Statistical Estimation	**Midterm 1**
ASEN 5044 Fall 2018	**Due Date: Oct 11, 2018**
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Problem 1

Inverted pendulum with equations of motion:

$$(M+m)\ddot{z} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = P$$
$$l\ddot{\theta} - g\sin\theta = \ddot{z}\cos\theta$$

Part (a)

The system's state equations can be expressed as follows:

$$\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{P - g\sin x_3\cos x_3 - mlx_4^2\sin x_3}{M+m\sin^2 x_3}\cos x_3 \\ x_4 \\ \frac{P\cos x_3 + (M+m)g\sin x_3 + mlx_4^2\sin x_3\cos x_3}{Ml + ml\sin^2 x_3}\end{bmatrix}$$

To demonstrate the system is in equilibrium at $\dot{z}=0,\ \theta=0,\ \dot{\theta}=0$, and $P(t)=0$ we note first that at the given conditions the equations of motion become

$$(M+m)\ddot{z} - ml\ddot{\theta} = 0$$
$$l\ddot{\theta} = \ddot{z}$$

If we plug the second equation back into the first we get

$$(M+m)\ddot{z} - m\ddot{z} = M\ddot{z} = 0$$

Because we know M is not equal to zero, this means $\ddot{z}$ must be equal to zero. Additionally, because $\ddot{z} = l\ddot{\theta}$ and $l \neq 0$ we can also conclude that $\ddot{\theta} = 0$. This means $\dot{x} = 0$ under the given conditions and the system is therefore in equilibrium.

Part (b)

Part (c)

Part (d)

Part (e)

Part (f)

Part (g)

Problem 2

Two 6-sided dice rolls with R_1 and R_2 denoting the outcome of the first and second die, respectively.

Part (a)

$P(R_1) = P(R_2) = \frac{1}{6}$ for all R_1 and R_2. Because the outcomes R_1 and R_2 are independent, $P(R_1, R_2) = P(R_1) * P(R_2)$. So

$$P(R_1, R_2) = \frac{1}{36},\ \forall R_1, R_2$$

Part (b)

The joint probabilities for X and Y are shown in table 1 below

Table 1: Joint Probabilities

X	Y=1	Y=2	Y=3	Y=4	Y=5	Y=6
1	1/36	2/36	2/36	2/36	2/36	2/36
2	0	1/36	2/36	2/36	2/36	2/36
3	0	0	1/36	2/36	2/36	2/36
4	0	0	0	1/36	2/36	2/36
5	0	0	0	0	1/36	2/36
6	0	0	0	0	0	1/36

Part (c)

The marginal probabilities of X obtained from the sum $\sum_y P(X = x, Y = y)$ and the marginal probabilities of Y obtained from the sum $\sum_x P(X = x, Y = y)$ are shown below in table 2.

Table 2: Marginal Probabilities

	X	Y
1	11/36	1/36
2	9/36	3/36
3	7/36	5/36
4	5/36	7/36
5	3/36	9/36
6	1/36	11/36

Part (d)

X and Y are not independent. Two variables are considered independent if the realization of one variable does not affect the probability of the other. This is not the case for X and Y. By the definitions of X and Y, the value of Y cannot be less than the value of X, since the maximum of R_1 and R_2 cannot be less than the minimum. So the $P(Y = 3, X = 5) = 0$, while $P(Y = 3, X = 1) = 2/36$.

Problem 3

A random variable X has the pdf $p(x) = k(1 - x^4)$ for $-1 \leq x \leq 1$ and $p(x) = 0$ elsewhere.

Part (a)

Because $\int_{-\infty}^{\infty} p(x)dx = 1$ we can find k as follows:

$$\begin{aligned}
\int_{-1}^{1} k(1-x^4)dx &= 1 \\
k\int_{-1}^{1} (1-x^4)dx &= \\
k\big[x - \frac{1}{5}x^5\big|_{-1}^{1} &= \\
k\Big(1 - \frac{1}{5} + 1 - \frac{1}{5}\Big) &= \\
\frac{8k}{5} &= 1 \\
k = \frac{5}{8}
\end{aligned}$$

Now we can calculate $E[x] = \int_{-\infty}^{\infty} xp(x)dx$ as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} xp(x)dx &= \frac{5}{8}\int_{-1}^{1} x(1-x^4)dx \\
&= \frac{5}{8}\int_{-1}^{1} (x-x^5)dx \\
&= \frac{5}{8}\left[\frac{1}{2}x^2 - \frac{1}{6}x^6\right|_{-1}^{1} \\
&= \frac{5}{8}\left(\frac{1}{2} - \frac{1}{6} - \frac{1}{2} + \frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Next we find $E[x^2] = \int_{-\infty}^{\infty} x^2p(x)dx$ as

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2p(x)dx &= \frac{5}{8}\int_{-1}^{1} x^2(1-x^4)dx \\
&= \frac{5}{8}\int_{-1}^{1} (x^2-x^6)dx \\
&= \frac{5}{8}\left[\frac{1}{3}x^3 - \frac{1}{7}x^7\right|_{-1}^{1} \\
&= \frac{5}{8}\left(\frac{1}{3} - \frac{1}{7} + \frac{1}{3} - \frac{1}{7}\right) \\
&= \frac{5}{21}
\end{aligned}$$

Finally we can find $\mathrm{var}(x) = E[x^2] - (E[x])^2 = \frac{5}{21} - 0 = \frac{5}{21}$.

Part (b)

The cumulative distribution function is defined as $P_X(\zeta) = \int_{-\infty}^{\zeta} p(x)dx$. So the cdf is:

$$\begin{aligned}
P_X(\zeta) &= \int_{-\infty}^{\zeta} p(x)dx \\
&= \frac{5}{8}\int_{-1}^{\zeta}(1-x^4)dx \\
&= \frac{5}{8}\Big[x - \frac{1}{5}x^5\Big|_{-1}^{\zeta} \\
&= \frac{5}{8}\Big(\zeta - \frac{1}{5}\zeta^5 + \frac{4}{5}\Big)
\end{aligned}$$

Part (c)

Because the pdf is symmetric about zero, $P(|X| < 0.5)$ is equivalent to $P(-0.5 < X < 0.5)$, which can be found as

$$P(-0.5 < X < 0.5) = P_X(0.5) - P_X(-0.5) = 0.7895$$

Problem 4

Blood alcohol tests on drivers given the conditional probabilities given in table 3:

Table 3: Conditional Probabilities

$P(T\|A)$	$A =$ drunk	$A =$ sober
$T =$ positive	0.99	0.001
$T =$ negative	0.01	0.999

Part (a)

We can find $P(A = \text{drunk}|T = \text{positive})$ through a straightforward application of Bayes' rule:

$$P(A = \text{drunk}|T = \text{positive}) = \frac{P(T = \text{positive}|A = \text{drunk}) * P(A = \text{drunk})}{P(T = \text{positive})}$$

Since we aren't given a number for $P(T = \text{positive})$ we can find it as $P(T = \text{positive}) = P(T = \text{positive}|A = \text{drunk})P(A = \text{drunk}) + P(T = \text{positive}|A = \text{sober})P(A = \text{sober}) = 0.99 + 0.001 = 0.991$. So the conditional probability is:

$$P(A = \text{drunk}|T = \text{positive}) = \frac{0.99 * 0.2}{0.991} = 0.1998$$

Part (b)

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Part (a)

Part (b)

Part (c)

Part (d)