Variational Bayes

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1 Motivation

Given that we are now in the Artificial Intelligence Generated Content (AIGC) era. The goal of AIGC is to be able to sample from the data distribution $p(\mathbf{x})$. For example, there may be a dataset of images $\mathcal{D}$, if we can build a distribution $p(\mathbf{x})$ reflecting the training data, then we are able to generate new images that are similar to $\mathcal{D}$ by sample from its distribution $p(\mathbf{x})$.

To do so, we must be able to find out the parameterised distribution of $p_\theta(\mathbf{x})$ first. However, we cannot possibly compute the distribution of $p(\mathbf{x})$ over $\mathcal{D}$ or even assume what type of distribution it is. Therefore, we only can approximate such with simple (or simpler) distributions $q_\phi(\mathbf{z})$, where ϕ are the parameters of the distribution.

1.1 notation

1.2 a starting point

However, let's start with a much simpler problem first. We assume that we know part of information of distribution $p(\mathbf{x})$, the prior $p(\mathbf{z})$ and the prior and the likelihood $p(\mathbf{x}|\mathbf{z})$ (hence we also know the joint density $p(\mathbf{x}, \mathbf{z})$). Although it appears that we also know the posterior distribution $p(\mathbf{z}|\mathbf{x})$, as we can use Bayes rule to compute the posterior distribution, i.e.,:

$$p(\mathbf{z}|\mathbf{x}) = \frac{\overbrace{p(\mathbf{x}|\mathbf{z})}^{\text{knows}} \overbrace{p(\mathbf{z})}^{\text{knows}}}{\underbrace{\int_{\mathbf{z}'} p(\mathbf{x}|\mathbf{z}')p(\mathbf{z}')\mathrm{d}\mathbf{z}'}_{\text{don't know analytically}}} \tag{1}$$

But unfortunately, we do not always neccessary to have access to the posterior distribution in closed form.

Also in this section, we do not worry about learning the parameters θ associated with $p_\theta(\mathbf{z}|\mathbf{x})$, which is something we will consider later for Variational Autoencoders (VAE).

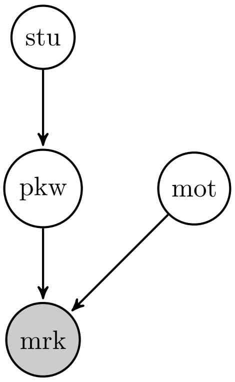
1.2.1 toy example

to illustate what is a observed data $\mathbf{x}$ and what is a latent space $\mathbf{z}$, let's look at the same "student mark" toy example you saw in both the probabilistic graphical models (which we will restrorpsectivley visit again) and MCMC (which we will not discuss in 2024) sections:

Variable	Description
$\mathbf{x}$	Data point
$\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$	Dataset
$p(\mathbf{x})$	Data distribution
$p_\theta(\mathbf{x})$	Parameterized data distribution
$q_\phi(\mathbf{z})$	Approximate distribution
ϕ	Parameters of the approximate distribution
$\mathbf{z}$	Latent variable
$p(\mathbf{z})$	Prior distribution
$p(\mathbf{x}\|\mathbf{z})$	Likelihood
$p(\mathbf{z}\|\mathbf{x})$	Posterior distribution
$\mathbb{KL}(q\\|p)$	Kullback-Leibler divergence
$\text{ELBO}(q)$	Evidence Lower Bound
μ	Mean of Gaussian distribution
τ	Precision (inverse of variance)
μ_0, λ_0	Prior (Gaussian) parameter for μ
a_0, b_0	Prior (Gamma) parameters for τ
μ_n, λ_n	Posterior (Gaussian) parameter for μ
a_n, b_n	Posterior (Gamma) parameters for τ
$q_\mu(\mu)$	Variational distribution for μ
$q_\tau(\tau)$	Variational distribution for τ

Table 1: List of Variables

1. "months of studies" (stu) z_1
2. "prior knowledge" (pkw) z_2
3. "motivation" (mot) z_3
4. "mark obtained" (mrk) $\mathbf{x}$



We have three latent variables, (stu), (pkw), (mot) and one observation (mrk), then if we want to perform posterior inference, i.e.,:

$$\Pr\left(\underbrace{\text{stu}, \text{pkw}, \text{mot}}_{\text{latent}} | \underbrace{\text{mrk}}_{\text{observation}}\right) \tag{2}$$

which allows us to compute things such as:

$$\mathbb{E}_{\Pr\left(\text{stu},\text{pkw},\text{mot}|\text{mrk}\right)}[\text{stu}, \text{pkw}, \text{mot}] \tag{3}$$

Unlike Markov Chain Monte Carlo (MCMC), in variational inference, instead of sampling, we propose a set of proposal distributions, $q_{\phi_{\text{stu}}}(\text{stu})$, $q_{\phi_{\text{pkw}}}(\text{pkw})$ and $q_{\phi_{\text{mot}}}(\text{mot})$ which aim to minimize between:

$$q(\text{stu}, \text{pkw}, \text{mot}) = q_{\phi_{\text{stu}}}(\text{stu}) \times q_{\phi_{\text{pkw}}}(\ \text{pkw}) \times q_{\phi_{\text{mot}}}(\text{mot}) \tag{4}$$

and

$$\Pr\left(\underbrace{\text{stu}}_{z_1}, \underbrace{\text{pkw}}_{z_2}, \underbrace{\text{mot}}_{z_3} | \underbrace{\text{mrk}}_{\mathbf{x}}\right) \tag{5}$$

Obviously, we need to optimize with respect to the parameters ϕ_{stu}, ϕ_{pkw} and ϕ_{mot}. then after that one can approximate:

$$\mathbb{E}_{\Pr\left(\text{stu},\text{pkw},\text{mot}|\text{mrk}\right)}[\text{stu}, \text{pkw}, \text{mot}] \approx \mathbb{E}_{q\left(\text{stu},\text{pkw},\text{mot}\right)}[\text{stu}, \text{pkw}, \text{mot}] \tag{6}$$

1.2.2 LDA example Optional - not examinable

For example, in the LDA example:

$$\begin{aligned} \text{observation:} \quad & \{w_{d\in\{1\ldots D\},n\in\{1\ldots N\}}\} \\ \text{latent variable:} \quad & \{\{\boldsymbol{\beta}_j\}_{j=1}^K, \{\boldsymbol{\theta}_d\}_{d=1}^D, \{z_{d\in\{1\ldots D\},n\in\{1\ldots N\}}\}\} \end{aligned} \tag{7}$$

Therefore, in LDA, the variational inference aims to:

$$\begin{aligned} & p\Big(\{\boldsymbol{\beta}_j\}_{j=1}^K, \{\boldsymbol{\theta}_d\}_{d=1}^D, \{z_{d\in\{1\ldots D\},n\in\{1\ldots N\}}\} \Big| \{w_{d\in\{1\ldots D\},n\in\{1\ldots N\}}\}\Big) \\ & \approx \prod_{j=1}^K q(\boldsymbol{\beta}_j|\boldsymbol{\lambda}_j) \prod_{d=1}^D q(\boldsymbol{\theta}_d|\boldsymbol{\gamma}_d) \prod_{d=1}^D \prod_{n=1}^N q(z_{d,n}|\boldsymbol{\phi}_{d,n}) \end{aligned} \tag{8}$$

it allows us to approximate:

$$\mathbb{E}_p\left[\left\{\{\boldsymbol{\beta}_j\}_{j=1}^K, \{\boldsymbol{\theta}_d\}_{d=1}^D, \{z_{d\in\{1\ldots D\},n\in\{1\ldots N\}}\}\right\}\right] \approx \mathbb{E}_q\left[\left\{\{\boldsymbol{\beta}_j\}_{j=1}^K, \{\boldsymbol{\theta}_d\}_{d=1}^D, \{z_{d\in\{1\ldots D\},n\in\{1\ldots N\}}\}\right\}\right] \tag{9}$$

we just need to optimize with respect to $\{\boldsymbol{\lambda}_j\}, \{\boldsymbol{\gamma}_d\}, \{\boldsymbol{\phi}_{d,n}\}$

1.3 A bit of history ...

This note started in 2010 when I was inspired to help people read Chapter 10 of Bishop [1] where I was trying to explain a few things in an oversimplified (hopefully!) way. I revamped it for the class. I also added exponential family distributions and an example on LDA when the model is fully conjugate [2]

2 The Variational Bayes Framework

now imagine we have some training data for us to compute the distribution of it, i.e, $p(\mathbf{x})$. Imagine we are able to compute its analytical form, then we can just sample from it, i.e., to create new data. However, in many cases, the distribution is intractable, i.e., we cannot compute it. Therefore, we need to approximate it.

Now, let's consider the following:

2.1 what is Evidence Lowerbound?

2.2 use Jensen Inequality

$$\begin{aligned}
\log p(\mathbf{x}) &= \log \int_{\mathbf{z}} p(\mathbf{x}, \mathbf{z}) \mathrm{d}\mathbf{z} \\
&= \log \int_{\mathbf{z}} \frac{p(\mathbf{x}, \mathbf{z})}{q_\phi(\mathbf{z}|\mathbf{x})} q_\phi(\mathbf{z}|\mathbf{x}) \mathrm{d}\mathbf{z} \\
&= \log \left[\mathbb{E}_{z \sim q_\phi(\mathbf{z}|\mathbf{x})} \left(\frac{p(\mathbf{x}, \mathbf{z})}{q_\phi(\mathbf{z}|\mathbf{x})} \right) \right] \\
&\geq \mathbb{E}_{z \sim q_\phi(\mathbf{z}|\mathbf{x})} \left[\log \left(\frac{p(\mathbf{x}, \mathbf{z})}{q_\phi(\mathbf{z}|\mathbf{x})} \right) \right] \qquad \text{by Jensen's inequality} \\
&= \mathbb{E}_{z \sim q_\phi(\mathbf{z}|\mathbf{x})} \left[\log(p(\mathbf{x}, \mathbf{z}) \right] - \mathbb{E}_{z \sim q_\phi(\mathbf{z}|\mathbf{x})} \left[\log(q_\phi(\mathbf{z}|\mathbf{x}) \right] \\
&= \mathrm{ELBO}(q)
\end{aligned} \tag{10}$$

2.3 another expansion

the above do not show what is the missing bit between $\log p(\mathbf{x}) - \mathrm{ELBO}(q)$. So what is it?

$$\begin{aligned}
\log\left(p(\mathbf{x})\right) &= \log \left(\frac{p(\mathbf{x}, \mathbf{z})}{p(\mathbf{z}|\mathbf{x})} \right) \\
&= \log\left(p(\mathbf{x}, \mathbf{z})\right) - \log\left(p(\mathbf{z}|\mathbf{x})\right) \\
&= \left[\log\left(p(\mathbf{x}, \mathbf{z})\right) - \log q_\phi(\mathbf{z})\right] - \left[\log\left(p(\mathbf{z}|\mathbf{x})\right) - \log q_\phi(\mathbf{z})\right] \qquad \because \pm q_\phi(\mathbf{z}) \\
&= \log \left(\frac{p(\mathbf{x}, \mathbf{z})}{q_\phi(\mathbf{z})} \right) - \log \left(\frac{p(\mathbf{z}|\mathbf{x})}{q_\phi(\mathbf{z})} \right)
\end{aligned} \tag{11}$$

now, let's taking the expectation on both sides, given $q_\phi(\mathbf{z})$:

$$\begin{aligned}\log\left(p(\mathbf{x})\right) &= \int q_\phi(\mathbf{z})\log\left(\frac{p(\mathbf{x},\mathbf{z})}{q_\phi(\mathbf{z})}\right)dz - \int q_\phi(\mathbf{z})\log\left(\frac{p(\mathbf{z}|\mathbf{x})}{q_\phi(\mathbf{z})}\right)\mathrm{d}\mathbf{z}\\ &= \int q_\phi(\mathbf{z})\log\left(\frac{p(\mathbf{x},\mathbf{z})}{q_\phi(\mathbf{z})}\right)dz + \int q_\phi(\mathbf{z})\log\left(\frac{q_\phi(\mathbf{z})}{p(\mathbf{z}|\mathbf{x})}\right)\mathrm{d}\mathbf{z}\\ &= \mathrm{ELBO}(q) + \mathbb{KL}(q\|p)\end{aligned} \tag{12}$$

2.3.1 name to both terms

$$\begin{aligned}\mathrm{ELBO}(q) &= \int q_\phi(\mathbf{z})\log\left(\frac{p(\mathbf{x},\mathbf{z})}{q_\phi(\mathbf{z})}\right)\mathrm{d}\mathbf{z} = \mathbb{E}_{z\sim q_\phi(\mathbf{z})}\big[\left(\frac{p(\mathbf{x},\mathbf{z})}{q_\phi(\mathbf{z})}\right)\big]\\ \mathbb{KL}(q\|p) &= \int q_\phi(\mathbf{z})\log\left(\frac{q_\phi(\mathbf{z})}{p(\mathbf{z}|\mathbf{x})}\right)\mathrm{d}\mathbf{z}\end{aligned} \tag{13}$$

now we can answer the question of why we do not minimize $\mathbb{KL}$ term directly? The key is that the $\mathbb{KL}$ term contains $p(z|x)$ and ELBO term contains $p(x|z)p(z)$! Why is this the problem? Because then one has to compute:

$$p(\mathbf{x}) = \int_{\mathbf{z}'} p(\mathbf{x}|\mathbf{z}')p(\mathbf{z}')\mathrm{d}\mathbf{z}' \tag{14}$$

since we can choose any $q_\phi(\mathbf{z})$ we'd like, and since we want $\mathbb{KL}(\cdot)$ to be minimized, there it's ideal to make:

$$q_\phi(\mathbf{z}) \equiv q_\phi(\mathbf{z}|\mathbf{x}) \tag{15}$$

i.e., it should also depend on x. Otherwise, it's highly unlikely that the $\mathbb{KL}\big(q\|p(\mathbf{z}|\mathbf{x})\big)$ will be minimized:

$$\mathbb{KL}(q\|p) = \int q_\phi(\mathbf{z}|\mathbf{x})\log\left(\frac{q_\phi(\mathbf{z}|\mathbf{x})}{p(\mathbf{z}|\mathbf{x})}\right)\mathrm{d}\mathbf{z} \tag{16}$$

