Statistical Estimation	Homework 2
ASEN 5044 Fall 2018	Due Date: Sep 20, 2018
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Exercise 1

In the game of blackjack, the player is initially dealt two cards from a deck of ordinary playing cards. Without going into all the game's details, it is enough to know the best possible hand for a player to receive on the initial deal is a combination of an ace of any suit and any face card or ten. What is the probability that a player will be dealt this combination?

The number of ways a single player can be dealt a blackjack (assuming there is only one player) is $\binom{4}{1}\binom{16}{1} = 64$. So the number of events in our event space (which we'll call A) is 64. The total number of ways to deal 2 cards to that player is $\binom{52}{2} = 1326$. So, the number of events in the whole sample space is 1326. the probability of a single player being dealt a blackjack is therefore

$$\frac{N_A}{N} = \frac{\binom{4}{1}}{\binom{16}{1}} = \frac{64}{1326} = 0.048$$

Exercise 2

Discrete and random variables X and Y can each take on integer values 1, 3, and 5. The joint probability table of X and Y is given below.

Table 1: Your first table.

X	Y=1	Y=3	Y=5
1	1/18	1/18	1/18
3	1/18	1/18	1/6
5	1/18	1/6	1/3

Problem (a)

Are the random variables X and Y independent?

No, the table clearly shows that the probability of X taking on certain values changes depending on the value of Y. For instance $P(X = 3|Y = 3) \neq P(X = 3|Y = 5)$ If X and Y were independent then $P(X = x|Y = y)$ would be the same for all values of y.

Problem (b)

Find the unconditional (marginal) probability $P(Y = 5)$.

The marginal probability $P(Y = 5) = P(Y = 5|X = 1) + P(Y = 5|X = 3) + P(Y = 5|X = 5) = (1/18) + (1/6) + (1/3) = (5/9)$.

Problem (c)

What is the conditional probability $P(Y = 5|X = 3)$?

Generally $P(A|B) = \frac{P(A,B)}{P(B)}$. This means we need to calculate the marginal probability $P(X = 3) = (1/18) + (1/18) + (1/6) = (5/18)$. With this information we can find

$$P(Y = 5|X = 3) = \frac{P(Y = 5, X = 3)}{P(X = 3)} = \frac{1/6}{5/18} = \frac{3}{5}$$

Exercise 3

Determine the value of a in the function

$$f_X(x) = \begin{cases} ax(1-x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

so that $f_X(x)$ is a valid probability density function.

For $f_X(x)$ to be a valid probability density function $\int_{-\infty}^{\infty} f_X(x)dx$ must be equal to one.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_0^1 ax(1-x)dx \\ &= a\int_0^1 (x - x^2)dx \\ &= a\Big[\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 \\ &= a\Big(\frac{1}{2} - \frac{1}{3}\Big) \\ 1 &= \frac{a}{6} \\ a &= 6 \end{aligned}$$

Exercise 4

The probability density function of an exponentially distributed random variable is defined as follows

$$f_X(x) = \begin{cases} ae^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $a \geq 0$.

Problem (a)

Find the probability distribution function of an exponentially distributed random variable.

$$
\begin{aligned}
P(a \leq X \leq b) &= \int_a^b f_X(x)dx \\
&= \int_a^b ae^{-ax}dx \\
&= -e^{-ax}|_a^b
\end{aligned}
$$

assuming $0 < a \leq b$.

Problem (b)

Find the mean of an exponentially distributed random variable.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf_X(x)dx \\
&= a\int_0^{\infty} xe^{-ax}dx \\
&= a\Bigg[\Big(-\frac{x}{a} - \frac{1}{a^2}\Big)e^{-ax}\Bigg|_0^{\infty} \\
&= \Bigg[\Big(-x - \frac{1}{a}\Big)e^{-ax}\Bigg|_0^{\infty} \\
&= \Big(-\infty - \frac{1}{a}\Big)e^{-a\infty} + \frac{1}{a}e^0
\end{aligned}
$$

Because $e^{-\infty} = 0$, $E[X] = \frac{1}{a}$.

Problem (c)

Find the second moment of an exponentially distributed random variable.

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\
&= \int_0^{\infty} x^2 ae^{-ax}dx \\
&= \frac{1}{a^2}\int_0^{\infty} t^2 e^{-}tdt \\
&= \frac{1}{a^2}\Gamma(2+1) = \frac{2!}{a^2} \\
&= \frac{2}{a^2}
\end{aligned}
$$

Problem (d)

Find the variance of an exponentially distributed random variable.

$$\begin{aligned}\sigma_x^2 &= E[X^2] - (E[X])^2) \\ &= \frac{2}{a^2} - \left(\frac{1}{a}\right)^2 \\ &= \frac{1}{a^2}\end{aligned}$$

Problem (e)

What is the probability that an exponentially distributed random variable takes on a value within one standard deviation of its mean?

The standard deviation is $\sqrt{\frac{1}{a}} = \frac{1}{a}$, we simply need to evaluate the probability distribution function from 0 to $\frac{2}{a}$:

$$\begin{aligned}P\left(0 < X \leq \frac{2}{a}\right) &= 1 - e^{-ax}\Big|_0^{2/a} \\ &= (1 - e^{-a\frac{2}{a}}) - (1 - e^0) \\ &= 1 - e^{-2}\end{aligned}$$

So the probability that X falls within one standard deviation of the mean is a constant, unrelated to the value of a.

Exercise 5

Exercise 6

AQ 1

AQ 2