ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 31: The DT Linearized KF and Extended KF (EKF)

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Announcements

- **HW 8 [due Tues 12/4]** = group assignment
  - Final project report **to be due Tues 12/18**: non-linear filtering and analysis

- Aerospace Seminar Tomorrow:

Dr. William Whitacre, Draper: “Multiple Hypothesis Navigation”

12 noon – 1 pm, DLC 1B70

(meet/greet reception from 1-1:30 pm)

Last Time…

- Introduction to non-linear optimal state estimation problem: still want $\hat{x}_k^+ = \arg\min E[e_k^{+,T} e_k^+] = \mathrm{tr}[P_k^+]$
- Basic idea: approximate the mean and covariance of posterior state pdf $p(x_k | y_{1:k})$ → $\hat{x}_k^+ = E[x_k | y_{1:k}]$
- Linearization-based approximations: use Taylor series approximation of DT state dynamics
- Mathematical Issues:
  - Generally end up with state-dependent LTV approximations from linearization!
  - How to compute the Jacobians for non-linear DT models?

$$\dot{x}(t) = \mathcal{F}[x(t), u(t), w(t)] \xrightarrow{\Delta T} x(k+1) = f[x(k), u(k), w(k)] \approx x_{nom}(k+1) + \tilde{F}_k \delta x_k + \tilde{G}_k \delta u_k + \tilde{\Omega}_k w_k$$

$$y(t) = \mathcal{H}[x(t), v(t)] \longrightarrow y(k+1) = h[x(k+1), v(k+1)] \approx y_{nom}(k) + \tilde{H}_{k+1} \delta x_{k+1} + v_{k+1}$$

$$\tilde{F}_k = \frac{\partial f}{\partial x_k}\Big|_{nom[k]} \qquad \tilde{G}_k = \frac{\partial f}{\partial u_k}\Big|_{nom[k]} \qquad \tilde{\Omega}_k = \frac{\partial f}{\partial w_k}\Big|_{nom[k]} \qquad \tilde{H}_k = \frac{\partial h}{\partial x_k}\Big|_{nom[k]}$$

$(\rightarrow \text{easy since } h = \mathcal{H}!)$

$$f[x_k, u_k, w_k] \text{ generally not closed form} \rightarrow \text{ DT } f \text{ Jacobians not closed form !!!}$$

$$\rightarrow \textbf{DT Jacobians must be computed numerically}$$

- Some useful basic facts for DT nonlinear dynamical state estimation
  - **Useful fact #1: the KF works for DT LTV systems also! (minor/obvious changes)**
  - **Useful fact #2: Calculating DT Jacobians from CT nonlinear models (for small ΔT)**

(Useful Fact #2 from end of last lecture): "Eulerized" DT Jacobians

- Use Euler integration to approximate DT state transition fxn for small ΔT
- Then take partial derivatives of this to approximate required DT Jacobians
- Naturally get to use CT Jacobians as part of result

Start with (mild) assumption that the CT nonlinear model can be generally written as

$$\dot{x}(t) = \mathcal{F}[x(t), u(t)] + \Gamma(t) \cdot \tilde{w}(t)$$

Euler approx: $x(t_{k+1}) \approx x(t_k) + \Delta T \cdot \dot{x}(t)\big|_{t=t_k}$

$\longrightarrow \quad x_{k+1} \approx x_k + \Delta T \cdot \dot{x}(t=t_k)$

$= x_k + \Delta T \cdot \{ \mathcal{F}[x(t_k), u(t_k)] + \Gamma(t_k) \cdot \tilde{w}(t_k) \} \approx \underline{f}(x_k, u_k, w_k)$

$\longrightarrow$ Then $\tilde{F}_k = \dfrac{\partial f}{\partial x_k}\Big|_{nom[k]} = \dfrac{\partial x_{k+1}}{\partial x_k}\Big|_{nom[k]} = \dfrac{\partial}{\partial x_k}\left(x_k + \Delta T \cdot \{ \mathcal{F}[x(t_k), u(t_k)] + \Gamma(t_k) \cdot \tilde{w}(t_k) \} \right)\Big|_{nom[k]}$

$= \dfrac{\partial}{\partial x_k}(x_k)\Big|_{nom[k]} + \dfrac{\partial}{\partial x_k}\{ \Delta T \cdot \mathcal{F}[\cdots] + \Gamma(t_k) \cdot \tilde{w}(t_k) \}\Big|_{nom[k]} = I + \Delta T \cdot \dfrac{\partial \mathcal{F}[\cdots]}{\partial x_k}\Big|_{t=t_k, nom[k]}$ $\longrightarrow$ CT Jacobian matrix!