We know that $p(\mathbf{x}) = \mathrm{ELBO}(q) + \mathbb{KL}(q\|p)$. We consider $\mathrm{ELBO}(q)$ is the lower bound of $p(\mathbf{x})$. Minimizing $\mathbb{KL}(q\|p)$ is the same as maximizing the lower bound $\mathrm{ELBO}(q)$, since the addition of the two becomes $p(\mathbf{x})$

3 The choice of $q(\mathbf{z})$: mean-field approximation

3.1 the update formula

By letting $q(\mathbf{z}) = \prod_{i=1}^{M} q_i(z_i)$, by fixing all the other distributions $\{q_{i \neq j}\}$,the following is the optimal update formula for $q_i(z_i)$ which we derived in 30:

$$q_j^*(z_j) = \exp(\mathbb{E}_{\mathbf{z} \setminus z_j}[\log{(p(\mathbf{x}, \mathbf{z}))}]) \tag{17}$$

in a sense that it will maximize the component-wise ELBO $\text{ELBO}(q_j)$, i.e.,:

$$\text{ELBO}(q_j) = -\mathbb{KL}\Big(q_j(z_j) \| \exp(\mathbb{E}_{\mathbf{z} \setminus z_j}[\log{(p(\mathbf{x}, \mathbf{z}))})\Big) \tag{18}$$

note however, the expection $\mathbb{E}_{\mathbf{z} \setminus z_j}[\log{(p(\mathbf{x}, \mathbf{z}))}]$ may not always be tractable. Therefore, it is not the general solution to VB. Also note that the optimality only occurs for the current component-wise ELBO, and not the global ELBO $\text{ELBO}(q)$.

3.2 derivation in section 3.1?

Since any $q(\mathbf{z})$ will work, therefore, we will choose the most simple form. Suppose let's choose $q(\mathbf{z})$, such that:

$$q(\mathbf{z}) = \prod_{i=1}^{M} q_i(z_i) \tag{19}$$

this is called mean-filed approximation.

$$\begin{aligned}
\text{ELBO}(q) &= \int q_\phi(\mathbf{z}) \log\left(\frac{p(\mathbf{x}, \mathbf{z})}{q_\phi(\mathbf{z})}\right) \text{d}\mathbf{z} \\
&= \int q_\phi(\mathbf{z}) \log(p(\mathbf{x}, \mathbf{z})) \text{d}\mathbf{z} - \int q_\phi(\mathbf{z}) \log(q_\phi(\mathbf{z})) \text{d}\mathbf{z} \\
&= \underbrace{\int \prod_{i=1}^{M} q_i(z_i) \log{(p(\mathbf{x}, \mathbf{z}))} \text{d}\mathbf{z}}_{-H(q,p)} + \underbrace{\Big(-\int \prod_{i=1}^{M} q_i(z_i) \sum_{i=1}^{M} \log{(q_i(z_i))} \text{d}\mathbf{z}\Big)}_{H(q)}
\end{aligned} \tag{20}$$

Since you have the objective function for $\text{ELBO}(q)$, a natural approach would be to optimize it repetitively using the parameters associated with each q.

3.3 component-wise of $-H(q, p)$:

first, let's consider the term $-H(q, p)$, which is the first term in $\text{ELBO}(q)$, and it is not the cross entropy.

$$
\begin{aligned}
-H(q,p) &= \int \prod_{i=1}^{M} q_i(z_i) \log\big(p(\mathbf{x},\mathbf{z})\big) \mathrm{d}\mathbf{z} \\
&= \int_{Z_1} \int_{Z_2} ... \int_{Z_M} \prod_{i=1}^{M} q_i(z_i) \log\big(p(\mathbf{x},\mathbf{z})\big) \mathrm{d}\mathbf{z}_1, \mathrm{d}\mathbf{z}_2, ...\mathrm{d}\mathbf{z}_M
\end{aligned}
\tag{21}
$$

Rearrange the expression by taking a particular $q_j(z_j)$ out of the integral. Note that unlike Part $H(q)$, we are not treating any terms to Const.. The component-wise of $-H(q,p)$ can be written as:

$$
\begin{aligned}
-H(q,p)_j &= \int_{z_j} q_j(z_j) \Big(\int_{Z_{i\neq j}} \cdots \int \prod_{i\neq j}^{M} q_i(z_i) \log\big(p(\mathbf{x},\mathbf{z})\big) \prod_{i\neq j}^{M} \mathrm{d}z_i \Big) \mathrm{d}z_j \\
&= \int_{z_j} q_j(z_j) \Big(\int_{Z_{i\neq j}} \cdots \int \log\big(p(\mathbf{x},\mathbf{z})\big) \prod_{i\neq j}^{M} q_i(z_i) \mathrm{d}z_i \Big) \mathrm{d}z_j
\end{aligned}
\tag{22}
$$

or, even more meaningfully, it can be put into an expectation function, and since $\prod_{i\neq j}^{M} q_i(z_i)$ is a joint probability density

$$
-H(q,p)_j = \int_{z_j} q_j(z_j) \big[\mathbb{E}_{\mathbf{z}\setminus z_j} \big[\log\big(p(\mathbf{x},\mathbf{z})\big) \big] \big] \mathrm{d}z_j \tag{23}
$$

therefore, by adding an exp to the $\mathbb{E}_{\mathbf{z}\setminus z_j}\big[\log\big(p(\mathbf{x},\mathbf{z})\big)\big]$ term, we then obtained a pseudo distribution:

$$
\exp\big(\mathbb{E}_{\mathbf{z}\setminus z_j} \big[\log\big(p(\mathbf{x},\mathbf{z})\big) \big] \big) \tag{24}
$$

3.4 component-wise of $H(q)$:

$$
H(q) = -\int \prod_{i=1}^{M} q_i(z_i) \sum_{i=1}^{M} \log\big(q_i(z_i)\big) \mathrm{d}\mathbf{z} \tag{25}
$$

Note that the above needs to integrate out all $\mathbf{z} = \{z_1, ..., z_M\}$, which is quite daunting. However, notice that each term in the sum, $\sum_{i=1}^{M} \log\big(q_i(z_i)\big)$ involves only a single i, therefore, we are able to simplify the above into the following:

$$
\begin{aligned}
H(q) &= \sum_{i=1}^{M} \Big(-\int_{z_i} q_i(z_i) \log\big(q_i(z_i)\big) \mathrm{d}z_i \Big) \\
&= \sum_{i=1}^{M} H\big(q_i(z_i)\big)
\end{aligned}
\tag{26}
$$

For a particular $p_j(z_j)$, the rest of the sum can be treated like a constant, therefore for $p_j(z_j)$ can be written as:

$$\begin{aligned} H(q)_j &= -\int_{z_j} q_i(z_i)\log\left(q_i(z_i)\right)\mathrm{d}z_j + \text{Const.} \\ &= H(q_i(z_i)) + \text{Const.} \end{aligned} \tag{27}$$

where Const. are the term does not involve z_j.

3.5 Putting $-H(q,p)$ and Part $H(q)$ together: component-wise ELBO

write ELBO(q) in terms of q_j, i.e., ELBO(q_j), in which we try to optimize q_j. The rest of the terms would also need to be optimized $\{q_i\}_{i\neq j}$:

$$\begin{aligned} \text{ELBO}(q_j) &= -H(q,p)_j + H(q)_j \\ &= \int_{z_j} q_j(z_j)\mathbb{E}_{\mathbf{z}\setminus z_j}\big[\log\left(p(\mathbf{x},\mathbf{z})\right)\big]\mathrm{d}z_j - \int_{z_j} q_j(z_j)\log\left(q_j(z_j)\right)\mathrm{d}z_j + \text{Const.} \end{aligned} \tag{28}$$

the key to realize is that we do not need to take derivative as one would normally do. All we need is to re-arrange the terms, and to realise it's the KL term, so we can just match the two distributions, between $\exp(\mathbb{E}_{\mathbf{z}\setminus z_j}\big[\log\left(p(\mathbf{x},\mathbf{z})\right)\big])$ and $q_j(z_j)$. Note that both are distributions of z_j

therefore Eq.(28) can be expressed equivalently as:

$$\begin{aligned} \text{ELBO}(q_j) &= \int_{z_j} q_j(z_j)\log\left[\frac{\exp(\mathbb{E}_{\mathbf{z}\setminus z_j}\big[\log\left(p(\mathbf{x},\mathbf{z})\right)\big])}{q_j(z_j)}\right] + \text{Const..} \\ &= -\mathbb{KL}\Big(q_j(z_j)\|\exp(\mathbb{E}_{\mathbf{z}\setminus z_j}[\log\left(p(\mathbf{x},\mathbf{z})\right)])\Big) \end{aligned} \tag{29}$$

Now this is the key: We can maximise ELBO(q), by minimising the KL divergence, where we can find approximate and optimal $q_j^*(z_j)$, by setting KL divergence in Eq.(29) to zero, i.e., $\exp(\mathbb{E}_{\mathbf{z}\setminus z_j}[\log\left(p(\mathbf{x},\mathbf{z})\right)]) = q_j^*(z_j)$. We also assume other $\{q_i(z_i)\}_{i\neq j}$ set is fixed.

$$\begin{aligned} q_j^*(z_j) &= \exp(\mathbb{E}_{\mathbf{z}\setminus z_j}[\log\left(p(\mathbf{x},\mathbf{z})\right)]) \\ \implies \log\left(q_j^*(z_j)\right) &= \mathbb{E}_{\mathbf{z}\setminus z_j}\big[\log\left(p(\mathbf{x},\mathbf{z})\right)\big] \end{aligned} \tag{30}$$

3.6 the update formula

therefore, the algorithm, imagine $\mathbf{z} = \{z_1, z_2, z_3\}$, and at iteration $t-1$, where we already have $q_1^{(t-1)}, q_2^{(t-1)}, q_3^{(t-1)}$

$$
\begin{aligned}
q^{(t)}(z_1, z_2, z_3) &: \begin{cases} q_1^{(t)}(z_1) & = \exp(\mathbb{E}_{z_2 \sim q_2^{(t-1)}, z_3^{(t-1)} \sim q_3^{(t-1)}}[\log(p(\mathbf{x}, z_1, z_2, z_3))]) \\ q_2^{(t)}(z_2) & = \exp(\mathbb{E}_{z_1 \sim q_1^{(t)}, z_3 \sim q_3^{(t-1)}}[\log(p(\mathbf{x}, z_1, z_2, z_3))]) \\ q_3^{(t)}(z_3) & = \exp(\mathbb{E}_{z_1 \sim q_1^{(t)}, z_2 \sim q_2^{(t)}}[\log(p(\mathbf{x}, z_1, z_2, z_3))]) \end{cases} \\
q^{(t+1)}(z_1, z_2, z_3) &: \begin{cases} q_1^{(t+1)}(z_1) & = \exp(\mathbb{E}_{z_2 \sim q_2^{(t)}, z_3^{(t)} \sim q_3^{(t)}}[\log(p(\mathbf{x}, z_1, z_2, z_3))]) \\ q_2^{(t+1)}(z_2) & = \exp(\mathbb{E}_{z_1 \sim q_1^{(t+1)}, z_3 \sim q_3^{(t)}}[\log(p(\mathbf{x}, z_1, z_2, z_3))]) \\ q_3^{(t+1)}(z_3) & = \exp(\mathbb{E}_{z_1 \sim q_1^{(t+1)}, z_2 \sim q_2^{(t+1)}}[\log(p(\mathbf{x}, z_1, z_2, z_3))]) \end{cases} \\
&\vdots
\end{aligned}
\tag{31}
$$

4 Example: Gaussian-Gamma (Conjugate) posterior

4.1 toy model

4.1.1 likelihood

Let $\mathcal{D} = \{x_1, \dots x_n\}$:

$$\begin{aligned} p(\mathcal{D}|\mu, \tau) &= \prod_{i=1}^{n} \left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} \exp\left(\frac{-\tau}{2}(x_i - \mu)^2\right) \\ &= \left(\frac{\tau}{2\pi}\right)^{\frac{n}{2}} \exp\left(\frac{-\tau}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \end{aligned} \tag{32}$$

4.1.2 prior

$$\begin{aligned} p(\mu|\tau) &= \mathcal{N}(\mu_0, (\lambda_0\tau)^{-1}) \propto \exp\left(\frac{-\lambda_0\tau}{2}(\mu - \mu_0)^2\right) \\ p(\tau) &= \text{Gamma}(\tau|a_0, b_0) \propto \tau^{a_0 - 1}\exp^{-b_0\tau} \end{aligned} \tag{33}$$

4.1.3 posterior

Of course, due to conjugacy, the solution can be found exactly:

$$\begin{aligned} p(\mu, \tau|\mathcal{D}) &\propto p(\mathcal{D}|\mu, \tau)p(\mu|\tau)p(\tau) \\ &= \mathcal{N}(\mu_n, (\lambda_n\tau)^{-1})\text{Gamma}(\tau|a_n, b_n) \end{aligned} \tag{34}$$

where:

$$\begin{aligned} \mu_n &= \frac{\lambda_0\mu_0 + n\bar{x}}{\lambda_0 + n} \\ \lambda_n &= \lambda_0 + n \\ a_n &= a_0 + n/2 \\ b_n &= b_0 + \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{\lambda_0 n(\bar{x} - \mu_0)^2}{2(\lambda_0 + n)} \end{aligned} \tag{35}$$

the exact derivation will be omitted and can be found from external sources easily.

