Statistical Estimation	**Homework 2**
ASEN 5044 Fall 2018	**Due Date: Sep 20, 2018**
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Exercise 1

Consider the equations of motion for a unit mass subjected to an inverse square law fore field,

$$\ddot{r} = r\dot{\theta}^2 + \frac{k}{r^2} + u_1(t)$$
$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2(t)$$

where r represents the radius from the center of the force field, θ gives the angle with respect to a reference direction in the orbital plane, k is a constant, and u_1 and u_2 represent radial and tanjectial thrusts, respectively. It is easily shown that for the initial conditions $r(0) = r_0$, $\theta(0) = 0$, $\dot{r}(0) = 0$, and $\dot{\theta}(0) = \omega_0$ with nominal thrusts $u_1(t) = 0$ and $u_2(t) = 0$ for all $t \geq 0$ the equations of motion have as a solution the circular orbit given by

$$r(t) = r_0 = \text{constant}$$
$$\dot{\theta}(t) = \omega_0 = \text{constant} = \sqrt{\frac{k}{r_0^3}}$$
$$\theta t = \omega_0 t + \text{constant}$$

Problem (a)

Pick a state vector for this system and express the original nonlinear ODEs in standard nonlinear state space form.

If we choose $x = [r,\ \theta, \dot{r}, \dot{\theta}]^T$ as our state vector and $y = [r,\ \theta]^T$ as our observation vector then we can express the original ODEs as

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 x_4^2 - \frac{k}{x_1^2} + u_1(t) \\ -\frac{2x_4 x_3}{x_1} + \frac{1}{x_1}u_2(t) \end{bmatrix}$$
$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem (b)

Linearize this system's nominal equations of motion about the nominal solution $r(t) = r_0$, $\dot{r}(0) = 0$, $\theta(t) = \omega_0 t + \text{constant}$ and $\dot{\theta}(t) = \omega_0$ with $u_1(t) = 0$ and $u_2(t) = 0$. Find (A, B, C, D) matrices for output $y(t) = [r(t), \theta(t)]^T$ for the linearized system of equations about the nominal solution.

If we take $x_{\text{nom}} = [r_0, 0, \omega_0 t + c, \omega_0]^T$ and $u_{\text{nom}} = [0, 0]^T$. We can say $x(t) = x_{\text{nom}}(t) + \tilde{x}(t)$ and $u(t) = u_{\text{nom}}(t) + \tilde{u}(t)$. We can define

$$\dot{\tilde{x}} = A_{\text{nom}}\tilde{x}(t) + B_{\text{nom}}\tilde{u}(t)$$

$$A_{\text{nom}} = \frac{\partial f}{\partial x}\bigg|_{x_{\text{nom}},u_{\text{nom}}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & -\frac{2\omega_0}{r_0} & 0 \end{bmatrix}$$

$$B_{\text{nom}} = \frac{\partial f}{\partial u}\bigg|_{x_{\text{nom}},u_{\text{nom}}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}$$

The observation function is already linear, so the C and D matrices do not need to be linearized:

$$y(t) = Cx(t) + Du(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem (c)

Convert the continuous time (A, B, C, D) matrices you found from part (b) into discrete time (F, G, H, M) matrices using a discretization step size of $\Delta t = 10\text{s}$ and setting $k = 398600\text{km}^3/\text{s}^2$ and $r_0 = 6678\text{km}$.

We start by reorganizing our ODE as $\dot{\tilde{x}}_a = \hat{A}[\tilde{x}, \tilde{u}]^T$ where

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 & 1 & 0 \\ 0 & 0 & -\frac{2\omega_0}{r_0} & 0 & 0 & \frac{1}{r_0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.00157 & 0 & 0 & 0.01788 & 1 & 0 \\ 0 & 0 & 4.01 \times 10^{-10} & 0 & 0 & 1.497 \times 10^{-4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that F will be the upper left $n \times n$ submatrix and G will be the the upper right $m \times n$ submatrix of $e^{\hat{A}\Delta t}$:

$$e^{\hat{A}\Delta t} = \begin{bmatrix} 1.079 & 0 & 10.26 & 0.906 & 50.65 & 4.496 \times 10^{-4} \\ 0 & 1 & 0 & 10 & 0 & 7.485 \times 10^{-3} \\ 0.0161 & 0 & 1.079 & 0.184 & 10.264 & 1.35 \times 10^{-4} \\ 0 & 0 & 0 & 1 & 0 & 1.497 \times 10^{-3} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.079 & 0 & 10.26 & 0.906 \\ 0 & 1 & 0 & 10 \\ 0.0161 & 0 & 1.079 & 0.184 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 50.65 & 4.496 \times 10^{-4} \\ 0 & 7.485 \times 10^{-3} \\ 10.264 & 1.35 \times 10^{-4} \\ 0 & 1.497 \times 10^{-3} \end{bmatrix}$$

The H and M matrices for the discretized system are simply equal to the C and D matrices for the continuous time system:

$$H = C$$
$$M = D$$

Problem (d)

Interpret the results for the STM in part (c), i.e. what is the physical meaning of each column vector that makes up F?

Each column vector F_i in F represents how $\tilde{x}_i(k)$ contributes to $\tilde{x}(k+1)$. In other words, each column vector describes how the corresponding entry in the $\tilde{x}$ vector at step k will affect the entire $\tilde{x}$ vector at step $k+1$.

Exercise 2

The linear position p of an object under constant acceleration is

$$p = p_0 + \dot{p}_0 t + \frac{1}{2}\ddot{p}_0 t^2$$

where p_0 is the initial position of the object.

Problem (a)

Define a state vector as $x = [p\ \dot{p}\ \ddot{p}]^T$ and write the state space equation $\dot{x} = Ax$ for this system.

Because the object is under constant acceleration, the derivative of the acceleration $\dddot{p} = 0$. So the state space equation is simply:

$$\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix}$$

Problem (b)

Use the first and last expressions in Equation (1.71) to find the state transition matrix for this system.

We'll first go through the infinite series method, calculating the powers of A first:

$$A\Delta t = \begin{bmatrix} 0 & \Delta t & 0 \\ 0 & 0 & \Delta t \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A\Delta t)^2 = \begin{bmatrix} 0 & 0 & \Delta t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A\Delta t)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Luckily for us, $(A\Delta t)^3$ is the zero matrix, so we don't have to extend the series beyond $i = 2$. This means the state transition matrix can be calculated as follows:

$$\begin{aligned} e^{A\Delta t} &= \sum_{i=0}^{2} \frac{(A\Delta t)^i}{i!} \\ &= I + \begin{bmatrix} 0 & \Delta t & 0 \\ 0 & 0 & \Delta t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & \Delta t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

When employing the third expression in equation 1.71, we find that $\hat{A} = A$ and $Q = I$. Since $\hat{A} = A$ we can't use the third expression to try to find $e^{\hat{A}\Delta t}$ as this would lead us into an infinite loop, so we'll use the result from above. Thus, the expression gives

$$\begin{aligned} e^{A\Delta t} &= Qe^{\hat{A}\Delta t}Q^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Problem (c)

Prove for the state transition matrix found above that $e^{A0} = I$.

$$e^{A0} = \begin{bmatrix} 1 & 0 & 0^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$