$(*)$ $\tilde{F}_k \approx \boxed{= I + \Delta T \cdot \tilde{A}\big|_{nom[k]}}$ $(*)$

Useful Fact #2 (cont'd): "Eulerized" DT Jacobians

- We can get approximations to remaining DT Jacobians in a similar fashion:

$$\tilde{G}_k\Big|_{nom[k]} = \frac{\partial f}{\partial u(t)}\Big|_{nom[k]} = \frac{\partial x_{k+1}}{\partial u_k}\Big|_{nom[k]} \Longrightarrow \Delta T \cdot \frac{\partial \tilde{\mathcal{F}}[\cdots]}{\partial u(t)}\Big|_{\substack{t=t_k \\ nom[k]}}$$

$$= \boxed{\Delta T \cdot \tilde{B}\Big|_{nom[k]} \approx \tilde{G}_k}$$

$$\tilde{\Omega}_k\Big|_{nom[k]} = \frac{\partial f}{\partial \tilde{w}(t)}\Big|_{nom[k]} = \frac{\partial x_{k+1}}{\partial \tilde{w}(t)}\Big|_{nom[k]} \Longrightarrow \boxed{\Delta T \cdot \Gamma(t=t_k)\Big|_{nom[k]} \approx \tilde{\Omega}_k\Big|_{nom[k]}}$$

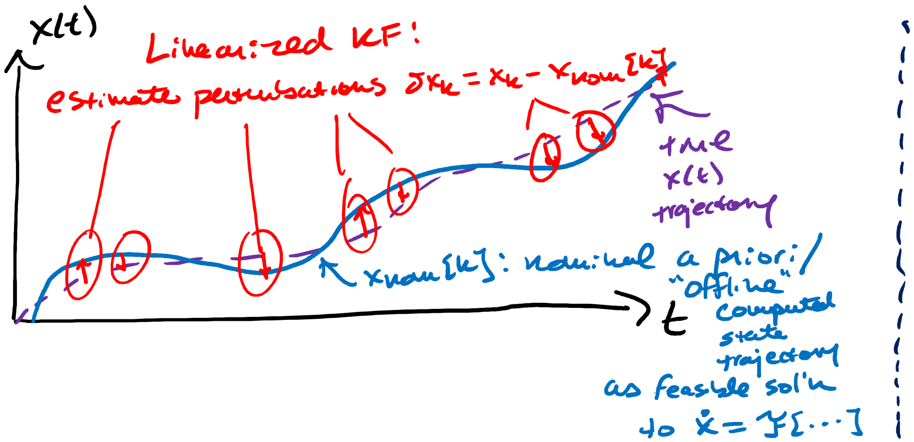
(Recall Lecture #5 for def. of $\underbrace{\tilde{A}\big|_{nom[k]}, \tilde{B}\big|_{nom[k]}}_{\text{CT Jacobians.}}$, etc.)

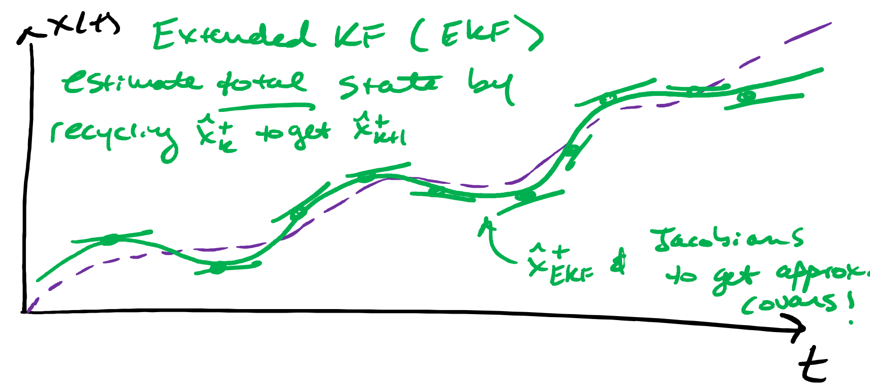
Today…

- Recap & Wrap-up DT Jacobian approximations for $\tilde{F}_k, \tilde{G}_k, \tilde{\Omega}_k$
- Approximately optimal DT state estimators based on linearization

