Math 6395: Hyperbolic Conservation Laws and Numerical Methods

Course structure

- Hyperbolic Conservation Laws are a type of PDEs arise from physics: fluid/gas dynamics.
- Distintive solution structures
  - Other types of PDEs we met before, e.g. the time independent elliptic equations, or time dependent parabolic equations. Solutions are smooth.
  - The solutions of hyperbolic problems have a distinctive feature: formation of discontinuities (shocks). Such feature together with others factors attract many attentions among the scientific and engineering community, leading to many interesting research work in the PDE level, and in the design of numerical schemes in the past 30-40 years or so.
- Structure of the course:
  - mathematical theory
  - numerical methods
    - methods for linear equations: local truncation error, accuracy, stability, convergence
    - methods for nonlinear scalar equations
      - low order schemes with good properties corresponding to those in the PDE theory
      - extension to high order schemes in the discontinuous Galerkin framework.

1 Introduction

Hyperbolic conservation laws:

$$\vec{u}_t + \nabla_x \cdot \vec{f}(\vec{u}) = 0, \quad (x,t) \in \mathcal{R}^d \times \mathcal{R}^+ \tag{1}$$

$$\vec{u}(\vec{x}, t=0) = \vec{u}_0(\vec{x}) \tag{2}$$

- $\vec{u}$: conserved quantities, $\mathcal{R}^d \times \mathcal{R}^+ \to \mathcal{R}^m$, with d being the dimension of the problem, and m being the number of components of $\vec{u}$.
- $\vec{f}(u)$: flux function, $\mathcal{R}^m \to \mathcal{R}^m$
- For example
  - The case of $m = 1$ and $d = 1$ corresponds to 1-D scalar conservation laws. In traffic flow problems, $\rho_t + f(\rho)_x = 0$ is of the form of (1). ρ is the density of the traffic, and $f(\rho) = \rho v(\rho)$ is the flux pass a location (mass multiple by the velocity of the traffic).
  - $m = 2 + d$. In gas/fluid dynamics, the compressible Euler equations in one dimension reads

$$\vec{u}_t + \nabla_x \cdot \vec{f}(\vec{u}) = 0, \tag{3}$$

    where $\vec{u} = (\rho, \rho v, E)^T$ are the conserved quantities (mass, momentum and energy) and $\vec{f}(\vec{u}) = (\rho v, \rho v^2 + p, (E+p)v)^T$ are the corresponding flux functions. Here ρ is the density, v is the velocity, ρv is the momentum, E is the energy and p is the pressure given as a function of other state variables, known as "equation of state".
- Hyperbolic equations are a type of time dependent linear/nonlinear PDEs arise from science and engineering. Most of the time, it is not possible to write down the exact solutions of the equations. People heavily rely on numerics to access the solution structures in future times.

Definition 1.1. (Hyperbolic for 1-D problem) The 1-D system (1) is **hyperbolic** if the $m \times m$ Jacobian matrix

$$J_{\vec{f}}(\vec{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ & \cdots & \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix}$$

of the flux function has the following property: For each value of $\vec{u}$ the eigenvalues of $J_{\vec{f}}(\vec{u})$ are real, and the matrix is diagonalizable, i.e. there is a complete set of m linearly independent eigenvectors.

Definition 1.2. (Hyperbolic for 2-D problem) The 2-D system in the form of

$$\vec{u}_t + \vec{f}(\vec{u})_x + \vec{g}(\vec{u})_y = 0 \tag{4}$$

with $\vec{u}$: $\mathcal{R}^2 \times \mathcal{R}^+ \to \mathcal{R}^m$, $\vec{f}, \vec{g} : \mathcal{R}^m \to \mathcal{R}^m$ is **hyperbolic** if any linear combination of $m \times m$ matrix $\alpha J_{\vec{f}}(\vec{u}) + \beta J_{\vec{g}}(\vec{u})$ of the flux Jacobians has real eigenvalues and the matrix is diagonalizable, i.e. there is a complete set of m linearly independent eigenvectors.

Remark 1.3. If the Jacobian matrices have distinct real eigenvalues, it follows that it's diagonalizable. In this case the system is called **strictly hyperbolic**. If the eigenvalues are real but not distinct, the Jacobian matrix may not be diagonalizable. Such a system is called **weakly hyperbolic**. If the Jacobian matrix is diagonalizable but has complex eigenvalues, the system is not hyperbolic.

