Artificial Intelligence I

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Contents

1 Recap

1.1 Sets

Def 1.1. Set: A collection of different things (called **elements** or **members** of the set). We can represent sets by listing the elements within curly brackets, e.g. $\{a, b, c\}$ or by describing the elements via a property, e.g. $\{x \mid x \mod 2 = 0\}$ (the set of even numbers). We can state element-hood ($a \in S$) or not ($b \notin S$)

Def 1.2. Set Equality: $A \equiv B :\equiv \forall x.x \in A \Leftrightarrow x \in B$

Def 1.3. Subset: $A \subseteq B :\equiv \forall x.x \in A \Rightarrow x \in B$

Def 1.4. Proper Subset: $A \subset B :\equiv (A \subseteq B) \wedge (A \not\equiv B)$

Def 1.5. Super Set: $A \supseteq B :\equiv \forall x.x \in B \Rightarrow x \in A$

Def 1.6. Proper Superset: $A \supset B :\equiv (A \supseteq B) \wedge (A \not\equiv B)$

Def 1.7. Set Union: $A \cup B := \{x \mid x \in A \vee x \in B\}$

Def 1.8. Set Intersection: $A \cap B := \{x \mid x \in A \wedge x \in B\}$

Def 1.9. Disjoint: Two sets A, B are **disjoint** iff $A \cap B = \Phi$

Def 1.10. Set Difference: $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$

Def 1.11. Power Set: $\mathcal{P}(A) := \{S \mid S \subseteq A\}$ (the set of all subsets)

Def 1.12. Cartesian Product: $A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$, (a, b) is a pair

Def 1.13. Family: A set of sets

Def 1.14. Topology: A family of open sets

Def 1.15. Indexing Function: Let I and X be sets. A function $f : I \to X$ is called an indexing function. The set I is called the index set. For each $i \in I$, we define $x_i := f(i)$, here i is called the index (or parameter) of x_i.

Def 1.16. Indexed Family: Given an indexing function $f : I \to X$, the set of values $f(I) = \{f(i) : i \in I\}$ is called an indexed family. It is often written as $(x_i)_{i\in I}$, $\langle x_i \rangle_{i \in I}$, or $\{x_i\}_{i\in I}$.

Def 1.17. Union over a Family: Let $(X_i)_{i\in I}$ be an indexed family, then $\bigcup_{i\in I} X_i := \{x \mid \exists i \in I.x \in X_i\}$

Def 1.18. Intersection over a Family: Let $(X_i)_{i\in I}$ be an indexed family, then $\bigcap_{i\in I} X_i := \{x \mid \forall i \in I.x \in X_i\}$

Def 1.19. n-fold Cartesian Product: $\prod_{i=1}^{n} X_i := X_1 \times \cdots \times X_n := \{\langle x_1, \cdots, x_n \rangle \mid \forall i.1 \le i \le n \Rightarrow x_i \in X_i\}$ where $\langle x_1, \cdots, x_n \rangle$ is called an n-tuple.

Def 1.20. n-dim Cartesian Space: $X^n := \{\langle x_1, \cdots, x_n \rangle \mid 1 \le i \le n \Rightarrow x_i \in X\}$ where $\langle x_1, \cdots, x_n \rangle$ is called a vector.

Def 1.21. Size of a Set: The size $\Gamma(A)$ (or $|A|$) of a set A is the number of elements in A.

1.2 Relations

Def 1.22. Relation: $R \subseteq A \times B$ is a (binary) relation between A and B

Def 1.23. Relation on: a relation $R \subseteq A \times B$ where $A = B$ is called a *relation on A*

Def 1.24. Total: A relation $R \subseteq A \times B$ is called *total* (*left total*) iff $\forall x \in A.\exists y \in B.(x, y) \in R$

Def 1.25. Converse Relation: $R^{-1} \subseteq B \times A := \{(y, x) \mid (x, y) \in R\}$ is the converse relation of R

Def 1.26. Relation Composition: The composition of $R \subseteq A \times B$ and $S \subseteq B \times C$ is defined as $R \circ S := \{(a, c) \in A \times C \mid \exists b \in B.(a, b) \in R \wedge (b, c) \in S\}$

A relation on A: $R \subseteq A \times A$ is called:

- **reflexive** on A, iff $\forall a \in A.(a, a) \in R$
- **irreflexive** on A, iff $\forall a \in A.(a, a) \notin R$
- **symmetric** on A, iff $\forall a, b \in A.(a, b) \in R \Rightarrow (b, a) \in R$
- **asymmetric** on A, iff $\forall a, b \in A.(a, b) \in R \Rightarrow (b, a) \notin R$
- **antisymmetric** on A, iff $\forall a, b \in A.(a, b) \in R \wedge (b, a) \in R \Rightarrow a = b$

- **transitive** on A, iff $\forall a, b, c \in A.(a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$
- **equivalence relation** on A, iff R is reflexive symmetric, and transitive

Def 1.27. Equality Relation: The equality relation is an equivalence relation on any set.

Def 1.28. Divides Relation: The *divides* relation on the integers is defined as $| \subseteq \mathbb{Z} \times \mathbb{Z}$, where $| = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid \exists k \in \mathbb{Z} : b = a \cdot k\}$. We write $a \mid b$ to denote $(a,b) \in |$ (read as *"a divides b"*).

Def 1.29. Congruence Modulo: For a fixed $n \in \mathbb{N}$ with $n \geq 1$, the *congruence modulo n* relation on the integers is defined as

$$\equiv_n \subseteq \mathbb{Z} \times \mathbb{Z}, \qquad \equiv_n = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid n \mid (a-b)\}.$$

We write $a \equiv b \pmod{n}$ to denote $(a,b) \in \equiv_n$ (read as *"a is congruent to b modulo n"*).

Def 1.30. Partial Ordering ($\preceq$): A relation $R \subseteq A \times A$ is called a (non-strict) **partial ordering** on A, iff R is reflexive, antisymmetric, and transitive on A.

Def 1.31. Strict Partial Ordering ($\prec$): A relation $R \subseteq A \times A$ is called a **strict partial ordering** on A, iff R is irreflexive, and transitive on A.

Def 1.32. Comparable: In a non-strict partial ordering, two elements a, b are comparable if either $a \preceq b$ or $b \preceq a$. If neither holds, they are incomparable.

Def 1.33. Linear Order: A partial ordering is called linear (or total) on A, iff all elements in A are comparable, i.e. if $(x,y) \in R$ or $(y,x) \in R$ for all $x, y \in A$. For example, the $\leq$ relation is a linear order on $\mathbb{N}$. However, the "divides" $(|)$ relation is a non-strict partial ordering on $\mathbb{N}$ that is not linear (e.g., neither $2 \mid 3$ nor $3 \mid 2$).

1.3 Functions

Def 1.34. Partial function: A relation $R \subseteq X \times Y$ is a *partial function* from X to Y ($f : X \rightharpoonup Y$) iff $\forall x \in X, \forall y_1, y_2 \in Y.\big((x,y_1) \in R \wedge (x,y_2) \in R \Rightarrow y_1 = y_2\big)$ (i.e. there is at most one $y \in Y$ with $(x,y) \in f$)

Def 1.35. dom, codom: We call X the domain ($\mathrm{dom}(f)$), and Y the codomain ($\mathrm{codom}(f)$)

Def 1.36. Function Application: Instead of writing $(x,y) \in f$ we write $f(x) = y$

Def 1.37. Undefined: we call a partial function $f : X \rightharpoonup Y$ undefined at $x \in X$, iff $\forall y \in Y.(x,y) \notin f$ ($f(x) = \perp$).

Def 1.38. Function: A partial function $f : X \rightharpoonup Y$ is called a *function* (or *total function*) from X to Y iff it is a total relation. ($\forall x \in X.\exists^1 y \in Y : (x,y) \in f$)

> **Note**
>
> A partial function guarantees uniqueness, but not existence. So each x has at most one y (some might have none). A total function guarantees both uniqueness and existence (at most and at least $\to$ exactly one)

A function $f : X \to Y$ is called

- **injective** iff $\forall x_1, x_2 \in X.f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- **surjective** iff $\forall y \in Y.\exists x \in X.f(x) = y$
- **bijective** iff f is injective and surjective

Def 1.39. Function Composition: If $f : A \to B$ and $g : B \to C$ are functions, then we call $g \circ f : A \to C; x \longmapsto g(f(x))$ the composition of g and f (read g after f).

Def 1.40. Image and Preimage: Let $f : X \to Y$ be a function, $X' \subseteq X$ and $Y' \subseteq Y$, then we call

- $f(X') := \{y \in Y \mid \exists x \in X'.(x,y) \in f\}$ the **image** of X' under f,
- $\mathrm{Im}(f) := f(X)$ the **image** of f, and
- $f^{-1}(Y') := \{x \in X \mid \exists y \in Y'.(x,y) \in f\}$ the **preimage** of Y' under f.

