

MMSP

The Mesoscale Microstructure Simulation Project

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Contents

1	Introduction	7
1.1	The MMSP concept	7
1.2	What MMSP does	8
1.3	What MMSP doesn't do	8
1.4	What MMSP requires	9
1.5	Terms of use	9
2	Getting started with MMSP	11
2.1	Download	11
2.2	Installation	11
2.3	Setup	12
2.4	Support	14
3	MMSP tutorials	15
3.1	A quick tutorial	15
3.1.1	The <code>Hello MMSP!</code> program	15
3.1.2	Compiling and running <code>Hello MMSP!</code>	16
3.2	A <code>grid</code> class example	17
3.3	More examples	20
3.3.1	Accessing node data	20
3.3.2	File input and output	21
3.3.3	Boundary conditions	22
3.4	The prototypical MMSP program	23
3.5	Application: the Allen-Cahn equation	25
3.6	Running the example codes	27
4	MMSP data classes	29
4.1	Introduction	29
4.2	Common features	29
4.3	Using MMSP data types	31
4.3.1	The <code>scalar</code> class	31
4.3.2	The <code>vector</code> class	31
4.3.3	The <code>sparse</code> class	33
4.4	Using built-in data types	34

4.5	Writing new data classes	34
5	The grid class	35
5.1	Introduction	36
5.2	grid class member functions	37
5.2.1	Constructors	37
5.2.2	Subscripting	37
5.2.3	File I/O	37
5.2.4	Buffer I/O	37
5.2.5	Accessing grid parameters	37
5.2.6	Setting grid parameters	38
5.2.7	Utility functions	38
5.2.8	Parallel communications	38
5.2.9	Multigrid functionality	38
5.3	grid class examples	38
5.3.1	Constructing a grid	38
5.3.2	Initializing a grid	39
5.3.3	Setting up grid parameters	39
5.3.4	Accessing grid parameters	39
5.3.5	Reading to and writing from files	39
5.3.6	Using grid in a parallel program	39
5.4	Specialized grid classes	39
5.4.1	CAGrid	39
5.4.2	FDgrid	39
5.4.3	MCgrid	39
5.4.4	PFgrid	39
5.4.5	sparsePF	39
6	MMSP Examples	41
6.1	Introduction	41
6.2	Algorithms	41
6.3	Beginner's Diffusion	42
6.4	Coarsening	42
6.4.1	Grain growth	42
6.4.2	Ostwald ripening	42
6.4.3	Zener pinning	42
6.5	Phase Transitions	42
6.5.1	Allen-Cahn	42
6.5.2	Cahn-Hilliard	42
6.5.3	Model A	43
6.5.4	Model B	43
6.5.5	Solidification	43
6.5.6	Spinodal	43
6.6	Differential Equations	43
6.6.1	Poisson	43
6.7	Statistical Mechanics	43

6.7.1	Heisenberg	43
6.7.2	Ising	43
6.7.3	Potts	43
7	Advanced MMSP Examples	45
7.1	Cahn-Hilliard	45
7.1.1	Convex splitting	45
7.1.2	Semi-implicit discretization	46
7.1.3	Jacobi iteration	47
7.1.4	Gauss-Seidel iteration	47
7.1.5	Successive over-relaxation	48
7.1.6	Numerical performance	48

Chapter 1

Introduction

The goal of the Mesoscale Microstructure Simulation Project (MMSP) is to provide a simple, consistent, and extensible programming interface for all grid and mesh based microstructure evolution methods. Simple means that the package has a very small learning curve, and for most routine simulations, only a minimal amount of code must be written. By consistent we mean, for example, that code for two-dimensional simulations is nearly identical to that for three-dimensional simulations, single processor programs are easily parallelized, and fundamentally different methods like Monte Carlo or phase field have the same look and feel. Finally, extensible means that it's straightforward to add new grid types or physical behaviors to the package. Other considerations include efficiency and portability (MMSP is written entirely in ISO compliant `c++`).

1.1 The MMSP concept

The design of MMSP is based on several observations about how mesoscale simulations are used by materials scientists: most mesoscale simulations discretize the spatial domain using a rectilinear grid. A data structure is associated with each grid node that has a particular size (scalar, vector, etc.) and value type (integer, floating point) depending on the simulation method. Most simulations update the data structure at each node in a way that falls under some common methodology (Monte Carlo, phase field, etc.) but has features unique to each given physical process. What this roughly means is that most mesoscale simulations use a common spatial discretization, but we usually need to tweak the details of how we represent spatial data and how we update it. The unfortunate truth is that typically, a researcher wishing to model a particular physical process produces code with a focus mainly on the particulars of the process itself (the “tweak”), largely ignoring the problem of how to design reusable data structures. What happens when they decide they should try a different simulation method, or when they realize that they need to use a parallel implementation? It then becomes apparent that more flexible data structures should have been

used in the first place. The purpose of **MMSP** is to provide the core functionality that we don't necessarily want to think about each time we program a new method. **MMSP** helps keep its users from reinventing grid data structures, file input and output, parallelization, handling boundary conditions, etc. while retaining enough flexibility to model a large number of physical processes.

Those familiar with similar code packages might have already noticed that the **MMSP** concept is a bit unusual. Other packages typically provide a very high level interface intended for use with a *single* computational method. Using the interface typically means learning package-specific methodology, classes, functions, methods, etc. In contrast, **MMSP** is meant to be used for any and all grid-based methods, and provides a much lower level interface. This results in a lot more flexibility in what **MMSP** can do. And while **MMSP** still requires some learning, users will find that they are able to leverage much more of their previous programming experience.

1.2 What **MMSP** does

MMSP is nothing more than a collection of `c++` header files that declare a number of `grid` objects (classes) and define how most of their methods (member functions) are implemented. Some things **MMSP** provides include:

- A simple, extensible programming interface
- Computational grids of arbitrary dimension
- Parallel implementations using MPI
- Automatic, optimal parallel mesh topologies
- Utility programs for grid visualization
- Classes for Monte Carlo methods
- Classes for cellular automata methods
- Classes for phase field methods (conventional)
- Classes for phase field methods (sparsePF)
- Classes for general finite difference PDE solvers
- Example simulation methods and grid objects

1.3 What **MMSP** doesn't do

MMSP is not the kind of software that reads a few parameters or an input file specified by the user and cranks out some generic computation. In fact, **MMSP** relies on the user to provide code for all of the real physics that the simulation

is intended to capture. This isn't as scary as it sounds. MMSP was designed to make this procedure as simple as possible. The takeaway message here is that MMSP makes programming materials simulation code easier, but it isn't a "black box" that can be used by a complete novice.

1.4 What MMSP requires

The MMSP interface is intended to look and feel very natural for most programmers with experience in scientific computing. While many of the most advanced features of `c++` have been used in creating the grid and data classes, the user need not be proficient in anything other than basic procedural programming. `Fortran`, `c`, and novice `c++` programmers alike will find that MMSP is quite easy to use. Basic requirements include:

- Minimal programming experience
- A `c++` compiler with ISO compliant libraries and headers (e.g. `gcc` 2.95 or later)
- MPI libraries are required if compiling parallel programs (e.g. OpenMPI)

1.5 Terms of use

MMSP is freely available for anyone performing non-profit scientific research; those interested in using MMSP for any other purposes should contact the author. We give no guarantees whatsoever about the capabilities of MMSP. If you use MMSP in your research, please tell others about it, send us any new code you'd like to see incorporated into the package, and above all, give us feedback!

Chapter 2

Getting started with MMSP

The following sections provide the necessary information for new users to obtain and set up MMSP. Typically, this involves downloading a source code archive, unpacking it, and running a few tests. Developers or those interested in maintaining an up-to-date version of the code should consider checking out a copy from the `git` repository.

2.1 Download

The source code for MMSP is available through a `git` repository, hosted on [GitHub](#). This requires the `git` version control software to be installed on the target machine, as well as a working internet connection. From the command line, type

```
git clone https://github.org/mesoscale/mmsp.git MMSP
```

If the checkout is successful, then there is no need to perform the steps for installation described below, and the user should move on to setup.

