ASEN 5044 Statistical Estimation for Dynamical Systems
Fall 2018
Midterm Exam 1

Out: Thursday 10/04/2018 (posted on Canvas)

Due: Thursday 10/11/2018 11 am (on Canvas)

This exam is open notes and open book. You may ask the TA and Prof. Ahmed for clarification only, ***but you may not consult with each other (CU Honor Code applies and will be strictly enforced)****. Show all your work and explain your reasoning for full credit.* ***You may not use computer software to solve any problems below, unless otherwise explicitly indicated in the problem statement.***

1. [35 pts] The *inverted pendulum on a cart* is a classical problem in control theory. This system model was originally developed to study the dynamics and control of vertical rocket stabilization before and during launch. This dynamics model is also the basis for the famous Segway human transport vehicle: the standing human passenger is the 'pendulum', while the platform is the 'cart'; the system achieves forward motion as the cart tries to stabilize the falling pendulum.

Figure 1 shows the basic set up of the inverted pendulum system, which consists of a massless rod of length l and mass m concentrated at the end. The cart has mass M, and rolls along the ground without any resistance according to a forcing input $u(t) = P(t)$. The system has the following non-linear equations of motion (EOMs), where z is the horizontal translation (in m), θ is the angular displacement (in rads), and $g = 9.81$ m/s^2,

$$(M+m)\ddot{z} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = P, \tag{1}$$

$$l\ddot{\theta} - g\sin\theta = \ddot{z}\cos\theta. \tag{2}$$

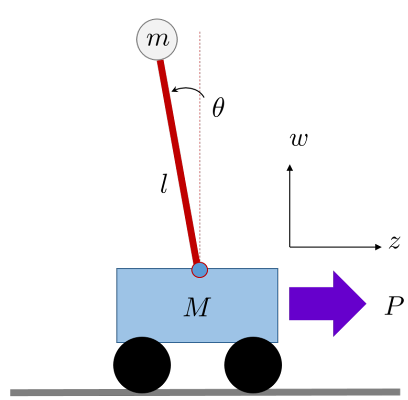

Figure 1: Inverted pendulum on a cart.

Answer the following questions (you can use computing software for parts (c)-(g) only):

(a) Put the non-linear EOMs into standard state space form using the state vector $x = [z, \dot{z}, \theta, \dot{\theta}]^T$, and show that the upright stationary state for the pendulum, i.e. where $\dot{z} = 0$, $\theta = 0$, and $\dot{\theta} = 0$ for $P(t) = 0$ N for all $t \geq 0$, is an equilibrium state .

(b) Suppose a sensor is attached to this system which reports the pendulum bob's horizontal displacement relative to the vertical, such that $y = z - l\sin\theta$. Linearize the equations of motion and this output equation about the stationary upright state, and put the result into LTI (A, B, C, D) form. What is the corresponding state vector $\tilde{x}(t)$?

(c) Suppose $l = 1$ m, $m = 1$ kg, and $M = 2$ kg. Transform the CT LTI linearized dynamics near equilibrium into an equivalent DT LTI (F, G, H, M) model using $\Delta T = 0.05$ sec, and assess the stability and observability of the DT system near the equilibrium.

(d) Suppose a full-state feedback control law is designed for the linearized CT system of the form $u(t) = -K\tilde{x}(t)$, where K is a 1×4 gain matrix whose values are provided in the array 'Kct' in the posted file `midterm1problem1data.mat`. Find the 'closed-loop' CT linearized system dynamics model for $\tilde{x}$ (i.e. the dynamics under feedback control) in LTI form $(A_{CL}, B_{CL}, C_{CL}, D_{CL})$, if this control law is used to define the forcing input $P(t)$ for $t \geq 0$ (with no other inputs).

(e) Convert the results from part (d) into DT LTI matrices $(F_{CL}, G_{CL}, H_{CL}, M_{CL})$ using the same sampling rate as before, and re-assess the closed-loop system's observability and stability – how does this compare to the the 'open-loop' dynamics from part (c)?