Def 1.41. Cardinality: We say that a set A is **finite** and has **cardinality** $\Gamma(A) \in \mathbb{N}$, iff there is a bijective function $f : A \to \{n \in \mathbb{N} \mid n < \Gamma(A)\}$.

Def 1.42. Countably Infinite: We say that a set A is countably infinite, iff there is a bijective function $f : A \to \mathbb{N}$.

Def 1.43. Countable: A set A is called countable, iff it is finite or countably infinite.

Def 1.44. Curried Function Type: Let X, Y, Z be sets. An *uncurried* function is written as

$$f : X \times Y \to Z, \quad f(x, y) = E(x, y).$$

The same function can equivalently be written in *curried* form as

$$f : X \to Y \to Z, \quad f(x)(y) = E(x, y).$$

Thus in the curried notation the function takes one argument at a time: for $x \in X$, the value $f(x)$ is itself a function $Y \to Z$, and applying it to $y \in Y$ yields $f(x)(y) \in Z$.

Def 1.45. Invertible: Let A and B be sets and $f : A \to B$ a function, then f is called invertible, iff there is a function $g : B \to A$, such that $f \circ g = \mathrm{Id}_B$. g is called the inverse function (or just inverse) of f and is written as f^{-1}.

1.4 Mathematical Structures

Def 1.46. Formulae: A mathematical formula can be a **mathematical statement** (clause) which can be true or false (e.g. $x > 5, 3 + 5 = 7$), or a **mathematical object** (e.g. $3, n, x^2 + y^2 + z^2, \int_1^0 x^{3/2}dx$)

Def 1.47. Mathematical Structure: A mathematical structure combines multiple mathematical objects (the components) into a new object. The components usually have names by which they can be referenced. Given a definition of a mathematical structure S, we say that any object that conforms to that is an instance of S.

Def 1.48. Group: A group is a mathematical structure $\langle G, \circ, e, .^{-1}\rangle$ that consists of:

- a base set G of objects,
- an operation $\circ : G \times G \to G$, such that $\forall a, b \in G.a \circ b \in G$ and $\forall a, b, c \in G.(a \circ b) \circ c = a \circ (b \circ c)$,
- a unit $e \in G$, such that $\forall a \in G.e \circ a = a \circ e = a$, and
- the inverse function $.^{-1} : G \to G$, such that $\forall a \in G.a \circ a^{-1} = a^{-1} \circ a = e$

An example of a group is the set $G = \mathbb{Z}$, with the operation $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$; then $e = 0$ and $.^{-1}$ is $\lambda x \in \mathbb{Z}. - x$. We write it as $\langle \mathbb{Z}, +, 0, -\rangle$.

Question: can we do the same for multiplication where $e = 1$? If we think about it, we would be stuck on finding the $.^{-1}$ component of the group. What can we always multiply with any integer and get $e = 1$ back? Take for example 1, we can multiply it with $\frac{1}{1}$ (great!). Take -1, we can multiply it with $\frac{1}{-1}$ (also great!). But take any other integer, e.g. 2, we must multiply it with $\frac{1}{2}$ to get 1 back, however, $\frac{1}{2} \notin \mathbb{Z}$, it is however $\in \mathbb{Q}$! The multiplicative group would only make sense iff $G = \mathbb{Q} \setminus \{0\}$. We write that as $\langle \mathbb{Q} \setminus \{0\}, *, 1, a \mapsto \frac{1}{a}\rangle$.

But we need some multiplication structure that works for $\mathbb{Z}$!

Def 1.49. Monoid: A monoid is a mathematical structure $\langle M, \circ, e\rangle$ that consists of:

- A base set M of objects,
- An operation $\circ : M \times M \to M$, such that for all $a, b \in M$, we have $a \circ b \in M$ (closure),
- Associativity: for all $a, b, c \in M$, $(a \circ b) \circ c = a \circ (b \circ c)$, and
- A unit element $e \in M$, such that for all $a \in M.e \circ a = a \circ e = a$.

Hence, the mathematical structure $\langle \mathbb{Z}, *, 1\rangle$ is a monoid (a group missing an inverse function).

We saw that $\langle \mathbb{Z}, +, 0, -\rangle$ is a group under addition. And $\langle \mathbb{Z}, *, 1\rangle$ is a monoid under multiplication. Now, we want to do everyday arithmetics with integers; we want expressions where we have additions and multiplication (e.g. $a * (b + c)$) which evaluates to $(a * b) + (a * c)$.

For that, we need a mathematical structure where both addition and multiplication live together.

Def 1.50. Ring: A ring is a mathematical structure $\langle R, +, 0, -, *, 1\rangle$ consisting of:

- A base set R.

- An addition operation $+ : R \times R \to R$ such that $\langle R, +, 0, - \rangle$ is a group (called abelian group).
- A multiplication operation $* : R \times R \to R$ such that $\langle R, *, 1 \rangle$ is a monoid.
- Distributivity: for all $a, b, c \in R, a * (b + c) = (a * b) + (a * c), (a + b) * c = (a * c) + (b * c)$

Hence, mixing components from the group $\langle \mathbb{Z}, +, 0, - \rangle$ and the monoid $\langle \mathbb{Z}, *, 1 \rangle$ leaves us with a ring $\langle \mathbb{Z}, +, 0, -, *, 1 \rangle$

Def 1.51. Magma: A magma is a mathematical structure $\langle M, \circ \rangle$ consisting of:

- A base set M, and
- A binary operation $\circ : M \times M \to M$.

Example: $\langle \mathbb{Z}, - \rangle$ (integers under subtraction). $\langle \mathbb{N}, + \rangle$ (natural numbres under addition)

Def 1.52. Semigroup: A semigroup (magma with associativity) is a mathematical structure $\langle S, \circ \rangle$ consisting of:

- A base set S,
- A binary opertation $\circ : S \times S \to S$, and
- Associativity: for all $a, b, c \in S, (a \circ b) \circ c = a \circ (b \circ c)$

Example: $\langle \mathbb{Z}, + \rangle$ (integers under addition).

1.5 Formal Languages and Grammars

Def 1.53. Alphabet: An alphabet is a finite set; we call each element $a \in A$ a **character**, and an n-tuple $s \in A^n$ a **string** (of *length* n over A). We often write a string $\langle c_1, \cdots, c_n \rangle$ as "$c_1 \cdots c_n$", for instance `"abc"` for $\langle a, b, c \rangle$

Def 1.54. Empty String: Let A be an alphabet, then $A^0 = \{\langle\rangle\}$, where $\langle\rangle$ is the unique 0-tuple. We consider $\langle\rangle$ as the string of length 0 and call it the **empty string** and denote it with ϵ.

Def 1.55. String Length: Given a string s, we denote its length with $|s|$.

Def 1.56. String Concatenation: The concatenation $\mathrm{conc}(s, t)$ of two strings $s = \langle s_1, \cdots, s_n \rangle \in A^n$ and $t = \langle t_1, \cdots, t_m \rangle \in A^m$ is defined as $s + t$ or simply `st`

Def 1.57. Kleene Plus/Star: Let A be an alphabet, then we define the sets:

- $A^+ := \bigcup_{i \in \mathbb{N}^+} A^i$ (*nonempty strings*), and
- $A^* := A^+ \cup \{\epsilon\}$ (*strings*).

Def 1.58. Formal Language: A set $L \subseteq A^*$ is called a *formal language* over A. For example, If $A = \{a, b\}$ then:

$$A^* = \bigcup_{n=0}^{\infty} A^n$$

where $A^0 = \epsilon$, $A^1 = \{a, b\}$, $A^2 = \{a, b, ab, ba\}$, and so on ...

$$A^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, \cdots\}$$
$$A^+ = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, \cdots\}$$

A formal language might be $L = \{a, b, aab, bbb, \cdots\}$

Def 1.59. Repeating Chars: We use $c^{[n]}$ for the string that consists of the character `c` repeated n times. **Examples:**

- Let $A = \{a, b\}$, $a^{[5]} =$ `"aaaaa"`.
- The set $M := \left\{ \mathrm{ba}^{[n]} \mid n \in \mathbb{N} \right\}$ of strings that start with character b followd by an arbitrary numbers of a's is a formal language over $A = \{a, b\}$

> **Note**
>
> A formal language might be infinite and even undecidable even if the alphabet A is finite. For example, let $A = \{a, b\}$ then the formal language $L = \{a, ab, bba\}$ is finite but the formal language $L = \{\, a^{[n]} \mid n \in \mathbb{N} \,\}$ is infinite.

Since we cannot list all strings in a formal language L because there are infinitely many, we need a **finite description** that can generate all strings in L. We need **Grammars**.