Having a local copy of MMSP makes it simple to keep your code up-to-date. From the root folder, simply type

```
git pull
```

to update a working copy with the latest version of the MMSP source code. See the `git` documentation for more details on version control.

2.2 Installation

After an appropriate source code archive is obtained as described above, the next step is to install MMSP. This should be as simple as unpacking the archive. Users with administrator privileges may choose to install the MMSP header files in a location that will be searched by their compiler's preprocessor, but we do

not describe how to do this here. The following paragraphs provide platform-specific instructions for a local installation.

Linux/Unix Local installation for Linux users should simply involve unpacking the archive. As most Linux systems have means to unpack both tarballs and zip files, there is likely no reason to prefer either. After downloading the archive file, move it to the directory where you want MMSP to reside, making sure that you have read access to the file as well as write access to the directory. Then issue a command to unpack the archive, e.g.

```
tar xzf MMSP.3.0.6.tar.gz
```

or

```
unzip MMSP.3.0.6.zip
```

This will unpack the contents of the archive into a folder named MMSP. Next, type

```
ls MMSP
```

which should indicate that folders such as MMSP/doc, MMSP/examples, etc. have been created. If either command fails or the folder MMSP is not created, check the `tar` or `unzip` documentation.

Mac OS Mac OS users will follow much of the same procedure as Linux users, so it is advisable to read the previous section on Linux installation. For those uninitiated, or who have never had any previous reason to use it, the **Terminal** application can be found under **Applications/Utilities**. Again, all steps described above for Linux installation should apply here as well.

Windows For those who insist on using Windows, it is still possible to use MMSP. The preferred way to do this is to use the [cygwin](#) environment. To use `cygwin` with MMSP, it is necessary that appropriate packages, such as `gcc` (the GNU compiler) and `make` (the GNU make utility), have been installed. These are optional packages that must be chosen manually during installation. If `cygwin` has been installed properly, MMSP may be installed by following the steps described above for Linux installations.

It is also possible to compile MMSP source code within a code development environment such as Visual Studio, however, MMSP code is typically so simple that any code management beyond command line or makefile compilation is only a hindrance.

2.3 Setup

Once MMSP has been installed, there are a few useful tests and utility programs that should be generated. First, enter the MMSP/test directory and type

```
make test
```

or, if `make` is not installed, type

```
g++ -I ../include test.cpp -o test
```

If this compiles with no problems, then you're in luck; issue the command

```
./test
```

which will generate a short message indicating successful operation.

If the test program fails to compile, it is most likely because either `make` or `g++` (the GNU `c++` compiler) is not installed on the system or is not configured properly. Of course, any other ISO-compliant `c++` compiler may be used instead. If there is a problem at this stage, users should check their configuration.

Next, enter the directory `MMSP/utility` and type

```
make utility
```

This will produce a number of conversion programs. In particular, it will produce several programs such as `mmsp2vti` which can be used to convert `MMSP` grid data files into formats that can be read by visualization software such as [ParaView](#). Because the programs provided in this directory are used so often, `MMSP` users may wish to add the `MMSP/utility` directory to their command path. This can be achieved by adding the following lines to their `$HOME/.bashrc` file,

```
PATH=$PATH:MMSP/utility
export PATH
```

for users of the `bash` shell, or

```
setenv PATH $PATH:MMSP/utility
```

for users of the `tcsh` shell.

Finally, those who plan to use `MMSP` with the MPI (Message Passing Interface) libraries should also take this opportunity to test their MPI configuration. To do this, return to the directory `MMSP/test` and type

```
make parallel
```

which, if successful, produces a parallel version of the test program. Once the code is compiled, run the program using an appropriate command for your MPI distribution, e.g.

```
mpirun -np 4 parallel
```

Note that for successful compilation, the MPI distribution is expected to provide a compilation script named `mpic++` and a the header file named `mpi.h`. If the program fails to compile, it may be that the user's MPI distribution provides `mpicxx`, `mpiCC`, or the like instead of `mpic++`, or that it provides `mpicxx.h` or something similar instead of `mpi.h`. In this case, the user should edit the

`Makefile` accordingly. Likewise, the appropriate command to run the compiled program may differ depending on the MPI distribution. This may take the form of, e.g. `mpiexec`, instead of `mpirun`. Unfortunately, MPI distributions do not adhere to a single standard with respect to compiling and running parallel programs, and so it is largely left to the user to determine what must be done for their particular system.

2.4 Support

MMSP is not commercial code and there are no guarantees or claims, stated or implied, pertaining to its fitness for any purpose. MMSP is intended solely for use in non-profit scientific research. In spite of this, the MMSP team is devoted to producing a quality product that addresses the needs of the scientific community. Please do not hesitate to contact our development team with any questions or suggestions. Contact information can be found on the MMSP page at [GitHub](#).

Chapter 3

MMSP tutorials

The following sections present a few short tutorials on writing, compiling, and running simple MMSP programs. For those totally confused by the syntax of the examples presented here we suggest consulting an introductory `c++` text first.

3.1 A quick tutorial

3.1.1 The Hello MMSP! program

Because every good programming language or interface tutorial starts with a “Hello World!” example program, we’ll do the same. For most MMSP applications, we include a header file named `MMSP.hpp`. Then we need a `main()` program and a few lines to print out our message. Here it is:

```
#include"MMSP.hpp"

int main(int argc, char* argv[])
{
    MMSP::Init(argc,argv);

    std::cout<<"Hello MMSP!"<<std::endl;

    MMSP::Finalize();

    return 0;
}
```

The only code here that should look out of the ordinary are the statements

```
MMSP::Init(argc,argv);
```

and

```
MMSP::Finalize();
```

What do these lines do? For single processor programs, they do absolutely nothing – they could actually be removed without any consequences. However, for programs that use the message passing interface (MPI), they act as wrappers for the similarly named `MPI::Init` and `MPI::Finalize` commands. It's useful to include them here because they'll allow us to write programs that may be compiled for both single or multiple processor environments.

Programmers familiar with `c++` will notice that there's obviously some `MMSP` namespace being used here. For those less familiar, namespaces are a somewhat recent addition to `c++` that are used as a means of avoiding naming conflicts. We can avoid using namespace resolution so frequently if we use an appropriate `using` statement, i.e.

```
#include"MMSP.hpp"
using namespace MMSP;

int main(int argc, char* argv[])
{
    Init(argc,argv);
}
```

Namespaces serve to prevent programming errors and to ensure code reusability, so naturally we should apply `using` statements with care. In the examples that follow, we'll sometimes assume a `using` statement has been issued for the sake of brevity.

The observant reader may notice that we've used some of the stream input/output functions of `c++` without including the requisite `<iostream>` header. In fact, this header and many other standard `c++` headers are included implicitly thorough the file `MMSP.hpp`. If you need a particular standard header for your application and aren't sure if it has been included by `MMSP.hpp`, you can always just `#include` it in your source code without any ill effects.

So that's it as far as the code goes. This is a source file that may be compiled for both single and parallel simulations of... nothing. In a moment we will move on to code that actually does something, but for now we should say a few words about compiling the code and running the executable.

3.1.2 Compiling and running Hello MMSP!

Compiling code is a task that is, unfortunately, fairly platform dependent. While `MMSP` programs should compile easily on any platform, the required steps to do so may not look like the method shown here. That said, let's look at how we would compile the previous example for a typical Linux or Unix setup. Suppose the above code has been saved to a file named `hello.cpp`, and that `MMSP` has been extracted to the same directory. A typical Linux machine will have at the least the GNU family of compilers installed, in which case we would type the command

```
g++ -I MMSP/include hello.cpp -o hello
```


which produces an executable file named `hello`. The compiler option `-I ...` suggests a directory to search for non-standard headers; if your `MMSP` headers reside somewhere else, you'll need to make the appropriate change. To run the program, we type

```
./hello
```

which should produce our message. Many other compilers on Linux and Unix machines use the same options as listed here, so this line may be very close to what you might use, even if you're not using `g++`.

