Review of Statistical Theory

Part 1

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Review of Statistical Theory

1. **The probability framework for statistical inference**
2. Estimation
3. Testing
4. Confidence Intervals

<u>The probability framework for statistical inference</u>

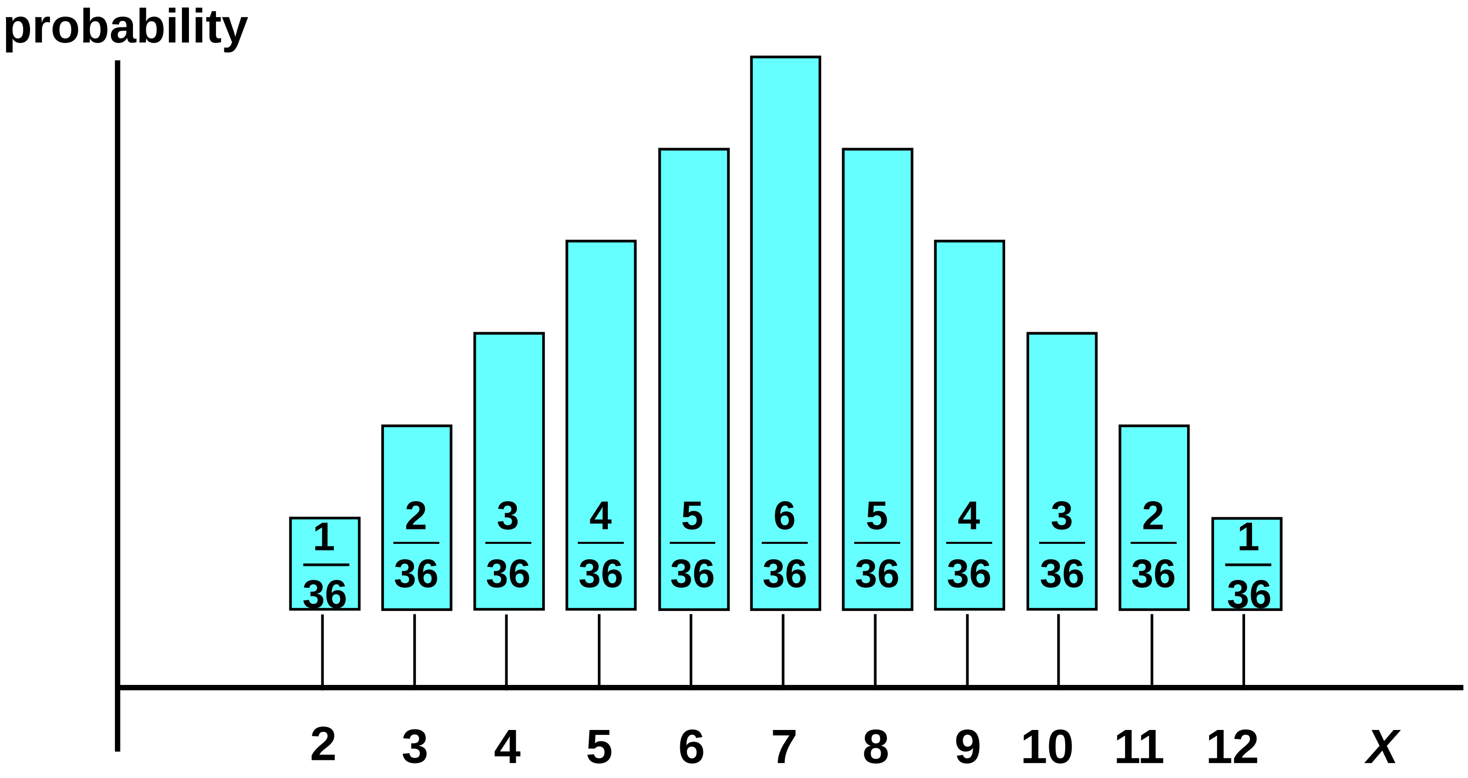
a) **Random variable, distribution**
b) **Moments of a distribution (mean, variance, standard deviation, covariance, correlation)**
c) **Conditional distributions and conditional means**
d) Distribution of a sample of data drawn randomly from a population: $Y_1,\ldots, Y_n$

