EECS 545: Machine Learning
Lecture 6. Classification 3

Honglak Lee

1/31/2024

UNIVERSITY OF MICHIGAN

Logistics/Announcements

- HW 2 due on Feb 13th
- Project Proposal due
  - No point deduction through Feb 2nd, Friday (3 late days with consuming your tokens afterward)
  - 1 point per day from Feb 5th onwards
- We will be providing CPU/GPU credits on Great Lakes
  - $60.91, per student (this may be good for starting point, but may be insufficient for the entire semester)
  - please stay tuned for an announcement

Outline

(grey: already covered)

- Probabilistic discriminative models
  - ✓ Logistic Regression
  - ✓ Softmax Regression
- Probabilistic generative models
  - ✓ Gaussian discriminant analysis
  - Naive Bayes (continued)
- Discriminant functions (non-probabilistic)
  - Fisher’s linear discriminant
  - Perceptron learning algorithm

Recap: Discriminative vs Generative Probabilistic Classifiers

- Goal: Learn the distributions $p(C_k \mid \mathbf{x})$.

(a) **Discriminative** models: Directly model $p(C_k \mid \mathbf{x})$ and learn parameters from the training set.

- Logistic regression
- Softmax regression

(b) **Generative** models: Learn joint densities $p(\mathbf{x}, C_k)$ by learning $p(\mathbf{x} \mid C_k)$ and priors $p(C_k)$, and then use Bayes rule for predicting the class C_k given $\mathbf{x}$:

- Gaussian Discriminant Analysis
- Naive Bayes

Naive Bayes classifier (continued)

Naive Bayes classifier

- Probability of class label:
  - $p(C_k)$: Constant (e.g., Bernoulli)
- Conditional probability of data given the class
  - Naive Bayes assumption: $p(\mathbf{x} \mid C_k)$ is factorized (Each coordinate of $\mathbf{x}$ is conditionally independent of other coordinates given the class label)

$$P(x_1, ..., x_M|C_k) = P(x_1|C_k) \cdots P(x_M|C_k) = \prod_{j=1}^{M} P(x_j|C_k)$$

- Classification: use Bayes rule

(binary) $P(C_1|\mathbf{x}) = \dfrac{P(C_1, \mathbf{x})}{P(\mathbf{x})} = \dfrac{P(C_1, \mathbf{x})}{P(C_1, \mathbf{x}) + P(C_2, \mathbf{x})}$

Naive Bayes classifier

- When classifying, we can simply find the class C_k that maximizes $P(C_k|\mathbf{x})$ using the Bayes rule:

$$\arg\max_k P(C_k|\mathbf{x}) = \arg\max_k P(C_k, \mathbf{x})$$

Naive Bayes classifier

- When classifying, we can simply find the class C_k that maximizes $P(C_k|\mathbf{x})$ using the Bayes rule:

$$\begin{aligned}\arg\max_k P(C_k|\mathbf{x}) &= \arg\max_k P(C_k, \mathbf{x}) \\ &= \arg\max_k P(C_k)P(\mathbf{x}|C_k)\end{aligned}$$

Naive Bayes classifier

- When classifying, we can simply find the class C_k that maximizes $P(C_k|\mathbf{x})$ using the Bayes rule:

$$\begin{aligned}\arg\max_k P(C_k|\mathbf{x}) &= \arg\max_k P(C_k, \mathbf{x}) \\ &= \arg\max_k P(C_k)P(\mathbf{x}|C_k) \\ &= \arg\max_k P(C_k)\prod_{j=1}^{M} P(x_j|C_k)\end{aligned}$$

Naive Bayes assumption

Example: Spam mail classification

- Label: y=1 (spam), y=0 (non-spam)
- Features:
  - x_j: *j*-th word in the mail, where M is the vocabulary size.
  - Multinomial variable (M-dimensional binary vector with only one coordinate with 1)
- Naive Bayes Assumption:
  - Given a class label y, each word in a mail is a independent multinomial variable.

Naive Bayes Spam classifier

- Model

$$P(\text{spam}) = Bernoulli(\phi)$$
$$P(\text{word}|\text{spam}) = Multinomial(\mu_1^s, \ldots, \mu_M^s)$$
$$P(\text{word}|\text{nonspam}) = Multinomial(\mu_1^{ns}, \ldots, \mu_M^{ns})$$

top words from my **non-spam** emails

top words from my **spam** emails

Naive Bayes Spam classifier

- Model

$$P(\text{spam}) = Bernoulli(\phi)$$
$$P(\text{word}|\text{spam}) = Multinomial(\mu_1^s, \ldots, \mu_M^s)$$
$$P(\text{word}|\text{nonspam}) = Multinomial(\mu_1^{ns}, \ldots, \mu_M^{ns})$$

- Goal

Find ϕ, μ^s, μ^{ns} that best fits the data $\{(x^{(1)}, y^{(1)}),\ldots, (x^{(N)}, y^{(N)})\}$

Naive Bayes Spam classifier

- Model

$$P(\text{spam}) = Bernoulli(\phi)$$
$$P(\text{word}|\text{spam}) = Multinomial(\mu_1^s, \dots, \mu_M^s)$$
$$P(\text{word}|\text{nonspam}) = Multinomial(\mu_1^{ns}, \dots, \mu_M^{ns})$$

- Goal

  Find ϕ, μ^s, μ^{ns} that best fits the data $\{(x^{(1)}, y^{(1)}), \dots, (x^{(N)}, y^{(N)})\}$ by maximizing the joint likelihood:

$$\prod_{i=1}^{N} P(\mathbf{x}^{(i)}, y^{(i)})$$

- Joint Likelihood (joint probability of inputs/labels)

