Statistical Estimation	**Homework 1**
ASEN 5044 Fall 2018	**Due Date: Sep 6, 2018**
Name: Andrew Kramer	**CU identitykey: ankr1041**

Exercise 1

Compute determinants for the following matrices by hand and state whether each one is invertible

Problem (a)

$$
\begin{aligned}
|A| &= 1\begin{vmatrix} 5 & 4 \\ 9 & 7 \end{vmatrix} - 2\begin{vmatrix} 6 & 4 \\ 8 & 7 \end{vmatrix} + 3\begin{vmatrix} 6 & 5 \\ 8 & 9 \end{vmatrix} \\
&= 1(-1) - 2(10) + 3(14) \\
&= 21
\end{aligned}
$$

Because $|A|$ is nonzero A is invertible.

Problem (b)

$$
\begin{aligned}
|A| &= 11\begin{vmatrix} 57 & 0 & 10 \\ 91 & 1 & 71 \\ 23 & 0 & 71 \end{vmatrix} - 26\begin{vmatrix} 64 & 0 & 10 \\ 83 & 1 & 71 \\ 54 & 0 & 71 \end{vmatrix} \\
&= 11(57(71) + 10(-23)) - 26(64(71) + 10(-54)) \\
&= 41978 - 109642 \\
&= -67655
\end{aligned}
$$

Because $|A|$ is nonzero A is invertible.

Problem (c)

Because $A_1 = -2A_3$ (where A_i refers to the i^{th} column of A), the columns of A are not linearly independent. This means A is not invertible and $|A| = 0$.

Problem (d)

Because the determinant of an upper triangular matrix is simply the product of its diagonal elements:

$$
\begin{aligned}
|A| &= 1 \times 8 \times 55 \times 233 \times 610 \\
&= 62537200
\end{aligned}
$$

Because $|A|$ is nonzero A is invertible.

Exercise 2

Prove each of the following statements:

Problem (a)

If a and b are non-zero $n \times 1$ vectors, then the matrix ab^T has rank 1.

Column i of the outer product of a and b is simply the vector a multiplied by the scalar b_i. This means that every column of ab^T is a scalar multiple of a, so none of the columns of ab^T are linearly independent. Thus, the rank of ab^T is always one if both a and b are nonzero.

Problem (b)

$\text{tr}(AB) = \text{tr}(BA)$ if A is an $m \times n$ matrix and B is $n \times m$.

The trace of AB can be expressed as

$$\begin{aligned}\text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}\end{aligned}$$

Similarly, the trace of BA is

$$\begin{aligned}\text{tr}(BA) &= \sum_{j=1}^{n} (BA)_{jj} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}\end{aligned}$$

Because $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$ is equal to $\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$ we can conclude that $\text{tr}(AB) = \text{tr}(BA)$.

Problem (c)

If A is invertible then $|A^{-1}| = \frac{1}{|A|}$.

If we start with $|AB| = |BA| = |A|\ |B|$ and replace B with A^{-1} we find that

$$\begin{aligned}|A|\ |A^{-1}| &= |AA^{-1}| \\ &= |I| \\ &= 1\end{aligned}$$

Because $|A|\ |A^{-1}| = 1$ it must be true that $|A^{-1}| = \frac{1}{|A|}$.

1 Problem 3

Consider the equations of motion for the coupled 2 mass 3 spring system like the one discussed in lecture. Find a set of A, B, C, D matrices for the state vector definition,

$$x = [q_1 - q_2,\ \dot{q}_1 - \dot{q}_2,\ q_1 + q_2,\ \dot{q}_1 + \dot{q}_2]^T$$

and for observations $y = [q_1,\ q_2]^T$ and inputs $u = [u_1,\ u_2]^T$.

The following solution makes the assumption, as was done in lecture, that $m_1 = m_2 = 1$kg and $k_1 = k_2 = k_3 = 1$N/m.

$$
\begin{aligned}
\dot{x} &= [\dot{q}_1 - \dot{q}_2,\ \ddot{q}_1 - \ddot{q}_2,\ \dot{q}_1 + \dot{q}_2,\ \ddot{q} + \ddot{q}]^T \\
\ddot{q}_1 &= -q_1 - q_1 + q_2 - u_1 \\
&= -2q_1 + q_2 - u_1 \\
\ddot{q}_2 &= -q_2 + q_1 - q_2 + u_1 + u_2 \\
&= -2q_2 + q_1 + u_1 + u_2 \\
\ddot{q}_1 - \ddot{q}_2 &= (-2q_1 + q_2 - u_1) - (q_1 - 2q_2 + u_1 + u_2) \\
&= -3q_1 + 3q_2 - 2u_1 - u_2 \\
&= -3(q_1 - q_2) - 2u_1 - u_2 \\
\ddot{q}_1 + \ddot{q}_2 &= (-2q_1 + q_2 - u_1) + (q_1 - 2q_2 + u_1 + u_2) \\
&= -q_1 - q_2 + u_2 \\
&= -(q_1 + q_2) + u_2
\end{aligned}
$$

From these results we can construct our A, B, C, and D matrices as follows:

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad
B = \begin{bmatrix} 0 & 0 \\ -2 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad
C = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix} \quad
D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$