(X + \beta X) - \ln(X)]$$

so $$\frac{\Delta Y}{Y} \cong \beta_1 \frac{\Delta X}{X}$$

or $$\beta_1 \cong \frac{\Delta Y / Y}{\Delta X / X} \text{ (small } \Delta X\text{)}$$

Log-log population regression function (2 of 2)

$$\ln(Y_i) = \beta_0 + \beta_1 \ln(X_i) + u_i$$

for small ΔX,

$$\beta_1 \cong \frac{\Delta Y / Y}{\Delta X / X}$$

Now $100 \times \frac{\Delta Y}{Y} =$ percentage change in Y, and $100 \times \frac{\Delta X}{X} =$ percentage change in X, so ***a 1% change in X is associated with a*** $\boldsymbol{\beta_1}$***% change in Y***.

In the log-log specification, $\boldsymbol{\beta_1}$ ***has the interpretation of an elasticity.***

Example: ln(TestScore) vs. ln(Income) (1 of 2)

- First defining a new dependent variable, ln(*TestScore*), ***and*** the new regressor, ln(*Income*)
- The model is now a linear regression of ln(*TestScore*) against ln(*Income*), which can be estimated by OLS:

$$\ln(\widehat{TestScore}) = \underset{(0.006)}{6.336} + \underset{(0.0021)}{0.0554} \times \ln(Income_i)$$

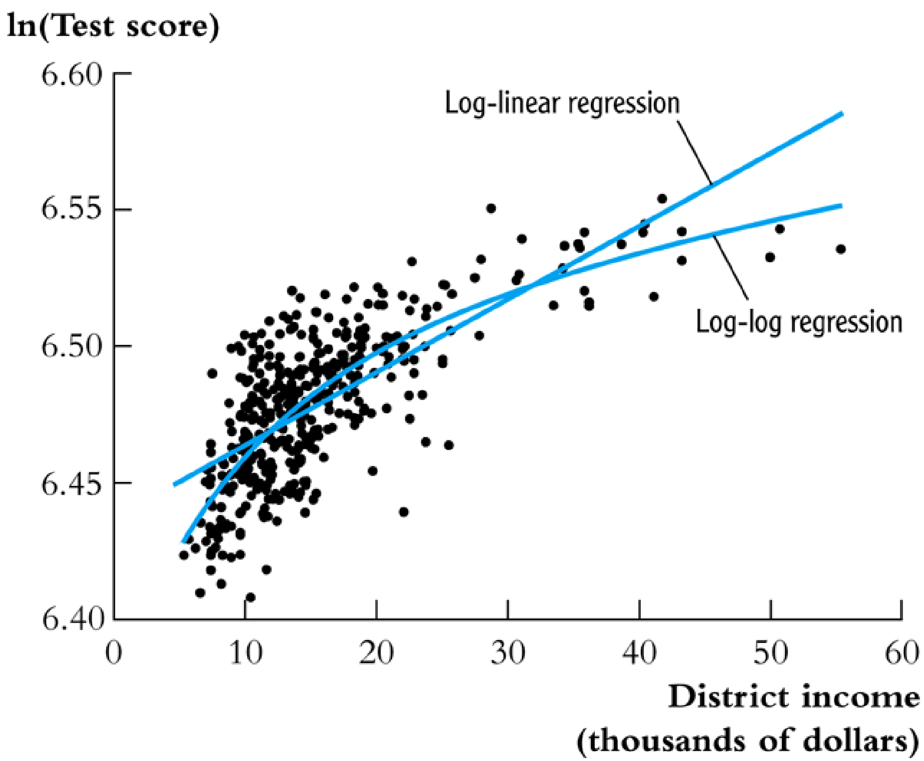
An 1% increase in *Income* is associated with an increase of .0554% in *TestScore* (*Income* up by a factor of 1.01, *TestScore* up by a factor of 1.000554)

Example: ln(TestScore) vs. ln(Income) (2 of 2)

$$\ln(\widehat{TestScore}) = 6.336 + 0.0554 \times \ln(Income_i)$$
$$(0.006) \quad (0.0021)$$

- For example, suppose income increases from \$10,000 to \$11,000, or by 10%. Then *TestScore* increases by approximately .0554 × 10% = .554%. If *TestScore* = 650, this corresponds to an increase of .00554 × 650 = 3.6 points.
- How does this compare to the log-linear model?

The log-linear and log-log specifications:



- Note vertical axis
- The log-linear model doesn't seem to fit as well as the log-log model, based on visual inspection.

Summary: Logarithmic transformations

- Three cases, differing in whether Y and/or X is transformed by taking logarithms.
- The regression is linear in the new variable(s) $\ln(Y)$ and/or $\ln(X)$, and the coefficients can be estimated by OLS.
- Hypothesis tests and confidence intervals are now implemented and interpreted "as usual."
- The interpretation of β_1 differs from case to case.

The choice of specification (functional form) should be guided by judgment (which interpretation makes the most sense in your application?), tests, and plotting predicted values