Statistical Estimation	**Homework 5**
ASEN 5044 Fall 2018	**Due Date: October 18, 2018**
Name: Andrew Kramer	**PhD Student**

Problem 1

Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_w σ_v, respectively. What is the standard deviation of the random variable $X = W + V$?

The variance of X can be expressed as

$$\begin{aligned}
\sigma_X^2 &= E(X^2) - E(X)^2 \\
&= E((W+V)^2) - E(W+V)^2 \\
&= E(W^2 + 2WV + V^2) - (E(W) + E(V))^2 \\
&= E(W^2) + 2E(WV) + E(V^2) - E(W)^2 - 2E(W)E(V) - E(V)^2
\end{aligned}$$

Because W and V are uncorrelated, $E(WV) = E(W)E(V)$. This means the above expression reduces to

$$\begin{aligned}
\sigma_X^2 &= E(W^2) - E(W)^2 + E(V^2) - E(V)^2 \\
&= \sigma_W^2 + \sigma_V^2
\end{aligned}$$

So the standard deviation of X is $\sqrt{\sigma_W^2 + \sigma_V^2}$.

Problem 2

Consider two scalar RVs X and Y.

Part a

Prove that if X and Y are independent their correlation coefficient $\rho = 0$.

For independent random variables $E(XY) = E(X)E(Y)$. Because of this their covariance $C_{XY} = E(XY) - E(X)E(Y) = 0$. This means their correlation coefficient is

$$\rho = \frac{C_{XY}}{\sigma_x \sigma_y} = \frac{0}{\sigma_x \sigma_y} = 0$$

Part b

Find an example of two RVs that are not independent but have a correlation coefficient of zero.

Assume $X = \mathcal{U}(-1, 1)$ and $Y = X^2$. Because $\rho = \frac{C_{XY}}{\sigma_X \sigma_Y}$ we just need to show that

$$C_{XY} = E(XY) - E(X)E(Y) = 0$$

to show $\rho = 0$. From the definition of the uniform distribution we know that $E(X) = \frac{1}{2}(-1+1) = 0$, so we know $E(X)E(Y) = 0$. We can now find

$$\begin{aligned} E(XY) &= E(X^3) \\ &= \int_{-1}^{1} x^3 dx \\ &= \frac{1}{4}x^4\Big|_{-1}^{1} \\ &= \frac{1}{4} - \frac{1}{4} = 0 \end{aligned}$$

So because $C_{XY} = E(XY) - E(X)E(Y) = 0 - 0E(Y) = 0$, ρ must also be equal to zero.

Part c

Prove that if Y is a linear function of X then $\rho = \pm 1$.

To show that $\rho = \pm 1$ when Y is a linear function of X we simply need to show that $|C_{XY}| = |\sigma_X \sigma_Y|$. We can do this by finding $E(Y)$, $E(Y^2)$, $E(XY)$, and σ_Y in terms of $E(X)$, $E(X^2)$, and σ_X.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} X dX = \frac{1}{2}X^2\Big|_{-\infty}^{\infty} \\ E(X^2) &= \int_{-\infty}^{\infty} X^2 dX = \frac{1}{3}X^3\Big|_{-\infty}^{\infty} \\ E(Y) &= E(AX) + E(B) \\ &= AE(X) + B \\ E(Y^2) &= E((AX+B)^2) \\ &= E(A^2X^2 + 2ABX + B^2) \\ &= A^2E(X^2) + 2ABE(X) + B^2 \\ E(XY) &= E(AX^2 + BX) \\ &= AE(X^2) + BE(X) \\ \sigma_Y &= E(Y^2) - (E(Y))^2 \\ &= A^2E(X^2) + 2ABE(X) + B^2 - (AE(X) + B)^2 \\ &= A^2E(X^2) + 2ABE(X) + B^2 - A^2(E(X))^2 - 2ABE(X) - B^2 \\ &= A^2(E(X^2) - (E(X)^2)) \\ &= A^2\sigma_X^2 \end{aligned}$$

