

Problem 1

Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_w σ_v , respectively. What is the standard deviation of the random variable $X = W + V$?

The variance of X can be expressed as

$$\begin{aligned}\sigma_X^2 &= E(X^2) - E(X)^2 \\ &= E((W + V)^2) - E(W + V)^2 \\ &= E(W^2 + 2WV + V^2) - (E(W) + E(V))^2 \\ &= E(W^2) + 2E(WV) + E(V^2) - E(W)^2 - 2E(W)E(V) - E(V)^2\end{aligned}$$

Because W and V are uncorrelated, $E(WV) = E(W)E(V)$. This means the above expression reduces to

$$\begin{aligned}\sigma_X^2 &= E(W^2) - E(W)^2 + E(V^2) - E(V)^2 \\ &= \sigma_W^2 + \sigma_V^2\end{aligned}$$

So the standard deviation of X is $\sqrt{\sigma_W^2 + \sigma_V^2}$.

Problem 2

Consider two scalar RVs X and Y .

Part a

Prove that if X and Y are independent their correlation coefficient $\rho = 0$.

For independent random variables $E(XY) = E(X)E(Y)$. Because of this their covariance $C_{XY} = E(XY) - E(X)E(Y) = 0$. This means their correlation coefficient is

$$\rho = \frac{C_{XY}}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0$$

Part b

Find an example of two RVs that are not independent but have a correlation coefficient of zero.

Assume $X = \mathcal{U}(-1, 1)$ and $Y = X^2$. Because $\rho = \frac{C_{XY}}{\sigma_X \sigma_Y}$ we just need to show that

$$C_{XY} = E(XY) - E(X)E(Y) = 0$$

to show $\rho = 0$. From the definition of the uniform distribution we know that $E(X) = \frac{1}{2}(-1+1) = 0$, so we know $E(X)E(Y) = 0$. We can now find

$$\begin{aligned} E(XY) &= E(X^3) \\ &= \int_{-1}^1 x^3 dx \\ &= \frac{1}{4}x^4 \Big|_{-1}^1 \\ &= \frac{1}{4} - \frac{1}{4} = 0 \end{aligned}$$

So because $C_{XY} = E(XY) - E(X)E(Y) = 0 - 0E(Y) = 0$, ρ must also be equal to zero.

Part c

Prove that if Y is a linear function of X then $\rho = \pm 1$.

To show that $\rho = \pm 1$ when Y is a linear function of X we simply need to show that $|C_{XY}| = |\sigma_X \sigma_Y|$. We can do this by finding $E(Y)$, $E(Y^2)$, $E(XY)$, and σ_Y in terms of $E(X)$, $E(X^2)$, and σ_X .

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} X dX = \frac{1}{2}X^2 \Big|_{-\infty}^{\infty} \\ E(X^2) &= \int_{-\infty}^{\infty} X^2 dX = \frac{1}{3}X^3 \Big|_{-\infty}^{\infty} \\ E(Y) &= E(AX) + E(B) \\ &= AE(X) + B \\ E(Y^2) &= E((AX + B)^2) \\ &= E(A^2X^2 + 2ABX + B^2) \\ &= A^2E(X^2) + 2ABE(X) + B^2 \\ E(XY) &= E(AX^2 + BX) \\ &= AE(X^2) + BE(X) \\ \sigma_Y &= E(Y^2) - (E(Y))^2 \\ &= A^2E(X^2) + 2ABE(X) + B^2 - (AE(X) + B)^2 \\ &= A^2E(X^2) + 2ABE(X) + B^2 - A^2(E(X))^2 - 2ABE(X) - B^2 \\ &= A^2(E(X^2) - (E(X))^2) \\ &= A^2\sigma_X^2 \end{aligned}$$

Given these preliminaries we can find

$$\begin{aligned}
 C_{XY} &= E(XY) - E(X)E(Y) \\
 &= AE(X^2) + BE(X) - E(X)(AE(X) + B) \\
 &= AE(X^2) + BE(X) - AE(X)^2 - BE(X) \\
 &= A(E(X^2) - E(X)^2) \\
 &= A\sigma_X^2 \\
 \sigma_X\sigma_Y &= \sigma_X\sqrt{A^2\sigma_X^2} \\
 &= A\sigma_X^2
 \end{aligned}$$

All of this shows that when Y is a linear function of X ,

$$\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{A\sigma_X^2}{A\sigma_X^2} = 1$$

Problem 3

Consider the following function

$$f_{XY} = \begin{cases} ae^{-2x}e^{-3y} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Part a

Find the value of a so that $f_{XY}(x, y)$ is a valid joint probability density function.

