Submodular Function Optimization

AU4606: Network Optimization

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Last few lectures

- Minimum spanning tree problems
    - Optimality conditions
    - Algorithms
        1. Kruskal's algorithm
        2. Prim's algorithm
        3. Sollin's algorithm
    - Matroids and greedy algorithms

Today

1. Submodular functions
2. Monotone submodular maximization
3. Unconstrained submodular minimization

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Greedy works for matroid optimization

An optimization problem over matroids

Let $(E, \mathcal{I})$ be a matroid and $w : E \to \mathbb{R}$ be a function that assigns a cost to each element of E.
Define $w(X) = \sum_{x \in X} w(x)$ for any nonempty subset $X \subseteq E$.
Find a maximal independent set that has the minimum cost.

Greedy works

If $(E, \mathcal{I})$ is a matroid, then greedy algorithm finds a maximal independent set that has the minimum cost.

For problems with special structure, they may be solved efficiently.

Greedy can fail

Problem setup

- Let $\{1, 2, \ldots, 2k+1\}$ be a ground set
- The function value
  1. $f(\{e\}) = 1$ for $e \in \{1, \ldots, k\}$
  2. $f(\{e\}) = \epsilon$ for $e \in \{k+1, \ldots, 2k\}$
  3. $f(\{2k+1\}) = 1 + \epsilon$
  4. For $S_1 \subseteq \{1, \ldots, k\}$ and $S_2 \subseteq \{k+1, \ldots, 2k\}$, we have $f(S_1 \cup S_2) = |S_1| + \epsilon |S_2|$
  5. For $S_1 \subseteq \{1, \ldots, k\}$ and $S_2 \subseteq \{k+1, \ldots, 2k\}$, we have $f(S_1 \cup S_2 \cup \{2k+1\}) = \max\{|S_1|, 1 + \epsilon\} + \epsilon |S_2|$

Pick a k-element set S to maximize $f(S)$?

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Pick a k-element set S to maximize $f(S)$? Pick $\{1, \ldots, k\}$ and obtain k

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Greedy works as follows

- Pick $\{2k+1\}$ in the first iteration
- Pick $k-1$ elements from $\{k+1, \ldots, 2k\}$
- Obtain a function value of $\epsilon(k-1) + 1 + \epsilon = 1 + \epsilon k$

Set function optimization

We are often interested in problems of the following form

$$\max\{f(S) : S \in \mathcal{F}\}$$
$$\min\{f(S) : S \in \mathcal{F}\}$$

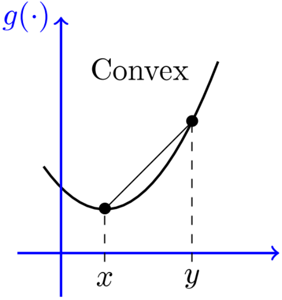
where $\mathcal{F} \subseteq 2^N$ is a collection of feasible subsets

Examples

1. min/max cut: $f(S)$ is cut value of cut set $(S, \bar{S})$ (N is node set)
2. min/max spanning tree: $f(S)$ is total arc weights ($\mathcal{F}$: spanning trees)
3. min vertex cover: $f(S) = |S|$, $\mathcal{F}$ is all sets of nodes that "cover" arcs

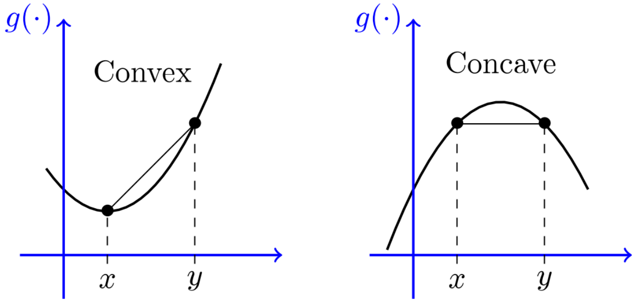
What properties should f have so that the problem is tractable?

What we know from continuous optimization



A function $g : \mathbb{R}^n \to \mathbb{R}$ can be minimized efficiently, if it is convex
A function $g : \mathbb{R}^n \to \mathbb{R}$ can be maximized efficiently, if it is concave

What we know from continuous optimization



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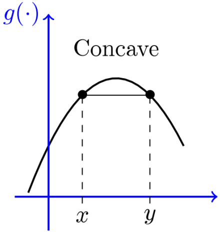
Can we have analogs in discrete optimization with $f : \{0,1\}^n \to \mathbb{R}$?