Now let's move on to parallel compilation. We assume that if you're not skipping over this part, you have the MPI libraries installed on your machine (as well as your cluster). With a typical MPI installation, a number of programs are included which effectively wrap your usual `c++` compiler with a script named something like `mpicxx`, `mpicxx`, `mpicc`, etc. With this in mind, we issue a command which may look like

```
mpicxx -I MMSP/include -include mpi.h hello.cpp -o hello
```

which again produces an executable named `hello`. This time, however, we need to run `hello` using an MPI command such as `mpirun` or `mpiexec` (see the documentation for your MPI distribution),

```
mpirun -np 4 hello
```

which in this case produces our message four times. The author sincerely hopes that your experience is this straightforward, but don't count on it!

3.2 A grid class example

In this section, we look at an example that uses the `MMSP grid` template class, and actually resembles part of a program you might actually use:

```
#include"MMSP.hpp"
using namespace MMSP;

int main(int argc, char* argv[])
{
    Init(argc,argv);

    grid<2,int> GRID(1,0,100,0,100);

    for (int x=x0(GRID); x<x1(GRID); x++)
        for (int y=y0(GRID); y<y1(GRID); y++)
            GRID[x][y] = GRID[x+1][y]-GRID[x-1][y];

    output(GRID,"testgrid");

    Finalize();
}
```

To compile, issue

```
g++ -I MMSP/include grid_example.cpp -o grid_example -lz
```

This program performs the amazing feat of generating a new 2-dimensional `grid` object, assigning to each node the sum of `x` and `y`, and then writing the final state of the `grid` to a file, the name of which is simply `testgrid`. Of course this isn't really anything special, but the imaginative reader may begin to see how this program might be used as a template to generate other, more useful `grid` objects.

Several new features have been introduced in this example. First, the line

```
grid<2,int> GRID(1,0,100,0,100);
```

constructs the `grid` object. Objects of the `grid` class take two template arguments (those arguments within `< >`), the first being the dimension (2) and the second being the data type (`int`). In MMSP, the dimension of a `grid` can be any (constant) integer greater than zero, while the data type can be any of `c++`'s built-in types or any of the MMSP data types (to be discussed later). In other words, MMSP allows for simulations in arbitrary spatial dimensions, and the type of data stored at each `grid` point is quite flexible.

The name of the `grid` comes next, which in this case is (`GRID`), and is followed by constructor arguments in parentheses. These arguments indicate the total number of fields to create at each node (relevant only for vector-like data types) and the lower and upper limits in the `x` and `y` directions, respectively. Like for `c++` arrays, lower `grid` limits are inclusive and upper limits are exclusive, so in this case we have that $0 \leq x, y < 100$, and a total number of 100 nodes along each coordinate axis. Unlike `c++` arrays, the limits may be either positive or negative, as long as the upper limit is always larger than the lower limit. Note that these limits are the *global* limits; they define the size of the entire `grid` object. The portion of a `grid` that is actually stored on a given machine is usually smaller than this in parallel programs.

To give a sense of how the above generalizes to other dimensions and data types, consider the following:

```
grid<3,double> GRID(1,0,128,0,256,0,256);
```

This statement declares a 3-dimensional `grid` object with $0 \leq x < 128$, $0 \leq y < 256$, $0 \leq z < 256$, and storage for the built-in data type `double` at each node.

Returning to the example code, the next lines iterate through the nodes of the `grid`,

```
for (int x=x0(GRID); x<x1(GRID); x++)
    for (int y=y0(GRID); y<y1(GRID); y++)
```

The functions `x0`, `x1` and `y0`, `y1` have been invoked to retrieve the *local* limits in each direction. As suggested above, the local grid size in a parallel program is typically smaller than the global grid size because parallelization in MMSP is achieved through spatial decomposition. Use of these functions ensures that we

iterate only through the nodes stored by the local process. While it isn't strictly necessary to use these functions in code meant to be run on a single processor, using them now will make parallelization completely trivial.

The line

```
GRID[x][y] = GRID[x+1][y]-GRID[x-1][y];
```

uses subscript operators `[]` to set and retrieve data stored at nodes of the `grid`. The usage of the subscript operator should look familiar to anyone who has programmed in c-style languages, and it has the expected effect – the subscript operator works the same way for MMSP grid objects as it does for subscripted arrays. However, we will later see that the MMSP subscript operators are “smarter” in the sense that they're cognizant of boundary conditions and adjust appropriately for calls made to nodes that are out of bounds. There are also more advanced ways to access `grid` data which we will discuss later.

Finally, the line

```
output(GRID,"testgrid");
```

seems fairly self-explanatory, and it is. This function passes the data contained in the `grid` class through `zlib` and writes the compressed data to a file; this requires the flag `-lz` appended to the compiler command. Here we also note that in the case of a parallel program, the `output` function performs the additional task of piecing back together the global `grid` object from all local `grid` objects before writing to a file.

As a point of reference, let's now look at how we might do something similar to the example above with the usual `c` or `c++` subscripted arrays:

```
int main(int argc, char* argv[])
{
    int GRID[100][100];

    for (int x=0; x<100; x++)
        for (int y=0; y<100; y++)
            GRID[x][y] = GRID[x+1][y]-GRID[x-1][y];

    // etc.
}
```

This may look considerably simpler, but consider the following: how do we run this code as a parallel process? The immediate answer is that we introduce MPI function calls, but then we also need to decide how to subdivide the grid, how to deal with boundary conditions, how to put the pieces back together when we're done, etc. Then again, what if we want to change the number of processors or number of nodes in each direction and maintain an optimal subdivision scheme? And what about boundary conditions? As written, the code will attempt to access data that is out of the bounds of the array. This will probably lead to runtime errors, or at least some unintentional results. Additional code needs to

be written to remedy this problem. Starting from scratch, these operations all require a great deal of code development, whereas all of these are done by MMSP automatically.

3.3 More examples

This section provides a number of additional examples demonstrating typical MMSP operations.

3.3.1 Accessing node data

In the previous section, we saw how data within a `grid` object may be accessed through the subscript operator, e.g.

```
// assign some node of GRID2D to "value"
double value = GRID2D[x][y];

// assign "value" to some node of GRID3D
GRID3D[x][y][z] = value;
```

There are two other ways to access data, both of which may be useful depending on the context. The first alternative is through integer subscripting using the parentheses operator `()`, e.g.

```
// assign the value of the ith node of GRID2D to "value"
double value = GRID2D(i);

// assign "value" as the value of the ith node of GRID3D
GRID3D(i) = value;
```

How is this operator useful? The answer is that it allows us to write code that works nicely for `grid` objects of arbitrary dimension:

```
for (int i=0; i<nodes(GRID); i++) GRID(i) = ... ;
```

Here, the function `nodes` tells us how many nodes¹ of the `grid` object are stored locally, so that we can simply iterate through them with a single loop. If we perform the same operations with each node regardless of dimension, then our code can be reused without adding additional loops, subscript operators, etc.

The second alternative is through `vector`-based subscripting. In this case, an `MMSP::vector` object is used:

¹The `nodes` function does not count “ghost” cells when such exist. This is not a problem because the parentheses operator skips these nodes anyway.

```

MMSP::grid<2,double> GRID(1,0,100,0,100);
...

for (int i=0; i<nodes(GRID); i++) {
    MMSP::vector<int> x = MMSP::position(GRID,i);
    GRID(x) = ... ;
}

```

This example shows the declaration of an `MMSP::vector` object, and its assignment to the coordinates of the i^{th} node of the `grid`. How might this be useful? Suppose we needed to perform some operation on all the nearest neighbors of a given node. With `vector`-based subscripting, we can use `vector` addition to make quick work of the problem,

```

for (int i=0; i<nodes(GRID); i++) {
    MMSP::vector<int> x = MMSP::position(GRID,i);

    for (int j=0; j<dim; j++) {
        MMSP::vector<int> dx(dim,0);
        dx[j] = 1;

        GRID(x+dx) = ... ;
        GRID(x-dx) = ... ;
    }
}

```

While this kind of operation could be handled quite easily with the more traditional subscript operator, note that this code would be much easier to extend to any number of spatial dimensions.