But we pretended that we canot compute the posterior, and we will use Variational Inference to approximate the posterior.

4.2 mean-field Variational Inference algorithm

we let $q(\mathbf{z})$ to be:

$$q(\mu, \tau) = q_\mu(\mu)q_\tau(\tau) \tag{36}$$

We use Variational Bayes formula from 30, $\log q_j^*(z_j) = \mathbb{E}_{\mathbf{z}\setminus z_j}[\log(p(\mathbf{x}, \mathbf{z}))]$

4.2.1 $\log\left(q^*_\mu(\mu)\right) = \mathbb{E}_{q_\tau(\tau)}\left[\log\left(p(\mu,\tau,\mathcal{D})\right)\right]$

$$
\begin{aligned}
\log\left(q^*_\mu(\mu)\right) &= \mathbb{E}_{q_\tau}\Big[\log\left(p(\mu,\tau,\mathcal{D})\right)\Big] \\
&= \mathbb{E}_{q_\tau}\Big[\log(p(\mathcal{D}|\mu,\tau)) + \log p(\mu|\tau)\Big] + \text{Const.} \qquad \text{leave out terms do NOT contain } \mu \\
&= \mathbb{E}_{q_\tau}\Big[\underbrace{\frac{n}{2}\log\left(\tau\right) - \frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2}_{\log(p(\mathcal{D}|\mu,\tau))} \underbrace{-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2}_{\log p(\mu|\gamma)}\Big] + \text{Const.} \\
&= -\frac{\mathbb{E}_{q_\tau}[\tau]}{2}\underbrace{\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right]}_{\text{terms taking out of } \mathbb{E}_\tau[\cdot]} + \text{Const.}
\end{aligned}
\tag{37}
$$

Completing the square for the μ terms (well, you guess it, we are going to trying to make it a guassian distribution $\mathcal{N}(\mu;\mu^\star,\tau^\star)$). So let's find out what $\mu^\star$ and $\tau^\star$ are:

$$
\begin{aligned}
\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2 &= n\mu^2 - 2n\mu\bar{x} + \lambda_0\mu^2 - 2\lambda_0\mu_0\mu + \text{Const.} \\
&= (n+\lambda_0)\mu^2 - 2\mu(n\bar{\mathbf{x}} + \lambda_0\mu_0) \\
&= (n+\lambda_0)\left(\mu^2 - \frac{2\mu(n\bar{\mathbf{x}}+\lambda_0\mu_0)}{(n+\lambda_0)}\right) \\
&= (n+\lambda_0)\left(\mu - \frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}\right)^2 + \text{Const.}
\end{aligned}
\tag{38}
$$

Therefore, we have:

$$
\begin{aligned}
\log\left(q^*_\mu(\mu)\right) &= -\frac{\mathbb{E}_{q_\tau}[\tau]}{2}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right] + \text{Const.} \\
&= -\frac{\mathbb{E}_{q_\tau}[\tau](n+\lambda_0)}{2}\left(\mu - \frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}\right)^2 + \text{Const.} \\
&= -\frac{1}{2}\underbrace{\mathbb{E}_{q_\tau}[\tau](n+\lambda_0)}_{\tau^\star}\left(\mu - \underbrace{\frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}}_{\mu^\star}\right)^2 + \text{Const.} \\
\implies\ q^*_\mu(\mu) &= \mathcal{N}\left(\frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}, \mathbb{E}_{q_\tau}[\tau](n+\lambda_0)\right) \qquad \because -\frac{\tau}{2}(x-\mu)^2
\end{aligned}
\tag{39}
$$

by the way, from Eq.(37), without the $\mathbb{E}_{q_\tau}\big[\,\cdot\,\big]$, we obtained the expression for $p(\mu|\mathcal{D},\tau)$:

$$\begin{aligned}
\log(p(\mathcal{D}|\mu,\tau)) + \log p(\mu|\tau) &= \underbrace{-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2}_{\log(p(\mathcal{D}|\mu,\tau))} \underbrace{-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2}_{\log p(\mu|\gamma)} + \text{Const.} \\
&= -\frac{\tau}{2}\Big[\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\Big] + \text{Const.} \\
&= -\frac{\tau(n+\lambda_0)}{2}\left(\mu - \frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}\right)^2 + \text{Const.} \\
\implies p(\mu|\mathcal{D},\tau) &= \mathcal{N}\left(\frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}, \tau(n+\lambda_0)\right) \qquad \text{Eq.(38)}
\end{aligned} \tag{40}$$

4.3 Computing $\log\left(q_i^*(\tau)\right) = \mathbb{E}_{q_\mu(\mu)}\left[\log\left(p(\mu,\tau,\mathcal{D})\right)\right]$

$$\begin{aligned}
\log\left(q_\tau^*(\tau)\right) &= \mathbb{E}_{q_\mu}\left[\log\left(p(\mu,\tau,\mathcal{D})\right)\right] \\
&= \mathbb{E}_{q_\mu}\left[\log(p(\mathcal{D}|\mu,\tau)) + \log p(\mu|\tau) + \log p(\tau)\right] + \text{Const.} \\
&= \mathbb{E}_{q_\mu}\Big[\underbrace{\frac{n}{2}\log(\tau) - \frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2}_{\log(p(\mathcal{D}|\mu,\tau))} \underbrace{-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2}_{\log p(\mu|\gamma)} \underbrace{+(a_0-1)\log(\tau) - b_0\tau}_{\log p(\tau)}\Big] + \text{Const.}
\end{aligned} \tag{41}$$

Bring terms without μ outside of the integral:

$$\begin{aligned}
&= \frac{n}{2}\log(\tau) + (a_0-1)\log(\tau) - b_0\tau - \frac{\tau}{2}\mathbb{E}_{q_\mu(\mu)}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right] + \text{Const.} \\
&= \Big(\underbrace{\frac{n}{2}+a_0}_{a_n}-1\Big)\log(\tau) - \tau\underbrace{\Big(b_0 + \frac{1}{2}\mathbb{E}_{q_\mu(\mu)}\Big[\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\Big]\Big)}_{b_n} + \text{Const.} \\
\implies q_\tau^*(\tau)) &= \text{Gamma}(a_n, b_n)
\end{aligned} \tag{42}$$

We can rewrite,

$$\begin{aligned}
b_n &= b_0 + \frac{1}{2}\mathbb{E}_{q_\mu}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right] \\
&= b_0 + \frac{1}{2}\mathbb{E}_{q_\mu}\left[-2\mu n\bar{x} + n\mu^2 + \lambda_0\mu^2 - 2\lambda_0\mu_0\mu\right] + \sum_{i=1}^{n}(x_i)^2 + \lambda_0\mu_0^2 \\
&= b_0 + \frac{1}{2}\left[(n+\lambda_0)\mathbb{E}_{q_\mu}[\mu^2] - 2\left(n\bar{x}+\lambda_0\mu_0\right)\mathbb{E}_{q_\mu}[\mu] + \sum_{i=1}^{n}(x_i)^2 + \lambda_0\mu_0^2\right]
\end{aligned} \tag{43}$$

We will compute $\mathbb{E}_{q_\mu}[\mu]$ and $\mathbb{E}_{q_\mu}[\mu^2]$ since we know of $q_\mu(\mu)$ from previously.

By the way, from Eq.(41), without the $\mathbb{E}_{q_\mu}[\cdot]$, we obtained expression for $p(\tau|\mathcal{D},\mu)$:

$$\begin{aligned}\log p(\tau|\mathcal{D},\mu) &= \log(p(\mathcal{D}|\mu,\tau)) + \log p(\mu|\tau) + \log p(\tau) + \text{Const.}\\ &= \underbrace{\frac{n}{2}\log(\tau) - \frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2}_{\log(p(\mathcal{D}|\mu,\tau))} \underbrace{-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2}_{\log p(\mu|\gamma)} \underbrace{+(a_0-1)\log(\tau) - b_0\tau}_{\log p(\tau)} + \text{Const.}\\ &= \Big(\underbrace{\frac{n}{2}+a_0}_{a_n} - 1\Big)\log(\tau) - \tau\underbrace{\Big(b_0 + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\Big)}_{b_n} + \text{Const.}\end{aligned} \tag{44}$$

$\implies p(\tau|\mathcal{D},\mu) = \text{Gamma}(a_n, b_n)$ where:

$$\begin{cases} a_n &= \frac{n}{2} + a_0 \\ b_n &= b_0 + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2 \end{cases} \tag{45}$$

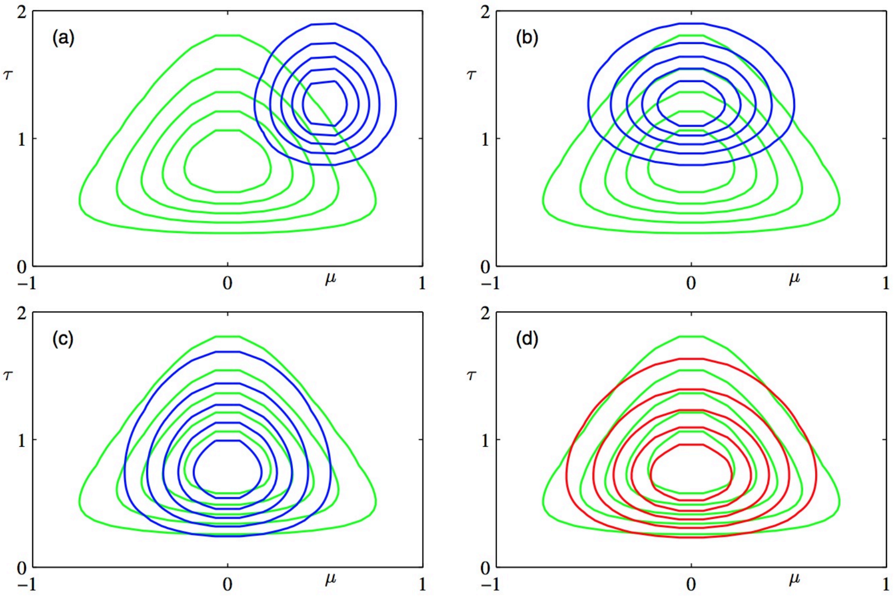


Figure 1: update for Normal Gamma: figure from [1]

4.3.1 update for $q_\mu(\mu)$ and $q_\tau(\tau)$

In summary, the update for $q_\mu(\mu)$ and $q_\tau(\tau)$ are:

$$\begin{aligned} q_\mu^*(\mu) &= \mathcal{N}\left(\frac{n\bar{\mathbf{x}}+\lambda_0\mu_0}{n+\lambda_0}, \mathbb{E}_{q_\tau}[\tau](n+\lambda_0)\right)\\ q_\tau^*(\tau) &= \text{Gamma}(a_n, b_n)\end{aligned} \tag{46}$$

where:

$$\begin{cases} a_n &= \frac{n}{2} + a_0 \\ b_n &= b_0 + \frac{1}{2} \left[(n+\lambda_0)\mathbb{E}_{q_\mu}[\mu^2] - 2\left(n\bar{x} + \lambda_0\mu_0\right)\mathbb{E}_{q_\mu}[\mu] + \sum_{i=1}^{n}(x_i)^2 + \lambda_0\mu_0^2 \right] \end{cases} \tag{47}$$

with:

$$\begin{aligned} \mathbb{E}_{q_\mu}[\mu] &= \frac{n\bar{x} + \lambda_0\mu_0}{n+\lambda_0} \\ \mathbb{E}_{q_\mu}[\mu^2] &= \mathrm{Var}_{q_\mu}[\mu] + (\mathbb{E}_{q_\mu}[\mu])^2 = \frac{1}{\mathbb{E}_{q_\tau}[\tau](n+\lambda_0)} + \left(\frac{n\bar{x} + \lambda_0\mu_0}{n+\lambda_0}\right)^2 \end{aligned} \tag{48}$$

Note that $\mathbb{E}_{q_\tau}[\tau] = \frac{a_n}{b_n}$ for the Gamma distribution.

4.4 Interactive Python Program for Variational Inference Visualization

To visualize the variational inference process for the Normal-Gamma distribution, we can create an interactive Python program using matplotlib. This program will show the true joint distribution p(,) and the variational approximations q( ) and q( ) after each iteration.