Experiment: Toss a Pair of "Fair" Dice and Calculate the Sum the Dots on the Two Upward Facing Sides			
Face Values - Outcomes	**Events Sum of Dots**	**Number of Outcomes that produce the Event**	**Probability**
(1,1)	2	1	1/36 = 0.0278
(1,2) (2,1)	3	2	2/36 = 0.0556
(1,3) (2,2) (3,1)	4	3	3/36 = 0.0833
(1,4) (2,3) (3,2) (4,1)	5	4	4/36 = 0.1111
(1,5) (2,4) (3,3) (4,2) (5,1)	6	5	5/36 = 0.1389
(1,6) (2,5) (3,4) (4,3) (5,2) (6,1)	7	6	6/36 = 0.1667
(2,6) (3,5) (4,4) (5,3) (6,2)	8	5	5/36 = 0.1389
(3,6) (4,5) (5,4) (6,3)	9	4	4/36 = 0.1111
(4,6) (5,5) (6,4)	10	3	3/36 = 0.0833
(5,6) (6,5)	11	2	2/36 = 0.0556
(6,6)	12	1	1/36 = 0.0278
Total # of outcomes	**1+2+3+…+2+1 = 36**		

The probability framework: nomenclature

- The mutually exclusive potential results of a random process are called the **outcomes**.
- The **probability** of an outcome is the proportion of the time that the outcome occurs in the long run.
- The set of all possible outcomes is called the **sample space**. An **event** is a subset of the sample space, that is, an event is a set of one or more outcomes.
- A **random variable** is a numerical summary of a random outcome. The number of times your computer crashes while you are writing a term paper is random and takes on a numerical value, so it is a random variable.
- Some random variables are **discrete** and some are **continuous**. As their names suggest, a discrete random variable takes on only a discrete set of values, like 0, 1, 2, c, whereas a continuous random variable takes on a continuum of possible values.

Probability distribution for dice example

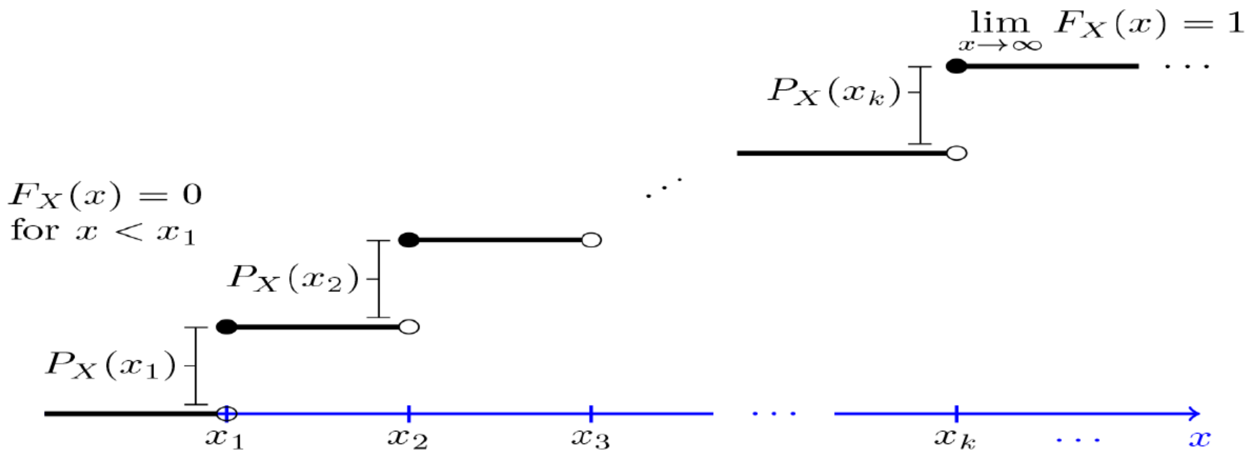


EXAMPLE: *X* IS THE SUM OF TWO DICE

Probability Distribution of a Discrete Random Variable

- The **probability distribution** of a discrete random variable is the list of all possible values of the variable and the probability that each value will occur. These probabilities sum to 1.
- The **cumulative probability distribution** is the probability that the random variable is less than or equal to a particular value. A cumulative probability distribution is also referred to as a cumulative distribution function, a C.D.F., or a cumulative distribution.

