ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 2: Rapid Linear Algebra Refresher

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Today

- Start reviewing important mathematical tools and concepts

- Quick refresher of linear algebra
  - highlights of n-dimensional matrix-vector concepts

START READING: Chapters 1.1 and 1.2 in Simon book

Vectors and vector operations in n-dimensions

- Vectors: v = ordered list of elements : $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, v_j is j^{th} scalar element

$v^T = [v_1, \dots, v_n]$

$v_j \in \mathbb{R}$, $v \in \mathbb{R}^n$

- Inner (dot) product :

is scalar $\alpha = a^T b$ for $a, b \in \mathbb{R}^n$ $(\mathbb{R}^{n\times 1})$

Notation: $\langle a, b \rangle = \alpha = \sum_{j=1}^{n} a_i b_i = a_1 b_1 + \dots + a_n b_n$

- Outer product $\rightarrow$ way of describing relative directions in n-dim space

if $b \in \mathbb{R}^n$ & $c \in \mathbb{R}^m$ ($m \neq n$ possibly), then

$$A = \underset{n\times 1}{b}\,\underset{1\times m}{c^T} = \begin{bmatrix} b_1 c_1 & b_1 c_2 & \cdots & b_1 c_m \\ b_2 c_1 & b_2 c_2 & \cdots & b_2 c_m \\ \vdots & & & \vdots \\ b_n c_1 & \cdots & \cdots & b_n c_m \end{bmatrix} = \text{outer product}$$

Matrices

- Matrix: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \in \mathbb{R}^{m \times n}$

  (*) always check this!!

  - if $A = BC$, then $B \in \mathbb{R}^{m \times p}$ & $C \in \mathbb{R}^{p \times n}$, where $p \geq 1$ [ie inner dims of B & C must match & outer must make sense]

  - Also: $A^T = (BC)^T = C^T B^T$ [recall: $BC \neq CB$ in general! non-commutative...]

- Trace: for square $n \times n$ matrix A:

  $$\text{trace}(A) = tr(A) = \sum_{i=1}^{n} a_{ii} \quad \text{(sum of diag. entries)}$$

  - Note: if $A \in \mathbb{R}^{n \times m}$ & $B \in \mathbb{R}^{m \times n}$, then $tr(AB) = tr(BA)$ [Also: $tr(ABC) = tr(CAB) = tr(BCA)$]

- Symmetric matrix: if A is $n \times n$, then A is sym. if $A = A^T$

Linear dependence/independence, rank

- Set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be **linearly dependent** if

linear combo

$$\exists \text{ scalars } \alpha_j \neq 0,\ j = 1, \ldots, n, \text{ s.t. } v_i = \sum_{j \neq i} \alpha_j v_j \text{ for at least } 1\ i = 1, \ldots, n$$

(i.e. at least one vector in the set equals a non-trivial linear combination of other vectors in the set)

- Vectors $\{v_1, v_2, \ldots, v_n\}$ are **linearly independent** if they are not linearly dependent

- Square nxn matrix A has rank = n (full rank) if its column (row) vectors are all linearly independent

$$A = [v_1, v_2, \ldots, v_n] = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}, \quad v_i \in \mathbb{R}^{n \times 1}, \quad r_i \in \mathbb{R}^{1 \times n}$$

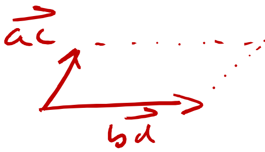
for non-square $A \in \mathbb{R}^{m \times n}$

$$\text{rank}(A) \leq \min(m, n)$$

(i.e. square A is just a stacked set of nx1 column vectors or 1xn row vectors; if rank(A)=n, then vectors LI)

Determinants of Square Matrices

- 2x2 case: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $det(A) = |A| = ad - bc$



$|A| =$ area of parallelogram

$$\rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ if } |A| \neq 0$$

- General case: define the **cofactor:** $c_{ij} = (-1)^{i+j}|M_{ij}|$ (determinant of minor)

  where the **minor** is: $M_{ij} = A$ with row i and column j removed

→ so $det(A) = |A| = \sum_{i=1}^{n} a_{ij}c_{ij}, \ \forall j = 1, \ldots, n$ (cofactor expansion)

3x3 example:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix}$$

expand along 1st col

$$|A| = \cancel{0 \cdot \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}} - (1)\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + \cancel{(0)\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}}$$

$= 4$

Singular/Non-Singular Square Matrices

- A is **singular** if $|A| = 0$
- A is **non-singular** if $|A| \neq 0$ (i.e. if all rows/cols of A are linearly indep)
- Also, if $|A| = 0$, then $\exists x \neq 0$ such that $Ax = 0$

$$\text{But if } |A| \neq 0, \text{ then } Ax = 0 \text{ if and only if } x = 0 \text{ (only trivial solution)}$$