Here's the Python code to implement this visualization:

5 Example of Gaussian Mixture Model Optional - not examinable

5.1 The joint density

$$\begin{aligned} p(X, Z, \mu, \Lambda, \pi) &= p(X|Z, \mu, \Lambda, \pi)p(Z|\mu, \Lambda, \pi)p(\mu|\Lambda, \pi)p(\Lambda|\pi)p(\pi) \\ &= p(X|Z, \mu, \Lambda)p(Z|\pi)p(\mu|\Lambda)p(\Lambda)p(\pi) \end{aligned} \tag{49}$$

5.2 Definitions for each probabilities

5.2.1 Definition for $p(Z|\pi)$:

first, is the probability of mixture indices, $Z = \{z_1, ..., z_N\}$, given weights π.

$$\begin{aligned} p(Z|\pi) &= \prod_{i=1}^{N} p(z_n|\pi) \\ &= \prod_{i=1}^{N} \prod_{k=1}^{K} \pi_k^{z_{ik}} \end{aligned} \tag{50}$$

The reason for which $\left(p(z_n|\pi) = \prod_{k=1}^{K} \pi_k^{z_{ik}}\right)$, or $\left(p(z|\pi) = \prod_{k=1}^{K} \pi_k^{z_k}\right)$, is because in Bishop, z is not represented in a scalar form, but rather in a vector of dimension K, which has a single element 1, and the rest are all 0s. For example, instead of using $p(z_n = 2|\pi = [0.2, 0.3, 0.5]) = 0.3$, Bishop uses $p(z_n = [0, 1, 0]|\pi = [0.2, 0.3, 0.5]) = 0.3$. In any case, this refers to the second element of π. Therefore, a more simpler and vocal representation for $p(z|\pi)$ is just the z^{th} value of π.

5.2.2 Definition for $p(X|Z, \mu, \Lambda)$:

$$p(X|Z, \mu, \Lambda) = \prod_{i=1}^{N} p(x_n|z_n, \mu, \Lambda)$$

In normal literatures, such as Bilmes, it is defined as:

$$= \prod_{i=1}^{N} \mathcal{N}(x_n|\mu_{z_n}, \Lambda_{z_n}^{-1}) \tag{51}$$

However, due to the vector representation of Bishop, the above is defined as:

$$= \prod_{i=1}^{N} \prod_{k=1}^{K} \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})^{z_{ik}}$$

However, the above two represent the same thing:

5.2.3 Definition for $p(\pi)$:

This is just a straight Dirichlet probability:

$$\begin{aligned} p(\pi|\alpha_0) &= \mathrm{Dir}(\pi|\alpha_0) \propto C(\alpha_0) \prod_{k=1}^{K} \pi_k^{\alpha_{0k}-1} \\ &\implies \log(\pi|\alpha_0) \propto (\alpha_0 - 1) \sum_{k=1}^{K} \log \pi_k \end{aligned} \tag{52}$$

5.2.4 Definition for $p(\mu|\Lambda)p(\Lambda)$:

This is almost always a Gaussian-Wishart distribution:

$$\begin{aligned} p(\mu, \Lambda) &= p(\mu|\Lambda)p(\Lambda) \\ &= \prod_{k=1}^{K} \mathcal{N}(\mu_k|m_0, (\beta_0\Lambda_k)^{-1})\mathcal{W}(\Lambda_k|W_0, v_0) \end{aligned} \tag{53}$$

5.3 Begin VB of GMM

5.3.1 The expression for $q^*(Z)$:

$$\begin{aligned} \log q^*(Z) &= \mathbb{E}_{\pi,\mu,\Lambda}\left[\log p(X, Z, \pi, \mu, \Lambda)\right] + \text{Const.} \\ &= \mathbb{E}_{\pi}\left[\log p(Z|\pi)\right] + \mathbb{E}_{\mu,\Lambda}\left[\log p(X|Z, \mu, \Lambda)\right] + \text{Const.} \\ &= \mathbb{E}_{\pi}\left[\log \prod_{i=1}^{N}\prod_{k=1}^{K} \pi_k^{z_{ik}}\right] + \mathbb{E}_{\mu,\Lambda}\left[\log \prod_{i=1}^{N}\prod_{k=1}^{K} \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})^{z_{ik}}\right] + \text{Const.} \\ &= \mathbb{E}_{\pi}\left[\sum_{i=1}^{N}\sum_{k=1}^{K} \log \pi_k^{z_{ik}}\right] + \mathbb{E}_{\mu,\Lambda}\left[\sum_{i=1}^{N}\sum_{k=1}^{K} \log \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})^{z_{ik}}\right] + \text{Const.} \\ &\text{given that:, } (\log a^b = b \log a) : \\ &= \mathbb{E}_{\pi}\left[\sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik} \log \pi_k\right] + \mathbb{E}_{\mu,\Lambda}\left[\sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik} \log \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})\right] + \text{Const.} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\mathbb{E}_{\pi}[\log \pi_k] + \sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\mathbb{E}_{\mu,\Lambda}[\log \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})] + \text{Const.} \\ &\text{Taking the common term to the left, } \sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik} : \\ &= \sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\left(\mathbb{E}_{\pi}[\log \pi_k] + \mathbb{E}_{\mu,\Lambda}[\log \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})]\right) + \text{Const.} \\ &\text{Bishop nominates a new term: } \log \rho_{ik} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\left(\log \rho_{ik}\right) + \text{Const.} \end{aligned} \tag{54}$$

Let's look at the expression for $\log \rho_{ik}$:

$$\begin{aligned}
\log \rho_{ik} &= \mathbb{E}_{\pi}[\log \pi_k] + \mathbb{E}_{\mu_k,\Lambda_k}[\log \mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})] \\
&= \mathbb{E}_{\pi}[\log \pi_k] + \mathbb{E}_{\mu_k,\Lambda_k}\left[\log\left(\frac{1}{(2\pi)^{(d/2)}}|\Lambda_k|^{1/2}\exp\left(-\frac{1}{2}(x_n-\mu_k)^{\top}\Lambda_k(x_n-\mu_k)\right)\right)\right] \\
&= \mathbb{E}_{\pi}[\log \pi_k] + \mathbb{E}_{\mu_k,\Lambda_k}\left[\log(2\pi)^{\frac{-d}{2}} + \frac{1}{2}\log|\Lambda_k| + \left(-\frac{1}{2}(x_n-\mu_k)^{\top}\Lambda_k(x_n-\mu_k)\right)\right] \\
&= \mathbb{E}_{\pi}[\log \pi_k] + \mathbb{E}_{\mu_k,\Lambda_k}\left[\frac{-d}{2}\log(2\pi) + \frac{1}{2}\log|\Lambda_k| - \left(\frac{1}{2}(x_n-\mu_k)^{\top}\Lambda_k(x_n-\mu_k)\right)\right] \\
&= \mathbb{E}_{\pi}[\log \pi_k] + \frac{-d}{2}\log(2\pi) + \frac{1}{2}\mathbb{E}_{\Lambda_k}\left[\log|\Lambda_k|\right] - \frac{1}{2}\mathbb{E}_{\mu_k,\Lambda_k}\left[(x_n-\mu_k)^{\top}\Lambda_k(x_n-\mu_k)\right]
\end{aligned} \tag{55}$$

Now, since $\log q^*(Z) = \log \rho_{ik}$

$$\begin{aligned}
\log q^*(Z) &= \sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\left(\log \rho_{ik}\right) + \text{Const.} \implies \\
q^*(Z) &= \exp\left(\sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\left(\log \rho_{ik}\right) + \text{Const.}\right) \\
&= C\prod_{i=1}^{N}\prod_{k=1}^{K}\exp(z_{ik}\left(\log \rho_{ik}\right)) = C\prod_{i=1}^{N}\prod_{k=1}^{K}\exp\left(\log \rho_{ik}^{z_{ik}}\right) = C\prod_{i=1}^{N}\prod_{k=1}^{K}\rho_{ik}^{z_{ik}}
\end{aligned} \tag{56}$$

Since $q^*(Z) = \prod_{i=1}^{N} q^*(z_n)$:

$$q^*(Z) = \prod_{i=1}^{N} C \prod_{k=1}^{K}\rho_{ik}^{z_{ik}} \tag{57}$$

In a way, $\rho_{ik}^{z_{ik}}$ plays the same role as π in $p(z_n|\pi)$, therefore, $\sum_{k=1}^{K}\pi_k = 1 \implies \sum_{k=1}^{K}\rho_{ik} = 1$:

$$\begin{aligned}
q^*(Z) = \prod_{i=1}^{N} q^*(z_i) &= \prod_{i=1}^{N}\left(\frac{1}{\sum_{j=1}^{K}\rho_{nj}}\prod_{k=1}^{K}\rho_{ik}^{z_{ik}}\right) \\
&= \prod_{i=1}^{N}\prod_{k=1}^{K}\frac{\rho_{ik}^{z_{ik}}}{\sum_{j=1}^{K}\rho_{nj}} = \prod_{i=1}^{N}\prod_{k=1}^{K} r_{nk}^{z_{ik}}
\end{aligned} \tag{58}$$

This is a multinomial distribution, therefore, $\mathbb{E}[z_i = k] = r_{ik}$

5.3.2 The expression for $q^*(\pi, \mu, \Lambda)$:

$$\begin{aligned}
\log q^*(\pi,\mu,\Lambda) &= \mathbb{E}_Z\left[\log p(X, Z, \pi, \mu, \Lambda)\right] + \text{Const.} \\
&= \mathbb{E}_Z\left[\log p(X|Z,\mu,\Lambda)\right] + \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \mathbb{E}_Z\left[\log p(\pi)\right] + \mathbb{E}_Z\left[\log p(\mu|\Lambda)\right] + \mathbb{E}_Z\left[\log p(\Lambda)\right] + \text{Const.} \\
&= \mathbb{E}_Z\left[\log p(X|Z,\mu,\Lambda)\right] + \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi) + \log p(\mu|\Lambda) + \log p(\Lambda) + \text{Const.}
\end{aligned} \tag{59}$$

Combine the mean and precision together:

$= \mathbb{E}_Z\left[\log p(X|Z,\mu,\Lambda)\right] + \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi) + \log p(\mu,\Lambda) + \text{Const.}$

And since each (μ_k, Λ_k) are independent, therefore:

$$
\begin{aligned}
&= \mathbb{E}_Z\left[\log \prod_{i=1}^{N}\prod_{k=1}^{K} \mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})^{z_{ik}}\right] + \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi) + \sum_{k=1}^{K}\log p(\mu_k,\Lambda_k) + \text{Const.}\\
&= \mathbb{E}_Z\left[\sum_{i=1}^{N}\sum_{k=1}^{K} \log(z_{ik})\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right] + \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi) + \sum_{k=1}^{K}\log p(\mu_k,\Lambda_k) + \text{Const.}\\
&= \sum_{k=1}^{K}\sum_{i=1}^{N} \mathbb{E}_Z[\log(z_{ik})]\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1}) + \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi) + \sum_{k=1}^{K}\log p(\mu_k,\Lambda_k) + \text{Const.}\\
&= \underbrace{\mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi)}_{\log q^*(\pi)} + \underbrace{\sum_{k=1}^{K}\sum_{i=1}^{N} \mathbb{E}_Z[\log(z_{ik})]\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1}) + \sum_{k=1}^{K}\log p(\mu_k,\Lambda_k)}_{\log q^*(\mu,\Lambda)} + \text{Const.}
\end{aligned}
\tag{60}
$$

For the part of $\log q^*(\pi)$:

$$
\begin{aligned}
\log q^*(\pi) &= \mathbb{E}_Z\left[\log p(Z|\pi)\right] + \log p(\pi)\\
&= \mathbb{E}_Z\left[\log \prod_{i=1}^{N}\prod_{k=1}^{K} \pi_k^{z_{ik}}\right] + \log p(\pi)\\
&= \mathbb{E}_Z\left[\sum_{i=1}^{N}\sum_{k=1}^{K} z_{ik}\log \pi_k\right] + \log p(\pi)\\
&= \sum_{i=1}^{N}\sum_{k=1}^{K} \log \pi_k \mathbb{E}_Z[z_{ik}] + (\alpha_0 - 1)\sum_{k=1}^{K}\log \pi_k + \text{Const.}\\
&= \sum_{k=1}^{K}\log \pi_k \sum_{i=1}^{N} r_{i,k} + (\alpha_0 - 1)\sum_{k=1}^{K}\log \pi_k + \text{Const.}\\
&= \left(\underbrace{\sum_{i=1}^{N} r_{i,k} + \alpha_0}_{a_n} - 1\right)\sum_{k=1}^{K}\log \pi_k + \text{Const.} = \text{DIR}\left(\pi|a_n\right)
\end{aligned}
\tag{61}
$$

For the part of $\log q^*(\mu,\Lambda)$:

$$
\log q^*(\mu,\Lambda) = \sum_{k=1}^{K}\sum_{i=1}^{N} \mathbb{E}_Z[\log(z_{ik})]\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1}) + \sum_{k=1}^{K}\log p(\mu_k,\Lambda_k)
\tag{62}
$$

We only have the expression for $\mathbb{E}_{q^*(Z)}[Z]$, but not $\mathbb{E}_{q^*(Z)}[\log(Z)]$:

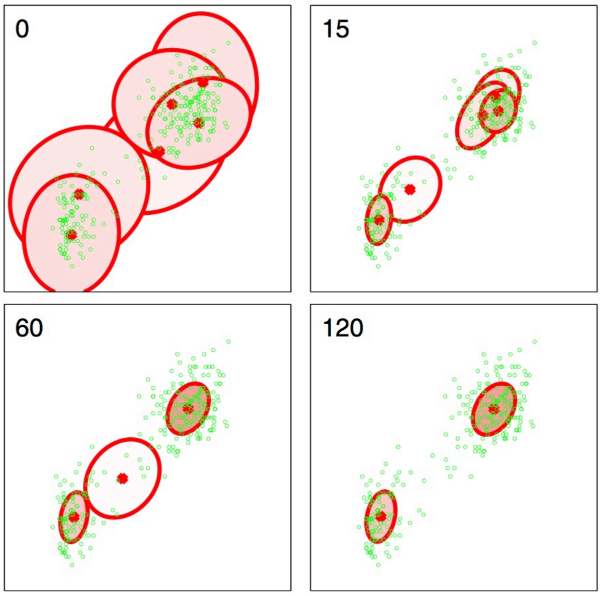


Figure 2: update for Gaussian Mixture Model: figure from [1]

6 Variational update in Exponential Family distribution

6.1 Big picture

Given both the prior and likelihood are exponential family distributions and they form a conjugacy pair, then the variational inference (also mean-field approximation), i.e., $q(\mathbf{z}) = \prod_i q_i(z_i)$ can have the following update formula:

$$\eta_j = \mathbb{E}_{q(\mathbf{z}\setminus z_j|\cdot)}[\eta_{\text{post}}(\mathbf{z} \setminus z_j)] \tag{63}$$

where $\eta_{\text{post}}(\mathbf{z} \setminus z_j)$ is the natural parameter associated with posterior distribution $p(z_j \mid \mathbf{z} \setminus z_j, \mathbf{x})$, i.e., we can express it as:

$$p_{\eta_{\text{post}}(\mathbf{z}\setminus z_j)} p(z_j \mid \mathbf{z} \setminus z_j, \mathbf{x}) \tag{64}$$

I know the probablity form of the above seems to be very complicated, however, you will see that it is no different to how we normally expressing distributions, $p_\theta(\mathbf{z}|\mathbf{x})$, where:

$$p \underbrace{\theta}_{\text{parameters}} (\underbrace{\mathbf{z}}_{\text{variable}} \mid \underbrace{\mathbf{x}}_{\text{conditioning variables}}) \tag{65}$$

It is very important to note that we are expressing a posterior distribution $p(\cdot)$ in terms of variable z_j. Therefore variables $\mathbf{z} \setminus z_j$ is now part of the parameters of the distribution, hence we use $\eta_{\text{post}}(\mathbf{z} \setminus z_j)$ to denote the natural parameter of the distribution $p(z_j|-)$.

