From FS to

Fourier Transform

&

DFT

Discrete

Preliminaries – complex numbers

for Fourier Series cool
for DFT

$\Rightarrow$ decomposing signal as a sum of functions oscillating at multiples of ~~some~~ fundamental frequency

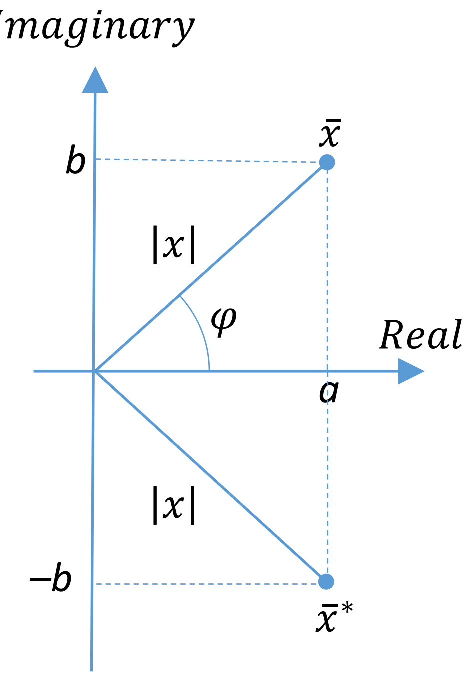
A complex number $\bar{x} = \underset{\mathbb{R}}{a} + \underset{\text{img}}{jb}$ is composed by a real (a) and a imaginary (jb) parts

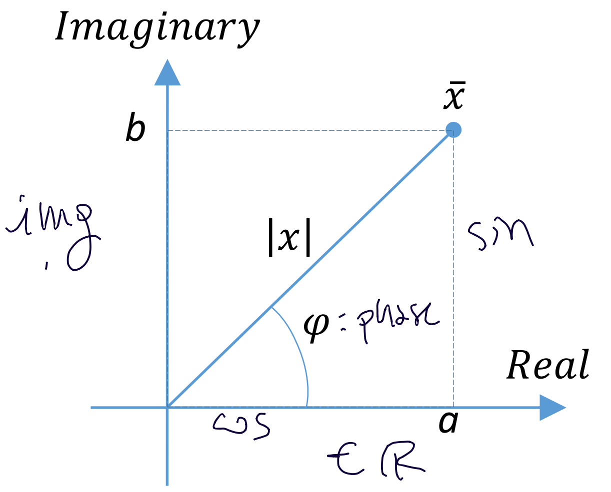
$j = \sqrt{-1}$

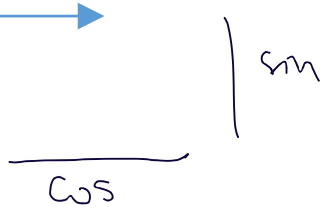
- where j is the imaginary number, square root of -1

It can be represented in the imaginary plane as a vector with modulus $|x|$ and phase φ

Conjugate of $\bar{x}$: $\bar{x}^* = a - jb$







- $\bar{x} = a + jb$
- $|x| = \sqrt{a^2 + b^2}$;
- $\varphi = \tan^{-1}\frac{b}{a}$
- $a = |x| \cos\varphi$
- $b = |x| \sin\varphi$

and also:

- $\bar{x} = |x| \cos\varphi + j|x| \sin\varphi$

Preliminaries – Euler's exponential

of euler #

The exponential $e^{j\varphi}$ is a complex number defined by Euler as :

$$\boxed{e^{\pm j\varphi} = \cos\varphi \pm j\sin\varphi}$$

From which we can derive the Euler's formulas:

$-\infty$ $+\infty$

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

$$\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

The complex $\boxed{\bar{x} = a + jb}$ can also be written as:

$\equiv$

$$\boxed{\bar{x} = |x|e^{j\varphi}}$$

- where $|x| = \sqrt{a^2 + b^2}$ and $\varphi = tan^{-1}\frac{b}{a}$,
- or also $a = |x|\,cos\,\varphi$; $b = |x|\,sin\,\varphi$

- by Euler's exponential, $|x|e^{j\varphi} = |x|\,cos\,\varphi + j|x|\,sin\,\varphi$

The conjugate of $\bar{x}$ is also represented as: $\bar{x}^* = |x|e^{-j\varphi}$

Fourier series with exponential base

The Fourier series is usually expressed in the complex numbers domain, to represent signals $s(t)$: $\mathbb{R} \to \mathbb{C}$

- this also includes signals $\mathbb{R} \to \mathbb{R}$
- it is a generalization of the Fourier series for real functions
- by means of the complex exponential
- it is generally used in this form

The base of Fourier is the set of functions:

$$\boxed{e^{j2\pi nFt} \quad \forall n \in \mathbb{Z}}$$

where the complex exponential e^{x+jy} is:

$$e^{x+jy} = e^x e^{jy} = e^x(\cos y + j \sin y)$$

hence,

$$e^{j2\pi nFt} = \cos(2\pi nFt) + j \sin(2\pi nFt)$$

⇒ inverse of period

→ Fundamental freq $\frac{1}{T}$

period of signal

Fourier series with exponential base

$P_s = \lim_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |s(t)|^2 \, dt < \infty$

$\Leftrightarrow$ its $E < +\infty$

$E = 0$

Given a **periodic** signal $s(t) = s(t+T) \;\; \forall t$

- $s(t)$ is a power signal → energy = 0, finite signal
- the frequency $F = 1/T$ is the first (fundamental) harmonic
  - expressed in Hertz ($= \sec^{-1}$)

We can represent the signal $s(t)$ with the linear combination:

Freq.

→ infinite sum of sin, cos for base

$$\boxed{s(t) = \sum_{n=-\infty}^{\infty} S_n e^{j2\pi nFt}}$$

Where S_n is the series:

Depend of signal

$$S_n = \frac{1}{T} \int_0^T s(t) e^{-j2\pi nFt} dt$$

AVG over the period

Here just 1 coeff.

⇒ evidence the contribution of the FUNDAMENTAL ARMONIC

You may find how the complex notation is derived at https://web.mit.edu/6.02/www/f2006/handouts/Fourier.pdf

La rappresentazione di Eulero permette di esprimere FT

come una serie ∞

Notes

∞

un

coeff S_n

⇓

s(t) composed by an infinite # of periodic signal

- $s(t)$ is fully represented by S_n, that is, the knowledge of the series S_n is equivalent to the knowledge of $s(t)$
  - i.e. the Fourier series gives the same values of $s(t)$ where $s(t)$ is continuous but not in discontinuities (Dirichlet theorem)
- The periodic signal $s(t)$ can be thought as composed by an infinite number of periodic signals
- The functions in the Fourier base have frequency nF multiple of the fundamental harmonic $(F = 1/T)$
- The terms $S_n e^{j2\pi nFt}$ $(n > 1)$ are called the harmonic components of $s(t)$
  - Each with frequency nF

AVG VALUE of our signal

- The coefficient $S_0 = \frac{1}{T}\int_0^T s(t)dt$ is the "continuous" component
  - It is the average value of $s(t)$

Geometric interpretation of Fourier series

harmonics

$$s(t) = \sum_{n=-\infty}^{\infty} \boxed{S_n}\boxed{e^{j2\pi nFt}} = \sum_{n=-\infty}^{\infty} \boxed{S_n}\boxed{(\cos(2\pi nFt) + j\sin(2\pi nFt))}$$

amplitude of the harmonics

called

- For $n = 0$ we have the continuous signal S_0, constant component, is Direct Current component DC
- For $|n| = 1$:
  - we have the fundamental frequency $f_0 = F$ with period $T = \frac{1}{F}$
  - S_1 and S_{-1} are the amplitudes of the fundamental frequency (armonic, multiples of fundamental frequency
- For $|n| > 1$:
  - we have the harmonics of frequency $f_n = nF$ with period T/n
  - S_n $(|n| > 1)$ are the amplitude of the harmonics

} Armonics, multiples of fundamental

FOR PERIODIC SIGNAL:

EASY
- To extend this to signals with limited support: => NON-PERIODIC SIGNALS, defined only in an interval [a, b] => $\emptyset$ outside

Fourier series of signals with limited support

You can obtain the Fourier series also for an aperiodic signal, with limited support (null out of an interval $[a, b]$)

defined only in an interval, $\emptyset$ otherwise

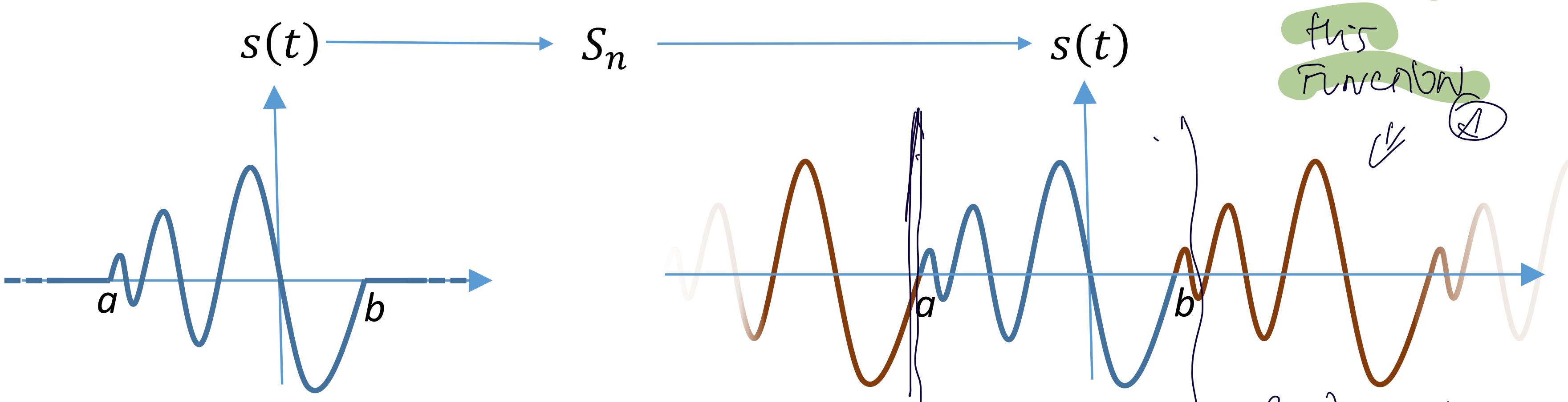
(→ continuous with finite # of discontinuities

However, the series assumes the signal is periodic anyway (takes as period the entire duration of the signal)

sinusoid

The consequence is that, if you revert (from S_n to $s(t)$ again), you obtain the periodic extension of the signal !!!

