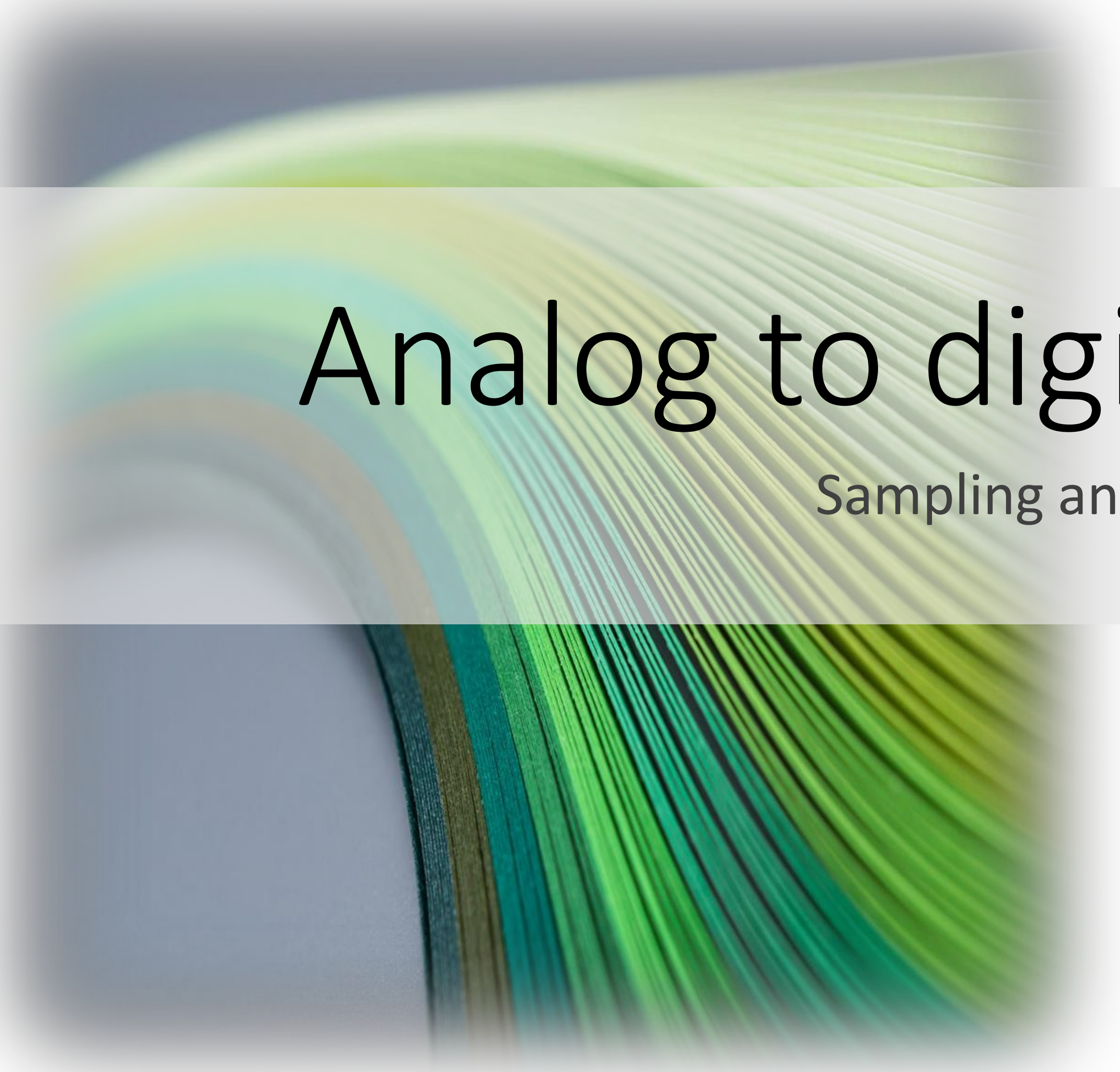
Analog to digital conversion

Sampling and quantization



take physica quantity (analog) → sample → have it signal sampled in time
(continuo) (in un instante di tempo)

Analog to digital conversion

Largely used in modern digital devices, music, videos, … are all converted in digital form

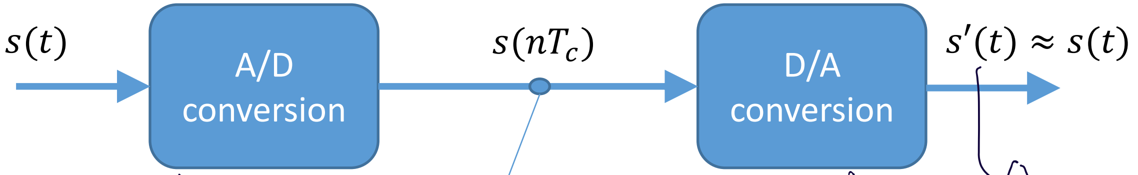
In digital form it is possible to process, manage and transmit information directly via software
without special purpose HW
- without special purpose hardware
- without altering the signal !!! no much modify ⇒ SAMPLING is ROBUST

A digital signal can be converted again into an analog signal (D/A conversion)
- It is important that the A/D process does not lose too much information of the original signal

- Analog to digital conversion transforms an analog signal $s(t)$ into a sequence of integer numbers
- This numeric sequence can be stored, processed and transmitted by a digital device
    - To this purpose it is, in turn, transformed into a binary stream.
- The elements of A/D conversion are ***sampling*** and ***quantization***

channel

ANALOG SIGNAL



get finite discreti valori

storage, transmission, duplication, …

to after, reconstruct the signal

Sampling and quantization

From cont. dom.
represent subset of
value of

↳ Done errors,
can't represent
all symbols

#bits ⇒ QUANT.
RESOLUTION

Sampling

↳ consider continuous signal
in interval of time

↓
RECONSTRUCTION

NON PERFECT ⇒ TRADEOFF

- Sampling a signal $s(t)$ means extracting from the signal a sequence of numeric values that the signal assumes at discrete time intervals
- Usually, sampling is performed at uniformely spaced time intervals, spaced of T_c i.e. the ***sampling period***
- Sampling returns a stream of real numbers…

• Se il periodo di sampling è Tc
⇒ prendo i valori del segnale a multipli
di Tc ⇒ nTc

Quantization

- Transforms the stream of real numbers produced by the sampling into a stream of numbers:
  - either floating point or integers
  - that can be managed by a digital device
- While sampling may preserve the original signal (under some hypothesis), quantization implies a loss of information
  - i.e. after quantization it is not possible to «perfectly» reconstruct the original signal

finite # of bits

↓ have to represent
this into our sys (floating point representation
or int ⇒ WANT INTEGER