Def 1.60. Phrase Structure Grammar: A phrase structure grammar (*type 0 grammar, unrestricted grammar, grammar*) is a tuple $\langle N, \Sigma, P, S\rangle$ where

- N is a finite set of **nonterminal symbols**,
- Σ is a finite set of **terminal symbols**. (members of $N \cup \Sigma$ are called symbols).
- P is a finite set of **production rules**: pairs $p := h \to b$ (also written as $h \Rightarrow b$), where $h \in (\Sigma \cup N)^* N (\Sigma \cup N)^*$ and $b \in (\Sigma \cup N)^*$. The string h is called the **head** of p and b the **body**.
- $S \in N$ is a distinguished symbol called the **start symbol** (also **sentence symbol**).

The sets N and Σ are assumed to be disjoint. Any word $w \in \Sigma^*$ is called a **terminal word**.

> Note
>
> Production rules map strings with at least one nonterminal to arbitrary other strings

Notation: If we have n production rule $h \to b_i$ sharing a head, we often write $h \to b_1 \mid \cdots \mid b_n$ instead.

Example: A simple grammar for english sentences:

$$\begin{aligned} \text{S} &::= \text{NP Vi} \\ \text{NP} &::= \text{Article N} \\ \text{Article} &::= \text{the} \mid \text{a} \mid \text{an} \\ \text{N} &::= \text{dog} \mid \text{teacher} \mid \cdots \\ \text{Vi} &::= \text{sleeps} \mid \text{smells} \mid \cdots \end{aligned}$$

Def 1.61. Lexical: A production rule whose head is a single nonterminal and whose body consists of a single terminal is called **lexical** or a **lexical insertion rule**.

Def 1.62. Lexicon of a Grammar: The subset of lexical rules of a grammar G is called the **lexicon** of G

Def 1.63. Vocabulary: The set of body symbols are called the vocabulary (or the alphabet).

Def 1.64. Lexical Categories of a Grammar: The nonterminals that appear in the heads of lexical rules are called lexical categories of the grammar.

Def 1.65. Structural Rules: The non lexicon production rules are called structural.

Def 1.66. Phrasal categories: The nonterminals in the heads of production rules that expand into other nonterminals are called phrasal (syntactic) categories.

Def 1.67. G-Derivation: Given a phrase structure grammar $G := \{N, \Sigma, P, S\}$, we say G **derives** $t \in (\Sigma \cup N)^*$ from $s \in (\Sigma \cup N)^*$ in one step, iff there is a production rule $p \in P$ with $p = h \to b$ and there are $u, v \in (\Sigma \cup N)^*$, such that $s = uhv$ and $t = ubv$. We write $s \to_G^p t$ (or $s \to_G t$ if p is clear from the context) and use $\to_G^*$ for the reflexive transitive closure of $\to_G$. We call $s \to_G^* t$ a G-derivation of t from s.

Def 1.68. Sentential Form: Given a phrase structure grammar $G := \{N, \Sigma, P, S\}$, we say that $s \in (\Sigma \cup N)^*$ is a **sentential form** of G, iff $S \to_G^* s$

Def 1.69. Sentence: A sentential form that does not contain nonterminals is called a sentence of G, we also say that G **accepts** s. We say that G **rejects** s, iff it is not a sentence of G.

Def 1.70. Language Generation: The language $L(G)$ of G is the set of its sentences. We say that $L(G)$ is **generated** by G.

Def 1.71. Equivalent Grammars: We call two grammars **equivalent**, iff they have the same languages.

Def 1.72. Universal Grammar: A grammar G is said to be **universal** if $L(G) = \Sigma^*$

Def 1.73. Syntactic Analysis: Syntactic analysis (parsing / syntax analysis) is the process of analyzing a string of symbols, either in a formal or a natural language by means of a grammar.

Note

The shape of the grammar determines the size of its language

Def 1.74. Context-Sensitive: We call a grammar context-sensitive (or type 1), if the bodies of production rules have no less symbols than the heads.

Def 1.75. Context-Free: We call a grammar context-free (or type 2), iff the heads have exactly one symbol.

Def 1.76. Regular: We call a grammar regular (or type 3, or REG), if additionally the bodies are empty or consist of a nonterminal, optionally followed by a terminal symbol.

Note

By extension, a formal language L is called *context-sensitive / context-free / regular*, iff it is the language of a respective grammar. Context-free grammars are sometimes `CFGs` and context-free languages `CFLs`.

1.6 Graphs and Trees

Def 1.77. Undirected Graph: An undirected graph is a pair $\langle V, E\rangle$ such that

- V is a set of **vertices (nodes)**, and
- $E \subseteq \{\, \{v, \bar{v}\} \mid v, \bar{v} \in V \wedge v \neq \bar{v} \,\}$ is the set of its **undirected edges**.

Def 1.78. Directed Graph: A directed graph (digraph) is a pair $\langle V, E\rangle$ such that

- V is a set of **vertices (nodes)**, and
- $E \subseteq V \times V$ is the set of its **directed edges**.

Def 1.79. Indegree: Given a directed graph $\langle V, E\rangle$, the indegree $\mathrm{indeg}(v)$ of a vertex $v \in V$ is defined as $\mathrm{indeg}(v) = \Gamma(\{\, w \mid (w, v) \in E \,\})$.

Def 1.80. Outdegree: Given a directed graph $\langle V, E\rangle$, the outdegree $\mathrm{outdeg}(v)$ (branching factor) of a vertex $v \in V$ is defined as $\mathrm{indeg}(v) = \Gamma(\{\, w \mid (v, w) \in E \,\})$.

Note

directed graphs are nothing else than relations.

Def 1.81. Initial vs Terminal Node: Let $G = \langle V, E\rangle$ be a directed graph, then we call a node $v \in V$

- **initial** (source of G), iff there is no $w \in V$ such that $(w, v) \in E$ (no predecessor)
- **terminal** (sink of G), iff there is no $w \in V$ such that $(v, w) \in E$ (no successor)

Def 1.82. Graph Isomorphism: Iff we can find a bijection $\psi : V \to \bar{V}$ between two graphs $G = \langle V, E\rangle$ and $\bar{G} = \langle \bar{V}, \bar{E}\rangle$ then we call them isomorphic. The bijection $\psi : V \to \bar{V}$ is defined as $(a, b) \in E \Leftrightarrow (\psi(a), \psi(b)) \in \bar{E}$ for directed graph. And for undirected graph it is defined as $\{a, b\} \in E \Leftrightarrow \{\psi(a), \psi(b)\} \in \bar{E}$

Def 1.83. Equivalent Graphs: Two graphs G and $\bar{G}$ are **equivalent** iff there is an isomorphism ψ between G and $\bar{G}$.

Def 1.84. Labeled Graph: A labeled graph G is a quadruple $\langle V, E, L, l\rangle$ where $\langle V, E\rangle$ is a graph and $l : V \cup E \rightharpoonup L$ is a partial function into a set L of labels.

Def 1.85. Paths in Graphs: Given a graph $G := \langle V, E\rangle$ we call a $n+1$-tuple $p = \langle v_0, \cdots, v_n\rangle \in V^{n+1}$ a **path** in G iff $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq n$ and $n > 0$.

- We say that v_i are nodes on p and that v_0 and v_n are **linked** by p.
- v_0 and v_n are called the **start** and **end** of p (write $\mathrm{start}(p)$ and $\mathrm{end}(p)$), the other v_i are called **inner nodes** of p.
- n is called the **length** of p (write $\mathrm{len}(p)$).
- We denote the set of paths in G with $\Pi(G)$.

Def 1.86. Cyclic Graphs: Given a directed graph $G = \langle V, E\rangle$, a path p is called **cyclic** iff $\mathrm{start}(p) = \mathrm{end}(p)$. A cycle $\langle v_0, \cdots, v_n\rangle$ is called **simple**, iff $v_i \neq v_j$ for $1 \leq i, j \leq n$ with $i \neq j$ (all inner nodes are distinct).

Def 1.87. DAG: A directed graph with no cycles is called **directed acyclic graph (DAG)**.

Def 1.88. Node Depth: Let $G = \langle V, E\rangle$ be a directed graph, then the depth $\mathrm{dp}(v)$ of a vertex $v \in V$ is defined to be 0, iff v is a source of G and the supremum $\sup(\{\, \mathrm{len}(p) \mid \mathrm{indeg}(\mathrm{start}(p)) = 0 \wedge \mathrm{end}(p) = v \,\})$ otherwise, i.e. the length of the longest path from a source of G to v (can be infinite).

Def 1.89. Graph Depth: Given a directed graph $G = \langle V, E\rangle$, the **depth** ($\mathrm{dp}(G)$) of G is defined as the supremum $\sup(\{\, \mathrm{len}(p) \mid p \in \Pi(G) \,\})$, i.e. the maximal path length in G.

Def 1.90. Tree: A tree is a DAG $G = \langle V, E\rangle$ such that

- There is exactly one initial node $v_r \in V$ (called the **root**)
- All nodes but the root have indegree of 1.

We call v the **parent** of w, iff $(v, w) \in E$ (w is a child of v). We call a node v a **leaf** of G, iff it is terminal, i.e. if it does not have children.