We can be made much clearly by using an example: if we have a posterior distribution concerning $z_1, \dots, z_3$, and we are interested in the posterior distribution of z_2 given z_1 and z_3 and $\mathbf{x}$, i.e.,

$$\begin{aligned} p(z_2 \mid \mathbf{z} \setminus z_2, \mathbf{x}) &= p(z_2 \mid z_1, z_3, \mathbf{x}) \\ &\equiv p_{\eta_{\text{post}}(z_1,z_3)}(z_2 \mid z_1, z_3, \mathbf{x}) \end{aligned} \tag{66}$$

then the update formula for $q_{\eta_2}(z_2)$ is:

$$\eta_2 = \mathbb{E}_{q(z_1,z_3)}\left[\eta_{\text{post}}(z_1, z_3)\right] \tag{67}$$

compare this with the generic update formula:

$$\log\left(q_i^*(z_i)\right) = \mathbb{E}_{i\neq j}\left[\log\left(p(\mathbf{x}, \mathbf{z})\right)\right] \tag{68}$$

using exponential family update formula Eq.(63), the update is directly applied to the parameter.

6.1.1 where is the catch?

Sounds too easy? Yes, it is! However, it is important to note that this is only applicable to exponential family distributions and conjugacy, and also η is the natural parameter when we are dealing with exponential family distributions. Then, obviously, we need to study a few things:

1. What is exponential family distribution?
2. What is natural parameter?
3. What is conjugacy?

Let's talk about exponential family distribution first.

6.2 What is Exponential Family?

Most of the distributions we are going to look at are from exponential family. They are expressed in terms of its natural parameter η:

$$h(x)\exp\left(T(x)^\top \eta - A(\eta)\right) \tag{69}$$

$$\begin{aligned} &\underbrace{\exp(-A(\eta))}_{\text{normalization}} h(x)\exp\{T(x)^\top \eta\} \\ \implies &\exp(-A(\eta))\int_x h(x)\exp\{T(x)^\top \eta\} = 1 \\ \implies &\int_x h(x)\exp\{T(x)^\top \eta\} = \exp(A(\eta)) \end{aligned} \tag{70}$$

6.2.1 example: 1-d Gaussian

$$\begin{aligned} \mathcal{N}(x;\mu,\sigma^2) &= (2\pi\sigma^2)^{-1/2}\exp^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ &= \exp\left(-\frac{x^2-2x\mu+\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right) \\ &= \exp\left(-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right) \\ &= \exp\left(\begin{bmatrix} x & x^2 \end{bmatrix}\begin{bmatrix} \frac{\mu}{\sigma^2} & -\frac{1}{2\sigma^2} \end{bmatrix}^\top - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right) \end{aligned} \tag{71}$$

$$\begin{aligned} T(\mathbf{x}) &= \begin{bmatrix} x & x^2 \end{bmatrix} \\ \boldsymbol{\eta} &= \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\mu}{\sigma^2} & -\frac{1}{2\sigma^2} \end{bmatrix} \end{aligned} \tag{72}$$

1. for η_2:

$$\eta_2 = -\frac{1}{2\sigma^2} \implies \sigma^2 = -\frac{1}{2\eta_2} \tag{73}$$

2. for η_1:

$$\begin{aligned}\eta_1 = \frac{\mu}{\sigma^2} \implies \mu &= \eta_1\sigma^2 \\ &= \eta_1\frac{-1}{2\eta_2} \\ &= \frac{-\eta_1}{2\eta_2}\end{aligned} \tag{74}$$

summarize, we have:

$$\theta = \begin{bmatrix}\mu \\ \\ \sigma^2\end{bmatrix} = \begin{bmatrix}\frac{-\eta_1}{2\eta_2} \\ \\ \frac{-1}{2\eta_2}\end{bmatrix} \tag{75}$$

6.2.2 in natural parameter form

now we can remove μ and σ^2:

$$\begin{aligned}\mathcal{N}_{\text{nat}}(x,\boldsymbol{\eta}) &= \exp\left(\begin{bmatrix}x & x^2\end{bmatrix}\begin{bmatrix}\eta_1 & \eta_2\end{bmatrix}^\top - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right) \\ &= \exp\left(\begin{bmatrix}x & x^2\end{bmatrix}\begin{bmatrix}\eta_1 & \eta_2\end{bmatrix}^\top - \frac{\left(\frac{-\eta_1}{2\eta_2}\right)^2}{2\left(\frac{-1}{2\eta_2}\right)} - \frac{1}{2}\log\left(2\pi\left(\frac{-1}{2\eta_2}\right)\right)\right) \\ &= \exp\left(T(x)^\top\boldsymbol{\eta} + \frac{\eta_1^2}{4\eta_2} - \frac{1}{2}\log\left(\frac{2\pi}{-2\eta_2}\right)\right) \\ &= \exp\left(T(x)^\top\boldsymbol{\eta} + \frac{\eta_1^2}{4\eta_2} + \frac{1}{2}\log(-2\eta_2) - \frac{1}{2}\log(2\pi)\right)\end{aligned} \tag{76}$$

now that the probability is fully in terms of the natural parameter

$$\mathcal{N}_{\text{nat}}(x,\boldsymbol{\eta}) = \exp\Big(T(x)^\top\boldsymbol{\eta} - \underbrace{\left(\frac{-\eta_1^2}{4\eta_2} - \frac{1}{2}\log(-2\eta_2)\right) - \frac{1}{2}\log(2\pi)}_{A(\boldsymbol{\eta})}\Big) \tag{77}$$

6.3 Python time: 2-d Gaussian

$$\mathcal{N}(\mathbf{z};\boldsymbol{\mu},\Sigma) = \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) - \frac{1}{2}\log|\Sigma| - \frac{d}{2}\log(2\pi)\right) \tag{78}$$

let $\Sigma = \begin{bmatrix}\sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2}\end{bmatrix}$, then we can obtain two conditional distribution:

1. $p(z_1|z_2)$
2. $p(z_2|z_1)$

6.3.1 Conditional distribution of z_1 given z_2, i.e., $p(z_1|z_2)$

1. The conditional distribution of z_1 given z_2 for a 2-D Gaussian is:

$$p(z_1|z_2) = \mathcal{N}\left(z_1; \mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(z_2 - \mu_2), \sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}\right) \tag{79}$$

   where $\sigma_{i,j}$ are the elements of the covariance matrix Σ.

2. The natural parameters of conditional distribution parameters correspond to the conditional distribution $p(z_1|z_2)$ is:

$$\begin{aligned}
\eta_1 &= \frac{\mu}{\sigma^2} \\
&= \frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(z_2 - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}} \\
\eta_2 &= -\frac{1}{2\sigma^2} \\
&= -\frac{1}{2(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}})}
\end{aligned} \tag{80}$$

   Note that $\sigma_{1,2} = \sigma_{2,1}$ in a symmetric covariance matrix.

3. therefore, the update formula for $q_{\eta_1}(z_1)$ is:

$$\begin{aligned}
\eta_1 &= \mathbb{E}_{q(z_2)}\left[\frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(z_2 - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}}\right] \\
&= \frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(\mathbb{E}_{q(z_2)}[z_2] - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}} \\
\eta_2 &= -\frac{1}{2(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}})}
\end{aligned} \tag{81}$$

4. Converting these natural parameters back to the standard Gaussian parameterization:

$$\begin{aligned}
\mu_1 = -\frac{\eta_1}{2\eta_2} &= -\frac{1}{2}\left(\frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(\mathbb{E}_{q(z_2)}[z_2] - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}}\right) \times \left((-2)\Big(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}\Big)\right) \\
&= \left(\frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(\mathbb{E}_{q(z_2)}[z_2] - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}}\right) \times \left(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}\right) \\
&= \mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(\mathbb{E}_{q(z_2)}[z_2] - \mu_2)
\end{aligned} \tag{82}$$

$$
\begin{aligned}
\sigma_1^2 &= -\frac{1}{2\eta_2} \\
&= -\frac{1}{2}\left(-\frac{1}{2(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}})}\right)^{-1} \\
&= -\frac{1}{2}\left(-2(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}})\right) \\
&= \sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}
\end{aligned} \tag{83}
$$

6.3.2 Conditional distribution of z_2 given z_1, i.e., $p(z_2|z_1)$

Basically, we can just swap the role of z_1 and z_2 in the above derivation.

1. The conditional distribution of z_1 given z_2 for a 2-D Gaussian is:

$$
p(z_1|z_2) = \mathcal{N}\left(z_1; \mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(z_2 - \mu_2), \sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}\right) \tag{84}
$$

   where $\sigma_{i,j}$ are the elements of the covariance matrix Σ. Likewise, the conditional distribution of z_2 given z_1 is:

2. For the conditional distribution $p(z_2|z_1)$, the corresponding natural parameters are:

$$
\begin{aligned}
\eta_1 &= \frac{\mu_2 + \frac{\sigma_{2,1}}{\sigma_{1,1}}(z_1 - \mu_1)}{\sigma_{2,2} - \frac{\sigma_{2,1}^2}{\sigma_{1,1}}} \\
\eta_2 &= -\frac{1}{2(\sigma_{2,2} - \frac{\sigma_{2,1}^2}{\sigma_{1,1}})}
\end{aligned} \tag{85}
$$

   Note that $\sigma_{1,2} = \sigma_{2,1}$ in a symmetric covariance matrix.

3. therefore, the update formula for $q_{\eta_1}(z_1)$ is:

   we can express the update formula for $q_{\eta_1}(z_1)$ in terms of the covariance matrix elements:

$$
\begin{aligned}
\eta_1 &= \mathbb{E}_{q(z_2)}\left[\frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(z_2 - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}}\right] \\
&= \frac{\mu_1 + \frac{\sigma_{1,2}}{\sigma_{2,2}}(\mathbb{E}_{q(z_2)}[z_2] - \mu_2)}{\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}}} \\
\eta_2 &= -\frac{1}{2(\sigma_{1,1} - \frac{\sigma_{1,2}^2}{\sigma_{2,2}})}
\end{aligned} \tag{86}
$$

4. Converting these natural parameters back to the standard Gaussian parameterization:

$$\begin{aligned}\mu_2 &= \mu_2 + \frac{\sigma_{2,1}}{\sigma_{1,1}}(\mathbb{E}_{q(z_1)}[z_1] - \mu_1) \\ \sigma_2^2 &= \sigma_{2,2} - \frac{\sigma_{2,1}^2}{\sigma_{1,1}}\end{aligned} \tag{87}$$

6.4 conjugate probabilities

conjugacy means that the prior and posterior are of the same type of distributions, for example:

$$\underbrace{p_{\eta_{\text{post}}}(\theta|\mathbf{x})}_{\text{same type}} \propto p(\mathbf{x}|\theta) \underbrace{p_{\eta_{\text{prior}}}(\theta)}_{\text{same type}} \tag{88}$$

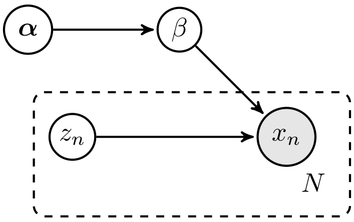
Note that prior and posterior are of the same type, but they are usually not the same distribution, otherwise, the likelihood $p(\mathbf{x}|\theta)$ is not useful at all!

Using exponential family distribution representation, when conjugacy is achieved, it means the prior and posterior have the same sufficient statistics $T(\theta)$ and $h(\theta)$, but different natural parameter, i.e., η_{post} and η_{prior}, and different log-normalizer for both.

