ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 7:
DT LTI Observability and Deterministic State Estimation

Prof. Nisar Ahmed (Nisar.Ahmed@Colorado.edu)

Tues 9/18/2018

University of Colorado Boulder

Aerospace Engineering

Announcements

- **HW 2 Thurs 9/21 at 11 am (before start of next lecture)**
- Submit to Canvas –
  - All submissions must be legible!!! – zero credit otherwise
  - All submissions must have your name on them!!! – zero credit otherwise
  - **SUBMIT ONLY PDFs!! (NO JPEGS, PNGs, BMPs, SVGs, GIFs, etc.)**
  - **ALL PAGES MUST BE SEPARATED WITHIN ONE SINGLE FILE!!! (do not submit separate files and do not mash all pages into one sheet…)**
  - Indicate PhD advanced question/extra credit per file naming instructions on Canvas and on first doc page

Last Time…

- Conversion of Continuous Time (CT) LTI SS → DT LTI SS
- Computing the DT G matrix
- Nyquist sampling rate

Today…

- Stability of CT/DT linear systems

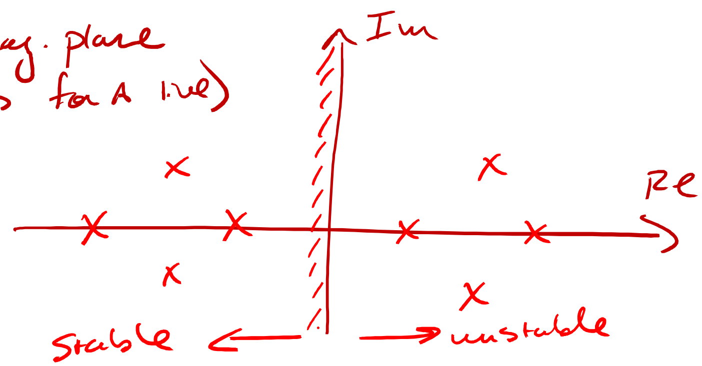
- Observability of DT linear systems and deterministic state estimation: how to find x(k=0) from *some* finite sequence of measurements y(0),y(1),…,y(K)?

READ: Chapter 2.1-2.2 in Simon book (probability)

Asymptotic Stability for CT LTI Systems

- Necessary & sufficient condition for **CT LTI asymptotic stability: e'vals of A matrix must all lie strictly in the left half plane** (e'vals must have strictly negative real part)

look @ imag. plane (where e'vals for A live)



$$eig(A) = \{\lambda_i\}_{i=1}^{n} \quad [n = \#\ \text{states}]$$

$$\lambda_i = \underset{Re}{\sigma_i} \pm j\,\underset{Im}{\omega_i}, \quad i = 1, \ldots, n$$

Intuition: if A were diagonalizable & $\dot{x} = Ax$, $x(0) = x_0$ & $u(t) = 0 \ \forall t$

$$\longrightarrow x(t) = e^{At} x(0) = \sum_{i=1}^{n} \alpha_i v_i \cdot e^{\lambda_i t}$$

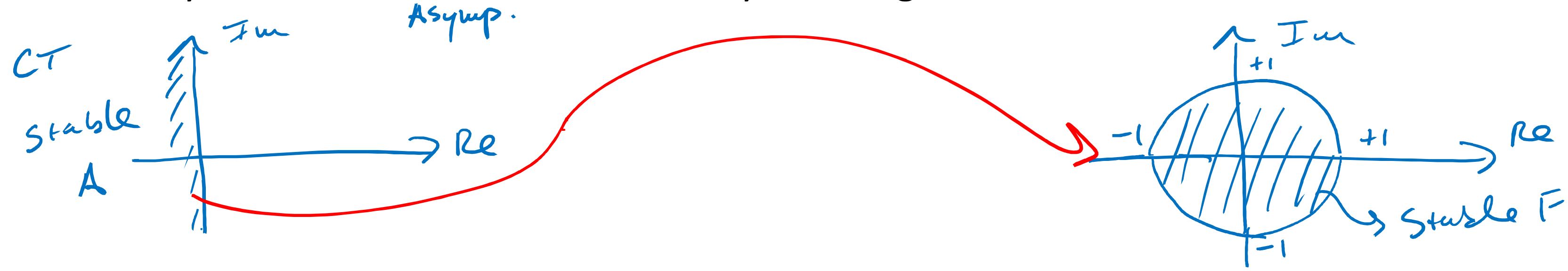
where α_i = some const. (depending on x_0)

v_i = e'vecs of A

(can be generalized to non diag. A case...)