From concavity to submodularity



$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$
is non-increasing in x

Let $f : 2^N \to \mathbb{R}$ and define

$$\Delta_e f(S) = f(S \cup \{e\}) - f(S)$$

Submodularity: we require that $\Delta_e f(S)$ is non-increasing in S

Submodular functions: definitions

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Submodular functions I

A function $f : 2^N \to \mathbb{R}$ is submodular if for every $A \subseteq B \subseteq N$ and $e \notin B$, it holds that

$$\Delta_e f(A) \geq \Delta_e f(B) \quad \Longleftrightarrow \quad f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B)$$

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Submodular functions II

A function $f : 2^N \to \mathbb{R}$ is submodular if for every $A, B \subseteq N$, it holds that

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$$

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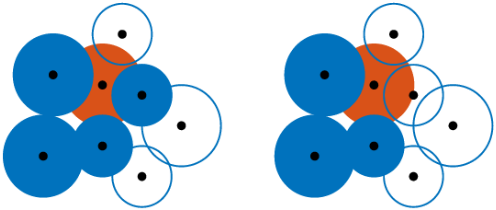
Why "submodular"? A function $f : 2^N \to \mathbb{R}$ is modular if

$$f(A \cup \{e\}) - f(A) = f(B \cup \{e\}) - f(B) \implies f(A) = \sum_{i \in A} f(\{i\}) + (|A| - 1) f(\emptyset)$$

Submodular functions: examples

Adding additional sensors has diminishing returns

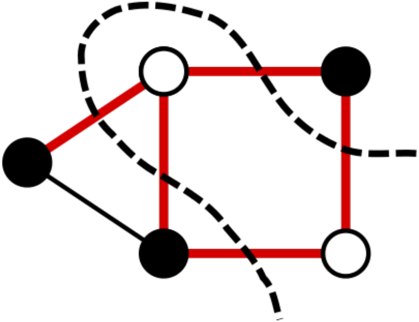


A sensor coverage example

$f : S \mapsto \mathbb{R}$ computes the area that sensors in set S cover

Submodular functions: examples

Including more nodes in a cut has diminishing returns



Cut function $f(S) = u(S, \bar{S}) = \sum_{i \in S, j \in \bar{S}} u_{ij}$

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Monotone submodular maximization & cardinality

Monotone function

A function $f : 2^N \to \mathbb{R}$ is monotone if for every $A \subseteq B \subseteq N$, $f(A) \leq f(B)$.

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$$\max\{f(S) : |S| \leq k\} \tag{1}$$

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Algorithm Greedy algorithm

1: $S \leftarrow \emptyset$
2: **while** $|S| < k$ **do**
3: Pick $i^* \in \text{argmax}_{i \in N \setminus S} f(S \cup \{i\}) - f(S)$
4: $S \leftarrow S \cup \{i^*\}$
5: **end while**

Monotone submodular maximization & cardinality

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Greedy is "good enough"

Suppose f in (1) is nonnegative, monotone submodular, then the greedy algorithm finds a solution S to (1) such that

$$f(S) \geq (1 - e^{-1}) f(S^*),$$

where S^* is an optimal solution.

Submodular maximization: more settings

Monotone maximization

- (1978) Under intersection of k matroids ($\frac{1}{k+1}$)
- (2004) Under knapsack constraint ($1 - e^{-1}$)

$$\max\{f(S) : \sum_{i \in S} c_i \leq B\}$$

  where c_i's are nonnegative integers
- (2007) Under a matroid constraint ($1 - e^{-1}$)
- ...

Non-monotone maximization

- (2007) Unconstrained ($\frac{1}{2}$)
- (2011) Under cardinality or matroid constraint (e^{-1})
- ...

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From discrete optimization to continuous optimization

$$\min\{f(S) : S \subseteq N\}$$

- $f : \{0,1\}^n \to \mathbb{R}$ is a function whose domain contains 2^n points
- Can we define a continuous extension $f^{\mathrm{E}} : [0,1]^n \to \mathbb{R}$?
- Hopefully
  1. f^{E} coincides with f at $\{0,1\}^n$
  2. f^{E} can be easily evaluated
  3. f^{E} can be efficiently optimized
  4. An optimizer S^* of f can be recovered from an optimizer x^* of f^{E}

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Extension of a set function

Given a set function $f : \{0,1\}^n \to \mathbb{R}$, an extension of f to $[0,1]^n$ is a function $f^{\mathrm{E}} : [0,1]^n \to \mathbb{R}$ satisfying $f^{\mathrm{E}}(x) = f(x)$ for $x \in \{0,1\}^n$.

Solve a continuous optimization problem with the extension of f.

Note $f : \{0,1\}^n \to \mathbb{R}$ is defined only at 2^n locations, for $x \in [0,1]^n$:

1. Find coefficients α_i's such that $x = \sum_i \alpha_i \mathbb{1}_{S_i}$
2. Define $f^{\mathrm{E}}(x) = \sum_i \alpha_i f(S_i)$

Continuous extensions

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- The combinations may not be unique

$$x = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad = \quad \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad = \quad \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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- If every $S_i \subseteq N$ appears in the sum, then exponentially many terms

The Lovász extension: definition

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Given $f : \{0,1\}^n \to \mathbb{R}^n$, its Lovász extension $f^{\mathsf{L}} : [0,1]^n \to \mathbb{R}$ is defined by

$$f^{\mathsf{L}}(x) = \mathbb{E}_\theta[f(\{i : x_i \geq \theta\})]$$

where θ is a random variable following a uniform distribution over $[0,1]$.

