# Assignment9 – First-Order Logic

**Problem 9.1 (Predicate Logic)**

Consider the following first-order signature $\Sigma$:

- binary function constant $f$
- unary predicate constant $p$

1. Give a closed formula over $\Sigma$ that uses all its constants.

   *Solution:* E.g., $\forall x.p(f(x,x))$

2. Give a first-order model for $\Sigma$

   *Solution:* E.g., universe $\mathbb{N}$, $\mathcal{I}(f)(u,v) = u+v$, $\mathcal{I}(p) = \{0\}$.

3. Consider the formula $A = p(x) \wedge p(y)$ and assume a first-order model with universe $\mathbb{N}$ and $\mathcal{I}(p) = \{0\}$.
   Give an assignment $\alpha$ such that $\mathcal{I}_\alpha(A) = \mathsf{T}$.

   *Solution:* $\alpha(x) = \alpha(y) = 0$

4. Prove or refute the following statement: A first-order model that satisfies $\forall x.p(x)$, satisfies all formulae.

   *Solution:* False. Any model that satisfies $\forall x.p(x)$ (of which there are many) does not satisfy $F$.

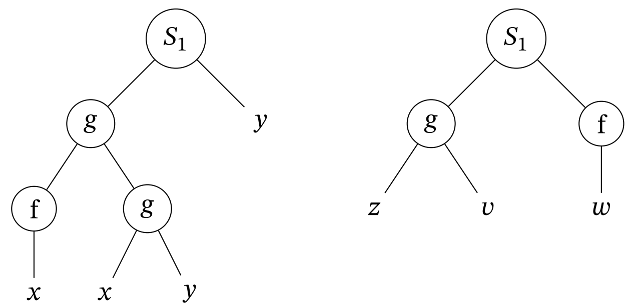
**Problem 9.2 (Unification)**

Let $S_1 \in \Sigma_2^f$, $S_2 \in \Sigma_3^f$, $f \in \Sigma_1^f$, $g \in \Sigma_2^f$, $c \in \Sigma_0^f$

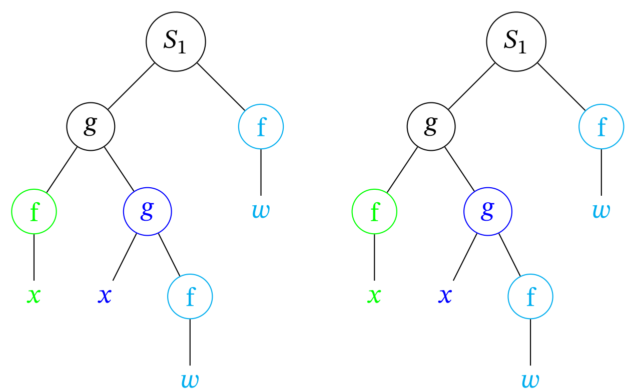
Decide whether (and how or why not) the following pairs of terms are unifiable.

1. $S_1(g(f(x), g(x,y)), y)$ and $S_1(g(z,v), f(w))$

   *Solution:* The term trees look like this:



Obviously, we need to perform the following substitutions to make the two trees equal:



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2. $S_2(g(f(x), g(x, u)), f(y), z)$ and $S_2(g(g(g(u, v), f(w)), f(c)), f(g(u, v)), f(c))$

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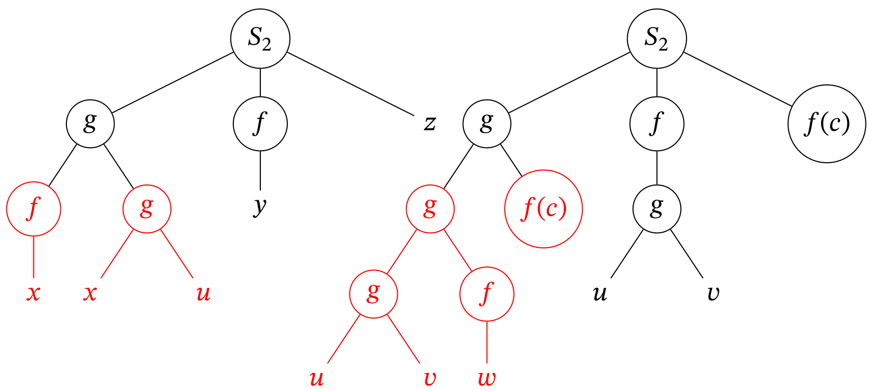
*Solution:* The term trees look like this:




Obviously, the red subtrees can't be unified.

