Homework #4 for MATH 6395

Due at the beginning of class, Oct. 30th, 2013

1. Consider linear advection equation,

$$u_t + u_x = 0, \quad x \in [-\pi, \pi]. \tag{1}$$

The third order conservative finite volume scheme with upwind flux for solving the equation reads,

$$\frac{d}{dt}\bar{u}_j(t) + \frac{1}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}) = 0, \tag{2}$$

where the upwind flux $\hat{f}_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}}$. A third order reconstruction of $u^-_{j+\frac{1}{2}}$ can be obtained by fitting a polynomial of degree 2, such that the reconstructed polynomial agrees with the given cell averages $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\}$. $u(x_{j+\frac{1}{2}})$ can be approximated with third order by

$$u^-_{j+\frac{1}{2}} = -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}. \tag{3}$$

The semi-discrete scheme (2) can be coupled with a third order total variation diminishing Runge-Kutta method for time discretization. Such Runge-Kutta for solving ODEs $\frac{d}{dt}y = f(y,t)$ reads,

$$\begin{aligned}
y^{(1)} &= y^n + \Delta t f(y^n, t^n) \\
y^{(2)} &= y^n + \frac{\Delta t}{4} f(y^n, t^n) + \frac{\Delta t}{4} f(y^{(1)}, t^{n+1}) \\
y^{n+1} &= y^n + \frac{\Delta t}{6} f(y^n, t^n) + \frac{2\Delta t}{3} f(y^{(2)}, t^{n+1/2}) + \frac{\Delta t}{6} f(y^{(1)}, t^{n+1})
\end{aligned}$$

- Derive that $u(x_{j+\frac{1}{2}})$ can be approximated with third order by (3).
- Implement the above finite volume scheme that is third order in both space and in time in Fortran/Matlab/your preferred computer language.
- Test the code with a smooth initial data $u_0(x) = \sin(x)$ and periodic boundary condition, by putting down an error table for numerical solutions at $T = 1.0$ with $N = 20, 40, 80$ for spatial meshes and $CFL = \frac{\Delta t}{\Delta x} = 0.5$.

2. Modify the code for solving the Burgers' equation,

$$u_t + (u^2/2)_x = 0, \quad x \in [-\pi, \pi], \tag{4}$$

with a smooth initial data $u_0(x) = \sin(x)$ and periodic boundary condition. Put down an error table for numerical solutions at $T = 0.2$ with $N = 20, 40, 80$ for spatial meshes and $CFL = \frac{\Delta t}{\Delta x} = 0.5$.