  - Note that the joint likelihood is conditioned on parameters ϕ, μ^s, μ^{ns}

Naive Bayes Spam classifier

- Model

$$P(\text{spam}) = Bernoulli(\phi)$$
$$P(\text{word}|\text{spam}) = Multinomial(\mu_1^s, \ldots, \mu_M^s)$$
$$P(\text{word}|\text{nonspam}) = Multinomial(\mu_1^{ns}, \ldots, \mu_M^{ns})$$

- Goal

Find ϕ, μ^s, μ^{ns} that best fits the data $\{(x^{(1)}, y^{(1)}), \ldots, (x^{(N)}, y^{(N)})\}$

- Likelihood - conditioned on parameters ϕ, μ^s, μ^{ns}

$$\prod_{i=1}^{N} P(\mathbf{x}^{(i)}, y^{(i)})$$
$$= \prod_{i=1}^{N} P(\mathbf{x}^{(i)}|y^{(i)}) P(y^{(i)})$$
$$= \underbrace{\boxed{\left(\prod_{i:y^{(i)}=1} P(\mathbf{x}^{(i)}|y^{(i)}) P(y^{(i)})\right)}}_{\text{Spam}} \underbrace{\left(\prod_{i:y^{(i)}=0} P(\mathbf{x}^{(i)}|y^{(i)}) P(y^{(i)})\right)}_{\text{Non-spam}}$$

Naive Bayes Spam classifier

- Likelihood - spam

$$\left(\prod_{i:y^{(i)}=1} \underline{P(\mathbf{x}^{(i)}|y^{(i)})}\,\underline{P(y^{(i)})}\right)$$

$$x_k^{(i)}$$

i-th mail

k-th word

- Naive Bayes assumption:

$$P(\text{spam}) = Bernoulli(\phi)$$
$$P(\text{word}|\text{spam}) = Multinomial(\mu_1^s, \ldots, \mu_M^s)$$

$$\underline{P(\mathbf{x}^{(i)}|y^{(i)}=1)} = \prod_{k=1}^{\text{len}(\mathbf{x}^{(i)})} P(x_k^{(i)}|y^{(i)}=1)$$
$$= \prod_{k=1}^{\text{len}(\mathbf{x}^{(i)})} \prod_{j=1}^{M} (\mu_j^s)^{\mathbb{I}(x_k^{(i)}=\text{``}j\text{''th word})}$$

$$\underline{P(y^{(i)}=1)} = \phi$$

Naive Bayes Spam classifier

- Likelihood - spam (cont')

$$\left(\prod_{i:y^{(i)}=1} P\left(\mathbf{x}^{(i)} \mid y^{(i)}\right) P\left(y^{(i)}\right)\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\text{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)}=\text{“}j\text{” th word}\right)} \phi\right)$$

Naive Bayes Spam classifier

- Likelihood - spam (cont')

$$\left(\prod_{i:y^{(i)}=1} P\left(\mathbf{x}^{(i)} \mid y^{(i)}\right) P\left(y^{(i)}\right)\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\operatorname{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)}=\text{``}j\text{''th word }\right)} \phi\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\operatorname{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)}=\text{``}j\text{''th word }\right)}\right) \left(\prod_{i:y^{(i)}=1}^{N} \phi\right)$$

Naive Bayes Spam classifier

- Likelihood - spam (cont')

$$\left(\prod_{i:y^{(i)}=1} P\left(\mathbf{x}^{(i)} \mid y^{(i)}\right) P\left(y^{(i)}\right)\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\mathrm{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)}=\text{“}j\text{” th word}\right)} \phi\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\mathrm{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)}=\text{“}j\text{” th word}\right)}\right) \left(\prod_{i:y^{(i)}=1}^{N} \phi\right)$$

$$= \left(\prod_{j=1}^{M} \left(\mu_j^s\right)^{\sum_{i:y^{(i)}=1}^{N} \sum_{k=1}^{\mathrm{len}\left(x^{(i)}\right)} \mathbb{I}\left(x_k^{(i)}=\text{“}j\text{” th word}\right)}\right) \left(\prod_{i:y^{(i)}=1}^{N} \phi\right)$$

Naive Bayes Spam classifier

- Likelihood - spam (cont')

$$\left(\prod_{i:y^{(i)}=1} P\left(\mathbf{x}^{(i)} \mid y^{(i)}\right) P\left(y^{(i)}\right)\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\mathrm{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)} = \text{“}j\text{” th word }\right)} \phi\right)$$

$$= \left(\prod_{i:y^{(i)}=1}^{N} \prod_{k=1}^{\mathrm{len}\left(x^{(i)}\right)} \prod_{j=1}^{M} \left(\mu_j^s\right)^{\mathbb{I}\left(x_k^{(i)} = \text{“}j\text{” th word }\right)}\right) \left(\prod_{i:y^{(i)}=1}^{N} \phi\right)$$

$$= \left(\prod_{j=1}^{M} \left(\mu_j^s\right)^{\sum_{i:y^{(i)}=1}^{N} \sum_{k=1}^{\mathrm{len}\left(x^{(i)}\right)} \mathbb{I}\left(x_k^{(i)} = \text{“}j\text{” th word }\right)}\right) \left(\prod_{i:y^{(i)}=1}^{N} \phi\right)$$

$$= \left(\prod_{j=1}^{M} \left(\mu_j^s\right)^{N_j^{\mathrm{spam}}}\right) \phi^{N^{\mathrm{spam}}}$$

