Solution set 1:

Scalar Conservation Laws

Exercise 1.1 The integral form of the scalar conservation law $u_t + f(u)_x = 0$ is given in Eq. 1 below.

$$\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt \tag{1}$$

For $f(u) = au$ in the conservation law we are left with the classic advection equation on integral form. We let $u_0(x)$ be any integrable function, i.e., $u_0(x)$ may include discontinuities.

$$\int_{x_1}^{x_2} u_0(x - at_2) - u_0(x - at_1)\,dx = a\int_{t_1}^{t_2} u_0(x_1 - at) - u(x_2 - at)\,dt \tag{2}$$

On the left side we have an integral in the x variable and on the left side we have an integral in the t variable. How do we compare? Well, let us try a change of variables so to make the left side look like the right side, let's set $x - at_1 = x_1 - at$ and $x - at_2 = x_2 - at$ so to get

$$\int_{x_1}^{x_2} u_0(x_2 - at) - u_0(x_1 - at)\,dx = a\int_{t_1}^{t_2} u_0(x_1 - at) - u(x_2 - at)\,dt \tag{3}$$

Then we realize that $\frac{d}{dt}(x - at_1) = \frac{d}{dt}(x_1 - at) \Rightarrow dx = -adt$ so to get

$$-a\int_{x_1}^{x_2} u_0(x_2 - at) - u_0(x_1 - at)\,dt = a\int_{t_1}^{t_2} u_0(x_1 - at) - u(x_2 - at)\,dt \tag{4}$$

Now for this integral to make sense we also need to change the limits to conform with the change of variable. For $x = x_1$ we get $x_1 - at_1 = x_1 - at \Rightarrow t = t_1$ while for $x = x_2$ we use that $x_2 - at_2 = x_2 - at \Rightarrow t = t_2$ so to arrive at

$$-a\int_{t_1}^{t_2} u_0(x_2 - at) - u_0(x_1 - at)\,dt = a\int_{t_1}^{t_2} u_0(x_1 - at) - u(x_2 - at)\,dt \tag{5}$$

From which we conclude that for any integrable function $u_0(x)$, $u_0(x - at)$ is a solution to the advection equation on integral form. The important thing to note here is that when working with the original integral equation derived from physical principles, there are no silly differentiability requirement and discontinuous solutions are admissible. These admissible solutions continue to exists as solutions to the differential equation derived from the integral form under assumption of smoothness, if a discontinuity arise in the solution, the assumption of smoothness is no longer valid and our differential model of the integral equation breaks down.

Exercise 1.2 From the theory we know that the solution of the Cauchy problem $u_t + f(u)_x = 0$ with $u(x,0) = u_0(x)$ can become singular only when the construction of the solution by the method of characteristics fail. Thus, we begin by constructing the solution according to the method of characteristics, when we will find where the procedure may fail. The projection of the characteristic curves onto the x,t plane is given by $x_t = f'(u)$, $x(\xi,0) = \xi$. On the characteristics, u satisfies $u_t = 0$ and $u(\xi,0) = u_0(\xi)$. Therefore, in a neighborhood of the initial condition we have

$$x(\xi,t) = f'(u_0(\xi))\,t + \xi,\ u(\xi,t) = u_0(\xi) \tag{6}$$

To get to the solution $u(x,t)$, we must extract $\xi(x,t)$ from Eq. 6. To do so, we employ the inverse function theorem which states that provided $x_\xi \neq 0$, there exists a smooth function $\xi(x,t)$ such that

$$x = f'(u_0(\xi(x,t)))\,t + \xi(x,t) \tag{7}$$

This procedure fails, if at some ξ and $t > 0$

$$0 = x_\xi = f''(u_0(\xi))\,u_0' t + 1 \tag{8}$$

Therefore, the solution $u(x,t)$ can become singular only at ξ and $t > 0$ where

$$t = \frac{-1}{f''(u_0(\xi))\,u_0'(\xi)} \tag{9}$$

Exercise 1.3 In this problem we considered the inviscid burgers equation where $f(u) = \frac{1}{2}u^2$ so that $u_t + uu_x = 0$ for two sets of initial data. First the solution for set **(a)**

$$u_0(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 1 \\ -1 & x > 1 \end{cases} \tag{10}$$

is presented. Figure 1 contains a sketch of the initial data profile $u_0(x)$ with characteristics evolving. We see that two shocks are present in the initial data. The task is to determine the exact solution $u(x,t)$ for all $t > 0$. We can determine the speed of the shocks by using the Rankine-Hugoniot jump condition, this reads