Note

For any node $v \in V$ except the root v_r, there is exactly one path $p \in \Pi(G)$ with $\mathrm{start}(p) = v_r$ and $\mathrm{end}(p) = v$.

2 Introduction

Def 2.1. Artificial intelligence (AI): The capability of computational systems to perform tasks typically associated with human intelligence, such as learning, reasoning, problem-solving, perception, and decision-making

Def 2.2. Symbolic AI: A subfield of Artificial Intelligence (AI) based on the assumption that many aspects of intelligence can be achieved by the manipulation of symbols, combining them into meaningful structures (expressions) and manipulating them (using processes) to produce new expressions.

Def 2.3. Statisitical AI: Remedies the two shortcomings of symbolic AI approaches: that all concepts represented by symbols are crisply defined, and that all aspects of the world are knowable/representable in principle. Statistical AI adopts sophisticated mathematical models of uncertainty and uses them to create more accurate world models and reason about them.

Def 2.4. Subsymbolic AI (a.k.a connectionism/neural AI): A subfield of AI that posits that intelligence is inherently tied to brains, where information is represented by a simple sequence pulses that are processed in parallel via simple calculations realized by neurons, and thus concentrates on neural computing.

Def 2.5. Embodied AI: Posits that intelligence cannot be achieved by reasoning about the state of the world (symbolically, statistically, or sub-symbolically), but must be embodied i.e. situated in the world, equipped with a "body" that can interact with it via sensors and actuators. Here, the main method for realizing intelligent behavior is by learning from the world.

Def 2.6. Reasoning: The process of producing valid arguments and predictive world models. There are three forms of reasoning:

- **deductive reasoning** to produce new knowledge from existing knowledge. *Example:* All humans are mortal. Socrates is a human. Therefore, Socrates is mortal.
- **inductive reasoning** to produce knowledge from perception. *Example:* The sun has risen every morning in recorded history. Therefore, the sun will rise tomorrow.
- **abductive reasoning** to produce explanations for observations and given knowledge. *Example:* The grass is wet this morning. If it rained last night, that would explain it. Therefore, it probably rained.

Def 2.7. Inference: The act or process of reaching a **conclusion** about something from known facts or evidence (jointly called **premises**)

Def 2.8. Formal Logic: The science of deductively valid inferences, meaning arguments whose conclusions necessarily follow from their premises by virtue of their form (their structure), regardless of the topic.

Def 2.9. Agent: An agent is a structure $A := \langle \mathcal{P}, \mathcal{A}, f \rangle$ where:

- $\mathcal{P}$ is a set of percepts
- $\mathcal{A}$ is a set of actions
- f is a function $f : \mathcal{P}^* \to \mathcal{A}$ that maps from percepts to actions.

In other words, an agent is anything that perceives its environment via sensors and acts on it with actuators.

Def 2.10. Performance Measure: A function that evaluates a sequence of environments.

Def 2.11. Rational: An agent is called **rational**, if it chooses whichever action maximizes the expected value of the performance measure given the percept sequence to date.

Def 2.12. PEAS: To design rational agents, we must specify the PEAS components: Performance Measure, Environment, Actuators, and Sensors.

Def 2.13. Environment Types: For an agent a we classify the environment e of a by its **type**, which is one of the following. We call e

1. *fully observable*, iff the a's sensors give it access to the complete state of the environment at any point in time; otherwise, we call it *partially observable*.
2. *deterministic*, iff the next state of the environment is completely determined by the current state and a's actions; otherwise, *stochastic*.
3. *episodic*, iff a's experience is divided into atomic episodes, where it perceives and then performs a single action, and the next episode does not depend on previous ones. Otherwise, *sequential*.
4. *dynamic*, iff the environment can change without an action performed by a; otherwise, *static*. If the environment does not change but a's performance measure does, we call e *semidynamic*.

5. *discrete*, iff the sets of e's state and a's actions are couuntable; otherwise, *continuous*.
6. *single-agent*, iff only a acts on e; otherwise *multi-agent*.

Def 2.14. Reflex: An agent $\langle \mathcal{P}, \mathcal{A}, f \rangle$ is called a **reflex agent**, iff it only takes the last percept into account when choosing an action, i.e. $f(p_1, \cdots, p_k) = f(p_k)\ \forall p_1 \cdots p_k \in \mathcal{P}$

Def 2.15. Model-Based Agent: A model-based agent $\langle \mathcal{P}, \mathcal{A}, \mathcal{S}, \mathcal{T}, s_0, S, a \rangle$ is an agent $\langle \mathcal{P}, \mathcal{A}, f \rangle$ whose actions depend on:

- a **world model**: a set $\mathcal{S}$ of possible *states*, and a start state $s_0 \in \mathcal{S}$.
- a **transition model** $\mathcal{T}$ that predicts a new state $\mathcal{T}(s, a)$ from a state s and an action a.
- a **sensor model** S that given a state s and a percept p determines a new state $S(s, p)$.
- an **action function** $a : \mathcal{S} \rightarrow \mathcal{A}$ that given a state $s \in \mathcal{S}$ selects the next action $a \in \mathcal{A}$.

Note

If the agent is in state s then it took action a and now perceives p, then the agent state will become $s' = S(p, \mathcal{T}(s, a))$ and accordingly take action $a' = a(s')$.

Def 2.16. Goal Based Agent: A goal based agent $\langle \mathcal{P}, \mathcal{A}, \mathcal{S}, \mathcal{T}, s_0, S, a, \mathcal{G} \rangle$ is a model based agent with an explicit set of goals. It consists of:

- a set of internal states $\mathcal{S}$ and an initial state $s_0 \in \mathcal{S}$,
- a transition model $\mathcal{T} : \mathcal{S} \times \mathcal{A} \rightarrow \mathcal{S}$,
- a state update function $S : \mathcal{P} \times \mathcal{S} \rightarrow \mathcal{S}$,
- a set of goals $\mathcal{G}$,
- a goal conditioned action function $a : \mathcal{S} \times \mathcal{G} \rightarrow \mathcal{A}$ selecting an action given a state and the goals.

Def 2.17. Utility-Based Agent: A utility-based agent uses a world model along with a utility function that models its preferences among the states of that world. It chooses the action that maximizes the expected utility.

Def 2.18. Learning Agent: A learning agent is an agent that can improve its own behavior through experience. It is composed of four main components:

- the performance element, which selects actions based on the agent's current percepts and represents its existing knowledge or behavior,
- the learning element, which improves the performance element over time by analyzing feedback and identifying how to make better decisions,
- the critic, which evaluates the agent's performance according to a given performance standard and provides feedback to the learning element,
- the problem generator, which suggests new and informative actions or experiences that help the agent explore and learn more effectively.

Def 2.19. State Representation: We call a state representation **atomic**, iff it has no internal structure (black box). However, iff each state is characterized by attributes and their values then the representation is **factored**. A **structured** state representation is when include representations of objects, their properties and relationships.

3 General Problem Solving

Def 3.1. Search Problem: A search problem $\Pi := \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{I}, \mathcal{G} \rangle$ consists of a set $\mathcal{S}$ of states, a set $\mathcal{A}$ of actions, and a transition model $\mathcal{T} : \mathcal{A} \times \mathcal{S} \to \mathcal{P}(\mathcal{S})$ that assigns to any action $a \in \mathcal{A}$ and state $s \in \mathcal{S}$ a set of successor states. Certain states in $\mathcal{S}$ are designated as goal (terminal) states ($\mathcal{G} \subseteq \mathcal{S}$ with $\mathcal{G} \neq \phi$) and initial states $\mathcal{I} \subseteq \mathcal{S}$.

Def 3.2. Action Application: We say that an action $a \in \mathcal{A}$ is applicable in state $s \in \mathcal{S}$, iff $\mathcal{T}(a, s) \neq \phi$ and that any $s' \in \mathcal{T}(a, s)$ is a result of applying action a to state s. We call $\mathcal{T}_a : \mathcal{S} \to \mathcal{P}(\mathcal{S})$ with $\mathcal{T}_a(s) := \mathcal{T}(a, s)$ the result relation for a and $\mathcal{T}_{\mathcal{A}} := \bigcup_{a \in \mathcal{A}} \mathcal{T}_a$ the result relation of Π.

Def 3.3. State Space: The graph $\langle \mathcal{S}, \mathcal{T}_{\mathcal{A}} \rangle$ is called the state space induced by Π.

Def 3.4. Solution: A solution for Π consists of a sequence $a_1, \cdots, a_n$ of actions such that for all $1 < i \leq n$:

- a_i is applicable to state s_{i-1} where $s_o \in \mathcal{I}$, and
- $s_i \in \mathcal{T}_{a_i}(s_{i-1})$, and $s_n \in \mathcal{G}$

Def 3.5. Cost Function: Often we add a cost function $c : \mathcal{A} \to \mathbb{R}_0^+$ that associates a step cost $c(a)$ to an action $a \in \mathcal{A}$. The cost of a solution is the sum of the step costs of its actions.

