ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 6: Discrete Time Linear State Space Systems

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Thurs 9/13/2018

University of Colorado **Boulder**

Aerospace Engineering

Announcements

HW 2 Posted – due Thurs 9/20 at 11 am (before start of next lecture)

- Submit to Canvas–
  - All submissions must be legible!!! – zero credit otherwise
  - All submissions must have your name on them!!! – zero credit otherwise
- please follow posted file naming instructions

- Advanced Questions:
  - required for PhD students
  - optional/extra credit for everyone else

Overview

Last time: Linearization of nonlinear to linear SS models

Today:

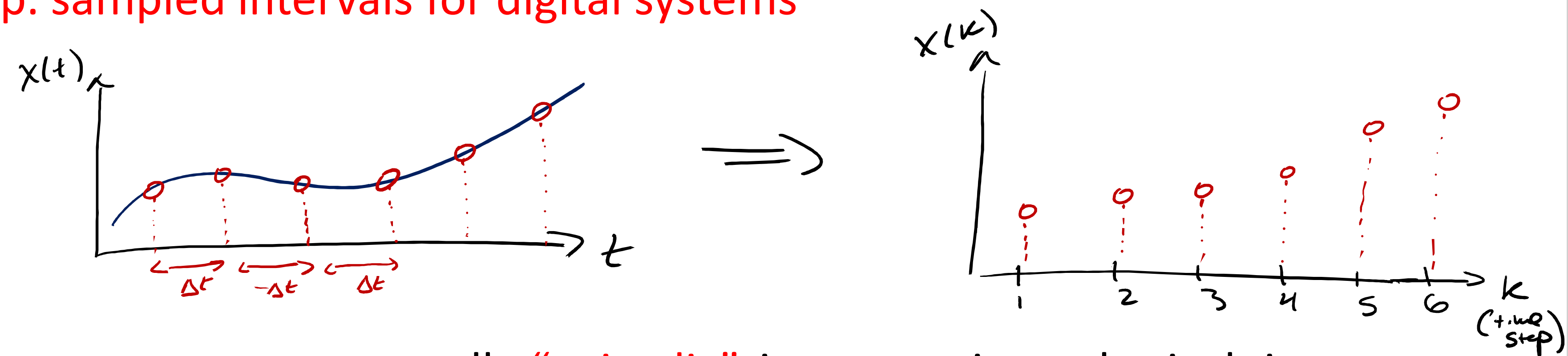
- **Discrete time (DT) linear systems,**
- **Converting continuous time (CT) systems to DT systems**

READ: Chapter 2.1 in Simon book (intro to probability)

Discrete Time Dynamic System Models

- State vector ="internal memory" of what system is doing at any given time

- In applications: only care to know what system is doing at fixed time instants, esp. sampled intervals for digital systems



- Some systems are naturally "episodic", i.e. agnostic to physical time
  - Baseball innings; rounds of poker, pool, squash, boxing, negotiation…
  - Finite state automata for computing, event-based systems
  - Often naturally described by finite difference equations (FDEs)

Discrete Time (DT) Dynamic System Models

- Convenient to specify dynamics as updates to internal system memory (i.e. state vector) from one discrete time step to another
- Linear DT models: matrices **summarize changes between integer time steps k**

$k = 0, 1, 2, 3, \ldots$ (integers)

$$\underset{\downarrow\; x_{k+1}}{x(k+1)} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \underline{\underline{F}}(k)\, \overset{x_k}{x(k)} + \underline{\underline{G}}(k)\, u(k), \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_m(k) \end{bmatrix}$$

Linear Time Varying (LTV) DT SS model

$$y(k+1) = \begin{bmatrix} y_1(k+1) \\ \vdots \\ y_p(k+1) \end{bmatrix} = \underline{\underline{H}}(k+1)\, x(k+1) + \underline{\underline{M}}(k+1)\, u(k+1)$$

$\overset{F_k}{F(k)}$ = State transition matrix $\in \mathbb{R}^{n \times n}$

$\overset{G_k}{G(k)}$ = Control effect matrix $\in \mathbb{R}^{n \times m}$

$H(k+1)$ = sensing matrix $\in \mathbb{R}^{p \times n}$

$M(k+1)$ = direct transmission $\in \mathbb{R}^{p \times m}$

If there is no dependence on k [time step]

$\rightarrow$ LTI DT SS:

F, G, H, M: $x(k+1) = F x(k) + G u(k)$ etc.

