Statistical Estimation	**Homework 1**
ASEN 5044 Fall 2018	**Due Date: Sep 6, 2018**
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Exercise 1

Compute determinants for the following matrices by hand and state whether each one is invertible

Problem (a)

$$\begin{aligned}|A| &= 1\begin{vmatrix}5 & 4\\ 9 & 7\end{vmatrix} - 2\begin{vmatrix}6 & 4\\ 8 & 7\end{vmatrix} + 3\begin{vmatrix}6 & 5\\ 8 & 9\end{vmatrix}\\ &= 1(-1) - 2(10) + 3(14)\\ &= 21\end{aligned}$$

Because $|A|$ is nonzero A is invertible.

Problem (b)

$$\begin{aligned}|A| &= 11\begin{vmatrix}57 & 0 & 10\\ 91 & 1 & 71\\ 23 & 0 & 71\end{vmatrix} - 26\begin{vmatrix}64 & 0 & 10\\ 83 & 1 & 71\\ 54 & 0 & 71\end{vmatrix}\\ &= 11(57(71) + 10(-23)) - 26(64(71) + 10(-54))\\ &= 41978 - 109642\\ &= -67655\end{aligned}$$

Because $|A|$ is nonzero A is invertible.

Problem (c)

Because $A_1 = -2A_3$ (where A_i refers to the i^{th} column of A), the columns of A are not linearly independent. This means A is not invertible and $|A| = 0$.

Problem (d)

Because the determinant of an upper triangular matrix is simply the product of its diagonal elements:

$$\begin{aligned}|A| &= 1 \times 8 \times 55 \times 233 \times 610\\ &= 62537200\end{aligned}$$

Because $|A|$ is nonzero A is invertible.

Exercise 2

Prove each of the following statements:

Problem (a)

If a and b are non-zero $n \times 1$ vectors, then the matrix ab^T has rank 1.

Column i of the outer product of a and b is simply the vector a multiplied by the scalar b_i. This means that every column of ab^T is a scalar multiple of a, so none of the columns of ab^T are linearly independent. Thus, the rank of ab^T is always one if both a and b are nonzero.

Problem (b)

$\text{tr}(AB) = \text{tr}(BA)$ if A is an $m \times n$ matrix and B is $n \times m$.

The trace of AB can be expressed as

$$\begin{aligned} \text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \end{aligned}$$

Similarly, the trace of BA is

$$\begin{aligned} \text{tr}(BA) &= \sum_{j=1}^{n} (BA)_{jj} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \end{aligned}$$

Because $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$ is equal to $\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$ we can conclude that $\text{tr}(AB) = \text{tr}(BA)$.

Problem (c)

If A is invertible then $|A^{-1}| = \frac{1}{|A|}$.

If we start with $|AB| = |BA| = |A|\ |B|$ and replace B with A^{-1} we find that

$$\begin{aligned} |A|\ |A^{-1}| &= |AA^{-1}| \\ &= |I| \\ &= 1 \end{aligned}$$

Because $|A|\ |A^{-1}| = 1$ it must be true that $|A^{-1}| = \frac{1}{|A|}$.

Exercise 3

Consider the equations of motion for the coupled 2 mass 3 spring system like the one discussed in lecture. Find a set of A, B, C, D matrices for the state vector definition,

$$x = [q_1 - q_2,\ \dot{q}_1 - \dot{q}_2,\ q_1 + q_2,\ \dot{q}_1 + \dot{q}_2]^T$$

and for observations $y = [q_1,\ q_2]^T$ and inputs $u = [u_1,\ u_2]^T$.

The following solution makes the assumption, as was done in lecture, that $m_1 = m_2 = 1$kg and $k_1 = k_2 = k_3 = 1$N/m.

$$\begin{aligned}
\ddot{q}_1 &= -q_1 - q_1 + q_2 - u_1 \\
&= -2q_1 + q_2 - u_1 \\
\ddot{q}_2 &= -q_2 + q_1 - q_2 + u_1 + u_2 \\
&= -2q_2 + q_1 + u_1 + u_2 \\
\ddot{q}_1 - \ddot{q}_2 &= (-2q_1 + q_2 - u_1) - (q_1 - 2q_2 + u_1 + u_2) \\
&= -3q_1 + 3q_2 - 2u_1 - u_2 \\
&= -3(q_1 - q_2) - 2u_1 - u_2 \\
\ddot{q}_1 + \ddot{q}_2 &= (-2q_1 + q_2 - u_1) + (q_1 - 2q_2 + u_1 + u_2) \\
&= -q_1 - q_2 + u_2 \\
&= -(q_1 + q_2) + u_2
\end{aligned}$$

From these results we can construct our A, B, C, and D matrices as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -2 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise 4

The linearized equations of motion for an orbiting satellite spinning with nominal angular rate p_o about the x axis are

$$\begin{aligned}
\Delta\dot{p} &= \frac{M_x}{I_x} \\
\Delta\dot{q} &= \frac{p_0(I_x - I_z)\Delta r + M_y}{I_y} \\
\Delta\dot{r} &= \frac{p_0(I_y - I_x)\Delta q + M_z}{I_z}
\end{aligned}$$

where I_x, I_y, and I_z are the moments of inertia about the roll, pitch, and yaw axes; M_x, M_y, and M_z are the corresponding input torques; and $\Delta p, \Delta q$, and Δr are perturbations in rolling, pitching, and yawing rates from the linearization point.