> **Note**
>
> - For deterministic environments, we have $|\mathcal{T}(a, s)| \leq 1$
> - For fully observable ones, we have $\mathcal{I} = \{s_0\}$

Def 3.6. Successor Function/State: In a search problem, $\mathcal{T}_a$ induces a partial function $S_a : \mathcal{S} \rightharpoonup \mathcal{S}$ whose natural domain is the set of states where a is applicable: $S_a(s) := s'$ if $\mathcal{T}_a = \{s'\}$ and undefined at s otherwise. We call S_a the **successor function** for a and $S_a(s)$ the **successor state** of s.

> **Note**
>
> - A search problem is called a **single-state problem** iff it is fully observable, deterministic, static, and discrete.
> - A search problem is called a **multi-state problem** iff it is partially observable.
> - A search problem is called a **contingency problem** iff the environment is non deterministic and the state space is unknown.

Def 3.7. Tree Search: Given a search problem $\Pi := \langle \mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{I}, \mathcal{G} \rangle$, the **tree search algorithm** consists of the simulated exploration of the state space $\langle \mathcal{S}, \mathcal{T}_{\mathcal{A}} \rangle$ in a search tree formed by successively expanding already explored states.

Def 3.8. Path Cost: We define the path cost of a node n in a search tree T to be the sum of the step costs on the path from n to the root of T.

The general structure of Tree Search algorithms is as follows:

```
TREE_SEARCH(start, INSERT):
    fringe ← empty list
    append start to fringe
    while fringe is not empty:
        node ← remove first element of fringe
        if is_goal(node):
            return node
        children ← expand(node)
        INSERT(fringe, children)
    return failure
```

Def 3.9. Strategy: A strategy is a function that picks a node from the fringe of a search tree

> **Note**
>
> Every algorithm has its own strategy, hence the implemention of the `INSERT` differs based on that strategy.

Def 3.10. Properties of Strategies:

- completeness: does it always find a solution if one exists?
- optimality: does it always find the optimal solution (least cost)?

- time complexity: number of nodes the algorithm explores (expands)
- space complexity: maximum number of nodes in memory

> **Note**
>
> Time and space complexity measured in terms of:
> - b: maximum branching factor of the search tree
> - d: minimal graph depth of a solution in the search tree
> - m: maximum graph depth of the search tree

Def 3.11. Breadth-First Search (BFS): The BFS strategy treats the fringe as a FIFO queue, i.e. successors go in at the end (back) of the fringe.

BFS Properties Analysis

- Time Complexity:

  assume we have a binary search tree ($b = 2$) and we found solution in depth $= 3$. In the 0^{th} level we explored $2^0 = 1$ nodes. In the 1^{st} level, $2^1 = 2$ nodes. In the 2^{nd} level, $2^2 = 4$ nodes. And finally $2^3 = 8$ nodes in the third level. Number of nodes the algorithm explored will be $2^0 + 2^1 + 2^2 + 2^3$.

  – Generally, for arbitrary b, d, we have $1+b+b^2+\cdots+b^d$, which means, worst time complexity is $\mathcal{O}(b^d)$ (exponential in d).

- The space complexity is also $\mathcal{O}(b^d)$.
- BFS is complete if: (i) b is finite, and (ii) the state space is finite or has a solution.
- BFS is optimal if we have a uniform constant cost for all actions.

Def 3.12. Uniform-Cost Search (UCS): UCS is the strategy where the fringe is ordered by increasing path cost. (*expands least cost first*)

> **Note**
>
> UCS it is equivalent to BFS if all costs are equal

UCS Properties Analysis

- Time and space complexity $\approx \mathcal{O}(b^d)$
- UCS is complete if:

  – b is finite

  – all actions costs are $\geq \epsilon > 0$

  – state space is finite or has a solution.

Def 3.13. Depth-First Search (DFS): DFS is the strategy where the fringe is organized as a LIFO stack, i.e. successors go in at the front of the fringe.

Def 3.14. Backtracking: Every node that is pushed to the stack is called a **backtrack point**. Popping a non-goal node from the stack and continuing the search with the new top element is called **backtracking**. For this reason, the DFS algorithm is also referred to as a **backtracking search**.

DFS Properties Analysis

- Time complexity is $\mathcal{O}(b^m)$ (exponential in the maximum depth)
- Space complexity is linear. While BFS keeps track of all nodes at the current level in memoty, DFS needs to only keep the current path (from root to m), and also needs to keep the remaining unexplored sibling root node for each node along that path (to backtrack). This means, for each level, we store the siblings, i.e. b nodes. We explore until we hit the max depth m and in each level we store b. Hence we have a linear space $\mathcal{O}(b \cdot m)$
- DFS is complete if the search tree is finite and acyclic.
- DFS is not optimal.

Def 3.15. Depth-Limited Search (DLS): DLS is a DFS with a depth limit l. We treat all nodes at depth l as if they have no children.

DLS Properties Analysis

- Time complexity $\rightsquigarrow \mathcal{O}(b^l)$
- Space complexity $\rightsquigarrow \mathcal{O}(b \cdot l)$
- Not complete (solution maybe beyond chosen l)
- Not optimal

Def 3.16. Iterative Deepening Search (IDS): IDS is a DLS with an ever-increasing depth limit. We call the difference between successive depth limits the step size.

> Note
>
> - IDS solves the problem of choosing a good l in DLS.
> - IDS tries all values until either a solution is found (depth is returned to DLS, we call that the cutoff value) or we return failure.
> - IDS combines benefits of DFS and BFS.

IDS Properties Analysis

It is similar to DFS in memory requirements (assuimg finite and acyclic search tree).

- $\mathcal{O}(b \cdot d)$ when there is a solution
- $\mathcal{O}(b \cdot m)$ when there is no solution

It is similar to BFS in terms of optimality, completeness, and time complexity. Optimal for problems where all actions have the same cost. Complete if b is finite and the state space is finite or has a solution.

With regard to time complexity, consider IDS finds a solution at depth d, then:

- nodes at the very bottom (d) are visited only once in the final iteration.
- nodes at $d-1$ are visited twice, once in the search with depth $d-1$ and once in the search with depth d.
- nodes at $d-2$ are visited three times
- and so son, until root node ($d = 0$) which is visited d times.

Generally, we have $(d)b^1 + (d-1)b^2 + (d-2)b^3 + \cdots + b^d$. However, the majority of nodes are at the max depth where there are b^d nodes. Those are only visited once and the extra cost of visiting shallower nodes does not significantly increase the overall count. Therefore, time complexity is bounded by b^d ($\mathcal{O}(b^d)$) because b^d dominates the total nodes count in a large search space.

Def 3.17. Graph Search: A graph search algorithm is a variant of a tree search algorithm that prunes nodes whose state has already been considrered (duplicate pruning), essentially using a DAG data strucutre.

The general structure of Graph Search algorithms is as follows:

```
GRAPH_SEARCH(start, INSERT):
    fringe ← empty list
    visited ← empty set
    append start to fringe
    add state(start) to visited
    while fringe is not empty:
        node ← remove first element of fringe
        if is_goal(node):
            return node
        children ← expand(node)
        for each child in children:
            if state(child) not in visited:
                add state(child) to visited
                INSERT(fringe, child)
    return failure
```

Def 3.18. Search Alrogrithm: We speak of a **search algorithm** when we do not want to distinguish whether it is a tree or a graph search algorithm - *difference considered as an implementation detail.*

Def 3.19. Informed: A search algorithm is called **informed**, iff it uses some form of external information to guide the search.

Def 3.20. Evaluation Function: An evaluation function assigns a desirability value to each node of the search tree. It is not part of the search problem, but must be added externally.

Def 3.21. Best-First Search: In best-first search, the fringe is a queue sorted in decreasing order of desirability.

Def 3.22. Heuristic: A heuristic is an evaluation function h on states that estimates the cost form n to the nearest goal state. We speak of **heuristic search** if the search algorithm uses a heuristic in some way.

Def 3.23. Greedy Search: A best-first search strategy where the fringe is organized as a queue sorted based on h value.

Greedy Search Properties

- Time and space complexity are both exponential in m (max depth) $\rightsquigarrow \mathcal{O}(b^m)$
- It is not optimal. But it is complete if the state space is finite.

Def 3.24. Heuristic Function: A heuristic funciton for search problem Π is a function $h : \mathcal{S} \to \mathbb{R}_0^+ \cup \{\infty\}$ so that $h(s) = 0 \quad \forall s \in \mathcal{G}$.