Asymptotic Stability for DT LTI Systems

- FACT: DT linear systems are ~~**BIBO**~~ Asymp. **stable** if and only if **the eigenvalues of the F matrix lie in the unit circle**



- Idea: x(k+1) = F*x(k) only "settles down" if F does not force magnitude of x(k) to grow as k→ infinity
- Simple example: consider if F is diagonal – each iteration through k scales elements of the state vectors:

$$F_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \text{ if } x(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{aligned} x(k+1) &= F\,x(k) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ x(k+2) &= F\,x(k+1) = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \\ x(k+3) &= F\,x(k+2) = \begin{bmatrix} 8 \\ 18 \end{bmatrix} \end{aligned}$$

→ as $k \to \infty$, $x(k)$ blows up! [not asymp. stable]

$$F_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \longrightarrow x(k+1) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

$$x(k+2) = \begin{bmatrix} 0.04 \\ 0.09 \end{bmatrix}$$

$\vdots$

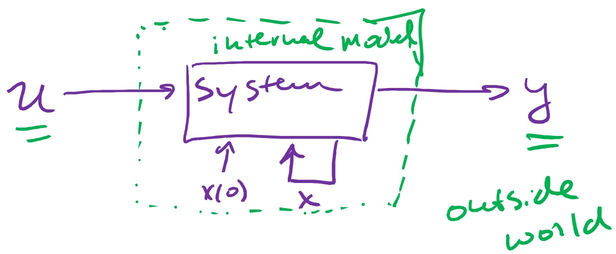
eventually $x(k) \to 0$ as $k \to \infty$ [asymp. stable]

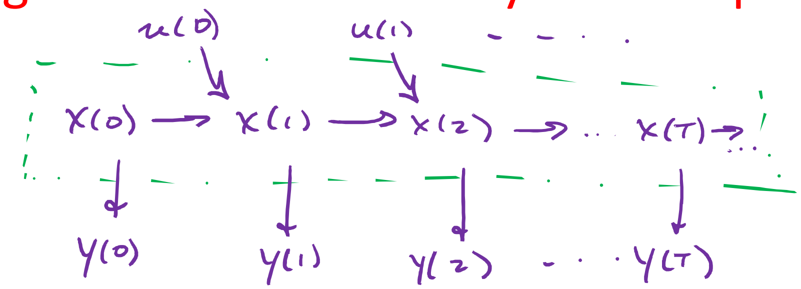
Deterministic DT LTI State Estimation

How to recover sequence of states that generated observed system outputs?

If $x(k+1) = F\,x(k) + G\,u(k)$

$y(k) = H\,x(k)$





$\Rightarrow$ We'd like to know which sequence of $x(k)$'s explains the observations $y(k)$ [hopefully/ideally: sequence of $x(k)$ is unique!]

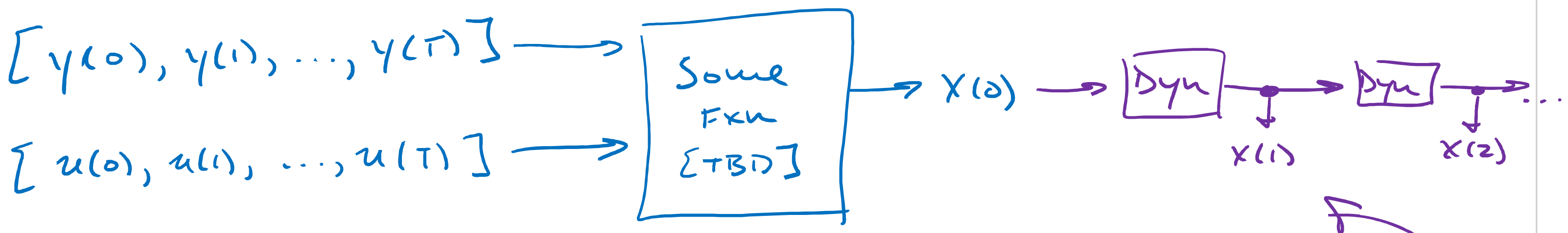
In practice, we must deal with some complications:

- model errors, nonlinearities
- sensor noise and state disturbances
- ability of states to "reveal" themselves via measurements and dynamics

- For now, assume LTI models and ignore random noise/state/model errors…
- When actually possible to uniquely compute internal state sequence?

Observability: DT Definition

- A system is **observable** if for any initial state x(0) and *some* final time T, the initial state x(0) can be **uniquely determined** from knowledge of u(k) and y(k) alone for k=0,1,…,T.



- Key idea: is it **always** possible to **perfectly reconstruct internal states** x from only inputs u and sensed outputs y over some finite time interval? (i.e. can there ever be enough info to “invert” the state space model?)

- Since state x(k) at any time k is initial condition to x(k+1), suffices to examine whether possible to recover any arbitrary x(0) for deterministic state estimation

DT Observability: Another View

Suppose $u(k) = 0$ for all $k \geq 0$ (ignore inputs for now).

A state $x(0) = x$ is *unobservable* for the system (F, H) if

$$y(k) = HF^k x(0) = HF^k x = 0 \text{ for every } k \geq 0.$$

Let $R_{\bar{o}}$ = set of all unobservable states x = *unobservable subspace* of (F, H).

System (F, H) is *observable* if $x = 0$ is only unobservable state, i.e. if $R_{\bar{o}} = \{0\}$.