Example: Linear Car Dealer Model (episodic LTI DT system)

- You are haggling with car dealer Gary Slick for a used Ferrari
- At negotiation round k, **Slick's offer = $x_1(k)$** and **your offer = $x_2(k)$**
- Offering algorithm (finite difference equation, FDE):
  - At each round k, you both lay down offers simultaneously
  - For round k+1, you update by adding fraction μ of difference to $x_2(k)$
  - For round k+1, Slick updates by subtracting fraction λ of difference from $x_1(k)$

FDE: let $\Delta_k = x_1(k) - x_2(k)$ [offer difference]

Slick's offer @ time k+1: $x_1(k+1) = x_1(k) - \lambda \cdot \Delta_k = x_1(k) - \lambda[x_1(k) - x_2(k)]$

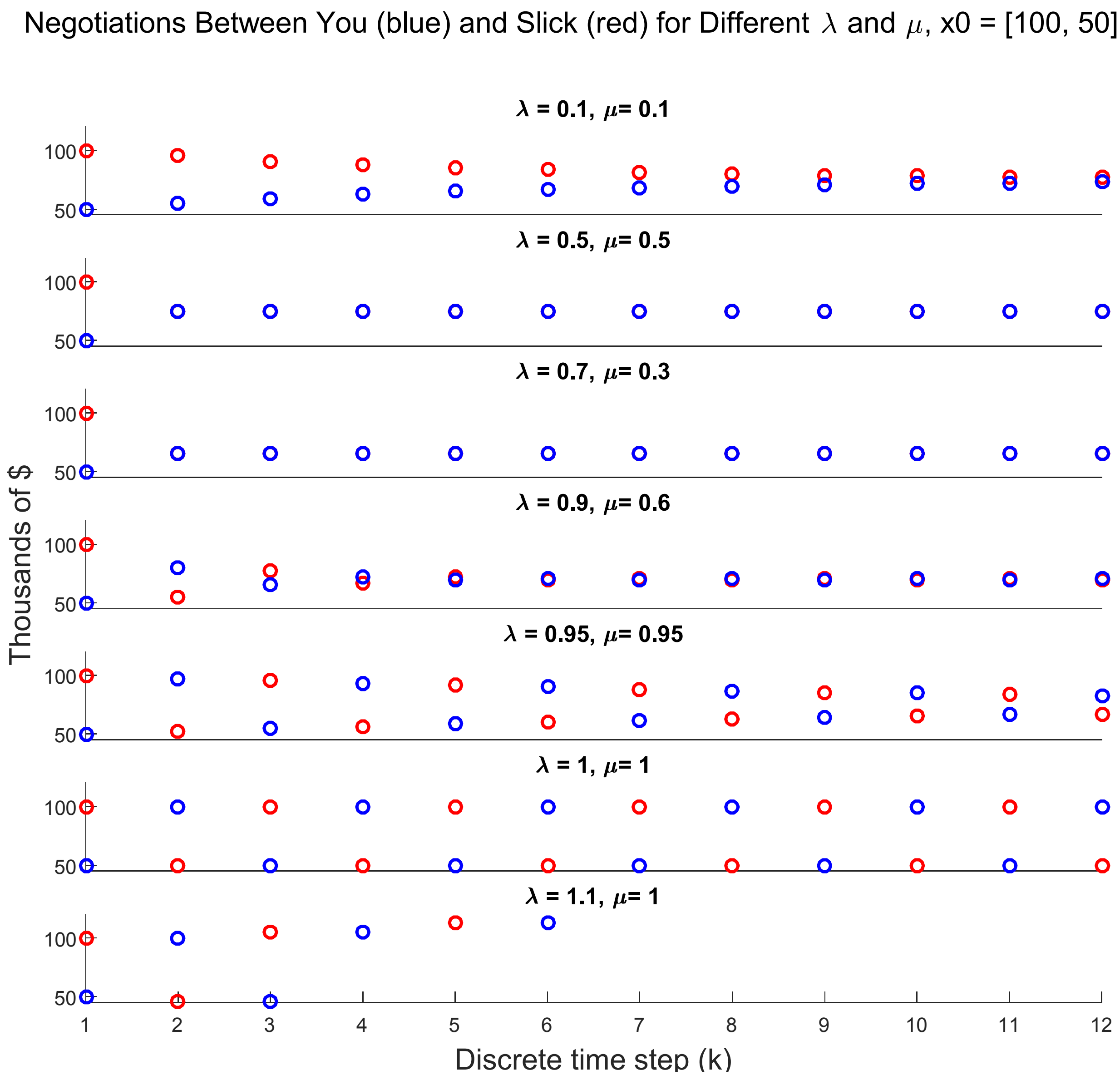
your offer @ time k+1: $x_2(k+1) = x_2(k) + \mu \cdot \Delta_k = x_2(k) + \mu[x_1(k) - x_2(k)]$

→ Simplify → Define $x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} (1-\lambda)x_1(k) + \lambda x_2(k) \\ \mu x_1(k) + (1-\mu)x_2(k) \end{bmatrix}$

into LTI DT SS model:

rewrite in F matrix times x(k) form:

$$x(k+1) = \underbrace{\begin{bmatrix} (1-\lambda) & \lambda \\ \mu & (1-\mu) \end{bmatrix}}_{\in \mathbb{R}^{2\times 2}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \underline{\underline{F}}\, x(k), \quad x(k) \in \mathbb{R}^2$$


Negotiations Between You (blue) and Slick (red) for Different λ and μ, x0 = [100, 50]
λ = 0.1, μ= 0.1
λ = 0.5, μ= 0.5
λ = 0.7, μ= 0.3
λ = 0.9, μ= 0.6
λ = 0.95, μ= 0.95
λ = 1, μ= 1
λ = 1.1, μ= 1
Thousands of $
Discrete time step (k)
100
50
1 2 3 4 5 6 7 8 9 10 11 12