Because $\int_X \int_Y f_{XY} dy dx = 1$ we can find a by the following:

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} f_{XY} dy \\
 &= ae^{-2x} \int_0^{\infty} e^{-3y} dy \\
 &= -\frac{a}{3} e^{-2x} e^{-3y} \Big|_0^{\infty} \\
 &= \frac{a}{3} e^{-2x} \\
 \int_0^{\infty} \frac{a}{3} e^{-2x} dx &= -\frac{a}{6} e^{-2x} \Big|_0^{\infty} \\
 &= \frac{a}{6} \\
 a &= 6
 \end{aligned}$$

Part b

Calculate \bar{x} and \bar{y} .

To find $E(X)$ and $E(Y)$ we do the following:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X dx \\ &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY} dy dx \\ &= \int_{-\infty}^{\infty} 2xe^{-2x} dx \\ &= \left. \frac{-2x-1}{2} e^{-2x} \right|_0^{\infty} \\ &= \frac{1}{2} \\ E(Y) &= \int_{-\infty}^{\infty} y f_Y dy \\ &= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY} dx dy \\ &= \int_{-\infty}^{\infty} 2ye^{-3y} dy \\ &= \left. \frac{-3y-1}{3} e^{-3y} \right|_0^{\infty} \\ &= \frac{1}{3} \end{aligned}$$

Part c

Calculate $E(X^2)$, $E(Y^2)$, and $E(XY)$.

$$\begin{aligned}
E(X^2) &= \int_X x^2 f_X dx \\
&= 2 \int_X x^2 e^{-2x} dx \\
&= \frac{-2x^2 - 2x - 1}{2} e^{-2x} \Big|_0^\infty \\
&= \frac{1}{2} \\
E(Y^2) &= \int_Y y^2 f_Y dy \\
&= 3 \int_Y y^2 e^{-3y} dy \\
&= \frac{-9y^2 - 6y - 2}{9} e^{-3y} \Big|_0^\infty \\
&= \frac{2}{9} \\
E(XY) &= \int_Y \int_X xy f_{XY} dx dy \\
&= 6 \int_Y \int_X e^{-2x} e^{-3y} dx dy \\
&= 6 \int_Y \left[\frac{-2x - 1}{4} e^{-2x} y e^{-3y} \right]_{x=0}^\infty dy \\
&= \frac{6}{4} \int_Y y e^{-3y} dy \\
&= \frac{6}{4} \frac{-3y - 1}{9} e^{-3y} \Big|_{y=0}^\infty \\
&= \frac{1}{6}
\end{aligned}$$

Part d

Calculate the autocorrelation matrix of the random vector $[X \ Y]^T$.

$$R_{XY} = \begin{bmatrix} E(X^2) & E(XY) \\ E(YX) & E(Y^2) \end{bmatrix} = \begin{bmatrix} 1/2 & 2/9 \\ 2/9 & 1/6 \end{bmatrix}$$

Part e

Calculate the variance σ_x^2 and σ_y^2 and the covariance C_{XY} .

$$\begin{aligned}\sigma_x^2 &= E(X^2) - E(X)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\ \sigma_y^2 &= E(Y^2) - E(Y)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9} \\ C_{XY} &= E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{1}{6} = 0\end{aligned}$$

Part f

Calculate the autocovariance matrix of the random vector $[X \ Y]^T$.

$$C = \begin{bmatrix} \sigma_X^2 & C_{XY} \\ C_{XY} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}$$

Part g

Calculate the correlation coefficient between X and Y .

Because the covariance $C_{XY} = 0$ the correlation coefficient $\rho = \frac{C_{XY}}{\sigma_x \sigma_y}$ is also equal to zero.

Problem 4

Prove the following two results used in lecture to derive the theoretical expectations for the Gaussian sampling experiment where $x \sim \mathcal{N}(\bar{x}, \sigma_x^2)$, $e \sim \mathcal{N}(0, \sigma_e^2)$, and $y = cx + de$.

Part a

$$\text{cov}(X, Y) = E[(x - \bar{x})(y - \bar{y})] = E[XY] - \bar{x}\bar{y}$$

$$\begin{aligned}\text{cov}(X, Y) &= E[(x - \bar{x})(y - \bar{y})] \\ &= E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}] \\ &= E(xy) - \bar{y}E(x) - \bar{x}E(y) + \bar{x}\bar{y} \\ &= E(xy) - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} \\ &= E(xy) - \bar{x}\bar{y}\end{aligned}$$