=> possible compute FS for those signals, can compute THE COEFFICIENT, RECONSTRUCT SIGNAL FROM THE COEFFICIENT => compute $\sum_{n=-\infty}^{+\infty} S_n e^{j2\pi n F t}$ and drawn this function ①



②

- Cut I'm summing periodic f(x), with ∞ duration, reconstructing the signal in interval [a, b]

periodic, with period b − a

Example

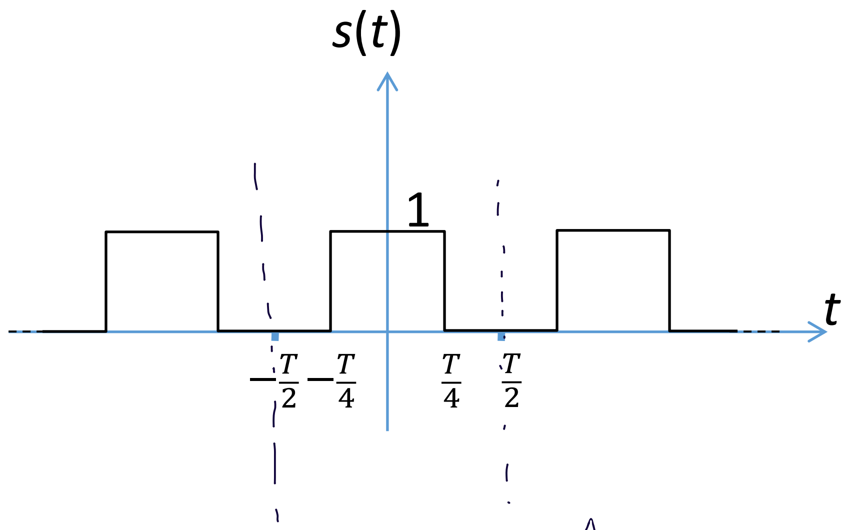
periodic

Consider the signal with period T and frequency $F = \frac{1}{T}$:

$$s(t) = \begin{cases} 1 & if \;\; |t| \leq \frac{T}{4} \\ 0 & if \;\; \frac{T}{4} < |t| \leq \frac{T}{2} \end{cases}$$

$T : [-\frac{T}{2}, \frac{T}{2}]$

$\Rightarrow$ period $2\frac{T}{2} = T$

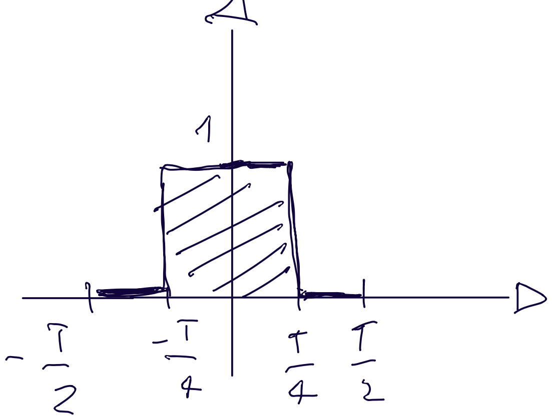


The coefficients of the Fourier series are:

$$S_n = \frac{1}{T}\int_0^T s(t)e^{-j2\pi nFt}dt = \frac{1}{T}\int_{-T/4}^{T/4} e^{-j2\pi nFt}dt =$$

$$= \begin{cases} \frac{1}{2} & if \; n = 0 \\ \frac{\sin(n\pi/2)}{n\pi} & if \; n \neq 0 \end{cases} = \begin{cases} \frac{1}{2} & if \; n = 0 \\ 0 & if \; n \; even \\ -\frac{(-1)^k}{(2k-1)\pi} & if \; n = 2k-1 \end{cases}$$

(pari) (odd)



Example

Hence:

$$s(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{k=\infty} \left(-\frac{(-1)^k}{(2k-1)\pi} \left(\cos(2\pi(2k-1)Ft) + j\sin(2\pi(2k-1)Ft) \right) \right)$$

recalling that $\sin a = -\sin -a$, we have that:

Coeff. are real (just a case)

$$\sum_{k=1}^{k=\infty} j\sin(2\pi(2k-1)Ft) = -\sum_{k=1}^{k=\infty} j\sin(-2\pi(2k-1)Ft) =$$

$$= -\sum_{k=1}^{k=\infty} j\sin(2\pi(-2k+1)Ft) = -\sum_{k=-1}^{k=-\infty} j\sin(2\pi(2k-1)Ft)$$

In this example the complex terms are null

• RECONSTRUCTION: • Exact: if we can do an ∞ Σ

• APPROXIMATION: if we sum a finite # of components of original signal

Example

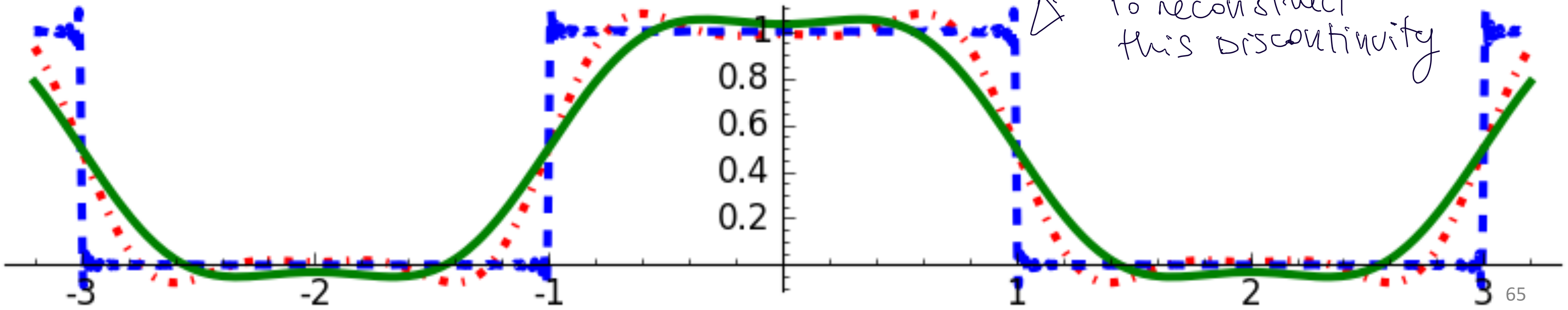
Hence,

$$s(t) = \frac{1}{2} + \sum_{k=-\infty}^{k=\infty} -\frac{(-1)^k}{(2k-1)\pi}\cos(2\pi(2k-1)Ft)$$

letting $T = 4$ $(F = 1/4)$ we have:

Here we need more freq. To reconstruct this discontinuity

Green: $k \in [-1,1]$
Red: $k \in [-2,2]$
Blue: $k \in [-100,100]$



Spectrum of a signal

$$s(t) = \sum_{n=-\infty}^{\infty} \boxed{S_n} \boxed{e^{j2\pi nFt}} = \sum_{n=-\infty}^{\infty} \boxed{S_n} \boxed{(\cos(2\pi nFt) + j\sin(2\pi nFt))}$$

harmonics

amplitude of the harmonics

Mod

→ this formula gives a way to compute the sequence of coefficients S_n

the weight of each armonics

called SPECTRUM OF THE SIGNAL

Given a signal $s(t)$, by Fourier we have a mean to find the coefficients of the series S_n

On the other hand, given the coefficient S_n we can reconstruct the signal $s(t)$

HAVING TH multiples of each armonic ⇒ spectrum

The sequence of ordered coefficients S_n from $-\infty$ to ∞ is called the ***spectrum*** of $s(t)$

- the spectrum can be seen as a discrete signal

↳ defined for finite discrete set of signal (values)

⇒ so from periodic signals ⇒ obtain discrete spectrum

⇒ spectrum defined for DISCRETE set of frequencies

Example

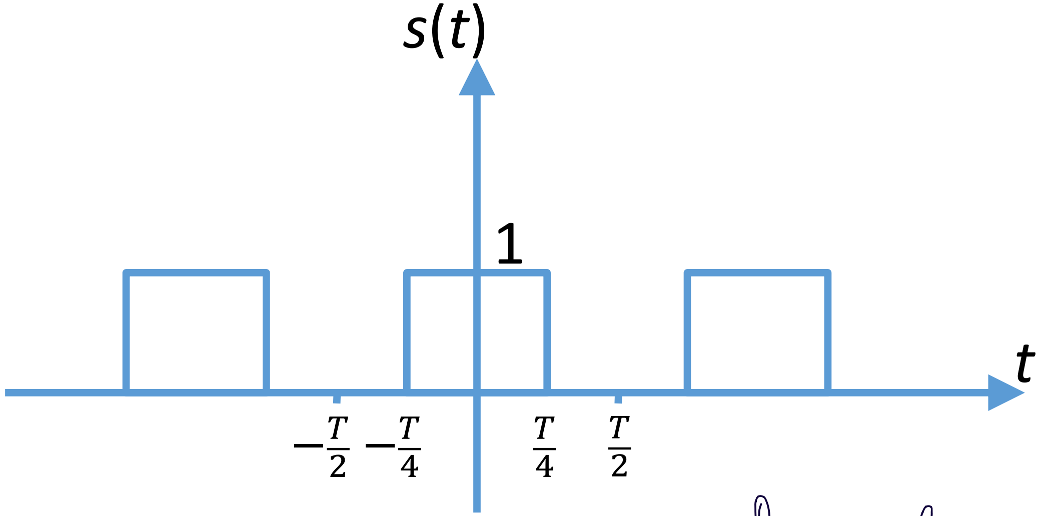
Reconsider the signal with period T and frequency $F = \frac{1}{T}$:

$$s(t) = \begin{cases} 1 & if \;\; |t| \leq \frac{T}{4} \\ 0 & if \;\; \frac{T}{4} < |t| \leq \frac{T}{2} \end{cases}$$

With:

$$S_n = \begin{cases} \frac{1}{2} & if \; n = 0 \\ 0 & if \; n \; even \\ -\frac{(-1)^k}{(2k-1)\pi} & if \; n = 2k-1 \end{cases}$$

1



Repr. of signal in terms of ~~time~~

periodic, continuos signal $\Leftarrow$

Example

①

① ② are equivalent ← if have all S_n ⇒ you reconstruct our periodic signal ⇐ here with the spectrum of signal

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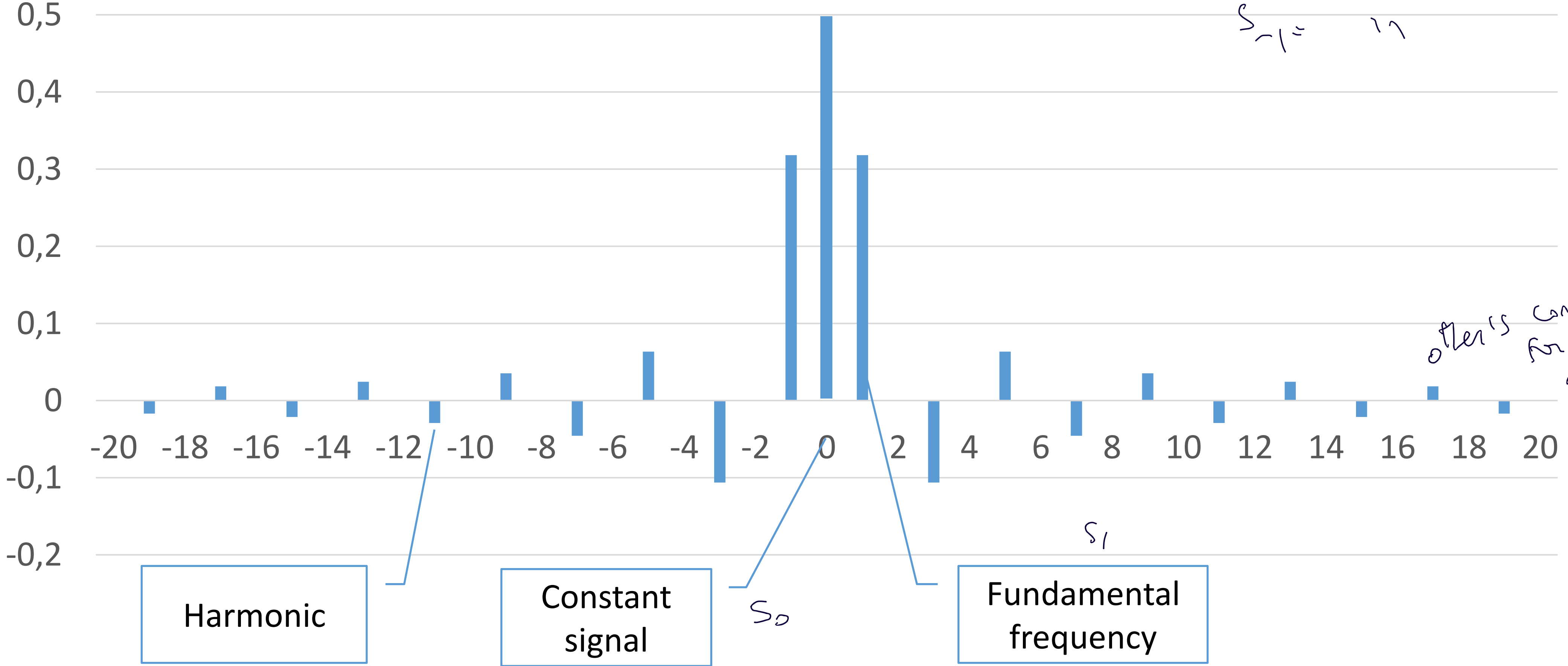
Spectrum of $s(t)$: value of coeff $S(n)$

