Statistical Estimation	**Homework 5**
ASEN 5044 Fall 2018	**Due Date: October 18, 2018**
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Problem 1

Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_w σ_v, respectively. What is the standard deviation of the random variable $X = W + V$?

The variance of X can be expressed as

$$\begin{aligned}
\sigma_X^2 &= E(X^2) - E(X)^2 \\
&= E((W+V)^2) - E(W+V)^2 \\
&= E(W^2 + 2WV + V^2) - (E(W) + E(V))^2 \\
&= E(W^2) + 2E(WV) + E(V^2) - E(W)^2 - 2E(W)E(V) - E(V)^2
\end{aligned}$$

Because W and V are uncorrelated, $E(WV) = E(W)E(V)$. This means the above expression reduces to

$$\begin{aligned}
\sigma_X^2 &= E(W^2) - E(W)^2 + E(V^2) - E(V)^2 \\
&= \sigma_W^2 + \sigma_V^2
\end{aligned}$$

So the standard deviation of X is $\sqrt{\sigma_W^2 + \sigma_V^2}$.

Problem 2

Consider two scalar RVs X and Y.

Part a

Prove that if X and Y are independent their correlation coefficient $\rho = 0$.

For independent random variables $E(XY) = E(X)E(Y)$. Because of this their covariance $C_{XY} = E(XY) - E(X)E(Y) = 0$. This means their correlation coefficient is

$$\rho = \frac{C_{XY}}{\sigma_x \sigma_y} = \frac{0}{\sigma_x \sigma_y} = 0$$

Part b

Find an example of two RVs that are not independent but have a correlation coefficient of zero.

Assume $X = \mathcal{U}(-1, 1)$ and $Y = X^2$. Because $\rho = \frac{C_{XY}}{\sigma_X \sigma_Y}$ we just need to show that

$$C_{XY} = E(XY) - E(X)E(Y) = 0$$

to show $\rho = 0$. From the definition of the uniform distribution we know that $E(X) = \frac{1}{2}(-1+1) = 0$, so we know $E(X)E(Y) = 0$. We can now find

$$\begin{aligned}
E(XY) &= E(X^3) \\
&= \int_{-1}^{1} x^3 dx \\
&= \frac{1}{4}x^4\Big|_{-1}^{1} \\
&= \frac{1}{4} - \frac{1}{4} = 0
\end{aligned}$$

So because $C_{XY} = E(XY) - E(X)E(Y) = 0 - 0E(Y) = 0$, ρ must also be equal to zero.

Part c

Prove that if Y is a linear function of X then $\rho = \pm 1$.

To show that $\rho = \pm 1$ when Y is a linear function of X we simply need to show that $|C_{XY}| = |\sigma_X \sigma_Y|$. We can do this by finding $E(Y)$, $E(Y^2)$, $E(XY)$, and σ_Y in terms of $E(X)$, $E(X^2)$, and σ_X.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} X dX = \frac{1}{2}X^2\Big|_{-\infty}^{\infty} \\
E(X^2) &= \int_{-\infty}^{\infty} X^2 dX = \frac{1}{3}X^3\Big|_{-\infty}^{\infty} \\
E(Y) &= E(AX) + E(B) \\
&= AE(X) + B \\
E(Y^2) &= E((AX+B)^2) \\
&= E(A^2X^2 + 2ABX + B^2) \\
&= A^2E(X^2) + 2ABE(X) + B^2 \\
E(XY) &= E(AX^2 + BX) \\
&= AE(X^2) + BE(X) \\
\sigma_Y &= E(Y^2) - (E(Y))^2 \\
&= A^2E(X^2) + 2ABE(X) + B^2 - (AE(X) + B)^2 \\
&= A^2E(X^2) + 2ABE(X) + B^2 - A^2(E(X))^2 - 2ABE(X) - B^2 \\
&= A^2(E(X^2) - (E(X)^2)) \\
&= A^2\sigma_X^2
\end{aligned}$$

Given these preliminaries we can find

$$\begin{aligned}
C_{XY} &= E(XY) - E(X)E(Y) \\
&= AE(X^2) + BE(X) - E(X)(AE(X) + B) \\
&= AE(X^2) + BE(X) - AE(X)^2 - BE(X) \\
&= A(E(X^2) - E(X)^2) \\
&= A\sigma_X^2 \\
\sigma_X\sigma_Y &= \sigma_X\sqrt{A^2\sigma_X^2} \\
&= A\sigma_X^2
\end{aligned}$$

All of this shows that when Y is a linear function of X,

$$\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{A\sigma_X^2}{A\sigma_X^2} = 1$$

Problem 3

Consider the following function

$$f_{XY} = \begin{cases} ae^{-2x}e^{-3y} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Part a

Find the value of a so that $f_{XY}(x, y)$ is a valid joint probability density function.