(f) Suppose the full closed-loop nonlinear inverted pendulum system is run from some unknown initial condition $\tilde{x}(0)$ using the control law from part (d) (with no process or sensor noise); the resulting non-linear sensor outputs $y(k)$ are stored in the data array 'yNLhist' at the discrete time steps k specified in 'thist', contained in `midterm1problem1data.mat`. Use the DT closed-loop linearized matrices from part (e) to estimate $\tilde{x}(0)$ at $k = 0$ from 'yNLhist' (explain your reasoning; report to at least 4 significant digits).

(g) Generate and plot the predicted sequence of measurements $y(k)$ produced by the linearized DT closed-loop dynamics model over the entire time interval in 'thist', using your estimated $x(0)$ as the initial condition. On the same plot, also plot all the original data from 'yNLhist' (using a different marker/color scheme to distinguish) and compare the results. How well do the outputs predicted by the linearized DT closed-loop model agree with the actual measurements in 'yNLhist' (explain why)?

2. [25 pts] Two 6-sided dice are tossed together; let R_1 and R_2 denote the outcome of the first die and second die, respectively. Note, that in this problem, we are **not** summing the outcome of the dice together.

(a) What is the joint probability distribution $P(R_1, R_2)$ for R_1 and R_2, if the tosses for each die are independent and the outcomes of each die are all equally probable?

(b) Let $X = \min(R_1, R_2)$ and $Y = \max(R_1, R_2)$ (e.g. if $R_1 = 5$ and $R_2 = 2$, then $X = 2$ and $Y = 5$). Find the joint probability distribution for X and Y.

(c) Find the marginal (i.e. unconditional) probability distributions for X and Y in (b).

(d) Are X and Y independent (justify)?

3. [25 pts] A random variable X has the pdf $p(x) = k(1-x^4)$ for $-1 \le x \le 1$ and $p(x) = 0$ elsewhere.

(a) Find k, $\mathbb{E}[x]$ and $\mathrm{var}(x)$.

(b) Find the cumulative distribution function (cdf) of X, $P_X(\zeta)$.

(c) Find $P(|X| < 0.5)$.

4. [15 pts] Do the following (show all work): Consider police blood tests on drivers suspected of having consumed too much alcohol. The table below shows the the conditional probabilities $P(T|A)$ for a particular test used by the police which returns $T \in \{\text{positive}, \text{negative}\}$ given the true driver state $A \in \{\text{drunk}, \text{sober}\}$ (note: $P(T = \text{negative}|A = \text{drunk}) = 0.01$ is the *false negative rate*, while $P(T = \text{positive}|A = \text{sober}) = 0.001$ is the *false positive rate*):

$P(T\|A)$	A = drunk	A = sober
T = positive	0.99	0.001
T = negative	0.01	0.999

(a) Suppose experience dictates that $P(A = \text{drunk}) = 0.20$. If a test for a driver returns $T = \text{positive}$, what is the probability that the driver is guilty of DUI?

(b) Then suppose some other statistics tell you that $P(A = \text{drunk}) = 0.001$ –what is now the probability that the same driver is guilty of DUI if $T = \text{positive}$?

Advanced Questions ***PhD students in the class MUST answer ALL questions below in addition to regular questions above*** – *non-PhD students are welcome to try any of these for extra credit (only given if all regular problems turned in on time as well). In either case, Submit your responses for these questions with rest of your exam, but make sure these are clearly labeled and start on separate pages –* ***indicate in the .pdf file name (per instructions on Canvas) and on the first page of your submission if you answered these questions (as a PhD student or for extra credit)***.

