ASEN5044 Assignment 2

Solutions

24 September 2018

1. (a) Define the state and input vectors as

$$\mathbf{x} \triangleq \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} \end{bmatrix}^T, \mathbf{u} \triangleq \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$

The system expressed in standard nonlinear state space form is then

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\ \dot{\theta} \\ -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \\ x_4 \\ -\frac{2x_4 x_2}{x_1} + \frac{1}{r}u_2 \end{bmatrix} \triangleq \mathbf{h}(\mathbf{x}, \mathbf{u})$$

(b) Beginning with the first order Taylor Series expansion of $\mathbf{h}$

$$\dot{\mathbf{x}} \approx \dot{\mathbf{x}}_0 + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{x} - \mathbf{x}_0) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{u} - \mathbf{u}_0)$$

Define new state $\delta\mathbf{x} \triangleq \mathbf{x} - \mathbf{x}_0 \implies \delta\dot{\mathbf{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0$ and observe that $\mathbf{u}_0 = \mathbf{0}$ to get

$$\delta\dot{\mathbf{x}} \approx \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} \delta\mathbf{x} + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} \mathbf{u}$$

The linearized system matrices are then given by

$$\mathbf{A} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} = \left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ \dot{\theta}^2 + \frac{2k}{r^3} & 0 & 0 & 2r\dot{\theta} \\ 0 & 0 & 0 & 1 \\ \frac{2\dot{\theta}\dot{r}}{r^2} - \frac{u_2}{r^2} & -\frac{2\dot{\theta}}{r} & 0 & -\frac{2\dot{r}}{r} \end{bmatrix}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} = \left.\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}$$

(c) Define

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$\mathbf{F}$ and $\mathbf{G}$ are then found by taking

$$\exp(\hat{\mathbf{A}}) = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{F} &= \begin{bmatrix} 1 & 9.999 & 0 & 772.5 \\ 1.548 \times 10^{-8} & 0.9997 & 0 & 154.5 \\ -8.942 \times 10^{-14} & -1.732 \times 10^{-5} & 1 & 9.999 \\ -2.682 \times 10^{-14} & -3.465 \times 10^{-6} & 0 & 0.9997 \end{bmatrix} \\
\mathbf{G} &= \begin{bmatrix} 50.00 & 0.3856 \\ 9.999 & 0.1157 \\ -5.775 \times 10^{-5} & 7.487 \times 10^{-3} \\ -1.732 \times 10^{-5} & 1.497 \times 10^{-3} \end{bmatrix}
\end{aligned}$$

(d) Each column $\mathbf{f}_i$ represents the relative contributions of the corresponding component of the deviation state δx_i at time $t = t_0$ to each of the state components at $t = t_0 + \Delta t$ under the linear approximation. For example, $\mathbf{f}_1$ represents the contribution of the deviation in the orbital radius from the nominal radius at $t = t_0$ to the radius, vertical speed, true anomaly, and angular velocity at $t = t_0 + \Delta t$. Alternatively, the columns of $\mathbf{F}$ represent the span of the possible perturbation states that can be reached at $t = t_0 + \Delta t$ under the linear approximation. Note that the linear approximation breaks down for large Δt, and thus nonphysical results (e.g. deviations in the orbital radius contributing to changes in true anomaly and angular velocity) if the discretization time step is too large.

2. (a) Given the state definition $\mathbf{x} = \begin{bmatrix} p & \dot{p} & \ddot{p} \end{bmatrix}^T$ and that $\ddot{p}$ is constant, observe that

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= 0
\end{aligned}$$

The system expressed in standard linear state space form is then

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}$$

(b) i. Using the infinite series definition of $e^{\mathbf{A}t}$,

$$e^{\mathbf{A}t} = \sum_{j=0}^{\infty} \frac{(\mathbf{A}t)^j}{j!}$$

Each of the powers of $\mathbf{A}$ are given by

$$\mathbf{A}^0 = \mathbf{I}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A}^j = \mathbf{0}, j \geq 3$$

Thus

$$e^{\mathbf{A}t} = \mathbf{I} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

ii. To find the Jordan canonical form $\hat{\mathbf{A}}$ of $\mathbf{A}$, first find the eigenvalues of $\mathbf{A}$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda^3 = 0 \implies \lambda_i = 0, i = 1, 2, 3$$

$\mathbf{A}$ has one eigenvalue with an algebraic multiplicity of 3, thus $\hat{\mathbf{A}}$ has one Jordan block and is given by