- Typical features of solutions for nonlinear hyperbolic systems.
  - Development of discontinuities or shocks in future solutions, even with smooth initial data. Because of this, solutions in classical sense could fail to exist.
  - If the solutions are defined in the weak distribution sense, there could be more than one weak solutions
  - Entropy solution is the uniquely exist physically relevant solution among weak solutions. In the PDE level, questions to ask/answer are:
    * How to define entropies solutions? (vanishing viscosity method, entropy inequalities)
    * What are appropriate spaces (norms) for entropy solutions? (BV norm, L^1 and L^∞)
    * What are criteria for selecting entropy solutions among weak solutions? (entropy conditions)
- Because of the above mentioned difficulties at the PDE level, numerical challenges in designing schemes for hyperbolic type PDEs are
  - how to design schemes with "good" accuracy and stability? ("good": high resolution/high order accuracy, stability in the sense of total variational diminishing or maximum principle preserving)
  - how to capture shocks without numerical artifacts such as oscillations?
  - how to know (or prove) if the numerical scheme approximates weak solutions, or the unique entropy solution among weak solutions?

2 The Derivation of Conservation Laws

We can model the traffic flow as a 1-D scalar conservation law. Denote density of the traffic as $\rho(x, t)$, then the evolution of the traffic flow density can be described by the following

$$\frac{\partial}{\partial t} \int_a^b \rho(x, t) dx = f(\rho(a, t)) - f(\rho(b, t)), \tag{5}$$

for any $a \leq b$. Equation (5): the rate of change of the density over an interval $[a, b]$ is due to the flux flowing into the region from the left boundary $x = a$, and the flux flowing out of the region from the right boundary $x = b$. The flux past a given point is give by

$$f(\rho(x, t)) = \rho(x, t) v(x, t),$$

where $v(x,t)$ is the velocity of the traffic flow, which could be dependent on ρ. For example, in a simple linear model, $v(\rho) = v_{max}(1 - \rho/\rho_{max})$. At zero density (empty road), the velocity is v_{max}; but decreases to zero as ρ approaches the maximum capacity of the road ρ_{max} (traffic jam).

If we assume $f(\rho(x,t))_x$ exists, then

$$\frac{\partial}{\partial t}\int_a^b \rho(x,t)dx = -\int_a^b f(\rho(x,t))_x dx \tag{6}$$

Hence

$$\int_a^b (\frac{\partial}{\partial t}\rho(x,t) + f(\rho(x,t))_x)dx = 0 \tag{7}$$

Since the above equation is true for any $a \le b$, then in terms of the differential equation, we have

$$\rho_t + f(\rho)_x = 0. \tag{8}$$

Remark 2.1. Note that

$$Equation(5) \overset{smooth f(\rho)}{\Longrightarrow} Equation(8); \qquad Equation(8) \Rightarrow Equation(5)$$

The integral equation does not require smoothness conditions on the solution.

Remark 2.2. System of conservation laws such as Euler equation in gas/fluid dynamics can be modeled in a similar (but more involved) way concerning the conservation of mass, momentum and energy. See Chap. 5 of the book.

3 Scalar Conservation Laws

3.1 Linear case

$$u_t + au_x = 0, \quad u(x, t = 0) = u_0(x), \quad -\infty < x < \infty, \tag{9}$$

- characteristics in $x - t$ plane in deriving the exact solution.

  Along characteristics $\frac{dx}{dt} = a$, the solution stays constant

  $$\frac{du}{dt} = u_t + u_x x_t = u_t + au_x = 0. \tag{10}$$

  Hence the exact solution is

  $$u(x,t) = u_0(x - at).$$

- finite speed of propagation (vs. infinite speed of propagation for parabolic equations)
- If $u_0(x)$ is a smooth function, then $u(x,t)$ is equally smooth in space and in time. If $u_0(x)$ is not smooth (the concept of classical solutions for differential equations fails), the solution of $u_0(x - at)$ satisfies the integral form of the equation, and is a solution in the weak/distribution sense.

3.2 Nonlinear case

$$u_t + f(u)_x = 0 \tag{11}$$

e.g. Burgers' equation $f(u) = u^2/2$.