○ **Linearized KF:** estimate *perturbations* around a priori nominal state trajectory:

- Uses linearization about nominal trajectory for both mean and covariance updates





○ **Extended KF (EKF):** estimate *total state* around online estimated trajectory:

- Uses full nonlinear model for predicted mean and predicted sensor measurements
- Uses linearization only to approximate matrix quantities (Kalman gain and covariances)

○ **READ SIMON BOOK, CHAPTERS 13.1-13.2 (nonlinear KFs)**

The Linearized KF

- Suppose nonlinear system stays near a nominal trajectory x*(t) for some u*(t) with 0 process noise input (desired equilibrium, or offline-calculated nonlinear ODE solution)

$$\dot{x}(t) = \mathcal{F}(x,u) + \Gamma(t)\tilde{w}(t), \text{ where } \tilde{w}, \tilde{v} \text{ are AWGN}$$

$$y(t) = h(x) + \tilde{v}(t),$$

$\rightarrow$ Nominal state satisfies: $\dot{x}^*(t) = \mathcal{F}(x^*(t), u^*(t))$ (deterministic solution **with no** process noise)

$\rightarrow$ Now consider actual state evolution **with** process noise present:

$$\begin{aligned} x(t) &= x^*(t) + \delta x(t), \qquad & \delta x(t) &= x(t) - x^*(t) \text{ (perturbation from } x^*(t)) \\ u(t) &= u^*(t) + \delta u(t), \qquad & \delta u(t) &= u(t) - u^*(t) \text{ (perturbation from } u^*(t)) \end{aligned}$$

$\rightarrow$ Plug into dynamics and measurement equation:

$$(\dot{x}^* + \dot{\delta x}) = \mathcal{F}(x^* + \delta x, u^* + \delta u) + \Gamma(t)\tilde{w}(t),$$

$$y(t) = h(x^* + \delta x) + \tilde{v}(t),$$

Linearization via Vector Taylor Series

- Now consider Taylor Series expansion of dynamics and measurement models near x*
- Using results of CT linearization from beginning of the course, we have that

for **small** δx and δu perturbations,

$$(\cancel{\dot{x}^*} + \dot{\delta x}) \approx \cancel{\mathcal{F}(x^*, u^*)} + \frac{\partial \mathcal{F}}{\partial x}\Big|_{(x^*,u^*)} \delta x(t) \; + \; \frac{\partial \mathcal{F}}{\partial u}\Big|_{(x^*,u^*)} \delta u(t) \; + \; \Gamma(t)\tilde{w}(t),$$

$$y(t) \approx h(x^*(t)) + \frac{\partial h}{\partial x}\Big|_{(x^*,u^*)} \delta x(t) \; + \; \tilde{v}(t),$$

$\rightarrow$ simplify using fact that $\dot{x}^*(t) = \mathcal{F}(x^*, u^*)$ and $\delta y(t) = y(t) - h(x^*)$:

$$\dot{\delta x}(t) \approx \frac{\partial \mathcal{F}}{\partial x}\Big|_{(x^*,u^*)} \delta x(t) \; + \; \frac{\partial \mathcal{F}}{\partial u}\Big|_{(x^*,u^*)} \delta u(t) \; + \; \Gamma(t)\tilde{w}(t)$$

$$\delta y(t) \approx \frac{\partial h}{\partial x}\Big|_{(x^*,u^*)} \delta x(t) + \tilde{v}(t)$$

$$\rightarrow \dot{\delta x}(t) \approx \tilde{A}|_{(x^*,u^*)} \delta x(t) + \tilde{B}|_{(x^*,u^*)} \delta u(t) + \Gamma(t)\tilde{w}(t),$$

$$\rightarrow \delta y(t) \approx \tilde{C}|_{(x^*,u^*)} \delta x(t) + \tilde{v}(t)$$