Sampling: continuous signals

INPUT

OUTPUT →

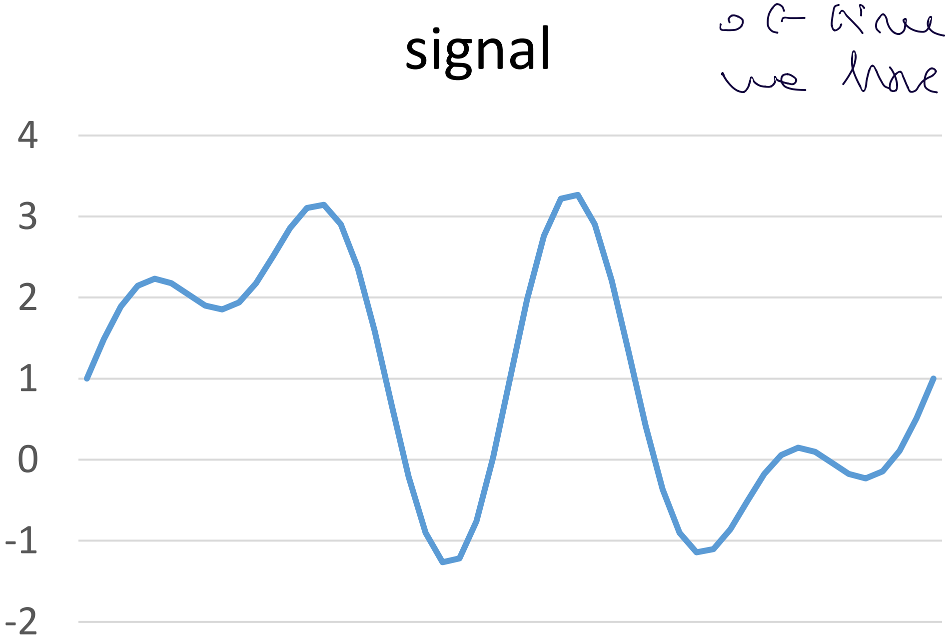
A discrete signal $g(nT), n \in \mathbb{Z}$ can be obtained with a uniform sampling of a continuous signal $s(t)$, where:

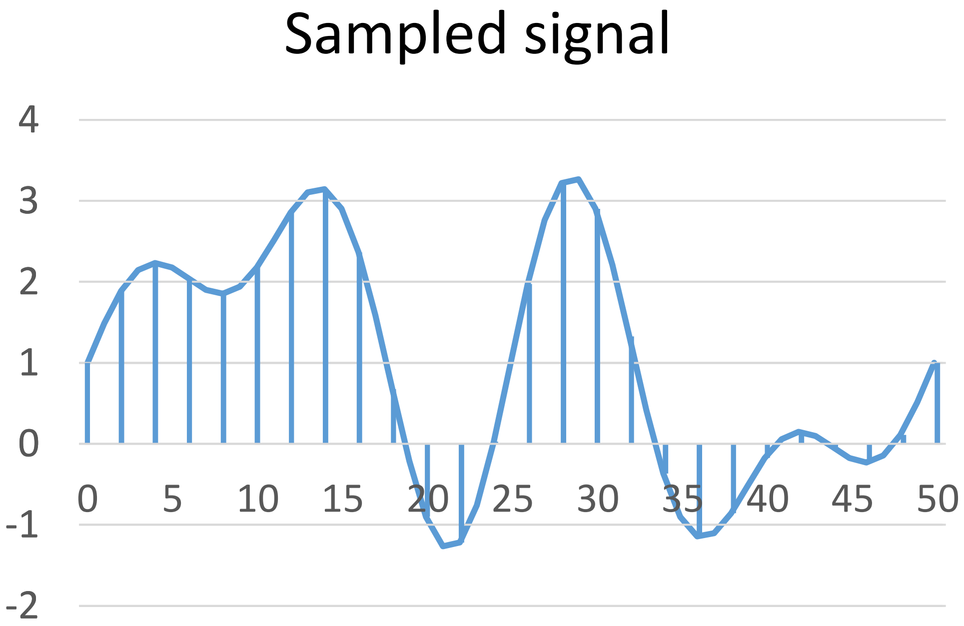
$$g(nT) = s(nT)$$

⇒ discrete instance of time when we have rep. of the signal

T is the sampling period

$\frac{1}{T}$ is the sampling frequency

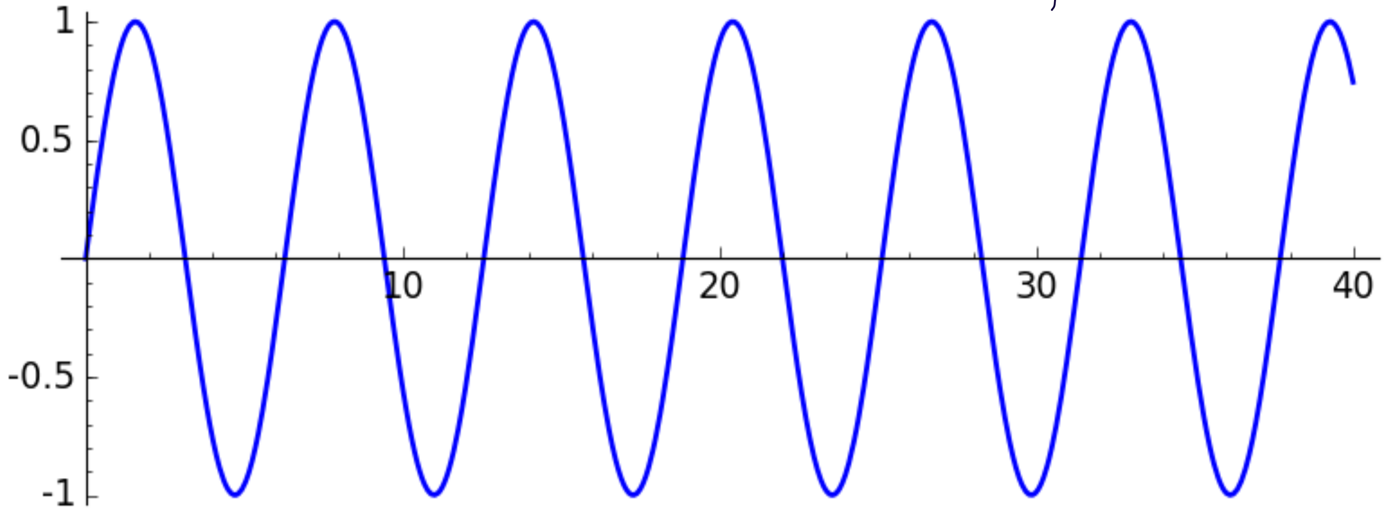




Example

$s(t) = \sin(t)$, frequency $f = \frac{1}{2\pi} \sim 0.159$

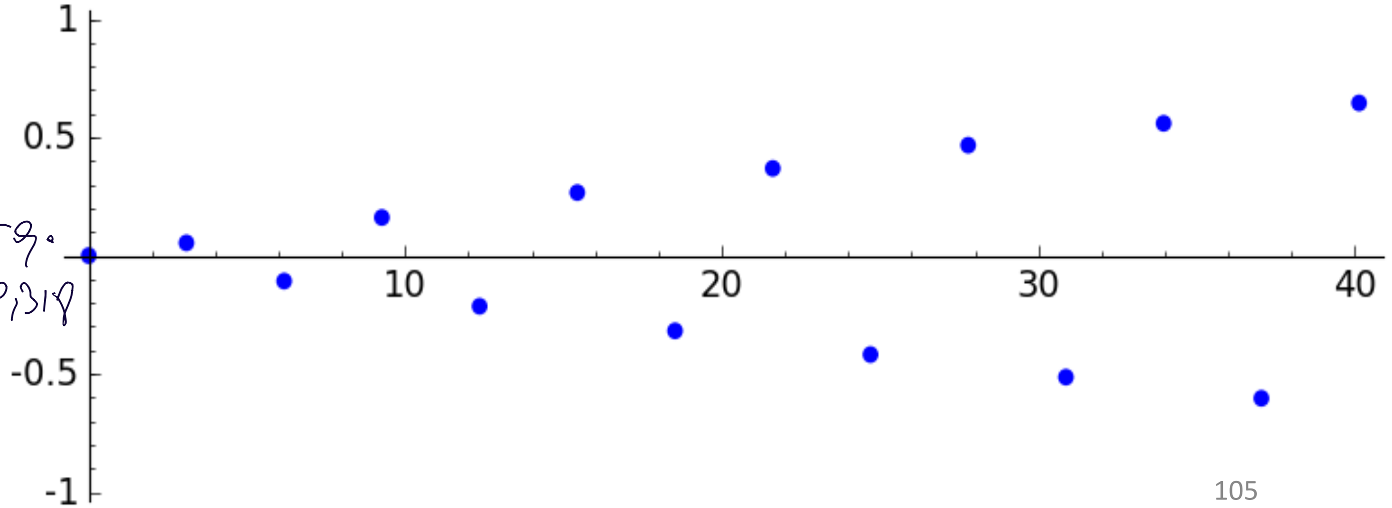
T



Represent a signal $s(t)$ by its samples, taken at discrete times

- a sample is a value of the signal at a precise time

$s(nT_c)$ with $T_c = 3.09$



The sampling is a signal itself, $s(nT_c)$, which takes a value for every multiple of the sampling period T_c

freq. of signal

$T_c = 3.09$ implies: $f_c \sim 0.324 > 2f$ $= 0,159$

$2 = 0,318$

faccio un sample

con un sampling period

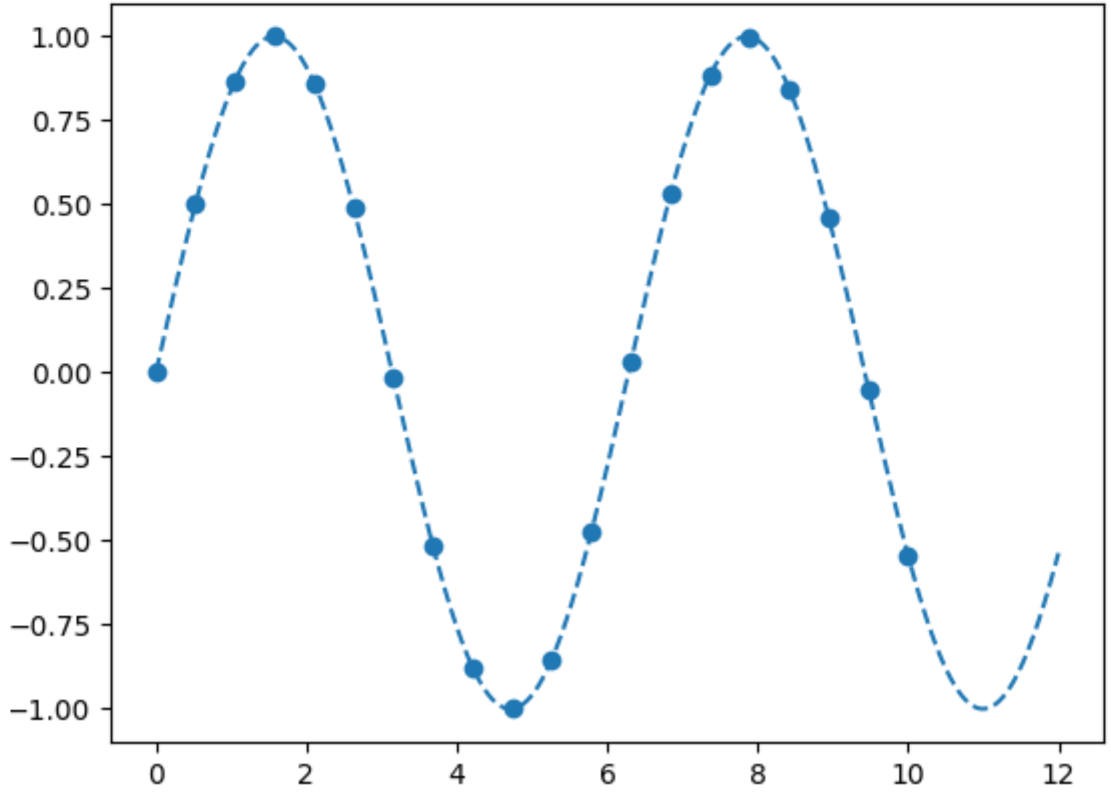
$f_c = \frac{1}{T_c} = \frac{1}{3,09} = 0,324$ ogni 3.09

Sampling and reconstruction

Sampl. THEOREM responds to

Is it possible to reconstruct a continuos signal s(t) from a sequence of samples?