Vocabulary:

- **Singular** = **non-invertible** = **rank deficient** (i.e. rank(A) < n)
- **Non-singular** = **invertible** = **full-rank**

→ If A is non-singular, then $|A| \neq 0$ and $\exists\ A^{-1} \in \mathbb{R}^{n \times n}$ s.t. $AA^{-1} = A^{-1}A = I$

$$\text{where the inverse of } A \text{ is } A^{-1} = \frac{adj(A)}{det(A)} = \frac{C_A^T}{det(A)} \; (C_A^T \text{ is matrix of cofactors})$$

Solutions to “Nice” Linear Systems of Equations

- If $A \in \mathbb{R}^{n\times n}$ and $|A| \neq 0$, and $b \in \mathbb{R}^n$, then we can solve

$$Ax = b \text{ for } x \in \mathbb{R}^n$$

$$\rightarrow \boxed{x = A^{-1}b}$$

Recall: this tells us that x is the unique solution in R^n, because:

- A represents a “1 to 1” and “onto” linear transformation from R^n to R^n

>>Range space of A is R^n

$$\text{Range}(A) = \{y \in \mathbb{R}^n |\ \exists x \in \mathbb{R}^n\ s.t.\ Ax = y\}$$

>>Null space of A is trivial (i.e. Null(A) only contains x=0)

$$\text{Null}(A) = \{x \in \mathbb{R}^n |\ Ax = 0\}$$

Solutions to "Not Nice" Linear Systems of Eqs.?

- Consider an "overdetermined" system of equations:

$$y = Mx, \text{ where } y \in \mathbb{R}^m, \; x \in \mathbb{R}^n, \; M \in \mathbb{R}^{m \times n}$$

So if $m > n$: if rank $(M) = n$ [Full col. rank], then easy to show the Gram matrix $G = M^T M \in \mathbb{R}^{n \times n}$ [square] also has rank $(G) = n$

$\rightarrow$ G [square] is invertible (i.e. $|G| \neq 0$) $\rightarrow$ G^{-1} exists $= (M^T M)^{-1}$

$\rightarrow$ how to solve for x? First multiply $y = Mx$ by M^T on LHS & RHS:

$\rightarrow M^T y = M^T M x \rightarrow$ since $M^T M = G$: $M^T y = G x$

$\rightarrow$ but since G^{-1} exists when M has rank(n) $\rightarrow G^{-1} M^T y = \underbrace{(G^{-1} G)}_{I} x$

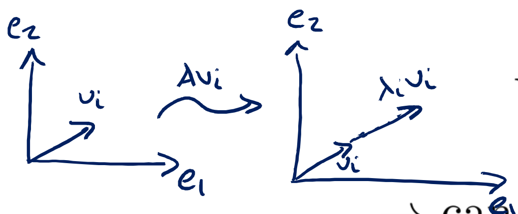
$\rightarrow G^{-1} M^T y = x \rightarrow \boxed{x = (M^T M)^{-1} M^T y = M_L^+ y}$

where $\boxed{M_L^+ = (M^T M)^{-1} M^T \text{ is left pseudoinverse}}$

Eigenvalues and Eigenvectors of Square Matrices

- **Given** $A \in \mathbb{R}^{n \times n}$, $\exists$ scalars λ_i (eigenvalues) (possibly complex numbers)

such that $\exists$ associated eigenvectors $v_i \in \mathbb{R}^n$ (possibly complex, if λ_i complex)



where $Av_i = \lambda_i v_i$, where $\underline{\underline{v_i \neq}}\, 0$ (by def.)

$\rightarrow$ can solve for these via $(A - \lambda_i I)v_i = 0$

$\rightarrow$ for non-trivial v_i, want matrix $(A - \lambda_i I) = Q(\lambda_i)$ to be singular,

i.e. want $Q(\lambda_i)v_i = 0$ for $v_i \neq 0$

$\rightarrow det(Q(\lambda_i)) = det(A - \lambda_i I) = 0$

$\rightarrow$ gives the **characteristic polynomial** for A (polynomial in λ of order n)

$\rightarrow$ roots of the characteristic polynomial = eigenvalues of A

$\rightarrow$ n (complex conjugate) eigenvalues always exist

Handy Dandy Facts About E'vals/E'vecs

- FACT 1: For real-valued <u>symmetric</u> square matrices

$$\text{i.e. if } A = A^T,$$

then n eigenvalues are all real
AND n eigenvectors exist which are all linearly independent and orthogonal

- FACT 2: For <u>any</u> square matrix A

So if $A \in \mathbb{R}^{n \times n}$ is singular,

$$\longrightarrow det(A) = |A| = \prod_{i=1}^{N} \lambda_i \text{ (determinant is product of e'vals)}$$

$$tr(A) = \sum_{i=1}^{N} \lambda_i \text{ (trace is the sum of e'vals)}$$

@ least one e'val is 0!