Def 3.25. Goal Distance Function: The function $h^* : \mathcal{S} \to \mathbb{R}_0^+ \cup \{\infty\}$, where $h^*(s)$ is the cost of a cheapest path from s to a goal state, or ∞ if no such path exists. (*the perfect heuristic which we do not know and cannot compute*)

Def 3.26. Admissible: For a search problem Π with states $\mathcal{S}$ and actions $\mathcal{A}$. We say that a heuristic h for Π is **admissible** if

$$h(s) \leq h^*(s) \quad \forall s \in \mathcal{S}$$

> **Note**
>
> Admissible heuristics never overestimates; our guess should be always optimistic or exact, never too high

Def 3.27. Consistent: For a search problem Π with states $\mathcal{S}$ and actions $\mathcal{A}$. We say that a heuristic h for Π is **consistent** if

$$h(s) - h(s') \leq c(a) \quad \forall s \in \mathcal{S}, a \in \mathcal{A}, s' \in \mathcal{T}(s, a)$$

> **Note**
>
> A consistent heuristic never drops faster than the real cost you pay when moving through the state space. Formally, for every transition $s \xrightarrow{a} s'$,
>
> $$h(s) \leq c(a) + h(s')$$
>
> This means the heuristic respects a triangle inequality with respect to the actual step costs. Consistency ensures that the estimated total cost along a path never decreases.

Def 3.28. A* Evaluation Function: Given a path cost function g and a heuristic h, the **A* evaluation function** is

$$f(n) \;=\; g(n) + h(n).$$

It estimates the total cost of a cheapest path from the start state to a goal via n.

Def 3.29. A* Search: A* search is the best-first search strategy that uses the A* evaluation function $f = g + h$ to order the fringe.

Theorem 3.1. A* Search with admissible heuristic is optimal

Def 3.30. Dominance: Let h_1 and h_2 be two admissible heuristic, we say that h_2 **dominates** h_1 if $h_2(s) \geq h_1(s) \quad \forall s \in \mathcal{S}$.

Theorem 3.2. If h_2 dominates h_1, then h_2 is better for search than h_1

Proof. If h_2 dominates h_1, then h_2 is closer to h^* than h_1 □

Def 3.31. Adversarial Search: An adversarial search problem is a search problem $\langle \mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{I}, \mathcal{G} \rangle$, where:

- $\mathcal{S} = \mathcal{S}^{\text{Max}} \uplus \mathcal{S}^{\text{Min}} \uplus \mathcal{G}$
- $\mathcal{A} = \mathcal{A}^{\text{Max}} \uplus \mathcal{A}^{\text{Min}}$
- For $a \in \mathcal{A}^{\text{Max}}$, if $s \xrightarrow{a} s'$, then $s \in \mathcal{S}^{\text{Max}}$ and $s' \in (\mathcal{S}^{\text{Min}} \cup \mathcal{G})$
- For $a \in \mathcal{A}^{\text{Min}}$, if $s \xrightarrow{a} s'$, then $s \in \mathcal{S}^{\text{Min}}$ and $s' \in (\mathcal{S}^{\text{Max}} \cup \mathcal{G})$

together with a **game utility function** $u : \mathcal{G} \to \mathbb{R}$

Def 3.32. Strategy: Let Θ be an adversarial search problem, and let $X \in \{\text{Max}, \text{Min}\}$. A **strategy** for X is a function $\sigma^X : \mathcal{S}^X \to \mathcal{A}^X$ so that $a \in \mathcal{A}^X$ is applicable to $s \in \mathcal{S}^X$ whenever $\sigma^X(s) = a$.

Def 3.33. Optimal Strategy: A strategy is called **optimal** if it yields the best possible utility for X assuming perfect opponent play.

> **Note**
>
> Assumptions for Adversarial Games:
> - Two players acting in strictly alternating turns
> - Fully observable and deterministic game states
> - Finite state space so the game tree is finite
> - Terminal states have a real-valued utility
> - The game is zero-sum: Max maximizes the utility, Min minimizes the same utility

Def 3.34. Minimax: Let Θ be an adversarial search problem with utility function $u : \mathcal{G} \to \mathbb{R}$. The **minimax value** is the function $\hat{u} : \mathcal{S} \to \mathbb{R}$ defined by:

$$\hat{u}(s) = \begin{cases} u(s) & \text{if } s \in \mathcal{G}, \\ \max\limits_{s' \in \mathcal{T}(a,s)} \hat{u}(s') & \text{if } s \in \mathcal{S}^{\text{Max}}, \\ \min\limits_{s' \in \mathcal{T}(a,s)} \hat{u}(s') & \text{if } s \in \mathcal{S}^{\text{Min}}, \end{cases}$$

where a ranges over all actions applicable to s. The **minimax decision** at a root state r is any action whose successor state achieves $\hat{u}(r)$.

Def 3.35. Minimax Algorithm: The minimax algorithm is given by the following recursive function whose arguments are a node (state) and whether it is Max's turn or Min's.

```
function MINIMAX(node, maximizing):
    if TERMINAL(node):
        return UTILITY(node)

    if maximizing:
        best = -infinity
        for each child in SUCCESSORS(node):
            val = MINIMAX(child, false)
            if val > best:
                best = val
        return best

    else:
        best = +infinity
        for each child in SUCCESSORS(node):
            val = MINIMAX(child, true)
            if val < best:
                best = val
        return best
```

Def 3.36. Evaluation Function: In practice, full-depth minimax is infeasible because the search tree is too large. We therefore impose a fixed *depth limit / horizon* d and stop the search at all states $\{ s \in \mathcal{S} \mid \text{dp}(s) = d \}$, called *cut-off states*.

An evaluation function is a function $f : \mathcal{S} \to \mathbb{R}$ that assigns a numerical estimate of the true minimax value $\hat{u}(s)$ to any nonterminal cut-off state s. If a cut-off state is terminal, its utility $u(s)$ is used instead of $f(s)$.

Def 3.37. Alpha Value: For each node n in a minimax search tree, the alpha value $\alpha(n)$ is the highest Max-node utility that search has encountered on its path from the root to n.

Def 3.38. Beta Value: For each node n in a minimax search tree, the beta value $\beta(n)$ is the highest Min-node utility that search has encountered on its path from the root to n.

Def 3.39. Alpha-Beta Pruning: Alpha-beta pruning is a variant of minimax that avoids exploring branches that cannot change the minimax value.

- At a Max-node n, if a successor s is found with utility $\hat{u}(s) \geq \beta(n)$, then no remaining successors of n can affect its value and they are pruned.
- At a Min-node n, if a successor s is found with utility $\hat{u}(s) \leq \alpha(n)$, then no remaining successors of n can affect its value and they are pruned.

The algorithm returns the same minimax value as full search while expanding fewer nodes.

```
function ALPHABETA(node, alpha, beta, maximizing):
    if TERMINAL(node):
        return UTILITY(node)

    if maximizing:
        best = -infinity
        for each child in SUCCESSORS(node):
            val = ALPHABETA(child, alpha, beta, false)
            if val > best:
                best = val
            if best > alpha:
                alpha = best
            if beta <= alpha:
                break
        return best

    else:
        best = +infinity
        for each child in SUCCESSORS(node):
            val = ALPHABETA(child, alpha, beta, true)
            if val < best:
                best = val
            if best < beta:
                beta = best
            if beta <= alpha:
                break
        return best
```

4 Constraint Satisfaction Problems

Def 4.1. Constraint Satisfaction Problem (CSP): A triple $\gamma := \langle V, D, C \rangle$ where

- V is a finite set of variables,
- D is an V-indexed family of domains: $(D_v)_{v \in V}$
- C is the set of constraints. For a subset $\{v_1, \cdots, v_k\} \subseteq V$, a constraint $C_{\{v_1,\cdots,v_k\}} \subset D_{v_1} \times \cdots \times D_{v_k}$

Def 4.2. Variable Assignment: We call a partial function:

$$\varphi : V \rightharpoonup \bigcup_{v \in V} D_v$$

a **variable assignment** if $\varphi(v) \in D_v \quad \forall v \in \mathrm{dom}(\varphi)$

Def 4.3. Satisfying Assignment: a variable assignment φ **satisfies** a constraint $C_{\{v_1,\cdots,v_k\}}$ iff $(\varphi(v_1), \cdots, \varphi(v_k)) \in C_{\{v_1,\cdots,v_k\}}$.

Def 4.4. Consistent Assignment: a variable assignment φ is called **consistent** iff it satisfies all constraints in C.

Def 4.5. Legal Assignment: A value $d \in D_v$ is called **legal** for a variable $v \in V$ iff $v \mapsto d$ is a consistent assignemnt; otherwise, **illegal**.

Def 4.6. Conflicted: A variable with an illegal value under assignment φ is called **conflicted**.

Def 4.7. CSP Solution: A variable assignment that is total (i.e. function) and consistent is a **solution** for the CSP.

Def 4.8. Satisfiable: A CSP γ is called **satisfiable** iff it has a solution (a total variable assignemnt φ that satisfies all constraints).

Def 4.9. Discrete: We call a CSP **discrete** iff all of the variables have countable domains; otherwise, **continuous**.