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The generalized eigenvectors are then given by

$$\begin{aligned}
(\lambda\mathbf{I} - \mathbf{A})\mathbf{q}_1 &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{q}_1 = \mathbf{0} \implies \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
(\lambda\mathbf{I} - \mathbf{A})^2\mathbf{q}_2 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{q}_2 = \mathbf{0} \implies \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
(\lambda\mathbf{I} - \mathbf{A})^3\mathbf{q}_3 &= \mathbf{0}\mathbf{q}_3 = \mathbf{0} \implies \mathbf{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \mathbf{I}$$

Alternatively, observe that $\mathbf{A}$ is already in Jordan canonical form, and hence $\mathbf{Q} = \mathbf{Q}^{-1} = \mathbf{I}$. The matrix exponential is then given by

$$e^{\mathbf{A}t} = \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1} = e^{\hat{\mathbf{A}}t} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

using results from 2(b)i.

(c)

$$e^{\mathbf{A}0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

3. Observe that

$$\begin{aligned}\dot{\mathbf{S}}(t) &= \begin{bmatrix} e^t & 0 \\ 0 & -2e^{-t} \end{bmatrix} \\ \mathbf{A}\mathbf{S}(t) &= \begin{bmatrix} e^t & 0 \\ 0 & -2e^{-t} \end{bmatrix} = \dot{\mathbf{S}}(t)\end{aligned}$$

but

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \neq \mathbf{S}(t)$$

4. (a) At the equilibrium,

$$\begin{aligned}\mathbf{x}_1(t) &= 0 \Longrightarrow \dot{\mathbf{x}}_1(t) = 0 \\ \mathbf{x}_2(t) &= 0 \Longrightarrow \dot{\mathbf{x}}_2(t) = 0\end{aligned}$$

Substituting into the second expression and rearranging gives

$$\frac{Ku}{x_3} = \frac{GM}{R^2}$$

Substituting for $u = -\dot{x}_3$ and rearranging then gives

$$\dot{x}_3 = -\frac{GM}{KR^2} x_3$$

The solution to this ordinary differential equation is given by

$$x_3 = x_{3,0} e^{-\frac{GM}{KR^2} t}$$

Thus,

$$u = -\dot{x}_3 = \frac{GM x_{3,0}}{KR^2} e^{-\frac{GM}{KR^2} t}$$

(b) From above,

$$x_3 = x_{3,0} e^{-\frac{GM}{KR^3} t}$$

(c) The nominal trajectory is defined by

$$\mathbf{x}_0 \triangleq \begin{bmatrix} 0 & 0 & x_{3,0} e^{-\frac{GM}{KR^3} t} \end{bmatrix}, u_0 \triangleq -\frac{GM x_{3,0}}{KR^2} e^{-\frac{GM}{KR^2} t}$$

Defining the nonlinear system as

$$\mathbf{h}(\mathbf{x}, u) \triangleq \begin{bmatrix} x_2 \\ \frac{Ku - g x_2}{x_3} - \frac{GM}{(R+x_1)^2} \\ -u \end{bmatrix}$$

the linearized system is given by

$$
\mathbf{A} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0,u_0)} = \left.\begin{bmatrix} 0 & 1 & 0 \\ \frac{2GM}{(R+x_1)^3} & \frac{-g}{x_3} & -\frac{(Ku-gx_2)}{x_3^2} \\ 0 & 0 & 0 \end{bmatrix}\right|_{(\mathbf{x}_0,u_0)}
$$

$$
= \begin{bmatrix} 0 & 1 & 0 \\ \frac{2GM}{R^3} & -\frac{g}{x_{3,0}} e^{\frac{GM}{KR^2}t} & -\frac{-GM}{x_{3,0}R^2} e^{\frac{GM}{KR^2}t} \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\mathbf{B} = \left.\frac{\partial \mathbf{h}}{\partial u}\right|_{(\mathbf{x}_0,u_0)} = \left.\begin{bmatrix} 0 \\ \frac{K}{x_3} \\ -1 \end{bmatrix}\right|_{(\mathbf{x}_0,u_0)} = \begin{bmatrix} 0 \\ \frac{K}{x_{3,0}} e^{\frac{GM}{KR^2}t} \\ -1 \end{bmatrix}
$$

(d) Plots of the altitude of the rocket computed using the nonlinear system and the linear system for $x_{3,0} = 1000$ are shown in Figures 1 to 3. The linearization begins to break down as the amplitude of the deviation from the nominal control input increases and as time increases due to the growing influence of the deviation from the nominal input.