Converting CT Linear Models to Sampled DT Linear Models

- How to translate from CT model (linear system of ODEs) for a given system?

$$\dot{x} = Ax(t) + Bu(t) \qquad \xrightarrow{??} \qquad x_{k+1} = Fx_k + Gu_k$$
$$y(t) = Cx(t) + Du(t) \qquad\qquad y_k = Hx_k + Mu_k$$

- Suppose u(k) follows a **zero-order hold (ZOH) discretization of u(t):**

$$u(t) = \text{some const.}, t \in [t_k, t_{k+1})$$

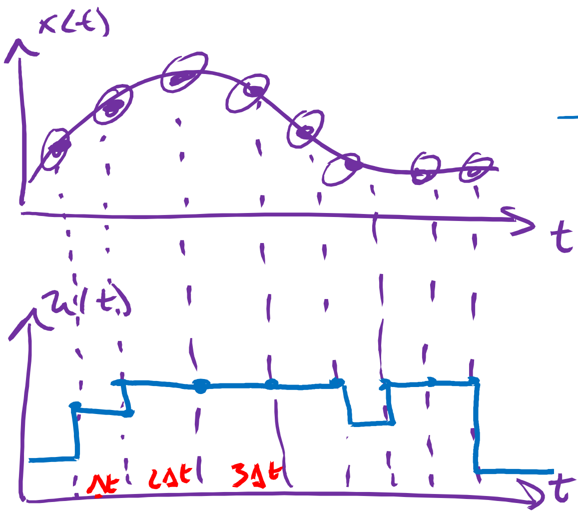
- Recall: general state solution x(t) is (for given $x(t_0)$):

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

→ If we use ZOH input $u(t)$ w/ fixed Δt sample time, then:

$$x(t) = \underline{x(t_0 + \Delta t)} = \boxed{\underbrace{e^{A \Delta t}}_{n \times n}}\, \underline{x(t_0)} + \boxed{\underbrace{\left[\int_{t_0}^{t} e^{A(t-\tau)} d\tau \cdot B\right]}_{n \times m}}\, u(t_0)$$

$$\underline{x(k+1)} = \boxed{F}\, \underline{x(k)} + \boxed{G}\, u(k)$$



Zero order Hold (ZOH)

Converting CT Linear Models to Sampled DT Linear Models

- **FACT:** if CT LTI SS model has ZOH input u(t) applied for fixed sample periods $\Delta t = t - t_0$, then can explicitly find DT LTI matrices (F,G,H,M) such that:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} x_{k+1} &= Fx_k + Gu_k \\ y_k &= Hx_k + Mu_k \end{aligned}}$$

$$\boxed{\begin{aligned} F &= e^{A\Delta t} \\ G &= \left[\int_{t_0}^{t} e^{A(t-\tau)} d\tau\right] B \\ H &= C \\ M &= D \end{aligned}}$$

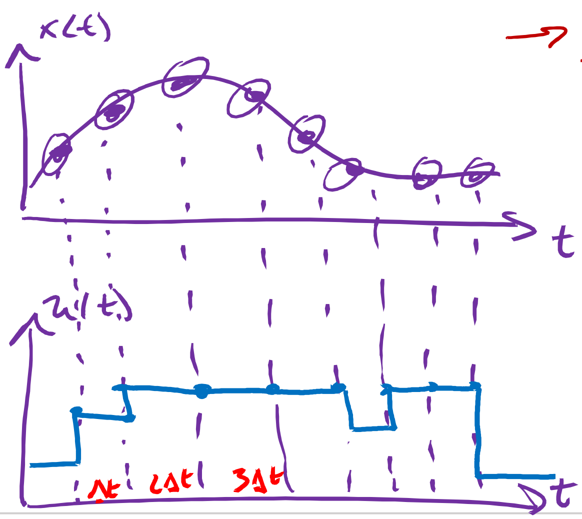
How to get this ??

→ In Simon book, Chap 1: gives formula for G if A is invertible

↳ But what if A is singular generally?

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow A^{-1}$ doesn't exist!

[from block mass problem in Lec 3!]



Zero order Hold (ZOH)

Computing the G matrix

- How to actually compute the G matrix integral? $G = \left[\int_{t_0}^{t} \underline{\underline{e}}^{A(t-\tau)} d\tau \right] \cdot B$
- First look at expansion of the integral:

$$\int_{t_0}^{t} e^{A(t-\tau)} d\tau = \int_{0}^{\Delta t} e^{A(\Delta t - \tau)} d\tau$$

plug in ∞ series def. of matrix exp.

$$= \int_0^{\Delta t} \sum_{i=0}^{\infty} A^i \frac{(\Delta t - \tau)^i}{i!} d\tau$$

$$= \sum_{i=0}^{\infty} \int_0^{\Delta t} A^i \frac{(\Delta t - \tau)^i}{i!} d\tau = \sum_{i=0}^{\infty} A^i \int_0^{\Delta t} \frac{(\Delta t - \tau)^i}{i!} d\tau$$

$$\xrightarrow[\text{to show}]{\text{easy}} \; = \sum_{i=1}^{\infty} A^{i-1} \frac{\Delta t^i}{i!}$$

$\longrightarrow$ $\underline{\text{So}}$: $G = \left[\sum_{i=1}^{\infty} A^{i-1} \frac{\Delta t^i}{i!} \right] \cdot B$ $\longrightarrow$ what does this converge to? how to compute?

