ASEN5044 Assignment 3

Solutions

24 September 2018

1. (a) Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad \mathbf{D} = \mathbf{0}$$

and $\Delta t = 0.05$ s, define

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

so that

$$\begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \exp \hat{\mathbf{A}} \Delta t$$

Hence,

$$\mathbf{F} = \begin{bmatrix} 0.9975 & 0.05 & 0.0012 & 0 \\ -0.0999 & 0.9975 & 0.0499 & 0.0012 \\ 0.0012 & 0 & 0.9975 & 0.05 \\ 0.0499 & 0.0012 & -0.0999 & 0.9975 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 0.0012 & 0 \\ -0.0499 & 0 \\ 0.0012 & 0.0012 \\ 0.0499 & 0.0500 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad \mathbf{M} = \mathbf{0}$$

The Nyquist limit is given by

$$\omega_N = 2\omega_{max}$$

The eigenvalues of $\mathbf{A}$ are

$$\lambda = \sigma + j\omega = \{\pm 1.7321j, \pm j\}$$

Thus,

$$\omega_N = 2 \cdot 1.7321 \text{ rad/s} = 3.4641 \text{ rad/s}$$

The corresponding sampling time is then

$$T_N = \frac{2\pi}{\omega_N} = 1.814 \text{ s} \gg \Delta t = 0.05 \text{ s}$$

(b) First compute the observability matrix

$$\mathbb{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \\ \mathbf{HF}^3 \end{bmatrix}$$

It can be shown (e.g. using MATLAB's `rank` function, or by Gaussian elimination) that rank $\mathbf{O} = 4 = n$, and thus the system is observable.

(c) The discretized linear state-space equations are given by

$$\begin{aligned} \mathbf{x}_k &= \mathbf{F}\mathbf{x}_{k-1} + \mathbf{G}\mathbf{u}_{k-1} \\ \mathbf{y}_k &= \mathbf{H}\mathbf{x}_k \end{aligned}$$

Observe that

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{H}\left(\mathbf{F}\mathbf{x}_0 + \mathbf{G}\mathbf{u}_0\right) \\ \mathbf{y}_1 - \mathbf{HG}\mathbf{u}_0 &= \mathbf{HF}\mathbf{x}_0 \end{aligned}$$

And similarly,

$$\begin{aligned} \mathbf{y}_2 - \mathbf{HG}\mathbf{u}_1 &= \mathbf{HF}\mathbf{x}_1 \\ &= \mathbf{HF}\left(\mathbf{F}\mathbf{x}_0 + \mathbf{G}\mathbf{u}_0\right) \\ \mathbf{y}_2 - \mathbf{HG}\mathbf{u}_1 - \mathbf{HFG}\mathbf{u}_0 &= \mathbf{HF}^2\mathbf{x}_0 \end{aligned}$$

It can be shown that for any k,

$$\mathbf{y}_k - \sum_{i=0}^{k-1} \mathbf{HF}^i\mathbf{G}\mathbf{u}_{k-1-i} = \mathbf{HF}^k\mathbf{x}_0$$

Stacking for $k = 1 \ldots N$ gives

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 - \mathbf{HG}\mathbf{u}_0 \\ \vdots \\ \mathbf{y}_N - \sum_{i=0}^{N-1} \mathbf{HF}^i\mathbf{G}\mathbf{u}_{N-1-i} \end{bmatrix}}_{\triangleq \mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{HF} \\ \vdots \\ \mathbf{HF}^N \end{bmatrix}}_{\triangleq \mathbf{L}} \mathbf{x}_0$$

The left pseudo-inverse solution is then given by

$$\hat{\mathbf{x}}_0 = \left(\mathbf{L}^T\mathbf{L}\right)^{-1}\mathbf{L}^T\mathbf{y}$$

(d) Using the formula derived in Part 1c, the initial state is estimated to be

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} 0.1 \\ 0.3 \\ -0.33 \\ -0.86 \end{bmatrix}$$

Plots of the state, predicted observations, and observation error vs. time are shown in Figures 1 to 3. Note that the differences between the predicted output and true output are on the order of machine precision.

