Statistical Estimation	**Homework 2**
ASEN 5044 Fall 2018	**Due Date: Sep 20, 2018**
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Exercise 1

Consider the 2-mass/3-spring system presented in lecture 4, where the continuous time state definition and inputs are the same, but the observed sensor inputs are now given by

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(t)$$

Problem (a)

Find the discrete time LTI representation for this system using a step size of $\Delta T = 0.05$ sec. How does this sampling rate compare with the system's Nyquist limit?

Recalling lecture 4, the state of the system is given by $x(t) = [q_1(t),\ \dot{q}_1(t),\ q_2(t),\ \dot{q}_2(t)]^T$ where q_i is the displacement of mass i and $u(t) = [u_1(t),\ u_2(t)]^T$ where u_1 is the force between the two masses and u_2 is the force on mass 2. The CT LTI system model is given by $\dot{x} = Ax(t) + bu(t)$ where

$$A = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ -2.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & -2.0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

This is relatively simple to convert to a discrete time system. We only need to find $e^{\hat{A}\Delta t}$ where

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The discretized system is described by $x(k+1) = Fx(k) + Gu(k)$ where F is the upper left 4×4 block of $e^{\hat{A}\Delta t}$ and G is the upper right 4×2 block:

$$F = \begin{bmatrix} 1.0 & 0.05 & 0.001 & 0.0 \\ -0.1 & 1.0 & 0.05 & 0.001 \\ 0.001 & 0.0 & 1.0 & 0.05 \\ 0.05 & 0.001 & -0.1 & 1.0 \end{bmatrix}$$

$$G = \begin{bmatrix} -0.001 & 0.0 \\ -0.05 & 0.0 \\ 0.001 & 0.001 \\ 0.05 & 0.05 \end{bmatrix}$$

To satisfy the Nyquist sampling criterion for this system the following inequality must hold $\frac{\pi}{\Delta t} > 2|\lambda_{\max}|$ where $|\lambda_{\max}|$ is the largest complex magnitude among all the eigenvalues of A. In this case that magnitude is 1.732.

$$\frac{\pi}{\Delta t} = 62.83$$
$$2|\lambda_{\max}| = 3.464$$

Clearly in this case the sampling frequency is well within the Nyquist limit for this system.

Problem (b)

Show that the DT system is observable.

We'll start by building candidates for our observability matrix $\mathbb{O}$ and checking if they are full rank. The simplest possible case, $\mathbb{O} = H$ is rank deficient, because column 4 of H is simply column 2 multiplied by -1.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

The next possibility, $[H,\ HF]^T$, is full rank.

$$\mathbb{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 \\ 1.0 & 0.05 & 0.001 & 0.0 \\ -0.15 & 1.0 & 0.15 & -1.0 \end{bmatrix}$$

This means the system is fully observable from two observations. This makes physical sense, as the displacement of mass 1, x_1 is directly measurable (as y_1) so we should be able to find the velocity of mass 1, x_2 with two measurements of its position. Then, if we know velocity of mass 1 and the relative velocity of the two masses, y_2 it is possible to calculate the velocity of mass 2, x_4. Lastly, knowing the velocity of mass 2 over one timestep will also allow us to calculate its displacement x_3 over that timestep.

Problem (c)

Suppose the system starts from some unknown initial condition $x(0)$ at $k = 0$ and is stimulated by an external set of ZOH inputs u at the $\Delta T = 0.05$ sec sampling rate from $t = 0$ to $t = 5$ seconds, where $u(t) = [\sin(t), 0.1\cos(t)]^T$, and the resulting output $y(k)$ at each sampling instant from $t = 0.05$ sec ($k = 1$) to $t = 5$ sec is recorded in `hw3problemdata.mat`. Derive a linear system of equations in matrix-vector form that would allow you to estimate the unknown initial condition $x(k = 0)$ using all the available logged y and u data.

We can regress $x(0)$ from the logged n measurements by stacking the measurements to form an $np \times 1$ column vector Y which will follow the relationship

$$Y = \mathbb{O}x(0)$$

where $\mathbb{O} = [HF^0,\ HF^1,\ HF^2,\ \ldots, HF^{n-1}]^T$. We can solve for $x(0)$ by multiplying both sides by the grammian of $\mathbb{O}$:

$$x(0) = (\mathbb{O}^T\mathbb{O})^{-1}\mathbb{O}^T Y$$

Problem (d)

Estimate $x(k = 0)$ and plot all the remaining states $x(k)$ for $k \geq 1$ vs time (in seconds) and separately plot their corresponding 'predicted' outputs $y(k)$ vs. time for all $k \geq 1$ in the recorded output time series. Validate your estimate by also separately plotting the differences between the 'predicted' and recorded $y(k)$ values vs. time.