Computing the G matrix

- Turns out there is a sneaky trick to computing this series for ZOH
- Note that ZOH assumption implies that for any $t \in [t_0, t_0 + \Delta t]$ } $\dot{u}(t) = 0$ [for $t \in [t_0, t_0+\Delta t]$]

Therefore, we have: $\dot{x}(t) = A x(t) + B u(t)$, $x(t_0) = x_0$

$\dot{u}(t) = 0$, $u(t_0) = u_0$ (const.)

Define: augmented state vector $x_a \triangleq \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$

s.t. $\dot{x}_a(t) = \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, $x_a(t_0) = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$

$\rightarrow \dot{x}_a = \hat{A} x_a$, where $\hat{A} \in \mathbb{R}^{[n+m] \times [n+m]}$, $n = \#$ states, $m = \#$ inputs

$\rightarrow$ Sol'n for $t_0 \rightarrow t = t_0 + \Delta t$: $x_a(t) = e^{\hat{A} \Delta t} \cdot x_a(t_0)$

Computing the G matrix

- But if we expand the matrix exponential in this case:

$$e^{\hat{A}\Delta t} = I + \hat{A}\Delta t + \hat{A}^2 \frac{\Delta t^2}{2!} + \hat{A}^3 \frac{\Delta t^3}{3!} + \cdots$$

where

$$\hat{A}^2 = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} A^2 & AB \\ \hline 0 & 0 \end{array}\right]$$

$$\hat{A}^3 = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^2 & AB \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} A^3 & A^2B \\ \hline 0 & 0 \end{array}\right]$$

$$\vdots$$

$$\hat{A}^i = \left[\begin{array}{c|c} A^i & A^{i-1}B \\ \hline 0 & 0 \end{array}\right]$$

plug into ∞ series:

$$e^{\hat{A}\Delta t} = \left[\begin{array}{c|c} e^{A\Delta t} & \left[\sum_{i=1}^{\infty} A^{i-1}\frac{\Delta t^i}{i!}\right] B \\ \hline 0 & I_{m \times m} \end{array}\right] = \left[\begin{array}{c|c} F & G \\ \hline 0 & I_{m\times m} \end{array}\right]$$

upper left $n\times n$ block

upper right $n\times m$ block

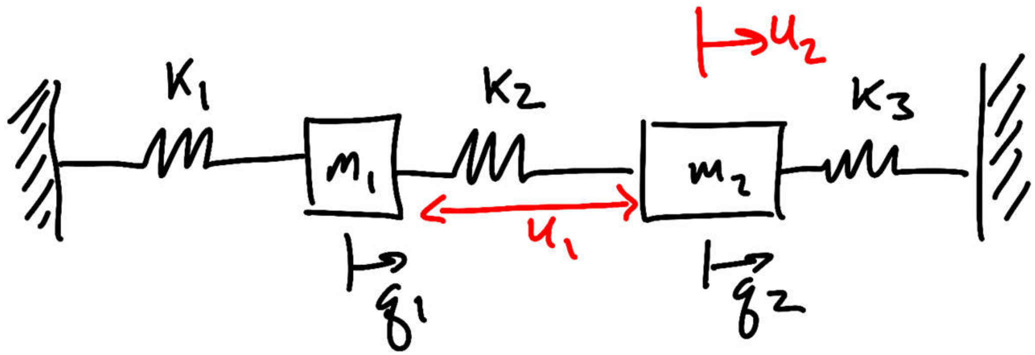
exactly equal to series expansion of $\int_{t_0}^{t} e^{A(t-\tau)}d\tau$ from before!

$= \text{expm}(\hat{A}\cdot\Delta t)$ [in Matlab]

$\in \mathbb{R}^{[n+m]\times[n+m]}$

Example: Convert CT SS model to DT SS model

- System of 2 masses and 3 springs: 2 actuator inputs u and 2 sensor outputs y



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

For $k_1 = k_2 = k_3 = 1$ N/m and $m_1 = m_2 = 1$ kg, use simple physics to get CT linear SS model

$$\dot{x} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$\longrightarrow$ eig(A) :

