ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 3:
State Space Models and Linear Dynamical Systems

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Overview

- Last time: Quick linear algebra refresher

- **Today:** State Space Models
  - motivation, examples

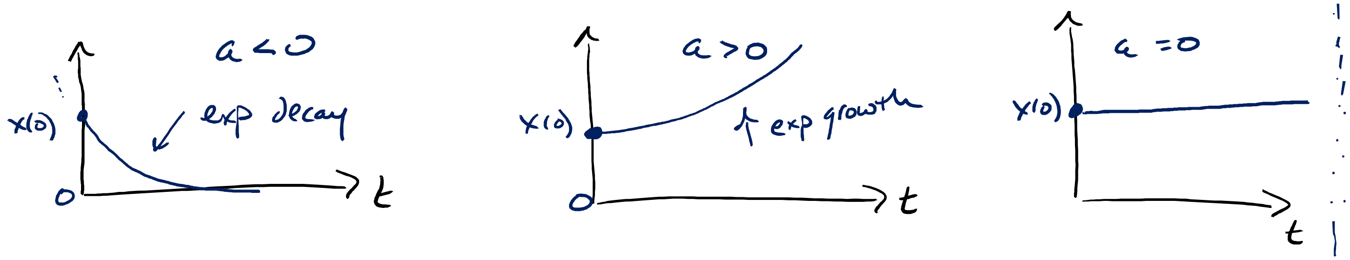
  →**(A,B,C,D) matrix parameters for continuous time (CT) linear dynamical systems**
  - **state transition matrix (STM) and matrix exponential solutions**

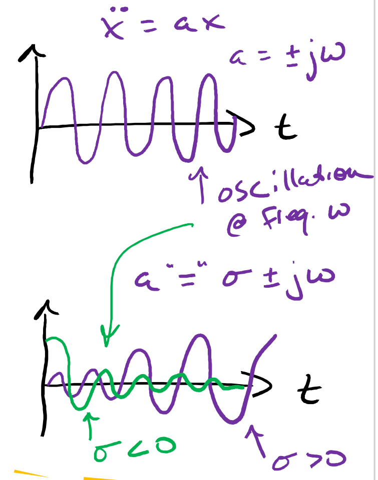
READ: Chapter 1.2-1.3 in Simon book

Refresher/Motivation: $\dot{x}(t) = ax(t) \iff x(t) = e^{at}x(0)$

- Given *a*, knowing x(0) <u>completely determines</u> x(t) **in future**
- Given x(0), knowing *a* <u>completely determines</u> how x(t) **behaves**

$\dot{x} = ax$





How to generalize these insights from scalar linear systems (ODEs) to more complex systems (ODEs) w/ vector variables & (linear) vector-matrix ODEs

e.g.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

- behavior of x_1 depends on behavior & initial conditions of $x_1, x_2, \ldots, x_n$
- likewise for $x_2, x_3, \ldots, x_n$

Solutions to all these ODEs are coupled generally!

State Space Models

- Idea of "state variable" (state vector):
  - Completely summarize information about condition of a dynamical system

  (i.e. summarizes results of past events leading to present)
  - Sufficient to **completely and uniquely** describe system at all future times (given an input + dynamics model)

  (i.e. the state is such that "knowing 'it' now is enough to tell you all about 'it' later")

Look @ scalar linear ODE again:

$$\dot{x}(t) = a x(t) \underbrace{(+ b u(t))}_{\text{forcing term}} \iff x(t) = \overbrace{\underbrace{e^{at} x(0)}_{\substack{\text{free response} \\ \text{(resp. to IC only)}}} \underbrace{\left(+ \int_0^t e^{a\tau} b u(\tau) d\tau \right)}_{\substack{\text{forced response} \\ \text{(resp. to } u(t) \text{ to only)}}}}^{\text{"Superposition"}}$$

if initial time is $t_0 = 0$

if $t_0 \neq 0$?