3.3.2 File input and output

Input and output of `grid` data to a file are performed as follows:

```

const char* filename = "...";
grid<2,double> GRID(1,0,100,0,100);

// Input grid from file
input(GRID,filename);

// Output grid to file
output(GRID,filename);

```

Note that the second argument to the `input` and `output` functions is a character array of type `char*`. Error checking does occur when a `grid` is input from a file; attempts to input a `grid` of the wrong dimension or data type will cause `MMSP` to produce an error message and exit the code.

To be sure, there's probably only rare occasions when we want to use `input` to overwrite an existing `grid`. Quite often, though, we want to create a new

`grid` object from `grid` data contained within a file. In this case, we use the `grid` constructor,

```
const char* filename = "...";
grid<2,double> GRID(filename);
```

which reads `grid` data from the specified file and uses it to construct the `grid` object.

In a later chapter, we discuss the `grid` class in more detail, and in particular, we note that most functions operating on a `grid` can be called in two ways. This also applies here,

```
// One way to output a grid...
output(GRID,filename);

// ... and an equivalent way.
GRID.output(filename);
```

The first case uses typical C style syntax, while the second case uses the class member function syntax of C++. For now, note that almost all functions that operate on the `grid` class or other MMSP data classes may be called both ways.

In case you were wondering, all of the code shown above can be used in a parallel program. In this case, the functions `input` and `output` automatically coordinate I/O tasks over all processors. The `grid` constructor and the `input` function also perform the task of determining the optimal² `grid` subdivision scheme.

3.3.3 Boundary conditions

Boundary conditions in MMSP are handled automatically. To show what this means, consider the following:

```
grid<2,double> f(1,0,100,0,100);

for (int x=x0(f); x<x1(f); x++)
    for (int y=y0(f); y<y1(f); y++) {
        double dfdx = (f[x+1][y]-f[x-1][y])/(2.0*dx(f));
        ...
    }
```

This code traverses each node in the `grid`, computing the derivative along the x axis, among a number of other possible operations. This is simple enough, but what happens at the `grid` boundaries? If similar code was written using built-in arrays, it would typically fail with a segmentation fault, and in any case, it certainly wouldn't produce the desired results. However, with MMSP `grid` objects, this isn't a problem; all of the data access operators in MMSP

²“Optimal” here means that the subdivision is chosen such that it will produce the fewest number of ghost cells possible, while maintaining as evenly balanced a load as possible.

are cognizant of domain boundaries and always enforce the chosen boundary conditions.

For the code example above, periodic boundary conditions are used. This is the default boundary condition in MMSP. A number of other boundary conditions are possible, including Dirichlet, Neumann, and mirror boundaries. Parallel processor boundaries are another special type of boundary condition which is not explicitly set by the user.

To change the boundary conditions used for a `grid`, one might use³

```
grid<2,double> GRID(1,0,100,0,100);

// set x-axis boundary conditions
b0(GRID,0) = Dirichlet;           // BC at lower x-axis boundary
b1(GRID,0) = mirror;             // BC at upper x-axis boundary

// set y-axis boundary conditions
b0(GRID,1) = periodic;           // BC at lower y-axis boundary
b1(GRID,1) = periodic;           // BC at upper y-axis boundary
```

This example demonstrates that the boundary conditions may not only be set independently for each coordinate axis, but that it's also possible to independently set the conditions at the upper and lower boundaries (within reason).

3.4 The prototypical MMSP program

MMSP was created with the intent that it would be used for mesoscale microstructure simulation. So let's look at how we would typically write code to do just that.

```
// prototype.cpp
#include"update.hpp"
using namespace MMSP;

int main(int argc, char* argv[])
{
    Init(argc,argv);

    grid<2,double> GRID(argv[1]);

    update(GRID,atoi(argv[3]));

    output(GRID,argv[2]);

    Finalize();
}
```

³Note, as usual, the boundary conditions and “set” functions all belong to namespace `MMSP` and must be used in the absence of a `using` statement.

In this example, in addition to the usual MMSP boilerplate code, we have a `grid` constructor that reads from a file with a name specified by `argv[1]`, a call to some function called `update` (more on this in a moment), and a call to `output` the `grid` object to a file with a name specified by `argv[2]`. After compiling this program, we would run it with the command line

```
./program initial.PF final.PF 100
```

or, if we compiled with the MPI libraries, we would run in with something similar to

```
mpirun -np 4 program initial.PF final.PF 100
```

Here, 100 is the intended number of time steps that the `update` function will perform.

Now that we have our prototype `main()` function, let's look at the `update` function. For this example, we would have a new header file named `update.hpp`,

```
// update.hpp
#include "MMSP.hpp"
using namespace MMSP;

void update(grid<2,double>& GRID, int steps)
{
    grid<2,double> update(GRID);

    for (int step=0; step<steps; step++) {
        for (int x=x0(GRID); x<x1(GRID); x++)
            for (int y=y0(GRID); y<y1(GRID); y++) {
                // replace this with a "real" computation...
                update[x][y] = GRID[x][y];
            }

        swap(GRID,update);
        ghostswap(GRID);
    }
}
```

Here we define the `update` function, with its first argument a `grid` object and its second argument the number of time steps to perform. The only thing we should bring special attention to here is the use of the ampersand (&) to force call-by-reference, which overrides c++'s usual convention of call-by-value (see a typical introductory text for more).

On the first line within the `update` function, we create a new `grid` object, itself called `update`. The new `grid` is created using a constructor with `GRID` as its argument. What this does is it generates a `grid` with the same spatial extent, number of fields, parallel topology, etc. In other words, the `grid` called `update` is a suitable workspace to store the values of `GRID` for the next time step as we compute them.

Next, we have a loop over all time steps. Within this loop we iterate through the nodes of `GRID` just as in the example of the previous section. The limiting values of `x` and `y` are obtained by appropriate function calls. At each node, we would normally perform some meaningful computation to determine the value of `GRID` at the next time step, then store the value in `update`. Here we simply copy the value of `GRID` at this node to `update`.

After computations are performed at each node, we have two more operations. First, we `swap` the data of `GRID` and `update`. `GRID` now contains the new values, while `update`, the workspace, contains the old. Next, a function called `ghostswap` is called with `GRID` as its argument. For single processor programs, the `ghostswap` function does nothing. For parallel programs, it performs a co-ordinated series of “ghost” cell data swaps, such that the ghosts associated with the local portion of `GRID` are filled with the appropriate data from all neighboring processors. And again, while it’s not critical to include this line when our intent is to write a single processor program, it will serve us well to include it here because it will allow us to easily parallelize the code later.