In solving $(\mathbb{O}^T\mathbb{O})^{-1}\mathbb{O}^T Y$ we find that $x(0) = [0.247,\ 0.434,\ -0.607,\ -1.697]$. See figure 1 below for a plot of the predicted states vs. time and figure 2 for the predicted outputs. Figure 3 shows the difference between the predicted and actual measurements. The predicted measurements match the actual measurements exactly in frequency, though their amplitude differs.

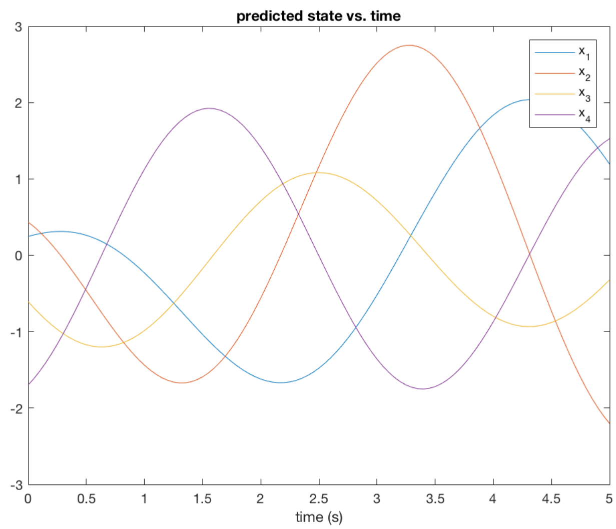

Figure 1: predicted states

Problem (e)

How many vector measurements $y(k)$ are actually needed to estimate $x(0)$, i.e. do you need to use all available measurements, or some smaller number? Is this consistent with an analysis of the observability matrix $\mathbb{O}$ and Grammian $\mathbb{O}^T\mathbb{O}$? Explain how and why the required number of vector measurements would theoretically change if the $y(k)$ data were instead given by three different position sensors for the first mass, where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem (f)

What happens to the observability of the system if only the first row of the output $y(k)$ is used for all $k \geq 1$? What if only the second row of the output vector $y(k)$ is used instead? Provide a physical explanation in each case.

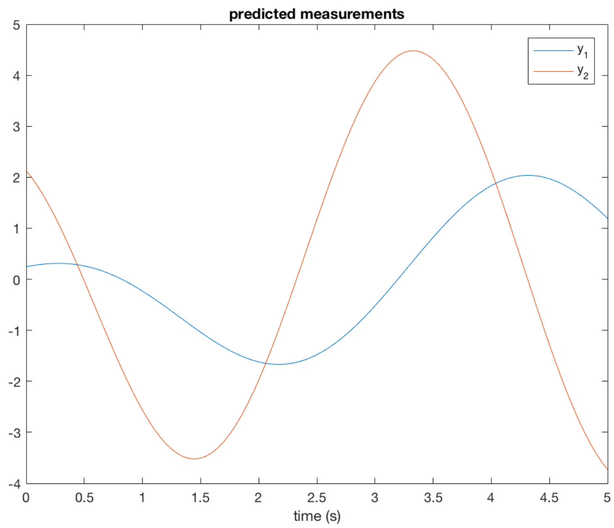

Figure 2: predicted measurements

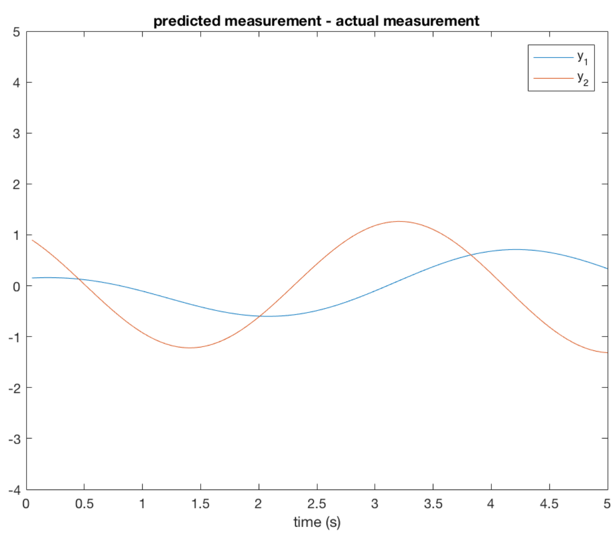

Figure 3: difference between predicted and actual measurements