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{11}$$

Applied to each shock present in the initial data we arrive at the two speeds

$$s_1 = \frac{\frac{1}{2}1^2 - \frac{1}{2}0^2}{1-0} = \frac{1}{2}, \; s_2 = \frac{\frac{1}{2}0^2 - \frac{1}{2}(-1)^2}{0-(-1)} = -\frac{1}{2} \tag{12}$$

The left shock is moving right and the right shock is moving left. From this we expect the two shocks to meet at $x = 0$ at time $t = 2$. Ok, so for a starters we know can write up the exact solution up until $t < 2$.

$$u(x,t) = \begin{cases} 1 & x < \frac{1}{2}(t-2) \\ 0 & \frac{1}{2}(t-2) < x < -\frac{1}{2}(t-2) \\ -1 & x > -\frac{1}{2}(t-2) \end{cases} \tag{13}$$

When the two shocks meet we are left with a Riemann problem, what will be the resulting speed? Again we turn to the Rankine-Hugoniot jump condition

$$s_3 = \frac{\frac{1}{2}1^2 - \frac{1}{2}(-1)^2}{1-(-1)} = 0 \tag{14}$$

When the two shocks meet they stop moving and for all time hereafter $t > 2$ we can write the solution as

$$u(x,t) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases} \tag{15}$$

Now we move on to the solution for initial data set **(b)** given in Eq. 16.

$$u_0(x) = \begin{cases} -1 & x < -1 \\ 0 & -1 < x < 1 \\ 1 & x > 1 \end{cases} \tag{16}$$

Figure 2 contains a sketch of the initial data profile $u_0(x)$ with characteristics evolving as a rarefaction wave. Why do we have a rarefaction wave solution and not simply another shock moving away from the center instead of towards it? This would lead to an entropy-violating shock, if you make another sketch (Try it!) you will notice how the characteristics of such a solution on this equation would evolve out of the shock and not towards it. On the sketch in figure 2 we do not have shocks forming where characteristics cross in time but rarefaction waves at $x = \pm 1$. A general solution model for rarefaction wave on a Riemann problem was supplied in the exercise description for initial data centered at zero, using the solution model with $f'(v(\xi)) = \xi \Rightarrow v(\xi) = \xi$ we arrive at

$$u(x,t) = \begin{cases} -1 & x < -t-1 \\ \frac{x}{t} & -t-1 < x < -1 \\ 0 & x > -1 \end{cases} \tag{17}$$

for the left rarefaction wave, and for the right rarefaction wave we have

$$u(x,t) = \begin{cases} 0 & x < 1 \\ \frac{x}{t} & 1 < x < t+1 \\ 1 & x > t+1 \end{cases} \tag{18}$$