$\lambda_{1,2} = \pm j\,1.73$

$\lambda_{3,4} = \pm j\,1.00$

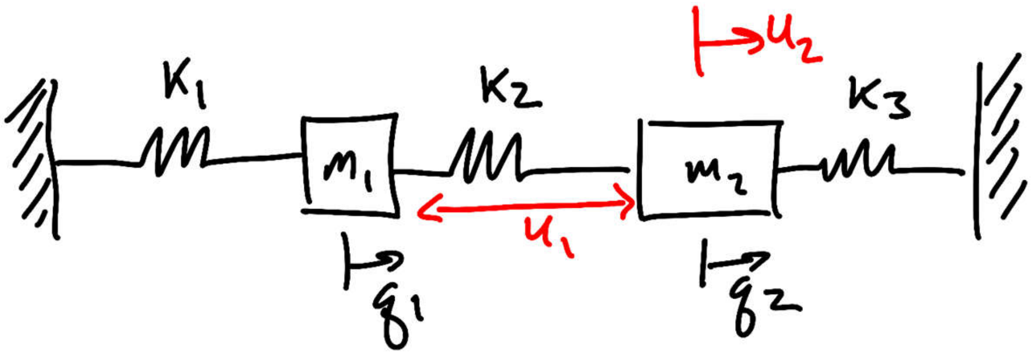
$$A = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ -2.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & -2.0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\longrightarrow \omega_{n,12} = 1.73$ rad/sec [2.72 Hz]

$\omega_{n,34} = 1.00$ rad/s [1.57 Hz]

Example: Convert CT SS model to DT SS model (cont'd)

- System of 2 masses and 3 springs: 2 actuator inputs u and 2 sensor outputs y



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

Converted to DT SS model **using the same state variables** with ZOH and sample rate $\Delta t = 0.2$ sec