⇒ Ricostruire un segnale continuo da un insieme di samples



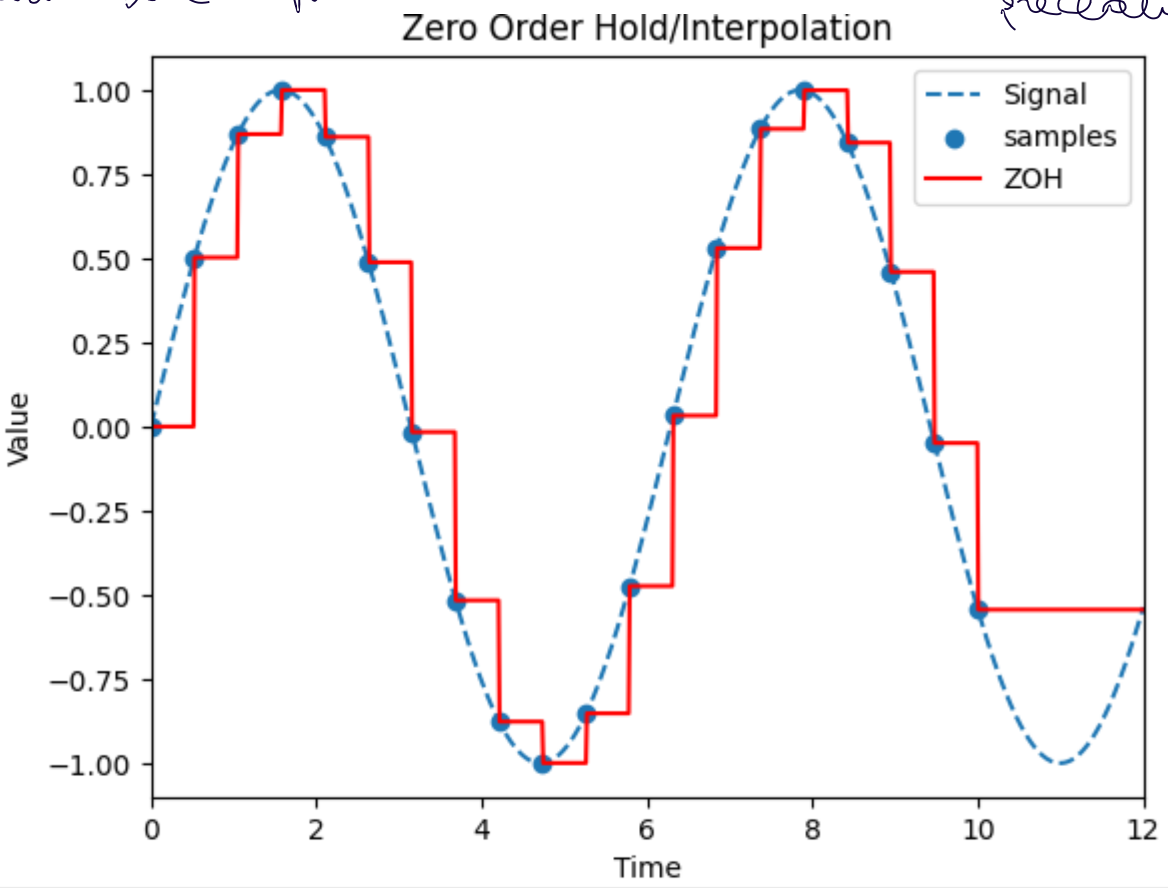
- QUANDO FACCIO SAMPLING ho solo i PUNTI, devo RICOSTRUIRE

⇒ INTERPOLATION

to reconstruct signal => we have infos about just the points
=> apply interp. to reconstruct the signal continuous

Interpolation - examples

otto il sample al tempo 0: decido i valori dei valori i valori della curva del segnale da 0 a T => prendo il valore del sample precedente (istante 0)



value from 0 To T

- Several methods for interpolation
- Several ways for reconstructing the signal:
  - Piecewise-constant signal, e.g.:
    - Zero-order hold, or
    - Nearest neighbour
  - Piece-wise linear, e.g.:
    - First-order hold reconstruction (constant signal)
  - Etc..

draw a line between 2 consecutive samples

=> I'm reconstructing signal approx

The Sampling Theorem

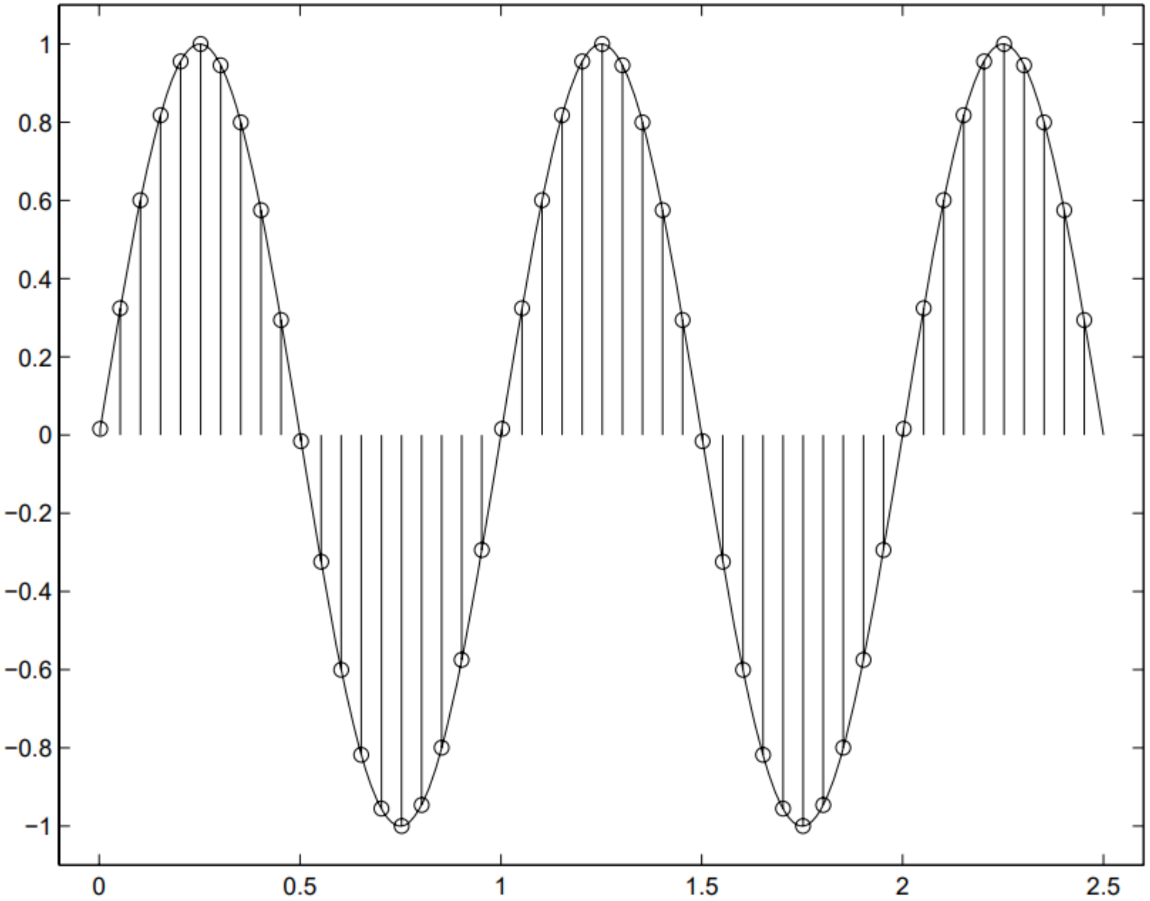
Se ho un segnale che cambia rapidamente
=> mi servono tanti samples,
Se cambia lentamente
=> pochi sample