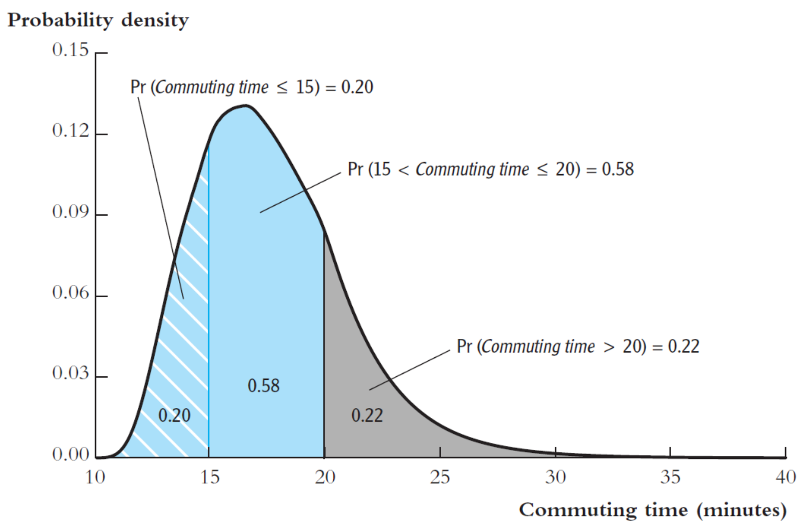
CDF of discrete random variables.



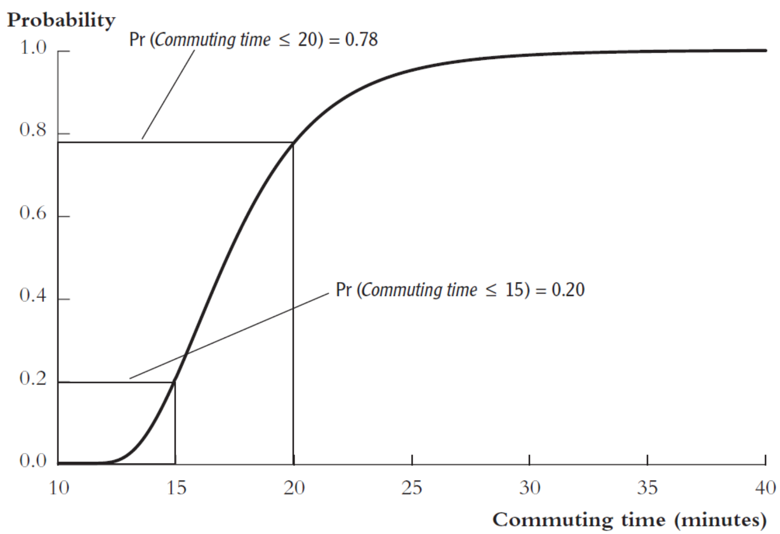
Probability Distribution of a Continuous Random Variable

Probability Density Function (PDF)

Cumulative Distribution Function (CDF)



(b) Probability density function of commuting time



(a) Cumulative distribution function of commuting time

(b) Moments of a population distribution: mean, variance, standard deviation, covariance, correlation

mean = expected value (expectation) of Y *a.k.a.* **first moment**

$= E(Y)$

$= \mu_Y$

= long-run average value of Y over repeated realizations of Y

variance $= E(Y - \mu_Y)^2$

$= \sigma_Y^2$

= measure of the squared spread of the distribution

= second central moment

$$\text{standard deviation} = \sqrt{\text{variance}} = \sigma_Y$$

(b) Moments of a population distribution: mean, variance, standard deviation, covariance, correlation (cont'd)

- The **expected value** of a random variable Y, denoted E(Y), is the long-run average value of the random variable over many repeated trials or occurrences.
- The expected value of a discrete random variable is computed as a weighted average of the possible outcomes of that random variable, where the weights are the probabilities of that outcome.