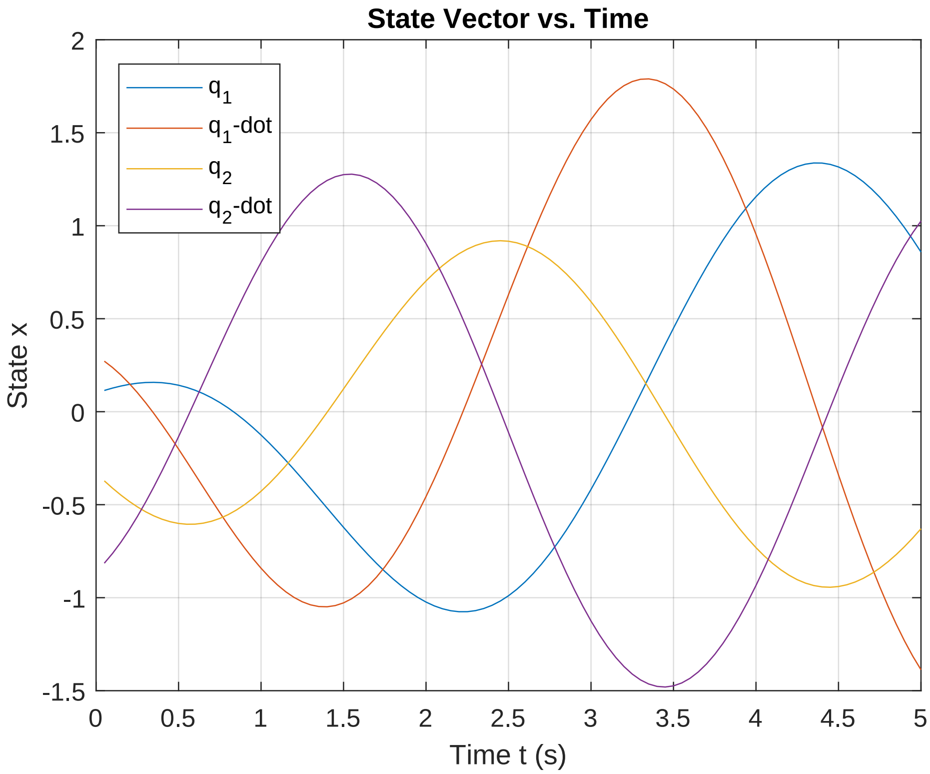

Figure 1: Propagated State Vector vs. Time

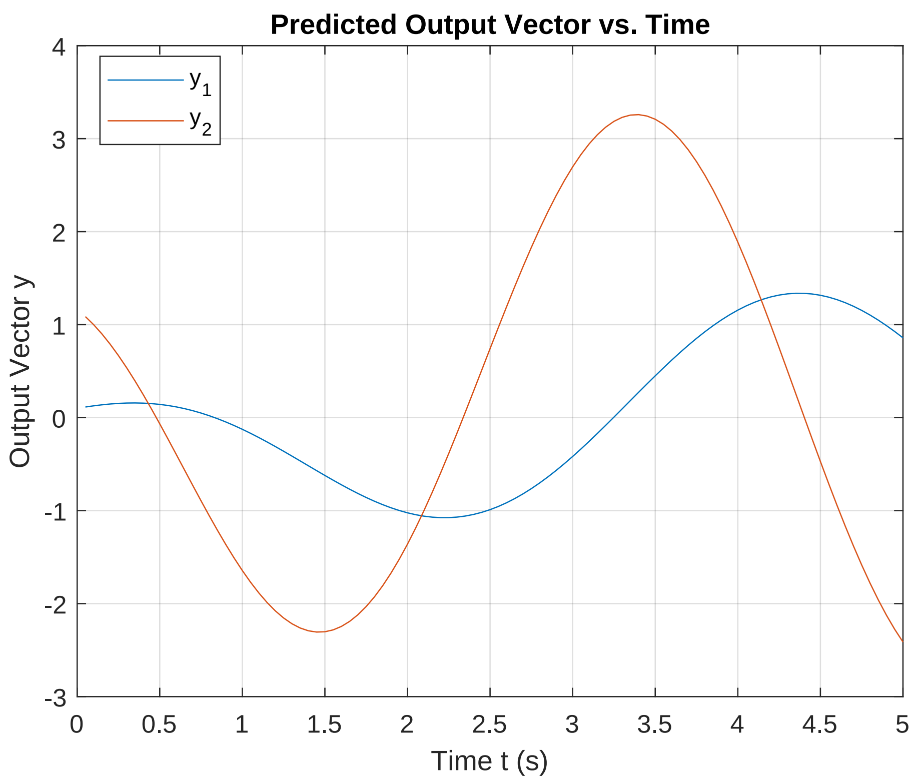

Figure 2: Predicted Output Vector vs. Time

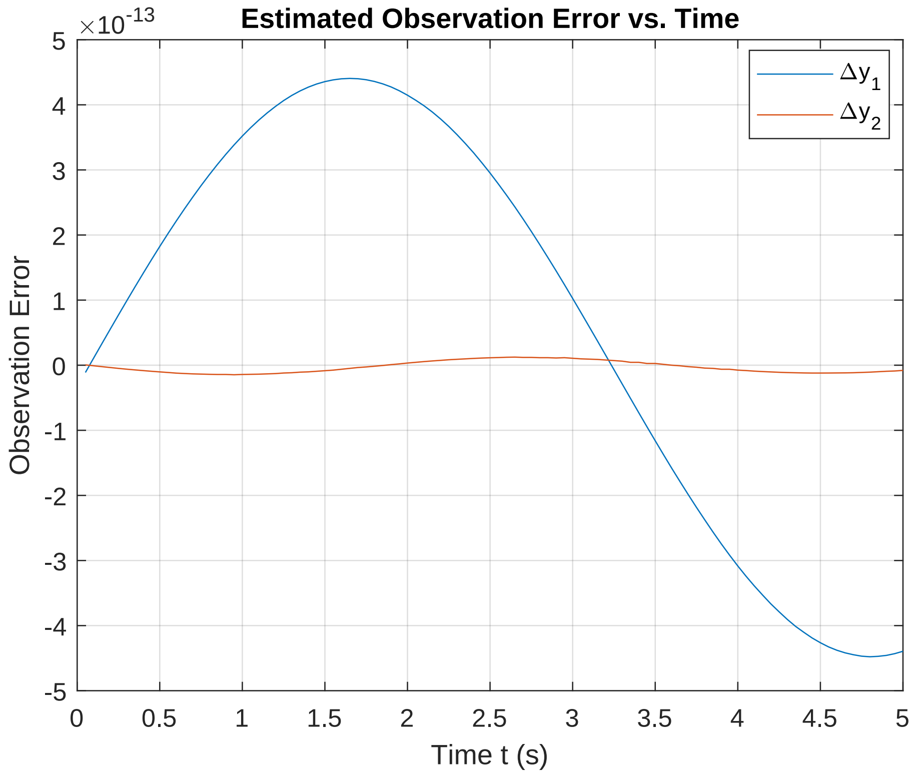

Figure 3: Difference Between Predicted and Measured Output vs. Time

(e) Observe that for $N = 2$ observations, the formula derived in Part 1c reduces to

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 - \mathbf{H}\mathbf{G}\mathbf{u}_0 \\ \mathbf{y}_2 - \mathbf{H}\mathbf{G}\mathbf{u}_0 - \mathbf{H}\mathbf{F}\mathbf{G}\mathbf{u}_1 \end{bmatrix}}_{\triangleq \mathbf{y}'} = \underbrace{\begin{bmatrix} \mathbf{H}\mathbf{F} \\ \mathbf{H}\mathbf{F}^2 \end{bmatrix}}_{\triangleq \mathbf{L}'} \mathbf{x}_0$$

$\mathbf{L}'$ is full rank and can be inverted directly to solve for $\mathbf{x}_0$, and therefore the minimum number of required observations in this case is $N = 2$. This is consistent with the observability analysis performed earlier. Because $\mathbf{L}'$ is full rank and the rows of $\mathbf{L}'$ are identical to the four middle rows of the observability matrix $\mathbb{O}$, $\mathbb{O}$ must therefore also be full rank. Conversely, the observability analysis found that the observability matrix has full column rank, which implies that the observability Grammian $\mathbb{O}^T\mathbb{O}$ is invertible. Since the observability Grammian is 4×4, this implies that at least 4 independent equations are needed.

With

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

the minimum number of observations needed is $N = 4$. The reason for this is that although this modified observation model provides an additional output at each time step, the three outputs are redundant, and therefore every three rows of the observation matrix are identical and only increase the rank of the observation matrix by 1. Alternatively, since this new observation model has only one independent output, four observations are needed in order to generate four independent equations to solve for the four states.

(f) Using only the first row of the original output, the observation matrix becomes

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing the resulting observability matrix reveals that rank $\mathbb{O} = 4$, and therefore the system is still observable.

Using only the second row of the original output results in an observation matrix of the form

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}$$

Analyzing the resulting observability matrix reveals that rank $\mathbb{O} = 2$, and therefore the system is not fully observable.

The eigenvectors of the system corresponding to the first eigenvalue pair $\lambda_{1,2} = \pm j$ are

$$\mathbf{v}_{1,2} = \left\{ \begin{bmatrix} -j \\ 1 \\ -j \\ 1 \end{bmatrix}, \begin{bmatrix} j \\ 1 \\ j \\ 1 \end{bmatrix} \right\}$$

For some constant c,

$$c\left(\mathbf{v}_1 + \mathbf{v}_2\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Modes 1 and 2 therefore correspond to an in-phase oscillation of the two masses.

The eigenvectors of the system corresponding to the second eigenvalue pair $\lambda_{3,4} = \pm 1.7321j$ are

$$\mathbf{v}_{1,2} = \left\{ \begin{bmatrix} 0.3536j \\ -0.6124 \\ -0.3536j \\ 0.6124 \end{bmatrix}, \begin{bmatrix} -0.3536j \\ -0.6124 \\ 0.3536j \\ 0.6124 \end{bmatrix} \right\}$$

For some constant d,

$$d\left(\mathbf{v}_3 + \mathbf{v}_4\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Modes 3 and 4 therefore correspond to an out-of-phase oscillation of the two masses.

Plots of the modal responses are shown in Figures 4 and 5.

Using only the first output, the first two modes are unobservable because the contribution of these two modes to the state are annihilated in the output. Therefore, the output of the system is independent of these modes, resulting in a family of