Basic idea: a signal that changes rapidly will need to be sampled much faster than a signal that changes slowly

The sampling Theorem tells us the minimum sampling frequency required to reconstruct the signal… in ideal conditions…

sample so have enough info
to reconstruct the
signal

Example: Sampling a sinusoid at a high rate



period

$$x(t) = sin\frac{2\pi}{T}t$$

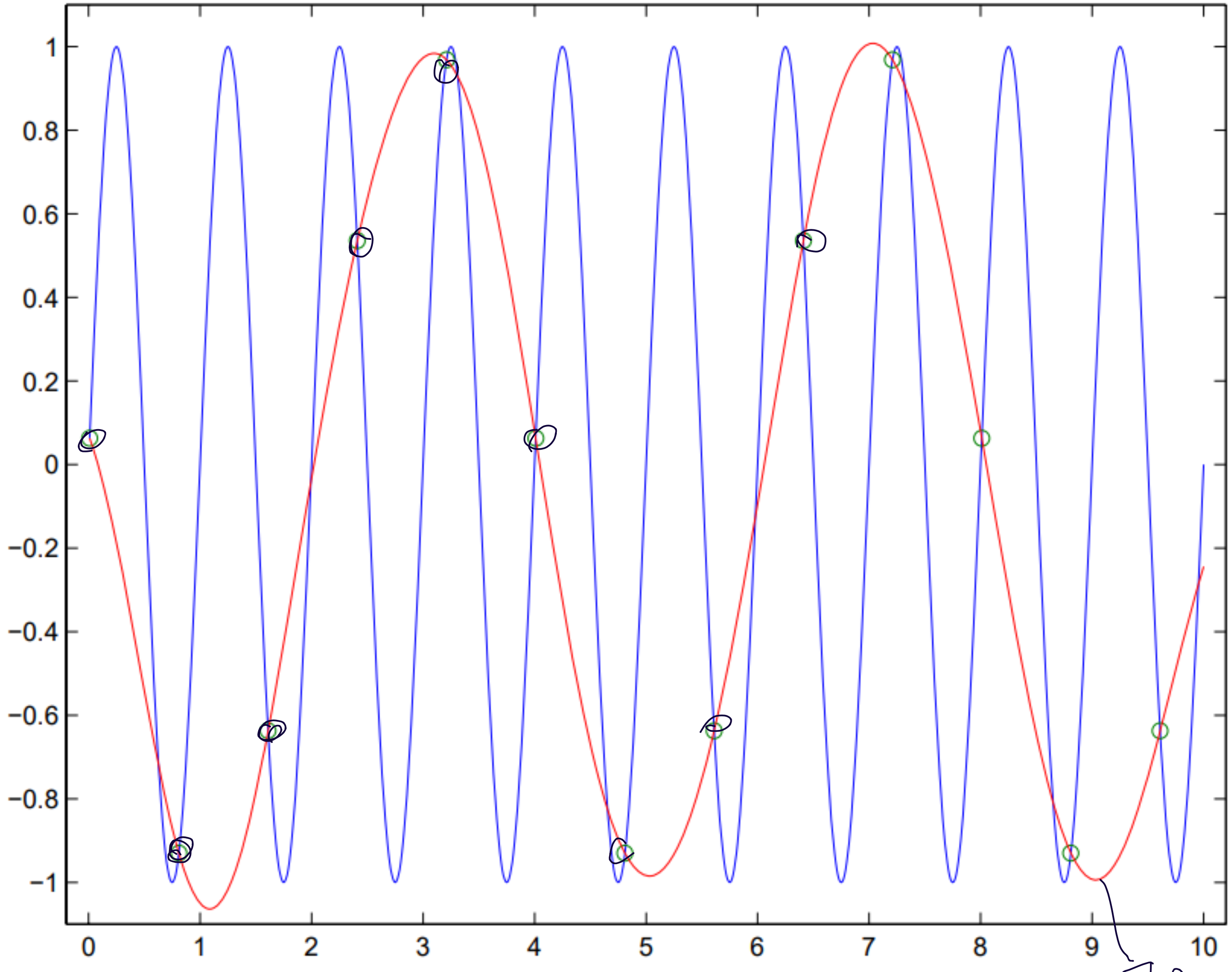
Samples in the example are taken at a frequency $f_c \gg \frac{1}{T}$

=> high # of samples

s(t) => blue ; sampling at low rate => frequenza di sampling f_s < 2 freq. del segnale s(t)

=> ALIASING

Example: Sampling a sinusoid at a low rate



If we sample (the blue signal) at a frequency which is too low

- we may not be able to reconstruct the signal
- We might reconstruct the lower frequency signal (red sinusoid)
- Aliasing:
  - different signals indistinguishable when sampled.
  - It depends on the sampling rate and frequency content of the signal

Luck: => can sample it at low rate

Freq sampl lower than freq of signal => don't represent good the signal low freq. => effect of sample at low freq => ALIASING

The Sampling Theorem

Consider a signal $s(t)$ with spectrum null at frequences above f_M

- … hence the band of frequencies of $s(t)$ is limited => LIMITED IN BAND (important) => spectrum defined until f_M => MAX freq of the signal
- f_M is the maximum frequency of the signal

$s(t)$ is **completely represented** by its samples taken:

- at regular intervals $t_n = nT_c$ (with $n \in \mathbb{Z}$)
- with a sampling period of $T_c \leq \frac{1}{2f_M}$ "multiple of sampling P"

i.e. from these samples it is possible to reconstruct $s(t)$ for any t

T_c freq. period $\leq \frac{1}{2 f_M}$ => sampling the signal at freq > 2 MAX freq of our signal

we're able to convert a cont signal to a discrete signal (composed by samples)

$$f_{c_{min}} = \frac{1}{T_{c_{max}}} = 2f_M$$

is called the ***Nyquist frequency***

to reconstruct => interpolation: start from a sample

The Sampling Theorem

The Nyquist frequency $f_{c_{min}}$ is the **minimum** frequency at which to sample a signal $s(t)$ limited in band

$f_{c_{min}}$ equals the double of the maximum frequency in $s(t)$

Under this condition $s(t)$ can be reconstructed using interpolation, for example (based on cardinal sin):

Frequency of samples

multiplo del sampling period

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT_c) \cdot \frac{\sin(\pi \cdot f_c \cdot (t - nT_c))}{\pi \cdot f_c \cdot (t - nT_c)}$$

RICOSTRUISCO IL segnale con una SOMMATORIA INFINITA

A sketch of the proof can be found in [1]

DEI miei samples $s(nT_c)$ • CARDINAL SIN

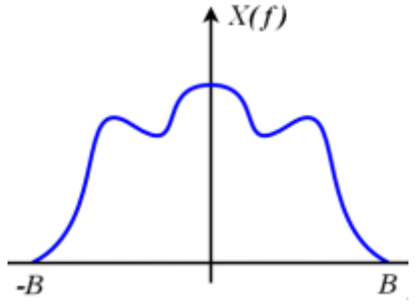
multiplo del sampling period

SHIFTATO DI $(t - nT_c)$

FT of a band limited signal

• ho un segnale nel tempo, il suo spettro limitato nella band

• Ho un segnale continuo
=> campiono => segnale discreto
(sampled signal)
=> Faccio la FT del segnale campionato
=> ottengo (per quanto riguarda la TRASFORMAZIONE in frequenze)
=> il segnale campionato ha uno SPETTRO formato da un SET di copie dello spettro del segnale continuo (discrete)



Fourier Transform of a band limited signal

continuo

• when a signal x(t) is sampled, the Fourier transform of the sampled signal consists of shifted and scaled copies of X(f)

FT of a band limited signal - sampled

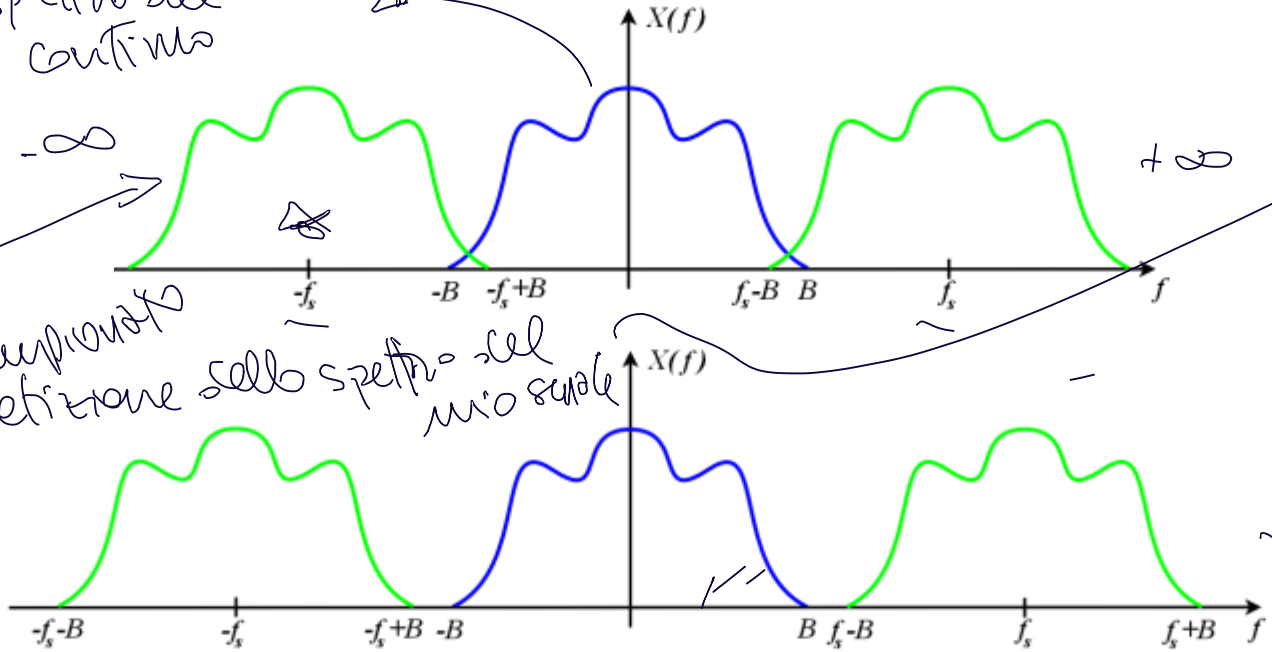
FT is a Σ => repliche sommate => le copie a frequenza

- The spacing between the replicas of X(f) depends on the sampling rate
  - **the faster we sample, the further apart are the replicas of X(f).**
  - If we sample every T seconds then the spacing between the replicas in frequency domain is equal to the sampling frequency $f_s = 1/T$

$f_s \geq 2$ Max freq of signal ⇓ aumento la distanza tra le repliche (copie dello spettro) ⇓ NO overlapping

spettro del segnale continuo

estratto dello spettro del segnale campionato => ripetizione dello spettro del mio segnale



⊕ centrato nei multipli della frequenza di campionamento $n f_s$

Increasing the sampling frequency

=> rappresenta il segnale continuo nel tempo

FT

time ⇔ freq. Domain

∫ dis operator represent the signal continuous

Aliasing

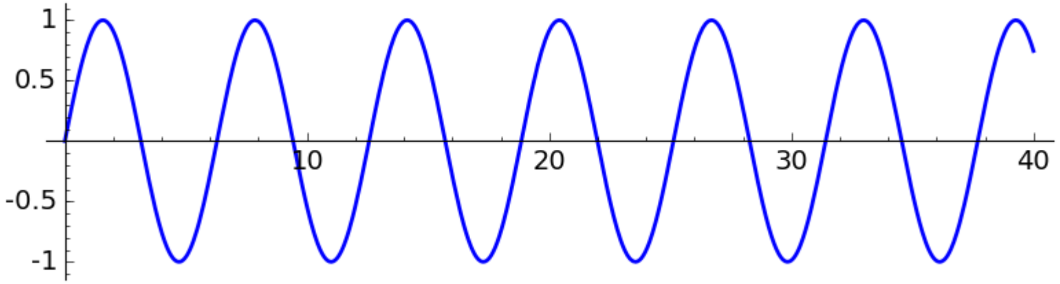
By ignoring what is between samples, the sampling process throws away lots of information