KEY CONCEPT 2.1

Expected Value and the Mean

Suppose the random variable Y takes on k possible values, $y_1, \ldots, y_k$, where y_1 denotes the first value, y_2 denotes the second value, and so forth, and that the probability that Y takes on y_1 is p_1, the probability that Y takes on y_2 is p_2, and so forth. The expected value of Y, denoted $E(Y)$, is

$$E(Y) = y_1 p_1 + y_2 p_2 + \cdots + y_k p_k = \sum_{i=1}^{k} y_i p_i, \tag{2.3}$$

where the notation $\sum_{i=1}^{k} y_i p_i$ means "the sum of $y_i p_i$ for i running from 1 to k." The expected value of Y is also called the mean of Y or the expectation of Y and is denoted μ_Y.

Expected value / weighted average in Excel

Using SUM()

	A	B	C	D	E
1	Y (Sum of Dots)	Number of Outcomes that produce the Event	Probability (colB / 36)	Probability weighted Y (colA*colC)	
2	2	1	0.0278	0.06	
3	3	2	0.0556	0.17	
4	4	3	0.0833	0.33	
5	5	4	0.1111	0.56	
6	6	5	0.1389	0.83	
7	7	6	0.1667	1.17	
8	8	5	0.1389	1.11	
9	9	4	0.1111	1.00	
10	10	3	0.0833	0.83	
11	11	2	0.0556	0.61	
12	12	1	0.0278	0.33	
13			E(Y)	=SUM(D2:D12)	7.00

Using SUMPRODUCT()

	A	B	C	D
1	Y (Sum of Dots)	Number of Outcomes that produce the Event	Probability (colB / 36)	Probability weighted Y (colA*colC)
2	2	1	0.0278	0.06
3	3	2	0.0556	0.17
4	4	3	0.0833	0.33
5	5	4	0.1111	0.56
6	6	5	0.1389	0.83
7	7	6	0.1667	1.17
8	8	5	0.1389	1.11
9	9	4	0.1111	1.00
10	10	3	0.0833	0.83
11	11	2	0.0556	0.61
12	12	1	0.0278	0.33
13			E(Y)	7.00
14			=SUMPRODUCT(A2:A12,C2:C12)	
15				

(b) Moments of a population distribution: mean, variance, standard deviation, covariance, correlation (cont'd)

The **variance** and **standard deviation** measure the dispersion or the "spread" of a probability distribution.

Variance and Standard Deviation

KEY CONCEPT 2.2

The variance of the discrete random variable Y, denoted σ_Y^2, is

$$\sigma_Y^2 = \text{var}(Y) = E[(Y - \mu_Y)^2] = \sum_{i=1}^{k}(y_i - \mu_Y)^2 p_i. \qquad (2.5)$$

The standard deviation of Y is σ_Y, the square root of the variance. The units of the standard deviation are the same as the units of Y.

Variance / Standard Deviation in Excel

Variance (Y)

	A	B	C	D	E	F	G	H
1	Y (Sum of Dots)	Number of Outcomes that produce the Event	Probability (colB / 36)	Probability weighted Y (colA*colC)	Difference From the Mean (colA - E(Y) or (colA - 7)	Difference From the Mean Squared (colE ^ 2)		
2	2	1	0.0278	0.06	-5	25		
3	3	2	0.0556	0.17	-4	16		
4	4	3	0.0833	0.33	-3	9		
5	5	4	0.1111	0.56	-2	4		
6	6	5	0.1389	0.83	-1	1		
7	7	6	0.1667	1.17	0	0		
8	8	5	0.1389	1.11	1	1		
9	9	4	0.1111	1.00	2	4		
10	10	3	0.0833	0.83	3	9		
11	11	2	0.0556	0.61	4	16		
12	12	1	0.0278	0.33	5	25		
13			E(Y)	7.00		Variance(Y)	5.83	
14			E(Y)	7.00		=SUMPRODUCT(F2:F12,C2:C12)		