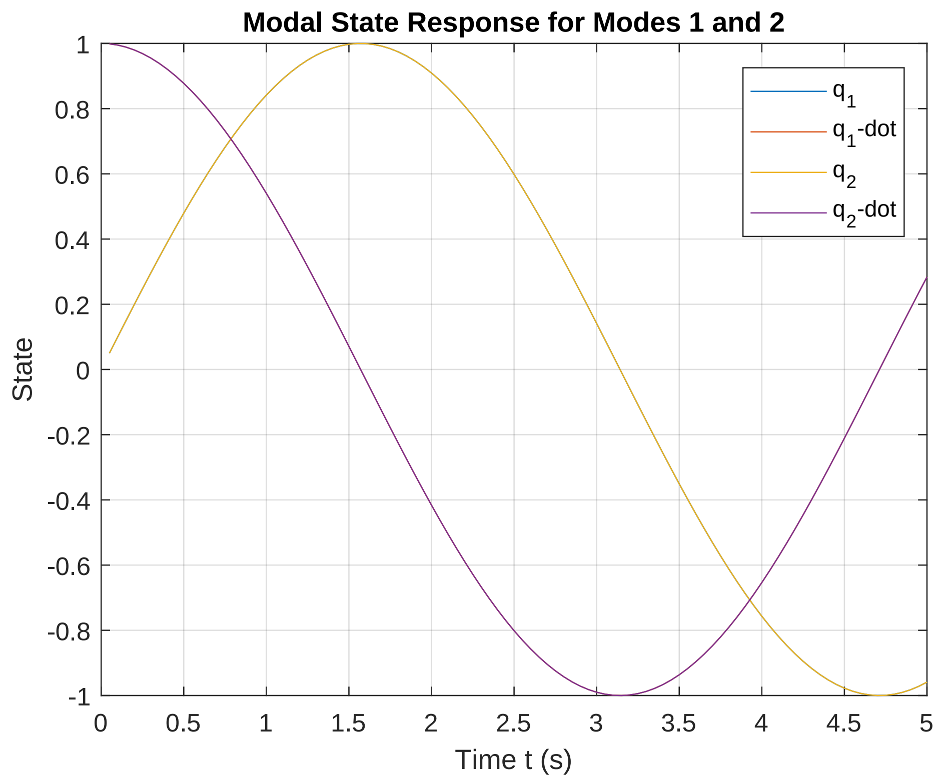

Figure 4: Modal State Response, Modes 1 and 2

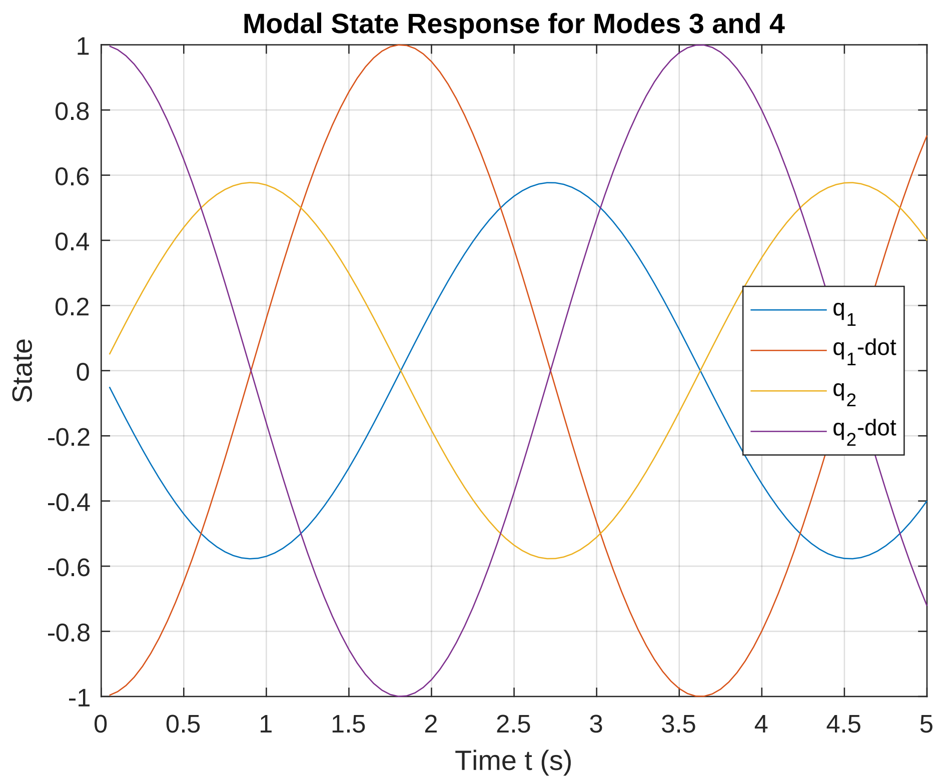

Figure 5: Modal State Response, Modes 3 and 4

state responses that produce identical outputs. Modes 1 and 2 are therefore unobservable modes of the system.

Alternatively, the eigenvectors can be used to transform the system into modal coordinates. Define $\hat{\mathbf{F}}$ as the diagonalized $\mathbf{F}$ matrix and $\mathbf{V}$ is a matrix whose columns are the eigenvectors of $\mathbf{F}$ so that

$$\hat{\mathbf{F}} = \mathbf{V}^{-1}\mathbf{F}\mathbf{V}$$

Now define $\hat{\mathbf{x}} \triangleq \mathbf{V}^{-1}\mathbf{x}$ and observe that

$$\begin{aligned} \mathbf{x} &= \mathbf{V}\hat{\mathbf{x}} \\ \dot{\mathbf{x}} &= \mathbf{V}\dot{\hat{\mathbf{x}}} \end{aligned}$$

Substituting these into the original discrete time state space model gives

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \underbrace{\mathbf{V}^{-1}\mathbf{F}\mathbf{V}}_{\hat{\mathbf{F}}}\hat{\mathbf{x}}_k + \underbrace{\mathbf{V}^{-1}\mathbf{G}}_{\hat{\mathbf{G}}}\mathbf{u}_k \\ \mathbf{y}_k &= \underbrace{\mathbf{H}\mathbf{V}}_{\hat{\mathbf{H}}}\hat{\mathbf{x}}_k \end{aligned}$$

The components of $\hat{\mathbf{x}}$ are then the modal coordinates of the system, and $\hat{\mathbf{F}}$, $\hat{\mathbf{G}}$, and $\hat{\mathbf{H}}$ are the modal system matrices. Now, observe that the modal $\hat{\mathbf{H}}$ for the case where only the second row of the output is used is given by

$$\hat{\mathbf{H}} = \begin{bmatrix} 0 & 0 & -1.2247 & -1.2247 \end{bmatrix}$$

Notice that the first two columns are zeros, indicating that the output is independent of the first two modes.

2. (a) Using the provided formulas for c_{k+1}, i_{k+1}, and y_k,

$$\begin{aligned} c_{k+1} &= \alpha y_k = \alpha\left(c_k + i_k + g_k\right) = \alpha\left(c_k + i_k\right) + \alpha g_k \\ i_{k+1} &= = \beta\left(c_{k+1} - c_k\right) = \beta\left[\alpha\left(c_k + i_k + g_k\right) - c_k\right] \\ &= \beta(\alpha - 1)c_k + \beta\alpha i_k + \beta\alpha g_k \\ y_k &= c_k + i_k + g_k \end{aligned}$$

Arranging these into matrix form gives

$$\begin{aligned} \begin{bmatrix} c_{k+1} \\ i_{k+1} \end{bmatrix} &= \begin{bmatrix} \alpha & \alpha \\ \beta(\alpha - 1) & \beta\alpha \end{bmatrix} \begin{bmatrix} c_k \\ i_k \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta\alpha \end{bmatrix} g_k \\ y_k &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} c_k \\ i_k \end{bmatrix} + g_k \end{aligned}$$

Substituting the provided definitions of $x_{1,k}$, $x_{2,k}$, and u_k provides

$$\begin{aligned} \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} &= \underbrace{\begin{bmatrix} \alpha & \alpha \\ \beta(\alpha - 1) & \beta\alpha \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}}_{\mathbf{x}_k} + \underbrace{\begin{bmatrix} \alpha \\ \beta\alpha \end{bmatrix}}_{\mathbf{G}} u_k \\ y_k &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + u_k \end{aligned}$$

(b) The eigenvalues and the rank of the observability matrix for each of the cases is shown in Table 1

Table 1: Eigenvalues and Observability Matrix Rank for Each Parameter Case

(α, β)	Eigenvalues	Magnitude of Eigenvalues	rank $\mathbb{O}$
(0.75, 1)	$0.75 \pm 0.433j$	0.866	2
(0.75, 1.5)	$0.9375 \pm 0.4961j$	1.0607	2
(1.25, 1)	$\{1.8090, 0.6910\}$	$\{1.8090, 0.6910\}$	2

The case $(\alpha, \beta) = (0.75, 1)$ is stable since the eigenvalues lie within the unit circle. The two other cases are unstable. All cases are fully observable, as indicated by the rank of the observability matrices.

Plots of the unit step response for each case are shown in Figures 6 to 8.

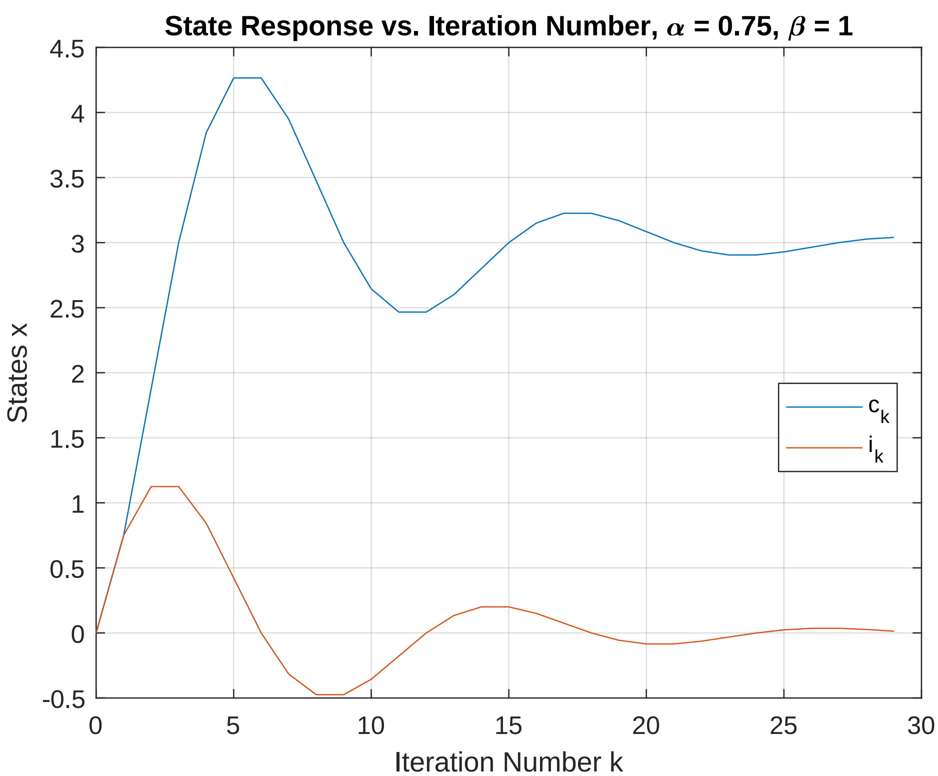

Figure 6: Unit Step Response, $\alpha = 0.75, \beta = 1$

(c) Figures 9 to 11 show the response of the system for $\mathbf{x}_0 = \begin{bmatrix} 5 & 1 \end{bmatrix}^T$. Once again, the responses demonstrate that only $(\alpha, \beta) = (0.75, 1)$ is stable.