S_0 = constant component

S_1 = ''

S_{-1} = ''



others components for the armonics

⇒ with ① ② to make the relation between:

signal in time & FT's spectrum

$s(t) \Longleftrightarrow S_n$

$\in \mathbb{R}$ — all coeff of FS $\in \mathbb{C}$

Fourier Transform

: is a function that goes from s(t) and provides this sequence S_n $s(t) \to S_n$

$S_n = F(s(t))$ (TRANSFORM)

$s(t) = F^{-1}(S_n)$ (INVERSE OF TRANSFORM)

⇓

from S_n (coeff) to signal in time s(t)

The transition from the continuous signal $s(t)$ to its spectrum, i.e. to its discrete signal S_n is called ***Fourier Transform*** (FT)

- It is denoted with

$$s(t) \underset{FT}{\Longleftrightarrow} S_n$$

$\in \mathbb{R}$ $\in \mathbb{C}$

from time D to spectrum D and viceversa

- Or even with:

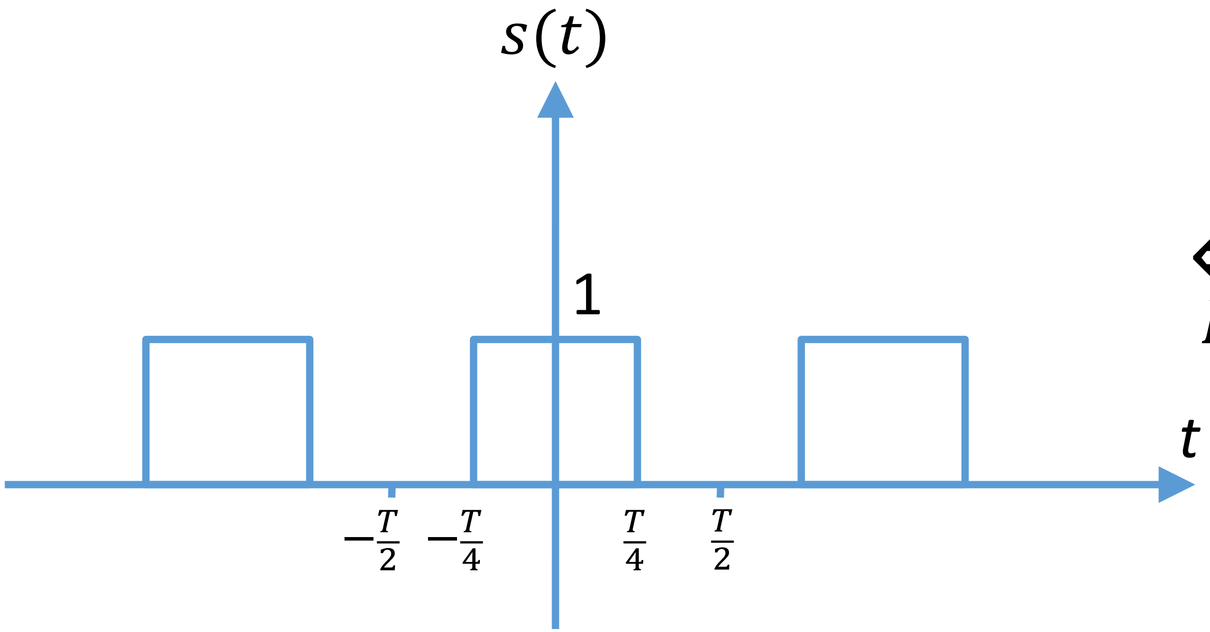
$$S_n = \mathcal{F}(s(t)) \qquad \text{(transform)}$$

$$s(t) = \mathcal{F}^{-1}(S_n) \qquad \text{(inverse transform)}$$

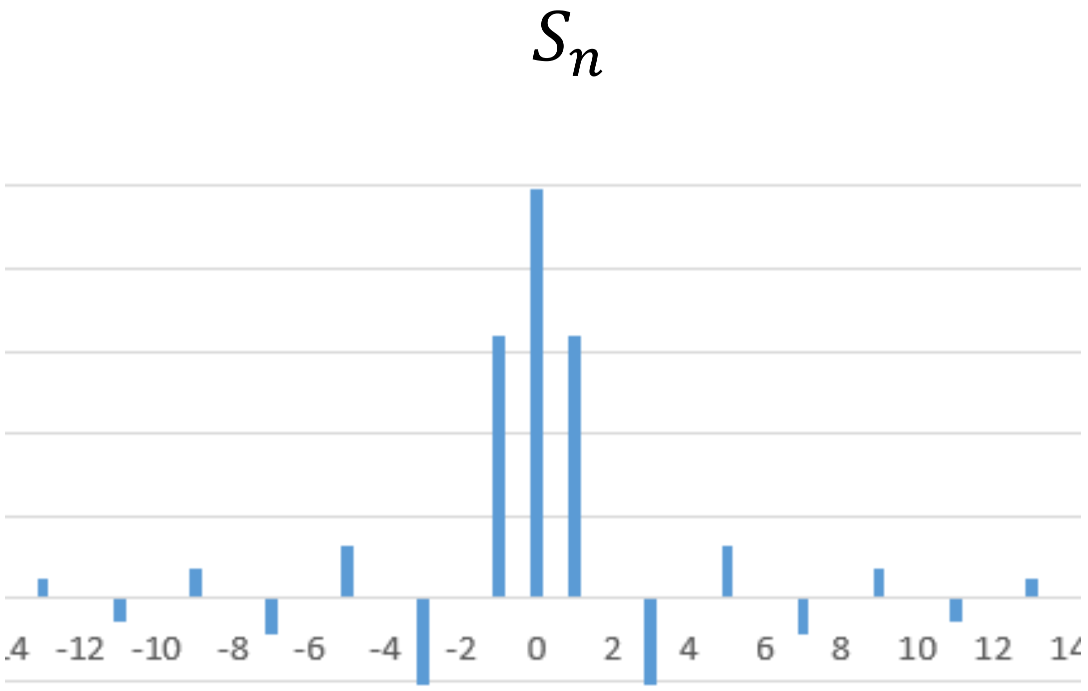
from coeff to signal in time ⇒ reconstruct signal

- The transform passes from the ***time domain*** of $s(t)$
- … to the ***frequency domain*** of S_n
  - S_n is a function of n, which identifies a harmonic and thus a frequency
  - Recall that S_n is the amplitude of the harmonic of frequency nF and period T/nF
- … more precisely it's a transform to the ***discrete frequency domain***
  - because we have a discrete set of frequencies

Fourier Transform



$$\underset{FT}{\Longleftrightarrow}$$



Analyze better

Spectrum of a signal

As discussed before, the coefficients S_n are, in general, complex numbers, even if s(t) is real => So can determine the amplitude of harmonics

- The previous example is a "lucky" case where both $s(t)$ and S_n are real

Hence it is not always possible to represent the spectrum with one 2D diagram…

… but we can represent the complex S_n by using the Euler's exponential:

Complex coeff $\rightarrow$ $$S_n = \boxed{|S_n|} \cdot \boxed{e^{j\theta_n}}$$

Since we can => determine amplitude

θ_n

Amplitude of harmonics

Phase of harmonics

The representation in frequency of $s(t)$ has hence two diagrams:

- Amplitude spectrum diagram
- Phase spectrum diagram

θ_n = phase of this coeff.

=> for a given harmonic, → n · FUNDAMENTAL HARMONICS

θ_n represent the phase

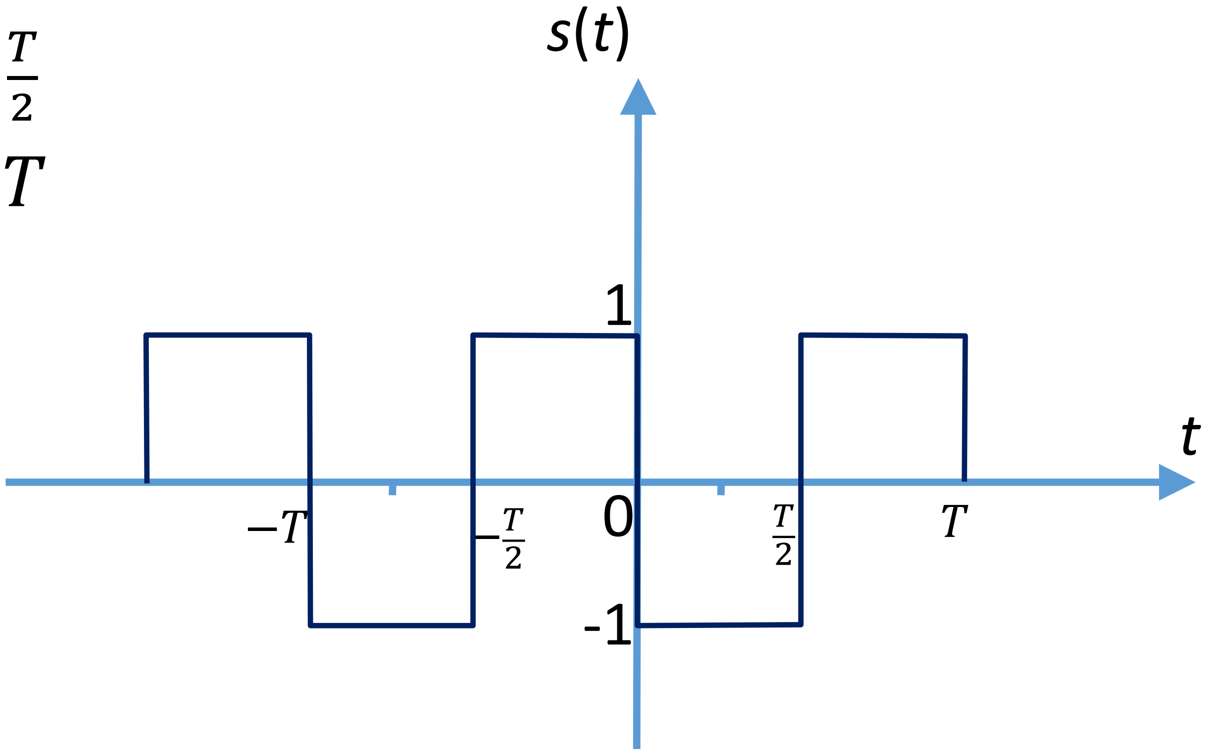
$f = \frac{1}{T}$

Example

in this ⇒ coeff will be complex numbers

Consider the signal with period T and frequency $F = \frac{1}{T}$:

$$s(t) = \begin{cases} -1 & if\ 0 < t \le \frac{T}{2} \\ 1 & if\ \frac{T}{2} < t \le T \end{cases}$$



