Geometric Factor Derivation

Adri Jackson, Aaron Russel, Laura Peticolas, Noé Lugaz

June 5, 2024

The derivation in Johnstone et al. begins with an equation for the density of particles in an elemental volume $d\mathbf{p}d\mathbf{q}$ of phase space,

$$N/G(\mathbf{p}) = f\left(\mathbf{p},\mathbf{q}\right)d\mathbf{p}d\mathbf{q} \tag{1}$$

where $\mathbf{q}$ is a position vector, $\mathbf{p}$ the conjugate momentum and $G(\mathbf{p})$ is the detector function. The counts in an accumulation bin of the sensor are,

$$N = \int \mathbf{v}\cdot\mathbf{A}f\left(\mathbf{v}\right)G\left(\mathbf{v}\right)d\mathbf{v}dt \tag{2}$$

where $\mathbf{A}$ is a vector whose magnitude is the aperture area and whose direction is along the normal to the aperture. $G\left(\mathbf{v}\right)$ has the property,

$$G\left(\mathbf{v}\right) = G\left(\mathbf{v}/v_0\right) = G\left(\mathbf{v}'\right) \tag{3}$$

where v_0 is the centre velocity in the passband. Therefore we may rewrite part of Equation 2,

$$\mathbf{v}\cdot\mathbf{A}G\left(\mathbf{v}\right)d\mathbf{v} = v_0^4[\mathbf{v}'\cdot\mathbf{A}G\left(\mathbf{v}'\right)v^{'2}d\mathbf{v}'d\Omega] \tag{4}$$

We may then define the bracketed quantity as the geometric factor,

$$[GF] = \int \mathbf{v}'\cdot\mathbf{A}G\left(\mathbf{v}'\right)v^{'2}d\mathbf{v}'d\Omega \tag{5}$$

which is independent of v_0 and can be obtained from calibration. If we assume $f\left(\mathbf{v}\right)$ is constant over the whole of the detector response, then we may write,

$$f\left(v_0\right) = N/\left(v_0^4[GF]dt\right) \tag{6}$$

To obtain the geometric factor from calibration we establish the number of counts in a measurement as,

$$N_b = Tv_0\left(\mathbf{v}_b^{'}\cdot\mathbf{A}\right)G\left(\mathbf{v}_b^{'}\right)\int f_b\left(\phi\right)d\left(\mathbf{v}\right) \tag{7}$$

where we take the distribution function of the beam to be like a delta function, and replace the integral over time by T, the accumulation time. The density of this beam may be obtained by measurement with a Faraday cup whose collecting area is A_f,

$$\int f_b\left(\mathbf{v}\right)dv = I_b/ev_b\mathbf{A}_f \tag{8}$$

where e is the charge of electron, the subscript b refers to the beam, and $I_b = I_0/S_b$ where I_0 is a standard current and S_b a weighting factor obtained from multiple measurements of the current. Plugging the right-hand side into Equation 7 yields,

$$N_b = Tv_0\left(\mathbf{v}_b^{'}\cdot\mathbf{A}\right)G\left(\mathbf{v}_b^{'}\right)I_b/ev_bA_f \tag{9}$$

We may rearrange this to,

$$\mathbf{v}\cdot\mathbf{A}G\left(\mathbf{v}\right)d\mathbf{v} = ev_bA_fN_bS_b/I_0Tv_0 \tag{10}$$

We may then plug this into the geometric factor, Equation 5, and replace the integral with a summation over the changes in velocity and angular elements,

$$[GF] = \sum_b\left(ev_bA_fN_bS_b/I_0Tv_0\right)v^{'2}\Delta v^{'}\left(sin\theta\right)\Delta\sin\Delta\phi \tag{11}$$

Pulling out constants from the summation, and using $I_b = I_0/S_b$ we may rearrange,

$$[GF] = \left(eA_f/I_0T\right)\left(v_b/v_0\right)\sum_b N_bS_bv^{'2}\Delta v^{'}\left(sin\theta\right)\Delta\sin\Delta\phi \tag{12}$$

If we make the steps in $\Delta v^{'}$ and $\Delta\theta$ constant, set $\Delta\phi = \Delta\phi^{'}/sin\theta =$ constant, and approximate $v_b/v_0 = 1, v^{'} = 1$, we may simplify to

$$GF = \left(eA_f\Delta v^{'}\Delta\theta\Delta\phi/I_0T\right)\sum_b N_bS_b \tag{13}$$

For our purposes we cannot approximate our beam energy as a delta function, our particle simulations used a uniform energy distribution from 10 eV to 150 eV, and therefore we must take a different route starting at Equation 7. After replacing the distribution function we should be able to follow similar final steps. We would be calculating a geometric factor based on data from testing in the lab combined with SIMION simulations performed by Dan Abel. For our velocity range we can trivially calculate that from the energy range in our simulations/testing. For the angular ranges we will take these to be the same as our estimated angular spread of particles in the lab. I have written code to calculate the instrument current response based on data provided by Sanj. I am currently not sure what we should set the accumulation time to, as this was not part of the simulation or recorded in the lab. The total counts can be obtained from the simulation results, and the weighting factors can be obtained by calculating a standard current, or set to 1 if we choose to treat the instrument response as standard.
Source:
https://iopscience.iop.org/article/10.1088/0022-3735/20/6/038