It turns out that the posterior inherits $h(\theta)$ from the prior, so we just need to make sure to put the appropriate criteria on the likelihood so that $T(\theta)$ is the same in both the prior and posterior

6.5 What is the criteria for likelihood to pair up with conjugate prior?

It's always better to have a discussion with a concrete example setup. So we have the following problem setup, described in [2], without actually defining what distributions they are using:



the joint density is of the form:

$$p(\mathbf{x}, \mathbf{z}, \boldsymbol{\beta}|\boldsymbol{\alpha}) = p(\boldsymbol{\beta}|\boldsymbol{\alpha}) \prod_{n=1}^{N} p(x_n, z_n|\boldsymbol{\beta}) \tag{89}$$

the conditionals are based on Exponential family:

$$\begin{aligned} p(\boldsymbol{\beta}|\mathbf{x}, \mathbf{z}, \alpha) &= h(\boldsymbol{\beta}) \exp\left\{T(\boldsymbol{\beta})^\top \eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha) - A_{\text{post}}(\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha))\right\} \\ p(z_{n,j}|x_n, z_{n,-j}, \boldsymbol{\beta}) &= h(z_{n,j}) \exp\left\{T(z_{n,j})\eta_{z_{n,j}}(x_n, z_{n,-j}, \boldsymbol{\beta}) - A_l\big(\eta_{z_{n,j}}(x_n, z_{n,-j}, \boldsymbol{\beta})\big)\right\} \end{aligned} \tag{90}$$

Think about why is this representation useful? Let's have look at a numerical example:

6.5.1 Conjugacy of exponential family distribution

Let's work through a concrete example of posterior $p(\boldsymbol{\beta}|x_n, z_n)$, instead of writing $\boldsymbol{\eta}_\beta$, we write $\boldsymbol{\beta}$ directly:

- prior:

$$p(\boldsymbol{\beta}|\boldsymbol{\alpha}) = h(\boldsymbol{\beta}) \exp\{T(\boldsymbol{\beta})^\top \boldsymbol{\alpha} - A_{\text{pri}}(\boldsymbol{\alpha})\} \tag{91}$$

  suppose the sufficient statistics of the prior can be written as:

$$\begin{aligned} T(\boldsymbol{\beta}) &= \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix} \\ \implies \boldsymbol{\alpha} &= \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \alpha_2 \end{bmatrix} \end{aligned} \tag{92}$$

  note that $\boldsymbol{\alpha}_1$ has the same dimensionality to $\boldsymbol{\beta}$, and α_2 is a scalar quantity. Then the prior itself can be written as:

$$p(\boldsymbol{\beta}) = h(\boldsymbol{\beta}) \exp\left\{ \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \alpha_2 \end{bmatrix} - A_{\text{pri}}(\boldsymbol{\alpha}) \right\} \tag{93}$$

- likelihood:
  and if the likelihood density (x_n, z_n) can be defined as:

$$p(x_n, z_n|\boldsymbol{\beta}) = h(x_n, z_n) \exp\left\{ T(x_n, z_n)^\top \boldsymbol{\beta} - A_l(\boldsymbol{\beta}) \right\} \tag{94}$$

- then posterior condition on a single data point:

$$\begin{aligned} p(\boldsymbol{\beta}|x_n, z_n, \boldsymbol{\alpha}) &\propto \underbrace{h(\boldsymbol{\beta}) \exp\{T(\boldsymbol{\beta})^\top \boldsymbol{\alpha}\}} \; \underline{\exp\{T(x_n, z_n)^\top \boldsymbol{\beta} - A_l(\boldsymbol{\beta})\}} \qquad \because h(x_n, z_n) \text{ is a constant} \\ &= \underbrace{h(\boldsymbol{\beta}) \exp\left\{ \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \alpha_2 \end{bmatrix} - A_{\text{pri}}(\boldsymbol{\alpha}) \right\}} \; \underline{\exp\{T(x_n, z_n)^\top \boldsymbol{\beta} - A_l(\boldsymbol{\beta})\}} \\ &\propto h(\boldsymbol{\beta}) \exp\left\{ \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \alpha_2 \end{bmatrix} \right\} \; \underline{\exp\{T(x_n, z_n)^\top \boldsymbol{\beta} - A_l(\boldsymbol{\beta})\}} \qquad \because A_{\text{pri}}(\boldsymbol{\alpha}) \text{ is a constant} \\ &= h(\boldsymbol{\beta}) \exp\left\{ \boldsymbol{\beta}^\top \boldsymbol{\alpha}_1 - \alpha_2 A_l(\boldsymbol{\beta}) + \boldsymbol{\beta}^\top T(x_n, z_n) - A_l(\boldsymbol{\beta}) \right\} \\ &= h(\boldsymbol{\beta}) \exp\left\{ \boldsymbol{\beta}^\top \left(\boldsymbol{\alpha}_1 + T(x_n, z_n) \right) - (\alpha_2 + 1) A_l(\boldsymbol{\beta}) \right\} \\ &= h(\boldsymbol{\beta}) \exp\left\{ \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\alpha}_1 + T(x_n, z_n) \\ \alpha_2 + 1 \end{bmatrix} \right\} \\ &= h(\boldsymbol{\beta}) \exp\left\{ T(\boldsymbol{\beta})^\top \begin{bmatrix} \boldsymbol{\alpha}_1 + T(x_n, z_n) \\ \alpha_2 + 1 \end{bmatrix} \right\} \end{aligned} \tag{95}$$

notice the posterior "inherited" $h(\boldsymbol{\beta})$ from the prior, so we only need to making sure that the $T(\boldsymbol{\beta})$ are the same for both prior and posterior.

6.5.2 posterior over n data points

1. Complete likelihood

$$\begin{aligned} p(\mathbf{x},\mathbf{z}|\beta) &= \prod_{n=1}^{N} \left\{ h(x_n,z_n)\exp\{\boldsymbol{\beta}^\top T(x_n,z_n) - A_l(\boldsymbol{\beta})\} \right\} \\ &= \prod_{n=1}^{N} h(x_n,z_n) \prod_{n=1}^{N} \exp\{\boldsymbol{\beta}^\top T(x_n,z_n) - A_l(\boldsymbol{\beta})\} \\ &= h(\mathbf{x},\mathbf{z})\exp\left\{ \sum_{n=1}^{N} \boldsymbol{\beta}^\top T(x_n,z_n) - N \times A_l(\boldsymbol{\beta}) \right\} \end{aligned} \tag{96}$$

2. Complete posterior

   now, look at:

$$\begin{aligned} p(\boldsymbol{\beta}|\mathbf{x},\mathbf{z},\boldsymbol{\alpha}) &\propto h(\boldsymbol{\beta})\exp\left\{ \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \alpha_2 \end{bmatrix} \right\} \underline{\exp\left\{ \sum_{n=1}^{N} \boldsymbol{\beta}^\top T(x_n,z_n) - N \times A_l(\boldsymbol{\beta}) \right\}} \\ &= h(\boldsymbol{\beta})\exp\left\{ \begin{bmatrix} \boldsymbol{\beta} \\ -A_l(\boldsymbol{\beta}) \end{bmatrix}^\top \underbrace{\begin{bmatrix} \boldsymbol{\alpha}_1 + \sum_{n=1}^{N} T(x_n,z_n) \\ \alpha_2 + N \end{bmatrix}}_{\eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha})} \right\} \end{aligned} \tag{97}$$

   When we use the expression and use $\eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha})$ instead:

$$\begin{aligned} p(\boldsymbol{\beta}|\mathbf{x},\mathbf{z},\boldsymbol{\alpha}) &= h(\boldsymbol{\beta})\exp\left\{ \eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha})^\top T(\boldsymbol{\beta}) - A_{\text{post}}(\eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha})) \right\} \\ \implies \eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha}) &= \begin{bmatrix} \boldsymbol{\alpha}_1 + \sum_{n=1}^{N} T(x_n,z_n) \\ \alpha_2 + N \end{bmatrix} \\ \implies \exp\left(A_{\text{post}}(\eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha})) \right) &= \int_{\boldsymbol{\beta}} h(\boldsymbol{\beta})\exp\left\{ \eta_{\text{post}}(\mathbf{x},\mathbf{z},\boldsymbol{\alpha})^\top T(\beta) \right\} \mathrm{d}\boldsymbol{\beta} \end{aligned} \tag{98}$$

6.5.3 Example: Posterior of Gaussian mean

suppose data x_i come from unit variance Gaussian. Compare with Section (6.2.1), we saved one parameter:

$$
\begin{aligned}
p(x|\mu) &= \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{1}{2}(x-\mu)^2 \Big\} \\
&= \underbrace{\frac{\exp\big(-x^2/2\big)}{\sqrt{2\pi}}}_{h(x)} \exp\Big\{ \underbrace{\mu}_{\beta} \underbrace{x}_{T(x)} - \underbrace{\frac{\mu^2}{2}}_{A_l(\beta)} \Big\}
\end{aligned} \tag{99}
$$

Therefore:

$$
\begin{aligned}
\beta &= \mu \\
T(x) &= x \\
A_l(\beta) &= \frac{\beta^2}{2} \\
h(x) &= \frac{\exp\big(-x^2/2\big)}{\sqrt{2\pi}}
\end{aligned} \tag{100}
$$

substitute into:

$$
p(x|\beta) = \frac{\exp\big(-x^2/2\big)}{\sqrt{2\pi}} \exp\Big\{ \beta x + \underbrace{-\frac{\beta^2}{2}}_{A_l(\beta)} \Big\} \tag{101}
$$

6.5.4 criteria for conjugate pair

A conjugate prior MUST be:

$$
\begin{aligned}
p(\beta|\alpha) &= h(\beta) \exp\Big\{ \alpha_1 \beta + \alpha_2 \underbrace{(-\beta^2/2)}_{A_l(\beta)} - A_g(\alpha) \Big\} \\
&= h(\beta) \exp\Big\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}^\top \begin{bmatrix} \beta \\ -\frac{\beta_2}{2} \end{bmatrix} - A_g(\alpha) \Big\}
\end{aligned} \tag{102}
$$

Wait, this doesn't look exactly in the form of Eq.(71), i.e.,:

$$
\mathcal{N}(x;\mu,\sigma^2) = \exp\left(\begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix}^\top \begin{bmatrix} x \\ x^2 \end{bmatrix} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2) \right) \tag{103}
$$

We can arrange Eq.(102) to look like, but with parameter $\begin{bmatrix} \alpha_1 & -\frac{\alpha_2}{2} \end{bmatrix}^\top$:

$$
p(\boldsymbol{\beta}|\alpha) = h(\boldsymbol{\beta}) \exp\left\{ \begin{bmatrix} \alpha_1 \\ -\frac{\alpha_2}{2} \end{bmatrix}^\top \begin{bmatrix} \beta \\ \beta^2 \end{bmatrix} - A_g(\alpha) \right\} \tag{104}
$$

From our knowledge, a distribution with sufficient statistics $T(\beta) = \begin{bmatrix} \beta & \beta^2 \end{bmatrix}$ is a Gaussian distribution.

Suppose the likelihood is an exponential family distribution. Every exponential family has a conjugate prior in theory. The natural parameter $\boldsymbol{\alpha} = \begin{bmatrix}\boldsymbol{\alpha}_1 & \alpha_2\end{bmatrix}^\top$ has dimension $\dim(\boldsymbol{\beta}) + 1$. The sufficient statistics of the prior are $\begin{bmatrix}\boldsymbol{\beta} & -A_l(\beta)\end{bmatrix}^\top$

6.6 Conjugate exponential family distribution: $\mathbb{E}_q[T(\beta)] = \nabla_\lambda A_g(\lambda)$

Lemma 1 Given that the exponential family distribution $q(\beta|\lambda)$ is of the form:

$$\begin{aligned} q(\beta|\lambda) &= h(\beta)\exp\{\lambda^\top T(\beta) - A_g(\lambda)\} \\ &= \frac{1}{\exp(A_g(\lambda))} h(\beta)\exp\{\lambda^\top T(\beta)\} \end{aligned} \tag{105}$$

then the expectation of the sufficient statistics $T(\beta)$ is:

$$\mathbb{E}_{q(\beta)}[T(\beta)] = \nabla_\lambda A_g(\lambda) \tag{106}$$

proof:

$$\begin{aligned} &\int_\beta q(\beta|\lambda)\mathrm{d}\beta = \int_\beta h(\beta)\exp\{\lambda^\top T(\beta) - A_g(\lambda)\}\mathrm{d}\beta = 1 \\ &\implies \nabla_\lambda \left(\int_\beta h(\beta)\exp\{\lambda^\top T(\beta) - A_g(\lambda)\}\mathrm{d}\beta\right) = \nabla_\lambda(1) = 0 \\ &\implies \int_\beta \nabla_\lambda \left(h(\beta)\exp\{\lambda^\top T(\beta) - A_g(\lambda)\}\right)\mathrm{d}\beta = 0 \\ &\implies \int_\beta h(\beta)\exp\left\{\lambda^\top T(\beta) - A_g(\lambda)\right\}(T(\beta) - \nabla_\lambda A_g(\lambda)) = 0 \\ &\implies \underbrace{\int_\beta h(\beta)\exp\left\{\lambda^\top T(\beta) - A_g(\lambda)\right\}T(\beta)}_{\mathbb{E}_{q(\beta)}[T(\beta)]} - \underbrace{\int_\beta h(\beta)\exp\left\{\lambda^\top T(\beta) - A_g(\lambda)\right\}}_{=1}\nabla_\lambda A_g(\lambda) = 0 \\ &\implies \mathbb{E}_{q(\beta)}[T(\beta)] - \nabla_\lambda A_g(\lambda) = 0 \end{aligned} \tag{107}$$

6.6.1 The choice of $q(\beta, \mathbf{z})$

We choose $q(\beta, \mathbf{z})$ to decouple β and $\mathbf{z}$ completely:

$$q(\beta, \mathbf{z}) = q(\beta|\lambda)\prod_{n=1}^{N}\prod_{j=1}^{J} q(z_{n,j}|\phi_{n,j}) \tag{108}$$

- $q(\beta|\lambda)$ is the SAME distribution type as $p(\beta|\mathbf{x}, \mathbf{z}, \alpha)$, they only differ in parameter. This means they have the same sufficient statistics $T(\beta)$:

$$\begin{aligned} q(\beta|\lambda) &= h(\beta)\exp\{\lambda^\top T(\beta) - A_g(\lambda)\} \\ \text{compare with:} \quad p(\beta|\mathbf{x}, \mathbf{z}, \alpha) &= h(\beta)\exp\left\{\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)^\top T(\beta) - A_{\text{post}}(\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha))\right\} \end{aligned} \tag{109}$$

- $q(z_{n,j}|\phi_{n,j})$ is the SAME distribution type as $p(z_{n,j}|x_n, z_{n,-j}, \beta)$, they only differ in parameter. This means they have the same sufficient statistics $T(z_{n,j})$:

$$\begin{aligned}
&q(z_{n,j}|\phi_{n,j}) = h(z_{n,j}) \exp\left\{\phi_{n,j}^\top T(z_{n,j}) - A_l(\phi_{n,j})\right\} \\
\text{compare with:} \quad &p(z_{n,j}|x_n, z_{n,-j}, \beta) = h(z_{n,j}) \exp\left\{\eta_l(x_n, z_{n,-j}, \beta)^\top T(z_{n,j}) - A_l(\eta_l(x_n, z_{n,-j}, \beta))\right\}
\end{aligned} \tag{110}$$

6.7 Proof for ELBO(λ) for $q(\beta|\lambda)$ Optional - not examinable

this section shows the proof for the update formula used in Eq.(63), i.e., $\eta_j = \mathbb{E}_{q(\mathbf{z}\setminus z_j|\cdot)}[\eta_{\text{post}}(\mathbf{z} \setminus z_j)]$, we will do so using an example from the setting described in this section.

