Review of Statistical Theory
Part 2

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Review of Statistical Theory

1. The probability framework for statistical inference
2. Estimation
3. Testing
4. Confidence Intervals

The probability framework for statistical inference

a) Random variable, distribution
b) Moments of a distribution (mean, variance, standard deviation, covariance, correlation)
c) Conditional distributions and conditional means
d) Distribution of a sample of data drawn randomly from a population: $Y_1,\ldots, Y_n$

(d) Distribution of a sample of data drawn randomly from a population: $Y_1,\ldots, Y_n$

Population

- The group or collection of all possible entities of interest (school districts). We will think of populations as infinitely large

We will assume simple random sampling

- Choose an individual (district, entity) at random from the population

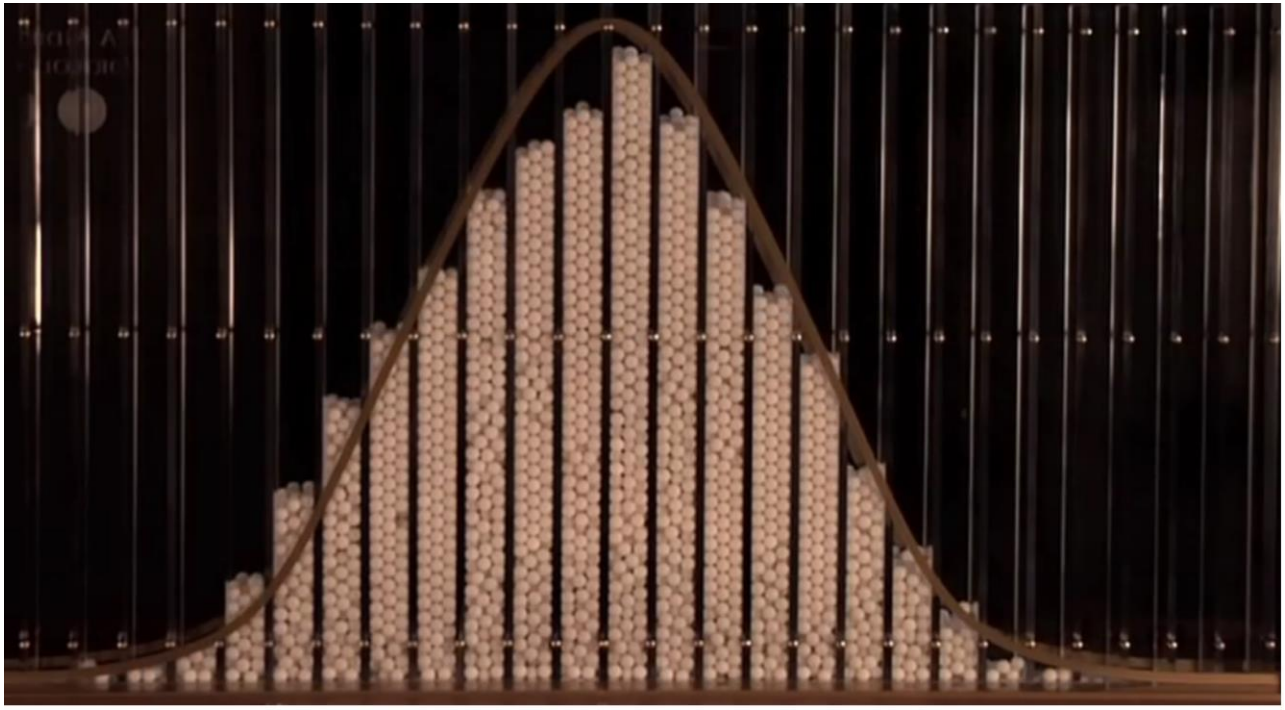
Randomness and data

- Prior to sample selection, the value of Y is random because the individual selected is random
- Once the individual is selected and the value of Y is observed, then Y is just a number – not random
- The data set is (Y_1, Y_2,…, Y_n), where Y_i = value of Y for the i^{th} individual (district, entity) sampled