3.5 Application: the Allen-Cahn equation

Now that we’ve seen much of the basic syntax used in `MMSP`, let’s apply it to a few realistic physical problems. The first application we’ll discuss is the numerical solution of the Allen-Cahn equation, which comes from a well-known model for order-disorder phase transformations in solids. In this model, the state of the system is described by a single parameter, ϕ , which represents the local atomic ordering. Two distinct, stable variants are represented when $\phi = -1$ and $\phi = 1$. The energy density of the system includes a penalty for states other than these as well as a term that depends on the gradient of ϕ . The total energy of the system takes the form

$$F = \int_V -\frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \frac{\kappa}{2}|\nabla\phi|^2 dV \quad (3.1)$$

where r , u , and κ are phenomenological constants. The time evolution of ϕ is assumed to take the form

$$\frac{d\phi}{dt} = -M \frac{\delta F}{\delta \phi} = -M(-r\phi + u\phi^3 - \kappa\Delta\phi). \quad (3.2)$$

Using a forward Euler time step, the above would be discretized as

$$\phi^{t+\Delta t} = \phi^t - \Delta t M(-r\phi^t + u(\phi^t)^3 - \kappa\Delta\phi^t). \quad (3.3)$$

Now, let’s look at an `MMSP` code representation of the above. Suppose we aim to write an `update` function, similar to what was shown in the previous section. This time, we’ll additionally use the `c++` template syntax to write a code that will work for `grid` objects of any dimension (and any scalar data type):

```

template <int dim, typename T>
void update(MMSP::grid<dim,T>& grid, int steps)
{
    MMSP::grid<dim,T> update(grid);

    double r = 1.0;
    double u = 1.0;
    double K = 1.0;
    double M = 1.0;
    double dt = 0.01;

    for (int step=0; step<steps; step++) {
        for (int i=0; i<nodes(grid); i++) {
            double phi = grid(i);
            update(i) =
                phi-dt*M*(-r*phi+u*pow(phi,3)-K*laplacian(grid,i));
        }
        swap(grid,update);
        ghostswap(grid);
    }
}

```

The `update` function declaration is equivalent to those shown before, although the use of template syntax affords us some greater generality. A temporary `grid` object called `update` is declared next, again with template syntax. The constants r , u , etc. are then declared and initialized. The remainder of the code performs the desired time steps. At each time step, the following computation is performed at each node:

```

double phi = grid(i);
update(i) = phi-dt*M*(-r*phi+u*pow(phi,3)-K*laplacian(grid,i));

```

The reader will notice that this looks nearly identical to the pseudocode given above. Here, the `c++` math library function `pow` is used to compute ϕ^3 and the `grid` class `laplacian` operator is used to compute $\Delta\phi$.

The Allen-Cahn equation is a fairly simple model, yet it demonstrates many of the powerful features that MMSP offers. Note that, with only a few unique lines of `c++`, we have produced a code that

1. Solves a realistic physical problem with minimal software development.
2. Works for `grids` of arbitrary spatial dimension.
3. Works for `grids` of arbitrary scalar data type.
4. Handles boundary conditions with no user intervention.
5. Produces both single processor and parallel code, depending only on compiler options.
6. Automatically produces an optimal parallel mesh topology, when required.

3.6 Running the example codes

Each example program in the `MMSP/examples` directory contains at least three files: the `example.hpp`, defining the `generate` and `update` functions; `example.cpp`, defining the `grid` types to be used; and a `Makefile`, which helps to simplify compiling MMSP code. The `hpp` and `cpp` files should be self-explanatory, based on the preceding discussion. A file named `notes` may also be included, identifying key research papers used in developing the program.

Let's take a closer look at the `Makefile` from the Allen-Cahn example code, available in `MMSP/examples/phase_transitions/allen-cahn/`:

```
# includes
incdir = ../../../include

# compilers/flags
compiler = g++
flags = -O3 -I $(incdir)
pcompiler = mpic++
pflags = -O3 -I $(incdir) -include mpi.h

# dependencies
core = $(incdir)/MMSP.main.hpp \
      $(incdir)/MMSP.utility.hpp \
      $(incdir)/MMSP.grid.hpp

# the program
allen-cahn: allen-cahn.cpp allen-cahn.hpp $(core)
$(compiler) $(flags) $< -o $@ -lz

parallel: allen-cahn.cpp allen-cahn.hpp $(core)
$(pcompiler) $(pflags) $< -o $@ -lz

clean:
rm -f allen-cahn parallel
```

The purpose of this file is to specify the compiler and paths to any file required to build the code. The folder `../../../include` specified in the first block equals `MMSP/include`, where the core MMSP header files are stored. The second block specifies the compiler and relevant flags, including optimization level. Options are set for both `g++` and `mpic++` here. The last definition block lists the specific MMSP headers to be used in the program. Note that not every header is included!

The blocks starting with `allen-cahn:` and `parallel:` specify the `make` commands and their output. Executing `make allen-cahn` will build the executable `allen-cahn` using `g++`; `make parallel` builds the executable `parallel` using `mpic++`. Finally, executing `make clean` will delete the executable files.

Any entry in the `Makefile` can be modified. For example, if you copy an example code as the basis for your own project, you can move it anywhere on

your hard drive so long as you edit the `incdir` variable appropriately. You may also change the executable names and compiler flags at will. The `Makefile` is a powerful tool for keeping track of lengthy compiler commands.

Chapter 4

MMSP data classes

4.1 Introduction

Aside from the dimension, the other defining characteristic of a `grid` object is the type of data stored at each node or cell. In MMSP, valid `grid` data may be either built-in variable types or one of several MMSP data classes. This chapter presents a more detailed look at using both kinds of data types.

4.2 Common features

MMSP data classes share a number of common features. Most importantly, the way we interact with them through functions and symbolic operators has been designed to be as intuitive and self-consistent as possible. In this section, we list the functions common to all MMSP data classes and built-in types, and in following sections we describe additional functions specific to each particular class.

The member functions common to every MMSP data class include

- `length()`
This function is used to determine the “length” of a given data object. For example, the length of a `scalar` is always one, the length of a `vector` object is the number of values stored in it, and so on.
- `resize(integral_value)`
This function is used to “resize” data types that contain more than one value, such as a `vector` object. When an object inherently has length one (e.g. `scalar` objects), this function does nothing at all.
- `copy(data_object)`
The `copy` function copies all data from the object in the parameter list to the calling object, i.e. the one that called the function.

- **swap**(*data_object*)
The **swap** function swaps all data of the object in the parameter list with the calling object.
- **buffer_size**()
This function returns the number of bytes that would be used if the calling object were stored in a character buffer.
- **to_buffer**(*character_buffer*)
This function packs the data of the calling object into the character buffer object given in the parameter list. It also returns the number of bytes that were used.
- **from_buffer**(*character_buffer*)
This function reads data from the character buffer object given in the parameter list and writes it to the calling function. It also returns the number of bytes that were read.
- **read**(*ifstream_object*)
The **read** function reads data from the given **c++ ifstream** object to the calling data object.
- **write**(*ofstream_object*)
The **write** function writes the data of the calling object to the the **c++ ofstream** object given in the parameter list.
- **operator=**(*data_object*)
The assignment operator is overloaded for all MMSP data classes to have the usual “copy” behavior.

Each of these functions may be called in either of two ways: as a class member function or as a globally defined function. In the first case, calls to a member function **f(...)** look like this:

data_object.f(parameter_list);

In the second case, the global function **f** is called with a similar syntax, but now the data object becomes the first function parameter:

f(*data_object, parameter_list*);

Globally defined functions have the same behavior as their associated grid member function. The user is free to choose either based on convenience or aesthetics. Symbolic operators are the only member functions that do not have associated functions.

It is, of course, also possible to use built-in data types such as **int**, **float**, and **double** anywhere an MMSP data object might be used. The functions listed above have been redefined to work for all built-in types, but note that because it's impossible to write member functions for built-ins, only the global function calls may be used.

4.3 Using MMSP data types

All MMSP data types are defined as template classes. Just as a declaration of a `grid` objects requires dimension and data type template parameters, each MMSP data class object requires a single template parameter.

Most of the time, the user won't work with MMSP data types directly, but through nodal data on `grids`. It's important to know some of the basic workings of the MMSP data classes for this reason.

4.3.1 The scalar class

The `scalar` class is essentially a wrapper for built-in data types. It was defined primarily for consistency, i.e. as a complement to the vector class discussed below. MMSP does not define any functions for `scalar` objects other than the common functions discussed in the previous section. However, because a `scalar` object is an instantiation of a class, it is possible to use both the member function and global function calling syntax.

```
// MMSP scalar example

// declaration of a "scalar" object
scalar<double> s;

// use a "scalar" as you would use the corresponding built-in
s = 1.2345;
double square = s*s;
std::cout<<"s = "<<s<<std::endl;

// the following two lines are equivalent
// and both write "1" to standard output
std::cout<<"length = "<<length(s)<<std::endl;
std::cout<<"length = "<<s.length()<<std::endl;

// how a "scalar" is used in a "grid"
grid<2,scalar<int> > GRID(1,0,10,0,10);
for (int x=xmin(GRID); x<xmax(GRID); x++)
    for (int y=ymin(GRID); y<ymax(GRID); y++)
        GRID[x][y] = x+y;

// this writes the value "1" to standard output
std::cout<<"fields = "<<fields(GRID)<<std::endl;
```

4.3.2 The vector class

The `vector` class is meant to be used primarily as a fixed-length container for nodal data. Just like a usual `c` or `c++` array, `vector` data is accessed by use of the subscript operator.