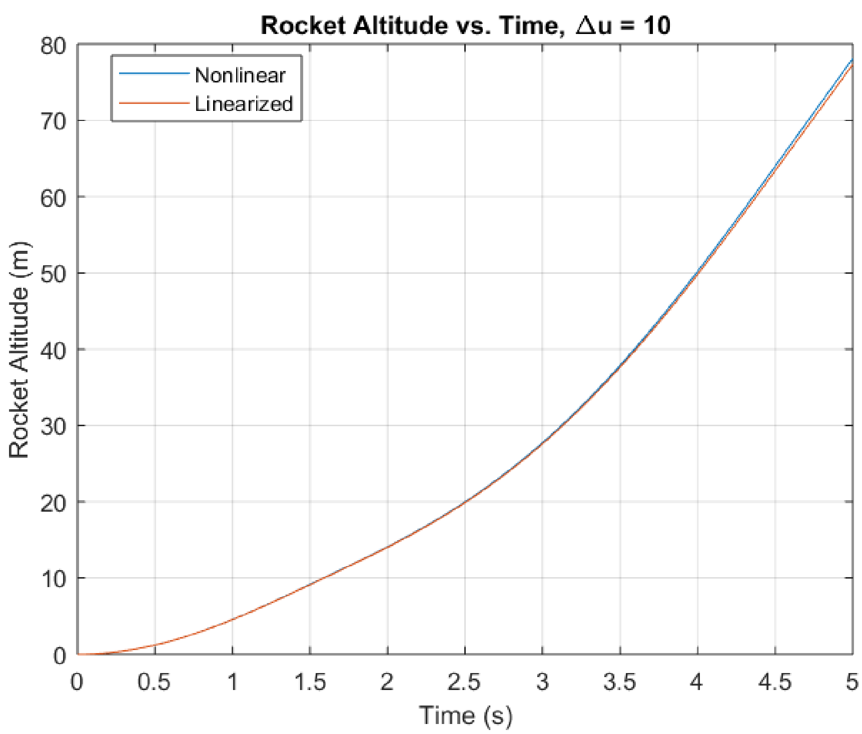

Figure 1: Altitude vs. Time, $\Delta u = 10$

AQ1.

$$
\begin{bmatrix} \mathbf{A} & \mathbf{A} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^2 + \mathbf{AC} \\ \mathbf{BA} + \mathbf{C}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}
$$

