ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 5: Nonlinear Systems and Linearization

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Announcements

- **HW 1 Due Thurs 9/13 at 11 am (before start of next lecture)**
- **Submit to Canvas –**
  - **All submissions must be legible!!! – zero credit otherwise**
  - **All submissions must have your name on them!!! – zero credit otherwise**

- Advanced Questions:
  - required for PhD students
  - optional/extra credit for everyone else

- Office hours today: 3 pm – 4:30 pm , ECAE 175

Overview

Last time: LTI State Space IVP Solutions and the Matrix Exponential

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \longrightarrow \qquad x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t_0) = x_0$$

$$e^{A(t-t_0)} = \sum_{i=0}^{\infty} A^i \frac{(t-t_0)^i}{i!} = \Phi(t,t_0)$$

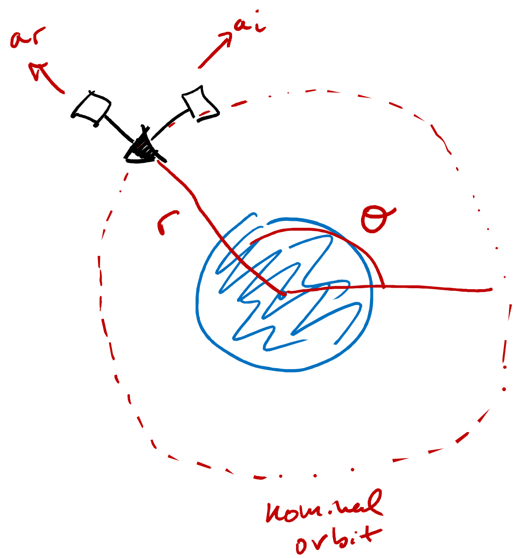
Today:

- Nonlinear systems and "standard form" nonlinear state space models
- Linearization and transformation to linear SS models

READ: Chapter 1.6-1.8 in Simon book

What to do about Nonlinear ODEs?

- Most systems have intrinsically nonlinear effects that are not obviously/easily modeled by linear physical relationships
- **What if a priori/first principles give nonlinear (NL) dynamics?**
- Example: equation for orbit plane motion of satellite



$$\ddot{r} - \underline{\underline{\dot{\theta}^2}}\, r = -\frac{\mu}{\underline{\underline{r^2}}} + a_r$$

$$\underline{\underline{r\ddot{\theta}}} + 2\,\underline{\underline{\dot{r}\dot{\theta}}} = a_i$$

a_i, a_r : in-track & radial accelerations

[due thrust, drag, SRP, grav. anomalies, etc.]

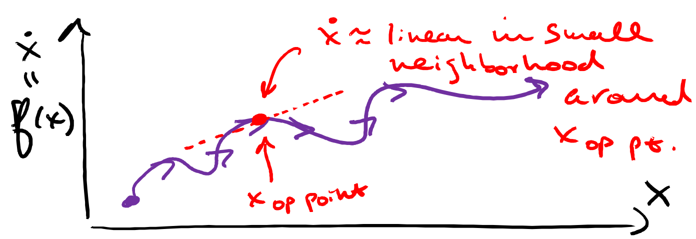
r, θ : states for s/c

NL systems can get very nasty and weird…

- For now, only focus on NL sys with "sufficiently smooth" nonlinearities
  - i.e. derivatives exist for state vars and are bounded

(Lipschitz continuity: for some $f(x)$, $\|f(x_1)-f(x_2)\| \leq c \cdot \|x_1 - x_2\| \;\; \forall\, x_1 \,\&\, x_2$ "sufficiently close" for some constant c)

- Most cases: want to keep NL sys near operating point/condition
  - Equilibrium: Set of x & u s.t. $\dot{x}=0$ (i.e. some $\{x_{eq}, u_{eq}\}$ s.t. $\dot{x} = f(x,u)\big|_{x_{eq},\,u_{eq}} = 0$)
  - Nominal trajectory: $\dot{x}_{nom}(t) = f(x,u)\big|_{x_{nom}(t),\,u_{nom}(t)}$ [where $\{x_{nom}(t), u_{nom}(t)\}$ are known solutions to the nonlinear ODE $\dot{x} = f(x,u)$]
- Can look at dynamics of "small" perturbations near op pt
  - If perturbations small enough, system behaves (almost) linearly!



