Solution set 4:

Godunov's method

Exercise 4.1 **(a)** The method first proposed by Godunov can be outlined in 3 steps. (1) reconstruct a piecewise polynomial function $\widetilde{u}^n(x, t_{n+1})$ defined for all x, from the cell averages U_i^n. In the simplest case this is a piecewise constant function that takes the value U_i^n in the ith grid cell. (2) Evolve the equation exactly(or approximately) with this initial data to obtain $\widetilde{u}^n(x, t_{n+1})$ a time Δt later. (3) Average this function over each grid cell to obtain new cell averages $U_i^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_i} \tilde{u}^n(x, t_{n+1})$. The whole process is then repeated in the next step **(b)** The main benefit of the Godunov's methods is that by solving the Riemann problems correctly in each step, upon convergence we are guaranteed to get the entropy satisfying solution.

Exercise 4.2 **(a)** If we apply the Rankine-Hugoniot jump condition, we get a stationary shock. As readily seen from the sketch in figure 2, this shock will not satisfy the entropy condition. As a shock is not admissible, the entropy solution must consists of rarefaction waves. The slope of the characteristics is -1 and 1 respectively on each side of the initial shock, see the sketch in figure 3. The true solution to the burgers equation with the initial data supplied must then be

$$u(x,t) = \begin{cases} -1 & x \leq -t+1 \\ \frac{x}{t} & -t+1 < x \leq t+1 \\ 1 & x > t+1 \end{cases} \tag{1}$$

(b)(c)(d) See Matlab/Octave code attached at the end of the paper. Experiment with different values of l. **(e)** We notice that in case of a, $k_l = \frac{1}{2l}$, for any choice of $l \in \mathbb{N}$, we get the entropy solution to the problem, and that the sequence of solutions converges as $l \to \infty$. In the case of sequence b, $k_l = \frac{1}{2l+1}$, for any choice of $l \in \mathbb{N}$ we get the entropy violating stationary shock. The sequence of solutions produced as $l \to \infty$ is Cauchy, and we converge to a weak solution of the problem, but unfortunately in this case we do not converge to the entropy solution. In the final case $k_l = \frac{1}{l}$, we observe solutions that oscillate between the cases (a) and (b), the sequence of solution produced as $l \to \infty$ is thus not Cauchy, we do not converge. The purpose of this exercise was to illustrate the Lax-Wendroff theorem, if we can find some sequence of solutions that converges as $k_l, h_l \to 0$, we are guaranteed to arrive at a weak solution to the problem, but we are not guaranteed to solution to be the entropy solution.

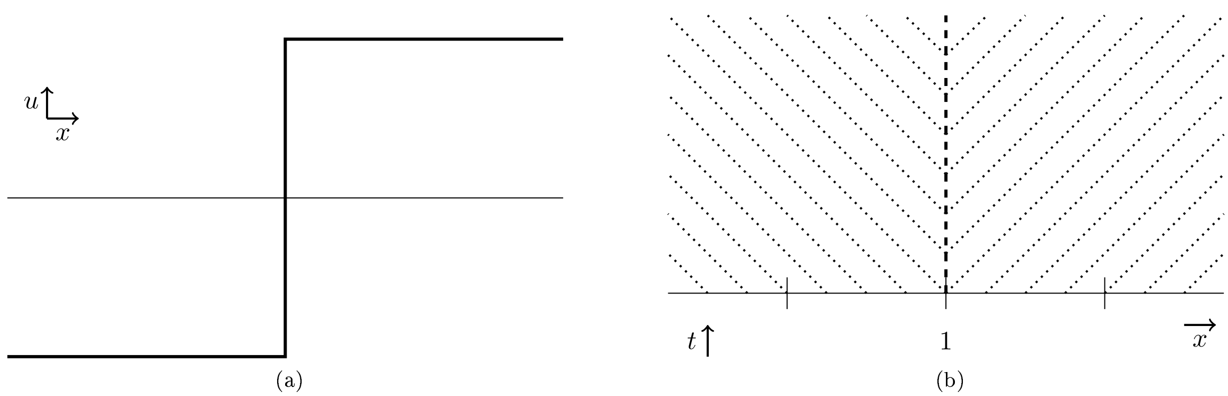

Figure 1: Exercise 1.3b. Initial data $u_0(x)$ Eq. 16 applied to the inviscid burgers equation. The two shocks move away from each other with equal speed and never merge. **(a)** Initial data. **(b)** Characteristics.

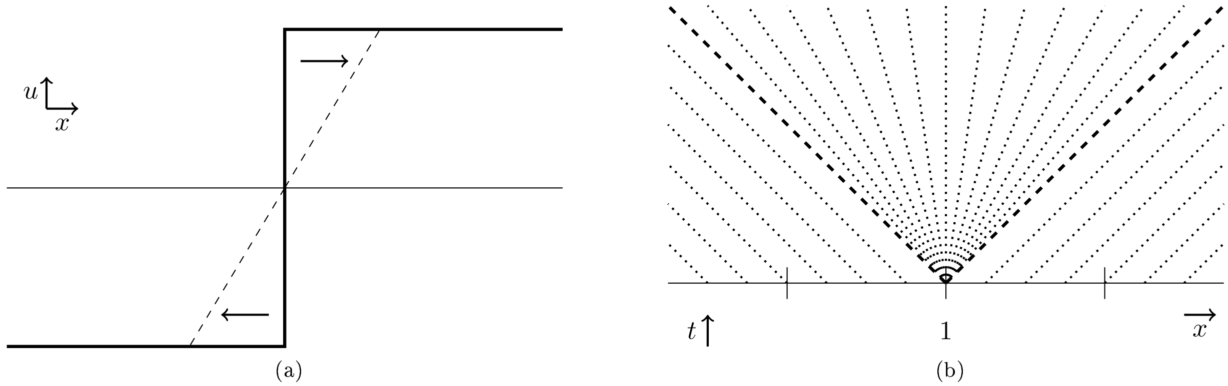

Figure 2: Exercise 1.3b. Initial data $u_0(x)$ Eq. 16 applied to the inviscid burgers equation. The two shocks move away from each other with equal speed and never merge. **(a)** Initial data. **(b)** Characteristics.

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% This script was written for EPFL MATH459, Numerical Methods for Conservation Laws
% and tested with Octave 3.6.4. Code for exercise 4.2.

% Function declarations
function F = flux(v,w)
    if v == w || (v^2-w^2)/(v-w) == 0;
        F = 0.5*v^2;
    elseif (v^2-w^2)/(v-w) > 0;
        F = 0.5*v^2;
    else (v^2-w^2)/(v-w) < 0;
        F = 0.5*w^2;
    end
end

% Sequence
l = 200;
k = 1/(2*l);
% Case A % k = 1/(2*l+1);
% Case B % k = 1/l;
% Case C

% Discretization
h = 2*k;
X = 0:h:2;
nX = numel(X);
T = 0:k:0.25;
nT = numel(T);

% Initial condition
U = zeros(nX,1);
U(find(X<1),1) = -1;
U(find(X==1),1) = 0;
U(find(X>1),1) = 1;

% Solve and visualize
for i = 1:nT
    Ut = U;
    for j = 2:nX-1
        U(j) = Ut(j) - k/h* (flux(Ut(j),Ut(j+1)) - flux(Ut(j-1),Ut(j)));
    end
    plot(X,U,'-b');grid on
    ylim([-1.25 1.25]);xlim([0 2]);
    set(gca,'XTick',-2:0.5:2);
    set(gca,'YTick',-1:0.5:1);
    drawnow
end
```