(general) Solution is:

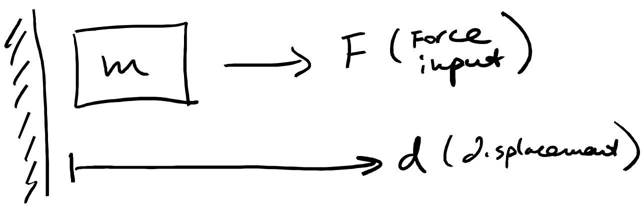
$$x(t - t_0) = e^{a(t-t_0)} x(t_0) \left(+ \int_{t_0}^t e^{a(\tau - t_0)} b u(\tau) d\tau \right)$$

"state" variable

(*) Minimal set of IC's needed to describe a unique sol'n to ODE (in absence of $u(t)$)

(*) How to extend "state" idea to more complex sys. w/ more vars?

Example #1: Mass with 1 Deg of Freedom



Time Domain Block Diagram

$u(t) \to F$ [input] $\longrightarrow \boxed{\frac{1}{m}\int dt} \longrightarrow v = \dot{d}$ [output]

"b" $\to \frac{1}{m}$ "a" "x(t)"

$\longrightarrow F = m\dot{v}$ (physical law) $\longleftrightarrow$ our ODE model $\Longrightarrow \dot{v}(t) = \frac{F(t)}{m} \left(+ \underline{\underline{0}} \cdot v(t) \right)$

$\longrightarrow$ want velocity [output] $v(t)$ as a fxn of time & external force $F(t)$ [input]

$\longrightarrow 1^{st}$ Fund. Thm. Calculus: $\int_{t_0}^{t} \dot{v}(\tau)\,d\tau = [v(t) - v(t_0)]$

$\longrightarrow$ re-arrange & solve for $v(t) = 1 \cdot \underline{\underline{v(t_0)}} + \frac{1}{m} \int_{t_0}^{t} F(\tau)\,d\tau$

but this looks like general sol'n to $\dot{x} = ax + bu$: $x(t) = e^{a(t-t_0)} \underline{\underline{x(t_0)}} + \int_{t_0}^{t} b\, e^{a(\tau - t_0)} u(\tau)\,d\tau$

$\longrightarrow$ so: State Variable is $x(t) = v(t)$ [1D state vector] $\longrightarrow$ For some known input $F(t)$ over $[t_0, t]$, the output $v(t) = \dot{d}(t)$ is different whenever $v(t_0)$ is different

Example #2: Same Mass, Slightly Different Model

Block Diagram: $F(t)$ [input] $\longrightarrow$ $\boxed{\;\cdots\;}$ $\longrightarrow$ $d(t)$ [ouput]

$F = m\dot{v}$ (ODE phys. law)

$\longrightarrow$ Different output: Now want displacement $d(t)$ as fxn of time & $F(t)$ [input]

Easy to show:

$$v(t) = \frac{1}{m}\int_{t_0}^{t} F(\tau)\,d\tau + \underline{v(t_0)}$$

&

$$d(t) = \int_{t_0}^{t} v(\tau)\,d\tau + \underline{d(t_0)}$$

Now we need 2 IC's hence, the state vector is 2D, i.e $x(t) = \begin{bmatrix} v(t) \\ d(t) \end{bmatrix}$

$\longrightarrow$ If we don't keep track of $v(t)$ & $d(t)$ together in our state $x(t)$, then we cannot uniquely specify output $d(t)$ for given $F(t)$ input

$\longrightarrow$ System has "memory": Need to integrate $F(t)$ & know $v(t_0)$ to get $v(t)$

+ integrate $v(t)$ & know $d(t_0)$ to get $d(t)$

Re-arrange Dynamics to Reflect States (want d vs. f)

- Let $x = \begin{bmatrix} v \\ d \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow[\text{elementwise}]{d/dt} \dot{x} = \begin{bmatrix} \dot{v} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = ?$

(state vector $\in \mathbb{R}^2$)

& $u(t) = F(t)$ (input $\in \mathbb{R}^1$)

Recall:

$$v(t) = v(t_0) + \frac{1}{m}\int_{t_0}^{t} F(\tau)\, d\tau$$

$$d(t) = d(t_0) + \int_{t_0}^{t} v(\tau)\, d\tau$$

$\frac{d}{dt}$

$$\dot{v}(t) = \frac{d}{dt}v(t) = \frac{1}{m}F(t) = \frac{1}{m}u(t)$$

$$\dot{d}(t) = \frac{d}{dt}d(t) = v(t) = x_1$$

$$\rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{v} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} \frac{1}{m}F(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{m}u(t) \\ x_1 \end{bmatrix} \rightarrow \dot{x} = \begin{bmatrix} \frac{1}{m}u(t) \\ x_1 \end{bmatrix}$$

where "sensed output" that we actually see is $y(t) = d(t) = x_2$

Re-arrange Dynamics to Reflect States (want d vs. f)