3.2.1 Characteristics and Shock Formation

Along characteristics $\frac{dx}{dt} = f'(u) = u$, the solution stays constant

$$\frac{du}{dt} = u_t + u_x x_t = u_t + f'(u)u_x = 0. \tag{12}$$

Hence the exact solution $u(x,t) = u_0(x^*)$, where x^* is the root of the characteristic equation

$$\frac{x - x^*}{t - 0} = f'(u_0(x^*)). \tag{13}$$

Example 3.1. Consider the Burgers' equation with initial condition $u_0(x) = \sin(x)$.

Plot for crossing of characteristics.

Shock formation time. Let $\frac{d}{dt} \doteq \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ be the differentiation along characteristics.

$$\frac{du}{dt} = 0.$$

Let $v = \frac{\partial u}{\partial x}$,

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + f'(u)\frac{\partial v}{\partial x} = \frac{\partial^2 u}{\partial t \partial x} + u\frac{\partial^2 u}{\partial x^2}.$$

Taking $x-$derivative on the Burgers' equation we have, $\frac{\partial^2 u}{\partial t \partial x} = -\frac{\partial^2 f(u)}{\partial x^2}$,

$$\frac{dv}{dt} = -\frac{\partial^2 f(u)}{\partial x^2} + u\frac{\partial^2 u}{\partial x^2} = -(u_x)^2 = -v^2.$$

Thus

$$\begin{aligned} d(\frac{1}{v}) &= dt \\ \frac{1}{v(T)} - \frac{1}{v(0)} &= T \\ v(T) &= \frac{v(0)}{Tv(0)+1} \end{aligned} \tag{14}$$

$v(T)$ goes to infinity when $T = -\frac{1}{v(0)}$. Shock forms when $T = \min_{x:u_0'(x)<0}\{-\frac{1}{u_0'(x)}\}$.

3.2.2 Weak Solutions

Let $\phi(x,t) \in C_0^1(\mathcal{R} \times \mathcal{R}^+)$ be a test function, where C_0^1 is the space of function that are continuously differentiable with compact support. If we multiply the equation (11) by the test function ϕ and integrate over space and time, we obtain

$$\int_0^\infty \int_{-\infty}^\infty \phi(u_t + f(u)_x)dxdt = 0. \tag{15}$$

Performing integration by parts yields

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u))dxdt + \int_{-\infty}^\infty \phi(x,0)u(x,0)dx = 0. \tag{16}$$

Definition 3.2. (Weak solution) The function $u(x,t)$ is called a weak solution of the conservation laws if equation (16) holds for all test function $\phi \in C_0^1(\mathcal{R} \times \mathcal{R}^+)$.

Remark 3.3. Mathematically the definition of weak solution in the integral form above is equivalent to the solution for the integral form of the equation over any choices of spatial and time interval. The weak form above could be easier to work with sometimes.

Special cases. If $u(x,t)$ is continuously differentiable, then $u(x,t)$ is a classical solution and a weak solution. In the $x-t$ plane, if $u(x,t) \in C^1(\mathcal{R} \times \mathcal{R}^+)$ and satisfies the strong/differential form of equation (11) except on a curve $x(t)$, then

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\int_a^b u(x,t)dx + f(u(b,t)) - f(u(a,t)) \\
&= \frac{\partial}{\partial t}\left(\int_a^{x(t)^-} u(x,t)dx + \int_{x(t)^+}^b u(x,t)dx\right) + f(u(b,t)) - f(u(a,t)) \\
&= \int_a^{x(t)^-} u_t(x,t)dx + u(x(t)^-,t)x'(t) + \int_{x(t)^+}^b u_t(x,t)dx - u(x(t)^+,t)x'(t) + f(u(b,t)) - f(u(a,t)) \\
&= \int_a^{x(t)^-} u_t(x,t)dx + f(u(x(t)^-,t)) - f(u(a,t)) + \int_{x(t)^+}^b u_t(x,t)dx + f(u(b,t)) - f(u(x(t)^+,t)) \\
&\quad -(f(u(x(t)^-,t)) - f(u(x(t)^+,t))) + u(x(t)^-,t)x'(t) - u(x(t)^+,t)x'(t) \\
&= -(f(u(x(t)^-,t)) - f(u(x(t)^+,t))) + u(x(t)^-,t)x'(t) - u(x(t)^+,t)x'(t) \qquad (17)
\end{aligned}$$

Let $[|f(u)|] \doteq -(f(u(x(t)^-,t)) + f(u(x(t)^+,t)))$ and $[|u|] \doteq -(u(x(t)^-,t) + u(x(t)^+,t))$ be the jump of f and u across discontinuity. From equation (17), we have

$$s \doteq x'(t) = \frac{[|f(u)|]}{[|u|]}. \qquad (18)$$

This is the so called "Rankine-Hugoniot" jump condition. $s \doteq x'(t)$ is the shock speed.