Given these preliminaries we can find

$$\begin{aligned}
C_{XY} &= E(XY) - E(X)E(Y) \\
&= AE(X^2) + BE(X) - E(X)(AE(X) + B) \\
&= AE(X^2) + BE(X) - AE(X)^2 - BE(X) \\
&= A(E(X^2) - E(X)^2) \\
&= A\sigma_X^2 \\
\sigma_X\sigma_Y &= \sigma_X\sqrt{A^2\sigma_X^2} \\
&= A\sigma_X^2
\end{aligned}$$

All of this shows that when Y is a linear function of X,

$$\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{A\sigma_X^2}{A\sigma_X^2} = 1$$

Problem 3

Consider the following function

$$f_{XY} = \begin{cases} ae^{-2x}e^{-3y} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Part a

Find the value of a so that $f_{XY}(x, y)$ is a valid joint probability density function.

Because $\int_X \int_Y f_{XY} dydx = 1$ we can find a by the following:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_{XY} dy \\
&= ae^{-2x} \int_0^{\infty} e^{-3y} dy \\
&= -\frac{a}{3}e^{-2x}e^{-3y}\Big|_0^{\infty} \\
&= \frac{a}{3}e^{-2x} \\
\int_0^{\infty} \frac{a}{3}e^{-2x} dx &= -\frac{a}{6}e^{-2x}\Big|_0^{\infty} \\
&= \frac{a}{6} \\
a &= 6
\end{aligned}$$

Part b

Calculate $\bar{x}$ and $\bar{y}$.

To find $E(X)$ and $E(Y)$ we do the following:

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X dx \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY} dy dx \\
&= \int_{-\infty}^{\infty} 2xe^{-2x} dx \\
&= \frac{-2x-1}{2} e^{-2x}\Big|_0^{\infty} \\
&= \frac{1}{2} \\
E(Y) &= \int_{-\infty}^{\infty} y f_Y dy \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY} dx dy \\
&= \int_{-\infty}^{\infty} 2ye^{-3y} dy \\
&= \frac{-3y-1}{3} e^{-3y}\Big|_0^{\infty} \\
&= \frac{1}{3}
\end{aligned}
$$

Part c

Calculate $E(X^2)$, $E(Y^2)$, and $E(XY)$.

Part d

Calculate the autocorrelation matrix of the random vector $[X\ Y]^T$.

Part e

Calculate the variance σ_x^2 and σ_y^2 and the covariance C_{XY}.

Part f

Calculate the autocovariance matrix of the random vector $[X\ Y]^T$.

Part g

Calculate the correlation coefficient between X and Y.

Problem 4

Prove the following two results used in lectire to derive the theoretical expectations for the Gaussian sampling experiment where $x \sim \mathcal{N}(\bar{x}, \sigma_x^2)$, $e \sim \mathcal{N}(0, \sigma_e^2)$, and $y = cx + d$.

Part a

$\text{cov}(X, Y) = E[(x - \bar{x})(y - \bar{y})] = E[XY] - \bar{x}\bar{y}$

Part b

$\text{var}(Y) = E[(y - \bar{y})^2] = c^2\sigma_x^2 + d^2\sigma_e^2$

Problem 5

Consider two continuous random variables x and y, where $y = \ln(x)$ and $x > 0$. Derive analytical closed-form expressions for each of the following:

Part a

$p(y)$ if $p(x) = \mathcal{U}[a, b]$ (i.e. if x has a uniform pdf for $0 < a \leq x \leq b$)

Part b

$p(y)$ if $p(\frac{1}{x}) = \mathcal{U}[c, d]$ (i.e. if $\frac{1}{x}$ has a uniform pdf $0 < c \leq \frac{1}{x} \leq d$)

Part c

$p(x)$ if $p(y) = \mathcal{U}[l, m]$ (i.e. if y has a uniform pdf for $l \leq y \leq m$)

Part d

$p(x)$ if $p(y) = \mathcal{N}(\mu_y, \sigma_y^2)$ (i.e. if y has a Gaussian pdf with mean μ_y and variance σ_y^2)