Can rewrite this all in general (Linear) matrix-vector ODE form:

$$\dot{x} = \begin{bmatrix} \dot{v}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} 0 \cdot v(t) + 0 \cdot d(t) + \frac{1}{m} F(t) \\ 1 \cdot v(t) + 0 \cdot d(t) + 0 \cdot F(t) \end{bmatrix}$$

(where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v \\ d \end{bmatrix}$ & $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{v} \\ \dot{d} \end{bmatrix}$)

$$= \begin{bmatrix} 0 \cdot x_1 + 0 \cdot x_2 + \frac{1}{m} u(t) \\ 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot u(t) \end{bmatrix}$$

expand: →

$m = \#\text{ inputs} = 1$

$$\boxed{\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{n \times 1} + \underbrace{\begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}}_{n \times m} \underbrace{u(t)}_{m \times 1}, \quad \text{w/ IC } x(0) = \begin{bmatrix} v(0) \\ d(0) \end{bmatrix}}$$

$n = 2$ (# states)

→ since we only sense $y = d(t) = x_2$:

$$\boxed{\begin{aligned} y(t) &= 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot u(t) \\ \rightarrow y(t) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{p \times n} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{p \times m} u(t) \end{aligned}}$$

$p = \#\text{ outputs} = 1$

Linear State Space Model

State Space Form of Linear Dynamical Systems

- In general, suppose we have state variables $x_1,\ldots,x_n$ obeying linear ODEs, with scalar u(t) (for now) $(m=1)$

$$\dot{x}_1 = a_{11} x_1(t) + a_{12} x_2(t) + \cdots + a_{1n} x_n(t) + b_1 u(t)$$

$$\vdots$$

$$\dot{x}_n = a_{n1} x_1(t) + a_{n2} x_2(t) + \cdots + a_{nn} x_n(t) + b_n u(t)$$

$$\Longleftrightarrow \quad \boxed{\dot{x} = Ax + Bu}$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{=\text{“A” } (n\times n \text{ state dyn matrix})} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x(t)\ (n\times 1 \text{ state vector})} + \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{=\text{“B” } (n\times m \text{ input-to-state matrix})} \underbrace{u(t)}_{m\times 1 \text{ input vector}}$$

Suppose also: only sensed outputs are:

$$y_1 = c_{11} x_1 + c_{12} x_2 + \cdots + c_{1n} x_n + d_1 u$$

$$\vdots$$

$$y_p = c_{p1} x_1 + c_{p2} x_2 + \cdots + c_{pn} x_2 + d_p u$$

(p = # of sensed outputs)

$$\Longleftrightarrow \quad \boxed{y(t) = Cx(t) + Du(t)}$$

where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ c_{p1} & \cdots & c_{pn} \end{bmatrix}}_{p\times n \text{ “C” output matrix}} x + \begin{bmatrix} d_1 \\ \vdots \\ d_p \end{bmatrix} u(t)$$

feed thru $p\times m$ “D” = matrix

General Linear State Space Models

Can generalize this formulation for *time-varying* or *time-invariant* dynamics:

- Linear Time Varying (LTV) State space model:

$$\left.\begin{aligned} \dot{x}(t) &= \underline{\underline{A(t)}}\, x(t) + \underline{\underline{B(t)}}\, u(t) \\ y(t) &= \underline{\underline{C(t)}}\, x(t) + \underline{\underline{D(t)}}\, u(t) \end{aligned}\right\}$$

matrix elements are fxn of time
(very powerful representation, but tricky to analyze/solve)

- Linear Time Invariant (LTI) SS models:

$$\left.\begin{aligned} \dot{x}(t) &= A\, x(t) + B u(t) \\ y(t) &= C\, x(t) + D\, u(t) \end{aligned}\right] \rightarrow [A, B, C, D] \text{ parameters are constant w.r.t. time}$$

$\rightarrow$ what if # inputs for u is ≥ 1? (i.e. m) $\rightarrow u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \rightarrow B \in \mathbb{R}^{n \times m} = \begin{bmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$