Linearization of NL ODEs via Multivariable Taylor Series to Get Linear SS Models

"standard form"

- Idea: express NL ODEs in standard (non-linear) state vector form → do Taylor expansion near operating point → drop higher order terms (HOTs)

Given set of NL ODEs & output relations, express in standard NL SS form (after we pick state vars)

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x,u,t) \\ f_2(x,u,t) \\ \vdots \\ f_n(x,u,t) \end{bmatrix} = f(x,u,t) \longrightarrow \text{stack of } n \text{ NL ODES}$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} h_1(x,u,t) \\ \vdots \\ h_p(x,u,t) \end{bmatrix} = h(x,u,t) \longrightarrow \text{stack of } p \text{ NL algebraic eqs.}$$

Also, let $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$ ($m \times 1$ input vector)

Linearization (cont'd)

- Suppose a nominal solution or operating point is known/given, i.e. $x_{nom}(t)$ & $u_{nom}(t)$

Define slight perturbations from nominal op. point:

perturbation vectors
$$\begin{cases} \delta x(t) \triangleq x(t) - x_{nom}(t) & \longrightarrow \quad x(t) = x_{nom}(t) + \delta x(t) \\ \delta u(t) \triangleq u(t) - u_{nom}(t) & \longrightarrow \quad u(t) = u_{nom}(t) + \delta u(t) \end{cases}$$

→ Plug in expressions for $x(t)$ & $u(t)$ into NL ODEs & do vector Taylor series expansion

$$\dot{x}(t) = f(x,u,t) \Longleftrightarrow \dot{x}(t) = \dot{x}_{nom}(t) + \delta\dot{x}(t) \Longleftrightarrow f(x,u,t) = f(x_{nom}(t)+\delta x(t), u_{nom}(t)+\delta u(t), t)$$

→ Vector Taylor series expansion:

$$\dot{x}(t) = f(x_{nom}+\delta x, u_{nom}+\delta u, t)$$
$$= f(x_{nom}, u_{nom}, t) + \left[\frac{\partial f}{\partial x}\bigg|_{\substack{x_{nom}\\u_{nom}}}\right] \cdot \delta x(t) + \left[\frac{\partial f}{\partial u}\bigg|_{\substack{x_{nom}\\u_{nom}}}\right] \cdot \delta u(t) + \text{HOTs}$$

(Higher order terms)

Likewise, for output relations:

$$y(t) = y_{nom} + \delta y = h(x_{nom}+\delta x, u_{nom}+\delta u, t)$$

$$\left(\substack{\text{Taylor}\\\text{exp}}\right) = h(x_{nom}, u_{nom}, t) + \left[\frac{\partial h}{\partial x}\bigg|_{nom}\right] \delta x + \left[\frac{\partial h}{\partial u}\bigg|_{nom}\right] \delta u + \text{HOTs}$$

Linearization (cont'd)

- Partial derivative matrices = **Jacobians** w.r.t. x and u

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x_{nom}\\u_{nom}}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\substack{x_{nom}\\u_{nom}}} \longrightarrow \underline{n \times n \text{ matrix}}$$

$$\left.\frac{\partial f}{\partial u}\right|_{\substack{x_{nom}\\u_{nom}}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{\substack{x_{nom}\\u_{nom}}} \longrightarrow \underline{n \times m \text{ matrix}}$$

$$\left(\text{Similarly for } \underset{[p \times n]}{\left.\frac{\partial h}{\partial x}\right|_{nom}} \ \ \& \ \ \underset{[p \times m]}{\left.\frac{\partial h}{\partial u}\right|_{nom}} \right)$$