Def 4.10. Boolean CSP: A discrete CSP is called **boolean** iff $|D_v| = 2 \quad \forall v \in V$.

> **Note**
>
> Discrete CSPs with domain size d and n variables has a search space of size d^n, so a naive solve (brute-force) is worst case $\mathcal{O}(d^n)$. In general, deciding solvability of a finite-domain CSP is NP-complete.

Def 4.11. Constraint Order (Arity): Let $\gamma = \langle V, D, C \rangle$ be a CSP. For a constraint

$$C_{\{v_1,\ldots,v_k\}} \subseteq D_{v_1} \times \cdots \times D_{v_k},$$

its **order** (or *arity*) is

$$\mathrm{ord}\big(C_{\{v_1,\ldots,v_k\}}\big) := k.$$

The order of γ is

$$\max_{C_{\{v_1,\ldots,v_k\}} \in C} k.$$

A constraint of order 1 is *unary*, order 2 is *binary*, and any constraint with order > 2 is *higher order*.

Def 4.12. Binary: A **binary CSP** is a CSP where each constraint is unary or binary.

Def 4.13. Constraint Graph: A binary CSP forms a graph called the **constraint graph** whose nodes are variables, and whose edges represent the constraints.

Def 4.14. Constraint Network: A constraint network is a CSP $\gamma := \langle V, D, C \rangle$ of order 2, hence, we write a unary constraint as C_v (representing $C_{\{v\}}$) and a binary constraint as C_{uv} (representing $C_{\{u,v\}}$). Note that $C_{uv} = C_{vu}$.

> **Note**
>
> The constraint graph where all constraints are binary is the undirected graph:
>
> $$\big\langle V, \big\{(u, v) \in V^2 \mid C_{uv} \neq D_u \times D_v\big\}\big\rangle$$

CSP as Search

We can induce a search problem $\Pi_\gamma := \langle \mathcal{S}_\gamma, \mathcal{A}_\gamma, \mathcal{T}_\gamma, \mathcal{I}_\gamma, \mathcal{G}_\gamma \rangle$ from the constraint network $\gamma := \langle V, D, C \rangle$.

We define the set of states $\mathcal{S}_\gamma$ as the set of all partial assignments:

$$\mathcal{S}_\gamma := \left\{ \varphi : V \rightharpoonup \bigcup_{v \in V} D_v \mid \varphi(v) \in D_v \quad \forall v \in \mathrm{dom}(\varphi) \right\}$$

We define goal states $\mathcal{G}_\gamma$ as the set of total and consistent assignments

$$\mathcal{G}_\gamma := \left\{ \varphi : V \to \bigcup_{v \in V} D_v \mid (\varphi(v) \in D_v \quad \wedge \quad (\varphi(u), \varphi(v)) \in C_{uv}) \quad \forall u, v \in V \right\}$$

The initial state is when we have no assignments:

$$\mathcal{I}_\gamma := \left\{ \varphi : V \rightharpoonup \bigcup_{v \in V} D_v \mid \mathrm{dom}(\varphi) = \emptyset \right\}$$

Actions represent assignments as pairs:

$$\mathcal{A}_\gamma := \{(v, d) \mid v \in V \wedge d \in D_v\}$$

Transition model, given a current state s and an action $a = (v, d)$ gives us the successor state s'.

$$\mathcal{T}_\gamma : \mathcal{A}_\gamma \times \mathcal{S}_\gamma \to \mathcal{P}(\mathcal{S}_\gamma)$$

Concretely: $\mathcal{T}_\gamma((v, d), s) := \{ \, s' \, \}$ where s' is the successor state (we are updating the variable assignment) which will evaluate as follows:

$$s'(w) := \begin{cases} d, & \text{if } w = v \\ s(w), & \text{if } w \in \mathrm{dom}(s) \text{ and } w \neq v \end{cases} \quad \text{and} \quad \mathrm{dom}(s') := \mathrm{dom}(s) \cup \{v\}.$$

Def 4.15. Backtracking Search: Backtracking search is the basic uninformed algorithm for solving CSPs. It is DFS with two improvements, namely:

1. One variable at a time
   - variable assignments are commutative, fixing the order saves us time!
   - i.e. starting from `state(WA=red)` $\rightarrow$ `state(WA=red, NT=green)` is the same as starting from `state(NT=green)` $\rightarrow$ `state(NT=green, WA=red)` [1]
   - We only need to consider assignments to a single variable at each step
2. Check constraints as you go
   - Consider only values which do not conflict previous assignments
   - That is not a goal test! We can think about it as incremental goal test.

The pseudocode for backtracking search is shown in Algorithm 1

Algorithm 1 Backtracking Search

```
function BACKTRACKING-SEARCH(csp)
    return RECURSIVE-BACKTRACKING({}, csp)
end function
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then
        return assignment
    end if
    var ← SELECT-UNASSIGNED-VARIABLE(Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given Constraints[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then
                return result
            end if
            remove {var = value} from assignment
        end if
    end for
    return failure
end function
```

> **Note**
>
> Note that the exact behaviors of the function: `SELECT-UNASSIGNED-VARIABLE` and `ORDER-DOMAIN-VALUES` are not specified yet

Def 4.16. Forward Checking: Forward checking is a filtering (inference) technique that improves the general uninformed backtracking seach. It propagates information about illegal values. Whenever a variable $v \in V$ is assigned by $d \in D_v$ ($\varphi(v) = d$), we delete all values that are inconsistent with $\varphi(v)$ from every D_u for all variables u connected with v by a constraint.

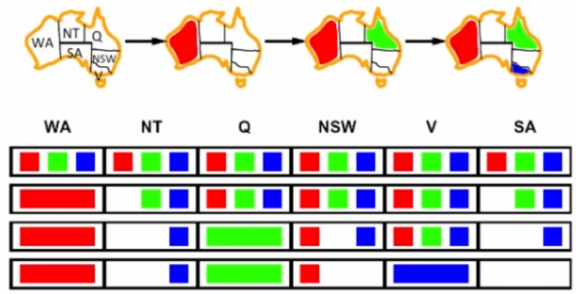
> **Note**
>
> The idea is to keep track of domains for unassigned variables and cross off bad options. In forward checking we cross off values that violate a constraint when added to the existing assignment (look ahead).

forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures. For example, (remember Australia map):

- Assume we chose to assign `WA=Red`
- Because `NT, SA` are neighbors, we remove `Red` from their domains

[1] I am referring to the known Australia map example mentioned in the lecture notes

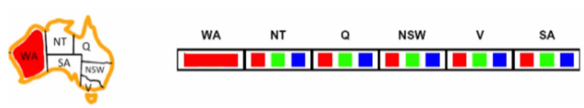
- Assume we pick next `Q=Green`
- Because `NT, SA, NSW` are neighbors, we remove `Green` from their domains
- At this point, forward checking thinks we are in good shape, but we are already doomed! *why?* Because `NT, SA` are neighbors and they both are left with only `Blue` in their domains!
- Assuming, next we choose `V=Green`, we cross off `Green` from `NSW, SA`, now `SA` is left with no colors, only then, we know we failed, so we backtrack.



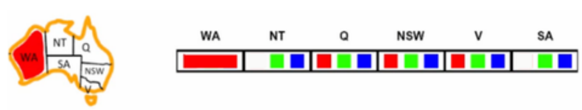
Def 4.17. Arc Consistency: Let $\gamma := \langle V, D, C\rangle$ be a constraint network, a variable $u \in V$ is **arc consistent** relative to another variable $v \in V$ if either $C_{uv} \notin C$, or for every value $d \in D_u$ there exist a value $d' \in D_v$ such that $(d, d') \in C_{uv}$. The constraint network γ is **arc consistent** if every variable $u \in V$ is **arc consistent** relative to every other variable $v \in V$.

> **Note**
>
> Note that when we are checking some arc $x \to y$ and find it inconsistent, we remove values from D_x to make it consistent. When doing so, we need to check every other arc $z \to x$ (where the *tail* of the arc we checked is the *head* of other arcs) even if it was checked before (because now we have fewer options, it is not necessarily consistent anymore!)