When a **vector** is used as the data associated with a **grid** node, its length is initialized to the number of fields assigned to the **grid**. Otherwise, a **vector**'s length must be initialized with the **resize** function.

If a **vector** of nonzero length is **resized** to a greater length, the original data it contains is preserved. Thus a **vector** may be used, e.g. to generate a list even when its final length is not known *a priori*. Trying to **resize** a **vector** being used as **grid** data is *not* recommended.

```
// MMSP vector example

// declaration of a "vector"
vector<double> v;

// a "vector" must be resized before use
resize(v,10);

// the following two lines are equivalent
// and both write "10" to standard output
std::cout<<"length = "<<length(v)<<std::endl;
std::cout<<"length = "<<v.length()<<std::endl;

// set and get values with the subscript operator
v[0] = 1.2345;
for (int i=1; i<length(v); i++)
    v[i] = 2.0*v[i-1];

// how a "vector" is used in a "grid"
grid<2,vector<int> > GRID(4,0,10,0,10);
for (int x=xmin(GRID); x<xmax(GRID); x++)
    for (int y=ymin(GRID); y<ymax(GRID); y++)
        for (int i=0; i<fields(GRID); i++)
            GRID[x][y][i] = x+y+i;

// this writes the value "4" to standard output
std::cout<<"fields = "<<fields(GRID)<<std::endl;
```


4.3.3 The sparse class

The MMSP `sparse` data class contains `vector`-like data, but rather than maintaining a fixed length of data, `sparse` objects grow as values are `set` by the user. `sparse` data is most useful in situations where a large number of fields are defined, but only a small number of them have values that differ from some “nominal” value.

```
// MMSP sparse example

// declaration of a "sparse"
sparse<double> s;

// the following two lines are equivalent
// and both write "0" to standard output
std::cout<<"length = "<<length(s)<<std::endl;
std::cout<<"length = "<<s.length()<<std::endl;

// all of the following lines
// write "0" to standard output
for (int i=0; i<10; i++)
    std::cout<<"s["<<i<<"] = "<<s[i]<<std::endl;

// use of the "set" function
set(s,0) = 1.2345;
set(s,1) = s[0];
set(s,2) = s[1];

// now the length is "3" ...
std::cout<<"length = "<<length(s)<<std::endl;

// ... and we can output only those values that are stored
// Note that both calls to "cout" produce the same output
for (int i=0; i<length(s); i++) {
    int ind = index(s,i);
    double val = value(s,i);
    std::cout<<"s["<<ind<<"] = "<<val<<std::endl;
    std::cout<<"s["<<ind<<"] = "<<s[ind]<<std::endl;
}

// how a "sparse" is used in a "grid"
grid<2,vector<int>> > GRID(0,0,10,0,10);
for (int x=xmin(GRID); x<xmax(GRID); x++)
    for (int y=ymin(GRID); y<ymay(GRID); y++)
        for (int i=0; i<10; i++)
            set(GRID[x][y],i) = x+y+i;
```

The **sparse** data class achieves its function by storing index/value pairs as they are **set** by the user. Obviously, this requires more memory than storing a single value, so that there's some point where, for a small enough number of fields, using **sparse** data becomes impractical.

The subscript operator applied to **sparse** data returns the value associated with the index given as the operator's argument. If the value for the given index has not been **set** by the user, the subscript operator returns zero.

The **length** function applied to a **sparse** object returns the number of values that have been **set**. The user may iterate through only those values that have been **set** by using the **value** and **index** functions. A synopsis of the functions unique to the **sparse** data type follow.

- **set**(*integral_value index*)
This function set the value of a particular index/value pair of a **sparse** object.
- **value**(*integral_value i*)
This function returns the value of the i^{th} index/value pair of a **sparse** object.
- **index**(*integral_value i*)
This function returns the index of the i^{th} index/value pair of a **sparse** object.

4.4 Using built-in data types

4.5 Writing new data classes

Chapter 5

The grid class

The MMSP base class `grid` is defined in the header `MMSP.grid.hpp`. The derived template classes `grid1D`, `grid2D`, and `grid3D` provide the base for the high level grid objects associated with various simulation methods. The template parameters for each grid object are a point data structure and a fundamental numeric type. The grid objects contain a large number of inherited methods (member functions) that perform common tasks such as file I/O, data storage and retrieval, etc. Member data for the grid classes determine the grid size, boundary conditions and point spacing along each coordinate axis. High level grid objects usually have a name that indicates both their intended use as well as their dimensionality, e.g. `PFgrid3D` and `MCgrid2D`. These are the grid objects that are used in performing simulations or other computations. Detailed information about each existing grid class is given in the reference section of this page. See also the source distribution. MMSP grid classes are designed so that programming with them is as intuitive as possible. For example, data can be accessed by subscript operators in exactly the same way we would use data stored in a subscripted array

```
// First create a grid object.
PFgrid2D grid(100,100,2);

// Assign a value to the phase field grid.
grid[10][20][1] = 1.0;

// Use a value from the grid in a computation.
float value = grid[10][20][1];
```

Because of this, C/C++ code that performs computations using "hard-coded" arrays automatically works for the corresponding MMSP grid object. This makes it trivial to insert MMSP grid objects into existing grid generation, simulation, or analysis code. Each grid object also provides a `neighbor()` function that returns a grid data structure using coordinates relative to a given position (x,y,z)

```
// First create a grid object.
MCgrid2D grid(100,100);

// Assign a value to the Monte Carlo grid.
grid[10][20] = 5;

// Access the value using the neighbor() function.
int spin = grid.neighbor(9,22,1,-2);
```

The `neighbor` function automatically accounts for the grid boundary conditions set by the user. See the next subsection for more details on programming with the grid classes. Finally, let's consider using grid objects in a parallel computing setting. In single processor programs, we typically create a single grid object within the `main()` function. This grid object is considered to be the entire simulation domain. In parallel programs, however, the grid object declared within `main()` is a subgrid of a larger, global simulation domain. MMSP can automatically decompose (or recombine) a global grid object into local subgrid "slabs" with cuts perpendicular to the x axis, a method which has a consistent (and simple) implementation regardless of grid dimension. MMSP allows the user to choose the number of ghost cells retained in each subgrid, and the boundary conditions of the global grid object are satisfied when the `neighbor()` function is used. The implementation of parallel grid functionality is contained within a separate header `MMSP.grid.parallel.hpp` for the convenience of users without a local MPI distribution.

5.1 Introduction

The `grid` class is the base class for all higher level grid classes...

A number of points about the grid class and its member functions are useful in the following.

- Most grid functions may be called in either of two ways: as a class member function or as a global friend function. In the first case, calls to a grid member function `f(...)` look like this:

grid_object.f(parameter_list);

In the second case, the global friend function `f` is called with a similar syntax:

f(grid_object, parameter_list);

Global friend functions have the same behavior as their associated grid member function. The user is free to choose either based on convenience or aesthetics. Note that the grid object now becomes the first argument to

the function, while the remainder of the parameter list is unchanged. The only functions that don't follow this convention are symbolic operators, e.g. the subscript operator:

```
grid_object[x][y] = rvalue;
lvalue = grid_object[x][y][z][index];
```

Symbolic operators are grid member functions that do not have associated friend functions.