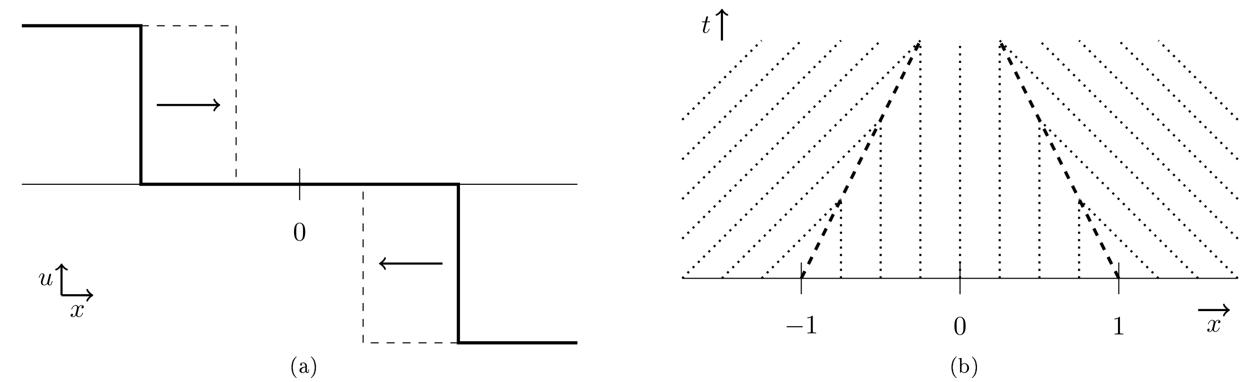

Figure 1: Exercise 1.3a. Initial data $u_0(x)$ Eq. 10 applied to the inviscid burgers equation. The two shocks move towards each other and merge at $x = 0$. At $x = 0$ they a new stationary shock is formed. **(a)** Initial data. **(b)** Characteristics.

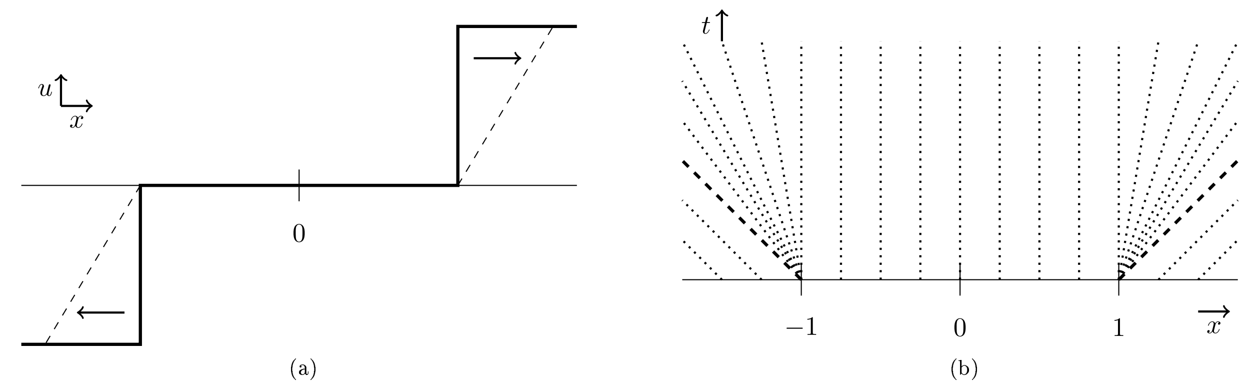

Figure 2: Exercise 1.3b. Initial data $u_0(x)$ Eq. 16 applied to the inviscid burgers equation. The two shocks move away from each other with equal speed and never merge. **(a)** Initial data. **(b)** Characteristics.

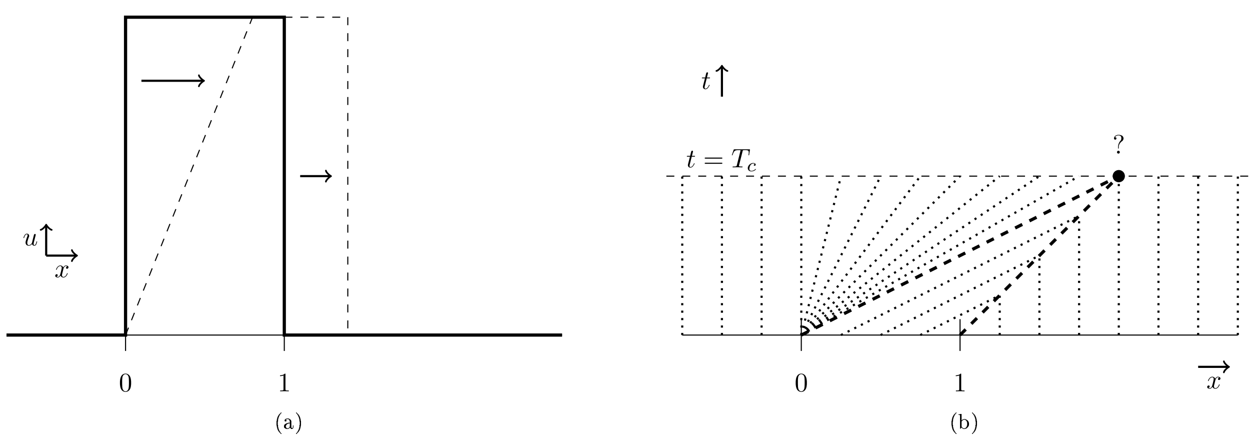

Figure 3: Exercise 1.4. Initial data $u_0(x)$ Eq. 19 applied to the inviscid burgers equation. The rarefaction wave and the shock both move in positive direction, the rarefaction wave moves faster than the shock and at some point in time $t = T_c > 0$ they meet and merge. **(a)** Initial data. **(b)** Characteristics.

