ASEN5044 Assignment 1

Solutions

11 September 2018

1. (a) $\det \mathbf{A} = 21 \implies \mathbf{A}$ is invertible.

   (b) Using Laplace's Formula,

$$\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{M}_{ij}$$

   where a_{ij} is the element of $\mathbf{A}$ at the i^{th} row and j^{th} column and $\mathbf{M}_{ij}$ is the (i, j) minor of $\mathbf{A}$, choose $j = 3$ to get

$$\det \mathbf{A} = \det \begin{bmatrix} 11 & 26 & 0 \\ 64 & 57 & 10 \\ 54 & 23 & 71 \end{bmatrix} = -62117 \implies \mathbf{A} \text{ is invertible}$$

   (c) Observe that column 3 is a scalar multiple of column 1 $\implies$ dim null $\mathbf{A} > 0 \implies \det \mathbf{A} = 0 \implies \mathbf{A}$ is not invertible.

   (d) $\mathbf{A}$ is a triangular matrix. In general, the determinant of a triangular matrix is given by the product of its diagonal elements, i.e.

$$\det \mathbf{A} = \prod_{i=1}^{n} a_{ii} = 62537200 \implies \mathbf{A} \text{ is invertible}$$

   Proof Given $\mathbf{A} \in^n \mathbb{R}^n$ a triangular matrix, for $n = 2$,

$$\det \mathbf{A} = \det \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = a_{11} a_{22}$$

   Assume by way of induction that for $n = k$,

$$\det \mathbf{A} = \prod_{i=1}^{n} a_{ii}$$

   holds. Then for $n = k + 1$,

$$\det \mathbf{A} = \sum_{j=1}^{k+1} (-1)^{i+j} a_{ij} \det \mathbf{M}_{ij}$$

Choose $i = 1$ so that for $j = 1$, $\mathbf{M}_{ij} \in^k \mathbb{R}^k$ is a triangular matrix, and hence

$$\det \mathbf{M}_{ij} = \prod_{i=2}^{k+1} a_{ii}$$

For $j \neq 1$, $\mathbf{M}_{ij}$ has a leading zero column, and thus

$$\det \mathbf{M}_{ij} = 0$$

Therefore,

$$\det \mathbf{A} = \sum_{j=1}^{k+1} (-1)^{i+j} a_{ij} \det \mathbf{M}_{ij} = a_{11} \det \mathbf{M}_{11} = \prod_{i=1}^{n} a_{ii}$$

2. (a) Let

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Then $\mathbf{a}\mathbf{b}^T = \begin{bmatrix} b_1\mathbf{a} & \dots & b_n\mathbf{a} \end{bmatrix} \implies$ the span (i.e. a sufficient set of basis vectors) of the column space of $\mathbf{a}\mathbf{b}^T$ consists only of $\{\mathbf{a}\} \implies$ rank $\mathbf{a}\mathbf{b}^T = 1$

(b) Let

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{a}_{r1} \\ \vdots \\ \mathbf{a}_{rm} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{c1} & \dots & \mathbf{a}_{cn} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{b}_{r1} \\ \vdots \\ \mathbf{b}_{rn} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{c1} & \dots & \mathbf{b}_{cm} \end{bmatrix} \end{aligned}$$

Then

$$\text{tr } \mathbf{AB} = \sum_{i=1}^{m} \mathbf{a}_{ri}\mathbf{b}_{ci} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} = \sum_{i=1}^{n} \mathbf{b}_{rj}\mathbf{a}_{cj} = \text{tr } \mathbf{BA}$$

(c)

$$\left|\mathbf{A}\mathbf{A}^{-1}\right| = |\mathbf{A}| \left|\mathbf{A}^{-1}\right| = |\mathbf{I}| = 1 \implies |\mathbf{A}| = \frac{1}{|\mathbf{A}^{-1}|}$$