Proposition 3.4. A function $u(x,t)$ is piecewise smooth and satisfies the PDE strong whenever $u \in C^1$. If the function satisfies the Rankine-Hugoniot jump condition along the discontinuity curve, then $u(x,t)$ is a weak solution of equation (11).

Remark 3.5. For curves on which the solution is continuous but not differentiable, equation (17) is satisfied. The solution is a weak solution.

3.2.3 Riemann Problem and Not-uniqueness of Weak Solutions

Example 3.6. Consider Burgers' equation with initial condition consist of two constant states u_l and u_r,

$$u_0(x) = \begin{cases} u_l = 1, & x < 0 \\ u_r = -1, & x > 0 \end{cases} \qquad (19)$$

Drawing characteristics: colliding into each other (formation of shocks). By Prop. 3.4, $u(x,t) = u_0(x)$ is the weak solution of the Burgers' equation.

Example 3.7. Consider Burgers' equation with initial condition consist of two constant states u_l and u_r,

$$u_0(x) = \begin{cases} u_l = -1, & x < 0 \\ u_r = 1, & x > 0 \end{cases} \qquad (20)$$

Drawing characteristics: spreading out. A weak solution is the rarefaction wave

$$u(x,t) = \begin{cases} -1, & x < -t \\ \frac{x}{t}, & -t < x < t \\ 1, & x > t \end{cases} \qquad (21)$$

Yet by Prop. 3.4, $u(x,t) = u_0(x)$ is also a weak solution of the Burgers' equation. In fact there are infinitely many weak solutions for this problem.

Remark 3.8. (from wiki) A Riemann problem, named after Bernhard Riemann, consists of a conservation law together with piecewise constant data having a single discontinuity. The Riemann problem is very useful for the understanding of hyperbolic partial differential equations like the Euler equations because all properties, such as shocks and rarefaction waves, appear as characteristics in the solution. It also gives an exact solution to some complex nonlinear equations, such as the Euler equations.

In numerical analysis, Riemann problems appear in a natural way in finite volume methods for the solution of equation of conservation laws due to the discreteness of the grid. For that it is widely used in computational fluid dynamics and in MHD simulations. In these fields Riemann problems are calculated using Riemann solvers.

3.2.4 Entropy Solution

We have seen that the classical/strong solution may fail to exist for hyperbolic equations; but weak solutions may not be unique. Additional criteria are needed to pick out the unique physically relevant solution, i.e. the entropy solution, among weak solutions. In physical models, the balance laws only come with some physical viscosity. For example, in the traffic flow modeling, the "viscosity" takes the form of slow response of drivers and automobiles; in the fluid dynamics, the viscosity corresponds to the informal notion of "thickness". For example, honey has a higher viscosity than water. Conservation laws with viscous terms provide more physically relevant models.

Definition 3.9. (Vanishing viscosity method for the entropy solution) Consider the viscous equation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \tag{22}$$

An entropy solution of (11) is the limit (a.e.) of u^ε of equation (22) when $\varepsilon \to 0$.

There are several issues to resolve to make sense of the above definition. For example, the solution for the viscous equation (22) *has to exist and is unique for each ε; there has to exist a limit of u^ε as $\varepsilon \to 0$; such limit should be one of the weak solutions (in fact the physically relevant entropy solution) for equation* (11)*. These questions have been answered in the PDE theory, but they are out of the scope of this course.*

Proposition 3.10. (Equivalent definition for entropy solution) A weak solution of (11) is an entropy solution if for all convex entropy function $U(u)$ with $U''(u) \geq 0$ and the associated entropy flux function $F(u)$ with $F'(u) = U'(u)f'(u)$, we have

$$U(u)_t + F(u)_x \leq 0, \tag{23}$$

in the distribution sense. That is for all $\phi \geq 0$, $\phi \in C_0^1(\mathcal{R} \times \mathcal{R}^+)$, we have

$$-\int_0^\infty \int_{-\infty}^\infty (\phi_t U(u) + \phi_x F(u))dxdt - \int_{-\infty}^\infty \phi(x,0)U(u(x,0))dx \leq 0. \tag{24}$$

Proof: see Section 3.8.1. of the textbook.