$$x_{k+1} = Fx_k + Gu_k$$

$$y_k = Hx_k + Mu_k$$

$$x_k = [q_1(k), \dot{q}_1(k), q_2(k), \dot{q}_2(k)]^T$$

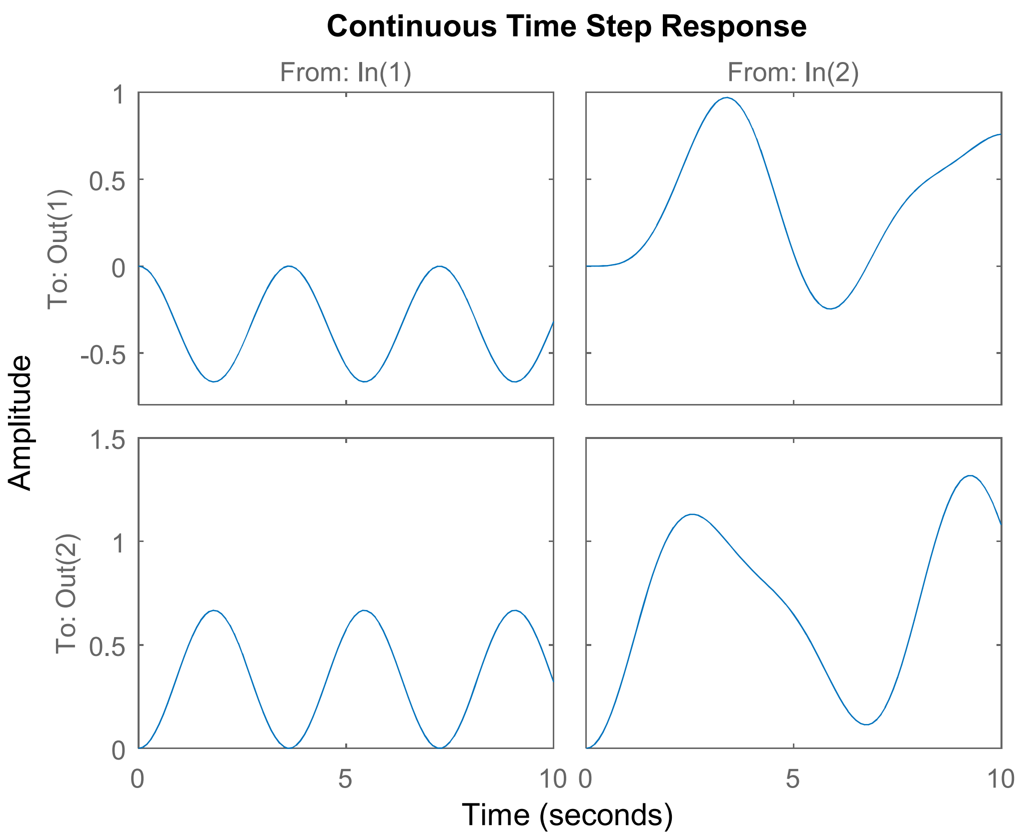
$$u_k = [u_1(k), u_2(k)]^T$$

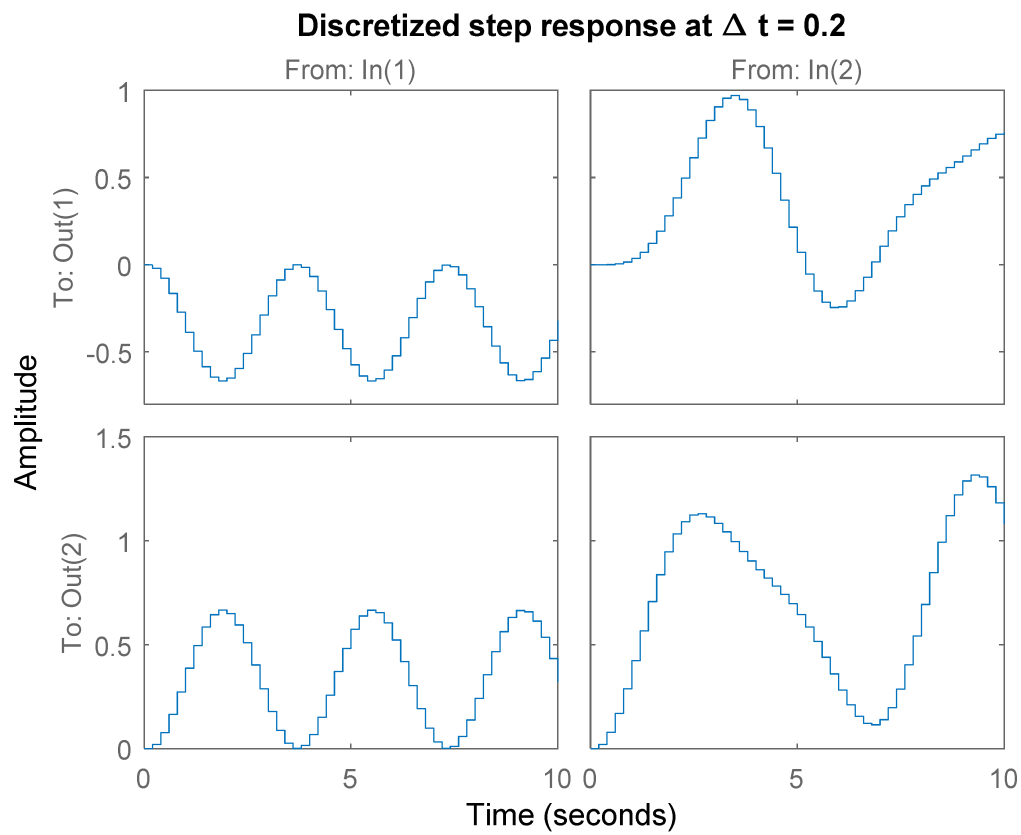
$$y_k = [q_1(k), q_2(k)]^T$$

$$F = \begin{bmatrix} 9.6033e-01 & 1.9735e-01 & 1.9734e-02 & 1.3227e-03 \\ -3.9337e-01 & 9.6033e-01 & 1.9470e-01 & 1.9734e-02 \\ 1.9734e-02 & 1.3227e-03 & 9.6033e-01 & 1.9735e-01 \\ 1.9470e-01 & 1.9734e-02 & -3.9337e-01 & 9.6033e-01 \end{bmatrix} \quad G = \begin{bmatrix} -1.9801e-02 & 6.6312e-05 \\ -1.9602e-01 & 1.3227e-03 \\ 1.9801e-02 & 1.9867e-02 \\ 1.9602e-01 & 1.9735e-01 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Sample Input Step Response Output from DT vs CT





