Solution set 6:

Roe's matrix and TVD methods

Exercise 6.1 **(a)** By the definition $z = \rho^{-1/2}u$, we have

$$\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}\rho^{1/2} \\ m\rho^{-1/2}\end{bmatrix} , \tag{1}$$

where $z = [\alpha, \beta]^T$. We solve for ρ and m to get

$$u(z) = \begin{bmatrix}\alpha^2 \\ \alpha\beta\end{bmatrix} . \tag{2}$$

Substituting this into the definition of f yields

$$\varphi(z) = f(u(z)) = \begin{bmatrix}\alpha\beta \\ a^2\alpha^2 + \beta^2\end{bmatrix} . \tag{3}$$

Finally, we have

$$f'(u) = \begin{bmatrix}0 & 1 \\ a^2 - (m/\rho)^2 & 2(m/\rho)\end{bmatrix} , \qquad \varphi'(z) = \begin{bmatrix}\beta & \alpha \\ 2a^2\alpha & 2\beta\end{bmatrix} , \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}z} = \begin{bmatrix}2\alpha & 0 \\ \beta & \alpha\end{bmatrix} . \tag{4}$$

(b) To compute $\hat{C}$, we notice that

$$\begin{aligned} f(u_r) - f(u_l) = \varphi(z_r) - \varphi(z_l) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\mu}\Big(\varphi(z(\mu))\Big)\,\mathrm{d}\mu \\ &= \int_0^1 \varphi'(z(\mu))\,\frac{\mathrm{d}z}{\mathrm{d}\mu}(\mu)\,\mathrm{d}\mu \\ &= \int_0^1 \varphi'(z(\mu))\,\mathrm{d}\mu\,\Big(z_r - z_l\Big) = \hat{C}(z_r - z_l) . \end{aligned} \tag{5}$$

Therefore, to calculate $\hat{C}$ we must compute the integrals of α and β over $(0, 1)$. We get

$$\int_0^1 \alpha(\mu)\,\mathrm{d}\mu = \int_0^1 (\alpha_l + (\alpha_r - \alpha_l)\mu)\,\mathrm{d}\mu = \frac{\alpha_r + \alpha_l}{2} =: \overline{\alpha} , \tag{6}$$

and similarly

$$\int_0^1 \beta(\mu)\,\mathrm{d}\mu = \frac{\beta_r + \beta_l}{2} =: \overline{\beta} , \tag{7}$$

which results in

$$\hat{C} = \begin{bmatrix}\overline{\beta} & \overline{\alpha} \\ 2a^2\overline{\alpha} & 2\overline{\beta}\end{bmatrix} . \tag{8}$$

(c) Similarly to the last derivation, we have

$$u_r - u_l = \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}z}(z(\mu))\,\mathrm{d}\mu\,\Big(z_r - z_l\Big) . \tag{9}$$

Let $\hat{B}$ be the matrix $\int_0^1 \frac{\mathrm{d}u}{\mathrm{d}z}(z(\mu))\,\mathrm{d}\mu$. Recalling (4), we get

$$\hat{B} = \begin{bmatrix}2\overline{\alpha} & 0 \\ \overline{\beta} & \overline{\alpha}\end{bmatrix} . \tag{10}$$

(d) Roe's matrix $\hat{A}$ for isothermal flow is given by

$$\hat{A} := \hat{C}\hat{B}^{-1} = \begin{bmatrix}0 & 1 \\ a^2 - \hat{v}^2 & 2\hat{v}\end{bmatrix} , \qquad \hat{v} = \frac{\overline{\beta}}{\overline{\alpha}} = \frac{m_r\sqrt{\rho_l} + m_l\sqrt{\rho_r}}{\sqrt{\rho_r\rho_l}\left(\sqrt{\rho_r} + \sqrt{\rho_l}\right)} . \tag{11}$$

Next we verify that Roe's conditions are satisfied:

1. By construction $\hat{A}$ satisfies
$$f(u_r) - f(u_l) = \hat{A}(u_r - u_l) \ . \tag{12}$$
2. The function $\hat{v}$ is continuous in ρ_r, ρ_l, m_r and m_l, and when $\rho_r = \rho_l = \rho$, and $m_r = m_l = m$, $\hat{v}$ satisfies $\hat{v} = m/\rho$. Therefore, $\lim_{u_r,u_l\to\overline{u}} \hat{A} = f'(\overline{u})$
3. The spectral decomposition of $\hat{A}$ is given by $\Lambda = R^{-1}\hat{A}R$, where
$$\Lambda = \begin{bmatrix} \hat{v}-a & 0 \\ 0 & \hat{v}+a \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 \\ \hat{v}-a & \hat{v}+a \end{bmatrix} \ . \tag{13}$$
In particular, $\hat{A}$ is diagonalizable and has only real eigenvalues.