Remark 3.11. If a solution is continuously differentiable (classical solution), then it is an entropy solution. The entropy inequality becomes an equality. If a solution $u(x,t) \in C^1(\mathcal{R} \times \mathcal{R}^+)$ and satisfies the strong/differential form of equation (11) except on a curve $x(t)$, then by using similar procedure in deriving the Rankine-Hugoniot condition, we have the following inequality for the entropy solution

$$-s[|U|] + [|F(U)|] \leq 0, \quad \text{for any entropy-entropy flux pairs} \tag{25}$$

The above definitions for entropy solutions are mathematically rigorous; however, it will be difficult to directly apply these definitions to select the entropy solution among weak solutions (e.g. for the example of Riemann problem above). Below, we look for simpler and more practical entropy conditions that one can directly apply in examples.

Proposition 3.12. (Oleinik entropy condition) A discontinuity propagating with speed $s = \frac{[|f(u)|]}{[|u|]}$ given by the Rankine-Hugoniot jump condition satisfies the Oleinik entropy condition if for all u between u_l and u_r,

$$\frac{f(u) - f(u_l)}{u - u_l} \geq s \geq \frac{f(u) - f(u_r)}{u - u_r} \tag{26}$$

where u_l and u_r are left and right state along the discontinuity respectively.

Proposition 3.13. (Lax entropy condition) A discontinuity propagating with speed s given by the Rankine-Hugoniot jump condition satisfies the Lax entropy condition if

$$f'(u_l) > s > f'(u_r), \tag{27}$$

where u_l and u_r are left and right state along the discontinuity respectively.

Remark 3.14. One can see that Oleinik entropy condition implies Lax entropy condition; the converse does not hold. Lax entropy condition is a necessary but not sufficient condition to single out the entropy condition. However, if $f(u)$ is strictly convex with $f''(u) > 0$ (or strictly concave with $f''(u) < 0$), then the Lax entropy condition is equivalent to the Oleinik entropy condition, and is sufficient to single out the entropy condition. In fact, under the convexity (or concavity) assumption on $f(u)$, the Lax entropy condition is reduced to $f'(u_l) > f'(u_r)$, i.e., the characteristics propagating into the shock, rather than diverging from the shock.

Entropy solutions of the scalar hyperbolic equation (11) are proved to exist and are unique in the class of $BV \cap L^1 \cap L^\infty$. Here BV stands for bounded variation. A function is of bounded variation on a given interval $[a, b]$, if

$$V_a^b(f) \doteq sup \sum_i |f(x_i) - f(x_{i+1})| < \infty, \quad \{x_i\}_i \quad \text{is any partition of} [a, b].$$

If $f(u)$ is continuously differentiable, then

$$V_a^b(f) \doteq \int_a^b |f'(x)| dx.$$

Proposition 3.15. (BV) If $u_0(x)$ is a function of locally bounded variation on $(-\infty, \infty)$, then for each $t > 0$, $u(\cdot, t)$ is also a function of locally bounded variation on $(-\infty, \infty)$, and

$$V_{-R}^R u(\cdot, t) \leq TV_{-R-st}^{R+st} u_0(\cdot),$$

where $s = \max_x |f'(u)|$.

Proposition 3.16. (L^1 contraction property) If $u(x, t)$ and $v(x, t)$ are solutions of the scalar hyperbolic equation (11) with initial data $u_0(x)$ and $v_0(x)$ respectively, then

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0(\cdot) - v_0(\cdot)\|_{L^1}.$$

Specifically, consider $v \equiv 0$, then $\|u(\cdot, t)\|_{L^1} \leq \|u_0(\cdot)\|_{L^1}$.

Proposition 3.17. (L^∞ maximum principle) If $u(x, t)$ is a solution of the scalar hyperbolic equation (11) with initial condition $u_0(x)$, then

$$\max_x u(x, t) \leq \max_x u_0(x), \quad \min_x u(x, t) \geq \min_x u_0(x),$$