Definition:

N^{spam}: total # examples for spam

N_j^{spam}: total # of word j from the entire spam emails

Naive Bayes Spam classifier

- Likelihood - non-spam

$$\left(\prod_{i:y^{(i)}=1} P(\mathbf{x}^{(i)}|y^{(i)})P(y^{(i)})\right)\left(\prod_{i:y^{(i)}=0} \underline{P(\mathbf{x}^{(i)}|y^{(i)})}\,\underline{P(y^{(i)})}\right)$$

- Similarly for non-spam mails,

$$P(\text{spam}) = Bernoulli(\phi)$$

$$P(\text{word}|\text{nonspam}) = Multinomial(\mu_1^{ns}, \ldots, \mu_M^{ns})$$

$$P(\mathbf{x}^{(i)}|y^{(i)}=0) = \prod_{k=1}^{\text{len}(\mathbf{x}^{(i)})} P(x_k^{(i)}|y^{(i)}=0)$$

$$= \prod_{k=1}^{\text{len}(\mathbf{x}^{(i)})} \prod_{j=1}^{M} (\mu_j^{ns})^{I(x_k^{(i)}=\text{“}j\text{” th word})}$$

$$P(y^{(i)}=0) = 1-\phi$$

Maximum likelihood estimation

- Putting together:

$$\prod_{i=1}^{N} P(\mathbf{x}^{(i)}, y^{(i)})$$

$$= \left(\prod_{i:y^{(i)}=1} P(\mathbf{x}^{(i)}|y^{(i)}) P(y^{(i)}) \right) \left(\prod_{i:y^{(i)}=0} P(\mathbf{x}^{(i)}|y^{(i)}) P(y^{(i)}) \right)$$

Maximum likelihood estimation

- Putting together:

$$\prod_{i=1}^{N} P(\mathbf{x}^{(i)}, y^{(i)})$$

$$= \left(\prod_{i:y^{(i)}=1} P(\mathbf{x}^{(i)}|y^{(i)})P(y^{(i)}) \right) \left(\prod_{i:y^{(i)}=0} P(\mathbf{x}^{(i)}|y^{(i)})P(y^{(i)}) \right)$$

$$= \left(\phi^{N^{spam}} \prod_{word\, j} \left(\mu_j^{s}\right)^{N_j^{spam}} \right) \left((1-\phi)^{N^{nonspam}} \prod_{word\, j} \left(\mu_j^{ns}\right)^{N_j^{nonspam}} \right)$$

Recall:
N^{spam}: total # examples for spam
$N^{nonspam}$: total # examples for non-spam

N_j^{spam}: total # word j from the entire spam emails
$N_j^{nonspam}$: total # word j from the entire nonspam emails

Maximum likelihood estimation

- Putting together:

$$
\begin{aligned}
&\prod_{i=1}^{N} P(\mathbf{x}^{(i)}, y^{(i)}) \\
= &\left(\prod_{i:y^{(i)}=1} P(\mathbf{x}^{(i)}|y^{(i)})P(y^{(i)})\right)\left(\prod_{i:y^{(i)}=0} P(\mathbf{x}^{(i)}|y^{(i)})P(y^{(i)})\right) \\
= &\left(\phi^{N^{spam}} \prod_{word\,j} \left(\mu_j^s\right)^{N_j^{spam}}\right)\left((1-\phi)^{N^{nonspam}} \prod_{word\,j} \left(\mu_j^{ns}\right)^{N_j^{nonspam}}\right)
\end{aligned}
$$

- Joint Log-likelihood

$$
\begin{aligned}
&\log P(\mathcal{D}) \\
= &\log \prod_{i=1}^{N} P(x^{(i)}, y^{(i)}) \\
= &N^{spam} \log \phi + \sum_{word\,j} N_j^{spam} \log \mu_j^s + N^{nonspam} \log(1-\phi) + \sum_{word\,j} N_j^{nonspam} \log \mu_j^{ns}
\end{aligned}
$$

Maximum likelihood estimation

- Joint Log-likelihood

$$
\begin{aligned}
& \log P(\mathcal{D}) \\
= \quad & \log \prod_{i=1}^{N} P(x^{(i)}, y^{(i)}) \\
= \quad & N^{spam} \log \phi + \sum_{word\, j} N_j^{spam} \log \mu_j^s + N^{nonspam} \log(1-\phi) + \sum_{word\, j} N_j^{nonspam} \log \mu_j^{ns}
\end{aligned}
$$

- Maximum-likelihood
  - Take the derivative of log-likelihood w.r.t. the parameters, and set it to zero.

Maximum likelihood estimation

- From $\frac{\partial l}{\partial \phi} = \frac{1}{\phi} N^{spam} - \frac{1}{1-\phi} N^{nonspam} = 0$

  We get $\phi = \frac{N^{spam}}{N^{spam} + N^{nonspam}}$

- Removing dependent variables:

$$\sum_{word\, j=1}^{M} N_j^{spam} \log \mu_j^s = \sum_{word\, j=1}^{M-1} N_j^{spam} \log \mu_j^s + N_M^{spam} \log(1 - \sum_{j=1}^{M-1} \mu_j^s)$$

$$\frac{\partial}{\partial \mu_j^s} \left(\sum_{word\, j=1}^{M} N_j^{spam} \log \mu_j^s \right) = \frac{N_j^{spam}}{\mu_j^s} - \frac{N_M^{spam}}{1 - \sum_{j=1}^{M-1} \mu_j^s} = 0$$

$$\text{s.t.} \quad \sum_j \mu_j^s = 1$$

$$\sum_j \mu_j^{ns} = 1$$

Maximum likelihood estimation

- From $\frac{\partial l}{\partial \phi} = \frac{1}{\phi} N^{spam} - \frac{1}{1-\phi} N^{nonspam} = 0$

