Solution set 10:

Discontinuous Galerkin methods

Exercise 10.1 **(a)** To develop a DG method for the PDE

$$u_t + a u_x = bu \tag{1}$$

we suppose the approximation u_h is given in the kth element $D^k = \left[x_l^k, x_r^k\right]$ by

$$u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\ \psi_n^k(x)\ , \tag{2}$$

where ψ_n^k is some basis of the space of polynomials of degree no grater than $N = N_p - 1$. We require that for each $k = 1, \dots, K$, the residual $\mathcal{R}_h = \left(\partial_t + a\partial_x - b\right) u_h$ satisfies

$$0 = \left(\mathcal{R}_h\ , \psi_j^k\right)_{D^k} = \int_{D^k} \mathcal{R}_h \psi_j^k \,\mathrm{d}x \qquad\qquad j = 1, \dots, N_p\ . \tag{3}$$

We use the divergence theorem and substitute the boundary terms by the numerical flux f^* to get

$$\sum_{n=1}^{N_p} \frac{\mathrm{d}\hat{u}_n^k}{\mathrm{d}t} \int_{D^k} \psi_n^k \psi_j^k \,\mathrm{d}x - \sum_{n=1}^{N_p} a\hat{u}_n^k \int_{D^k} \psi_n^k \frac{\mathrm{d}\psi_j^k}{\mathrm{d}x} \,\mathrm{d}x - \sum_{n=1}^{N_p} b\hat{u}_n^k \int_{D^k} \psi_n^k \psi_j^k \,\mathrm{d}x = -\left[\psi_j^k f^*\right]_{x_l^k}^{x_r^k}\ . \tag{4}$$

Equations (4) with $j = 1, \dots, N_p$, can also be written as a linear system

$$\hat{\mathcal{M}}^k \frac{\mathrm{d}}{\mathrm{d}t} \hat{\boldsymbol{u}}_h^k - a\left(\hat{\mathcal{S}}^k\right)^T \hat{\boldsymbol{u}}_h^k - b\hat{\mathcal{M}}^k \hat{\boldsymbol{u}}_h^k = -\left[\boldsymbol{\psi}^k f^*\right]_{x_l^k}^{x_r^k} \tag{5}$$

where $\hat{\boldsymbol{u}}_h^k = \left(\hat{u}_1^k, \dots, \hat{u}_{N_p}^k\right)^T$, and $\boldsymbol{\psi}^k = \left(\psi_1^k, \dots, \psi_{N_p}^k\right)^T$.

(b) In this exercise, we suppose that $a > 0$ and use the upwind flux

$$f_{k,k+1}^* = a u_r^k\ . \qquad\qquad u_r^k = u_h^k\left(x_r^k, \cdot\right)\ . \tag{6}$$

We also write $u_h^k\left(x_l^k, \cdot\right) = u_l^k$. Multiplying (5) by $\hat{\boldsymbol{u}}_h^k$, and summing over the elements provides

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_h\right\|^2 - b\left\|u_h\right\|^2 = \sum_{k=1}^{K}\left[a\left(u_r^k\right)^2 - a\left(u_l^k\right)^2 - 2u_r^k f_{k,k+1}^* + 2u_l^k f_{k-1,k}^*\right]. \tag{7}$$

Thus, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_h\right\|^2 - b\left\|u_h\right\|^2 = -\sum_{k=1}^{K-1}\Big[& -a\left(u_r^k\right)^2 + 2u_r^k f_{k,k+1}^* + a\left(u_l^{k+1}\right)^2 - 2u_l^{k+1} f_{k,k+1}^*\Big] \\ & + a\left(u_r^K\right)^2 - 2u_r^K f_{K,K+1}^* - a\left(u_l^1\right)^2 + 2u_l^1 f_{0,1}^*\ .\end{aligned} \tag{8}$$

Since the terms

$$+a\left(u_r^K\right)^2 - 2u_r^K f_{K,K+1}^* - a\left(u_l^1\right)^2 + 2u_l^1 f_{0,1}^*$$

correspond to the boundary conditions, we will not address them here. Therefore, it is left to show that for each $k = 1, \dots, K-1$, the term

$$\delta^k = -a\left(u_r^k\right)^2 + 2u_r^k f_{k,k+1}^* + a\left(u_l^{k+1}\right)^2 - 2u_l^{k+1} f_{k,k+1}^* \tag{9}$$

is nonnegative. Since clearly,

$$\delta^k = -a\left(u_r^k\right)^2 + 2a\ u_r^k u_r^k + a\left(u_l^{k+1}\right)^2 - 2a u_r^k u_l^{k+1} = a\left(u_r^k - u_l^{k+1}\right)^2 \geq 0 \tag{10}$$

the scheme is stable.

Exercise 10.2 **(a)** In this exercise you were asked to propose a DG method for the non-homogeneous advection equation. To prevent confusion, we denote the forcing function by η (instead of the original notation in the exercise f), and reserve the letter f for the flux. The construction of a DG scheme for the PDE

$$u_t + a u_x = \eta \tag{11}$$

is similar to what we have seen. Here, for each $k = 1, \ldots, K$, we get

$$\left(\partial_t u_h^k \ , \ \psi_j^k\right)_{D^k} - a\left(u_h^k \ , \ \partial_x \psi_j^k\right)_{D^k} = -\left[f^* \psi_j^k\right]_{x_l^k}^{x_r^k} + \hat{\eta}_j^k \qquad j = 1, \ldots, N_p \ , \tag{12}$$

where

$$\hat{\eta}_j^k = \left(\eta, \psi_j^k\right)_{D^k} \ . \tag{13}$$

This can be also written as a system

$$\hat{\mathcal{M}}^k \frac{\mathrm{d}}{\mathrm{d}t} \hat{\boldsymbol{u}}_h^k - a\left(\hat{\mathcal{S}}^k\right)^T \hat{\boldsymbol{u}}_h^k = -\left[f^* \boldsymbol{\psi}^k\right]_{x_l^k}^{x_r^k} + \hat{\boldsymbol{\eta}}^k \tag{14}$$

(b) We take the dot product of (14) and $\hat{\boldsymbol{u}}_h^k$, and sum over $k = 1, \ldots, K$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_h\|^2 = \sum_{k=1}^{K} \left[a\left(u_r^k\right)^2 - a\left(u_l^k\right)^2 - 2u_r^k f_{k,k+1}^* + 2u_l^k f_{k-1,k}^*\right] + 2\left(\eta, u_h\right) \ . \tag{15}$$

To prove the stability of this method, we suppose that the numerical flux f^* corresponds to a stable DG method for the homogeneous advection equation $v_t + a v_x = 0$. Then, by the Cauchy-Schwarz inequality we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u_h\|^2 \leq (\eta, u_h) \leq \|\eta\| \, \|u_h\| \ , \tag{16}$$

which leads to the desired bound on the norm of u_h.