Problem (a)

Using the state vector $x = [\Delta p, \Delta q, \Delta r]^T$, input vector $u = [M_x, M_y, M_z]^T$, and output vector $y = x$, put this system into state space form.

The state space form of the given system is as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{p_o(I_x - I_z)}{I_y} \\ 0 & \frac{p_o(I_y - I_x)}{I_z} & 0 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix} \quad C = \mathbb{I}_3 \quad D = 0_3$$

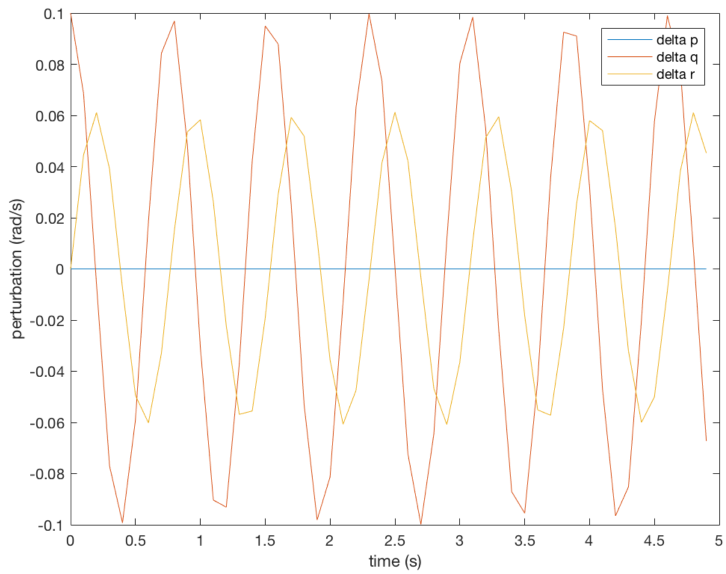

Figure 1: State variables plotted over time (exercise 4)

Problem (b)

Use Matlab's `expm` function to compute the state transition matrix for this system assuming that $I_y = 750\text{kg m}^3$, $I_z = 1000\text{kg m}^3$, $I_x = 500\text{kg m}^3$, $p_0 = 20\text{rad/s}$, and $\Delta t = 0.1\text{s}$.

$$e^{A\Delta t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.685 & -1.19 \\ 0 & 0.446 & 0.685 \end{bmatrix}$$

Problem (c)

Use the state transition matrix to compute and plot the state time history for 5s, assuming zero inputs and assuming initial states $\Delta q(0) = 0.1\text{rad/s}$, and $\Delta p(0) = \Delta r = 0$. What can you say about the behavior of this system in terms of stability?

As figure 1 above shows, the system is asymptotically stable in Δp and marginally stable in Δq and Δr. Δq is bounded between -0.1 and 0.1 and Δr is bounded between -0.06 and 0.06.

Exercise 5

Problem (a)

Consider the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Recall that the eigenvalues af A are found by finding the roots of the polynomial $P(\lambda) = |\lambda I - A|$. Show that $P(A) = 0$.

The characteristic polynomial $p(\lambda)$ of A is $|\lambda I - A| = \lambda^2 + \lambda(-a-c) + ac - b^2$. So

$$\begin{aligned} p(A) &= A^2 + A(-a-c) + I(ac-b^2) \\ &= \begin{bmatrix} a^2+b^2 & ab+bc \\ ab+bc & b^2+c^2 \end{bmatrix} + \begin{bmatrix} -a^2-ac & -ab-bc \\ -ab-bc & -ac-c^2 \end{bmatrix} + \begin{bmatrix} ac-b^2 & 0 \\ 0 & ac-b^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Problem (b)

Consider the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where a, b, and c are real and a and c are nonnegative. Compute the solutions of the characteristic polynomial of A to prove that the eigenvalues of A are real. Also, for what values of b is A positive semidefinite?

The characteristic polynomial $p(\lambda)$ of A is $|\lambda I - A| = \lambda^2 + \lambda(-a-c) + ac - b^2$. The roots of this polynomial are given by the quadratic formula:

$$\begin{aligned} \lambda &= \frac{-(-a-c) \pm \sqrt{(-a-c)^2 - 4(ac-b^2)}}{2} \\ &= \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2} \end{aligned}$$

Notice that the polynomial's discriminant (the expression under the radical) is the sum of the squares of real numbers. This means the discriminant is always positive and hence the roots of the characteristic polynomial (which are also the eigenvalues of A) are always real.
A is positive semidefinite if $z^T A z \geq 0$ for every $z \in \mathbb{R}^2$.