Example

$$s(t) = \begin{cases} -1 & if\ 0 < t \le \frac{T}{2} \\ 1 & if\ \ \frac{T}{2} < t \le T \end{cases}$$

The coefficients of the Fourier series are:

$$S_n = \frac{1}{T}\int_0^T s(t)e^{-j2\pi nt/T}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-j2\pi nt/T}dt - \frac{1}{T}\int_0^{\frac{T}{2}} e^{-j2\pi nt/T}dt =$$

$$= \begin{cases} 0 & if\ n = 0 \\ j\dfrac{1-\cos n\pi}{n\pi} & if\ n \neq 0 \end{cases} = \begin{cases} 0 & if\ n\ even \\ j\dfrac{2}{n\pi} & if\ n\ odd \end{cases}$$

Example

$S_n = |S_n| \cdot e^{j\theta_n}$

Spectrum of harmonics

phase

Recall now that a complex $\bar{x} = a + jb$ can also be written as $\bar{x} = |x|e^{j\theta}$

- where $|x| = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\frac{b}{a}$,
- or also $a = |x|\, cos\, \theta$; $b = |x|\, sin\, \theta$

and, by the Euler's exponential, we have: $|x|e^{j\theta} = |x|\, cos\, \theta + j|x|\, sin\, \theta$

Hence, if n odd, $S_n = j\frac{2}{n\pi}$ (where $a = 0, b = \frac{2}{n\pi}, \theta_n = tan^{-1}\frac{b}{a}$) can be written as:

$$S_n = |S_n|e^{j\theta_n}$$

→ phase

amplitude

Example

$$S_n = |S_n| e^{j\theta_n}$$

PHASE

if $n > 0$, n odd: $a = 0, b > 0$, hence $\theta_n = tan^{-1}\frac{b}{a} = \frac{\pi}{2}$

if $n < 0$, n odd: $a = 0, b < 0$, hence $\theta_n = tan^{-1}\frac{b}{a} = -\frac{\pi}{2}$

then:

$$S_n = \begin{cases} 0 & if\ n\ even \\ \frac{2}{n\pi} e^{j\frac{\pi}{2}} & if\ n > 0, n\ odd \\ -\frac{2}{n\pi} e^{-j\frac{\pi}{2}} & if\ n < 0, n\ odd \end{cases}$$

Example

modulo

How can expen the ~~model~~

Amplitude:

$$|S_n| = \begin{cases} 0 & if\ n\ even \\ \dfrac{2}{n\pi} & if\ n > 0, n\ odd \\ -\dfrac{2}{n\pi} & if\ n < 0, n\ odd \end{cases}$$

Phase:

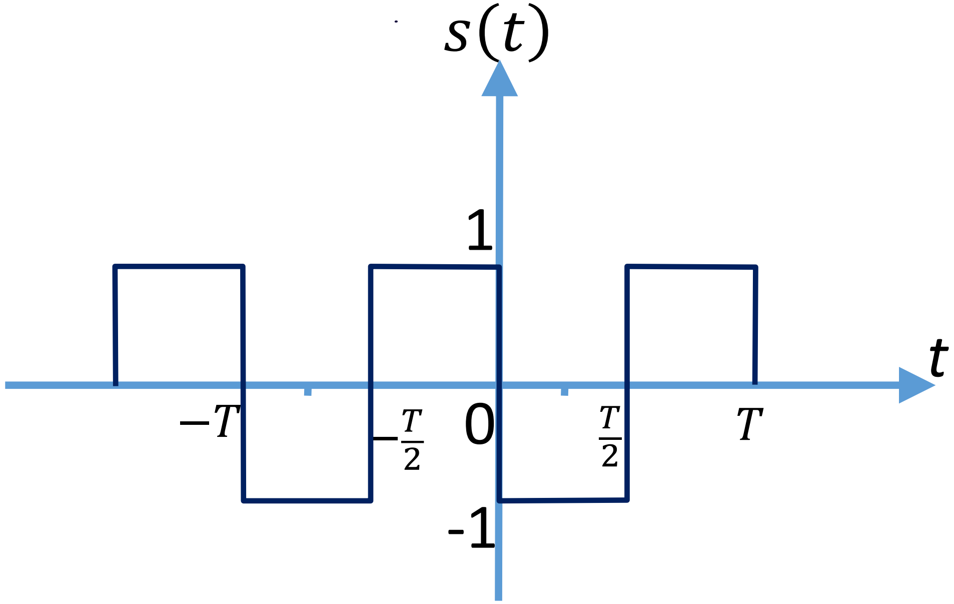
$$\theta_n = \begin{cases} 0 & if\ n\ even \\ \pi/2 & if\ n > 0, n\ odd \\ -\pi/2 & if\ n < 0, n\ odd \end{cases}$$

spectrum, frequency

IMPORTANT: Since spectrum is a complex number can represent it by 2 infos:

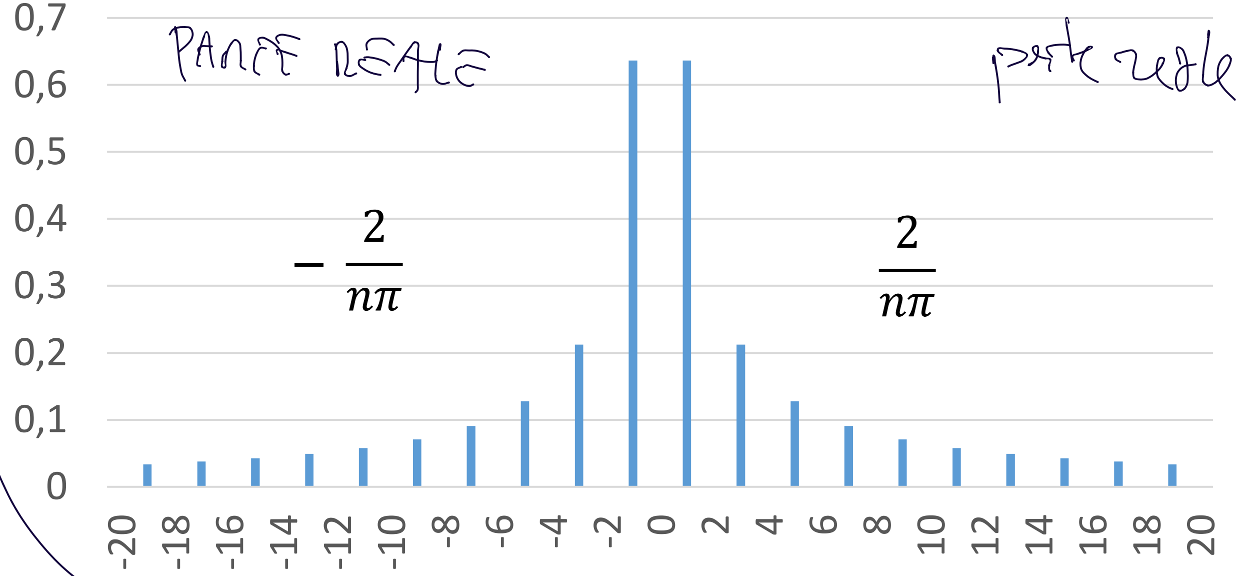
Example

- AMPLITUDE
- PHASE

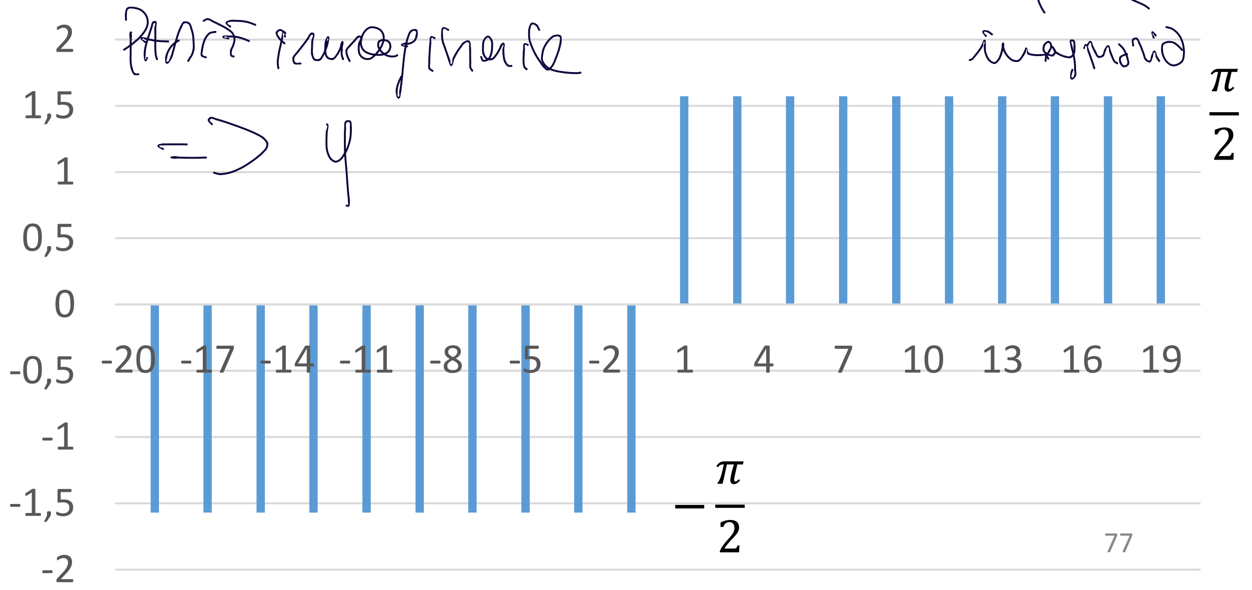


$$S_n = \begin{cases} 0 & if\ n\ even \\ \frac{2}{n\pi} e^{j\frac{\pi}{2}} & if\ n > 0, n\ odd \\ -\frac{2}{n\pi} e^{-j\frac{\pi}{2}} & if\ n < 0, n\ odd \end{cases}$$

Amplitude spectrum ~~for the freq.~~



Phase of the spectrum



Fourier Transform for non-periodic signals

=> apply FT to periodic signals with period T, with T -> ∞.
(FT before was computed on multiples of fundamental frequency -> 1/T)

=> goal: consider non-periodic signals: since we want to derive a non-periodic => increase value of T

defined in period [a,b]

discrete series for

think to a periodic signal with period T -> ∞ (↑T, => signal become less periodic limited time frame)

The transform so far applies only to periodic signals

- It can also be applied to **finite signals (in a time frame of lenght T)** FT
- In this case however the counter-transform returns a periodic signal with period T

how signal of period T, increase T => treat non-periodic => periodic

non periodic by T -> ∞ => signal becomes less periodic

What happens if $T \to \infty$?