Because $\int_X \int_Y f_{XY}dydx = 1$ we can find a by the following:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_{XY}dy \\
&= ae^{-2x}\int_0^{\infty} e^{-3y}dy \\
&= -\frac{a}{3}e^{-2x}e^{-3y}\Big|_0^{\infty} \\
&= \frac{a}{3}e^{-2x} \\
\int_0^{\infty} \frac{a}{3}e^{-2x}dx &= -\frac{a}{6}e^{-2x}\Big|_0^{\infty} \\
&= \frac{a}{6} \\
a &= 6
\end{aligned}$$

Part b

Calculate $\bar{x}$ and $\bar{y}$.

To find $E(X)$ and $E(Y)$ we do the following:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X dx \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY} dy dx \\
&= \int_{-\infty}^{\infty} 2xe^{-2x} dx \\
&= \frac{-2x-1}{2} e^{-2x} \Big|_0^{\infty} \\
&= \frac{1}{2} \\
E(Y) &= \int_{-\infty}^{\infty} y f_Y dy \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY} dx dy \\
&= \int_{-\infty}^{\infty} 2ye^{-3y} dy \\
&= \frac{-3y-1}{3} e^{-3y} \Big|_0^{\infty} \\
&= \frac{1}{3}
\end{aligned}$$

Part c

Calculate $E(X^2)$, $E(Y^2)$, and $E(XY)$.

$$
\begin{aligned}
E(X^2) &= \int_X x^2 f_X dx \\
&= 2\int_X x^2 e^{-2x} dx \\
&= \frac{-2x^2 - 2x - 1}{2} e^{-2x}\Big|_0^\infty \\
&= \frac{1}{2} \\
E(Y^2) &= \int_Y y^2 f_Y dy \\
&= 3\int_Y y^2 e^{-3y} dy \\
&= \frac{-9y^2 - 6y - 2}{9} e^{-3y}\Big|_0^\infty \\
&= \frac{2}{9} \\
E(XY) &= \int_Y \int_X xy f_{XY} dx dy \\
&= 6\int_Y \int_X e^{-2x} e^{-3y} dx dy \\
&= 6\int_Y \Big[\frac{-2x-1}{4} e^{-2x} y e^{-3y}\Big|_{x=0}^\infty dy \\
&= \frac{6}{4}\int_Y y e^{-3y} dy \\
&= \frac{6}{4}\frac{-3y-1}{9} e^{-3y}\Big|_{y=0}^\infty \\
&= \frac{1}{6}
\end{aligned}
$$

Part d

Calculate the autocorrelation matrix of the random vector $[X\ Y]^T$.

$$
R_{XY} = \begin{bmatrix} E(X^2) & E(XY) \\ E(YX) & E(Y^2) \end{bmatrix} = \begin{bmatrix} 1/2 & 2/9 \\ 2/9 & 1/6 \end{bmatrix}
$$

Part e

Calculate the variance σ_x^2 and σ_y^2 and the covariance C_{XY}.

$$\sigma_x^2 = E(X^2) - E(X)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$
$$\sigma_y^2 = E(Y^2) - E(Y)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$
$$C_{XY} = E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{1}{6} = 0$$

Part f

Calculate the autocovariance matrix of the random vector $[X\ Y]^T$.

$$C = \begin{bmatrix} \sigma_X^2 & C_{XY} \\ C_{XY} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}$$

Part g

Calculate the correlation coefficient between X and Y.

Because the covariance $C_{XY} = 0$ the correlation coefficient $\rho = \frac{C_{XY}}{\sigma_x \sigma_y}$ is also equal to zero.

Problem 4

Prove the following two results used in lectire to derive the theoretical expectations for the Gaussian sampling experiment where $x \sim \mathcal{N}(\bar{x}, \sigma_x^2)$, $e \sim \mathcal{N}(0, \sigma_e^2)$, and $y = cx + de$.

Part a

$\text{cov}(X, Y) = E[(x - \bar{x})(y - \bar{y})] = E[XY] - \bar{x}\bar{y}$

$$\begin{aligned} \text{cov}(X, Y) &= E[(x - \bar{x})(y - \bar{x})] \\ &= E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}] \\ &= E(xy) - \bar{y}E(x) - \bar{x}E(y) + \bar{x}\bar{y} \\ &= E(xy) - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} \\ &= E(xy) - \bar{x}\bar{y} \end{aligned}$$