Standard Deviation (Y)

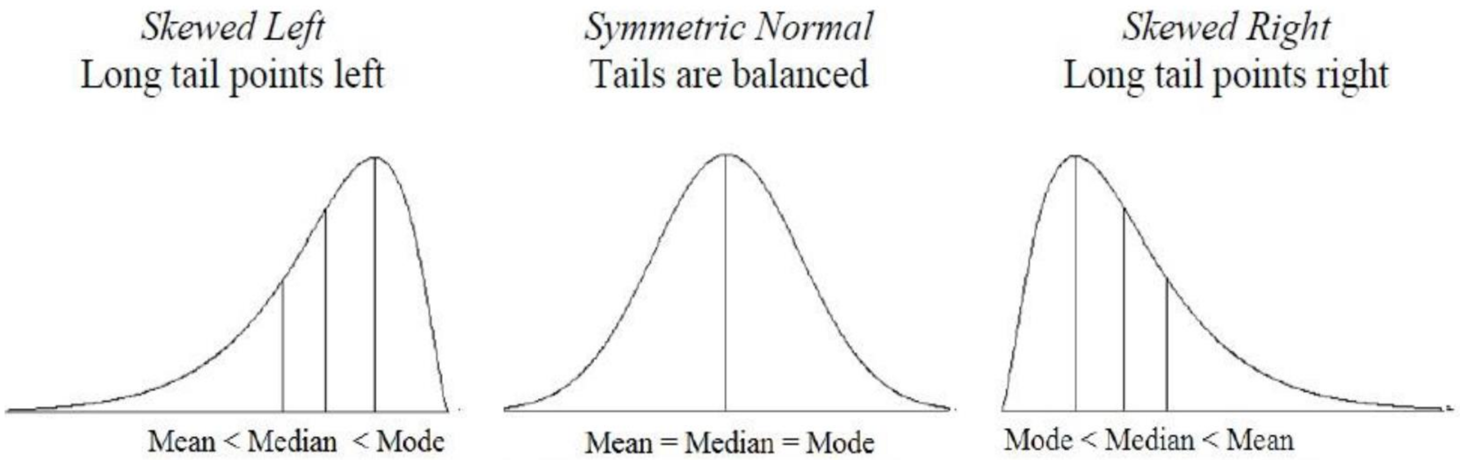
	A	B	C	D	E	F
1	Y (Sum of Dots)	Number of Outcomes that produce the Event	Probability (colB / 36)	Probability weighted Y (colA*colC)	Difference From the Mean (colA - E(Y) or (colA - 7)	Difference From the Mean Squared (colE ^ 2)
2	2	1	0.0278	0.06	-5	25
3	3	2	0.0556	0.17	-4	16
4	4	3	0.0833	0.33	-3	9
5	5	4	0.1111	0.56	-2	4
6	6	5	0.1389	0.83	-1	1
7	7	6	0.1667	1.17	0	0
8	8	5	0.1389	1.11	1	1
9	9	4	0.1111	1.00	2	4
10	10	3	0.0833	0.83	3	9
11	11	2	0.0556	0.61	4	16
12	12	1	0.0278	0.33	5	25
13			E(Y)	7.00		Variance(Y)
14			E(Y)	7.00		5.83
15						
16						2.42
17						=SQRT(F14)
18						StdDev(Y)

(b) Moments of a population distribution: mean, variance, standard deviation, covariance, correlation (cont'd)

$$\text{skewness} = \frac{E\left[(Y-\mu_Y)^3\right]}{\sigma_Y^3}$$

= third central moment; measure of asymmetry of a distribution

- *skewness* = 0: distribution is symmetric
- *skewness* > 0: distribution has long right tail (if <0 then long left tail)