-
-
-

5.2 grid class member functions

5.2.1 Constructors

```
grid(int fields, int x0, int x1, ...);
grid(int fields, int min[dim], int max[dim]);
grid(const grid& GRID);
grid(char* filename);
```

5.2.2 Subscripting

5.2.3 File I/O

5.2.4 Buffer I/O

5.2.5 Accessing grid parameters

```
int x0(const grid& GRID, int i);
int x1(const grid& GRID, int i);
int x0(const grid& GRID);
int x1(const grid& GRID);
int y0(const grid& GRID);
int y1(const grid& GRID);
int z0(const grid& GRID);
int z1(const grid& GRID);
```

5.2.6 Setting grid parameters

5.2.7 Utility functions

5.2.8 Parallel communications

5.2.9 Multigrid functionality

5.3 grid class examples

5.3.1 Constructing a grid

```
#include"MMSP.grid.hpp"

int main(int argc, char* argv[])
{
    MMSP::Init(argc,argv);

    MMSP::grid<1,double> GRID(1,0,100);

    MMSP::Finalize();
}

#include"MMSP.grid.hpp"

int main(int argc, char* argv[])
{
    MMSP::Init(argc,argv);

    char* filename = argv[1];

    MMSP::grid<2,double> GRID(filename);

    MMSP::Finalize();
}

#include"MMSP.grid.hpp"

int main(int argc, char* argv[])
{
    MMSP::Init(argc,argv);

    MMSP::grid<3,MMSP::vector<double> > GRID();

    MMSP::Finalize();
}
```

5.3.2 Initializing a grid

5.3.3 Setting up grid parameters

5.3.4 Accessing grid parameters

5.3.5 Reading to and writing from files

5.3.6 Using grid in a parallel program

5.4 Specialized grid classes

5.4.1 CAggrid

5.4.2 FDgrid

5.4.3 MCgrid

5.4.4 PFgrid

5.4.5 sparsePF

Chapter 6

MMSP Examples

6.1 Introduction

The MMSP source code includes a large number of examples implementing canonical problems in diffusion, phase transformation, and statistical mechanics. These are provided to demonstrate the best practices for writing efficient MMSP code and to provide starting points for your research. These examples are documented as clearly as possible in the `c++` files themselves, with references to academic research papers for further details.

MMSP provides a common back-end for solving partial differential equations on spatial domains. Writing the equation of motion is up to the user: MMSP itself is agnostic to your realm of inquiry or numerical techniques. Example folder may contain several related approaches: kinetic Monte Carlo vs. phase field, isotropic vs. anisotropic, vector vs. sparse data types.

These scripts are intended to showcase MMSP, not to test the physics *per se*. You are encouraged to adapt an example for your research purposes, but check our work against the literature. For consistency, every example compiles against `MMSP/include/MMSP.main.hpp`, which serves as a useful template for your own `main()` function.

6.2 Algorithms

The folder `MMSP/algorithms` contains miscellaneous algorithms for initializing and analyzing simulation domains, especially for grain growth models. Some highlights include:

- `tessellate.hpp` Algorithms to initialize an empty grid with a Poisson-Voronoi tessellation, a commonly used starting point for grain growth simulations.
- `grainsize.hpp` Algorithms to compute grainsize.
- `topology` Algorithms to extract discrete topological properties from diffuse interfaces.

6.3 Beginner's Diffusion

Solves the canonical diffusion-type equation,

$$\frac{\partial C}{\partial t} = -D\nabla^2 C.$$

6.4 Coarsening

6.4.1 Grain growth

Phase field

Evolves co-existing sub-domains $\phi_i \in \vec{\phi}$ according to non-conserved Allen-Cahn dynamics with mobility L ,

$$\begin{aligned} \frac{\delta \mathcal{F}}{\delta \phi_i} &= -\epsilon^2 \nabla^2 \phi_i + \phi_i^3 - \phi_i^2 + \phi_i \sum_{j \neq i} \phi_j^2 = -\epsilon^2 \nabla^2 \phi_i - \phi_i^2 + \phi_i \sum_{j=1}^N \phi_j^2 \\ \frac{\partial \phi_i}{\partial t} &= -L \frac{\delta \mathcal{F}}{\delta \phi_i}. \end{aligned}$$

6.4.2 Ostwald ripening

Evolves coupled equations for non-conserved (phase ϕ) and conserved (composition c) quantities. Due to surface curvature, larger domains consume smaller ones to reduce the total system energy.

6.4.3 Zener pinning

6.5 Phase Transitions

6.5.1 Allen-Cahn

6.5.2 Cahn-Hilliard

Applies Cahn-Hilliard dynamics to the spinodal defined by

$$f(c) = \frac{C^4}{4} - \frac{C^2}{2},$$

producing a fourth-order PDE:

$$\frac{\partial C}{\partial t} = D\nabla^2 \left(\frac{\partial f}{\partial C} - \epsilon^2 \nabla^2 C \right).$$

Explicit

Discretizes the Cahn-Hilliard equation explicitly, coupling two second-order equations in the old value (C^n) and new value (C^{n+1}) to recover the fourth-order dynamics:

$$\begin{aligned}\mu &= \frac{\partial f}{\partial C^n} - \epsilon^2 \nabla^2 C^n \\ C^{n+1} &= C^n + D\Delta t \nabla^2 \mu\end{aligned}$$

Convex splitting

This advanced example is discussed in [Chapter 7](#).

6.5.3 Model A**6.5.4 Model B****6.5.5 Solidification****6.5.6 Spinodal****6.6 Differential Equations****6.6.1 Poisson****6.7 Statistical Mechanics****6.7.1 Heisenberg****6.7.2 Ising****6.7.3 Potts**

Chapter 7

Advanced MMSP Examples

7.1 Cahn-Hilliard

This section describes the mathematical foundations of the example code in `MMSP/examples/phase_transitions/cahn_hilliard/convex_splitting`.

Spinodal decomposition is readily modeled using the Helmholtz free energy functional

$$\mathcal{F} = \int [F(C) + \kappa |\nabla C|^2] \, dV,$$

with free energy density

$$F(C) = \frac{C^4}{4} - \frac{C^2}{2}.$$

The first derivative

$$f(C) = \frac{dF}{dC} = C^3 - C^2$$

has two terms, one of which is convex in its arguments $\left(\frac{d^2(C^3)}{dC^2} \Big|_{C \geq 0} \geq 0 \right)$ and the other concave $\left(\frac{d^2(-C^2)}{dC^2} < 0 \right)$. From a numerical perspective, the convex term tends to smooth, while the concave term amplifies discontinuity: the convex term is contractive in C , the concave expansive. When used in the Cahn-Hilliard equation of motion,

$$\frac{\partial C}{\partial t} = D \nabla^2 [f(C) - \kappa \nabla^2 C],$$

antagonistic initial conditions with improperly discretized equations of motion will increase the system energy, which is not physical.