  We get $\phi = \frac{N^{spam}}{N^{spam} + N^{nonspam}}$

- Removing dependent variables:

$$\sum_{word\, j=1}^{M} N_j^{spam} \log \mu_j^s = \sum_{word\, j=1}^{M-1} N_j^{spam} \log \mu_j^s + N_M^{spam} \log(1 - \sum_{j=1}^{M-1} \mu_j^s)$$

$$\frac{\partial}{\partial \mu_j^s} \left(\sum_{word\, j=1}^{M} N_j^{spam} \log \mu_j^s \right) = \frac{N_j^{spam}}{\mu_j^s} - \frac{N_M^{spam}}{1 - \sum_{j=1}^{M-1} \mu_j^s} = 0$$

$$\frac{N_j^{spam}}{\mu_j^s} = constant, \forall j$$

$$\mu_j^s = \frac{N_j^{spam}}{\sum_j N_j^{spam}}$$

Maximum likelihood estimation

- Summary:

$$P(spam) = \phi = \frac{N^{spam}}{N^{spam} + N^{nonspam}}$$

$$P(word = j|spam) = \mu_j^s = \frac{N_j^{spam}}{\sum_j N_j^{spam}}$$

$$P(word = j|non - spam) = \mu_j^{ns} = \frac{N_j^{nonspam}}{\sum_j N_j^{nonspam}}$$

Recall:
N^{spam}: total # examples for spam
$N^{nonspam}$: total # examples for non-spam

N_j^{spam}: total # word j from the entire spam emails
$N_j^{nonspam}$: total # word j from the entire nonspam emails

Laplace Smoothing

- Maximum likelihood is problematic when a specific word count is 0
  - Leads to probability of 0!
- Solution: Put "imaginary" counts for each word
  - prevent zero probability estimates (overfitting)!
  - E.g.: Adding "1" as imaginary count for each word

$$P(spam) = \phi = \frac{N^{spam}}{N^{spam} + N^{nonspam}}$$

$$P(word = j|spam) = \mu_j^s = \frac{N_j^{spam} \boxed{+1}}{\sum_j N_j^{spam} \boxed{+M}}$$

$$P(word = j|non - spam) = \mu_j^{ns} = \frac{N_j^{nonspam} \boxed{+1}}{\sum_j N_j^{nonspam} \boxed{+M}}$$

Outline

(grey: already covered)

- Probabilistic discriminative models
  - ✓ Logistic Regression
  - ✓ Softmax Regression
- Probabilistic generative models
  - ✓ Gaussian discriminant analysis
  - ✓ Naive Bayes
- Discriminant functions (non-probabilistic)
  - Fisher's linear discriminant
  - Perceptron learning algorithm

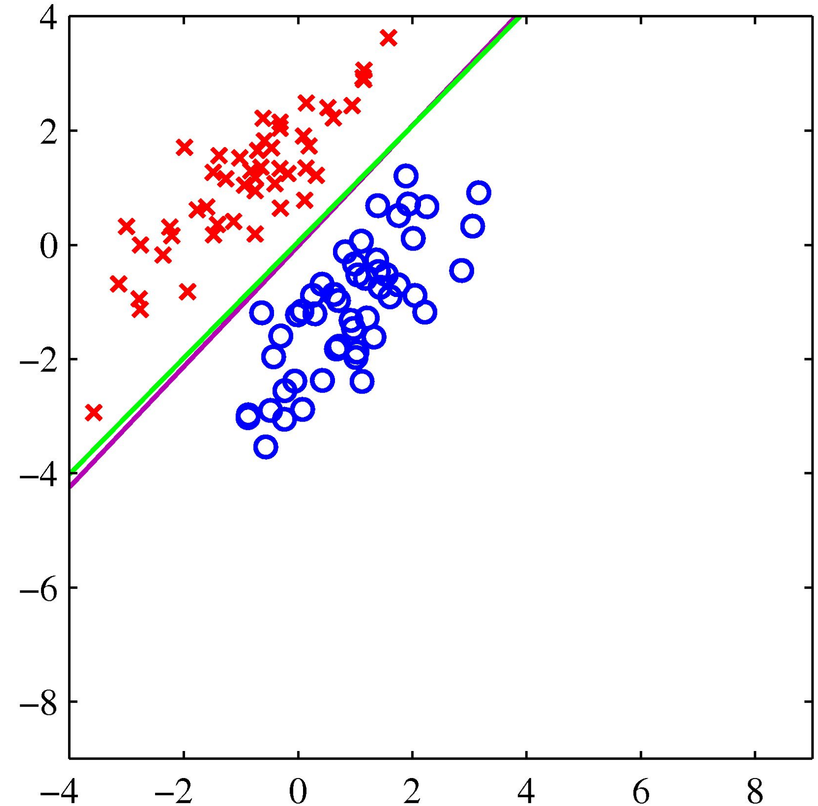
Discriminant Functions

Linear Discriminant functions: Discriminating two classes

- Specify a weight vector $\mathbf{w}$ and a bias w_0

$$h(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

- Assign $\mathbf{x}$ to C_1 if $h(x) \geq 0$ and to C_0 otherwise.
- Q: How to pick $\mathbf{w}$?