AQ1. [20 pts] Consider a Bayesian analysis of the following fundamental question in astrobiology: what is the probability of abiogenesis (i.e. the original evolution of living organisms from inorganic substances) occurring on an Earth-like planet, given that life emerged sometime in Earth's past? A 2012 paper by Spiegel and Turner [1] suggests that the probability of life arising n times in time t, for $t_{min} < t < t_{max}$ is

$$P[\lambda, n, t] = Poisson[\lambda, n, t] = e^{-\lambda(t-t_{min})} \frac{\{\lambda(t - t_{min})\}^n}{n!},$$

where λ defines the probability per unit time of abiogenesis occurring on an 'Earth-like' planet, t_{min} is the time following planet formation after which conditions are suitable for the possibility of life emerging, and t_{max} is the time after which conditions on the planet will no longer allow life to arise (e.g. following the death of its parent star). The difficulty with

[1] D.S. Spiegel and E.L. Turner, 'Bayesian analysis of the astrobiological implications of life's early emergence on Earth,' Proc. National Academy of Sciences, vol. 109, no. 2, pp. 395-400, Jan. 10, 2012.

using this model to estimate the abundance of life in the universe is that λ is unknown. Yet, while t_{min}, and t_{max} are also generally not precisely known, Spiegel and Turner argue that these time values can be reasonably constrained and thus used as evidence for narrowing down likely values of $\lambda > 0$, based on the fact that life most definitely arose on Earth within a limited period of time (i.e. early enough to allow us to ponder abiogenesis).

(a) Show that, under the above model, the probability of life occurring at least once within some time t of planet formation is given by,

$$P_{life} = 1 - e^{-\lambda(t - t_{min})}.$$

(b) Suppose t_{emerge} is the age of the Earth from when the earliest extant evidence of life remains, and also suppose t_{req} is maximum age that the Earth could have had at the origin of life in order for humanity to have a chance of showing up by the present, where $t_{min} < t_{emerge} < t_{req}$. Define two binary events E and R with the following interpretations: $E = 1$ means 'life emerged at least once within time t_{emerge}' ($E = 0$ otherwise), and $R = 1$ means 'life emerged at least once within time t_{req}' ($R = 0$ otherwise). Use Bayes' rule to show (for some given set of values λ, t_{min}, t_{emerge}, and t_{req}) that it must therefore follow that

$$P(E = 1|R = 1, \lambda, t_{min}) = \frac{1 - e^{-\lambda(t_{emerge} - t_{min})}}{1 - e^{-\lambda(t_{req} - t_{min})}}.$$

(c) Suppose you distill your knowledge of geophysics, paleobiology, and theories on the origin and evolution of intelligent life into a prior pdf $p(y)$ for $y = \log_{10} \lambda$. You then read a newly published study about abiogenesis on Earth which shows $t_{min} = 0.61$ GigaYears (GYr), $t_{emerge} = 0.78$ GYr, and $t_{req} = 1.4$ GYr. You recognize this means the conditional probability $P(E = 1|R = 1, \lambda, t_{min})$ given above can be easily transformed into an observation likelihood function $P(E = 1|R = 1, y, t_{min})$ as a function of y, thus letting you derive a posterior pdf $p(y|E = 1, R = 1, t_{min})$ over y via Bayes' rule. Derive the likelihood function $P(E = 1|R = 1, y, t_{min})$ in terms of y, and then use it to numerically evaluate and plot the posterior pdf $p(y|E = 1, R = 1, t_{min})$ vs. y using a grid-based approximation on the interval $-3 \leq y \leq 3$ for each of the following possible $p(y)$ cases (use 10^4 evenly spaced grid points for y in this interval, and label carefully):

Case 1: $p(y) = \mathcal{U}[-3, 3]$ (i.e. uniform on $y = \log \lambda$ for $-3 \leq y \leq 3$),

Case 2: $p(y) = \dfrac{|\ln(10) \cdot 10^{y}|}{(10^3 - 10^{-3})}$ (i.e. uniform on $10^y = \lambda$ for $10^{-3} \leq \lambda \leq 10^3$),

Case 3: $p(y) = \dfrac{|-\ln(10) \cdot 10^{-y}|}{(10^3 - 10^{-3})}$ (i.e. uniform on $10^{-y} = \lambda^{-1}$ for $10^{-3} \leq \lambda^{-1} \leq 10^3$).

(d) For which of the 3 cases evaluated above does the posterior show the greatest change from the prior? Discuss the implications for the problem choosing a prior – is any one of the priors above 'better' than the others (explain)?