7.1.1 Convex splitting

Convex splitting aims to guarantee energy stability $\left(\frac{d\mathcal{F}}{dt} \leq 0 \right)$ by discretizing the equation semi-implicitly, evaluating contractive terms at the next timestep

(index $n + 1$) and expansive terms at the old timestep (index n):

$$\begin{aligned} f_c(C) &= C^3 \\ f_e(C) &= -C^2 \\ C^{n+1} &= C^n + kD\Delta_h\mu^{n+1} \\ \mu^{n+1} &= f_c(C^{n+1}) + f_e(C^n) - \kappa\Delta_h C^{n+1}, \end{aligned}$$

with common symbolic substitutions for timestep ($k = \Delta t$) and discrete Laplacian operator (Δ_h) with grid spacing $h = \Delta x$. Collecting old and new terms,

$$\begin{aligned} C^{n+1} - kD\Delta_h\mu^{n+1} &= C^n \\ \kappa\Delta_h C^{n+1} - f_c(C^{n+1}) + \mu^{n+1} &= f_e(C^n). \end{aligned} \tag{7.1}$$

7.1.2 Semi-implicit discretization

Let's prepare to solve the system of equations iteratively. In the Jacobi method, elements along the matrix diagonal (i.e., the grid point of interest) are evaluated at the next guess (index $l + 1$), and off-diagonal elements belong to the last guess (index l). The nonlinear source term C^3 is linearized, so

$$\begin{aligned} \ell_c(C_l^{n+1}) &= (C_l^{n+1})^2 \\ f_c(C) &= \ell_c(C_l^{n+1})C_{l+1}^{n+1}. \end{aligned}$$

Let's separate the discrete Laplacian Δ_h into its central (matrix-diagonal) and fringe (off-diagonal) components. The d -dimensional formula is readily generalized from the 5-point stencil in 2-dimensions given $\Delta x = \Delta y = h$:

$$\begin{aligned} \Delta_h C(x, y) &= \frac{C(x+1, y) - 2C(x, y) + C(x-1, y) + C(x, y+1) - 2C(x, y) + C(x, y-1)}{h^2} \\ &= \frac{-4C(x, y)}{h^2} + \frac{C(x+1, y) + C(x-1, y) + C(x, y+1) + C(x, y-1)}{h^2} \\ \Delta_h C(\vec{x}) &= \frac{-2dC(\vec{x})}{h^2} + \Delta_o C(\vec{x}), \end{aligned}$$

with the “fringe Laplacian” operator Δ_o introduced as convenient shorthand.

Finally, we have the semi-implicit discretization properly indexed for iteration:

$$\begin{aligned} C_{l+1}^{n+1} + \frac{2dkD}{h^2}\mu_{l+1}^{n+1} &= C^n + kD\Delta_o\mu_l^{n+1} \\ -\frac{2d\kappa}{h^2}C_{l+1}^{n+1} - \ell_c(C_l^{n+1})C_{l+1}^{n+1} + \mu_{l+1}^{n+1} &= f_e(C^n) - \kappa\Delta_o C_l^{n+1}, \end{aligned} \tag{7.2}$$

or in matrix form

$$\begin{bmatrix} 1 & \frac{2dkD}{h^2} \\ -\frac{2d\kappa}{h^2} - \ell_c(C_l^{n+1}) & 1 \end{bmatrix} \begin{bmatrix} C_{l+1}^{n+1} \\ \mu_{l+1}^{n+1} \end{bmatrix} = \begin{bmatrix} C^n + kD\Delta_o\mu_l^{n+1} \\ f_e(C^n) - \kappa\Delta_o C_l^{n+1} \end{bmatrix}.$$

This system of linear equations is solved using Cramer's rule for 2×2 systems $\mathbf{Ax} = \mathbf{b}$:

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}}, \quad x_2 = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{21} a_{12}}.$$

For clarity, the code implementing Cramer's rule for the equations of motion retains the a_{ij} , x_i , b_i notation.

7.1.3 Jacobi iteration

To determine values for the next timestep C^{n+1} and μ^{n+1} , seed the matrix using the old values C^n and μ^n , then iterate until the system of equations is true, meaning the error

$$\eta = \|\mathbf{b}' - (\mathbf{Ax})'\|,$$

using the L_2 vector norm $\|\cdot\|$, falls tolerably close to zero. The matrix terms are dashed (') for a reason: the residual does not depend on the iteration scheme chosen, so $(\mathbf{Ax})' = \mathbf{b}'$ represents the equations of motion prior to l -indexing for Jacobi iteration, Eqn. 7.1 rather than Eqn. 7.2:

$$\begin{aligned} \mathbf{Ax} &= \begin{bmatrix} C_{l+1}^{n+1} + \frac{2dkD}{h^2} \mu_{l+1}^{n+1} \\ -\frac{2d\kappa}{h^2} C_{l+1}^{n+1} - \ell_c(C_l^{n+1}) C_{l+1}^{n+1} + \mu_{l+1}^{n+1} \end{bmatrix}, & (\mathbf{Ax})' &= \begin{bmatrix} C_{l+1}^{n+1} - kD\Delta_h \mu_{l+1}^{n+1} \\ \kappa\Delta_h C_{l+1}^{n+1} - f_c(C_{l+1}^{n+1}) + \mu_{l+1}^{n+1} \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} C^n + kD\Delta_o \mu_l^{n+1} \\ f_e(C^n) - \kappa\Delta_o C_l^{n+1} \end{bmatrix}, & \mathbf{b}' &= \begin{bmatrix} C^n \\ f_e(C^n) \end{bmatrix} \end{aligned}$$

Note that $(\mathbf{Ax})' = \mathbf{b}'$ depends only on the old *value* and the latest *guess*.

Since the values at $n + 1$ are interdependent, the local errors are gathered up in the residual:

$$r = \sqrt{\sum_N \frac{\|\mathbf{b}' - (\mathbf{Ax})'\|}{2N\|\mathbf{b}'\|}},$$

in which N is the number of nodes in the grid and 2 is the number of equations being solved at each node. Calculating the residual is as expensive as a full iteration, and is done at regular intervals rather than after each iteration. An appropriate tolerance for convergence is $r_{tol} \leq 10^{-14}$: if $r > r_{tol}$, guess again.

7.1.4 Gauss-Seidel iteration

Jacobi iteration requires allocating an extra grid to store intermediate guesses. In the interests of speed and memory footprint, red-black Gauss-Seidel is a better choice. In this scheme, the grid points are labeled according to a checkerboard pattern with alternating "red" and "black" tiles. The new guess is calculated on each red tile first, using the previous guess stored in adjacent black tiles; then the new guess is calculated on each black tiles, using the new guess stored in adjacent red tiles. This scheme also enables seamless switching between serial

and parallel architectures, since cells adjacent across parallel boundaries can never be both red or both black.

Note that the change from Jacobi to Gauss-Seidel modifies the order in which the guesses are made, with minimal effect on the values computed.

7.1.5 Successive over-relaxation

Successive over-relaxation increases the rate of convergence of Gauss-Seidel iteration by weighted averaging the outcome of Cramer's rule (representing the new guess) with the last guess. Instead of

$$\begin{aligned} C_{l+1}^{n+1} &= x_1 \\ \mu_{l+1}^{n+1} &= x_2, \end{aligned}$$

we record

$$\begin{aligned} C_{l+1}^{n+1} &= \omega x_1 + (1 - \omega) C_l^{n+1} \\ \mu_{l+1}^{n+1} &= \omega x_2 + (1 - \omega) \mu_l^{n+1}. \end{aligned}$$

The relaxation parameter ω must necessarily satisfy $0 < \omega < 2$ for convergence. Successive over-relaxation requires $\omega > 1$, and $\omega = 1.1$ is recommended for this specific system. $\omega = 1$ recovers Gauss-Seidel, and $\omega < 1$ is *successive under-relaxation*, which is not recommended.

7.1.6 Numerical performance

This example is intended to show that MMSP can handle numerical schemes far beyond forward Euler. Stability analysis of the explicit scheme requires

$$\Delta t_E < \frac{(\Delta x)^4}{16\kappa D} \text{Co},$$

with Courant-Friedrichs-Lewy number $\text{Co} \ll 1$. Taking $\text{Co} = 1/4$ and $\kappa = D = 1$, we have $\Delta t_E = 1/64$, and integration to $t = 50,000$ (adequate time to reach equilibrium) takes 3.2×10^6 explicit steps.

The semi-implicit scheme discussed here is unconditionally numerically stable, but the larger the step, the more iterations are necessary to converge. Early on, when the system is rapidly evolving from initial conditions, the convex splitting scheme can be much more expensive than forward Euler. As the simulation approaches equilibrium, however, convergence is achieved in many fewer iterations per timestep. With $\Delta t = 1$ and $\omega = 1$, the semi-implicit code takes 2.6×10^6 iterations in total to reach $t = 50,000$. Assuming one semi-implicit iteration incurs the same computational cost as one explicit step, this means Gauss-Seidel gives us $1.2\times$ speedup. Successive over-relaxation with $\omega = 1.1$ does a little better, with a $1.4\times$ boost.