(d) Plots of the observations for each case are shown in Figures 12 to 14. Errors between observations generated using the true initial state and those generated using the least-squares estimate of the initial state are shown in Figures 15 to 17. Note that the observation error for the final case is more significant than others

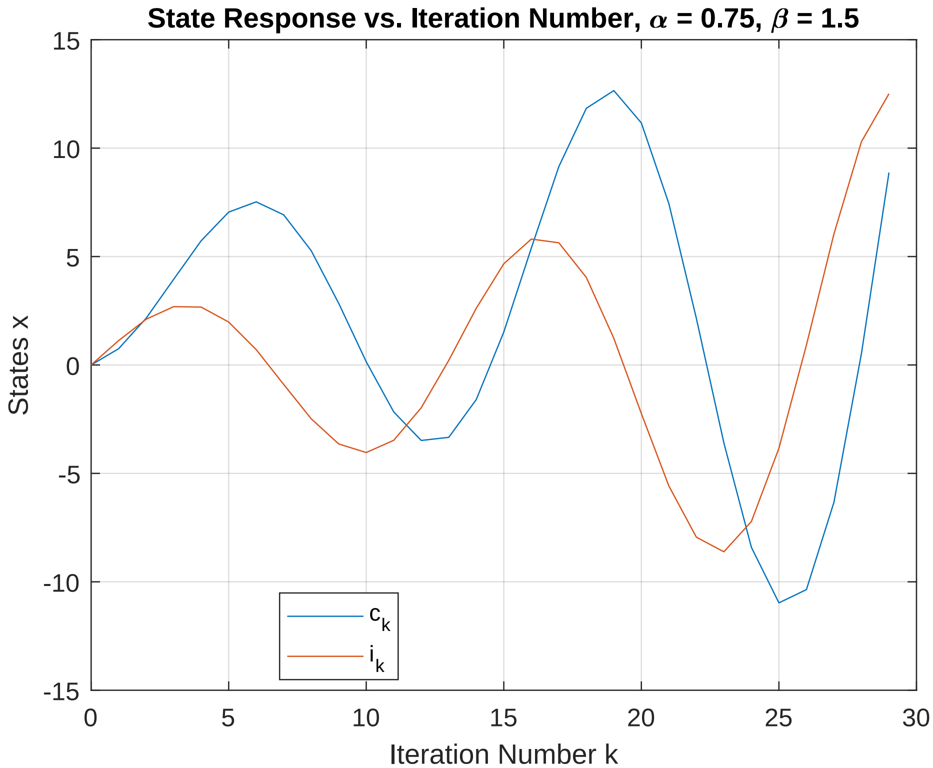

Figure 7: Unit Step Response, $\alpha = 0.75, \beta = 1.5$

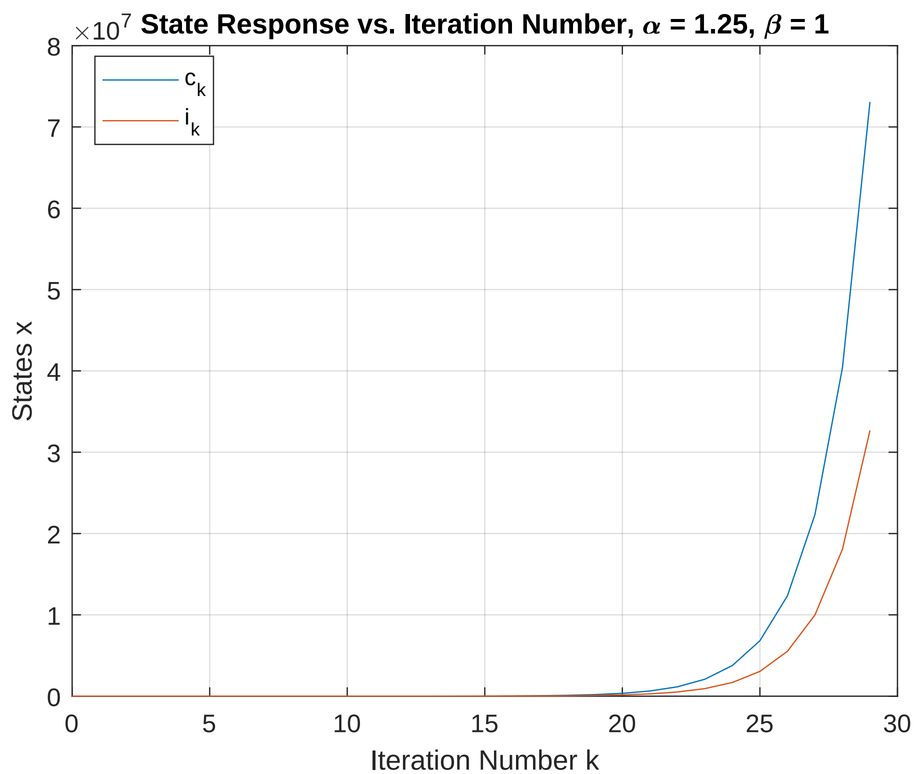

Figure 8: Unit Step Response, $\alpha = 1.25, \beta = 1$

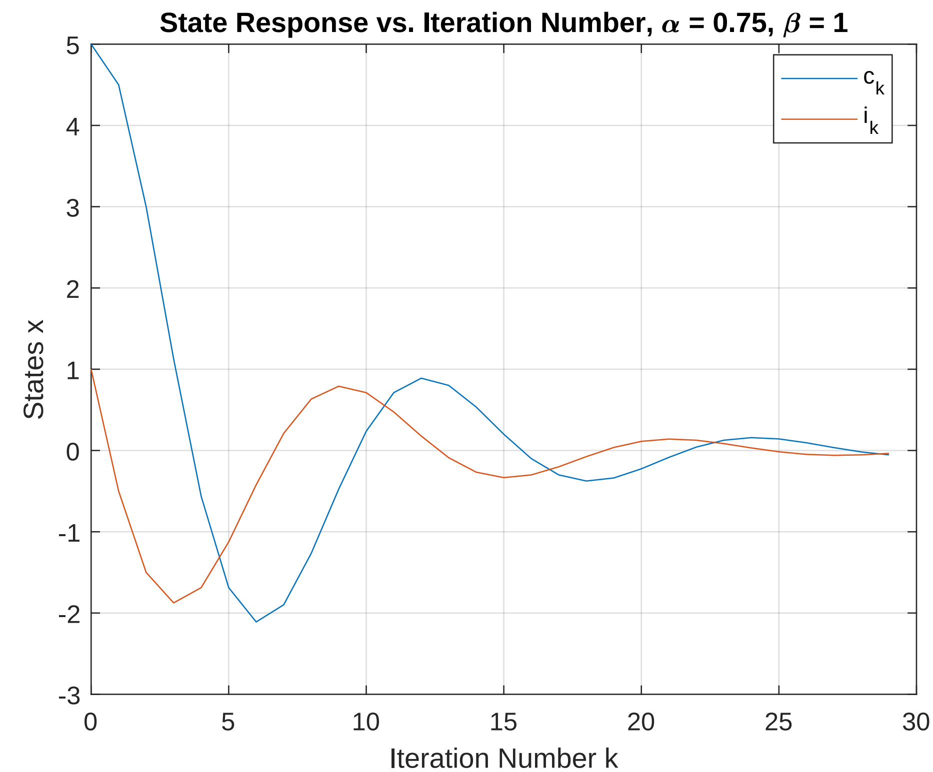

Figure 9: Initial Condition Response, $\alpha = 0.75, \beta = 1$

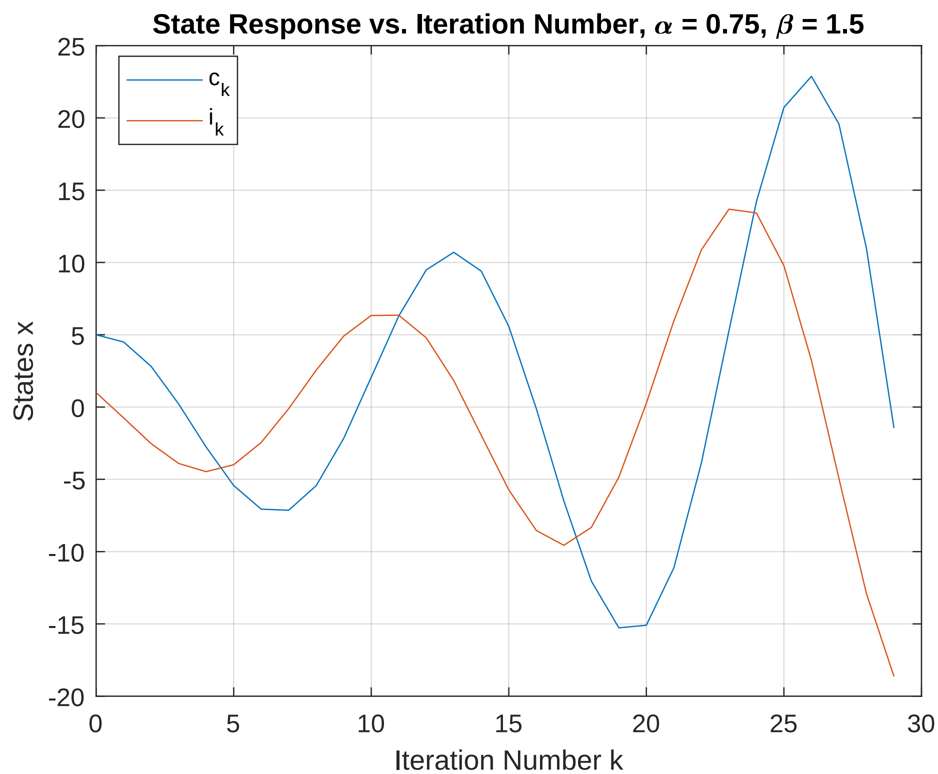

Figure 10: Initial Condition Response, $\alpha = 0.75, \beta = 1.5$

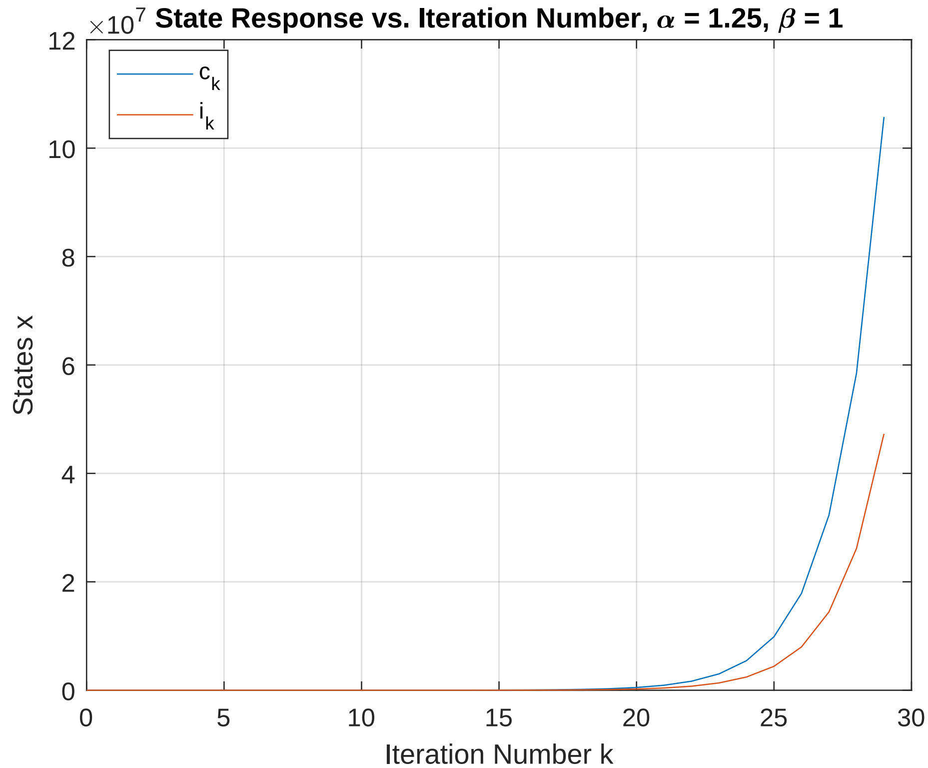

Figure 11: Initial Condition Response, $\alpha = 1.25, \beta = 1$

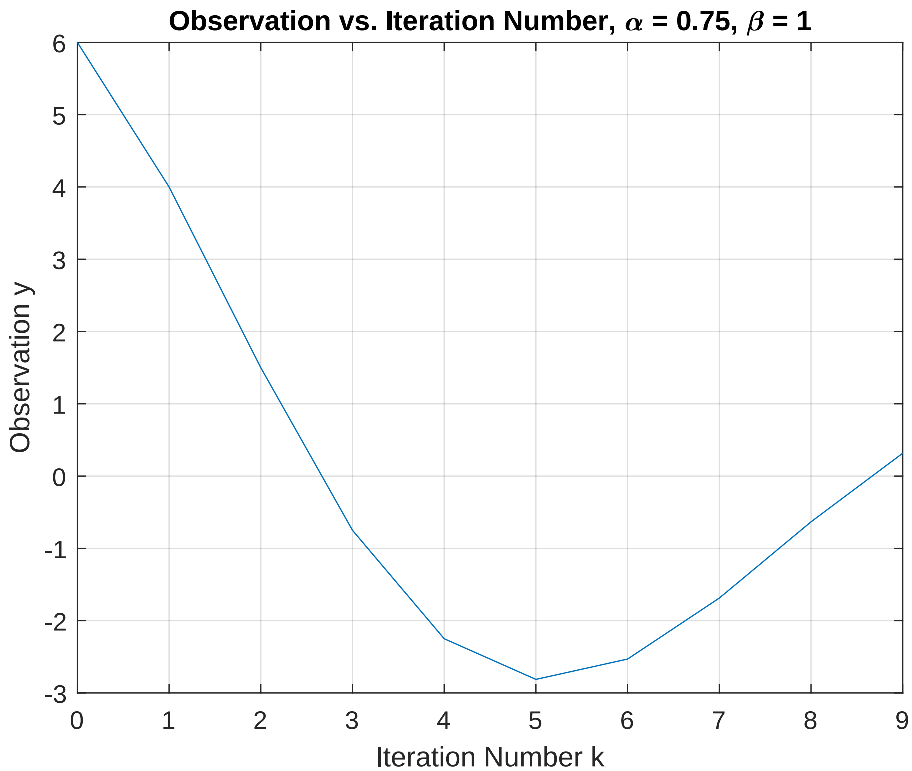

Figure 12: Observation vs. Time, $\alpha = 0.75, \beta = 1$

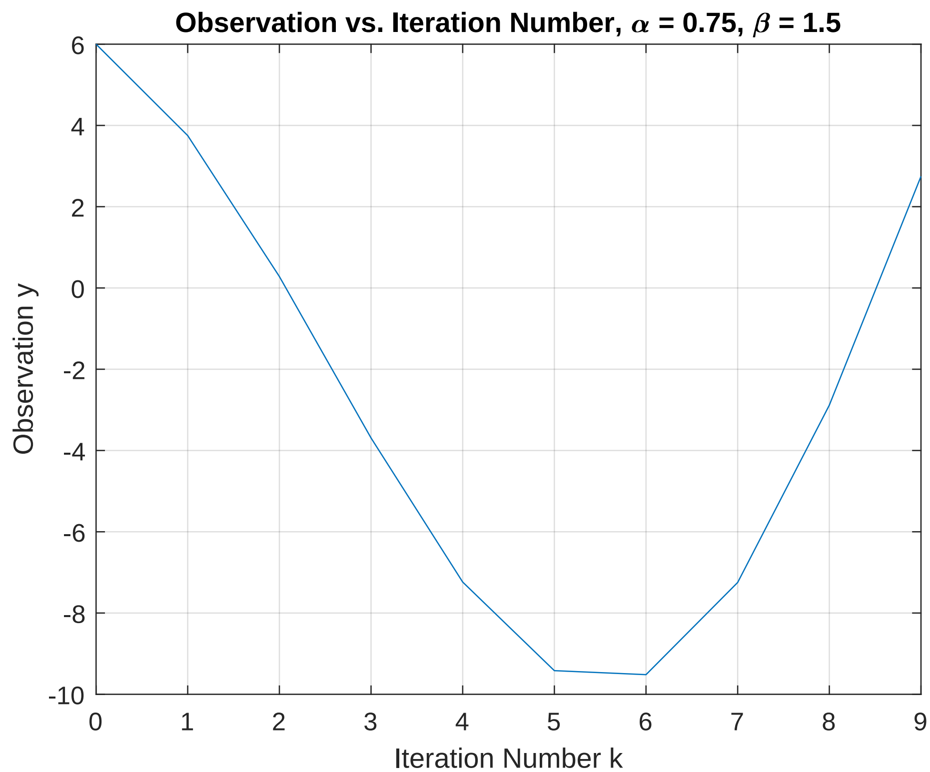

Figure 13: Observation vs. Time, $\alpha = 0.75, \beta = 1.5$

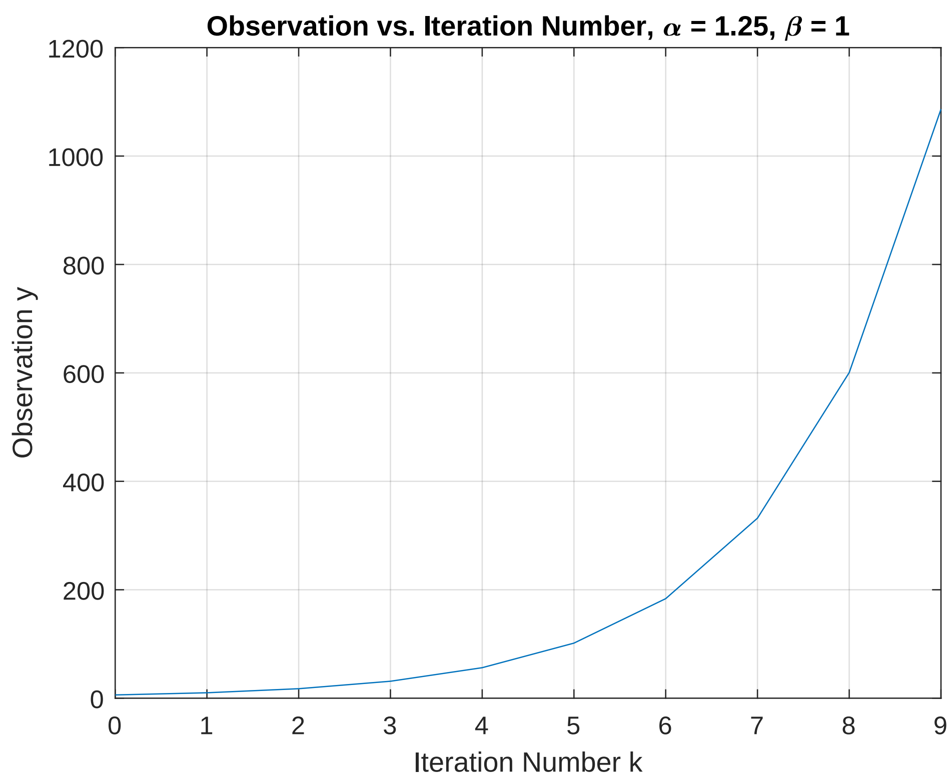

Figure 14: Observation vs. Time, $\alpha = 1.25, \beta = 1$

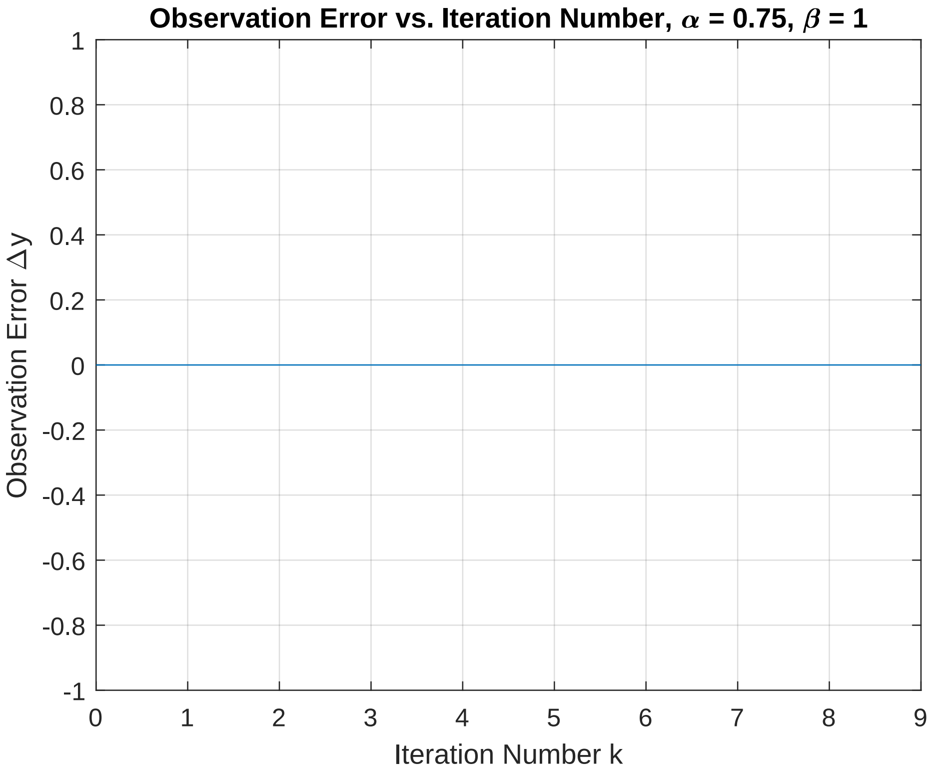

Figure 15: Observation Error vs. Time, $\alpha = 0.75, \beta = 1$

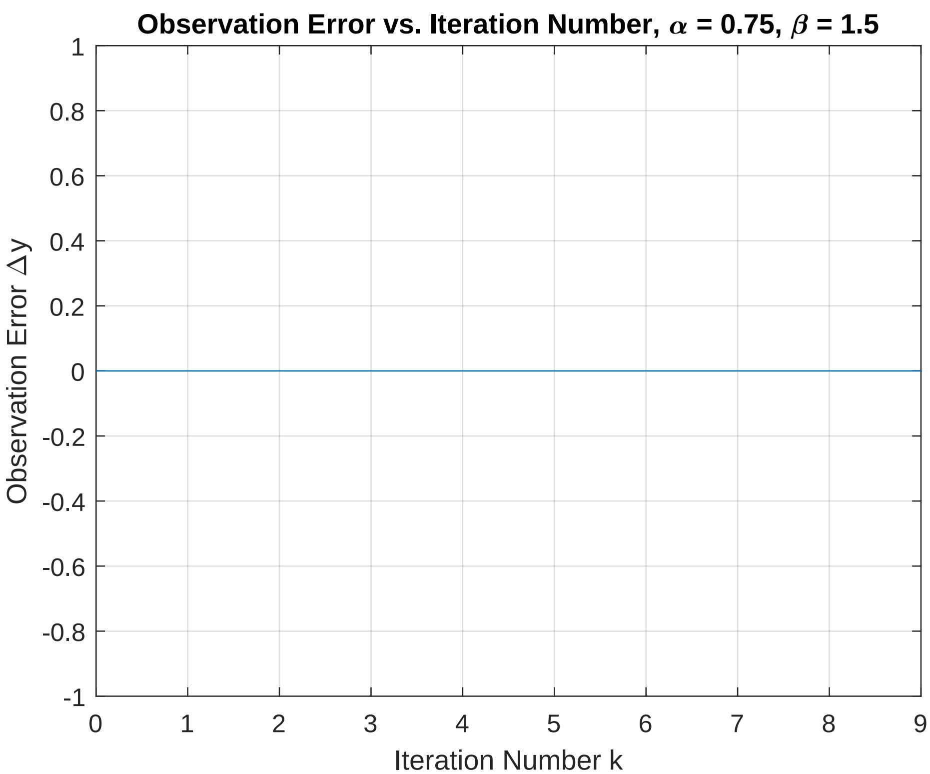

Figure 16: Observation Error vs. Time, $\alpha = 0.75, \beta = 1.5$

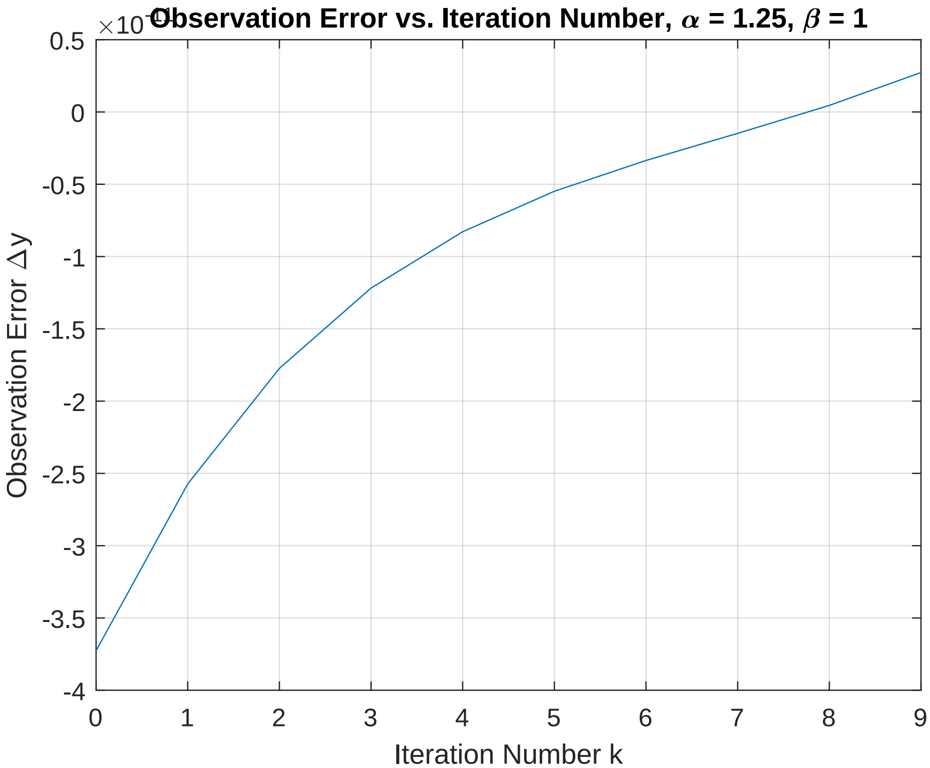

Figure 17: Observation Error vs. Time, $\alpha = 1.25, \beta = 1$

due to the larger magnitude of the observations for large k, but is still primarily driven by floating-point precision.

The initial state is estimated by using similar arguments as in Problem 1c to get