The Linearized KF Model

- Thus:

$$\left.\begin{aligned}\dot{\delta x}(t) &\approx \tilde{A}|_{(x^*,u^*)}\delta x(t) + \tilde{B}|_{(x^*,u^*)}\delta u(t) + \Gamma(t)\tilde{w}(t),\\ \delta y(t) &\approx \tilde{C}|_{(x^*,u^*)}\delta x(t) + \tilde{v}(t),\end{aligned}\right\}\ \text{CT perturbation dynamics model}$$

where $\tilde{A}, \tilde{B}, \tilde{C}$ are the CT Jacobian matrices evaluated at (x^*, u^*)

$\rightarrow$ now convert CT perturbation model into DT model:

$$\delta x(k+1) \approx \tilde{F}_k|_{nom[k]}\delta x(k) \ + \ \tilde{G}_k|_{nom[k]}\delta u(k) \ + \ \tilde{\Omega}_k w(k),$$

$$\delta y(k+1) \approx \tilde{H}_{k+1}|_{nom[k+1]}\delta x(k+1) + v(k+1)$$

where we already showed earlier that: (for sufficiently small ΔT):

$$\tilde{F}_k|_{nom[k]} \approx I + \Delta T \cdot \tilde{A}|_{(x^*,u^*,t=t_k)}, \qquad \tilde{G}_k|_{nom[k]} \approx \Delta T \cdot \tilde{B}|_{(x^*,u^*,t=t_k)},$$

$$\tilde{\Omega}_k|_{nom[k]} \approx \Delta T \cdot \Gamma(t)|_{(t=t_k)}, \qquad \tilde{H}_{k+1}|_{nom[k+1]} = \tilde{C}|_{(x^*,u^*,t=t_{k+1})} = \frac{\partial h}{\partial x}|_{(x^*,u^*,t=t_{k+1})}$$

The Linearized KF Algorithm

- So now we can estimate the total state as follows:

Deterministic

$$\hat{x}^+_{k+1} \approx \underline{x^*_{k+1}} + \underline{\hat{\delta x}^+_{k+1}} \rightarrow \text{random correction}$$

where $x^*_{k+1} = x^*(t = t_{k+1})$ and $\hat{\delta x}_{k+1}$ is estimated using LTV KF for $\underline{\delta x_{k+1}}$ and $\underline{\delta y_{k+1}}$:

online

Time update/prediction step for time k+1:

$$\hat{\delta x}^-_{k+1} = \tilde{F}_k \hat{\delta x}^+_k + \tilde{G}_k \delta u_k$$

$$P^-_{k+1} = \tilde{F}_k P^+_k \tilde{F}^T_k + \tilde{\Omega}_k Q_k \tilde{\Omega}^T_k$$

$$\delta u_{k+1} = u_{k+1} - u^*_{k+1}$$

Measurement update/correction step for time k+1:

$$\hat{\delta x}^+_{k+1} = \hat{\delta x}^-_{k+1} + K_{k+1}(\delta y_{k+1} - \underline{\tilde{H}_{k+1} \hat{\delta x}^-_{k+1}})$$

$\rightarrow \delta y^-_{k+1} =$ predicted perturbed meas.

$$P^+_{k+1} = (I - K_{k+1}\tilde{H}_{k+1})P^-_{k+1}$$

$$K_{k+1} = P^-_{k+1}\tilde{H}^T_{k+1}[\tilde{H}_{k+1}P^-_{k+1}\tilde{H}^T_{k+1} + R_{k+1}]^{-1}$$

$$\delta y_{k+1} = y_{k+1} - y^*_{k+1} = y_{k+1} - h(x^*_{k+1})$$

Actual received sensor measurement at time k+1

Computed nominal sensor measurement at time k+1

where $\tilde{F}_k, \tilde{G}_k, \tilde{\Omega}_k, \tilde{H}_k$ eval'd along (x^*, u^*) nom. sol'n at each time step k

Pros/Cons of the Linearized KF

- Pros:
  - Easy to program and numerically fast [can compute all required Jacobians offline]
  - Good for predictable systems with small/low process noise inputs

- Cons:
  - Will break if actual true system x(t) trajectory deviates too far from nominal x*(t)

$$(\text{i.e. if } \delta x(t) \text{ and } \delta u(t) \text{ get too big} \Rightarrow \hat{\delta x}(t) \text{ will have large errors } \rightarrow \text{possibly unrecoverable!!})$$

(DT Jacobians will be wrong!) Filter divergence

- Alternative: what if we kept estimating total state (not just perturbation) using most recent online total state estimate as prior for linearization (instead of fixed nominal trajectory)?