For small enough δx, δu, can neglect HOTs

- Get linearized eqs **for dynamics of perturbations** δx, δy w.r.t. δu "near" nominal op point:

$$(i)\quad \dot{x}(t) = \cancel{\dot{x}_{nom}(t)} + \delta\dot{x} \approx \cancel{f(x_{nom}, u_{nom}, t)} + \left[\frac{\partial f}{\partial x}\right|_{nom}\delta x + \left[\frac{\partial f}{\partial u}\right|_{nom}\delta u + \cancel{HOTS}$$

cancel w/ RHS — cancel w/ LHS — ignore for small δx, δu

$$(ii)\quad y(t) = \cancel{y_{nom}(t)} + \delta y \approx \cancel{h(x_{nom}, u_{nom}, t)} + \left[\frac{\partial h}{\partial x}\right|_{nom}\delta x + \left[\frac{\partial h}{\partial u}\right|_{nom}\delta u + \cancel{HOTS}$$

cancel

But: $\left.\begin{aligned}\dot{x}_{nom}(t) &= f(x_{nom}, u_{nom}, t)\\ y_{nom}(t) &= h(x_{nom}, u_{nom}, t)\end{aligned}\right\}$ so these terms cancel in LHS & RHS above!

Left with →

$$\boxed{\begin{aligned}\delta\dot{x} &= \left[\frac{\partial f}{\partial x}\right|_{nom}\delta x(t) + \left[\frac{\partial f}{\partial u}\right|_{nom}\delta u(t)\\ \delta y &= \left[\frac{\partial h}{\partial x}\right|_{nom}\delta x(t) + \left[\frac{\partial h}{\partial u}\right|_{nom}\delta u(t)\end{aligned}}$$

perturbation vector dynamics & output relations → only valid if δx & δu "small enough"

Let $\tilde{x} = \delta x$, $\tilde{y} = \delta y$, $\tilde{u} = \delta u$ →

"$A|_{nom}$" $= \left[\frac{\partial f}{\partial x}\right|_{nom}$

"$B|_{nom}$" $= \left[\frac{\partial f}{\partial u}\right|_{nom}$

"$C|_{nom}$" $= \left[\frac{\partial h}{\partial x}\right|_{nom}$

"$D|_{nom}$" $= \left[\frac{\partial h}{\partial u}\right|_{nom}$

$$\boxed{\begin{aligned}\dot{\tilde{x}} &= A|_{nom}\tilde{x}(t) + B|_{nom}\tilde{u}(t)\\ \tilde{y} &= C|_{nom}\tilde{x}(t) + D|_{nom}\tilde{u}(t)\end{aligned}}$$

Looks like linear SS model! But [A, B, C, D] matrices now depend on $x_{nom}(t)$ & $u_{nom}(t)$!!

Example: 2nd order NL ODE (no input)

$$\ddot{z} + (1+z)\dot{z} - 2z + 0.5z^3 = 0 \quad \left(\rightarrow \ddot{z} = 2z - 0.5z^3 - (1+z)\dot{z} \right)$$

rewrite w/ x_1 & x_2

Step 1: define state variables and put into standard NL SS form:

$$\begin{matrix} x_1 = z \\ x_2 = \dot{z} \end{matrix} \rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1 - 0.5x_1^3 - (1+x_1)x_2 \end{bmatrix}$$

$$\rightarrow f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1 - 0.5x_1^3 - (1+x_1)x_2 \end{bmatrix} = \dot{x} \quad \left(\text{std. NL SS form} \right)$$

Define output $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$ No linearization needed!

suppose now we linearize dynamics around equilibrium points . . .