3 big questions:

- How do we know if a given (F, H) is observable?
- Once we know this, how do we recover $x(0)$? [from $y(0), \ldots, y(T), \ldots$]
- If (F, H) is unobservable, how to know what $R_{\bar{o}}$ is?

Assessing DT Observability

- Consider zero input case for finding x(0) from y(0),…,y(n-1) [n sequential px1 measurements]

Start w/ $\boxed{y(0) = H\,x(0)}$ & fact that $x(k+1) = F\,x(k)$ & $x(0) \in \mathbb{R}^{n\times 1}$

$y(k) \in \mathbb{R}^{p\times 1}$

→ Since $x(k=1) = F\,x(0)$, we have that

$$\boxed{y(1) = H\cdot x(1) = H\cdot F\,x(0)}$$

Likewise: $y(2) = H\,x(2) = H\cdot F\,x(1)$ [from dynamics]

but we also know $x(1) = F\,x(0)$, so it follows that

$$\boxed{y(2) = H F\,x(1) = HF\cdot F\,x(0) = HF^2\,x(0)}$$

→ using similar reasoning: easy to show that

$$\boxed{y(3) = HF^3\,x(0)}$$

$$\vdots$$

$$\boxed{y(n-1) = HF^{n-1}\,x(0)}$$

re-arrange …

Solution to Deterministic State Estimation Problem

- Now stack up all the eqs. for $y(0), y(1), \ldots, y(n-1)$:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} H\,x(0) \\ HF\,x(0) \\ \vdots \\ HF^{n-1}\,x(0) \end{bmatrix} \longrightarrow \mathcal{Y} = \underbrace{\begin{bmatrix} y(0) \\ \vdots \\ y(n-1) \end{bmatrix}}_{(n\cdot p)\times 1} = \underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}}_{(n\cdot p)\times n} \underbrace{x(0)}_{n\times 1}$$

→ looks like an overdetermined sys. of linear eqs! (more eqs. than unknowns → see Lecture 2)

→ Rewrite as: $\mathcal{Y} = \mathcal{O}\, x(0)$, where $\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \in \mathbb{R}^{p\cdot n \times n}$

→ Recall: that unique solution to $x(0)$ is obtainable if $\text{rank}(\mathcal{O}) = n$ [full col rank]

→ If this the case, then $\boxed{\underbrace{x(0)}_{n\times 1} = \underbrace{(\mathcal{O}^T\mathcal{O})^{-1}}_{n\times n}\, \underbrace{\mathcal{O}^T}_{n\times np}\, \underbrace{\mathcal{Y}}_{np\times 1}}$ ✓

→ If $\text{rank}\,\mathcal{O} < n$, then $\text{Null}(\mathcal{O})$ is non-trivial → $\text{Null}(\mathcal{O}) = \bar{R}$ = (unobservable subspace)

LTI Observability Matrix

- A test for **LTI observability**: examine rank of **observability matrix**:

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} = \text{DT LTI obsv matrix for system } (\underline{\underline{F}}, G, \underline{\underline{H}}, M)$$

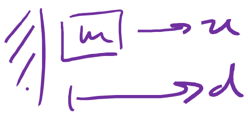
$$x(k+1) = Fx(k) + Gu(k)$$
$$y(k) = Hx(k) + Mu(k)$$

- If $\text{rank}(\mathcal{O}) = n$ [# states], then (F,H) is fully state observable & $\mathcal{Z}_{\mathcal{O}} = \{0\}$
- If $\text{rank}(\mathcal{O}) < n$, then (F,H) is <u>NOT</u> fully observable & $\mathcal{Z}_{\mathcal{O}} = \text{Null}\,\mathcal{O}$

- If the system (F,H) is observable, then it is possible to solve for any x(0) using <u>at most</u> n vector measurements y(0), y(1),…,y(n-1)
- Corollary: if (F,H) observable, then possible to solve for any x(k) by either solving for initial condition x(0) and propagating solution forward, or by using k as initial time and measurements up to no more than y(k+n-1).

Example 1: 1D Block Mass

- For DT LTI model, we have $x(k) = \begin{bmatrix} v(k) \\ d(k) \end{bmatrix} = \begin{bmatrix} \text{velocity @ time } k \\ \text{pos'n " " "} \end{bmatrix}$



→ Convert from CT to DT: (for some fixed Δt sample time)

$$x(k+1) = F\,x(k) + G\,u(k)$$
$$y(k) = H\,x(k)$$

$, \; F = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} = \text{STM}$ (see Lec 4)

→ Suppose $u(k) = 0 \;\; \forall k \geq 0$ & we have

$H = \begin{bmatrix} 0 & 1 \end{bmatrix}$ (pos'n sensor) $\longrightarrow \mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta t & 1 \end{bmatrix}$

→ $\text{rank}(\mathcal{O}) = 2$ → sys. is obsv!

→ Now suppose $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (vel. sensor) $\longrightarrow \mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

→ $\text{rank}(\mathcal{O}) = 1$ → sys. not obsv!