Fundamental series is decreasing

- Ideally, the Fourier Transform interprets the signal with a fundamental frequency $F = \frac{1}{T} \to 0$ ...
- ... the harmonics tend become infinitely close to each other...
- ... and the periodicity progressively disappears

less, less periodic => ↓ frequencies becomes close to each other =>

from periodic: discrete set freq. to non-p => continuous set of freq

~~Like~~

Continuous Fourier Transform

A non-periodic signal $s(t)$ can be expressed as:

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$$

where f is the "continuous" frequency, ranging in $[-\infty, \infty]$, and $S(f)$ is the amplitude of the frequency f, given by:

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$$

- In this case, the spectrum $S(f)$ is a continuous signal

The passage from the non-periodic signal $s(t)$ to its spectrum, i.e. to $S(f)$ is called the ***Continuous Fourier Transform*** (CFT)

- It is denoted with

$$s(t) \underset{CFT}{\Longleftrightarrow} S(f)$$

- Or also with:

$$S(f) = \mathcal{F}_c(s(t)) \qquad \text{(transform)}$$

$$s(t) = \mathcal{F}_c^{-1}(S(f)) \qquad \text{(inverse transform)}$$

$F = \frac{1}{T}$ $T \to \infty$

=> $F = 0, \ldots$

=> fundamental F decreases

=> signal goes less, less periodic until is non-periodic

(decreasing the freq $\frac{1}{T}, \frac{1}{nT}, \frac{1}{2T} \ldots$

=> Frequencies comes close to each other

continuos frequencies

armonics decrease continuosly

So from discrete set of freq (periodic signal) to non-periodic signal => continuos set of frequencies

Relation with slide 61

5:55

S(f) is amplitude of frequency F

=> So $\int$ not a Σ

F is continuous now

spectrum (Fourier transf)

amplitude of frequency F

=> continuos domain of freq

=> continuos in time & in freq

in domain of frequencies

in domain of freq

TIME DOMAIN

transform

Frequencies domain

NOW WE ARE IN CONTINOUS DOMAIN OF FREQUENCIES

of Fourier transf

from freq dom to time

Example

⇒ rectangular signal

– in discrete ⇒ regular repeats

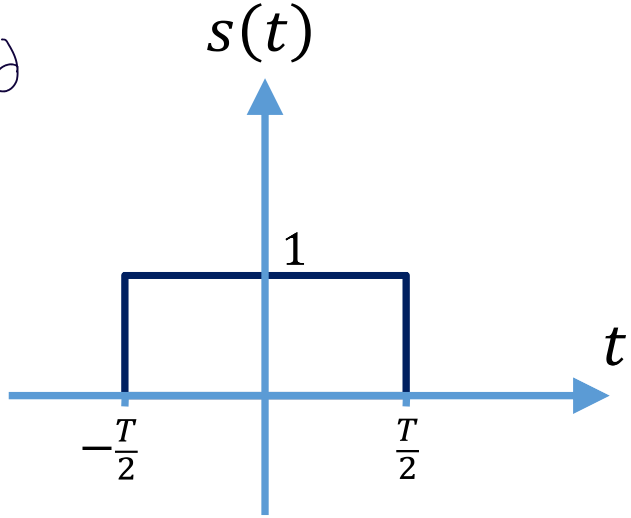
– here rect. signal

so in time interval

– ∅ fouri

Consider the pulse signal:

$$s(t) = \begin{cases} 1 & if \ \ |t| \leq \frac{T}{2} \\ 0 & otherwise \end{cases}$$



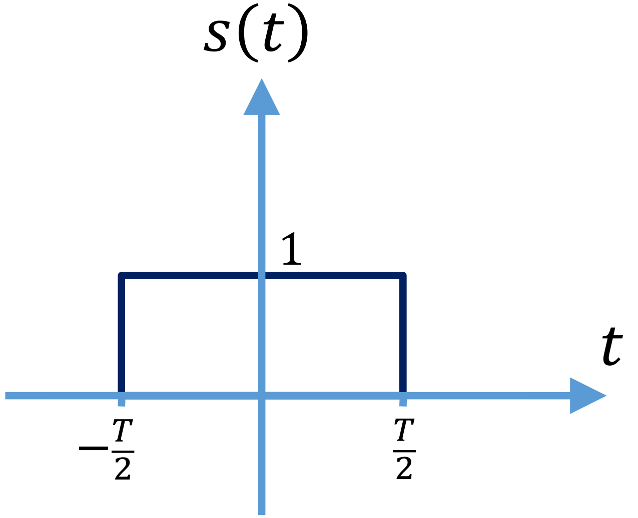
want to model this signal

The coefficients of the Fourier series are:

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt = \int_{-T/2}^{T/2} e^{-j2\pi f t} dt = \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-T/2}^{T/2} = \cdots$$

Example

$$s(t) = \begin{cases} 1 & if\ \ |t| \le \frac{T}{2} \\ 0 & otherwise \end{cases}$$



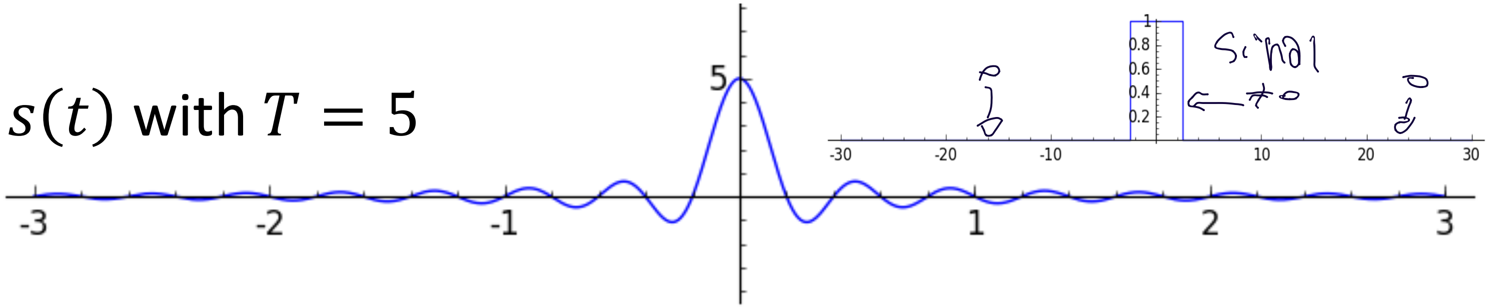
The coefficients of the Fourier series are:

$$S(f) = \frac{e^{\frac{j2\pi fT}{2}}}{j2\pi f} - \frac{e^{-\frac{j2\pi fT}{2}}}{j2\pi f} = \frac{\sin(\pi fT)}{\pi f}$$

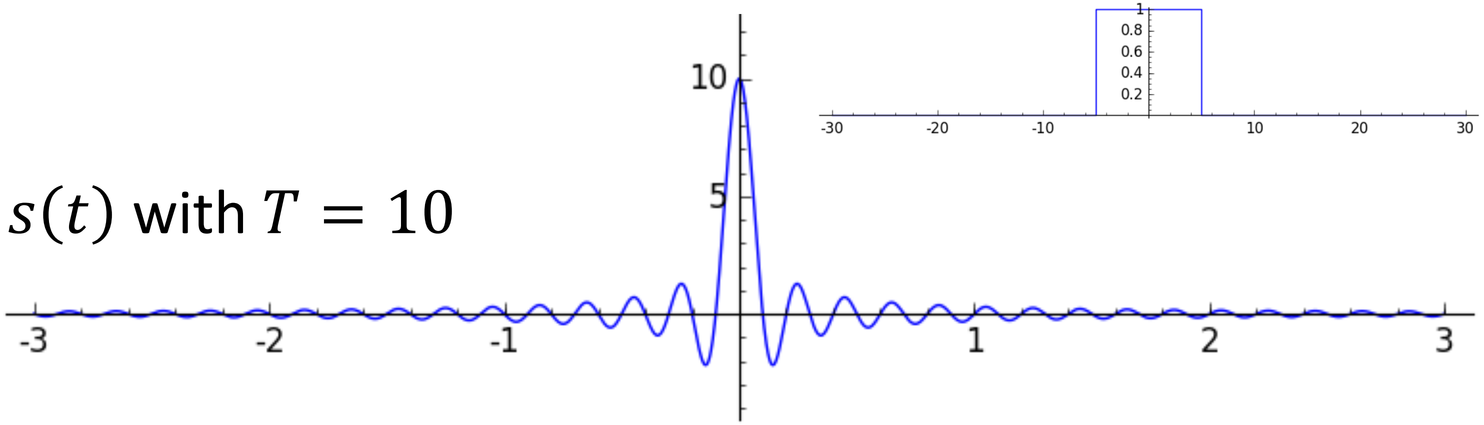
real #

Example

Spectrum of $s(t)$ with $T = 5$



Spectrum of $s(t)$ with $T = 10$

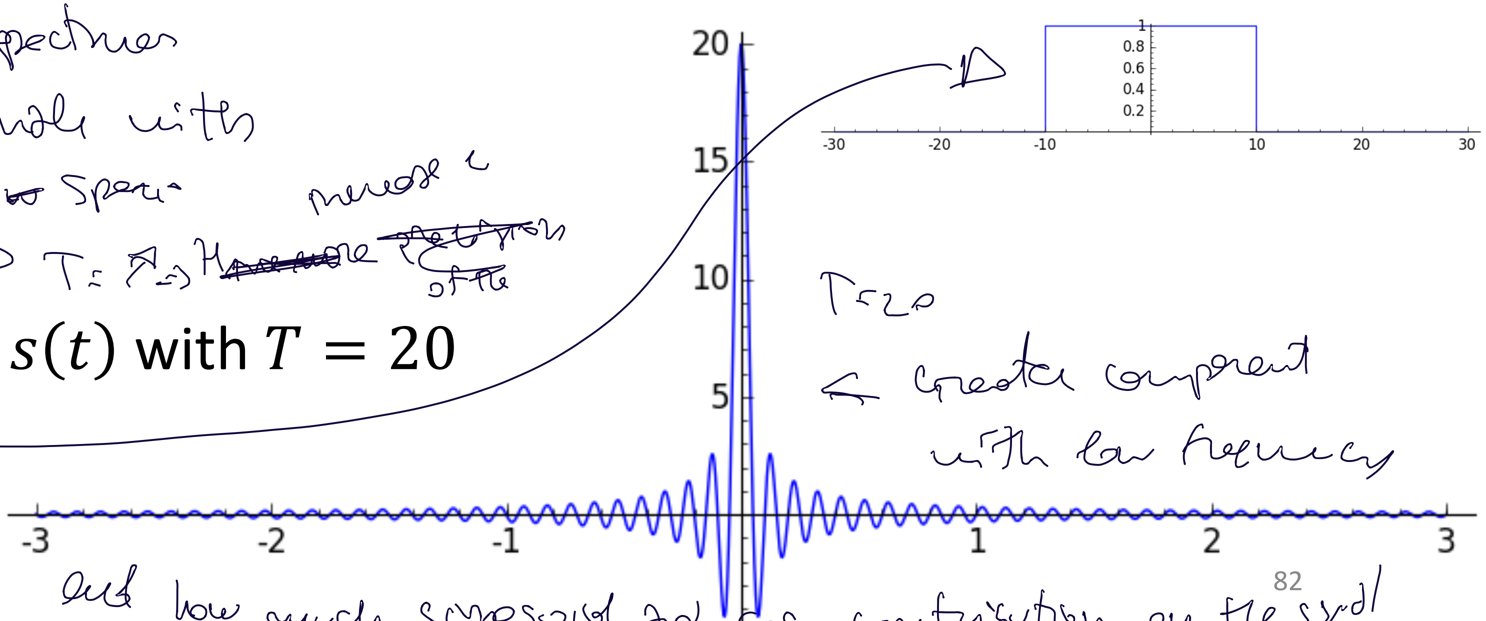


Note: in this particular case, where $S(f)$ is real, the phase spectrum is constant and not represented

if compute spectrum of rect. signals with increase ~~w~~ spread
increase
=> T ↗ => ~~...~~ ~~...~~ of the

↑T => greater component on low frequencies
=> contrib on lower freq
(=> getting more important
as signal becomes flatter
=> more contribution of lower freq.