Part b

$\text{var}(Y) = E[(y - \bar{y})^2] = c^2\sigma_x^2 + d^2\sigma_e^2$

The expected value of a linear combination of gaussians is $\sum_i c_i\bar{x}_i$ so the expected value of y is $\bar{y} = c\bar{x} + d\bar{e}$. But since $\bar{e} = 0$, $\bar{y} = c\bar{x}$. With this we can find $\text{var}(Y)$ as follows:

$$\begin{aligned}
\text{var}(Y) &= E((y-\bar{y})^2) \\
&= E((cx + de - c\bar{x})^2) \\
&= c^2E(x^2) + cdE(ex) - c^2\bar{x}E(x) + cdE(ex) + d^2E(e^2) - dc\bar{x}E(e) \\
&\quad - c^2\bar{x}E(x) - cd\bar{x}E(e) + c^2\bar{x}^2 \\
&\text{since } x \text{ and } e \text{ are independent} \\
&= c^2(E(x^2) - \bar{x}^2) + d^2E(e^2) - 2cdE(e)E(x) - 2dc\bar{x}E(e) \\
&\text{and since } E(e) = 0 \\
&= c^2(E(x^2) - \bar{x}^2) + d^2(E(e^2) - \bar{e}^2) \\
&= c^2\sigma_x^2 + d^2\sigma_e^2
\end{aligned}$$

Problem 5

Consider two continuous random variables x and y, where $y = \ln(x)$ and $x > 0$. Derive analytical closed-form expressions for each of the following:

Part a

$p(y)$ if $p(x) = \mathcal{U}[a, b]$ (i.e. if x has a uniform pdf for $0 < a \leq x \leq b$)

Generally, if $Y = g(X)$ then $p(y) = \left|\frac{d}{dy}g^{-1}(y)\right|p(g^{-1}(y))$. So for the given distribution for X,

$$\begin{aligned}
g^{-1}(y) &= e^y \\
\frac{d}{dy}g^{-1}(y) &= e^y \\
f(g^{-1}) &= \begin{cases} \frac{1}{\ln(b)-\ln(a)} & \ln(a) \leq y \leq \ln(b) \\ 0 & \text{otherwise} \end{cases} \\
p(y) &= \begin{cases} \frac{e^y}{\ln(b)-\ln(a)} & \ln(a) \leq y \leq \ln(b) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Part b

$p(y)$ if $p(\frac{1}{x}) = \mathcal{U}[c, d]$ (i.e. if $\frac{1}{x}$ has a uniform pdf $0 < c \leq \frac{1}{x} \leq d$)

We are not directly given the distribution for X. Instead we are given $p(z) = \mathcal{U}[c, d]$ where $z = 1/x$, so we need to derive the distribution for X.

$$
\begin{aligned}
z = h(x) &= \frac{1}{x} \\
h^{-1}(z) &= \frac{1}{z} \\
\frac{d}{dz} &= -\frac{1}{z^2} \\
p(z) &= \left|\frac{d}{dz}h^{-1}(z)\right| p(h^{-1}(z)) \\
\frac{1}{z^2}p(h^{-1}(z)) &= \begin{cases} \frac{1}{d-c} & c \leq z \leq d \\ 0 & \text{otherwise} \end{cases} \\
p(h^{-1}(z)) &= \begin{cases} \frac{z^2}{d-c} & c \leq z \leq d \\ 0 & \text{otherwise} \end{cases} \\
p(x) &= \begin{cases} \frac{(1/x)^2}{(1/d)-(1/c)} & 1/c \leq x \leq 1/d \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Given this distribution for $p(x)$ we can calculate $p(y) = \left|\frac{d}{dy}g^{-1}(y)\right| p(g^{-1}(y))$.

$$
\begin{aligned}
p(y) &= \begin{cases} \frac{(1/e^y)^2}{\ln(1/d)-\ln(1/c)}e^y & \ln(1/d) \leq y \leq \ln(1/c) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{e^{-y}}{\ln(c)-\ln(d)} & -\ln(d) \leq y \leq -\ln(c) \end{cases}
\end{aligned}
$$

Part c

$p(x)$ if $p(y) = \mathcal{U}[l, m]$ (i.e. if y has a uniform pdf for $l \leq y \leq m$)

$$
\begin{aligned}
x = g(y) &= e^y \\
y = g^{-1}(x) &= \ln(x) \\
\frac{d}{dx}g^{-1}(x) &= \frac{1}{x} \\
\mathcal{U}[l, m] &= \left|\frac{d}{dx}g^{-1}(x)\right| p(g^{-1}(x)) \\
\mathcal{U}[l, m] &= \frac{1}{x}p(g^{-1}(x)) \\
p(g^{-1}) &= \begin{cases} \frac{x}{m-l} & l \leq x \leq m \\ 0 & \text{otherwise} \end{cases} \\
p(y) &= \begin{cases} \frac{e^y}{\ln(m)-\ln(n)} & \ln(l) \leq y \leq \ln(m) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Part d

$p(x)$ if $p(y) = \mathcal{N}(\mu_y, \sigma_y^2)$ (i.e. if y has a Gaussian pdf with mean μ_y and variance σ_y^2)

The probability distribution for X is given by

$$
\begin{aligned}
p(x) &= \left|\frac{d}{dx} g^{-1}(x)\right| p_Y(g^{-1}(x)) \\
&= \frac{1}{x} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(\ln(x)-\mu_y)^2}{2\sigma_y^2}}
\end{aligned}
$$

which is the log-normal distribution.