Example: 2^{nd} order NL ODE (cont'd)

Step 2: look for eq. points to use as x_{nom} op. point:

⊗ Eq. points: sol'ns of $\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array}\right]\Bigg|_{(x_1,x_2)} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array}\right] = \left[\begin{array}{c} x_2 \\ (\cdots) \end{array}\right]$

→ So we must have: $x_2 = 0$

& $2x_1 - 0.5\,x_1^3 - (1+x_1)\,x_2 = 0$

→ Solve for the roots of the 2^{nd} equation [since $x_2 = 0$ is known via 1^{st} eq.]

→ get 3 equilib. pts.:

$$x_{eq,1} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \quad \left(= \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right)$$

$$x_{eq,2} = \left[\begin{array}{c} 2 \\ 0 \end{array}\right]$$

$$x_{eq,3} = \left[\begin{array}{c} -2 \\ 0 \end{array}\right]$$

Now we take Jacobians of NL ODE w.r.t. x & evaluate @ these different x_{nom} op. pts.

Example: 2^{nd} order NL ODE (cont'd)

Step 3: Find Jacobians at x_{nom} points

$$\left[\frac{\partial f}{\partial x}\right]\Big|_{x_{eq}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x_{eq}} = \begin{bmatrix} 0 & 1 \\ 2 - \frac{3}{2}x_1^2 - x_2 & -(1+x_1) \end{bmatrix}\Bigg|_{x_{eq}}$$

$[2 \times 2]$

$\rightarrow$ for $x_{eq,1}$: $A_1|_{nom} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$

$x_{eq,2}$: $A_2|_{nom} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}$

$x_{eq,3}$: $A_3|_{nom} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix}$

$\rightarrow$ $\dot{\tilde{x}} = A_i \tilde{x}$ for $\tilde{x} = \delta x$ (where $i = 1, 2, 3$ for eq. pts) $\Big\}$ 3 <u>different</u> linearized ODEs for <u>different</u> <u>eq. pts</u>

Example: 2nd order NL ODE (cont'd)

Step 4: Put into LTI SS form: what is the state vector?

$\tilde{x} = \delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$ = perturbation state vector $\neq$ total state vector $x(t)$!

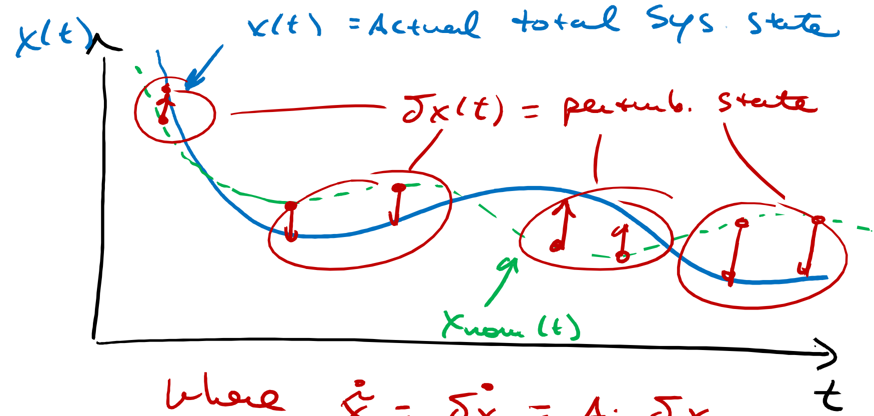
$\rightarrow$ what is Actual total state of NL system @ any time (w.r.t. op. pt.)?

$$x(t) = x_{nom}(t) + \delta x(t)$$

$$= \begin{bmatrix} x_{nom,1}(t) \\ x_{nom,2}(t) \end{bmatrix} + \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

$$= \begin{bmatrix} z_{nom,i} \\ \dot{z}_{nom,i} \end{bmatrix} + \begin{bmatrix} \delta z \\ \delta \dot{z} \end{bmatrix}$$

(for eq. pt. i in example)



where $\dot{\tilde{x}} = \delta \dot{x} = A_i \, \delta x$