Test, DG short course

One hour exam. Open book, open notes

1. For the conservation law
$$u_t + f(u)_x = 0, \quad 0 \le x < 1$$
with periodic boundary condition, the discontinuous Galerkin method with an implicit time discretization is defined as follows. Find $u_h^{n+1} \in V_h$, such that
$$\int_{I_j} \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h \, dx - \int_{I_j} f(u_h^\theta)(v_h)_x \, dx + \hat{f}^\theta_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - \hat{f}^\theta_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0 \tag{1}$$
for all $v_h \in V_h$ and all j. Here,
$$u_h^\theta \equiv \theta u_h^{n+1} + (1-\theta) u_h^n, \quad \hat{f}^\theta_{j+\frac{1}{2}} = \hat{f}\left((u_h^\theta)^-_{j+\frac{1}{2}}, (u_h^\theta)^+_{j+\frac{1}{2}} \right),$$
the numerical flux $\hat{f}(u^-, u^+)$ is a monotone flux (non-increasing in the second argument) which is consistent with the flux f (that is, $\hat{f}(u,u) = f(u)$), and
$$V_h = \{ v_h : (v_h)|_{I_j} \in P^k(I_j), \forall j \}, \tag{2}$$
I_j is a partition of the computational domain $[0,1]$, $P^k(J_j)$ denotes the collection of polynomials of degree up to k in the element I_j.

   For $\theta \ge \frac{1}{2}$, prove the following cell entropy inequality for the convex entropy $U(u) = \frac{u^2}{2}$:
$$\int_{I_j} \frac{U(u_h^{n+1}) - U(u_h^n)}{\Delta t} \, dx + \hat{F}^\theta_{j+\frac{1}{2}} - \hat{F}^\theta_{j-\frac{1}{2}} \le 0$$
for some numerical entropy flux $\hat{F}^\theta_{j+\frac{1}{2}} = \hat{F}\left((u_h^\theta)^-_{j+\frac{1}{2}}, (u_h^\theta)^+_{j+\frac{1}{2}} \right)$ which is consistent with the entropy flux $F(u) = \int^u U'(u) f'(u) \, du$.

2. For the original DG scheme solving the one-dimensional steady hyperbolic equation
$$u_x = f(x), \quad 0 \le x < 1$$
with the boundary condition
$$u(0) = a,$$
defined as: find $u_h \in V_h$, such that for all $v \in V_h$ and all j, we have
$$-\int_{I_j} u_h(x) v_x(x) \, dx + (u_h)^-_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (u_h)^-_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = \int_{I_j} f(x) v(x) \, dx,$$
prove $u_h = Pu$, where P is the Gauss-Radau projection that we introduced in class, i.e., $Pu \in V_h$ is defined as the unique function in V_h satisfying, for all j,
$$(u - Pu)^-_{j+\frac{1}{2}} = 0,$$
and
$$\int_{I_j} (u - Pu) v \, dx = 0, \quad \forall v \in P^{k-1}(I_j).$$
Hint: Look at the proof the optimal error estimates in class in which we decomposed the error $e = u - u_h$ into $e = \eta - \xi$, where $\eta = u - Pu$ and $\xi = Pu - u_h$. Try to prove $\xi = 0$.