3. Define $\mathbf{x}' \triangleq \begin{bmatrix} \mathbf{q}_1 & \dot{\mathbf{q}}_1 & \mathbf{q}_2 & \dot{\mathbf{q}}_2 \end{bmatrix}^T$. The state space representation of the system for this

state is given by

$$\begin{aligned}
\dot{\mathbf{x}}' &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{bmatrix}}_{\mathbf{A}'} \mathbf{x}' + \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{B}} \mathbf{u} \\
\mathbf{y} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{C}'} \mathbf{x}'
\end{aligned}$$

Now observe that

$$\mathbf{x} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \mathbf{x}'$$

or

$$\mathbf{x}' = \mathbf{T}^{-1}\mathbf{x}$$

Making substitutions for $\mathbf{x}'$ in the original state space model,

$$\begin{aligned}
\dot{\mathbf{x}}' &= \mathbf{A}'\mathbf{x}' + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}'\mathbf{x}' + \mathbf{D}\mathbf{u}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}^{-1}\dot{\mathbf{x}} &= \mathbf{A}'\mathbf{T}^{-1}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\dot{\mathbf{x}} &= \mathbf{T}\mathbf{A}'\mathbf{T}^{-1}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}'\mathbf{T}^{-1}\mathbf{x} + \mathbf{D}\mathbf{u}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{A} &= \mathbf{T}\mathbf{A}'\mathbf{T}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+2k_2}{2m_1} - \frac{2k_2+k_3}{2m_2} & 0 & \frac{k_1}{2m_1} - \frac{k_3}{2m_2} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+2k_2}{2m_1} + \frac{2k_2+k_3}{2m_2} & 0 & \frac{k_1}{2m_1} + \frac{k_3}{2m_2} & 0 \end{bmatrix} \\
\mathbf{C} &= \mathbf{C}'\mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}
\end{aligned}$$

$\mathbf{B}$ and $\mathbf{D} = \mathbf{0}$ remain the same.

4. (a) By inspection

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{p_0(I_x-I_z)}{I_y} \\ 0 & \frac{p_0(I_y-Ix)}{I_z} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \mathbf{I}\mathbf{x}
\end{aligned}$$

(b)

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6848 & -1.1900 \\ 0 & 0.4463 & 0.6848 \end{bmatrix}$$

(c) See Figure 1. The system is stable, but not asymptotically stable, i.e. neither wants nor does not want to stay near linearization point.