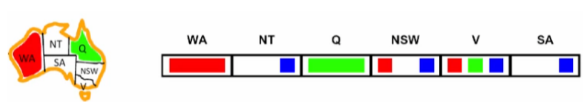
Example:



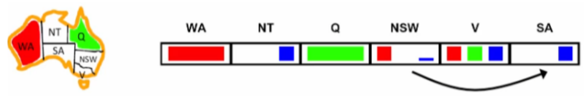
- Checking `NT -> WA` $\rightsquigarrow$ removing `Red` from the domain of `NT`
- Checking `SA -> WA` $\rightsquigarrow$ removing `Red` from the domain of `SA`
- All arcs are consistent now



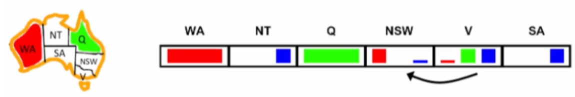
Assume we pick next `Q=Green`, and therefore remove `Green` from `NT, NSW, SA`:



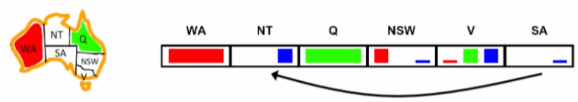
- Looking at `V -> NSW` $\rightsquigarrow$ consistent
- Looking at `SA -> NSW` $\rightsquigarrow$ consistent
- Looking at `NSW -> SA` $\rightsquigarrow$ not consistent! If we choose `NSW = Blue`, nothing I can choose in `SA` that would not violate the constraint!
- We make `NSW -> SA` consistent by removing `Blue` from the domain of `NSW`



- Now, we check again the arc where `NSW` was the head.
- Looking at `V -> NSW` $\rightsquigarrow$ not consistent



- Looking at `SA -> NSW` $\rightsquigarrow$ consistent
- Continue checking arcs. Assume looking at `SA -> NT` $\rightsquigarrow$ not consistent



- Now `SA` domain is empty, we know we failed, so we backtrack!
- Notice that we detect failure earlier than forward checking!

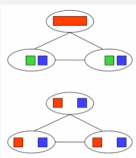
The pseudocode for the arc consistency algorithm (called `AC-3`) is shown in Algorithm 2

Algorithm 2 AC-3

```
function AC-3(csp)
    inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
    local variables: queue, a queue of arcs, initially all the arcs in csp
    while queue is not empty do
        (X_i, X_j) ← REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES((X_i, X_j)) then
            for each X_k in Neighbors[X_i] do
                add (X_k, X_i) to queue
            end for
        end if
    end while
    return csp
end function
function REMOVE-INCONSISTENT-VALUES((X_i, X_j))
    removed ← false
    for each x in Domain[X_i] do
        if no value y in Domain[X_j] allows (x, y) to satisfy the constraint X_i ↔ X_j then
            delete x from Domain[X_i]
            removed ← true
        end if
    end for
    return removed
end function
```

- In the worst case, we have a fully connected constraint network of n variables, hence we will have a total of $n(n-1)$ arcs.
- For one arc $x \rightarrow y$, if the number of values each variable can take is d, to make this arc consistent, we have one variable that has d possibilities (*the tail*) and the *head* too (another d possibilities), and we want to make sure for everything in the *tail* there is some value in the *head* that we can assign. That requires $d \cdot d$ checks in the worst case $= d^2$.
- Moreover, it is not enough to check each arc once. We `enqueue` an already `dequeued` arc whenever we eliminate a value from the domain of the arc's *head*. In the worst case, we might eliminate all the values of a variable, i.e. we put an arc back d times.

- Therefore, the total run time for `AC-3` is $\mathcal{O}((n(n-1)) \cdot d^2 \cdot d) = \mathcal{O}(n^2 d^3)$

> **Note**
>
> Notice that the run time ≈ polynomial and yet detecting all possible future problems is NP-Hard because of backtracking. Arc consistency does not guarantee finding a solution. After enforcing arc consistency, we might have:
> - one solution left
> - multiple solutions left
> - no solutions left (and not know it!)
>
> Moreover, the algorithm still runs inside backtracking search.
>
> 

Def 4.18. Minimum Remaining Values: The **MRV** heuristic for backtracking search (*a.k.a Most Constrained Variable First*) selects the unassigned variable with the fewest legal values given the current partial assignment φ. Formally, it chooses the variable v that minimizes $\big|\{ d \in D_v \mid \varphi \cup \{v \mapsto d\} \text{ is consistent } \}\big|$.

> **Note**
>
> MRV $\mapsto$ we want to fail early! Hence, MRV ordering is also called fail-fast ordering.

Def 4.19. Least Constraining Value: Given a variable v, the LCV heuristic chooses the least constraining value for v, i.e. the one that rules out the fewest values in the remaining variables. For the current partial assignment φ and a chosen variable v, we pick a value $d \in D_v$ that minimizes $|d' \in D_u \mid u \notin \text{dom}(\varphi), C_{uv} \in C, \text{ and } (d', d) \notin C_{uv}|$

Def 4.20. Independent Subproblems: Assume we have independent connected components of a constraint graph. We treat each component as a separate CSP since it has no constraints outside its variables.

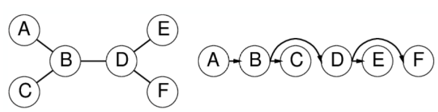
Remember, a naive solver (brute-force) solves a discrete CSPs with domain size d and n variables in $\mathcal{O}(d^n)$ (exponential in n). If that same problem has independent subproblems each with c variables, then we have $\frac{n}{c}$ subproblems where each one in the worst case is $\mathcal{O}(d^c)$. So the total time will be $\mathcal{O}(\frac{n}{c} d^c)$, that is, linear in n and exponential in c. That is very fast for small c.

Def 4.21. Decomposition: The process of decomposing a constraint network into components.

Def 4.22. Tree-Structured CSP: We call a CSP **tree-structured** iff its constraint graph is acyclic.

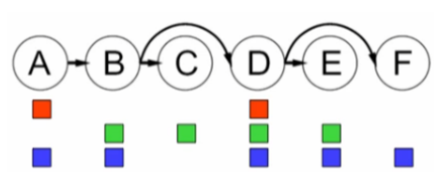
Theorem 4.1. A Tree-Structured CSP can be solved in $\mathcal{O}(nd^2)$

To solve a Tree-Structured CSP, we choose a variable as root and order the variables from root to leaves such that every node's parent precedes it in the ordering:



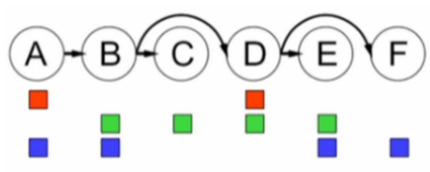
- Backward pass: For $i = n$ and down to 2, apply `Remove-Inconsistent-Values`$((\texttt{Parent}(X_i), X_i))$
- Forward pass: For $i = 1$ and up to n, assign X_i consistently with $\texttt{Parent}(X_i)$

For example, consider the following domains:

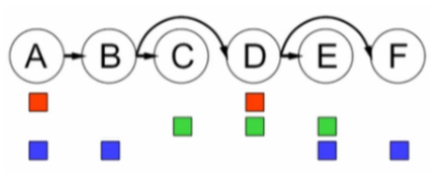


Backward Pass:

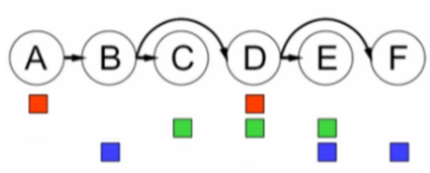
- Start with F, checking $D \to F \rightsquigarrow$ not consistent:



- Next is E, checking $D \to E \rightsquigarrow$ consistent.
- Next is D, checking $B \to D \rightsquigarrow$ consistent.
- Next is C, checking $B \to C \rightsquigarrow$ not consistent:



- Next is B, checking $A \to B \rightsquigarrow$ not consistent:



Forward Pass:

$\varphi(A) = \texttt{Red}$, $\varphi(B) = \texttt{Blue}$, $\varphi(C) = \texttt{Green}$, $\varphi(D) = \texttt{Green}$, $\varphi(E) = \texttt{Blue}$, $\varphi(F) = \texttt{Blue}$

> **Note**
>
> After the backward pass on a tree-structured CSP, every root-to-leaf arc becomes consistent, and no arc ever needs to be enqueued again. The key reason is the direction in which consistency is enforced. Starting from the leaves and moving upward, each arc $x \to y$ is processed only after all of x's children have been handled. When enforcing consistency on $x \to y$, we only remove values from the domain of the tail x. Since all arcs of the form $v \to x$ were already processed earlier, there will be no later step that removes values from the head y of any previously processed arc. In contrast to general AC algorithms, where pruning the head forces re-checking incoming arcs, the tree structure guarantees that once an arc's head has been finalized, it is never modified again. This is why no arc needs to be re-enqueued: domain reductions always flow upward toward the root, never back down.

> **Note**
>
> If root-to-leaf arcs are consistent, the forward pass will not backtrack and will go straight to the solution!

Def 4.23. Conditioning: Instantiate a variable, prune its neighbors' domains

Def 4.24. Cutset conditioning: Instantiate in all ways a set of variables such that the remaining constraint graph is a tree.

> **Note**
>
> A cutset of size c gives runtime $\mathcal{O}(d^c(n-c)d^2)$:
> - Because we are left with a tree of size $n-c$, we have $\mathcal{O}((n-c)d^2)$
> - Because we have to look at all possible assignments of the cutset, we have $\mathcal{O}(d^c)$
>
> and that is very fast for small c!