Statistical Estimation	**Homework 1**
ASEN 5044 Fall 2018	**Due Date: Sep 6, 2018**
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Exercise 1

Compute determinants for the following matrices by hand and state whether each one is invertible

Problem (a)

$$\begin{aligned}|A| &= 1\begin{vmatrix}5 & 4\\ 9 & 7\end{vmatrix} - 2\begin{vmatrix}6 & 4\\ 8 & 7\end{vmatrix} + 3\begin{vmatrix}6 & 5\\ 8 & 9\end{vmatrix}\\ &= 1(-1) - 2(10) + 3(14)\\ &= 21\end{aligned}$$

Because $|A|$ is nonzero A is invertible.

Problem (b)

$$\begin{aligned}|A| &= 11\begin{vmatrix}57 & 0 & 10\\ 91 & 1 & 71\\ 23 & 0 & 71\end{vmatrix} - 26\begin{vmatrix}64 & 0 & 10\\ 83 & 1 & 71\\ 54 & 0 & 71\end{vmatrix}\\ &= 11(57(71) + 10(-23)) - 26(64(71) + 10(-54))\\ &= 41978 - 109642\\ &= -67655\end{aligned}$$

Because $|A|$ is nonzero A is invertible.

Problem (c)

Because $A_1 = -2A_3$ (where A_i refers to the i^{th} column of A), the columns of A are not linearly independent. This means A is not invertible and $|A| = 0$.

Problem (d)

Because the determinant of an upper triangular matrix is simply the product of its diagonal elements:

$$\begin{aligned}|A| &= 1 \times 8 \times 55 \times 233 \times 610\\ &= 62537200\end{aligned}$$

Because $|A|$ is nonzero A is invertible.

Exercise 2

Prove each of the following statements:

Problem (a)

If a and b are non-zero $n \times 1$ vectors, then the matrix ab^T has rank 1.

Column i of the outer product of a and b is simply the vector a multiplied by the scalar b_i. This means that every column of ab^T is a scalar multiple of a, so none of the columns of ab^T are linearly independent. Thus, the rank of ab^T is always one if both a and b are nonzero.

Problem (b)

$\text{tr}(AB) = \text{tr}(BA)$ if A is an $m \times n$ matrix and B is $n \times m$.

The trace of AB can be expressed as

$$\begin{aligned} \text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \end{aligned}$$

Similarly, the trace of BA is

$$\begin{aligned} \text{tr}(BA) &= \sum_{j=1}^{n} (BA)_{jj} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \end{aligned}$$

Because $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$ is equal to $\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$ we can conclude that $\text{tr}(AB) = \text{tr}(BA)$.

Problem (c)

If A is invertible then $|A^{-1}| = \frac{1}{|A|}$.

If we start with $|AB| = |BA| = |A|\ |B|$ and replace B with A^{-1} we find that

$$\begin{aligned} |A|\ |A^{-1}| &= |AA^{-1}| \\ &= |I| \\ &= 1 \end{aligned}$$

Because $|A|\ |A^{-1}| = 1$ we must conclude that $|A^{-1}| = \frac{1}{|A|}$.