$$\begin{bmatrix} y(0) - u_0 \\ \vdots \\ y(9) - \sum_{i=0}^{8} \mathbf{H}\mathbf{F}^i \mathbf{G} u_{8-i} - u_9 \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \vdots \\ \mathbf{H}\mathbf{F}^8 \end{bmatrix} \mathbf{x}_0$$

Given $u_k = 0 \ \forall k$,

$$\underbrace{\begin{bmatrix} y(0) \\ \vdots \\ y(9) \end{bmatrix}}_{\triangleq \mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{H} \\ \vdots \\ \mathbf{H}\mathbf{F}^8 \end{bmatrix}}_{\triangleq \mathbf{L}} \mathbf{x}_0$$

The exact left pseudo-inverse solution is then given by

$$\hat{\mathbf{x}}_0 = \left(\mathbf{L}^T \mathbf{L}\right)^{-1} \mathbf{L}^T \mathbf{y}$$

3. (a) Using the provided definition for $\mathbf{y}(k+1)$ and the system dynamics for $\mathbf{z}$, the

observation equations can be obtained by observing that

$$\begin{aligned}\mathbf{y}(k+1) &= \begin{bmatrix} z_1(k+1) - z_1(k) \\ z_2(k+1) - z_2(k) \end{bmatrix} = \begin{bmatrix} \lambda\left(z_1(k) - z_2(k)\right) \\ \mu\left(z_1(k) - z_2(k)\right) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} z_1(k) - z_2(k) & 0 \\ 0 & z_1(k) - z_2(k) \end{bmatrix}}_{\mathbf{H}_k} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}\end{aligned}$$

Since λ and μ do not change with time, the dynamics are simply given by

$$\begin{bmatrix} \lambda_{k+1} \\ \mu_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} \lambda_k \\ \mu_k \end{bmatrix}$$

The input matrices are simply $\mathbf{G} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$.

Note that $\mathbf{H}_k$ is time-varying, and therefore the methods for LTI observability analysis derived in class cannot be used here without modification.

(b) Again, using similar arguments as in Problem 1c, it can be shown that