Linear Discriminant functions: Discriminating K>2 classes

- Instead each class C_k gets its own function

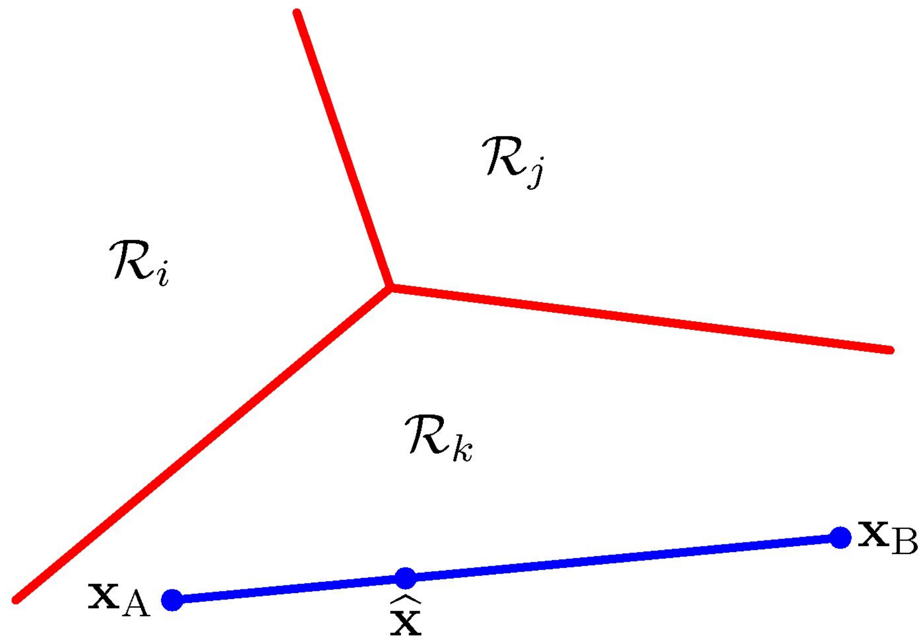
$$h_k(\mathbf{x}) = \mathbf{w}_k^\top \mathbf{x} + w_{k,0}$$

  - Assign $\mathbf{x}$ to C_k if

$$h_k(\mathbf{x}) > h_j(\mathbf{x}) \text{ for all } j \neq k$$

- The decision regions are convex polyhedra.

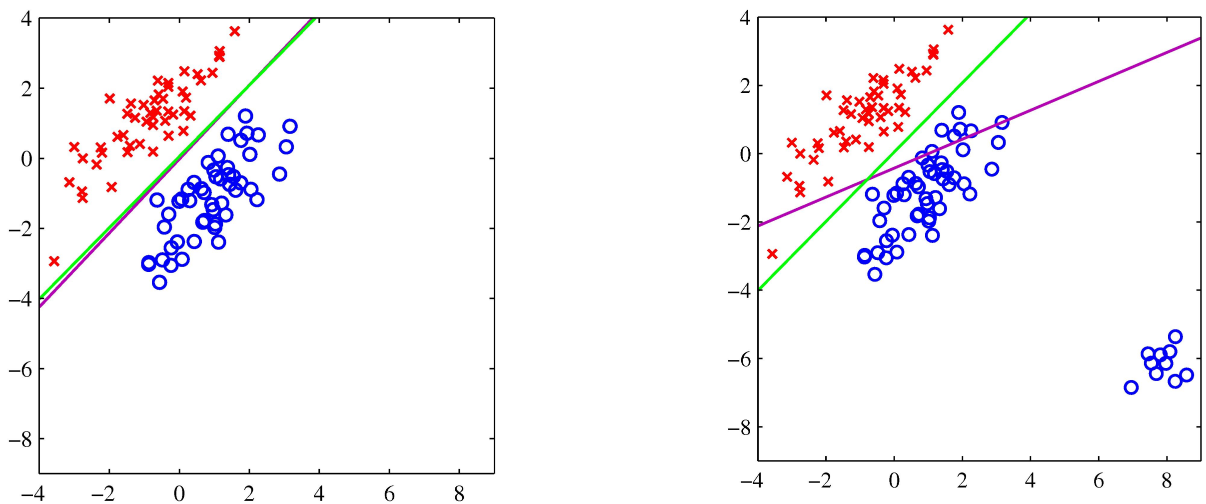
Decision Regions



- Decision regions are convex, with piecewise linear boundaries.

How do we set the weights w?

- How about $\mathbf{w}$ that minimizes squared error?
  - Label $\mathbf{y}$ versus linear prediction $h(\mathbf{w})$.
  - Least squares is too sensitive to outliers. (why?)



Read Bishop book

Learning Linear Discriminant Functions

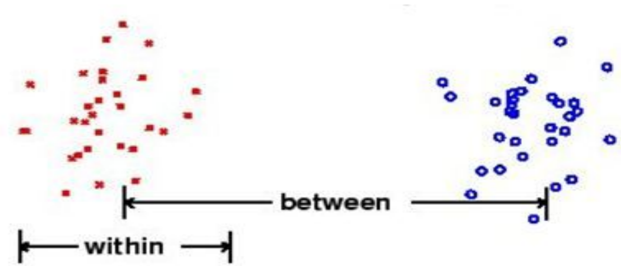
- Fisher's linear discriminant
- Perceptron learning algorithm

Fisher's Linear Discriminant

- Let's consider binary classification case.
- Use $\mathbf{w}$ to project $\mathbf{x}$ to one dimension.

$$\text{if} \quad \mathbf{w}^\top \mathbf{x} \geq -w_0 \quad \text{then} \quad C_1 \quad \text{else} \quad C_0$$