Exercise 1.4 In this problem we again considered the inviscid burgers equation, now with the set of initial data

$$u_0(x) = \begin{cases} 2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

For exercise 1.4 **(a)** the figure 3 contains a sketch of the initial data profile $u_0(x)$ with characteristics evolving. At the shock at $x = 0$ we expect a rarefaction wave to arise. What direction will things be moving? To determine the speed of the shock we return to the Rankine-Hugoniot jump condition to get

$$s_{shock} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}2^2 - \frac{1}{2}(0)^2}{2 - (0)} = 1 \tag{20}$$

How about the rarefaction? From the general solution to the rarefaction wave problem we have that this wave will move with a speed of $s_{rf} = f'(u_r) = 2$. The rarefaction wave thus moves to the left twice as fast as the shock wave and at some point in time the waves must meet. **(b)** First we seek to determine the exact solution for $0 < t < T_c$, where T_c is the time when the rarefaction wave catches up with the shock as given in the exercise. The time T_c is trivial to find by writing two equations for the position of the rarefaction front wave and the position of the shock, $x_{shock} = t + 1$, $x_{rf} = 2t$ from which we have $T_c + 1 = 2T_c \Rightarrow T_c = 1$. Up until time T_c we can find the exact solution using the same approach as in exercise 1.3 to get

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < 2t \\ 2 & 2t < x < t+1 \\ 0 & x > t+1 \end{cases} \tag{21}$$

(c) But what about when $t > T_c$? We can again use the Rankine-Hugoniot jump condition, now to construct an ODE for the position of the shock x_s after the rarefaction and shock wave have merged.

$$\frac{dx_s(t)}{dt} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}\left(\frac{x_s(t)}{t}\right)^2 - \frac{1}{2}(0)^2}{\frac{x_s(t)}{t} - (0)} = \frac{1}{2}\frac{x_s(t)}{t} \tag{22}$$

This ODE has the general solution $x_s(t) = C\sqrt{t}$. At $t = 1$ we know that $x_s = 2$ so $C = 2$. But what about the profile of $u(x,t)$? The integral over $u(x,t)$ for x at any fixed t must be conserved, remember we are working with a conservation law. When the two waves meet they merge and continue with the speed $x_s/2$, on the right side there is a shock and on the left side the rarefaction. If $u(x_s(t), t),\ t > T_c$ remains constant the integral over $u(x,t)$ will not be conserved at 2. Thus $2 = \frac{1}{2}x_s(t)\,u(x_s(t), t) \Rightarrow u(x_s(t), t) = \frac{2}{\sqrt{t}}$. With this information, and a the parametrization $\frac{2}{\sqrt{t}}\frac{x}{2\sqrt{t}} = \frac{x}{t}$,we can write the solution for time $t > T_c$ as

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < 2\sqrt{t} \\ 0 & x > 2\sqrt{t} \end{cases} \tag{23}$$

The above is the same solution as one would have obtained by applying the general expression for the solution of the convex scalar equation rarefaction wave problem, instead we obtained the solution by remembering that u is a conserved variable.