[corresp e'vecs are basis for Null(A)!]

Positive Definite Matrices

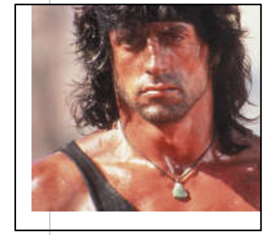
- **Definition:** Matrix $P \in \mathbb{R}^{n\times n}$ is positive definite (posdef) if: $\}\ (P \succ 0)$

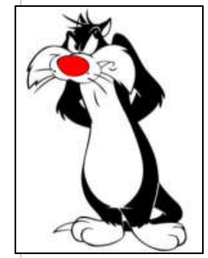
$$x^T P x > 0 \quad \text{for all} \quad x \neq 0 \in \mathbb{R}^n$$

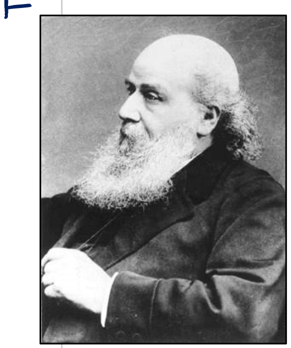
Note: if P is posdef, then all e'vals of P are positive, i.e. $\lambda_i(P) > 0,\ i = 1, \cdots, n$

$\rightarrow$ We often need to verify that a matrix is posdef in computation,

but we don't necessarily want to compute all the e'vals of P to do so (expensive)!







Sylvester's method: check posdef'ness of P by examining if the principal minors of P are all positive $\rightarrow$ if so, P is posdef

ie if $P \in \mathbb{R}^{n\times n}$, w/ $P = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix}$, then check:

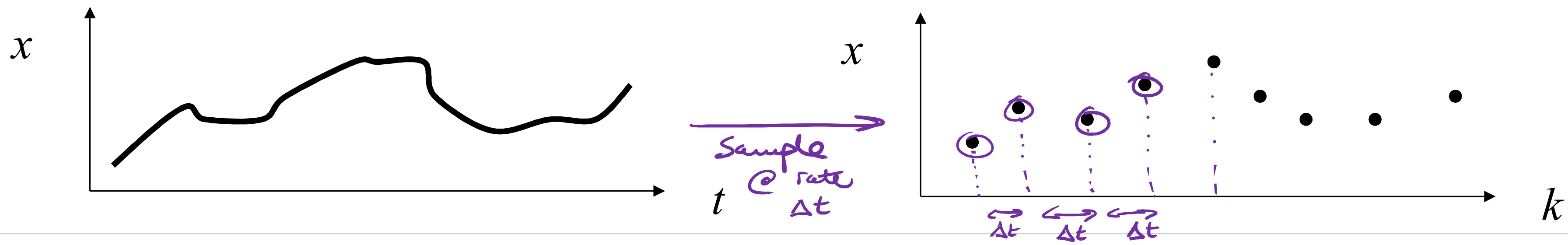
$P_{11} \overset{?}{>} 0$ (1st principal minor), $\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} \overset{?}{>} 0$ (2nd principal minor), $\ldots$, $|P| \overset{?}{>} 0$ (n^{th} prin. minor)

Onto Dynamical State Space Systems

- Want to study how **vector quantities change over time**, especially when the **vector elements are related to each other**
  - Vehicle state: position, velocity, attitude, attitude rate,…
  - Physiological state: blood pressure, heart rate, O_2 level…
  - Economic state: GNP, GDP, national debt,…

- Such time-varying variables can define the ***state*** **x** of a system over time
- **Continuous time (CT) systems:** continuous state x(t) depends on **continuous** t time variable
- **Discrete time (DT) systems:** continuous state x(k) depends on **integer** k time variable
- **Often, the state x itself cannot actually be observed – but only some sensed variable y related to x**

$t \in \mathbb{R}$ (Real numbers) $k \in \mathbb{Z}$ (Integers)



The Big Picture

- Goal: analysis, control and estimation of dynamical systems
  - Need to understand behavior over time (so we can influence/change it)
  - Work with mathematical models first…
  - …then go test/implement on real thing

- Interested in (physical) dynamical systems that obey **differential equations**
  - Ex.: scalar linear **ordinary differential equation** (ODE):

$$\frac{dx(t)}{dt} \rightarrow \underbrace{\dot{x}(t)}_{\text{where system going}} = a\,\underbrace{x(t)}_{\text{where system is @ time } t \text{ (i.e. state)}} \longrightarrow x(t) = \underbrace{x(0)}_{\text{where system started}} \cdot \underbrace{e^{at}}_{\text{tells us how system behaves}}$$