Statistical Estimation	**Homework 2**
ASEN 5044 Fall 2018	**Due Date: Sep 20, 2018**
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Exercise 1

Consider the equations of motion for a unit mass subjected to an inverse square law fore field,

$$\ddot{r} = r\dot{\theta}^2 + \frac{k}{r^2} + u_1(t)$$
$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2(t)$$

where r represents the radius from the center of the force field, θ gives the angle with respect to a reference direction in the orbital plane, k is a constant, and u_1 and u_2 represent radial and tanjectial thrusts, respectively. It is easily shown that for the initial conditions $r(0) = r_0$, $\theta(0) = 0$, $\dot{r}(0) = 0$, and $\dot{\theta}(0) = \omega_0$ with nominal thrusts $u_1(t) = 0$ and $u_2(t) = 0$ for all $t \geq 0$ the equations of motion have as a solution the circular orbit given by

$$r(t) = r_0 = \text{constant}$$
$$\dot{\theta}(t) = \omega_0 = \text{constant} = \sqrt{\frac{k}{r_0^3}}$$
$$\theta t = \omega_0 t + \text{constant}$$

Problem (a)

Pick a state vector for this system and express the original nonlinear ODEs in standard nonlinear state space form.

If we choose $x = [r,\ \theta, \dot{r}, \dot{\theta}]^T$ as our state vector and $y = [r,\ \theta]^T$ as our observation vector then we can express the original ODEs as

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 x_4^2 - \frac{k}{x_1^2} + u_1(t) \\ -\frac{2x_4x_3}{x_1} + \frac{1}{x_1}u_2(t) \end{bmatrix}$$
$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem (b)

Linearize this system's nominal equations of motion about the nominal solution $r(t) = r_0$, $\dot{r}(0) = 0$, $\theta(t) = \omega_0 t + \text{constant}$ and $\dot{\theta}(t) = \omega_0$ with $u_1(t) = 0$ and $u_2(t) = 0$. Find (A, B, C, D) matrices for output $y(t) = [r(t), \theta(t)]^T$ for the linearized system of equations about the nominal solution.

If we take $x_{\text{nom}} = [r_0, 0, \omega_0 t + c, \omega_0]^T$ and $u_{\text{nom}} = [0, 0]^T$. We can say $x(t) = x_{\text{nom}}(t) + \tilde{x}(t)$ and $u(t) = u_{\text{nom}}(t) + \tilde{u}(t)$. We can define

$$\dot{\tilde{x}} = A_{\text{nom}}\tilde{x}(t) + B_{\text{nom}}\tilde{u}(t)$$

$$A_{\text{nom}} = \left.\frac{\partial f}{\partial x}\right|_{x_{\text{nom}},u_{\text{nom}}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & -\frac{2\omega_0}{r_0} & 0 \end{bmatrix}$$

$$B_{\text{nom}} = \left.\frac{\partial f}{\partial u}\right|_{x_{\text{nom}},u_{\text{nom}}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}$$

The observation function is already linear, so the C and D matrices do not need to be linearized:

$$y(t) = Cx(t) + Du(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem (c)

Convert the continuous time (A, B, C, D) matrices you found from part (b) into discrete time (F, G, H, M) matrices using a discretization step size of $\Delta t = 10\text{s}$ and setting $k = 398600\text{km}^3/\text{s}^2$ and $r_0 = 6678\text{km}$.

We start by reorganizing our ODE as $\dot{\tilde{x}}_a = \hat{A}[\tilde{x}, \tilde{u}]^T$ where

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 & 1 & 0 \\ 0 & 0 & -\frac{2\omega_0}{r_0} & 0 & 0 & \frac{1}{r_0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.00157 & 0 & 0 & 0.01788 & 1 & 0 \\ 0 & 0 & 4.01 \times 10^{-10} & 0 & 0 & 1.497 \times 10^{-4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that F will be the upper left $n \times n$ submatrix and G will be the the upper right $m \times n$ submatrix of $e^{\hat{A}\Delta t}$:

$$e^{\hat{A}\Delta t} = \begin{bmatrix} 1.079 & 0 & 10.26 & 0.906 & 50.65 & 4.496 \times 10^{-4} \\ 0 & 1 & 0 & 10 & 0 & 7.485 \times 10^{-3} \\ 0.0161 & 0 & 1.079 & 0.184 & 10.264 & 1.35 \times 10^{-4} \\ 0 & 0 & 0 & 1 & 0 & 1.497 \times 10^{-3} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.079 & 0 & 10.26 & 0.906 \\ 0 & 1 & 0 & 10 \\ 0.0161 & 0 & 1.079 & 0.184 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 50.65 & 4.496 \times 10^{-4} \\ 0 & 7.485 \times 10^{-3} \\ 10.264 & 1.35 \times 10^{-4} \\ 0 & 1.497 \times 10^{-3} \end{bmatrix}$$

The H and M matrices for the discretized system are simply equal to the C and D matrices for the continuous time system:

$$H = C$$
$$M = D$$

Problem (d)

Interpret the results for the STM in part (c), i.e. what is the physical meaning of each column vector that makes up F?

Each column vector F_i in F represents how $\tilde{x}_i(k)$ contributes to $\tilde{x}(k+1)$. In other words, each column vector describes how the corresponding entry in the $\tilde{x}$ vector at step k will affect the entire $\tilde{x}$ vector at step $k+1$.

Exercise 2

The linear position p of an object under constant acceleration is

$$p = p_0 + \dot{p}_0 t + \frac{1}{2}\ddot{p}_0 t^2$$

where p_0 is the initial position of the object.

Problem (a)

Define a state vector as $x = [p\ \dot{p}\ \ddot{p}]^T$ and write the state space equation $\dot{x} = Ax$ for this system.

Because the object is under constant acceleration, the derivative of the acceleration $\dddot{p} = 0$. So the state space equation is simply:

$$\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix}$$

Problem (b)

Use all three expressions in Equation (1.71) to find the state transition matrix for this system.

Problem (c)

Prove for the state transition matrix found above that $e^{A0} = I$.