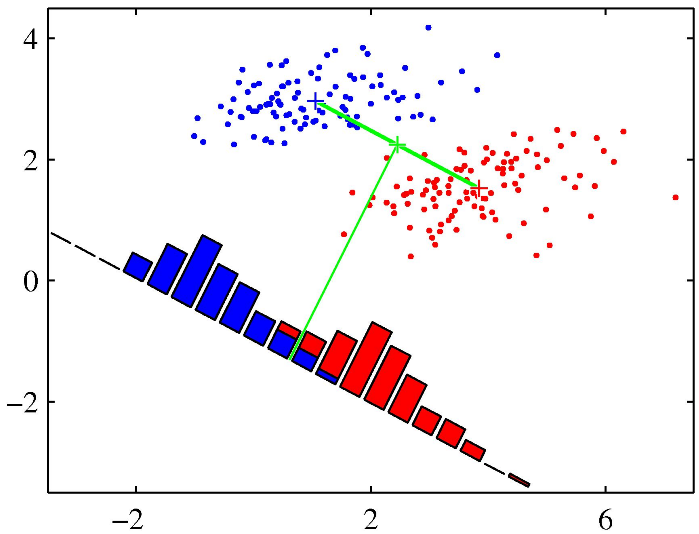
- Select w that best separates the classes.
- By "separating", the algorithm simultaneously
  - maximizes between-class (inter-class) variances
  - minimizes within-class (intra-class) variances

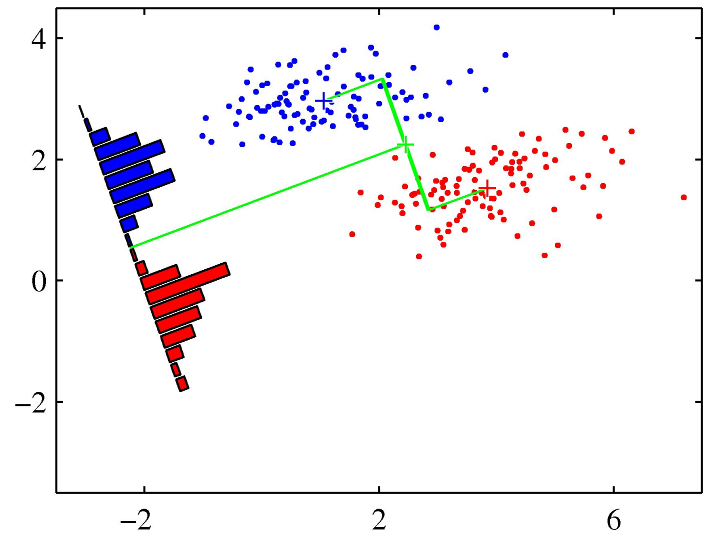


Read Bishop book

Fisher’s Linear Discriminant

- Maximizing separation alone is not enough.
  - Minimizing class variance is a big help.





Read Bishop book

Objective function

- We want to maximize the “distance between classes”

$$\underbrace{m_2}_{\text{Projected mean}} - m_1 \equiv \mathbf{w}^\top(\underbrace{\mathbf{m}_2}_{\text{Mean}} - \mathbf{m}_1) \qquad \text{where } \mathbf{m}_k = \frac{1}{N_k}\sum_{n \in C_k} \mathbf{x}_n$$

Read Bishop book

Objective function

- We want to maximize the "distance between classes"

$$\underbrace{m_2}_{\text{Projected mean}} - m_1 \equiv \mathbf{w}^\top(\underbrace{\mathbf{m}_2}_{\text{Mean}} - \mathbf{m}_1) \qquad \text{where } \mathbf{m}_k = \frac{1}{N_k}\sum_{n \in C_k} \mathbf{x}_n$$

- While minimizing the "distance within each class"

$$s_1^2 + s_2^2 \equiv \sum_{n \in C_1} (\mathbf{w}^\top \mathbf{x}_n - m_1)^2 + \sum_{n \in C_2} (\mathbf{w}^\top \mathbf{x}_n - m_2)^2$$

Read Bishop book

Objective function

- We want to maximize the "distance between classes"

$$\underline{m_2} - m_1 \equiv \mathbf{w}^\top(\underline{\mathbf{m}_2} - \mathbf{m}_1) \qquad \text{where } \mathbf{m}_k = \frac{1}{N_k}\sum_{n \in C_k} \mathbf{x}_n$$

Projected mean Mean

- While minimizing the "distance within each class"

$$s_1^2 + s_2^2 \equiv \sum_{n \in C_1} (\mathbf{w}^\top \mathbf{x}_n - m_1)^2 + \sum_{n \in C_2} (\mathbf{w}^\top \mathbf{x}_n - m_2)^2$$

- Objective function: $J(\mathbf{w}) = \dfrac{(m_2 - m_1)^2}{s_1^2 + s_2^2}$

Read Bishop book

Derivation of objective

- Numerator: $m_2 - m_1 \equiv \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1)$

$$\|m_2 - m_1\|^2 = \mathbf{w}^\top \underbrace{(\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^\top}_{= S_B} \mathbf{w}$$

Read Bishop book

Derivation of objective

- Numerator: $m_2 - m_1 \equiv \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1)$

  $$\|m_2 - m_1\|^2 = \mathbf{w}^\top \underbrace{(\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^\top}_{= S_B} \mathbf{w}$$