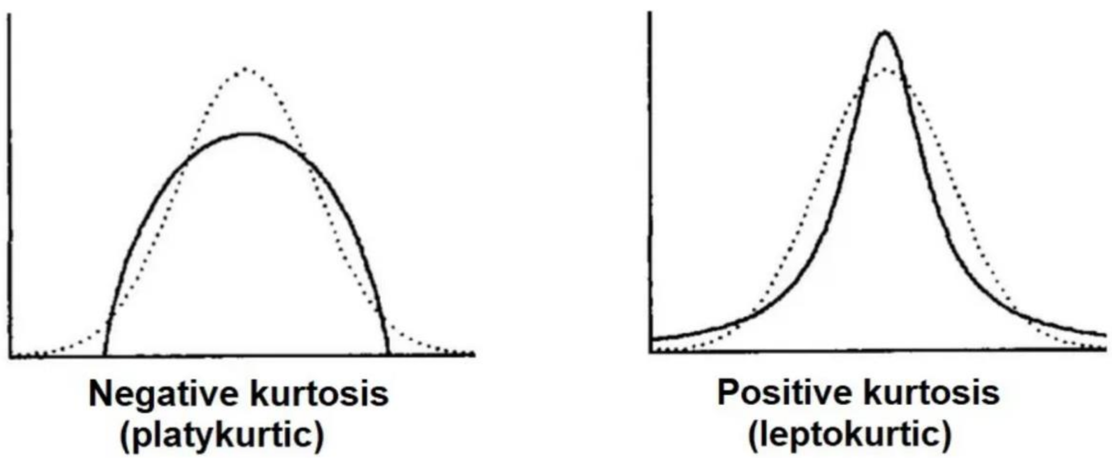
Sketches showing general position of mean, median, and mode in a population.

(b) Moments of a population distribution: mean, variance, standard deviation, covariance, correlation (cont'd)

$$\text{kurtosis} = \frac{E[(Y-\mu_Y)^4]}{\sigma_Y^4}$$

= fourth central moment; measure of mass in tails
= measure of probability of large values

- *kurtosis* = 3: normal distribution
- *kurtosis* > 3: heavy tails ("***leptokurtotic***")



2 random variables: joint distributions and covariance

- Random variables X and Y have a ***joint distribution***
- The joint probability distribution can be written as the function

$$\Pr(X = x, Y = y) = f(x,y)$$

- The ***covariance*** between X and Y is

$$\text{cov}(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sigma_{XY}$$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y) \equiv E(XY) - \mu_X \mu_Y = \sigma_{XY}$$

- The covariance is a measure of the linear association between X and Y; its units are units of X × units of Y
- $\text{cov}(X,Y) > 0$ means a positive relation between X and Y
- If X and Y are independently distributed, then $\text{cov}(X,Y) = 0$ (but not vice versa!!)
- The covariance of a r.v. with itself is its variance:

$$\text{cov}(X,X) = E[(X - \mu_X)(X - \mu_X)] = E[(X - \mu_X)^2] = \sigma_X^2$$

Means, Variances, and Covariances of Sums of Random Variables

Let X, Y, and V be random variables, let μ_X and σ_X^2 be the mean and variance of X, let σ_{XY} be the covariance between X and Y (and so forth for the other variables), and let a, b, and c be constants. Equations (2.29) through (2.35) follow from the definitions of the mean, variance, and covariance:

$$E(a + bX + cY) = a + b\mu_X + c\mu_Y, \tag{2.29}$$

$$\text{var}(a + bY) = b^2\sigma_Y^2, \tag{2.30}$$

$$\text{var}(aX + bY) = a^2\sigma_X^2 + 2ab\sigma_{XY} + b^2\sigma_Y^2, \tag{2.31}$$

$$E(Y^2) = \sigma_Y^2 + \mu_Y^2, \tag{2.32}$$

$$\text{cov}(a + bX + cV, Y) = b\sigma_{XY} + c\sigma_{VY}, \tag{2.33}$$

$$E(XY) = \sigma_{XY} + \mu_X\mu_Y, \tag{2.34}$$

$$|\text{corr}(X, Y)| \leq 1 \text{ and } |\sigma_{XY}| \leq \sqrt{\sigma_X^2\sigma_Y^2} \text{ (correlation inequality).} \tag{2.35}$$