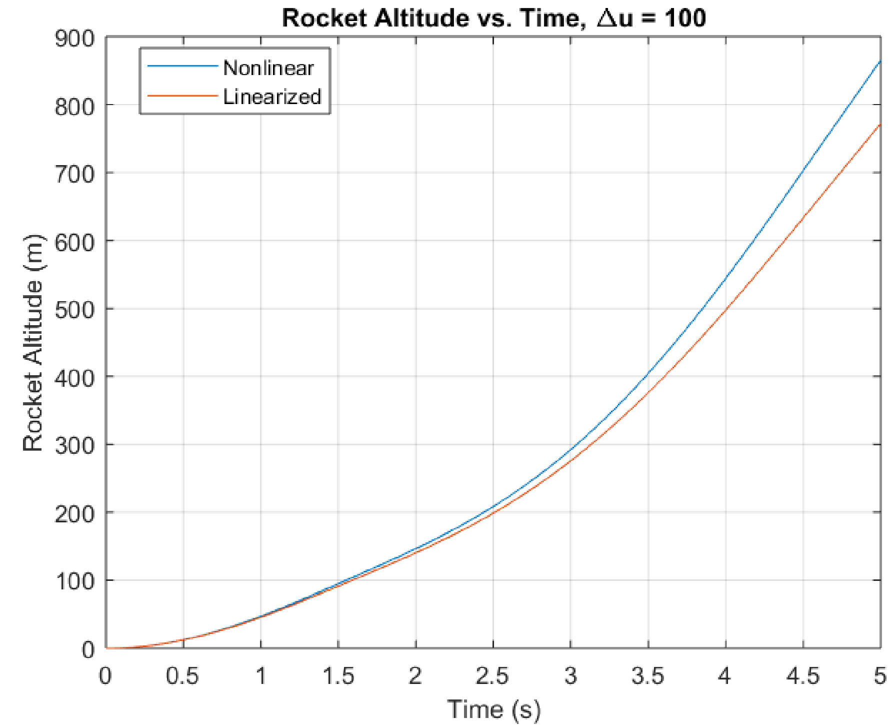

Figure 2: Altitude vs. Time, $\Delta u = 100$

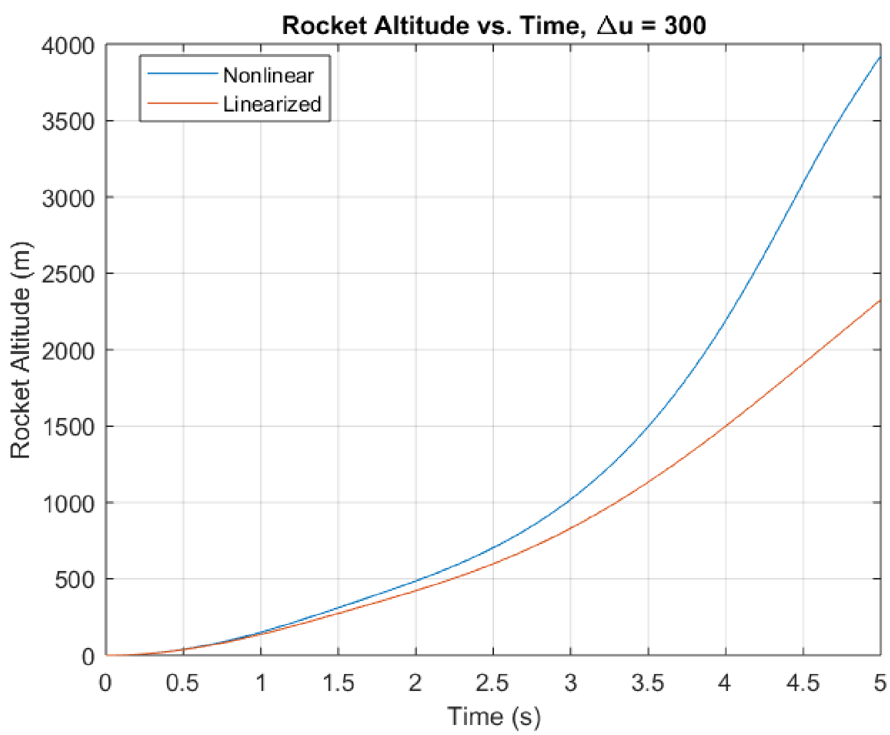

Figure 3: Altitude vs. Time, $\Delta u = 300$

$$\begin{aligned}\mathbf{A}^2 + \mathbf{AC} &= \mathbf{0} \implies \mathbf{A}^2 = -\mathbf{AC} \implies \mathbf{C} = -\mathbf{A} \\ \mathbf{BA} + \mathbf{AC} &= \mathbf{BA} - \mathbf{A}^2 = \mathbf{I} \implies \mathbf{B} = \mathbf{A}^{-1} + \mathbf{A}\end{aligned}$$

AQ2. To prove this result, it is necessary first to prove the following lemma.

Lemma The product of two upper triangular matrices is also an upper triangular matrix, and the elements of the main diagonal of the resulting matrix are given by the product of the corresponding main diagonal elements of the original matrices.

Proof Given $\mathbf{A} \in^n \mathbb{R}^n, \mathbf{B} \in^n \mathbb{R}^n$ upper triangular matrices, for $n = 2$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} \\ 0 & a_{22}b_{22} \end{bmatrix}$$

Assuming by way of induction that for $n = k$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11}b_{11} & \cdots & \\ & \ddots & \vdots \\ \mathbf{0} & & a_{kk}b_{kk} \end{bmatrix}$$

holds. Then for $n = k + 1$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \mathbf{a}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & \mathbf{b}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix}$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11}b_{11} & a_{11}\mathbf{b}_{12} + a_{12}\mathbf{B}_{22} \\ 0 & \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

$\mathbf{A}_2 2 \in^k \mathbb{R}^k$ and $\mathbf{B}_2 2 \in^k \mathbb{R}^k$ are upper triangular matrices $\implies$ $\mathbf{C}$ is an upper triangular matrix with $c_{ii} = a_{ii}b_{ii}, i = 1 \ldots k + 1$. ■

Now observe that

$$\det e^{\mathbf{A}t} = \det\left(\mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1}\right) = \det \mathbf{Q} \det e^{\hat{\mathbf{A}}t} \det \mathbf{Q}^{-1} = \det \mathbf{Q} \det e^{\hat{\mathbf{A}}t} \frac{1}{\det \mathbf{Q}} = \det e^{\hat{\mathbf{A}}t}$$

