Assignment7 – Calculi

Problem 7.1 (FOL-Signatures)

1. Represent the syntax of integers using operations for 0 ,1, $x+y$, $-x$, and $x < y$ as a FOL signature. (FOL and PLNQ signatures are the same.)

   Solution:
   - We have constants (= nullary functions, elements of Σ_0^f) called `zero` and `one`.
   - We have a binary function symbol (element of Σ_2^f) called `plus`.
   - We have a unary function symbol (element of Σ_1^f) called `minus`.
   - We have a binary predicate symbol (element of Σ_2^p) called `less`.

Now consider the signature given by

- $\Sigma_0^f = \{a, b\}$
- $\Sigma_1^f = \{f, g\}$
- $\Sigma_2^f = \{h\}$
- $\Sigma_0^p = \{p\}$
- $\Sigma_1^p = \{q\}$
- $\Sigma_2^p = \{r\}$
- all other sets empty

2. Give a term over this signature that uses all function symbols.

   Solution: E.g., $t = h(f(a), g(b))$.

   ---

3. Give a formula over this signature that uses all function and predicate symbols.

   Solution: E.g., $r(t \wedge t) \wedge q(t) \wedge p$ where t is as above.

Problem 7.2 (FOL Models)

1. Consider the syntax of integers using the symbols `zero`, `one`, `plus`, `minus` (unary), and `less`. Give the model that assigns the usual interpretations in the integers.

Solution: The model is $\langle \mathbb{Z}, I \rangle$ (i.e., the domain/universe is the set $\mathbb{Z}$ of integers) where the interpretations $I(-)$ of the symbols are:

- $I(\texttt{zero}) = 0$
- $I(\texttt{one}) = 1$
- $I(\texttt{plus})(u, v) = u + v$
- $I(\texttt{minus})(u) = -u$
- $I(\texttt{less}) \ni \langle u, v \rangle$ iff $u < v$

for $u, v \in \mathbb{Z}$

Now consider the signature given by

- $\Sigma^f_0 = \{a\}$
- $\Sigma^f_1 = \{f\}$
- $\Sigma^f_2 = \{h\}$
- $\Sigma^p_0 = \{p\}$
- $\Sigma^p_2 = \{r\}$
- all other sets empty

2. Give a model for it.

Solution: Lots of options. A trivial choice is $M = \langle \{0\}, I \rangle$. Because the universe is a singleton set, the interpretation of all function symbols is already determined:

- $I(a) = 0$
- $I(f)(0) = 0$
- $I(h)(0, 0) = 0$

The easiest interpretation of predicate symbols is to make them always false:

- $I(p) = \emptyset$
- $I(r) = \emptyset$

A non-trivial example is $N = \langle \{-1, 0, 1\}, J \rangle$ with

- $J(a) = 0$
- $J(f)(u) = -u$
- $J(h)(u, v) = u \cdot v$
- $J(p) = \{\langle\rangle\}$ (the set containing the empty tuple, i.e., the nullary predicate symbol is true)
- $J(r) \ni \langle u, v \rangle$ iff $u \leq v$

---

3. Give a formula over this signature that uses all function and predicate symbols and is satisfied by your model.

Solution: E.g., $\neg p \wedge \neg r(f(a), h(a,a))$ is satisfied in M.
$p \wedge r(f(a), h(a,a))$ is satisfied in N.

Problem 7.3 (Natural Deduction)

Prove the following formula using the propositional Natural Deduction calculus.

$$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C) \Rightarrow C$$

Solution:

(1)	1	$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)$	Assumption
(2)	1	$A \vee B$	$\wedge E_l$ (on 1)
(3)	1	$(A \Rightarrow C) \wedge (B \Rightarrow C)$	$\wedge E_r$ (on 1)
(4)	1	$A \Rightarrow C$	$\wedge E_l$ (on 3)
(5)	1	$B \Rightarrow C$	$\wedge E_r$ (on 3)
(6)	1,6	A	Assumption
(7)	1,6	C	$\Rightarrow E$ (on 4 and 6)
(8)	1,8	B	Assumption
(9)	1,8	C	$\Rightarrow E$ (on 5 and 8)
(10)	1	C	$\vee E$ (on 2, 7 and 9)
(11)		$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C) \Rightarrow C$	$\Rightarrow I$ (on 1 and 10)

Problem 7.4 (Proving in Tableau Calculus)

We use the propositional variables P, Q, and R and define formulae A, B, and C by

$$A = Q \wedge (Q \Rightarrow R)$$
$$B = P \Rightarrow A$$
$$C = P \Rightarrow R$$

Prove the formula $B \Rightarrow C$ using the propositional tableau calculus $\mathcal{T}_0$.

Solution:

$(B \Rightarrow C)^{\mathsf{F}}$

B^{T}
C^{F}
P^{T}
R^{F}

P^{F}	A^{T}	
close on P	Q^{T}	
	$(Q \Rightarrow R)^{\mathsf{T}}$	
	Q^{F}	R^{T}
	close on Q	close on R

$\perp$ for closing is acceptable.

Problem 7.5 (Logical Systems)

Fix a set V of propositional variables. We define a logical system $\langle L, K, \models\rangle$. (Note: This logical system is different from the ones in the lecture and only used here as an exercise.)

- L is the powerset of V, i.e., a formula is a set of propositional variables.
- K is the set of functions $V \to \{F, T\}$.
- For $A \in L$ and $M \in K$, $M \models A$ holds if $M(p) = T$ for all $p \in A$.

1. Try to give examples of formulas that are
    1. satisfiable
    2. falsifiable
    3. unsatisfiable
    4. valid

Solution: Assume some $p \in V$.

1. $\{p\}$ (satisfied if $M(p) = T$)
2. $\{p\}$ (falsified if $M(p) = F$)
3. No such formula exists
4. $\emptyset$ is the only valid formula

---

2. Give a sound and complete calculus for this logical system.

Solution: Because $\emptyset$ is the only valid formula. It suffices to have a single rule

$$\overline{\emptyset}$$

---

3. Consider the relation $H \vdash A$ holding if $A = \bigcup_{h \in H} h$. Check if $\vdash$ is a derivation relation.

Solution: It is not (unless $|V| \leq 1$). For example, put $H = \{\{p\}, \{q\}\}$ and $A = \{p\}$. Then $\bigcup_{h \in H} h = \{p, q\}$ and thus $H \nvdash A$ even though $A \in H$.
It becomes a derivation relation if we use $\subseteq$ instead of $=$ in the definition.