Distribution of $Y_1,\ldots, Y_n$ under simple random sampling

- Because individuals #1 and #2 are selected at random, the value of Y_1 has no information content for Y_2. Thus:
  - Y_1 and Y_2 are ***independently distributed***
  - Y_1 and Y_2 come from the same distribution, that is, Y_1, Y_2 are ***identically distributed***
  - That is, under simple random sampling, Y_1 and Y_2 are independently and identically distributed (***i.i.d.***).
  - More generally, under simple random sampling, $\{Y_i\}$, $i = 1,\ldots, n$, are i.i.d.

I.I.D. sampling intuition



Galton board

- Imagine we stamp the bin number on each of these balls, then drop them all in a bag
- Sampling: grabbing one from the bag at random a.k.a.
- Note: more likely to get a ball stamped “12” than a “1” or “21”
- Dependent v. **independent**: before we mix them, balls 1-7 are heated, balls 14-21 are cooled; instead of grabbing a ball randomly you grab one that matches temperature to previous sample
- **Identical distributed**: we sample from the same bag each time

This framework allows rigorous statistical inferences about moments of population distributions using a sample of data from that population…

1. The probability framework for statistical inference
2. **Estimation**
3. Testing
4. Confidence Intervals

Estimation

The sample mean ($\bar{Y} = \frac{\sum Y}{n}$) is the natural estimator of the mean. But:

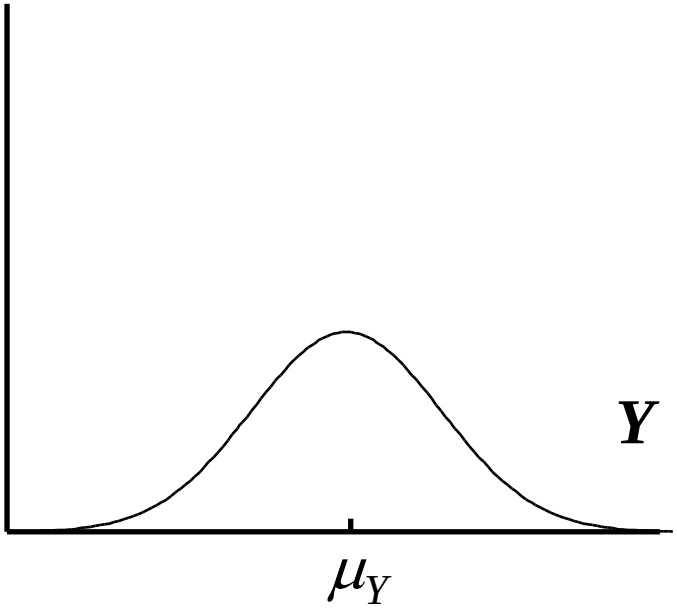
a) What are the properties of $\bar{Y}$?

b) Why should we use $\bar{Y}$ rather than some other estimator?

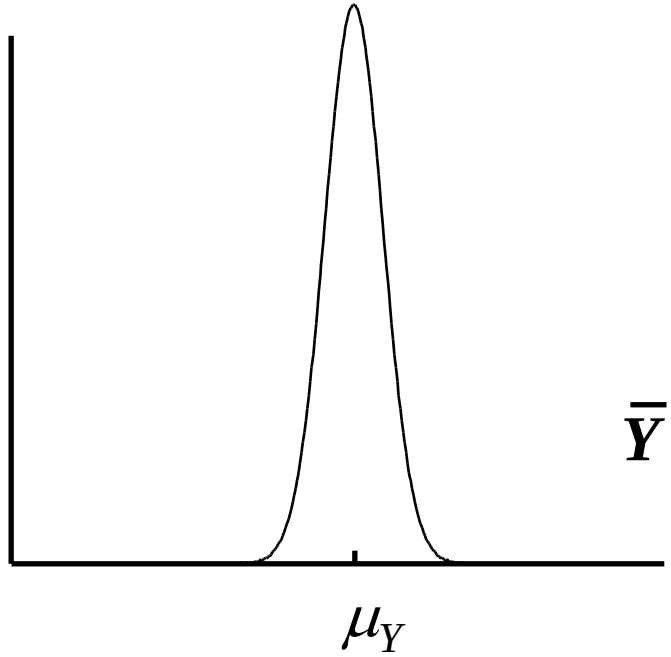
- Y_1 (the first observation)
- maybe unequal weights – not simple average
- median(Y_1,…, Y_n)