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Let $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ be a permutation of x such that

$$x_{\pi(0)} = 1 \geq x_{\pi(1)} \geq \cdots \geq x_{\pi(n)} \geq 0 = x_{\pi(n+1)}$$

- When $x_{\pi(i+1)} < \theta \leq x_{\pi(i)}$, we have $\{i : x_i \geq \theta\} = \{\pi(1),\ldots,\pi(i)\}$
- The expectation defining $f^{\mathsf{L}}(x)$ is then a sum of $n+1$ terms
- f^{L} and f coincide at $x \in \{0,1\}^n$

The Lovász extension as a continuous extension

Note $f : \{0,1\}^n \to \mathbb{R}$ is defined only at 2^n locations, for $x \in [0,1]^n$:

1. Find coefficients α_i's such that $x = \sum_i \alpha_i \mathbb{1}_{S_i}$
2. Define $f^{\mathrm{E}}(x) = \sum_i \alpha_i f(S_i)$

Let $x \in [0,1]^n$, suppose $x_1 \leq x_2 \leq \cdots \leq x_n$, note

$$x = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} + (x_2 - x_1) \begin{bmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} + (x_3 - x_2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} + \cdots + (x_n - x_{n-1}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} + (1 - x_n) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Thus,

$$f^{\mathrm{L}}(x) = x_1 f(S) + (x_2 - x_1) f(S \setminus \{1\}) + \cdots + (x_n - x_{n-1}) f(\{n\}) + (1 - x_n) f(\emptyset)$$

- $f(S) = \min\{1, |S|\}$, that is $f(\emptyset) = 0$ and $f(S) = 1$ elsewhere

The Lovász extension: examples

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- Let $G = (N, E)$ be an undirected graph, for $A \subseteq V$, define

$$f(A) = |\{(i,j) \in E \mid i \in A, j \in \bar{A}\}|$$

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- Let $G = (N, E)$ be an undirected graph, for $A \subseteq V$, define

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Then the Lovász extension is given by

$$f^{\mathsf{L}}(x) = \sum_{(i,j)\in E} |x_i - x_j|$$

Submodular minimization: setups

$$\min\{f(S) : S \subseteq N\}$$

- $f : \{0,1\}^n \to \mathbb{R}$ is submodular
- No monotonicity is assumed
- No constraint is imposed

Key idea: Lovász extension

$$\begin{array}{c}\text{Submodular function:}\\ f : \{0,1\}^n \to \mathbb{R}\\ \Downarrow \\ \text{Convex continuous function:}\\ f^{\mathrm{L}} : [0,1]^n \to \mathbb{R}\end{array}$$

László Lovász (1948-)

- If f^{L} is convex, then the extension can be solved efficiently
- A minimizer of f^{L} can be converted into a minimizer of $f(S)$

The Lovász extension: why useful

The Lovász extension: definition

Given $f : \{0,1\}^n \to \mathbb{R}^n$, its Lovász extension $f^{\mathrm{L}} : [0,1]^n \to \mathbb{R}$ is defined by

$$\begin{aligned} f^{\mathrm{L}}(x) &= \sum_{i=0}^{n} \lambda_i(x) f(S_i(x)) \\ &= f(\emptyset) + \sum_{i=1}^{n} x_{\pi(i)}\big(f(S_i(x)) - f(S_{i-1}(x))\big) \end{aligned}$$

1. $\lambda_i(x) = x_{\pi(i)} - x_{\pi(i+1)}$
2. $S_{i+1} = S_i \cup \{\pi(i+1)\}$ with $S_0 = \emptyset$

The Lovász extension is a piece-wise linear function of x!

The Lovász extension: why useful

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Convexity and submodularity

A function $f : \{0,1\}^n \to \mathbb{R}^n$ is submodular if and only if its Lovász extension $f^{\mathsf{L}} : [0,1]^n \to \mathbb{R}$ is convex.

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- We can recover an optimal set S^* from a minimizer x^* of $f^{\mathsf{L}}(x)$

$$f^{\mathsf{L}}(x^*) = \sum_{i=0}^{n} \lambda_i(x^*) f(S_i(x^*))$$

  where $\sum_{i=0}^{n} \lambda_i(x^*) = 1$

We did not discuss

- Why is $f^{\mathsf{L}}(x)$ convex when f is submodular?
- How to solve the convex problem?

Upcoming

Week 1-8 (AU4606 & AI4702):

- Introduction (1 lecture)
- Preparations (3 lectures)
  - basics of graph theory
  - algorithm complexity and data structure
  - graph search algorithm
- Shortest path problems (3 lectures)
- Maximum flow problems (5 lectures)
- Minimum cost flow problems (3 lectures)
- Introduction to multi-agent systems (1 lecture)
- Introduction to cloud networks (1 lecture)

Week 9-16 (AU4606):

- Simplex and network simplex methods (1 lecture)
- Global minimum cut problems (1.5 lectures)
- Minimum spanning tree problems (1.5 lectures)
- Submodular function optimization (2 lectures)
- Optimal assignments and matching (2 lectures)