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**Problem 9.3 (Natural Deduction)**

Let $R \in \Sigma_2^p$, $P \in \Sigma_1^p$, $c \in \Sigma_0^f$.
Prove the following formula in Natural Deduction:

$$(\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)) \wedge (\exists Y.R(c,Y)) \Rightarrow P(c)$$

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*Solution:*

| | |
|---|---|
| $1(\text{Assumption})^1$ | $(\forall X.\forall Y.R(Y,X) \Rightarrow P(Y), \exists Y.R(c,Y))$ |
| $2 \wedge E_l(1)$ | $\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)$ |
| $3 \wedge E_r(1)$ | $\exists Y.R(c,Y)$ |
| $4 \forall E(2)$ | $\forall Y.R(Y,X) \Rightarrow P(Y)$ |
| $5 \forall E(4)$ | $R(c,X) \Rightarrow P(c)$ |
| $6 \forall I(5)$ | $\forall X.R(c,X) \Rightarrow P(c)$ |
| $7(\text{Assumption})^2$ | $R(c,d)$ |
| $8 \forall E(6)$ | $R(c,d) \Rightarrow P(c)$ |
| $9 \Rightarrow E(8,7)$ | $P(c)$ |
| $10 \exists E^2(3,9)$ | $P(c)$ |
| $11 \Rightarrow I^1(10)$ | $((\forall X.\forall Y.R(Y,X)) \Rightarrow P(Y)) \wedge (\exists Y.R(c,Y)) \Rightarrow P(c)$ |

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**Problem 9.4 (First-Order Tableaux)**

Prove or refute the following formula using the first-order free variable tableaux calculus. We have $P, Q \in \Sigma_1^p$.

$$(\forall X.P(X) \Rightarrow Q(X)) \Rightarrow (((\forall X.P(X)) \Rightarrow (\forall X.Q(X))))$$

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*Solution:*

$$\begin{array}{c|c}
\multicolumn{2}{c}{((\forall X.P(X) \Rightarrow Q(X)) \Rightarrow ((\forall X.P(X)) \Rightarrow (\forall X.Q(X))))^{\mathsf{F}}} \\
\multicolumn{2}{c}{(\forall X.P(X) \Rightarrow Q(X))^{\mathsf{T}}} \\
\multicolumn{2}{c}{((\forall X.P(X)) \Rightarrow (\forall X.Q(X)))^{\mathsf{F}}} \\
\multicolumn{2}{c}{(P(Y) \Rightarrow Q(Y))^{\mathsf{T}}} \\
\multicolumn{2}{c}{(\forall X.P(X))^{\mathsf{T}}} \\
\multicolumn{2}{c}{(\forall X.Q(X))^{\mathsf{F}}} \\
\multicolumn{2}{c}{P(Z)^{\mathsf{T}}} \\
\multicolumn{2}{c}{Q(c)^{\mathsf{F}}} \\
P(Y)^{\mathsf{F}} & Q(Y)^{\mathsf{T}} \\
\bot : [Y/Z] & \bot : [c/Y]
\end{array}$$

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**Problem 9.5 (First-Order Resolution)**

Let $P, Q \in \Sigma_1^p$, $R \in \Sigma_2^p$, $c, d \in \Sigma_0^f$. Prove the following formula using the first-order resolution calculus $\mathcal{R}_1$.

$$\exists X.\forall Y.\exists W.\exists Z.\neg((R(Z,Y)\vee\neg P(Z))\wedge(\neg Q(d)\vee P(c))\wedge(Q(d)\vee\neg P(c))\wedge(\neg R(Z,Y)\vee\neg P(W)\vee\neg Q(X))\wedge P(c))$$

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*Hint:* Note, that the formula is already (close to) a negated CNF, so if you spend any significant amount of time transforming the formula, you are most likely doing something wrong.

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*Solution:* We negate:

$$\forall X.\exists Y.\forall W.\forall Z.(R(Z,Y)\vee\neg P(Z))\wedge(\neg Q(d)\vee P(c))\wedge(Q(d)\vee\neg P(c))\wedge(\neg R(Z,Y)\vee\neg P(W)\vee\neg Q(X))\wedge P(c)$$

Substituting bound variables:

$$(R(Z,f_Y(X))\vee\neg P(Z)\vee\neg Q(d)\vee P(c)\vee Q(d)\vee\neg P(c)\vee\neg R(Z,f_Y(X))\vee\neg P(W)\vee\neg Q(X)\vee P(c))$$

Resolution:

$$\{Q(d)^{\mathsf{T}}, P(C)^{\mathsf{F}}\} + \{P(c)^{\mathsf{T}}\} \Longrightarrow \{Q(d)^{\mathsf{T}}\}$$

$$\{Q(d)^{\mathsf{T}}\} + \{R(Z, f_Y(X))^{\mathsf{F}}, P(W)^{\mathsf{F}}, Q(X)^{\mathsf{F}}\}[d/X] \Longrightarrow \{R(Z, f_Y(d))^{\mathsf{F}}, P(W)^{\mathsf{F}}\}$$

$$\{P(c)^{\mathsf{T}}\} + \{R(Z, f_Y(d))^{\mathsf{F}}, P(W)^{\mathsf{F}}\}[c/W] \Longrightarrow \{R(Z, f_Y(d))^{\mathsf{F}}\}$$

$$\{P(c)^{\mathsf{T}}\} + \{R(Z, f_Y(X))^{\mathsf{T}}, P(Z)^{\mathsf{F}}\}[c/Z] \Longrightarrow \{R(c, f_Y(X))^{\mathsf{T}}\}$$

$$\{R(c, f_Y(X))^{\mathsf{T}}\}[d/X] + \{R(Z, f_Y(d))^{\mathsf{F}}\}[c/Z] \Longrightarrow \emptyset$$

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