Sampling and estimators

probability density function of Y



probability density function of $\overline{Y}$



In this illustration, Y and $\overline{Y}$ are both centered around μ_Y, but the dispersion differs.

The mean and variance of the sampling distribution of $\bar{Y}$ (1 of 3)

- General case – that is, for Y_i i.i.d. **from any distribution**

- mean: $E(\bar{Y}) = E(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n}\sum_{i=1}^{n} \mu_Y = \mu_Y$

- Variance:
$$\begin{aligned} \text{var}(\bar{Y}) &= E[\bar{Y} - E(\bar{Y})]^2 \\ &= E[\bar{Y} - \mu_Y]^2 \\ &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) - \mu_Y\right]^2 \\ &= E\left[\frac{1}{n}\sum_{i=1}^{n} (Y_i - \mu_Y)\right]^2 \end{aligned}$$

The mean and variance of the sampling distribution of $\bar{Y}$ (2 of 3)

so

$$\text{var}(\bar{Y}) = E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_Y)\right]^2$$

$$= E\left\{\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_Y)\right] \times \left[\frac{1}{n}\sum_{j=1}^{n}(Y_j - \mu_Y)\right]\right\}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left[(Y_i - \mu_Y)(Y_j - \mu_Y)\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \text{cov}(Y_i, Y_j)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \sigma_Y^2$$

$$= \frac{\sigma_Y^2}{n}$$

The mean and variance of the sampling distribution of $\bar{Y}$ (3 of 3)

$$E(\bar{Y}) = \mu_Y$$

$$\text{var}(\bar{Y}) = \frac{\sigma_Y^2}{n}$$

Implications:

1. $\bar{Y}$ is an *unbiased* estimator of μ_Y (that is, $E(\bar{Y}) = \mu_Y$)
2. var($\bar{Y}$) is inversely proportional to n
    1. the spread of the sampling distribution is proportional to $1/\sqrt{n}$
    2. Thus the sampling uncertainty associated with $\bar{Y}$ is proportional to $1/\sqrt{n}$ (larger samples, less uncertainty, but square-root law)

(a) The sampling distribution of $\bar{Y}$

$\bar{Y}$ is a random variable, and its properties are determined by the ***sampling distribution*** of $\bar{Y}$

- The individuals in the sample are drawn at random.
- Thus the values of $(Y_1, \ldots, Y_n)$ are random
- Thus functions of $(Y_1, \ldots, Y_n)$, such as $\bar{Y}$, are random: had a different sample been drawn, they would have taken on a different value
- The distribution of $\bar{Y}$ over different possible samples of size n is called the ***sampling distribution*** of $\bar{Y}$.
- The mean and variance of $\bar{Y}$ are the mean and variance of its sampling distribution, $E(\bar{Y})$ and var($\bar{Y}$).
- The concept of the sampling distribution underpins all of econometrics.

Things we want to know about the sampling distribution:

- What is the mean of $\bar{Y}$?
  - If $E(\bar{Y}) = \mu$, then $\bar{Y}$ is an ***unbiased*** estimator of μ
- What is the variance of $\bar{Y}$?
  - How does var($\bar{Y}$) depend on n (famous $1/n$ formula)
- Does $\bar{Y}$ become close to μ when n is large?
  - Law of large numbers: $\bar{Y}$ is a ***consistent*** estimator of μ
- $\bar{Y} - \mu$ appears bell shaped for n large…is this generally true?
  - In fact, $\bar{Y} - \mu$ is approximately normally distributed for n large (Central Limit Theorem)

The sampling distribution of $\bar{Y}$ when n is large

For small sample sizes, the distribution of $\bar{Y}$ is complicated, but if n is large, the sampling distribution is simple!

