CS2601 Linear and Convex Optimization: Project

SJTU 2024 Fall

Dec. 06, 2024

Submission Guideline

Deadline: 23:59pm, Sunday, Dec. 29, 2024

Submissions later than the deadline will be discounted:

(a) within 0-24 hours, 20% off;

(b) within 24-48 hours, 50% off;

(c) larger than 48 hours, not acceptable.

Acceptable submission formats:

(1) You should submit a **zipped** file containing both the report (in PDF format) and the code on Canvas.

(2) The report can be written in either English (preferred) or Chinese.

(3) There are no specific requirements for the code, but adding comments is encouraged.

1 About the Project

The goal of the project is to develop a program that implements the *Water-filling* Problem introduced below.

1.1 Environment Description

We consider the convex optimization problem

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1, \end{array}$$

where $\alpha_i > 0$. This problem arises in information theory, in allocating power to a set of n communication channels. The variable x_i represents the transmitter power allocated to the i th channel, and $\log(\alpha_i + x_i)$ gives the capacity or communication rate of the channel, so the problem is to allocate a total power of one to the channels, in order to maximize the total communication rate.

Introducing Lagrange multipliers $\lambda^\star \in \mathbf{R}^n$ for the inequality constraints $x^\star \succeq 0$, and a multiplier $\nu^\star \in \mathbf{R}$ for the equality constraint $\mathbf{1}^T x = 1$, we obtain the KKT conditions

$$\begin{gathered} x^\star \succeq 0, \quad \mathbf{1}^T x^\star = 1, \quad \lambda^\star \succeq 0, \quad \lambda_i^\star x_i^\star = 0, \quad i = 1, \ldots, n, \\ -1/(\alpha_i + x_i^\star) - \lambda_i^\star + \nu^\star = 0, \quad i = 1, \ldots, n. \end{gathered}$$

We can directly solve these equations to find $x^\star, \lambda^\star$, and $\nu^\star$. We start by noting that $\lambda^\star$ acts as a slack variable in the last equation, so it can be eliminated, leaving

$$\begin{gathered} x^\star \succeq 0, \quad \mathbf{1}^T x^\star = 1, \quad x_i^\star (\nu^\star - 1/(\alpha_i + x_i^\star)) = 0, \quad i = 1, \ldots, n, \\ \nu^\star \geq 1/(\alpha_i + x_i^\star), \quad i = 1, \ldots, n. \end{gathered}$$

If $\nu^\star < 1/\alpha_i$, this last condition can only hold if $x_i^\star > 0$, which by the third condition implies that $\nu^\star = 1/(\alpha_i + x_i^\star)$. Solving for $x_i^\star$, we conclude that $x_i^\star = 1/\nu^\star - \alpha_i$ if $\nu^\star < 1/\alpha_i$. If $\nu^\star \geq 1/\alpha_i$, then $x_i^\star > 0$ is impossible, because it would imply $\nu^\star \geq 1/\alpha_i > 1/(\alpha_i + x_i^\star)$, which violates the complementary slackness condition. Therefore, $x_i^\star = 0$ if $\nu^\star \geq 1/\alpha_i$. Thus we have

$$x_i^\star = \begin{cases} 1/\nu^\star - \alpha_i & \nu^\star < 1/\alpha_i \\ 0 & \nu^\star \geq 1/\alpha_i \end{cases}$$

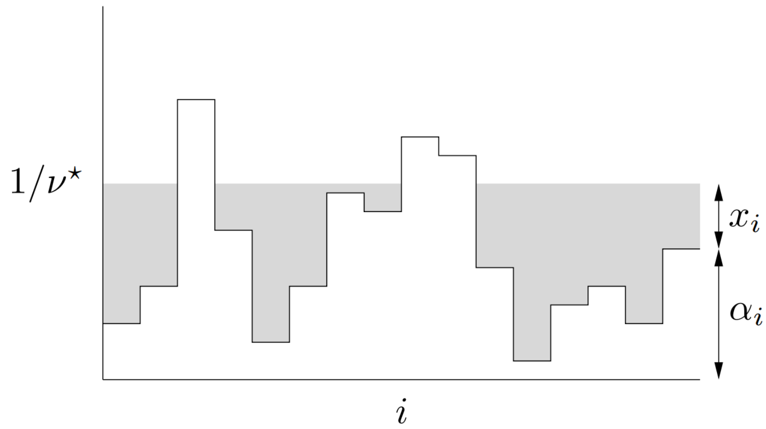

Figure 1: **Illustration of water-filling algorithm.** The height of each patch is given by α_i. The region is flooded to a level $1/\nu^\star$ which uses a total quantity of water equal to one. The height of the water (shown shaded) above each patch is the optimal value of $x_i^\star$.

or, put more simply, $x_i^\star = \max\{0, 1/\nu^\star - \alpha_i\}$. Substituting this expression for $x_i^\star$ into the condition $\mathbf{1}^T x^\star = 1$ we obtain

$$\sum_{i=1}^{n} \max\{0, 1/\nu^\star - \alpha_i\} = 1$$

The lefthand side is a piecewise-linear increasing function of $1/\nu^\star$, with breakpoints at α_i, so the equation has a unique solution which is readily determined.

This solution method is called water-filling for the following reason. We think of α_i as the ground level above patch i, and then flood the region with water to a depth $1/\nu$, as illustrated in figure 1. The total amount of water used is then $\sum_{i=1}^{n} \max\{0, 1/\nu^\star - \alpha_i\}$. We then increase the flood level until we have used a total amount of water equal to one. The depth of water above patch i is then the optimal value $x_i^\star$.

2 Requirements

- The experimental data must be generated randomly. Specific requirements and ranges of the data can refer to the provided implementation.
- Use Python to write the program independently. Plagiarism is strictly prohibited and will result in a score of 0 for the experiment.
- The report should include the following:
  - **Objectives of the Experiment:** Clearly outline the goals you aim to achieve.
  - **Experimental Process:** Describe the methodology in detail, such as the algorithms used to solve the problem, the implementation details of your code, etc.
  - **Results and Analysis:** Present the final results of the experiment along with your analysis, including screenshots of your local implementation if possible.

3 Evaluation and Bonus

The project will be evaluated based on the following criteria, with a maximum score of 100 points. *Optional* items worth additional points, but the total score will not exceed 100. You may choose not to include them.

- (40 points) Comprehensive description of the experimental process and results.
- (40 points) Excellent analysis of the results.
- (20 points) Code readability.
- (5 points, *Optional*) Visualization of the Water-filling Environment.
- (5 points, *Optional*) Code implementations that achieve better/faster/accurate performance.