Discrete Bivariate Probability Distribution

X \ Y	-4	1	3	Marginal Distr. of X $\sum_y f(x,y)$
1	0.15	0.20	0.05	0.40
2	0.10	0.25	0.25	0.60
Marginal Distr. of Y $\sum_x f(x,y)$	0.25	0.45	0.30	1.00

$\sum_X \sum_Y f(x,y) = 1$

cov(X,Y) = $E(XY) - E(X)E(Y)$

$E(XY) = X * Y * f(x,y)$ = 1(-4)(0.15) + 1(1)(.2)+1(3)(.05)+

+ 2(-4)(.1) + 2(1)(.25) + 2(3)(.25)

= -0.6 + 0.2 + 0.015 + -0.8 + 0.5 + 1.5

= -0.25 + 1.2 = 0.95

$E(X)$ = (1)(0.4) + 2 (0.6) = 1.6

$E(Y)$ = (-4)(0.25) + (1)(0.45) + (3)(0.3) = 0.35

$E(X)E(Y)$ = 1.6 * 0.35 = 0.56

cov(X,Y) = 0.95 – 0.56 = 0.39

The *correlation coefficient* is defined in terms of the covariance:

$$\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = r_{XY}$$

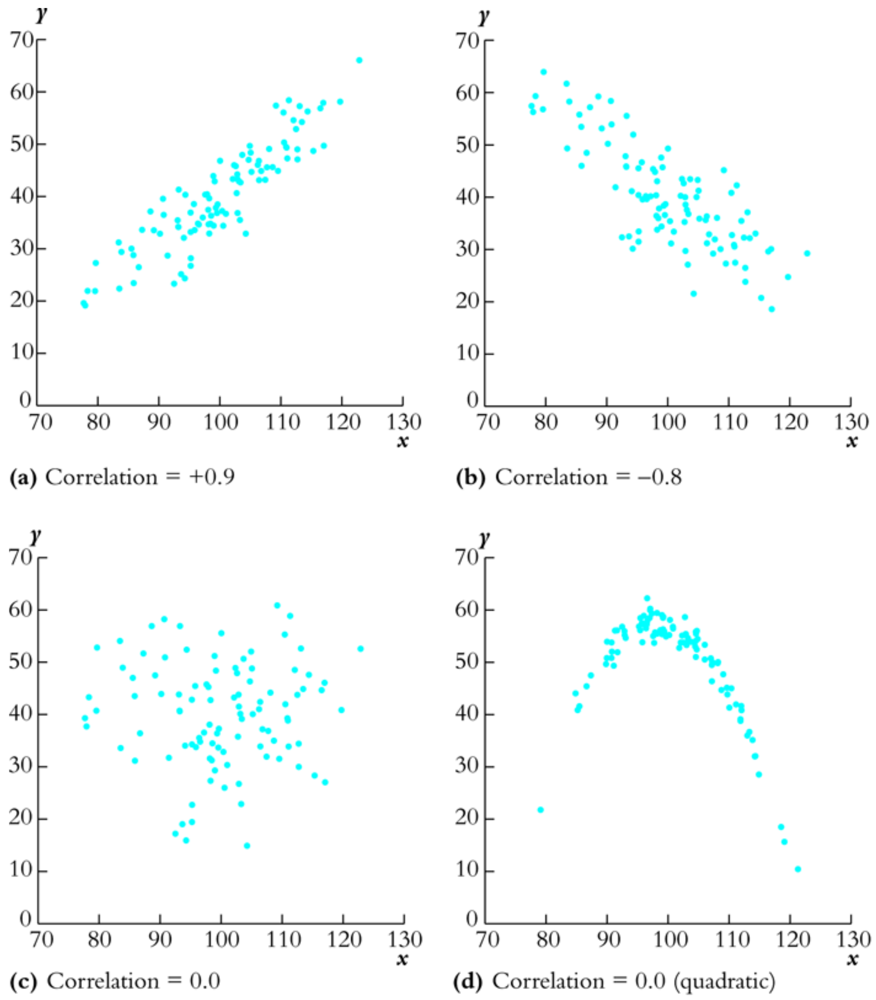
- $-1 \leq \text{corr}(X,Y) \leq 1$
- corr(X,Y) = 1 mean perfect positive linear association
- corr(X,Y) = –1 means perfect negative linear association
- corr(X,Y) = 0 means no linear association

