EXAM

1 : Let

$$E(u,v) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 + F(u,v)\mathrm{d}x$$

with

$$F(u,v) = \frac{1}{4}(u^2-1)^2 + \frac{1}{4}(v^2-1)^2 + u^2 v.$$

Consider the gradient flow

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta \dfrac{\delta E}{\delta u}, \quad \dfrac{\partial u}{\partial n}\big|_{\partial\Omega} = \dfrac{\partial v}{\partial n}\big|_{\partial\Omega} = 0 \\ \dfrac{\partial v}{\partial t} = \Delta \dfrac{\delta E}{\delta v}, \quad \dfrac{\partial \Delta u}{\partial n}\big|_{\partial\Omega} = \dfrac{\partial \Delta v}{\partial n}\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}$$

Construct a first-order SAV scheme for (0.1) and prove the energy stability.

Solution for # 1:

1st-step: Owing to Young's inequality $u^2 v \le \frac{1}{8}u^4 + 2v^2$, we have

$$F(u,v) \ge \frac{1}{8}(u^4 - 4u^2) + \frac{1}{4}(v^4 - 4v^2) + \frac{1}{2} \ge -1.$$

So we can take a constant $C_0 > -1$ s.t. $\int_\Omega F(u,v) + C_0 > 0$.

2rd-step: (**SAV scheme**) By variational derivatives

$$\frac{\delta F}{\delta u} = (u^2-1)u + 2uv, \quad \frac{\delta F}{\delta v} = (v^2-1)u + u^2$$

and denoting $r(t) = \sqrt{\int_\Omega F(u,v)\mathrm{d}x + C_0}$.

The SAV scheme can be read as

$$(1) \quad \frac{u^{n+1}-u^n}{\Delta t} = \Delta \mu^{n+1}$$

$$(2) \quad \mu^{n+1} = -\Delta u^{n+1} + \frac{r^{n+1}}{\sqrt{\int_\Omega F^n \ \mathrm{d}x + C_0}}(\frac{\delta F}{\delta u})^n$$

$$(3) \quad \frac{v^{n+1}-v^n}{\Delta t} = \Delta w^{n+1}$$

$$(4) \quad w^{n+1} = -\Delta v^{n+1} + \frac{r^{n+1}}{\sqrt{\int_\Omega F^n \ \mathrm{d}x + C_0}}(\frac{\delta F}{\delta v})^n$$

$$(5) \quad \frac{r^{n+1}-r^n}{\Delta t} = \frac{1}{\sqrt{\int_\Omega F^n \ \mathrm{d}x + C_0}} \int_\Omega (\frac{\delta F}{\delta u})^n \frac{u^{n+1}-u^n}{\Delta t} + (\frac{\delta F}{\delta v})^n \frac{v^{n+1}-v^n}{\Delta t} \mathrm{d}x$$

3rd-step: The energy stability can be proved by

$$\Big((1), \mu^{n+1}\Big) + \Big((2), -\frac{u^{n+1}-u^n}{\Delta t}\Big) + \Big((3), w^{n+1}\Big) + \Big((4), -\frac{v^{n+1}-v^n}{\Delta t}\Big) + \Big((5), 2r^{n+1}\Big)$$

$\#\ 2$: Consider the following velocity-conection scheme for time dependent Stokes equation:

$$\begin{cases} \dfrac{\tilde{u}^{n+1} - u^n}{\Delta t} - \nu \Delta u^n + \nabla p^{n+1} = 0 \\ \nabla \cdot \tilde{u}^{n+1} = 0 \\ \tilde{u}^{n+1} \cdot n\big|_{\partial\Omega} = 0 \end{cases} \tag{0.2}$$

and

$$\frac{u^{n+1} - \tilde{u}^n}{\Delta t} - \nu \Delta (u^{n+1} - u^n) = 0, \quad u^{n+1}\big|_{\partial\Omega} = 0. \tag{0.3}$$

Prove that (0.2)-(0.3) is unconditionally energy stable.

Proof for # 2:

1st-step: Owing to $\nabla \cdot \tilde{u}^{n+1} = 0$ and $\tilde{u}^{n+1} \cdot n\big|_{\partial\Omega} = 0$, it's easy to derive that $(\nabla p^{n+1}, \tilde{u}^{n+1}) = 0$. Then we have

$$\begin{aligned} 0 =& \Big(\frac{\tilde{u}^{n+1} - u^n}{\Delta t} - \nu \Delta u^n + \nabla p^{n+1}, 2\Delta t \tilde{u}^{n+1} \Big) \\ =& \|\tilde{u}^{n+1}\|^2 - \|u^n\|^2 + \|\tilde{u}^{n+1} - u^n\|^2 - 2\nu\Delta t (\Delta u^n, \tilde{u}^{n+1}) \end{aligned}$$

2rd-step: Rewrite (0.3) as

$$u^{n+1} - \nu \Delta t \Delta u^{n+1} = \tilde{u}^{n+1} - \nu \Delta t \Delta u^n.$$

By taking inner product with itself on both side, we have

$$\begin{aligned} &\|u^{n+1}\|^2 + 2\nu\Delta t \|\nabla u^{n+1}\|^2 + \nu^2 \Delta t^2 \|\Delta u^{n+1}\|^2 \\ =& \|\tilde{u}^{n+1}\|^2 + \nu^2 \Delta t^2 \|\Delta u^n\|^2 - 2\nu\Delta t (\Delta u^n, \tilde{u}^{n+1}) \end{aligned}$$

3th-step: We end the proof by combining the above identities.