Part b

$$\text{var}(Y) = E[(y - \bar{y})^2] = c^2\sigma_x^2 + d^2\sigma_e^2$$

The expected value of a linear combination of gaussians is $\sum_i c_i \bar{x}_i$ so the expected value of y is $\bar{y} = c\bar{x} + d\bar{e}$. But since $\bar{e} = 0$, $\bar{y} = c\bar{x}$. With this we can find $\text{var}(Y)$ as follows:

$$\begin{aligned}
 \text{var}(Y) &= E((y - \bar{y})^2) \\
 &= E((cx + de - c\bar{x})^2) \\
 &= c^2 E(x^2) + cdE(ex) - c^2 \bar{x}E(x) + cdE(ex) + d^2 E(e^2) - dc\bar{x}E(e) \\
 &\quad - c^2 \bar{x}E(x) - cd\bar{x}E(e) + c^2 \bar{x}^2 \\
 &\text{since } x \text{ and } e \text{ are independent} \\
 &= c^2(E(x^2) - \bar{x}^2) + d^2 E(e^2) - 2cdE(e)E(x) - 2dc\bar{x}E(e) \\
 &\text{and since } E(e) = 0 \\
 &= c^2(E(x^2) - \bar{x}^2) + d^2(E(e^2) - \bar{e}^2) \\
 &= c^2\sigma_x^2 + d^2\sigma_e^2
 \end{aligned}$$

Problem 5

Consider two continuous random variables x and y , where $y = \ln(x)$ and $x > 0$. Derive analytical closed-form expressions for each of the following:

Part a

$p(y)$ if $p(x) = \mathcal{U}[a, b]$ (i.e. if x has a uniform pdf for $0 < a \leq x \leq b$)

Generally, if $Y = g(X)$ then $p(y) = \left| \frac{d}{dy} g^{-1}(y) \right| p(g^{-1}(y))$. So for the given distribution for X ,

$$\begin{aligned}
 g^{-1}(y) &= e^y \\
 \frac{d}{dy} g^{-1}(y) &= e^y \\
 f(g^{-1}) &= \begin{cases} \frac{1}{\ln(b) - \ln(a)} & \ln(a) \leq y \leq \ln(b) \\ 0 & \text{otherwise} \end{cases} \\
 p(y) &= \begin{cases} \frac{e^y}{\ln(b) - \ln(a)} & \ln(a) \leq y \leq \ln(b) \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Part b

$p(y)$ if $p(\frac{1}{x}) = \mathcal{U}[c, d]$ (i.e. if $\frac{1}{x}$ has a uniform pdf $0 < c \leq \frac{1}{x} \leq d$)

We are not directly given the distribution for X . Instead we are given $p(z) = \mathcal{U}[c, d]$ where $z = 1/x$, so we need to derive the distribution for X .

$$\begin{aligned}
 z &= h(x) = \frac{1}{x} \\
 h^{-1}(z) &= \frac{1}{z} \\
 \frac{d}{dz} &= -\frac{1}{z^2} \\
 p(z) &= \left| \frac{d}{dz} h^{-1}(z) \right| p(h^{-1}(z)) \\
 \frac{1}{z^2} p(h^{-1}(z)) &= \begin{cases} \frac{1}{d-c} & c \leq z \leq d \\ 0 & \text{otherwise} \end{cases} \\
 p(h^{-1}(z)) &= \begin{cases} \frac{z^2}{d-c} & c \leq z \leq d \\ 0 & \text{otherwise} \end{cases} \\
 p(x) &= \begin{cases} \frac{(1/x)^2}{(1/d)-(1/c)} & 1/c \leq x \leq 1/d \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Given this distribution for $p(x)$ we can calculate $p(y) = \left| \frac{d}{dy} g^{-1}(y) \right| p(g^{-1}(y))$.

$$\begin{aligned}
 p(y) &= \begin{cases} \frac{(1/e^y)^2}{\ln(1/d)-\ln(1/c)} e^y & \ln(1/d) \leq y \leq \ln(1/c) \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} \frac{e^{-y}}{\ln(c)-\ln(d)} & -\ln(d) \leq y \leq -\ln(c) \end{cases}
 \end{aligned}$$

Part c

$p(x)$ if $p(y) = \mathcal{U}[l, m]$ (i.e. if y has a uniform pdf for $l \leq y \leq m$)

$$\begin{aligned}
 x &= g(y) = e^y \\
 y &= g^{-1}(x) = \ln(x) \\
 \frac{d}{dx} g^{-1}(x) &= \frac{1}{x} \\
 p_X(x) &= \left| \frac{d}{dx} g^{-1}(x) \right| p_Y(g^{-1}(x)) \\
 &= \frac{1}{x} p_Y(g^{-1}(x)) \\
 &= \begin{cases} \frac{1}{x(\ln(m)-\ln(l))} & \ln(l) \leq x \leq \ln(m) \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Part d

$p(x)$ if $p(y) = \mathcal{N}(\mu_y, \sigma_y^2)$ (i.e. if y has a Gaussian pdf with mean μ_y and variance σ_y^2)

The probability distribution for X is given by

$$\begin{aligned} p(x) &= \left| \frac{d}{dx} g^{-1}(x) \right| p_Y(g^{-1}(x)) \\ &= \frac{1}{x} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(\ln(x)-\mu_y)^2}{2\sigma_y^2}} \end{aligned}$$

which is the log-normal distribution.