Example:

σ_X = 2.05, σ_Y =1.50, cov (X,Y) = 2.24

r_{XY} = 2.24 / (2.05)(1.5) = 0.73

The correlation coefficient measures linear association



(c) Conditional distributions and conditional means

Conditional distributions

- The distribution of Y, given value(s) of some other random variable, X
- Ex: the distribution of test scores, given that STR < 20

Conditional expectations and conditional moments

- *conditional mean* = mean of conditional distribution

  $= E(Y \mid X = x)$ (***important concept and notation***)

- *conditional variance* = variance of conditional distribution
- *Example*: $E(\textit{Test score} \mid STR < 20)$ = the mean of test scores among districts with small class sizes

The difference in means is the difference between the means of two conditional distributions:

(c) Conditional distributions and conditional means (cont'd)

$\Delta = E(\textit{Test score} \mid STR < 20) - E(\textit{Test score} \mid STR \geq 20)$

Other examples of conditional means:

- Wages of all female workers (Y = wages, X = sex)
- Mortality rate of those given an experimental treatment (Y = live/die; X = treated/not treated)
- If $E(X|Z)$ = const, then corr(X,Z) = 0 (not necessarily vice versa however)

The conditional mean is a (possibly new) term for the familiar idea of the group mean

(c) Conditional distributions and conditional means (cont'd)

The conditional mean plays a key role in prediction:

- Suppose you want to predict a value of Y, and you are given the value of a random variable X that is related to Y. That is, you want to predict Y given the value of X.
  - For example, you want to predict someone's income, given their years of education.
- A common measure of the quality of a prediction m of Y is the mean squared prediction error (MSPE), given X, $E[(Y - m)^2|X]$
- Of all possible predictions m that depend on X, the conditional mean $E(Y|X)$ has the smallest mean squared prediction error (*optional proof is in Appendix 2.2*).

(c) Conditional distributions and conditional means (cont'd)

Example. Suppose that (X, Y) is a bivariate discrete random variable such that the point $(1, 2)$ occurs with probability 1/8, $(1, 3)$ with probability 3/8, $(2, 3)$ with probability 1/4, and $(3, 1)$ with probability 1/4. Then (X, Y) assumes as values

	$Y = 1$	$Y = 2$	$Y = 3$	marginal of X
$X = 1$	0	1/8	3/8	1/2
$X = 2$	0	0	1/4	1/4
$X = 3$	1/4	0	0	1/4
marginal of Y	1/4	1/8	5/8	1

(c) Conditional distributions and conditional means (cont'd)

We compute the **conditional probability** function of Y given $X = 1$. Note that $P[Y = y \mid X = 1] = 0$ except for $y = 2,3$. Thus,

$$P[Y = 2 \mid X = 1] = \frac{P[X = 1, Y = 2]}{P[X = 1]} = \frac{1/8}{1/2} = 1/4;$$
$$P[Y = 3 \mid X = 1] = \frac{P[X = 1, Y = 3]}{P[X = 1]} = \frac{3/8}{1/2} = 3/4.$$

Note that once again $\sum_y P[Y = y \mid X = 1] = 1$.

We compute the **conditional mean** of Y given that $X = 1$:

$$E[Y \mid X = 1] = 2 \cdot 1/4 + 3 \cdot 3/4 = 11/4.$$

Also, we compute the conditional mean of X given that $Y = 3$. The conditional distribution of X given $Y = 3$:

$$p_{X|Y=3}(1) = 3/5; p_{X|Y=3}(2) = 2/5; p_{X|Y=3}(3) = 0.$$

Thus $E[X \mid Y = 3] = 1 \cdot 3/5 + 2 \cdot 2/5 = 7/5$.