$$\underbrace{\begin{bmatrix} \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(5) \end{bmatrix}}_{\triangleq \mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{H}_0 \\ \vdots \\ \mathbf{H}_4 \end{bmatrix}}_{\triangleq \mathbf{L}} \mathbf{x}$$

The exact left pseudo-inverse solution is then given by

$$\hat{\mathbf{x}}_0 = \left(\mathbf{L}^T\mathbf{L}\right)^{-1} \mathbf{L}^T \mathbf{y}$$

As noted previously, $\mathbf{H}_k$ is time-varying, and therefore classical methods for observability analysis cannot be used here. However, the solution is nevertheless found by solving an overdetermined system of equations, which requires inverting $\mathbf{L}^T\mathbf{L}$. Thus the state variables λ and μ can be identified iff rank $\mathbf{L} = 2$, which is true as long as $z_1(k) - z_2(k) \neq 0$ for at least one value of $k = 0 \ldots 4$. The practical interpretation of this is that these parameters cannot be identified if the parties already agree on the price.

(c) Using the provided observations, the exact left pseudo-inverse estimates for the two parameters are $\lambda = -0.7042$ and $\mu = 0.2410$.

AQ1. (a) The rank of the controllability matrix computed using the provided formula is 4, therefore the system is fully state-reachable.

(b) The system is not fully state reachable with only the first input and is fully state reachable with only the second input. With only the first input, the controller is only capable of exciting Modes 3 and 4 (the out-of-phase oscillation).

Alternatively, examining the system in modal coordinates as in any Problem 1f, the modal input matrix $\hat{\mathbf{G}}$ with only the first input has the form

