Homework #3 for MATH 6395

Due at the beginning of class, Oct. 16th, 2013

1. Consider Burgers' equation,
$$u_t + (u^2/2)_x = 0, \quad x \in [-\pi, \pi],$$
with the initial condition $u_0(x)$.

   Code the first order monotone scheme with Lax-Friedrichs flux for solving the Burgers' equation,
$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}), \tag{1}$$
where the Lax-Friedrich flux reads
$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(\bar{u}_j, \bar{u}_{j+1}) = \frac{1}{2}(f(\bar{u}_j) + f(\bar{u}_{j+1})) + \frac{\alpha}{2}(\bar{u}_j - \bar{u}_{j+1}), \tag{2}$$
with $\alpha = \max_u |f'(u)|$. The initial condition is
$$\bar{u}_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx, \quad \forall j.$$

   - Implement the scheme in Fortran/Matlab/your preferred computer language.
   - Test the code with a smooth initial data $u_0(x) = \sin(x)$ and periodic boundary condition, by putting down an error table for numerical solutions at $T = 0.2$ with $N = 10, 20, 40, 80$ for spatial meshes and $CFL = \frac{\Delta t}{\Delta x} = 0.5$. *Hint: the exact solution can be numerically obtained in the same way as in HW #1.*
   - Test the code with Riemann initial data with,
$$u_0(x) = \begin{cases} u_l = 1, & x \le 0, \\ u_r = -0.5, & x > 0, \end{cases}$$
by plotting the numerical solution with $N = 160$ equally spaced mesh, versus the exact solution at $T = 0.2$. *Hint: the exact solution can be obtained as in HW #2.*
   - Test the code with Riemann initial data with,
$$u_0(x) = \begin{cases} u_l = -0.5, & x \le 0, \\ u_r = 1.0, & x > 0, \end{cases}$$
by plotting the numerical solution with $N = 160$ equally spaced mesh, versus the exact solution at $T = 0.2$.