- Denominator:
  - $s_k^2 = \sum_{n \in C_k} (\mathbf{w}^\top \mathbf{x}_n - m_k)^2$
    $= \sum_{n \in C_k} \mathbf{w}^\top (\mathbf{x}_n - \mathbf{m}_k)(\mathbf{x}_n - \mathbf{m}_k)^\top \mathbf{w}$
  - $s_1^2 + s_2^2 = \mathbf{w}^\top \left[\underbrace{\sum_{k=1,2} \sum_{n \in C_k} (\mathbf{x}_n - \mathbf{m}_k)(\mathbf{x}_n - \mathbf{m}_k)^\top}_{= S_W} \right] \mathbf{w}$

Read Bishop book

Derivation of objective

- Numerator: $m_2 - m_1 \equiv \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1)$

  $$\|m_2 - m_1\|^2 = \mathbf{w}^\top \underbrace{(\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^\top}_{= S_B} \mathbf{w}$$

- Denominator:
  - $s_k^2 = \sum_{n \in C_k} (\mathbf{w}^\top \mathbf{x}_n - m_k)^2$
    $= \sum_{n \in C_k} \mathbf{w}^\top (\mathbf{x}_n - \mathbf{m}_k)(\mathbf{x}_n - \mathbf{m}_k)^\top \mathbf{w}$
  - $s_1^2 + s_2^2 = \mathbf{w}^\top \left[\underbrace{\sum_{k=1,2} \sum_{n \in C_k} (\mathbf{x}_n - \mathbf{m}_k)(\mathbf{x}_n - \mathbf{m}_k)^\top}_{= S_W} \right] \mathbf{w}$

- After definition of terms, we get

  $$J(\mathbf{w}) = \frac{\mathbf{w}^\top S_B \mathbf{w}}{\mathbf{w}^\top S_W \mathbf{w}}$$

  - Solution: $\mathbf{w} \propto S_W^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$

Read Bishop book

Fisher’s Linear Discriminant Analysis: Pros and Cons

Pros:

- Simple and effective approach for classification.
- Can effectively handle correlations between features
- Minimal assumptions about the underlying data distribution.
- Easy to interpret and explain

Cons:

- Only suitable for two-class classification problems
- Can be sensitive to outliers and may produce suboptimal results when the data has noisy features/labels

The Perceptron

- A "generalized linear function"

$$h(\mathbf{x}) = f(\mathbf{w}^\top \phi(\mathbf{x}))$$

where

$$f(a) = \begin{cases} +1, & a \geq 0 \\ -1, & a < 0 \end{cases}$$

- Uses target code: *y*=+1 for C_1, *y*=-1 for C_2.
- This means that we always want:

$$\mathbf{w}^\top \phi(\mathbf{x}^{(n)}) y^{(n)} > 0$$

The Perceptron Criterion

- Only count errors from misclassified points:

$$E_P(\mathbf{w}) = - \sum_{\mathbf{x}^{(n)} \in \mathcal{M}} \mathbf{w}^\top \phi(\mathbf{x}^{(n)}) y^{(n)}$$

  – where $\mathcal{M}$ is the set of **misclassified** points.

- Stochastic gradient descent:
  – Update the weight vector according to the each misclassified sample (i.e., take gradient per sample):

$$\mathbf{w}^{(\tau+1)} = \mathbf{w}^{(\tau)} - \eta \nabla E_P(\mathbf{w}) = \mathbf{w}^{(\tau)} + \eta \phi(\mathbf{x}^{(n)}) y^{(n)}$$

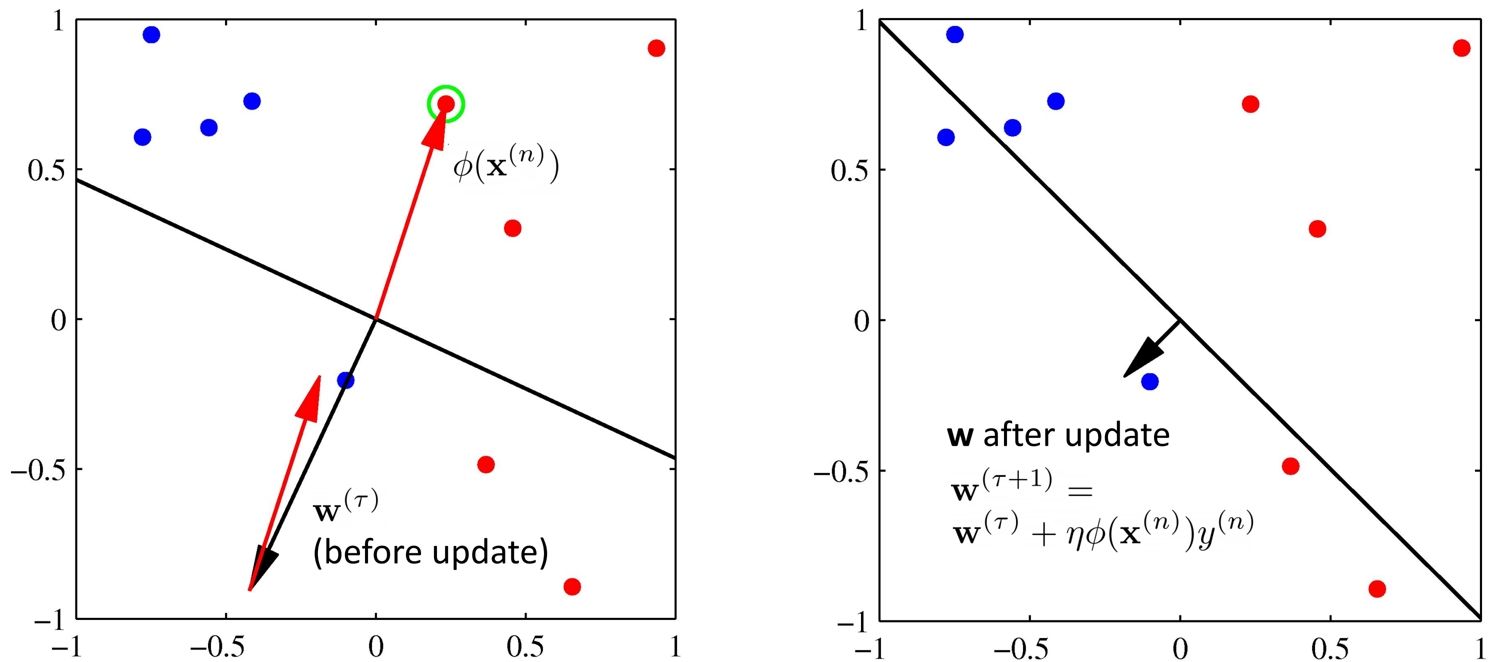
Note: update only for misclassified examples