Our goal is to maximize the ELBO, i.e.,

$$\text{ELBO}(q) \triangleq \mathbb{E}_{q(\beta,\mathbf{z})}[\log p(\mathbf{x}, \mathbf{z}, \beta|\alpha)] - \mathbb{E}_{q(\beta,\mathbf{z})}[\log q(\mathbf{z}, \beta)] \tag{111}$$

Note that q used here is $q(\beta, \mathbf{z})$ not just $q(\beta|\lambda)$

$$\begin{aligned}
\text{ELBO}(\lambda) &= \mathbb{E}_{q(\beta,\mathbf{z})}[\log p(\beta|\mathbf{x}, \mathbf{z}, \alpha)] + \mathbb{E}_{q(\beta,\mathbf{z})}[\log p(\mathbf{x}, \mathbf{z})] - \mathbb{E}_{q(\beta,\mathbf{z})}[\log q(\beta)] \\
&= \mathbb{E}_{q(\beta,\mathbf{z})}\big[\log p(\beta|\mathbf{x}, \mathbf{z}, \alpha)\big] - \mathbb{E}_q\big[\log q(\beta)\big] + \text{Const.} \\
&= \mathbb{E}_{q(\beta,\mathbf{z})}\left[\log\left(h(\beta) \exp\left\{\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)^\top T(\beta) - A_{\text{post}}(\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha))\right\}\right)\right] - \mathbb{E}_q[\log q(\beta)] + \text{Const.} \\
&= \mathbb{E}_{q(\beta,\mathbf{z})}[\log(h(\beta))] + \underline{\mathbb{E}_q[\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)^\top T(\beta)]} - \mathbb{E}_q\left[\log\left(h(\beta) \exp\{\lambda^\top T(\beta) - A_{\text{pri}}(\lambda)\}\right)\right] + \text{Const.} \\
&= \underbrace{\mathbb{E}_{q(\beta,\mathbf{z})}[\log(h(\beta))]}_{\text{doesn't contain } \lambda} + \underline{\mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)]^\top \mathbb{E}_{q(\beta|\lambda)}[T(\beta)]} - \underbrace{\mathbb{E}_{q(\beta,\mathbf{z})}[\log h(\beta)]}_{\text{doesn't contain } \lambda} - \mathbb{E}_q[\lambda^\top T(\beta)] + A_{\text{pri}}(\lambda) + \text{Const.} \\
&= \mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_{\text{post}}(x, z, \alpha)]^\top \mathbb{E}_{q(\beta|\lambda)}[T(\beta)] - \lambda^\top \mathbb{E}_{q(\beta|\lambda)}[T(\beta)] + A_{\text{pri}}(\lambda) + \text{Const.} \qquad \because A_{\text{pri}}(\lambda) \text{ contains } \lambda
\end{aligned} \tag{112}$$

now let's substitute $\mathbb{E}_{q(\beta|\lambda)}[T(\beta)] = \nabla_\lambda A_{\text{pri}}(\lambda)$ into ELBO(λ):

$$\text{ELBO}(\lambda) = \mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_{\text{post}}(x, z, \alpha)]^\top \nabla_\lambda A_{\text{pri}}(\lambda) - \lambda^\top \nabla_\lambda A_{\text{pri}}(\lambda) + A_{\text{pri}}(\lambda) + \text{Const.} \tag{113}$$

Maximize ELBO(λ) we get:

$$\begin{aligned}
\nabla_\lambda \text{ELBO}(\lambda) &= \mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_g(\mathbf{x}, \mathbf{z}, \alpha)]^\top \nabla_\lambda^2 A_{\text{pri}}(\lambda) - \nabla_\lambda A_{\text{pri}}(\lambda) - \lambda^\top \nabla_\lambda^2 A_{\text{pri}}(\lambda) + \nabla_\lambda A_{\text{pri}}(\lambda) = 0 \\
&= \mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_g(\mathbf{x}, \mathbf{z}, \alpha)]^\top \nabla_\lambda^2 A_{\text{pri}}(\lambda) - \lambda^\top \nabla_\lambda^2 A_{\text{pri}}(\lambda) = 0 \\
&\implies \nabla_\lambda^2 A_{\text{pri}}(\lambda) \left(\mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)]^\top - \lambda^\top\right) = 0
\end{aligned} \tag{114}$$

$$\lambda = \mathbb{E}_{q(\mathbf{z}|\Phi)}[\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)] \tag{115}$$

in words, when we try to update λ for $q(\beta|\lambda)$, it find the corresponding posterior $p(\beta|\mathbf{x}, \mathbf{z}, \alpha)$, and its natural parameter $\eta_{\text{post}}(\mathbf{x}, \mathbf{z}, \alpha)$, then computes the expectation with all the $q(\cdot)$ that its natural parameter has random variable for.

6.8 Update for ELBO($\phi_{n,j}$) for $q(z_{n,j}|\phi_{n,j})$

In a very similar fashion to $\mathcal{L}(\lambda)$, we can prove:

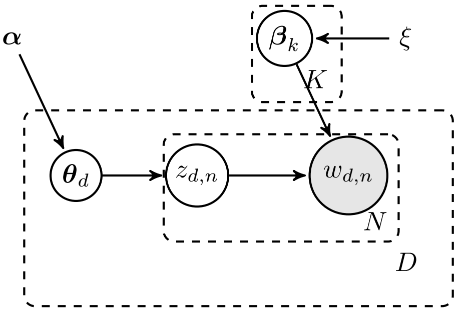
$$\nabla_{\phi_{n,j}} \text{ELBO}(\phi_{n,j}) = \nabla^2_{\phi_{n,j}} A_l(\phi_{n,j}) \big(\mathbb{E}_{q(\lambda)}[\eta_l(x_n, z_{n,-j}, \beta)]^\top - \phi_{n,j}^\top \big) = 0 \tag{116}$$

$$\phi_{n,j} = \mathbb{E}_{q(\lambda)} \big[\eta_l(x_n, z_{n,-j}, \beta) \big] \tag{117}$$

in words, when we try to update $\phi_{n,j}$ for $q(z_{n,j}|\phi_{n,j})$, it find the corresponding posterior $p(z_{n,j}|x_n, z_{n,-j})$, and its natural parameter $\eta_l(x_n, z_{n,-j})$, then computes the expectation with all the $q(\cdot)$ that its natural parameter has random variable for.

7 apply variational inference (Exponential family version) to Latent Dirichlet Allocation

let's visit Latent Dirichlet Allocation again [3]:



- $\boldsymbol{\beta}_k \sim \text{Dir}(\xi, \dots, \xi)$ for $k \in \{1, ..., K\}$.
- For each document d:
  $\boldsymbol{\theta}_d \sim \text{Dir}(\alpha, \dots, \alpha)$
  For each word $w \in \{1, ..., N\}$:
  $z_{dn} \sim \text{Mult}(\boldsymbol{\theta}_d)$
  $w_{dn} \sim \text{Mult}(\beta_{z_{dn}})$

7.1 define corresponding $q(\cdot)$

1. $q(z_{d,n})$

$$\begin{aligned} & q(z_{d,n}) = \text{Mult}(\boldsymbol{\phi}_{d,n}) \\ \implies & q(z_{d,n} = k) = \phi_{d,n}^k \end{aligned} \tag{118}$$

2. $q(\boldsymbol{\beta}_k)$

$$q(\boldsymbol{\beta}_k) = \text{Dir}(\boldsymbol{\lambda}_k) \tag{119}$$

3. $q(\boldsymbol{\theta}_d)$

$$q(\boldsymbol{\theta}_d) = \text{Dir}(\boldsymbol{\gamma}_d) \tag{120}$$

7.1.1 Facts about Dirichlet Distribution

$$\begin{aligned} & \theta \sim \text{Dir}(\gamma_1, \dots \gamma_K) \\ \implies & \mathbb{E}[\log(\theta_k)|\gamma] = \Psi(\gamma_k) - \Psi\left(\sum_{i=1}^{K} \gamma_i\right) \quad \text{for component } k \end{aligned} \tag{121}$$

where:

$$\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln \big(\Gamma(x)\big) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{122}$$

7.2 Updating $q(z_{d,n}|\boldsymbol{\phi}_{d,n})$: optimize $\boldsymbol{\phi}_{d,n}$

7.2.1 find natural parameter of posterior $p(z_{dn} = k|\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}, w_{d,n})$

$$\begin{aligned} p(z_{d,n} = k|\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}, w_{d,n}) &\propto p(z_{d,n} = k|\boldsymbol{\theta}_d) p(w_{d,n}|z_{d,n} = k, \boldsymbol{\beta}_{1:K}) \\ &= \theta_{d,k} \times \beta_{k,w_{d,n}} \\ &\propto \exp\Bigg(\underbrace{\log(\theta_{d,k}) + \log(\beta_{k,w_{d,n}})}_{\eta_l(\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}, w_{d,n})} \times \underbrace{1}_{T(z_{d,n})} \Bigg) \end{aligned} \tag{123}$$

for the last line, we substitute the exponential family form of multinomial distribution. Expressing it using "normal " multinomial distribution, then its parameter is:

$$p(z_{d,n}|\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}, w_{d,n}) = \mathrm{Mult}\big(\theta_{d,1} \times \beta_{1,w_{d,n}}, \ldots, \theta_{d,K} \times \beta_{K,w_{d,n}}\big) \tag{124}$$

diagrammatically, $\boldsymbol{\beta}_{1:K}$ is represented as a matrix of:

$$\boldsymbol{\beta}_{1:K} \equiv \begin{bmatrix} -- & \boldsymbol{\beta}_1 & -- \\ \vdots & \vdots & \vdots \\ -- & \boldsymbol{\beta}_k & -- \end{bmatrix} \equiv \begin{bmatrix} \beta_{1,w_1} & \cdots & \beta_{1,w_{|\mathcal{V}|}} \\ \vdots & \vdots & \vdots \\ \beta_{K,w_1} & \cdots & \beta_{K,w_{|\mathcal{V}|}} \end{bmatrix} \tag{125}$$

7.2.2 optimize $\boldsymbol{\phi}_{d,n}$

apply the update formula, in which we need the natural parameter for $p(z_{d,n}|\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}, w_{d,n})$ in the exception:

$$\begin{aligned} \eta(\phi^k_{d,n}) = \log(\phi^k_{d,n}) &\propto \mathbb{E}_{q(\boldsymbol{\theta}_d)q(\boldsymbol{\beta}_k)}\left[\eta_l\left(\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}, w_{d,n}\right)\right] \\ &= \mathbb{E}_{q(\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K})}\left[\log(\theta_{d,k})\right] + \mathbb{E}_{q(\boldsymbol{\beta}_k)}\left[\log(\beta_{k,w_{d,n}})\right] \\ &= \Psi(\gamma_{d,k}) - \Psi\Big(\sum_{k=1}^{K} \gamma_{d,k}\Big) + \Psi\Big(\lambda_{k,w_{d,n}}\Big) - \Psi\Big(\sum_{v} \lambda_{k,v}\Big) \end{aligned} \tag{126}$$

compare this with Eq.(63), i.e., $\eta_j = \mathbb{E}_{q(\mathbf{z}\setminus z_j)}[\eta_{\text{post}}(\mathbf{z} \setminus z_j)]$, you can see easily that:

$$\mathbf{z} \setminus z_j \equiv \{\boldsymbol{\theta}_d, \boldsymbol{\beta}_{1:K}\} \tag{127}$$

to obtain $\boldsymbol{\phi}_{d,n}$:

$$
\begin{aligned}
\implies \phi_{d,n}^{k} &\propto \exp\left[\Psi(\gamma_{d,k}) - \Psi\Big(\sum_{k=1}^{K}\gamma_{d,k}\Big) + \Psi\Big(\lambda_{k,w_{d,n}}\Big) - \Psi\Big(\sum_{v}\lambda_{k,v}\Big)\right] \\
&\propto \exp\left[\Psi(\gamma_{d,k}) + \Psi\Big(\lambda_{k,w_{d,n}}\Big) - \Psi\Big(\sum_{v}\lambda_{k,v}\Big)\right] \quad \because \sum_{k=1}^{K}\gamma_{d,k} \text{ has same value irrespective of } k
\end{aligned}
\tag{128}
$$