Using the infinite series definition of the matrix exponential to expand $e^{\hat{\mathbf{A}}t}$ gives

$$e^{\hat{\mathbf{A}}t} = \sum_{i=0}^{\infty} \frac{\left(\hat{\mathbf{A}}t\right)^i}{i!}$$

$\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_p \end{bmatrix}$$

where p is the number of unique eigenvalues of $\mathbf{A}$. Each $\mathbf{J}_k \in^{q_k} \mathbb{R}^{q_k}$ is a Jordan block of the form

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & & 1 \\ \mathbf{0} & & & \lambda_k \end{bmatrix}, k = 1 \dots p$$

where q_k is the algebraic multiplicity of the k^{th} eigenvalue. $\hat{\mathbf{A}}$ is therefore an upper triangular matrix. Using the lemma proved earlier,

$$e^{\hat{\mathbf{A}}t} = \sum_{i=0}^{\infty} \frac{\left(\hat{\mathbf{A}}t\right)^i}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{bmatrix} (\lambda_1 t)^i & \cdots & \\ & \ddots & \vdots \\ \mathbf{0} & & (\lambda_p t)^i \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & \\ & \ddots & \vdots \\ \mathbf{0} & & e^{\lambda_p t} \end{bmatrix}$$

The determinant of a triangular matrix is equal to the product of its diagonal elements (see Assignment 1, Problem 1(d)). Hence,

$$\det e^{\hat{\mathbf{A}}t} = \det \begin{bmatrix} e^{\lambda_1 t} & \cdots & \\ & \ddots & \vdots \\ \mathbf{0} & & e^{\lambda_p t} \end{bmatrix} = e^{\left(\sum_{k=1}^{p} \lambda_k\right)t} = e^{\operatorname{tr} \hat{\mathbf{A}}t}$$

A MATLAB Script

```
%% ASEN5044 Assignment 2 Solutions
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% 7 September 2018

%% Problem 1

% Part C

dt = 10;                    % Discretization time (s)
k = 3.986e5;                 % Gravitational parameter (km^3/s^2)
r0 = 6678;                   % Nominal orbital radius (km)
om0 = sqrt(k/r0^3);          % Nominal angular rate (rad/s)

A = [0 1 0 0;
    om0^3 0 0 2*r0*om0;
    0 0 0 1;
    0 -2*om0/r0 0 0];

B = [0 0; 1 0; 0 0; 0 1/r0];

Fh = expm([A B; zeros(2, 6)]*dt);

F = Fh(1:4, 1:4);
G = Fh(1:4, 5:6);

%% Problem 4

% Part D

% Define constants
K = 1000;               % Thrust constant
g = 50;                 % Drag constant
G = 6.673e-11;          % Universal gravitational parameter (N m^2/kg^2)
M = 5.98e24;            % Mass of the Earth (kg)
R = 6.37e6;             % Radius of the Earth (m)
m0 = 1000;              % Initial mass (kg)

% Define u
unl = @(t, du) G*M*m0/(K*R^2)*expm(-G*M/(K*R^2)*t) + du*abs(cos(t));
ulin = @(t, du) du*abs(cos(t));

% Define nonlinear and linearized systems
hnl = @(t, x, du) [x(2); (K*unl(t, du) - g*x(2))/x(3) - G*M/(R + x(1))^2; -unl(t, du)];
hlin = @(t, x, du) [0 1 0; 2*G*M/R^3 -g/m0*expm(G*M/(K*R^2)*t) -G*M/(m0*R^2)*expm(G*M/(K*R

% Simulate nonlinear and linear system
tin = 0:0.01:5;                         % Simulation timesteps (s)
x0nl = [0; 0; m0];
x0lin = [zeros(2, length(tin)); m0*exp(-G*M/(K*R^2)*tin)];
for du = [10 100 300]
```

```
    hnlwrap = @(t, x) hnl(t, x, du);
    [tout, xoutnl] = ode45(hnlwrap, tin, x0nl);

    hlinwrap = @(t, x) hlin(t, x, du);
    [tout, xoutlin] = ode45(hlinwrap, tin, [0; 0; 0]);

    figure;
    plot(tout, xoutnl(:, 1), tout, xoutlin(:, 1) + x0lin(1, :)');
    xlabel('Time (s)');
    ylabel('Rocket Altitude (m)');
    legend('Nonlinear', 'Linearized', 'location', 'best');
    title(sprintf('Rocket Altitude vs. Time, \\Deltau = %g', du));
    grid on;
end
```