Homework #5 for MATH 6395

Due at the beginning of class, Nov. 13th, 2013

1. Consider solving the linear advection equation,

$$u_t + u_x = 0, \quad x \in [-\pi, \pi]. \tag{1}$$

Use the third order conservative finite volume scheme coupled with third order Runge-Kutta method as in HW4 to solve (1) with the initial condition,

$$u_0(x) = 0, \quad [-\pi, 0]; \quad u_0(x) = 1, \quad [0, \pi],$$

and periodic boundary conditions

- Plot the numerical solution and the exact solution at $T = 2$ on the same figure; and describe what you observe.
- Modify the scheme to total variation diminishing schemes with

$$u^-_{j+\frac{1}{2}} = \bar{u}_j + minmod(u^-_{j+\frac{1}{2}} - \bar{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}), \tag{2}$$

where the minmod function is defined as

$$minmod(a, b, c) = \begin{cases} sign(a)min(|a|, |b|, |c|), & \text{if} \quad sign(a) = sign(b) = sign(c), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

  - Plot the numerical solution and the exact solution at $T = 2$ on the same figure; and describe what you observe.
  - Test the code with a smooth initial data $u_0(x) = \sin(x)$ and periodic boundary condition, by putting down an error table for numerical solutions at $T = 1.0$ with $N = 20$, 40, 80 for spatial meshes and $CFL = \frac{\Delta t}{\Delta x} = 0.5$. How do you compare the results with those in HW4?

- Modify the scheme to total variation bounded schemes with

$$u^-_{j+\frac{1}{2}} = \bar{u}_j + \overline{minmod}(u^-_{j+\frac{1}{2}} - \bar{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}), \tag{4}$$

where

$$\overline{minmod}(a, b, c) = \begin{cases} a & \text{if} \quad |a| \le M\Delta x^2, \\ minmod(a, b, c), & \text{otherwise.} \end{cases} \tag{5}$$

  - Find a suitable M, such that the third order TVB scheme maintains third order accuracy for smooth solutions and produce discontinuous solutions without much oscillations. Please show numerical evidences to support your estimate on M.

2. Modify the code with TVB limiters for solving the Burgers' equation,

$$u_t + (u^2/2)_x = 0, \quad x \in [-\pi, \pi], \tag{6}$$

with a smooth initial data $u_0(x) = \sin(x)$ and periodic boundary condition.

- Find a suitable M, such that the third order TVB scheme maintains third order accuracy when the solutions are still smooth ($T = 0.2$) and produces discontinuous solutions without much oscillations when the shock develops ($T = 2.0$). Please show numerical evidences to support your estimate on M.