Perceptron Learning (1)

red: positve (y= +1)
blue: negative (y= -1)

- If $\mathbf{x}^{(n)}$ is misclassified, add $\phi(\mathbf{x}^{(n)})$ into $\mathbf{w}$.

green circle: misclassified sample

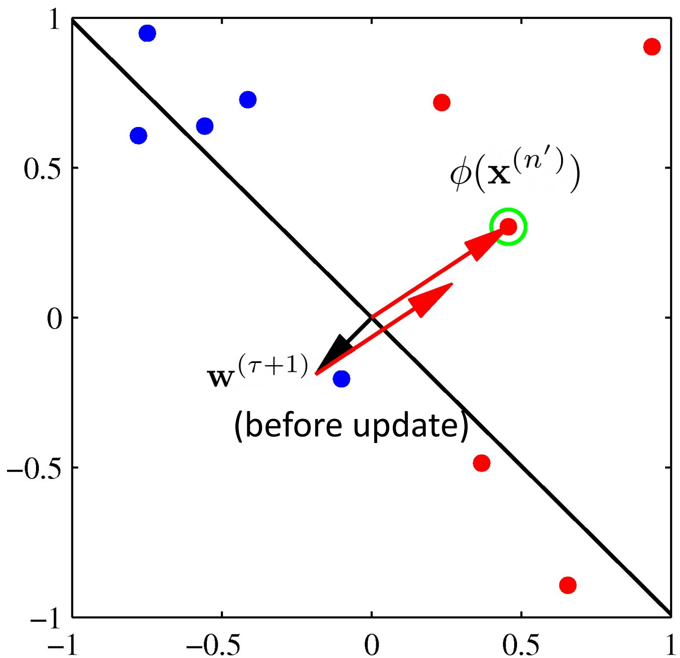


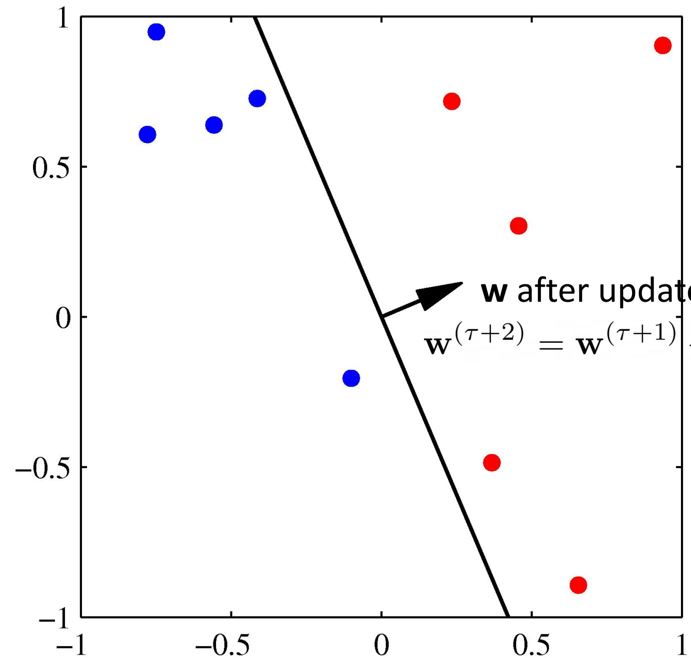
Perceptron Learning (2)

red: positve (y= +1)
blue: negative (y= -1)

- If $\mathbf{x}^{(n)}$ is misclassified, add $\phi(\mathbf{x}^{(n)})$ into $\mathbf{w}$.

green circle: misclassified sample





Perceptron Learning

- Perceptron Convergence Theorem (Block, 1962, and Novikoff, 1962):
  - If there exists an exact solution (i.e., if the training data is linearly separable)
  - then the learning algorithm will find it in a finite number of steps.
- Limitations of perceptron learning:
  - The convergence can be very slow.
  - If dataset is not linearly separable, it won’t converge.
  - Does not generalize well to K>2 classes.

Next class

- Kernel methods
- Remember to submit your project proposals!

Any feedback (about lecture, slide, homework, project, etc.)?

(via **anonymous** google form: https://forms.gle/99jeftYTaozJvCEF8)

Change Log of lecture slides:
https://docs.google.com/document/d/e/2PACX-1vRKx40eOJKACqrKWraio0AmlFS1_xBMINuWcc-jzpfo-ySj_gBuqTVdfHy8v4HDmqDJ3b3TvAW1FVuH/pub