7.3 Updating $q(\boldsymbol{\theta}_d|\boldsymbol{\gamma}_d)$: optimize $\boldsymbol{\gamma}_d$

7.3.1 find natural parameter of posterior $p(\boldsymbol{\theta}_d|\mathbf{z}_d)$

$$
\begin{aligned}
p(\boldsymbol{\theta}_d|\mathbf{z}_d) &= p(\boldsymbol{\theta}_d|\boldsymbol{\alpha})\prod_{n=1}^{N} p(z_{d,n}|\boldsymbol{\theta}_d) = \text{Dir}(\boldsymbol{\alpha}) \times \prod_{n=1}^{N}\text{Mult}(z_{d,n}|\boldsymbol{\theta}_d) \\
&= \prod_{k}\Big(\theta_{d,k}^{\alpha_k-1}\prod_{n=1}^{N}\theta_{d,k}^{\mathbb{1}(z_{d,n}=k)}\Big) \\
&= \exp\Big[\log\Big(\prod_{k}\Big(\theta_{d,k}^{\alpha_k-1}\prod_{n=1}^{N}\theta_{d,k}^{\mathbb{1}(z_{d,n}=k)}\Big)\Big)\Big] \\
&= \exp\left[\sum_{k}\log\Big(\theta_{d,k}^{\alpha_k-1}\prod_{n=1}^{N}\theta_{d,k}^{\mathbb{1}(z_{d,n}=k)}\Big)\right] \\
&= \exp\left[\sum_{k}\left(\log\theta_{d,k}^{\alpha_k-1} + \sum_{n=1}^{N}\log\Big(\theta_{d,k}^{\mathbb{1}(z_{d,n}=k)}\Big)\right)\right] \\
&= \exp\left[\sum_{k}\left((\alpha_k-1)\log\theta_{d,k} + \sum_{n=1}^{N}\mathbb{1}(z_{d,n}=k)\log\theta_{d,k}\right)\right] \\
&= \exp\left[\sum_{k}\left(\alpha_k - 1 + \sum_{n=1}^{N}\mathbb{1}(z_{d,n}=k)\right)\log\left(\theta_{d,k}\right)\right] \\
=&\exp\left(\underbrace{\begin{bmatrix}(\alpha_1-1+n_1)\\ \dots \\ (\alpha_K-1+n_K)\end{bmatrix}^{\top}}_{\eta_l(\alpha,z_d)}\underbrace{\begin{bmatrix}\log(\theta_{d,1})\\ \dots \\ \log(\theta_{d,K})\end{bmatrix}}_{T(\boldsymbol{\theta}_d)}\right) \qquad \text{by letting } n_k = \sum_{n=1}^{N}\mathbb{1}(z_{d,n}=k) \\
=&\text{Dir}(\alpha_1+n_1,\dots,\alpha_K+n_K)
\end{aligned}
\tag{129}
$$

It turns out that the Dirichlet distribution is the only distribution with exactly the same ”normal” and natural parameters.

7.3.2 optimize $\boldsymbol{\gamma}_d$

let’s use $q(\eta(\boldsymbol{\gamma}_d)) = \text{Dir}((\eta(\boldsymbol{\gamma}_d))$ to approximate $p(\boldsymbol{\theta}_d|\mathbf{z}_d)$ (another Dirichlet distribution), we apply for the exponential family update formula Eq.(63):

$$\begin{aligned}\eta(\boldsymbol{\gamma}_d) &= \mathbb{E}_{q(z_{d,n}|\boldsymbol{\phi}_{d,n})}\left[\eta_l\left(\alpha, z_d\right)\right] \\ &= \mathbb{E}_{q(z_{d,n}|\boldsymbol{\phi}_{d,n})}\big[(\alpha_1 - 1 + n_1) \quad \ldots \quad (\alpha_K - 1 + n_K)\big] \quad \text{substitute Eq.(129)}\end{aligned} \tag{130}$$

compare this with Eq.(63), i.e., $\eta_j = \mathbb{E}_{q(\mathbf{z}\setminus z_j)}[\eta_{\text{post}}(\mathbf{z} \setminus z_j)]$, you can see easily that:

$$\mathbf{z} \setminus z_j \equiv \{z_{d,n}\} \tag{131}$$

how to compute $\mathbb{E}_{q(z_{d,n}|\boldsymbol{\phi}_{d,n})}\big[n_1)\big]$? Firstly, let's recognize that $\boldsymbol{\phi}_{d,n}$ itself is a probability vector:

$$\begin{aligned}\boldsymbol{\phi}_{d,n} &= \big[\boldsymbol{\phi}_{d,n}^1 \quad \ldots \quad \boldsymbol{\phi}_{d,n}^K\big] \\ &= \big[q(z_{d,n} = 1) \quad \ldots \quad q(z_{d,n} = K)\big]\end{aligned} \tag{132}$$

since:

$$\begin{aligned}\mathbb{E}\big[\sum_{n=1}^{N} \mathbb{1}(z_{d,n} = k)\big] &= \sum_{n=1}^{N} \mathbb{E}[\mathbb{1}(z_{d,n} = k)] \\ &= \sum_{n=1}^{N} q(z_{d,n} = k) \\ &= \sum_{n=1}^{N} \boldsymbol{\phi}_{d,n}^k\end{aligned} \tag{133}$$

continue with Eq.(132), we have:

$$\eta(\boldsymbol{\gamma}_d) = \Big[\alpha_1 - 1 + \textstyle\sum_{n=1}^{N} \boldsymbol{\phi}_{d,n}^k \quad \ldots \quad \alpha_K - 1 + \textstyle\sum_{n=1}^{N} \boldsymbol{\phi}_{d,n}^k\Big] \tag{134}$$

to obtain the Dirichlet distribution parameter $\boldsymbol{\gamma}_d$, luckily, for Dirichlet distribution, the natural and "normal" parameter are the same.

$$\begin{aligned}\boldsymbol{\gamma}_d &= \Big[\Big(\alpha_1 + \textstyle\sum_{n=1}^{N} \phi_{d,n}^1\Big) \quad \ldots \quad \Big(\alpha_K + \textstyle\sum_{n=1}^{N} \phi_{d,n}^K\Big)\Big] \\ &= \boldsymbol{\alpha} + \sum_{n=1}^{N} \boldsymbol{\phi}_{d,n}\end{aligned} \tag{135}$$

7.4 Updating $q(\boldsymbol{\beta}_k|\boldsymbol{\lambda}_k)$ optimize $\boldsymbol{\lambda}_k$

7.4.1 find natural parameter of posterior $p(\boldsymbol{\beta}_k|\mathbf{z},\mathbf{w})$

$$\begin{aligned}
p(\boldsymbol{\beta}_k|\mathbf{z},\mathbf{w}) &= p(\boldsymbol{\beta}_k|\xi)\prod_{d=1}^{D}\prod_{n=1}^{N} p\left(w_{d,n}|\boldsymbol{\beta}_k\right)^{\mathbb{1}(z_{d,n}=k)} = \mathrm{Dir}(\eta) \times \prod_{d=1}^{D}\prod_{n=1}^{N} \beta_{k,w_{d,n}}^{\mathbb{1}(z_{d,n}=k)} \\
&\propto \exp\left(\underbrace{\left(\xi - 1 + \sum_{d=1}^{D}\sum_{n=1}^{N} w_{d,n}\mathbb{1}(z_{d,n}=k)\right)}_{\eta_l(\eta,Z,W)} \times \underbrace{\log(\boldsymbol{\beta}_k)}_{T(\boldsymbol{\beta}_k)}\right) \\
&= \mathrm{Dir}(\eta_l\left(\xi, Z, W\right))
\end{aligned} \tag{136}$$

7.4.2 optimize $\boldsymbol{\lambda}_k$

let's use $q(\eta(\boldsymbol{\lambda}_k)) = \mathrm{Dir}((\eta(\boldsymbol{\lambda}_k))$ to approximate $p(\boldsymbol{\beta}_k|\mathbf{z},\mathbf{w})$ (another Dirichlet distribution), we apply for the exponential family update formula Eq.(63):

$$\begin{aligned}
\eta(\boldsymbol{\lambda}_k) &= \mathbb{E}_{\prod_{d=1}^{D}\prod_{n=1}^{N} q(z_{d,n}|\phi_{d,n}^{k})}\left[\eta_l(\xi,\mathbf{z},\mathbf{w})\right] \\
&= \mathbb{E}_{\prod_{d=1}^{D}\prod_{n=1}^{N} q(z_{d,n}|\phi_{d,n}^{k})}\left[\xi - 1 + \sum_{d=1}^{D}\sum_{n=1}^{N} w_{d,n}\mathbb{1}(z_{d,n}=k)\right] \\
&= \eta - 1 + \sum_{d=1}^{D}\sum_{n=1}^{N} w_{d,n}\phi_{d,n}^{k}
\end{aligned} \tag{137}$$

$$\boldsymbol{\lambda}_k = \xi + \sum_{d=1}^{D}\sum_{n=1}^{N} w_{d,n}\phi_{d,n}^{k} \tag{138}$$

8 Collapsed Variational Inference Optional - not examinable

$$q(z_{d,n}) = \mathrm{Mult}(\boldsymbol{\phi}_{d,n}) \text{ or } q(z_{d,n} = k) = \phi^k_{d,n} \qquad q(\boldsymbol{\beta}_k) = \mathrm{Dir}(\boldsymbol{\lambda}_k) \qquad q(\boldsymbol{\theta}_d) = \mathrm{Dir}(\boldsymbol{\gamma}_d) \tag{139}$$

$$\begin{aligned} \implies q(Z, \theta_1 \ldots \boldsymbol{\theta}_d, \beta_1 \ldots \boldsymbol{\beta}_k) &= \left(\prod_{d=1}^{d=D}\prod_{n=1}^{N} q(z_{d,n}|\boldsymbol{\phi}_{d,n})\right) \prod_{d=1}^{D} q(\boldsymbol{\theta}_d|\boldsymbol{\gamma}_d) \prod_{k=1}^{K} q(\theta_k|\boldsymbol{\lambda}_k) \\ \text{now change to: } &= \underbrace{\left(\prod_{d=1}^{d=D}\prod_{n=1}^{N} q(z_{d,n}|\boldsymbol{\phi}_{d,n})\right)}_{q(Z)} q(\Theta, \beta|Z) \end{aligned} \tag{140}$$

Maximize ELOB, it becomes: (remove X for clarity)
Let $U = \{\Theta, \beta\}$:

$$\begin{aligned} \mathrm{ELBO}(q) &\triangleq \mathbb{E}_{q(U,Z)}[\log p(Z,U)] - \mathbb{E}_{q(U,Z)}[\log q(Z,U)] \\ &= \mathbb{E}_{q(U,Z)}[\log p(Z,U)] - \mathbb{E}_{q(U,Z)}[\log q(U|Z) - \log q(Z)] \\ &= \mathbb{E}_{q(Z)}\left(\mathbb{E}_{q(U|Z)}[\log p(Z,U)]\right) - \mathbb{E}_{q(Z)}\left(\mathbb{E}_{q(U|Z)}[\log q(U|Z)]\right) - \mathbb{E}_{q(Z,U)}[\log q(Z)] \\ &= \mathbb{E}_{q(Z)}\left(\underbrace{\mathbb{E}_{q(U|Z)}\left([\log p(Z,U)] - [\log q(U|Z)]\right)}_{\mathcal{L}(q(U|Z))}\right) - \mathbb{E}_{q(Z)}[\log q(Z)] \end{aligned} \tag{141}$$

Think this as treating Z as X.
(removed X for clarity)

$$\begin{aligned} \underset{q(U|Z)}{\arg\max}(\mathrm{ELBO}(q)) &= \underset{q(U|Z)}{\arg\max}\left[\mathbb{E}_{q(Z)}\left(\underbrace{\mathbb{E}_{q(U|Z)}\left([\log p_X(Z,U)] - [\log q(U|Z)]\right)}_{\mathcal{L}(q(U|Z))}\right) - \mathbb{E}_{q(Z)}[\log q(Z)]\right] \\ &= \mathbb{E}_{q(Z)}\left(\underbrace{\underset{q(U|Z)}{\arg\max}\left[\mathbb{E}_{q(U|Z)}\left([\log p(Z,U)] - [\log q(U|Z)]\right)\right]}\right) - \mathbb{E}_{q(Z)}[\log q(Z)] \\ &= \mathbb{E}_{q(Z)}[\underbrace{p(Z)}] - \mathbb{E}_{q(Z)}[\log q(Z)] \end{aligned} \tag{142}$$

$$\underset{q(U|Z)}{\arg\max}\left[\mathbb{E}_{q(U|Z)}\left([\log p(Z,U)] - [\log q(U|Z)]\right)\right] = p(Z) \tag{143}$$

maximum occur when $q(U|Z) = p(U|Z) \implies \mathbb{KL}\left(q(U|Z)\|p(U|Z)\right) = 0$

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[3] David M Blei, Andrew Y Ng, and Michael I Jordan, “Latent dirichlet allocation,” Journal of machine Learning research, vol. 3, no. Jan, pp. 993–1022, 2003.