Exercise 1.5 In this problem we considered the traffic flow equation where $f(q) = u_{max}\left(q - q^2\right)$

$$q_t + \left(u_{max}\left(q - q^2\right)\right)_x = 0 \tag{24}$$

where $q(x,t)$ is the density of cars and u_{max} the maximum speed of these cars at any given point. We wish to determine what the conditions are for a shock to be admissible in this equation. **(a)** First we apply the entropy condition, the speed of such a shock across a discontinuity (q_l, q_r) is again given by the Rankine-Hugoniot jump condition

$$s = \frac{f(q_l) - f(q_r)}{q_l - q_r} = \frac{u_{max}\left(q_l - q_l^2\right) - u_{max}\left(q_r - q_r^2\right)}{q_l - q_r} = u_{max}\left(1 - (q_r + q_l)\right) \tag{25}$$

we apply the entropy condition to get $f'(q_l) > s > f'(q_r)$

$$u_{max}\left(1 - q_l\right) > u_{max}\left(1 - (q_r + q_l)\right) > u_{max}\left(1 - q_r\right) \tag{26}$$

which reduces to

$$q_l - q_r < 0 \tag{27}$$

from which it is clear that we must have $q_l < q_r$ for a shock to be admissible. **(b)** The flux function for our traffic equation is $f(q) = u_{max}\left(q - q^2\right)$, choosing an entropy function $\eta(q) = q^2$, by $\psi'(q) = \eta'(q) f'(q)$ we deduce the entropy flux to be $\psi(u) = u_{max}\left(q^2 - \frac{4}{3}q^3\right)$. Inserting in

$$s\left(\eta(q_r) - \eta(q_l)\right) > \psi(q_r) - \psi(q_l) \tag{28}$$

we find that

$$u_{max}\left(1 - (q_r + q_l)\right)\left(q_r^2 - q_l^2\right) > u_{max}\left(q_r^2 - \frac{4}{3}q_r^3\right) - u_{max}\left(q_l^2 - \frac{4}{3}q_l^3\right) \tag{29}$$

which after quite a few operations reduces to

$$0 > \frac{1}{3}\left(q_l - q_r\right)^3 \tag{30}$$

and we reach the same conclusion that for a shock to be admissible we must have $q_l < q_r$. As an example of such a non admissible shock, we present Figurer 4. Figure 4 contains a sketch of a weak solution of the traffic flow equation on the Riemann problem Eq. 31 with $q_l > q_r$.

$$u_0(x) = \begin{cases} 1.5 & x < 0 \\ 0.5 & x > 0 \end{cases} \tag{31}$$

Notice how, on this particular weak solution, the characteristics flow out of the shock. Such a shock formation is called an entropy-violating shock. If time permits, you should create this sketch yourself by remembering how the speed of the characteristic for the traffic flow equation is given by $\frac{dX(t)}{dt} = f'(q) = u_{max}(1 - 2q)$.

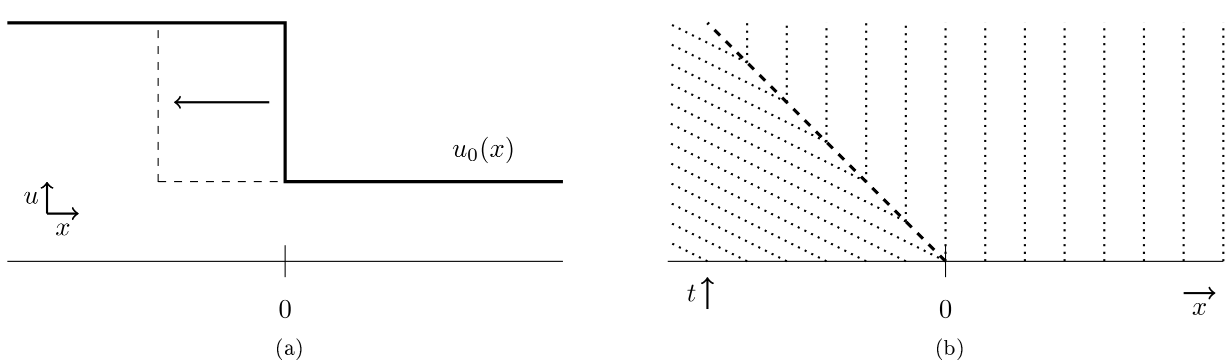

Figure 4: Exercise 1.5. Initial data Eq. 31 applied to the traffic flow equation. As seen from the behavior of the characteristics, this particular weak solution contains an entropy-violating shock. This is no surprice as the entropy-violation for such a set of initial data was already predicted in the exercise. **(a)** Initial data. **(b)** Characteristics.