$$\hat{\mathbf{G}} = \begin{bmatrix} 0 \\ 0 \\ 0.0408 + 0.0018j \\ 0.0408 - 0.0018j \end{bmatrix}$$

Notice that the first two rows of the modal input matrix are zeros, showing that the first input has no effect on Modes 1 and 2. That is, any linear combination of the modal responses that can be created with only the first input will not contain Modes 1 and 2. The first and second modes are called uncontrollable modes of the system and lie outside the controllable subspace, defined as the image space of the controllability matrix.

(c) Again using similar reasoning as that in Problem 1c, it can be shown that

$$\mathbf{x}_N = \mathbf{F}^N \mathbf{x}_0 + \sum_{i=1}^{N-1} \mathbf{F}^i \mathbf{G} \mathbf{u}_{i-1}$$

Rearranging and expressing as a matrix-vector equation gives

$$\underbrace{\begin{bmatrix} \mathbf{F}^{N-1}\mathbf{G} & \dots & \mathbf{F}\mathbf{G} & \mathbf{G} \end{bmatrix}}_{\triangleq \mathbf{L}} \underbrace{\begin{bmatrix} \mathbf{u}_N \\ \vdots \\ \mathbf{u}_0 \end{bmatrix}}_{\triangleq \mathbf{u}} = \underbrace{\mathbf{x}_N - \mathbf{F}^N \mathbf{x}_0}_{\triangleq \xi}$$

(d) The expression derived in Part 3c has more variables than equations, and therefore has infinitely many solutions. This reflects the fact that there are infinitely many trajectories to get from $\mathbf{x}_0$ to $\mathbf{x}_N$.

The solution that results in the "smallest" inputs is known as the *least norm solution* or right pseudo-inverse solution, and is given by

$$\hat{\mathbf{u}} = \mathbf{N}^T \left(\mathbf{N} \mathbf{N}^T \right)^{-1} \xi$$

Note that similarly to the argument for observability, $\mathbf{N}\mathbf{N}^T$ is guaranteed to be invertible because $\mathbf{N}$ has full row rank per the controllability matrix.

