Final Exam – FEM

Problem sets

1. Consider the boundary value problem

$$\begin{aligned} -\Delta u + b \cdot \nabla u + cu &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

   where $b = (b_1, b_2)$ is a *constant* vector and $c = c(x) \geq c_0 > 0$ in Ω.

   (a) Derive a weak formulation for the problem and specify V, $a(\cdot,\cdot)$, and $F(\cdot)$.

   (b) Prove that there is a unique solution to the weak formulation using Lax-Milgram Theorem.

2. State and prove Ceá lemma for Abstract Galerkin Method.

3. Consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := [0,1] \times [0,1] \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

   where f and g are two given continuous functions. Let $\mathcal{T}_h$ denotes the square/rectangular mesh of size $h = \frac{1}{N+1}$ over Ω, where N is a positive integer. If the above Poisson problem is solved by the *bilinear finite element method*,

   (a) Find the local stiffness matrix on the element on $[0,h] \times [0,h]$;

   (b) Find explicitly the global stiffness matrix A. (The mesh points are labeled by row.)

   Choose one out of the following two problems

4. Prove that the 2-norm condition number of A (defined in Problem 3) satisfies $cond_2(A) = \mathcal{O}(h^{-2})$.

5. Let $\mathcal{T}_h$ be a triangular mesh of a polygonal domain $\Omega \subset \mathbb{R}^2$. Prove that a picewise r-th order ($r \geq 1$) polynomial function $v_h \in H^1(\Omega)$ if and only if $v_h \in C^0(\overline{\Omega})$.