Exercise 6.2 **(a)** The three numerical fluxes are equivalent. To see this, recall that $\hat{A}$ satisfies $f(u_r) - f(u_l) = \hat{A}(u_r - u_l)$.**(b)** Recall that $\hat{A}^+ = R\Lambda^+R^{-1}$, $\hat{A}^- = R\Lambda^-R^{-1}$ and $\left|\hat{A}\right| = R\left|\Lambda\right|R^{-1}$. You can thus use what you found in exercise 6.2 to find $\hat{A}^+$, $\hat{A}^-$ and $\left|\hat{A}\right|$ trough trivial matrix matrix multiplication.

Exercise 6.3 **(a)** For the LF scheme, by rearranging the terms we get
$$\begin{aligned} U_{j+1}^{n+1} - U_j^{n+1} = &\frac{1}{2}\left[\left(U_{j+2}^n - U_{j+1}^n\right) - \frac{k}{h}\left(f\left(U_{j+2}^n\right) - f\left(U_{j+1}^n\right)\right)\right] \\ &+ \frac{1}{2}\left[\left(U_j^n - U_{j-1}^n\right) + \frac{k}{h}\left(f\left(U_j^n\right) - f\left(U_{j-1}^n\right)\right)\right] \ . \end{aligned} \tag{14}$$
By the mean value theorem, for each $j \in \mathbb{Z}$ there exists a number w_{j-1}^n between U_j^n and U_{j-1}^n such that
$$f\left(U_j^n\right) - f\left(U_{j-1}^n\right) = f'\left(w_{j-1}^n\right)\left(U_j^n - U_{j-1}^n\right) \ . \tag{15}$$
By substituting this into (14) and summing over $j \in \mathbb{Z}$, we get the estimate
$$\begin{aligned} TV\left(U^{n+1}\right) \le &\frac{1}{2}\sum_{j=-\infty}^{\infty}\left|1 - \frac{k}{h}f'\left(w_{j+1}^n\right)\right| \cdot \left|U_{j+2}^n - U_{j+1}^n\right| \\ &+ \frac{1}{2}\sum_{j=-\infty}^{\infty}\left|1 + \frac{k}{h}f'\left(w_{j-1}^n\right)\right| \cdot \left|U_j^n - U_{j-1}^n\right| \ , \end{aligned} \tag{16}$$
which implies
$$TV\left(U^{n+1}\right) \le \frac{1}{2}\sum_{j=-\infty}^{\infty}\left(\left|1 - \frac{k}{h}f'\left(w_j^n\right)\right| + \left|1 + \frac{k}{h}f'\left(w_j^n\right)\right|\right) \cdot \left|U_{j+1}^n - U_j^n\right| \ . \tag{17}$$
Thus, the estimate $TV\left(U^{n+1}\right) \le TV(U^n)$ holds for each $k > 0$ for which
$$\frac{k}{h}\left|f'(w)\right| \le 1 \qquad \forall w, \quad \min_{j\in\mathbb{Z}} U_j^n \le w \le \max_{j\in\mathbb{Z}} U_j^n \ . \tag{18}$$
To complete the proof we must show that for each initial data U^0 with finite total variation $TV\left(U^0\right)$, there exists $k_0 > 0$ such that (18) holds for each $0 < k \le k_0$ and $n \ge 0$. Since f' is bounded on compact intervals, it suffices to show that there exist some $k_0 > 0$ and $R > 0$ such that for each $0 < k \le k_0$ and $n \ge 0$
$$\left|U_j^n\right| \le R \qquad \forall j \in \mathbb{Z} \ . \tag{19}$$
This can be shown by induction if we choose, for example,
$$R = \frac{1}{2}\,TV\left(U^0\right) + \frac{\left|U_\infty + U_{-\infty}\right|}{2} \tag{20}$$
and $k_0 > 0$ such that $\left|f'(w)\right| k_0/h \le 1$ for all $|w| \le R$, where $U_{\pm\infty} = \lim_{j\to\pm\infty} U_j^0$. It should be remarked that this condition, while convenient for this proof, is not very efficient. In fact, it is possible to show that if (18) is satisfied at $n = 0$, then it will also be satisfied at every $n \ge 0$.
(b) The LF scheme is conservative and consistent, and it follows from **(a)** that (in the scalar case) there exist $k_0 > 0$ and $R > 0$ such that
$$TV(U^n) \le R \qquad \forall n, k \quad \text{with} \quad 0 < k < k_0 \ , \quad nk \le T. \tag{21}$$
Therefore, the LF scheme is TV-stable.