A MATLAB Script

```
%% ASEN5044 Assignment 3 Solutions
% Y. Shen
% 14 September 2018

%% Problem 1

% Part A

dt = 0.05;                              % Discretization time (s)

A = [0 1 0 0; -2 0 1 0; 0 0 0 1; 1 0 -2 0];
B = [0 0; -1 0; 0 0; 1 1];

Fh = expm([A B; zeros(2, 6)]*dt);

F = Fh(1:4, 1:4);
G = Fh(1:4, 5:6);

% Part B

H = [1 0 0 0; 0 1 0 -1];

O = [H; H*F; H*F^2; H*F^3];
rank(O)

% Part D

% Load data and transpose for convenience
load('hw3problem1data');
Udata = Udata';
Ydata = Ydata';

% Construct y vector and L matrix
y = reshape(Ydata, size(Ydata, 1)*size(Ydata, 2), 1);
L = zeros(size(Ydata, 1)*size(Ydata, 2), 4);
for k = 1:size(Ydata, 1)
    for i = 0:k - 1
        y(2*k - 1:2*k) = y(2*k - 1:2*k) - H*F^i*G*Udata(:, k - i);
    end
    L(2*k - 1:2*k, :) = H*F^k;
end

% Find least-squares solution of x0
x0 = (L'*L)\(L'*y);

x = zeros(4, size(Udata, 2));
y = zeros(size(Ydata));
x(:, 1) = x0;

t = dt:dt:5;
```

```
for k = 2:size(Udata, 2)
    x(:, k) = F*x(:, k - 1) + G*Udata(:, k - 1);
    y(:, k - 1) = H*x(:, k);
end

figure;
plot(t, x(:, 2:end));
title('State Vector vs. Time');
ylabel('State x');
xlabel('Time t (s)');
legend('q_1', 'q_1-dot', 'q_2', 'q_2-dot', 'location', 'best');
grid on;

figure;
plot(t, y);
title('Predicted Output Vector vs. Time');
ylabel('Output Vector y');
xlabel('Time t (s)');
legend('y_1', 'y_2', 'location', 'best');
grid on;

figure;
plot(t, y - Ydata);
title('Estimated Observation Error vs. Time');
ylabel('Observation Error');
xlabel('Time t (s)');
legend('\Deltay_1', '\Deltay_2', 'location', 'best');
grid on;

% Part E

% Test observability
H = [1 0 0 0];

O = [H; H*F; H*F^2; H*F^3];
rank(O)

H = [0 1 0 -1];

O = [H; H*F; H*F^2; H*F^3];
rank(O)

% Compute eigenvalues and eigenvector matrix and transform into modal
% coordinates
[V, D] = eig(A);
Hhat = H*V;
Ghat = V\G;

% Compute modal system responses and plot
x12 = zeros(4, length(t));
x12(:, 1) = [0; 1; 0; 1];
x34 = zeros(4, length(t));
x34(:, 1) = [0; -1; 0; 1];
for k = 2:size(Udata, 2)
```

```
    x12(:, k) = F*x12(:, k - 1);
    x34(:, k) = F*x34(:, k - 1);
end

figure;
plot(t, x12(:, 2:end));
title('Modal State Response for Modes 1 and 2');
ylabel('State');
xlabel('Time t (s)');
legend('q_1', 'q_1-dot', 'q_2', 'q_2-dot', 'location', 'best');
grid on;

figure;
plot(t, x34(:, 2:end));
title('Modal State Response for Modes 3 and 4');
ylabel('State');
xlabel('Time t (s)');
legend('q_1', 'q_1-dot', 'q_2', 'q_2-dot', 'location', 'best');
grid on;

%% Problem 2

% Part B

% Define alpha and beta for each of the parameter cases
al = [0.75; 0.75; 1.25];
be = [1; 1.5; 1];

% Define final k and u
kend = 30;
u = 1;

% Compute eigenvalues, state responses for each of the parameter cases
for i = 1:3
    % Allocate states and define initial condition
    x = zeros(2, kend);
    x(:, 1) = [0; 0];

    % Define system matrices
    F = [al(i) al(i); be(i)*(al(i) - 1) be(i)*al(i)];
    G = [al(i); be(i)*al(i)];
    H = [1 1];
    M = 1;

    % Compute observability matrix
    O = [H; H*F];

    fprintf('Eigenvalues of F for case (%g, %g):\n', al(i), be(i));
    disp(eig(F));
    fprintf('Magnitude of eigenvalues of F for case (%g, %g):\n', al(i), be(i));
    disp(abs(eig(F)));
    fprintf('Rank of the observability matrix for case (%g, %g):\n', al(i), be(i));
    disp(rank(O));
```

```matlab
    % Compute states
    for k = 2:kend
        x(:, k) = F*x(:, k - 1) + G*u;
    end

    figure;
    plot(0:kend - 1, x);
    title(sprintf('State Response vs. Iteration Number, \\alpha = %g, \\beta = %g', al(i),
    xlabel('Iteration Number k');
    ylabel('States x');
    legend('c_k', 'i_k', 'location', 'best');
    grid on;
end

% Part C
u = 0;
for i = 1:3
    % Allocate states and define initial condition
    x = zeros(2, kend);
    x(:, 1) = [5; 1];

    % Define system matrices
    F = [al(i) al(i); be(i)*(al(i) - 1) be(i)*al(i)];
    G = [al(i); be(i)*al(i)];
    H = [1 1];
    M = 1;

    % Compute states
    for k = 2:kend
        x(:, k) = F*x(:, k - 1) + G*u;
    end

    figure;
    plot(0:kend - 1, x);
    title(sprintf('State Response vs. Iteration Number, \\alpha = %g, \\beta = %g', al(i),
    xlabel('Iteration Number k');
    ylabel('States x');
    legend('c_k', 'i_k', 'location', 'best');
    grid on;
end

% Part D
u = 0;
kend = 10;
for i = 1:3
    % Allocate states and define initial condition
    x = zeros(2, kend);             % True state
    xh = zeros(2, kend);            % Estimated state
    x(:, 1) = [5; 1];

    % Define system matrices
    F = [al(i) al(i); be(i)*(al(i) - 1) be(i)*al(i)];
    G = [al(i); be(i)*al(i)];
    H = [1 1];
```

```
    M = 1;

    % Allocate observations
    y = zeros(1, kend);                 % True observations
    yh = zeros(1, kend);                % Estimated observations
    y(1) = H*x(:, 1) + M*u;

    % Compute true states and observations
    for k = 2:kend
        x(:, k) = F*x(:, k - 1) + G*u;
        y(k) = H*x(:, k) + M*u;
    end

    % Construct L matrix
    L = zeros(length(y), 2);
    for k = 1:length(y)
        L(k, :) = H*F^(k - 1);
    end

    % Find least squares solution of x0
    x0h = (L'*L)\(L'*y');

    % Compute states and observations based on estimated initial condition
    xh(:, 1) = x0h;
    yh(1) = H*xh(:, 1) + M*u;
    for k = 2:kend
        xh(:, k) = F*xh(:, k - 1) + G*u;
        yh(k) = H*xh(:, k) + M*u;
    end

    figure;
    plot(0:kend - 1, y);
    title(sprintf('Observation vs. Iteration Number, \\alpha = %g, \\beta = %g', al(i), be
    xlabel('Iteration Number k');
    ylabel('Observation y');
    grid on;

    figure;
    plot(0:kend - 1, y - yh);
    title(sprintf('Observation Error vs. Iteration Number, \\alpha = %g, \\beta = %g', al(i
    xlabel('Iteration Number k');
    ylabel('Observation Error \Deltay');
    grid on;
end

%% Problem 3

% Part C

% Generate observations and L matrix
z = [100 43.6658 40.5785 40.4093 40.4000 40.3995;
    20 39.2815 40.3382 40.3961 40.3993 40.3995];
y = reshape((z(:, 2:end) - z(:, 1:end - 1)), size(z, 1)*(size(z, 2) - 1), 1);
```

```
L = zeros(size(y, 1), 2);
for k = 1:size(z, 2) - 1
    L(2*k - 1:2*k, :) = eye(2)*(z(1, k) - z(2, k));
end

x = (L'*L)\L'*y;

%% Advanced Question 1

% Part A

dt = 0.05;                          % Discretization time (s)

A = [0 1 0 0; -2 0 1 0; 0 0 0 1; 1 0 -2 0];
B = [0 0; -1 0; 0 0; 1 1];

Fh = expm([A B; zeros(2, 6)]*dt);

F = Fh(1:4, 1:4);
G = Fh(1:4, 5:6);

C = [G F*G F^2*G F^3*G];
fprintf('Rank of controllability matrix with both inputs: %g\n', rank(C));

% Part B

G1 = G(:, 1);

C = [G1 F*G1 F^2*G1 F^3*G1];
fprintf('Rank of controllability matrix with only first input: %g\n', rank(C));

G2 = G(:, 2);

C = [G2 F*G2 F^2*G2 F^3*G2];
fprintf('Rank of controllability matrix with only second input: %g\n', rank(C));
```