$$z^T A z = z_1^2 a + 2 z_1 z_2 b + z_2^2 c$$

Since a and c are nonnegative $z_1^2 a \geq 0$ and $z_2^2 c \geq 0$. This means for A to be positive semidefinite we only need to guarantee that $z_1 z_2 b \geq 0$. This is true if z_1 and z_2 have the same sign and $b \geq 0$ or if z_1 and z_2 have opposite signs and $b \leq 0$. So the only value of b that guarantees that $z^T A z \geq 0$ for all $z \in \mathbb{R}^2$ is 0. There are other values of b that result in a positive semidefinite A matrix. A is positive semidefinite if its eigenvalues are nonnegative. We can find the values of b which result in nonnegative roots to the characteristic polynomial:

$$(a+c) \pm \sqrt{(a-c)^2 + 4b^2} \geq 0$$

We first find the values of b for which the above inequality is true when the square root is subtracted:

$$\begin{aligned} -\sqrt{(a-c)^2 + 4b^2} &\geq -(a+c) \\ \sqrt{(a-c)^2 + 4b^2} &\leq (a+c) \\ (a-c)^2 + 4b^2 &\leq (a+c)^2 \\ 4b^2 &\leq (a+c)^2 - (a-c)^2 \\ 4b^2 &\leq a^2 + 2ac + c^2 - a^2 + 2ac - c^2 \\ 4b^2 &\leq 4ac \\ b &\leq \sqrt{ac} \end{aligned}$$

Then we find the values of b for which the inequality holds when the square root is added:

$$\begin{aligned}
\sqrt{(a-c)^2+4b^2} &\geq -(a+c) \\
(a-c)^2+4b^2 &\geq (-a-c)^2 \\
4b^2 &\geq (-a-c)^2-(a-c)^2 \\
4b^2 &\geq a^2+2ac+c^2-a^2+2ac-c^2 \\
b &\geq \sqrt{ac}
\end{aligned}$$

So A is positive semidefinite when $b \geq \sqrt{ac}$, $b \leq \sqrt{ac}$, and $b = 0$.

Exercise AQ1

Explain why matrix element $(4,4)$ in problem 1c belongs in the last column.

The number 42 is the answer to life, the universe, and everything. Because a single, numerical answer to life, the universe, and everything is an irrational concept it belongs in the column composed of irrational numbers.

Exercise AQ2

Using the infinite series definition of the matrix exponential, find the analytical expression for the state transition matrix of the spinning satellite system in problem 4, assuming a generic value for Δt and using the values for I_x, I_y, I_z, and p_0 given in that problem. Your answer should be a function of Δt.

When we expand the first few terms of the infinite series for $e^{A\Delta t}$ we get

$$\begin{aligned}
\sum_{i=0}^{4}\frac{(A\Delta t)^i}{i!} &= I + A\Delta t + \frac{(A\Delta t)^2}{2!} + \frac{A\Delta t)^3}{3!} + \frac{(A\Delta t)^4}{4!} \\
&= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -13.333\Delta t\\ 0 & 5\Delta t & 0\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}0 & 0 & 0\\ 0 & 66.667\Delta t^2 & 0\\ 0 & 0 & 66.667\Delta t^2\end{bmatrix} \\
&\quad + \frac{1}{3!}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 888.9\Delta t^3\\ 0 & -333.3\Delta t^3 & 0\end{bmatrix} + \frac{1}{4!}\begin{bmatrix}0 & 0 & 0\\ 0 & 4444\Delta t^4 & 0\\ 0 & 0 & 4444\Delta t^4\end{bmatrix}
\end{aligned}$$

We can see the nonzero terms in the sum above, which I'll refer to as ϕ, are

$$\begin{aligned}
\phi_{1,1} &= 1 \\
\phi_{2,2} = \phi_{3,3} &= 1 - \frac{66.666}{2!}\Delta t^2 + \frac{4444}{4!}\Delta t^4 \\
&= 1 - \frac{8.17^2}{2!}\Delta t^2 + \frac{8.17^4}{4!}\Delta t^4 \\
\phi_{2,3} &= -13.333\Delta t + \frac{888.9}{3!}\Delta t^3 \\
&= -1.63\Big[8.17\Delta t + \frac{8.17^3}{3!}\Delta t^3\Big] \\
\phi_{3,2} &= 5\Delta t - \frac{333.3}{3!}\Delta t^3 \\
&= 0.612\Big[8.17\Delta t + \frac{8.17^3}{3!}\Delta t^3\Big]
\end{aligned}$$

Noting that the Taylor series expansion of $a\cos(bx) = a[1 - \frac{(bx)^2}{2!} + \frac{(bx)^4}{4!} - \dots]$ and the expansion of $a\sin(bx) = a[bx - \frac{(bx)^3}{3!} + ...]$ we can deduce that $e^{A\Delta t}$ is equal to

$$e^{A\Delta t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(8.17\Delta t) & -1.63\sin(8.17\Delta t) \\ 0 & 0.612\sin(8.17\Delta t) & \cos(8.17\Delta t) \end{bmatrix}$$