Homework #6 for MATH 6395

1. Consider using the Runge-Kutta discontinuous Galerkin (DG) method to solve the linear advection equation,
$$u_t + u_x = 0, \quad x \in [-\pi, \pi], \tag{1}$$
with following two sets of initial condition.
$$\text{Smooth:} \qquad u_0(x) = \sin(x), \tag{2}$$
$$\text{step function:} \qquad u_0(x) = 0, \quad [-\pi, 0]; \quad u_0(x) = 1, \quad [0, \pi], \tag{3}$$
and periodic boundary conditions.
   - Implement the DG scheme with P^k ($k = 0, 1, 2$) polynomial spaces coupled with the third order TVD Runge-Kutta method. Test the code with a smooth initial data by putting down an error table of L^1 and L^∞ error and the corresponding order of convergence for numerical solutions at $T = 1.0$ with $N = 20$, 40, 80 for spatial meshes and $CFL = \frac{\Delta t}{\Delta x} = \frac{1}{2k+1}$, where k is the polynomial degree of DG solution spaces.
   - Test the code with the step function initial condition. Plot the numerical solution and the exact solution at $T = 2$ on the same figure; and describe what you observe.
   - Apply the total variation bounded (TVB) limiters to the RKDG scheme.
     - Test the code again via a smooth initial data by putting down an error table of L^1 and L^∞ error and the corresponding order of convergence for numerical solutions at $T = 1.0$.
     - Test the code with the step function initial condition. Plot the numerical solution and the exact solution at $T = 2$ on the same figure; and describe what you observe.
   - Modify the code with TVB limiters for solving the Burgers' equation,
   $$u_t + (u^2/2)_x = 0, \quad x \in [-\pi, \pi], \tag{4}$$
   with a smooth initial data $u_0(x) = \sin(x)$ and periodic boundary condition. Numerically verify the RKDG code with TVB limiters: if it maintains third order accuracy when the solutions are still smooth ($T = 0.2$) and produces discontinuous solutions without much oscillations when the shock develops ($T = 2.0$).