The Extended Kalman Filter (EKF) Algorithm

- Step 1: Initialization: start with some initial estimate of total state and covariance

$$\hat{x}^+(0),\ \hat{P}^+(0)$$

- Step 2: set k=0

- **Step 3: Time update/prediction step for time k+1:**

using Runge-Kutta/ode45 (i.e numerical integration)

$$\hat{x}^-_{k+1} = f[\hat{x}^+_k, u_k, w_k = 0] \quad \text{(deterministic nonlinear DT dyn. fxn eval. on } \hat{x}^+_k\text{)}$$

$$P^-_{k+1} = \tilde{F}_k P^+_k \tilde{F}^T_k + \tilde{\Omega}_k Q_k \tilde{\Omega}^T_k, \quad \text{(approx. predicted covar. via dyn. linearization about } \hat{x}^+_k\text{)}$$

$$\text{i.e. } \hat{x}^-_{k+1} \approx \hat{x}^+_k + \Delta T \cdot \mathcal{F}[\hat{x}^+_k, u_k]$$

use best available est. @ time k for pred.!

only for covar/matrix approx.

$$\text{where} \quad \tilde{F}_k|_{\hat{x}^+_k, u_k, w_k=0} \approx I + \Delta T \cdot \tilde{A}|_{(\hat{x}^+_k, u(t_k), w(t_k)=0)},$$

$$\tilde{\Omega}_k| \approx \Delta T \cdot \Gamma(t)|_{(t=t_k)},$$

The Extended Kalman Filter (EKF)

- Step 4: **Measurement update/correction step for time k+1:**

Compute:

$$\hat{y}^-_{k+1} = h[\hat{x}^-_{k+1}, v_{k+1} = 0] \quad \text{(deterministic nonlinear fxn evaluation)}$$

$$\tilde{H}_{k+1} = \frac{\partial h}{\partial x}\Big|_{\hat{x}^-_{k+1}} \quad \text{(meas. fxn Jacobian at predicted state)}$$

again, using linearization about "best" available estimate!

$$\tilde{e}_{y_{k+1}} = y_{k+1} - \hat{y}^-_{k+1} \quad \text{(nonlinear meas. innovation: actual data minus predicted)}$$

$$\tilde{K}_{k+1} = P^-_{k+1}\tilde{H}^T_{k+1}[\tilde{H}_{k+1}P^-_{k+1}\tilde{H}^T_{k+1} + R_{k+1}]^{-1} \quad \text{(approx. KF gain from meas. linearization)}$$

$$\Rightarrow \quad \hat{x}^+_{k+1} = \hat{x}^-_{k+1} + \tilde{K}_{k+1}\tilde{e}_{y_{k+1}} \quad \text{(updated total state estimate)}$$

$$P^+_{k+1} = (I - \tilde{K}_{k+1}\tilde{H}_{k+1})P^-_{k+1} \quad \text{(approx. updated covar. via linearization)}$$

- Step 5: recursion: go back to step 3 and repeat cycle for next time step…

The "1st Order" EKF Algorithm: Important Features

Useful to remember some key ideas for the EKF:

- Finding approx. Gaussian joint pdf for state and measurements from "best available guess" of total nonlinear system behavior/state at each time k
- Only use best available estimate of state at any point in time to compute required Jacobian matrices and nonlinear function evaluations at that time

  → do not need to know nominal trajectory in advance!!! (figuring it out online)

- We only need 1st order Taylor series/linearization of dynamics and measurements to get predicted covariance P^{-}_{k+1} , updated covariance P^{+}_{k+1}, and EKF gain $\tilde{K}_{k+1}$

  ➔ all of these matrix quantities are obtained via Jacobians

  (similar to vanilla KF, except now matrices are time-varying and depend on $\hat{x}^{+}_{k}$!)

- **DO NOT use linearization/Jacobians to get predicted state $\hat{x}^{-}_{k+1}$ or measurement $\hat{y}_{k+1}$**

  ➔ predicted vectors come directly from integrating/evaluating nonlinear CT fxns!