ASEN 5044 Statistical Estimation for Dynamical Systems
Fall 2018
Homework 3

Out: Thursday 09/20/2018 (posted on Canvas)

Due: Thursday 09/27/2018 (Canvas - **no credit for illegible submissions**)

Show all your work and explain your reasoning.

1. Consider the 2-mass/3-spring system presented in Lecture 4, where the CT state definition and inputs are the same, but the observed sensor outputs are now given by

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(t).$$

(a) Find the discrete time (DT) LTI representation for this system using a step size of $\Delta T = 0.05$ sec. How does this sampling rate compare to the system's Nyquist limit?

(b) Show that the DT system is observable.

(c) Suppose the system starts from some unknown initial condition $x(0)$ at $k = 0$ and is stimulated by an external set of ZOH inputs u at the $\Delta T = 0.05$ sec sampling rate from $t = 0$ to $t = 5$ secs, where $u(t) = [\sin(t),\ 0.1 \cdot \cos(t)]^T$, and the resulting output $y(k)$ at each sampling instant from $t = 0.05$ sec ($k = 1$) to $t = 5$ sec is recorded in the posted data log `hw3problem1data.mat` (the input sequence for $u(k)$ starting from $k = 0$ is also included). Derive a linear system of equations in matrix-vector form that would allow you to estimate the unknown initial condition $x(k = 0)$ using all the available logged y and u data.

(d) Estimate $x(k = 0)$ (report the vector value) and plot all the remaining states $x(k)$ for $k \geq 1$ vs. time (in secs) and separately plot their corresponding 'predicted' outputs $y(k)$ vs. time, for all $k \geq 1$ in the recorded output time series. Validate your estimate by also separately plotting the differences between the 'predicted' and recorded $y(k)$ values vs. time.

(e) How many vector measurements $y(k)$ are actually needed to estimate $x(0)$, i.e. do you need to use all available measurements, or some smaller number? Is this consistent with an analysis of the observability matrix $\mathbb{O}$ and Gramian $\mathbb{O}^T\mathbb{O}$? Explain how and why the required number of vector measurements would theoretically change if the $y(k)$ data were instead given by three different position sensors for the first mass, where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(f) What happens to the observability of the system if only the first row of the output $y(k)$ is used for all $k \geq 1$? What if only the second row of the output vector $y(k)$ is used instead? Provide a physical explanation for the results in each case. (**Hint:** consider how the modified outputs relate to the system's natural modal behaviors – to see the natural modes, try visualizing the state response with zero inputs for initial conditions corresponding to $x_0 = c(v_1 + v_2)$ to excite the first mode and $x_0 = d(v_3 + v_4)$ to excite the second mode, where $v_{1,2}$ and $v_{3,4}$ are the complex conjugate pairs of eigenvectors corresponding to the system's eigenvalues, and c and d are any non-zero scalar constants. Note that a diagonalizable system's state response to any initial condition is a superposition of its modal state responses, since any initial condition can be represented by a linear combination of eigenvectors. What does this mean for observability in the case of the modified outputs? A certain basis transformation can help show the effect.)

2. (Luenberger, 1979) discusses a simple model for the national income dynamics. The national income y_k in year k in terms of consumer expenditure c_k, private investment i_k and government expenditure g_k is assumed to be given by $y_k = c_k + i_k + g_k$, where the interrelations between these quantities are specified by $c_{k+1} = \alpha y_k$ and $i_{k+1} = \beta(c_{k+1} - c_k)$. The constant α is called the *marginal propensity to consume*, while β is a growth coefficient. Typically, $0 < \alpha < 1$ and $\beta > 0$.

(a) Show that these relations can be re-arranged into the following discrete time state space model with (F, G, H, M) matrix parameters:

$$x_{k+1} = \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} \alpha & \alpha \\ \beta(\alpha - 1) & \beta\alpha \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta\alpha \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + u_k$$

where $x_{1,k} \equiv c_k$, $x_{2,k} \equiv i_k$ and $u_k \equiv g_k$.

(b) Let the parameters (α, β) take on the following pairs of values: (0.75,1), (0.75,1.5), and (1.25,1). For each case, determine the eigenvalues of the F matrix, and also plot the states for $0 \leq k \leq 30$ when u_k is a unit step input (i.e. $u_k = 1$ for all $k > 0$) and $x_0 = [0, 0]^T$. Comment on the stability and observability of the system in each case.

(c) For each of the parameter cases in (b), plot the states for $k \geq 0$ when $u_k = 0$ for all time k and $x_0 = [5, 1]^T$. Comment on your results.

(d) For each of the parameter cases in (c), simulate a sequence of observations $y(0), ..., y(9)$ and use all the data to estimate the initial condition x_0 as if it were unknown (assume u_k is the same as in case c). Show a plot of your observation sequence vs. time, and report your resulting estimates in each case (be sure to explain the approach used to get the estimates, and validate your estimates by plotting the resulting predicted $y(k)$ outputs vs. time against the 'real' $y(k)$ data you generated in each case).