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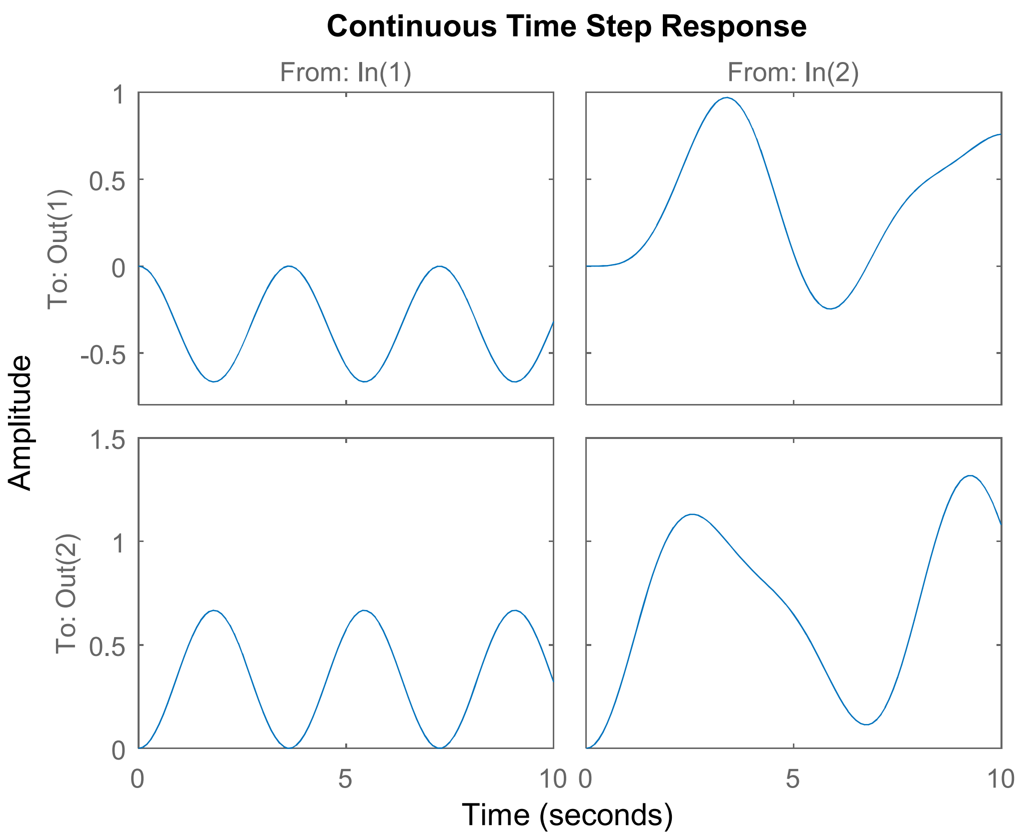
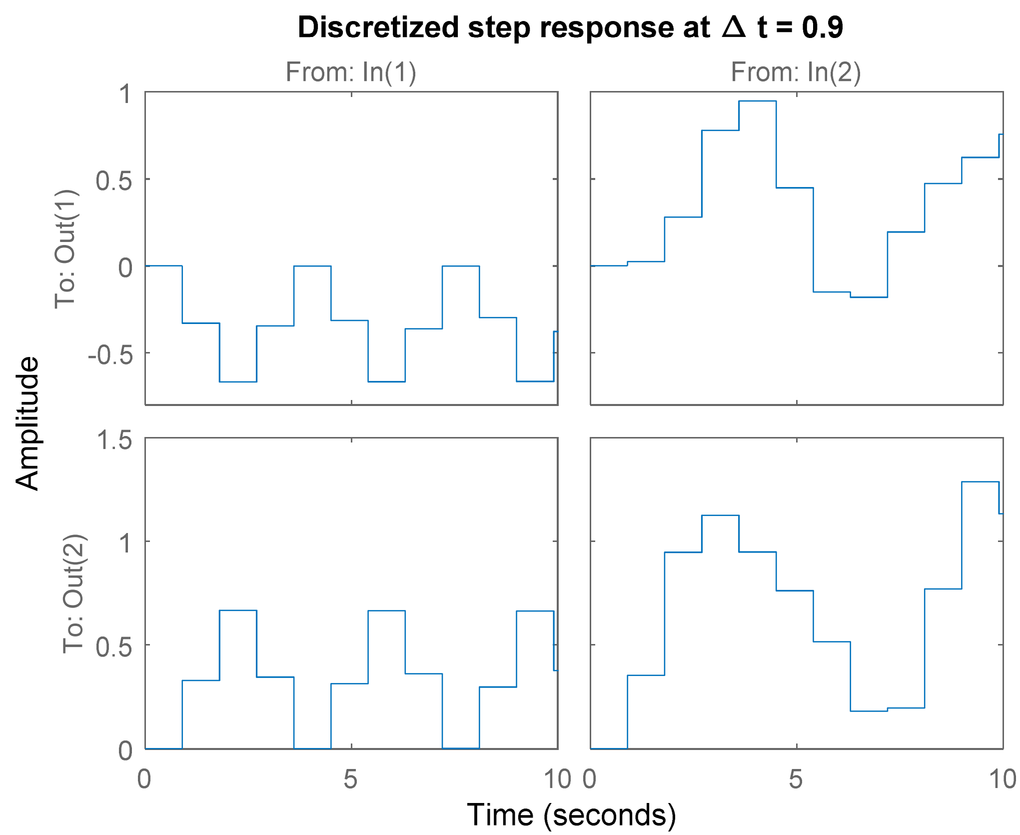
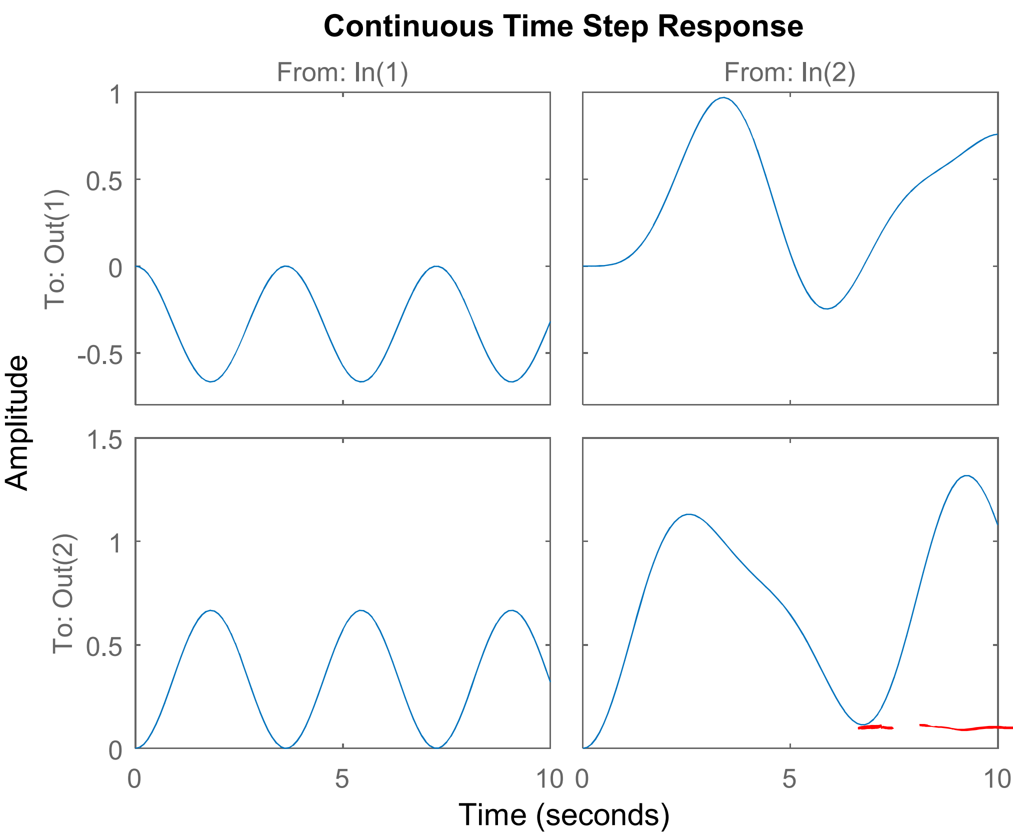
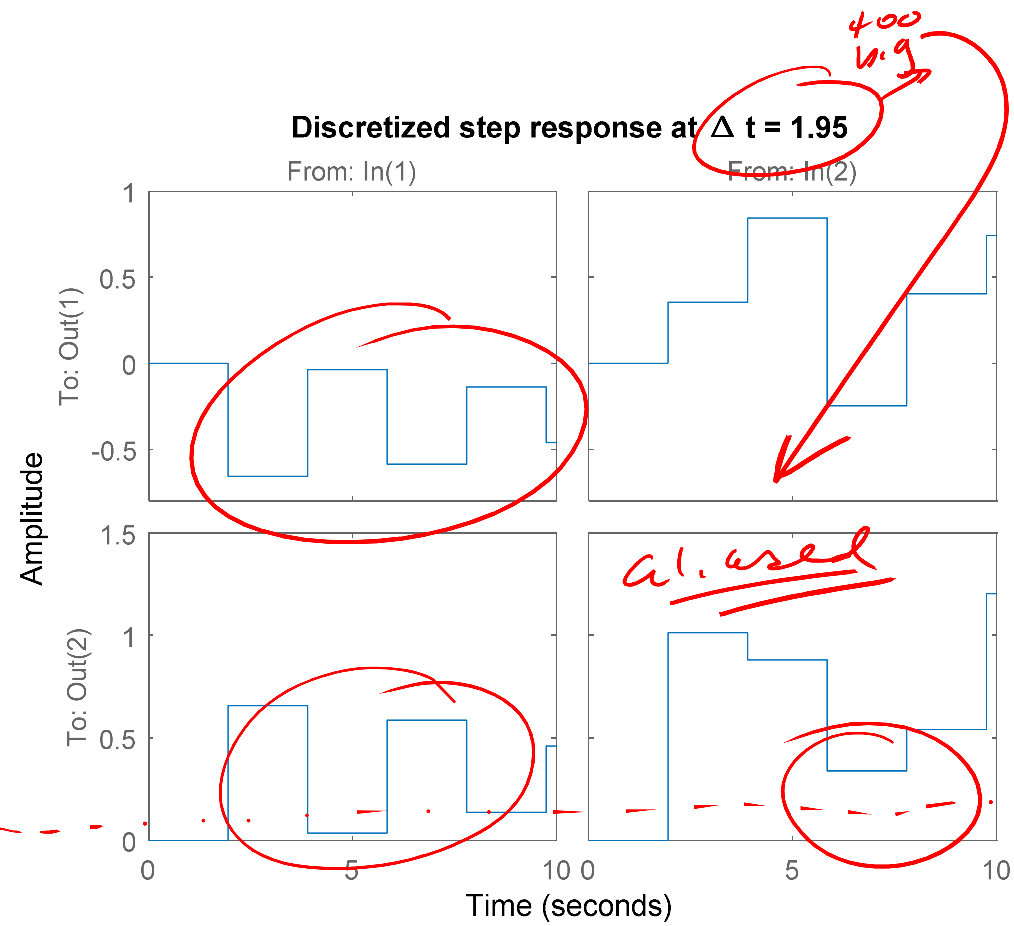




Sample Input Step Response Output from DT vs CT





Nyquist Rate and CT System Natural Frequencies

- **WARNING FOR CT → DT conversions: cannot just pick any old Δt !!!**
- For LTI systems: fundamental upper bound on how large Δt should be

Nyquist sampling criterion: Need $\left(\frac{\pi}{\Delta t}\right) > 2 \cdot |\lambda_A^{max}|$

where $|\lambda_{max}^{A}|$ is the largest complex magnitude (max. freq. / +ive const.) among all eigenvalues of A