Spectrum of $s(t)$ with $T = 20$



and how much sinusoid and cos contribution on the signal

Band of a signal

In general, the spectrum of a signal spans all frequencies in $(-\infty, +\infty)$

- while, if the signal is periodic, the spectrum contains only the harmonics of the main frequency (which is the fundamental harmonic)

If we filter some of the frequencies of the spectrum we obtain a signal limited in bandwidth

→ to select limited interval of frequencies (band) for that spectrum

Usually the filter can:

- exclude the high frequencies (low-pass filter)
- exclude the low frequencies (high-pass filter)
- Exclude low and high frequencies and keep the intermediate frequencies (bandpass filter)

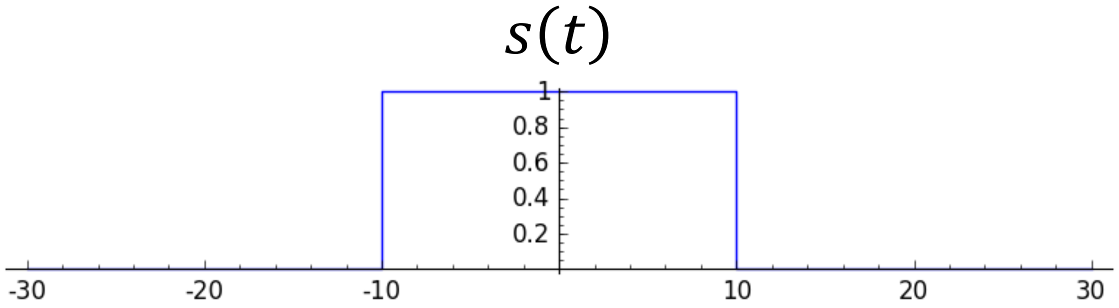
Example

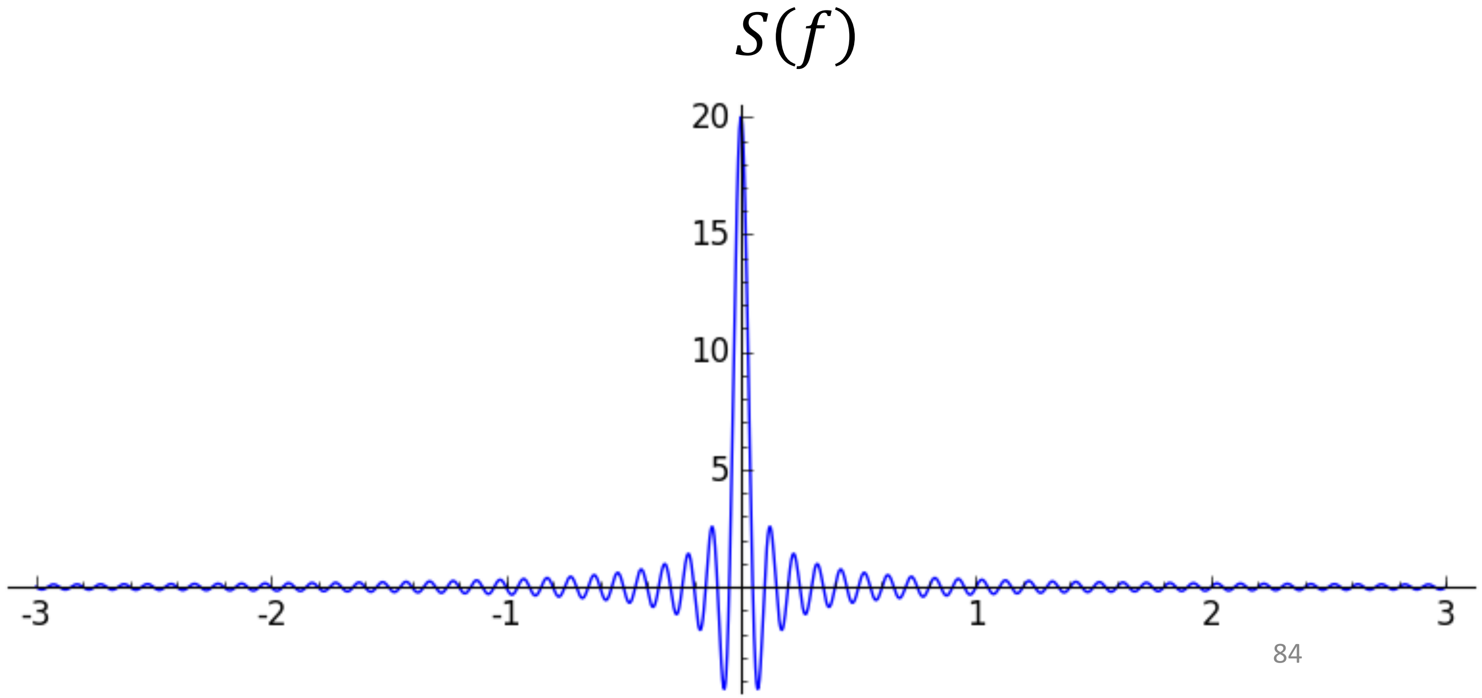
Consider again the pulse signal with $T = 20$:

$$s(t) = \begin{cases} 1 & if \ \ |t| \leq 10 \\ 0 & otherwise \end{cases}$$

and its spectrum:

$$S(f) = \frac{\sin(20 \cdot \pi \cdot f)}{\pi \cdot f}$$





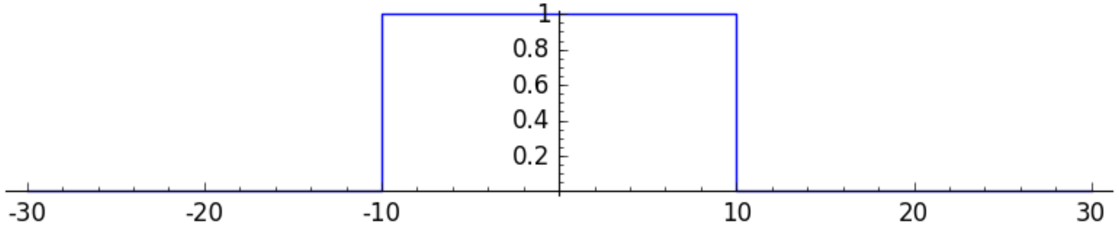
Example

If we apply a low-pass filter

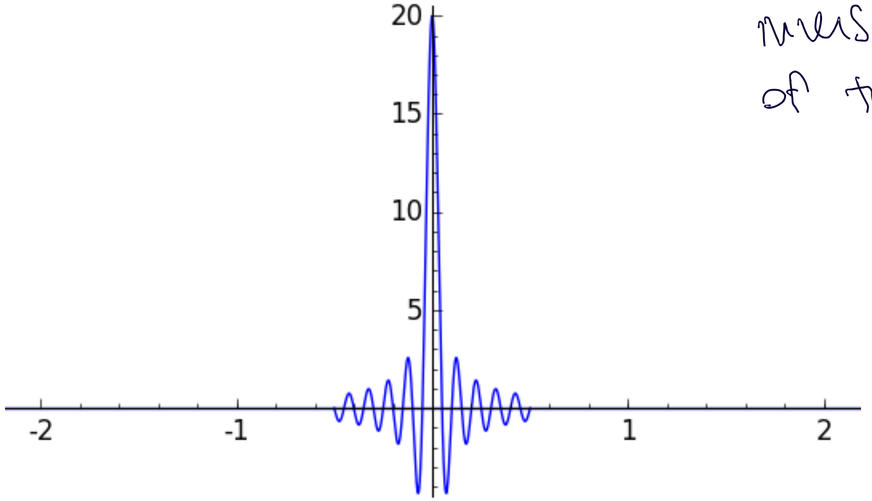
for frequencies below 0.5, we obtain:

cut freq ~~≤~~ 0.5

$s(t)$



$S(f)$

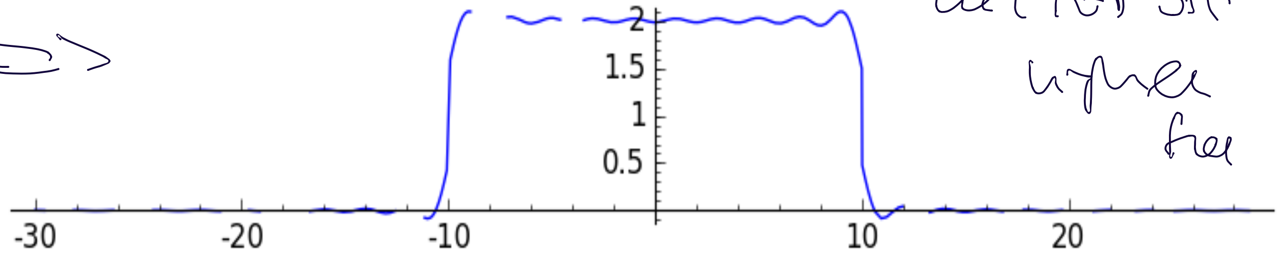


=> applying inverse of them fourier transform

Low => Approx of our signal

=>

Reconstruction of $s(t)$
with available frequencies low pass

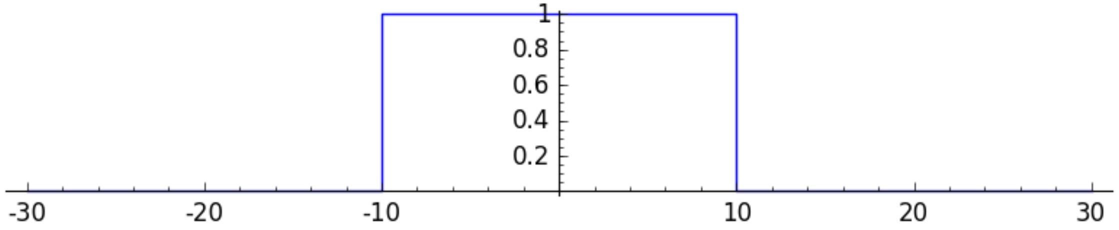


cutting off higher freq

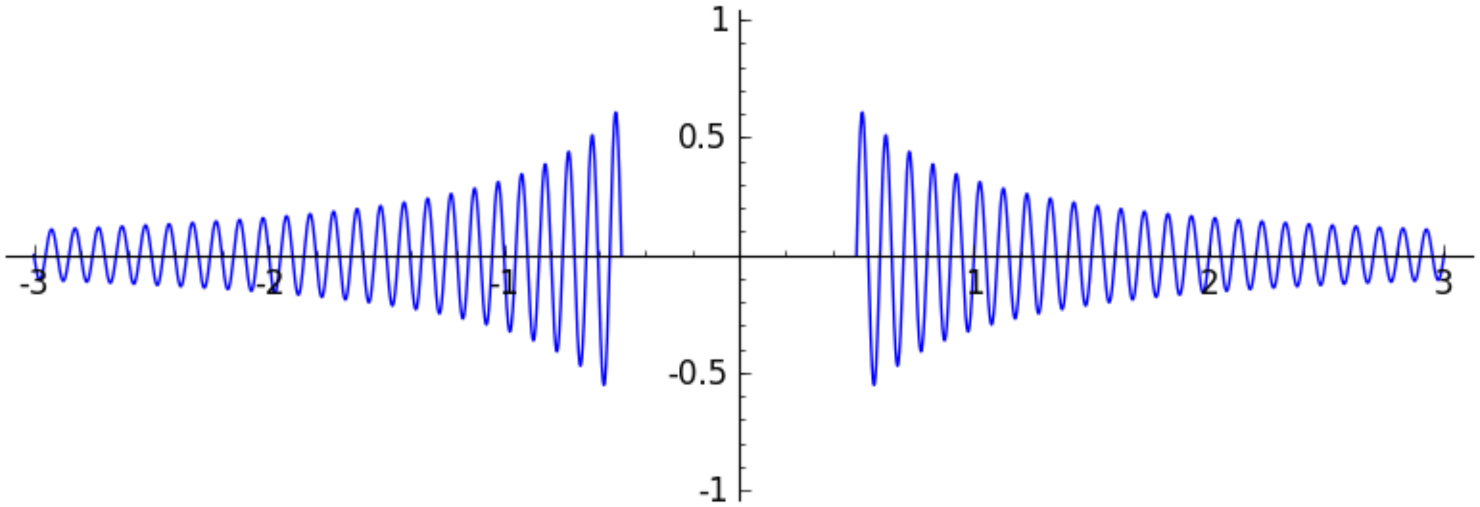
Example

If we apply a high-pass filter
for frequencies above 0.5, we obtain:

$s(t)$



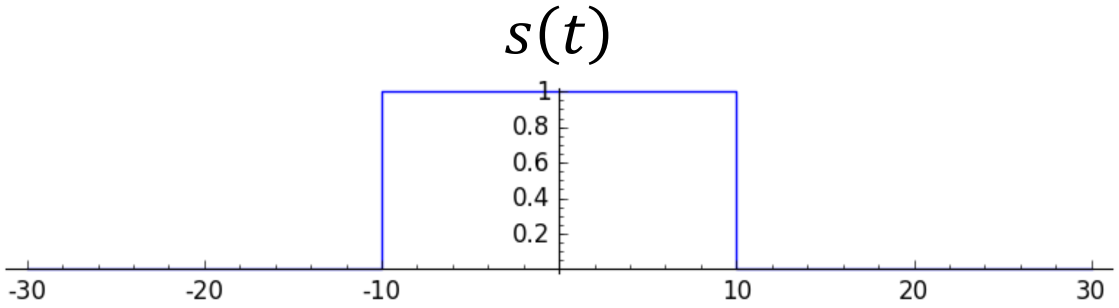
$S(f)$

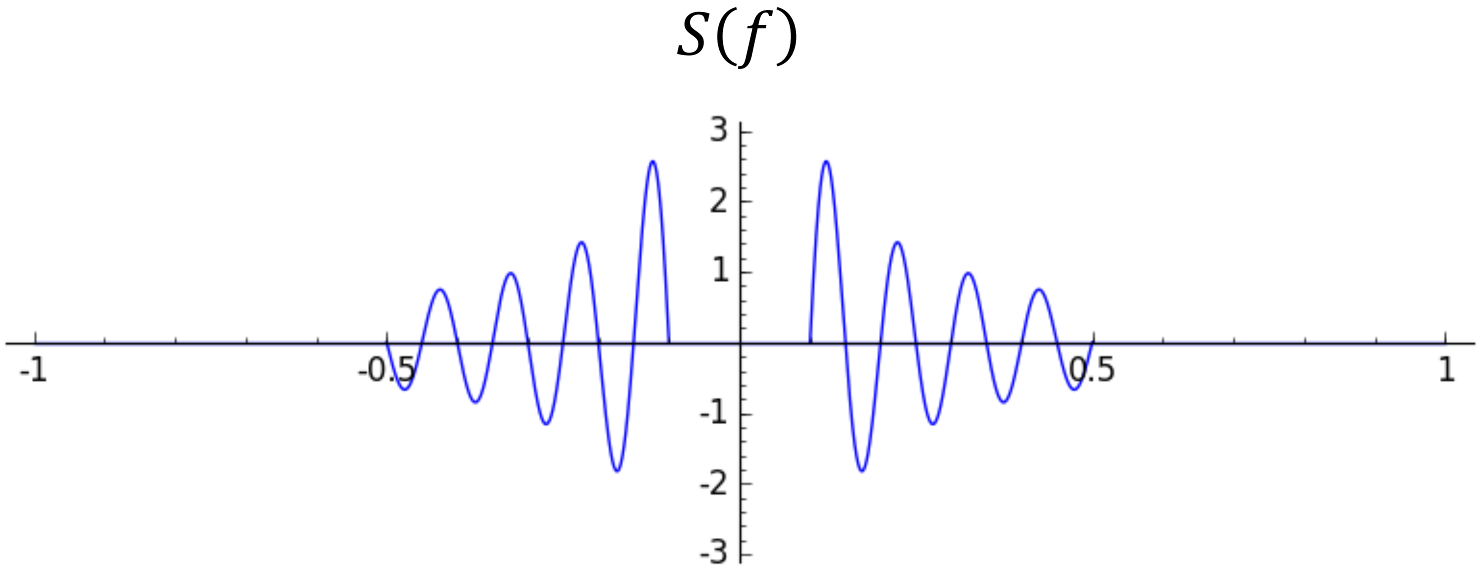


Filter
[illegible]
a given
band of
freqs
⇒ equalize
to [illegible]

Example

If we apply a band-pass filter
for frequencies in $[0.1, 0.5]$, we obtain:





Question

like

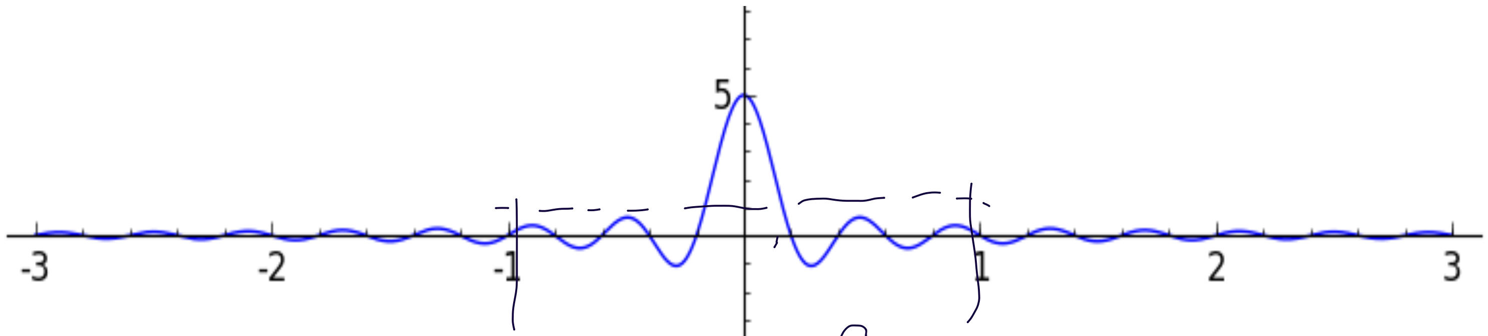
defined for ∞ set of freq ?

⇒ not limited

freq, to ∞

Considering the amplitude spectrum $S(f) = \frac{\sin(5\pi f)}{\pi f}$ of a signal $s(t)$, as shown in the figure. Explain whether this is limited in band and, in case, to which band. Then, depict the spectrum filtered with a low-pass filter with threshold frequency set at 1 Hz.

No



~~Cut~~ low pass

Now apply FT (shift from time domain to freq. domain) to DISCRETE TIME SIGNALS => like human speech
(since now, have seen continuos signals (periodic or non-periodic)
- continuos sample
- take some samples

From FT to Discrete Fourier Transform (DFT)

(to apply digital transmiss)

SITUA => START FROM a continuos signal (where t is $\mathbb{R}$), consider ONLY multiples of a fixed value T => our

(A)

- A type of FT for **discrete time signals**, i.e. signals known only at N instants separated by sample times T (i.e., a finite sequence of data)

domain becomes a discrete set (time)

(B) Signal is known at n-instant of time

separated by sample time T

human, sound, continuos sound → take samples of this to apply DIGITAL ADC

- Example
  - Discrete time signals are typically obtained from continuous signals with a domain restriction from R into $\mathbb{Z}(T)$.

discrete signals for discrete instances of time

T is real and consider only multiples of T

  - This operation, called sampling, is stated by the simple relationship

$$sc(nT) = s(nT),\ nT \in \mathbb{Z}(T)$$

where $s(t), t \in \mathbb{R}$, is the reference continuous signal and $sc(nT),\ nT \in \mathbb{Z}(T)$, is the discrete signal obtained by the sampling operation.

signal known at n-instant of times
known at sample time nT

domain not continuos but discrete set

Discrete Time Signal

- A discrete-time signal is a complex function of a discrete variable

from set of multiples of T 0T 1T 2T

$$f(t) : \mathbb{Z}(T) \longrightarrow \mathfrak{C}$$

Z'm

↳ from multiples of T to C

- where the domain $\mathbb{Z}(T)$ is the set of the multiples of T $\mathbb{Z}(T)$ = {...,−T, 0,T, 2T,...}, T> 0.
- The signal will usually be denoted in the forms

$$s(nT), nT \in \mathbb{Z}(T) \; or \; s(t), t \in \mathbb{Z}(T)$$

Discret Faktor T for on wich we know just a finite set of samples

il fralke $f(x)$
for finite # of sample

From Fourier Transform… operates on finite # of values of signal (X)

96

- Let $s(t)$, $t \in \mathbb{R}$, be the continuous signal which is the source of the data. Let N samples be denoted by $s(0), s(1), s(2), \ldots, s(N-1)$

- The FT of the original signal would be

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$$

and $S(f)$ is the amplitude of the frequency f, given by:

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$$

… to Discrete Fourier Transform 1/2

POTREMMO CONSIDERARE

VALUE of our sample

- We could regard each sample $s(t)$ as an impulse having area $s(t)$. Then, since the integrand exists only at the sample points:

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt = s(0)e^{-j2\pi f 0} + s(1)e^{-j2\pi f} +$$

constant component

other comp

$$s(2)e^{-j2\pi f 2} + \ldots .\ s(N-1)e^{-j2\pi f (N-1)}$$

$$\text{i.e. } S(f) = \sum_{0}^{N-1} s(k)\, e^{-j2\pi f k}, k \in \{0, \ldots . N-1\}$$

finite sum of sin, cos

… to Discrete Fourier Transform 2/2

- Countable
- Discrete set of freq
- Finite # of samples

FT

why ~~Disc~~ DFT considers only a finite # of frequencies X

- since there are only a finite number of input data points, the DFT treats the data as if it were periodic
  - $s(N)$ to $s(2N-1)$ is the same as $s(0)$ to $s(N-1)$

N: finite # of sample

value of sample

- Since the operation treats the data as if it were periodic, we evaluate the DFT equation for the fundamental frequency (1/~~N~~T) and its harmonics, i.e.

compute FT only for multiple of fundamental freq

$$f = 0, \frac{1}{\cancel{N}T}, \frac{2}{\cancel{N}T}, \dots, \frac{N-1}{\cancel{N}T}$$

the DFT computes a finite number of DFT coefficients (N)

IN 96 spectrum in 1 cycle

=> have finite Σ of armonics before ∞ Σ of armonics

⊗ Why DFT considers only a finite number of frequencies?