3. You are waiting around Gary Slick's dealership while your new Ferrari is being serviced, when you notice local folk hero Pladimir Vutin walk in and start haggling with Slick over a vintage GAZ-13 Chaika that is prominently on display. You can't help but overhear their intense discussion, and record the following discrete time series for Slick's offers (z_1, in tens of thousands of dollars) and Vutin's offers (z_2, in tens of thousands of dollars), stacked into the vector $z(k) = [z_1(k), z_2(k)]^T$,

$$z(0) = \begin{bmatrix} 100.0000 \\ 20.0000 \end{bmatrix}, z(1) = \begin{bmatrix} 43.6658 \\ 39.2815 \end{bmatrix}, z(2) = \begin{bmatrix} 40.5785 \\ 40.3382 \end{bmatrix},$$
$$z(3) = \begin{bmatrix} 40.4093 \\ 40.3961 \end{bmatrix}, z(4) = \begin{bmatrix} 40.4000 \\ 40.3993 \end{bmatrix}, z(5) = \begin{bmatrix} 40.3995 \\ 40.3995 \end{bmatrix}.$$

Having conducted your own negotiation with Slick recently, you assume the dynamics of $z(k)$ could be reasonably modeled as

$$\begin{aligned} z_1(k+1) &= z_1(k) + \lambda[z_1(k) - z_2(k)] \\ z_2(k+1) &= z_2(k) + \mu[z_1(k) - z_2(k)], \end{aligned}$$

where the constants λ and μ describe Slick's and Vutin's update parameters, respectively. You don't know what values of λ and μ apply for this particular negotiation – but, you'd sure like to figure out how Vutin galloped away with Slick's shirt on this deal...

(a) Suppose the 'unknown state' to be estimated in this case is $x(k) = [\lambda, \mu]^T$, and (based on what you recorded) the 'observed output' for $k = 0, 1, 2, ...$ is taken to be $y(k+1) = [g(k+1), p(k+1)]^T$, where $g(k+1) = z_1(k+1) - z_1(k)$ and $p(k+1) = z_2(k+1) - z_2(k)$. Find a suitable DT linear state space system model for this problem, and discuss any interesting features of the (F, G, H, M) matrix parameters for the model (e.g. which are time invariant or time varying?).

(b) Under what conditions can the λ and μ parameters be identified? Give a careful precise mathematical justification and interpretation.

(c) If the conditions hold for identifying λ and μ, set up a linear system of equations that could be solved to determine the parameters. What is your estimate, based on the observed sequence of bids recorded above? If the required conditions do not hold, explain why they do not hold in this case.

Advanced Questions ***PhD students in the class MUST answer ALL questions below in addition to regular homework questions above*** – *non-PhD students are welcome to try any of these for extra credit (only given if all regular problems turned in on time as well). In either case, Submit your responses for these questions with rest of your homework, but make sure these are clearly labeled and start on separate pages* – ***indicate on the top of the front page of your assignment if you answered these questions (as a PhD student, or for extra credit) so they can be spotted, graded and recorded more easily***.

AQ1. A system is said to be *fully state reachable* if it is always possible to find a sequence of inputs $u(k), k = 0, 1, 2, ...$ that takes the system from an initial state x_s at time $k = 0$ to any other arbitrary state x_f in a finite amount of time. LTI DT system reachability can be assessed by examining the rank of the reachability (or controllability) matrix $\mathbb{C} \in \mathbb{R}^{n \times nm}$, which is defined analogously to the observability matrix $\mathbb{O}$ as

$$\mathbb{C} = \begin{bmatrix} G, & FG, & F^2G, & \cdots, & F^{n-1}G \end{bmatrix}.$$

(a) Show that the 2-mass/3-spring system from Problem 1 is fully state reachable.

(b) What happens to the system's reachability when either one of the actuator inputs u_1 or u_2 is removed, but the other control input remains? As with Problem 1(e), use insights about the system modes to help explain what's going on in each case.

(c) Derive a linear system of equations in matrix-vector form that would theoretically allow you to solve for a sequence of control inputs $u(0), u(1), ..., u(N-1)$ to take the system from an initial state $x(0) = x_s$ at time $k = 0$ to any other arbitrary state $x(N) = x_f$ in a finite number of time steps (assuming both actuators are available and that both are operated in open-loop, i.e. not using any kind of feedback).

(d) How many solutions exist for the system of equations in (c) (justify mathematically)? Is it possible to find a 'smallest' solution, i.e. one that leads to input vectors whose norms are as small as possible? If so, provide a formula for obtaining it – you need not derive it completely from scratch, but do explain how it relates to the equations in part (c) and why it works mathematically.