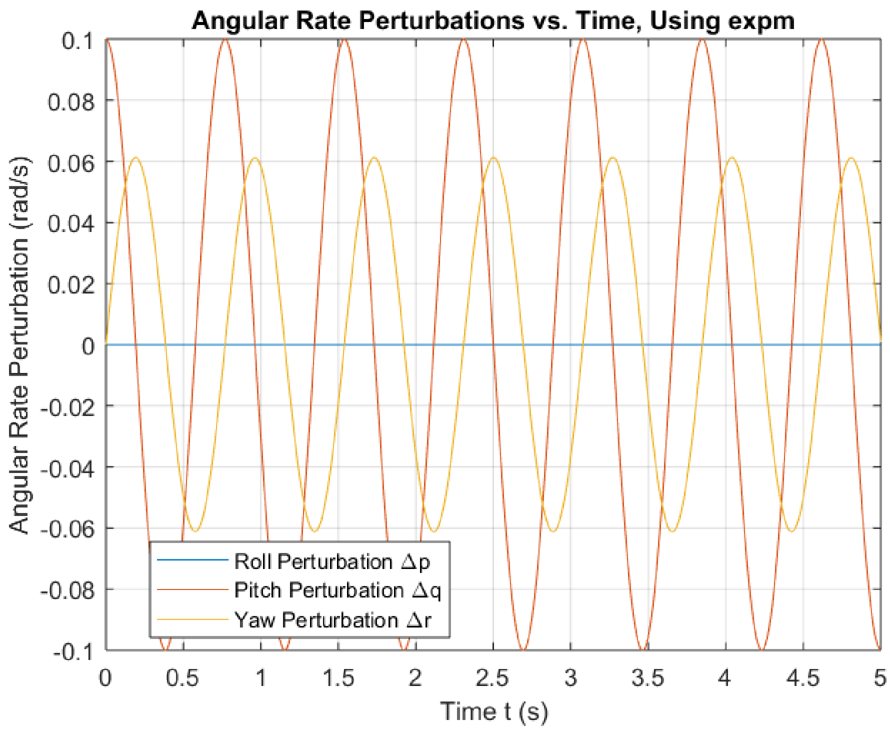

Figure 1: Attitude Deviation vs. Time

5. (a) The characteristic equation of $\mathbf{A}$ is given by

$$\begin{aligned} P(\lambda) &= \det(\lambda \mathbf{I} - \mathbf{A}) \\ &= \det \begin{bmatrix} \lambda - a & -b \\ -b & \lambda - c \end{bmatrix} \\ &= (\lambda - a)(\lambda - c) - b^2 \\ P(\lambda) &= \lambda^2 - (a+c)\lambda + ac - b^2 \end{aligned}$$

Substituting $\mathbf{A}$ for λ,

$$\begin{aligned} P(\mathbf{A}) &= \mathbf{A}^2 - (a+c)\mathbf{A} + (ac - b^2)\mathbf{I} \\ &= \begin{bmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{bmatrix} - (a+c) \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} ac - b^2 & 0 \\ 0 & ac - b^2 \end{bmatrix} = \mathbf{0} \end{aligned}$$

(b) i. From 5(a),

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

The eigenvalues are given by

$$\begin{aligned} \lambda &= \frac{a+c \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} \\ &= \frac{a+c \pm \sqrt{(a-c)^2 - 4b^2}}{2} \\ (a-c)^2 + 4b^2 &\geq 0 \implies \lambda \in \mathbb{R} \end{aligned}$$

ii. Using Sylvester's Criterion,

$$\mathbf{A} \geq \mathbf{0} \Leftrightarrow c \geq 0 \text{ and } \det \mathbf{A} \geq 0$$

Therefore, $\mathbf{A}$ is positive semi-definite if

$$\det \mathbf{A} = ac - b^2 \geq 0 \implies b^2 \geq ac$$

AQ1. 42 is an irrational answer to the Question of Life, the Universe, and Everything, just as all the other numbers in that column are irrational. Equally irrational is the fact that the Question was "What is 7×9?" But what is one to expect? This work was all done by mice.

AQ2.

$$\mathbf{\Phi} = \exp(\mathbf{A}\Delta t) = \sum_{i=0}^{\infty} \mathbf{A}^i \frac{\Delta t^i}{i!}$$

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & b & 0 \end{bmatrix}$$

The powers of $\mathbf{A}$ are then given by

$$\mathbf{A}^i = \begin{cases} \mathbf{I}, & i = 0 \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a^{\frac{i-1}{2}+1} b^{\frac{i-1}{2}} \\ 0 & a^{\frac{i-1}{2}} b^{\frac{i-1}{2}+1} & 0 \end{bmatrix}, & i \text{ odd} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a^{\frac{i}{2}} b^{\frac{i}{2}} & 0 \\ 0 & 0 & a^{\frac{i}{2}} b^{\frac{i}{2}} \end{bmatrix}, & i \text{ even} \end{cases}$$

Define

$$j \triangleq \frac{i-1}{2}, k \triangleq \frac{i}{2} \implies i = 2j+1 = 2k$$

The nonzero terms of $\mathbf{\Phi}$ are then given by

$$\begin{aligned}
\Phi_{23} &= \sum_{j=0}^{\infty} a^{j+1} b^{j} \frac{\Delta t^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} \frac{\sqrt{ab}\Delta t^{2j+1}}{(2j+1)!} \sqrt{\frac{a}{b}} \\
\Phi_{32} &= \sum_{j=0}^{\infty} a^{j} b^{j+1} \frac{\Delta t^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} \frac{\sqrt{ab}\Delta t^{2j+1}}{(2j+1)!} \sqrt{\frac{b}{a}} \\
\Phi_{22} &= \Phi_{33} = \sum_{k=0}^{\infty} a^{k} b^{k} \frac{\Delta t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{\sqrt{ab}\Delta t^{2k}}{(2k)!}
\end{aligned}$$