=> Since have only finite # of samples (N), DFT consider the signal as PERIODIC

=> $s(N)$ and $s(2N-1)$ is the same values of samples

and computes the FT only for multiples of the FONDAMENTAL FREQ

$$\Downarrow$$
$$\frac{1}{T}$$

Concepts

- discrete set of freq.
- finite # of samples
- in principle for FSeries, consider an $\sum_{\infty}^{+\infty}$ of armonics

$$f = 0, \frac{1}{T}, \frac{2}{T}, \dots, \frac{N-1}{T}$$

Here => finite Σ of armonics

=> considering only N frequencies

=> allowed by transformation simmetry   time $\Longleftrightarrow$ frequencies

- from periodic signal in time to freq we have discrete freq.
- from discrete samples in time to freq => periodic spectrum

- since you've periodic spectrum => with DFT I'm considering the samples of the spectrum in 1 cycle

WAY TO GO Time → freq dom

Discrete Fourier Transform

operates on finite number of values of the signal where N n0 ... N-1

led discrete FT is computed

- Given a finite length sequence composed of N values s_n with n= 0, 1, ..., N− 1, we can calculate the Discrete Fourier Transform

samples

B → # of samples of signal

• Summing up sn

$$S_f = \sum_{n=0}^{N-1} s_n e^{-j\frac{2\pi f}{N}n} \quad with\ f = 0, 1, \ldots, N-1$$

SN

Samples of signal

→ coeff not for ∞ # of freq BUT FINITE = # of samples in input

• computed for a finite # of frequencies = # of samples in input 0 ... N-1

also moves

- An anti-trasform operation is capable of returning temporal samples s_n

$$s_n = \frac{1}{N} \sum_{0}^{N-1} S_f \ e^{j2\pi f\frac{n}{N}}$$

=> obtain original sample sn

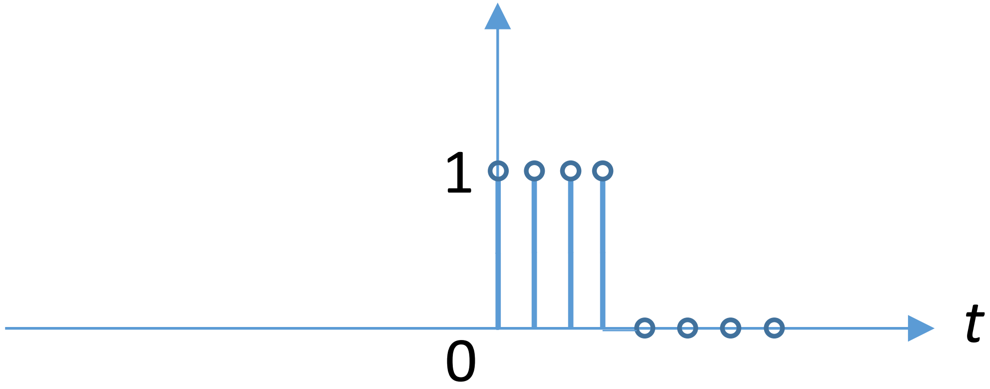
=> Start from discrete signal, finite # of sample, DFT compute contrib of some multiples in some freq => find multiples are finite

Fast Fourier Transform

- The computational complexity of the Discrete Fourier Transform is $O(N^2)$, N is the data set size
- Fast Fourier Transform (FFT): class of algorithms for the calculation of DFT and its inverse, characterized by the use of a much reduced number of operations, thus making the numerical processing of signals computationally feasible
  - $O(N \log_2 N)$

Example

• Discrete time signal   N = 8



4 samples value 1

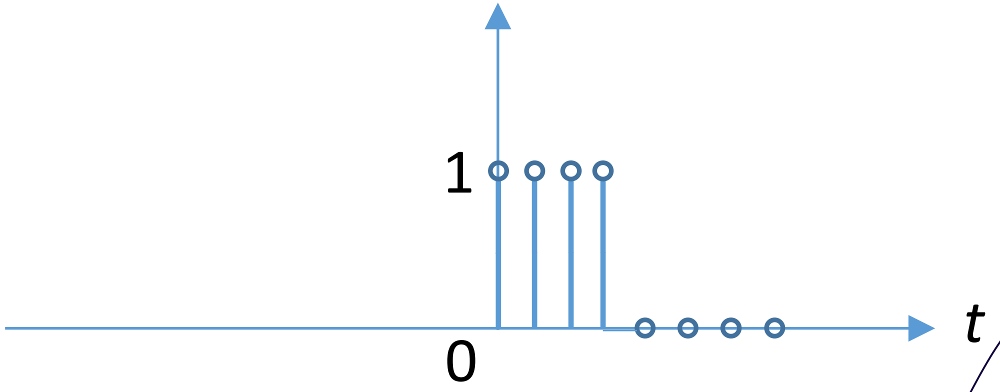
$$s(n) = \begin{cases} 1 \; if \;\; n = 0,1,2,3 \\ 0 \; if \; n = 4,5,6,7 \end{cases}$$

4 samples values 0

$$S_f = \sum_{n=0}^{N-1} s_n e^{-j\frac{2\pi f}{N}n} \; with \; f = 0, 1, \dots, N-1$$

Compute the DFT

Example



Component for null freq = 4

$$s(n) = \begin{cases} 1 \; if \;\; n = 0,1,2,3 \\ 0 \;\; if \; n = 4,5,6,7 \end{cases}$$

values of each freq

8 values

with

$$S_0 = \sum_{n=0}^{7} s(n) = 4$$

$$S_1 = s(0) + s(1)\, e^{-j\frac{2\pi}{8}} + s(2)e^{-j\frac{2\pi}{8}2} + s(3)e^{-j\frac{2\pi 1}{N}3} \;\; 1 - \; j\, 2.414$$

$$S_3 = s(0) + s(1)\, e^{-j\frac{2\pi 3}{8}} + s(2)e^{-j\frac{2\pi 3}{8}2} + s(3)e^{-j\frac{2\pi 3}{N}3} = 1 - \; j\, 0.414$$

$$S_f = \sum_{n=0}^{N-1} s_n e^{-j\frac{2\pi f}{N}n} \; with \; f = 0, 1, \ldots, N-1$$

$$S_5 = s(0) + s(1)\, e^{-j\frac{2\pi 5}{8}} + s(2)e^{-j\frac{2\pi 5}{8}2} + s(3)e^{-j\frac{2\pi 5}{N}3} = 1 + \; j\, 0.414$$

don't have a set of freq but vary f from 0 to 7

=> 8 values for our coefficient ~~S(f)~~ S_f

$$S_7 = s(0) + s(1)\, e^{-j\frac{2\pi 7}{8}} + s(2)e^{-j\frac{2\pi 7}{8}2} + s(3)e^{-j\frac{2\pi 7}{N}3} = 1 + \; j\, 2.414$$

complex

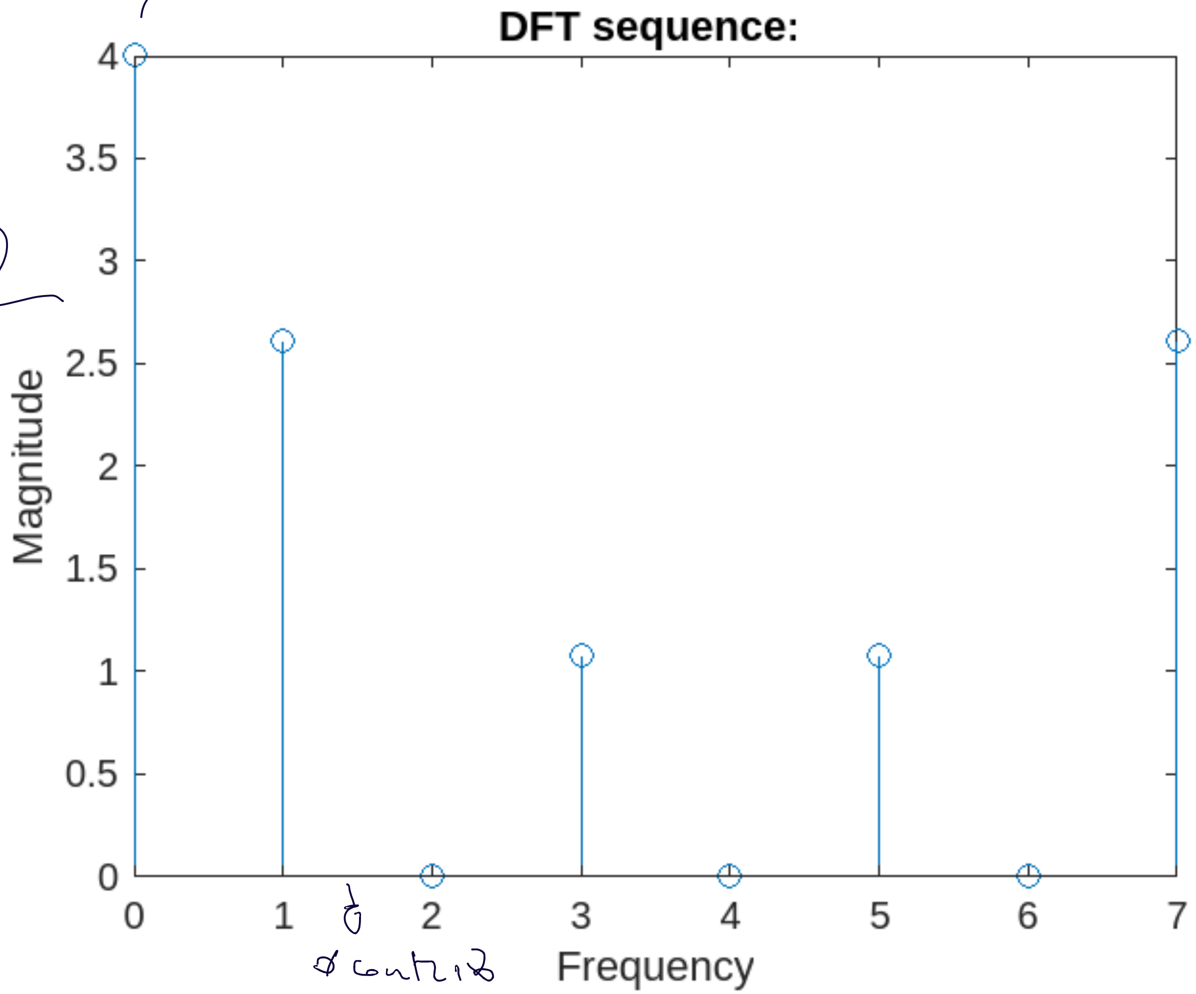
$$S_f = 0 \; for \; f = 2,4,6$$

Example

Amplitude of the DFT

always comes from null freq

$\in \mathbb{R}$



∅ contrib

for even freq