1. As n increases, the distribution of $\bar{Y}$ becomes more tightly centered around μ_Y (the *Law of Large Numbers*)
2. Moreover, the distribution of $\bar{Y} - \mu_Y$ becomes normal (the *Central Limit Theorem*)

The *Law of Large Numbers*:

An estimator is ***consistent*** if the probability that its falls within an interval of the true population value tends to one as the sample size increases.

If $(Y_1,\ldots,Y_n)$ are i.i.d. and $\sigma_Y^2 < \infty,$ then $\bar{Y}$ is a consistent estimator of $\mu_Y,$ that is,

$$\Pr[|\bar{Y} - \mu_Y| < \mu] \to 1 \text{ as } n \to \infty$$

which can be written, $\bar{Y} \xrightarrow{p} \mu_Y$

("$\bar{Y} \xrightarrow{p} \mu_Y$" means "$\bar{Y}$ converges in probability to μ_Y").

(*the math*: as $n \to \infty$, $\text{var}(\bar{Y}) = \dfrac{\sigma_Y^2}{n} \to 0,$ which implies that $\Pr[|\bar{Y} - \mu_Y| < \varepsilon] \to 1$.)

The *Central Limit Theorem* (CLT)

If $(Y_1,\ldots,Y_n)$ are i.i.d. and $0<\sigma_Y^2<\infty,$ then when n is large the distribution of $\bar{Y}$ is well approximated by a normal distribution.

- $\bar{Y}$ is approximately distributed $N(\mu_Y, \frac{\sigma_Y^2}{n})$ ("normal distribution with mean μ_Y and variance σ_Y^2/n")
- $\sqrt{n}\ (\bar{Y}-\mu_Y)/\sigma_Y$ is approximately distributed $N(0,\ 1)$ (standard normal)
- **That is, "standardized"** $\bar{Y} = \frac{\bar{Y}-E(\bar{Y})}{\sqrt{\text{var}(\bar{Y})}} = \frac{\bar{Y}-\mu_Y}{\sigma_Y/\sqrt{n}}$ **is approximately distributed as** $N(0,\ 1)$
- **The larger is *n*, the better is the approximation.**

Summary: The Sampling Distribution of $\bar{Y}$

For $Y_1, \ldots, Y_n$ i.i.d. with $0 < \sigma_Y^2 < \infty$,

- The exact (finite sample) sampling distribution of $\bar{Y}$ has mean μ_Y ("$\bar{Y}$ is an unbiased estimator of μ_Y") and variance σ_Y^2/n
- Other than its mean and variance, the exact distribution of $\bar{Y}$ is complicated and depends on the distribution of Y (the population distribution)
- When n is large, the sampling distribution simplifies:
  - $\bar{Y} \xrightarrow{p} \mu_Y$ (Law of large numbers)
  - $\dfrac{\bar{Y} - E(\bar{Y})}{\sqrt{\text{var}(\bar{Y})}}$ is approximately $N(0,1)$ (CLT)

(b) Why Use $\bar{Y}$ To Estimate μ_Y?

- $\bar{Y}$ is unbiased: $E(\bar{Y}) = \mu_Y$
- $\bar{Y}$ is consistent: $\bar{Y} \xrightarrow{p} \mu_Y$
- $\bar{Y}$ has a smaller variance than all other *linear unbiased* estimators:

  consider the estimator, $\hat{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n} a_i Y_i$, where $\{a_i\}$ are such that $\hat{\mu}_Y$ is unbiased; then $\text{var}(\bar{Y}) \leq \text{var}(\hat{\mu}_Y)$ (proof: SW, Ch. 17)

- $\bar{Y}$ isn't the only estimator of μ_Y – can you think of a time you might want to use the median instead