Observe that $I_x < I_y < I_z$ so that

$$\begin{aligned}
a &= \frac{p_0 \left(I_x - I_z\right)}{I_y} < 0 \\
b &= \frac{p_0 \left(I_y - I_x\right)}{I_z} < 0
\end{aligned}$$

Define $c \triangleq -a$. Then

$$\begin{aligned}
\Phi_{23} &= \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\sqrt{cb}\Delta t^{2j+1}}{(2j+1)!} \sqrt{\frac{c}{b}} = -\sqrt{\frac{c}{b}} \sin \sqrt{cb} \Delta t \\
&= -\sqrt{\frac{I_z \left(I_z - I_x\right)}{I_y \left(I_y - I_x\right)}} \sin \left(p_0 \sqrt{\frac{\left(I_z - I_x\right)\left(I_y - I_x\right)}{I_y I_z}} \Delta t \right) \\
\Phi_{32} &= \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\sqrt{cb}\Delta t^{2j+1}}{(2j+1)!} \sqrt{\frac{c}{b}} = \sqrt{\frac{c}{b}} \sin \sqrt{cb} \Delta t \\
&= \sqrt{\frac{I_y \left(I_y - I_x\right)}{I_z \left(I_z - I_x\right)}} \sin \left(p_0 \sqrt{\frac{\left(I_z - I_x\right)\left(I_y - I_x\right)}{I_y I_z}} \Delta t \right) \\
\Phi_{22} &= \sum_{k=0}^{\infty} (-1)^{k} \frac{\sqrt{cb}\Delta t^{2k}}{(2k)!} = \cos \sqrt{cb} \Delta t \\
&= \cos \left(p_0 \sqrt{\frac{\left(I_z - I_x\right)\left(I_y - I_x\right)}{I_y I_z}} \Delta t \right)
\end{aligned}$$

A MATLAB Script

```
%% ASEN5044 Assignment 1 Solutions
% Y. Shen
% 31 August 2018

%% Problem 4

% Part B
Ix = 500;                % kg m^3
Iy = 750;                % kg m^3
Iz = 1000;               % kg m^3

p0 = 20;                 % rad/s
dt = 0.1;

A = [0 0 0; 0 0 p0*(Ix - Iz)/Iy; 0 p0*(Iy - Ix)/Iz 0];

Phi = expm(A*dt);

% Part C
t = 0:0.01:5;               % Times (s)
x0 = [0; 0.1; 0];            % Initial conditions (rad/s)

x = zeros(3, length(t));     % States (rad/s)

for i = 1:length(t)
    x(:, i) = expm(A*t(i))*x0;
end

figure;
plot(t, x);
xlabel('Time t (s)');
ylabel('Angular Rate Perturbation (rad/s)');
title('Angular Rate Perturbations vs. Time, Using expm');
legend('Roll Perturbation \Deltap', 'Pitch Perturbation \Deltaq', 'Yaw Perturbation \Delta
grid on;

%% Advanced Question 2

c = p0*(Iz - Ix)/Iy;
b = p0*(Iy - Ix)/Iz;

Phi = @(t) [0 0 0;
    0 cos(sqrt(c*b)*t) -sqrt(c/b)*sin(sqrt(c*b)*t);
    0 sqrt(b/c)*sin(sqrt(c*b)*t) cos(sqrt(c*b)*t)];

for i = 1:length(t)
    x(:, i) = Phi(t(i))*x0;
end

figure;
```

```
plot(t, x);
xlabel('Time t (s)');
ylabel('Angular Rate Perturbation (rad/s)');
title('Angular Rate Perturbations vs. Time, Using Analytical Solution');
legend('Roll Perturbation \Deltap', 'Pitch Perturbation \Deltaq', 'Yaw Perturbation \Delta
grid on;
```