Se comprimo a serie frequenze

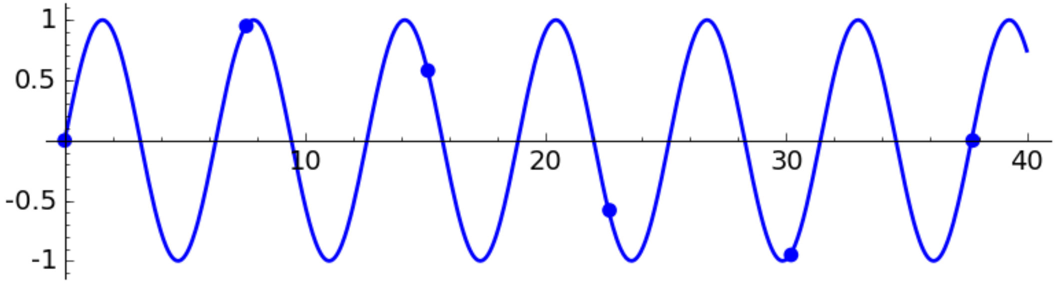
- … however, if the signal is perfectly band-limited then no information is lost!
  … you should consider this in your system design…

No overlapping ⇒ ALIASING

Consider for example a sinusoidal signal of frequency $f = 1/2\pi$:



If you know the frequency you can predict precisely the sampled signal:

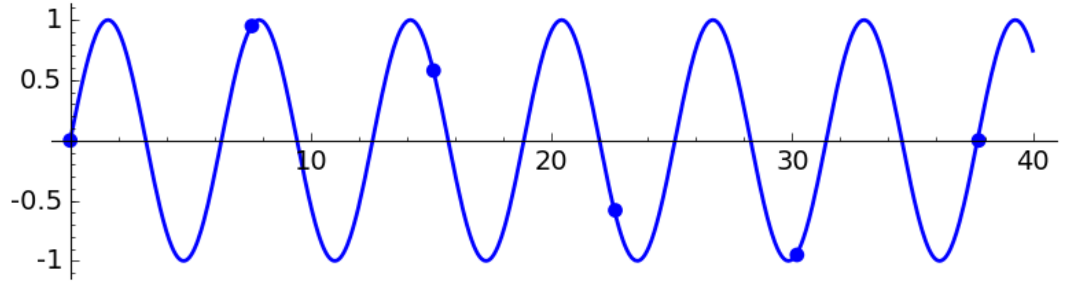


o Ho un segnale

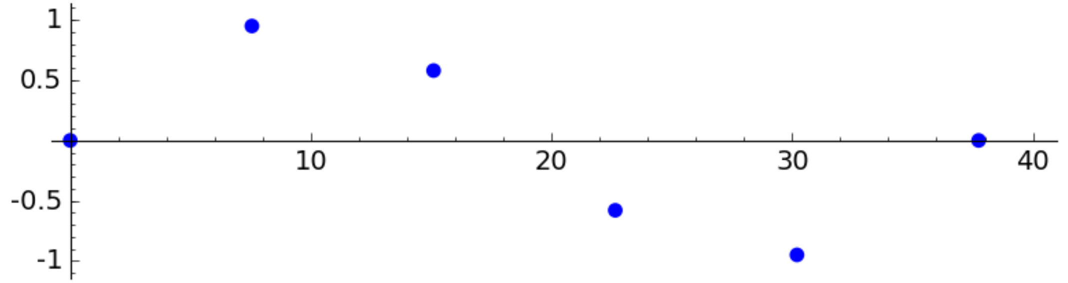
o Campiono il segnale e ricostruisco il segnale se so la frequenza del mio segnale

Aliasing

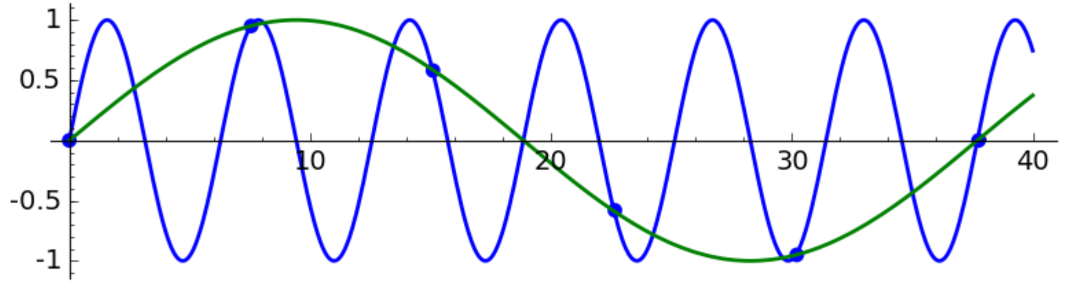
=> se campiono a basso rate => dal BW ottengo il verde che sono diversi NON => ricostruisco



… but from the sampled signal alone you cannot necessarily find the frequency…



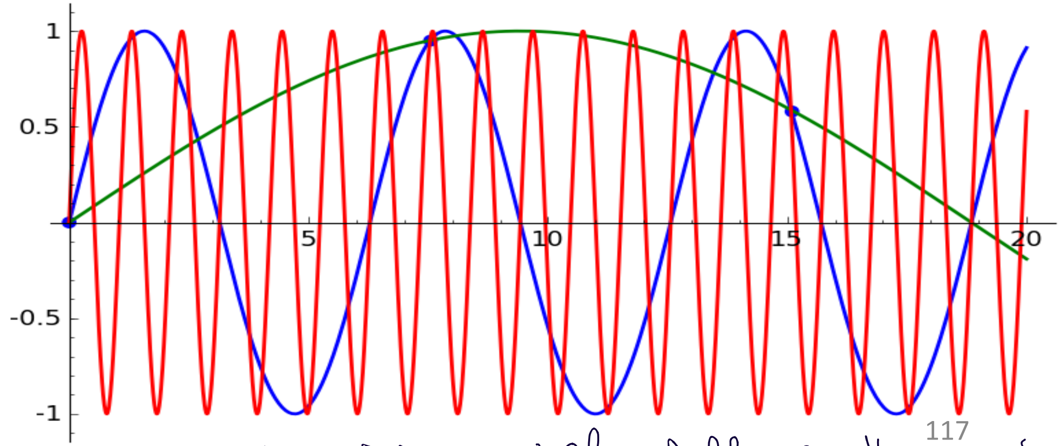
If you take another sinusoid at frequency $f' = f/6$ that's what you get:



This is known as the ***aliasing*** effect …

=> posso ricostruire solo il verde

=> no rosso o BW



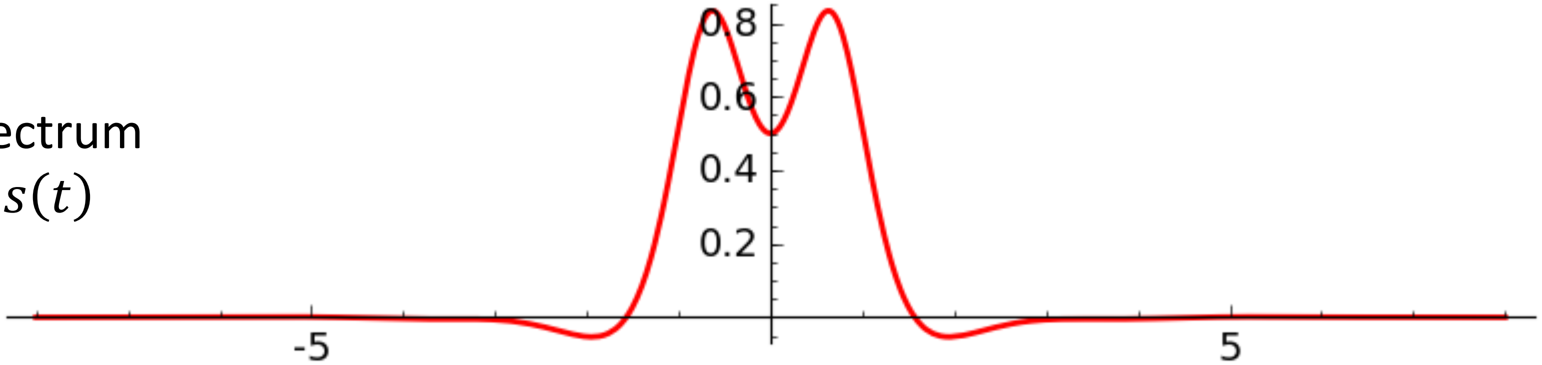
a bassa frequenza di campionamento => ho repliche dello spettro centrate in multipli dello spettro

Aliasing

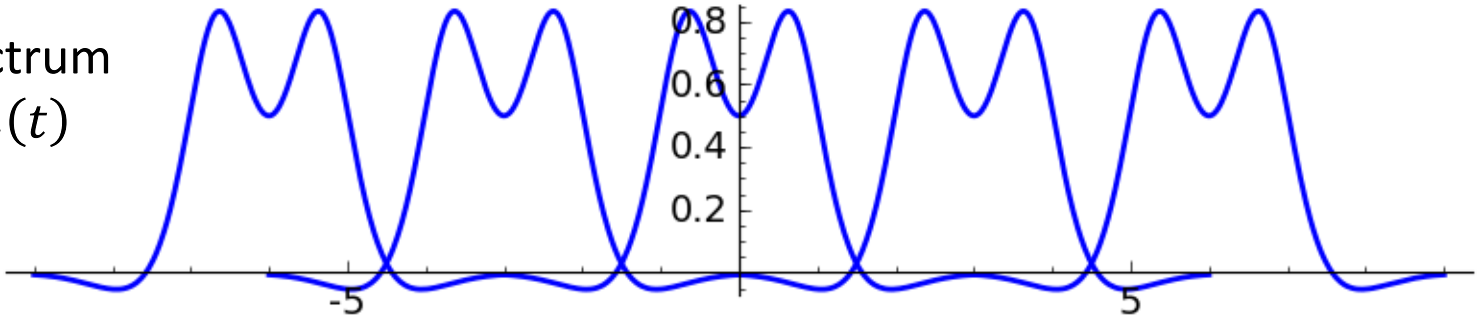
The example seen before when sampling a sinusoid holds also for more complex signals.

In general, the spectrum of the sampled signal $s_c(t)$ consists of infinite replicas of the spectrum of of $s(t)$ centered at frequences multiple of $f_c = 3$

Spectrum of $s(t)$



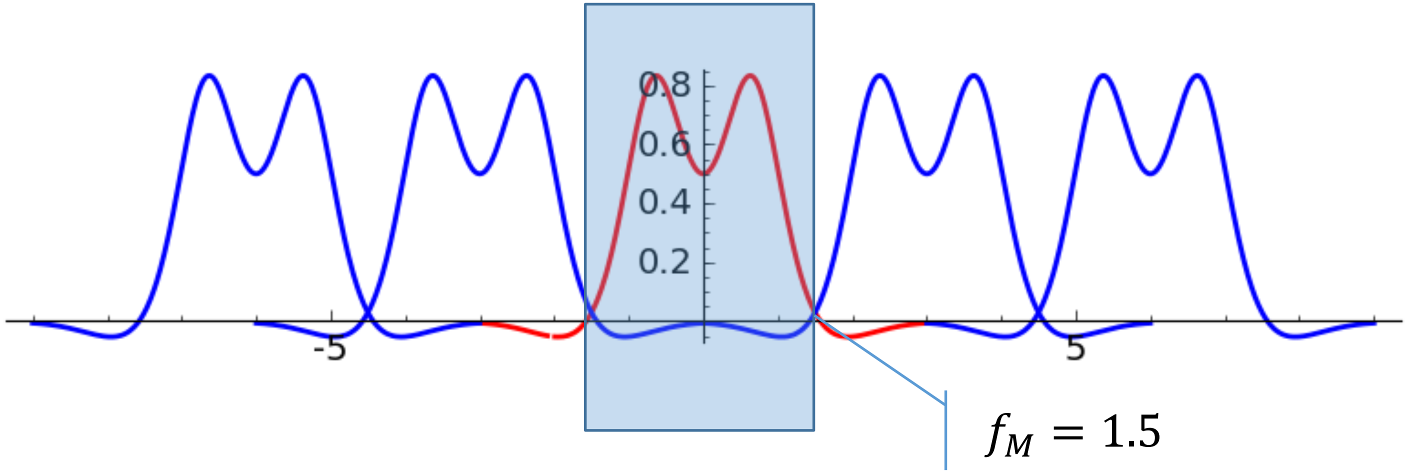
Spectrum of $s_c(t)$



Aliasing

That's why Nyquist requires that $s(t)$ is limited in band!

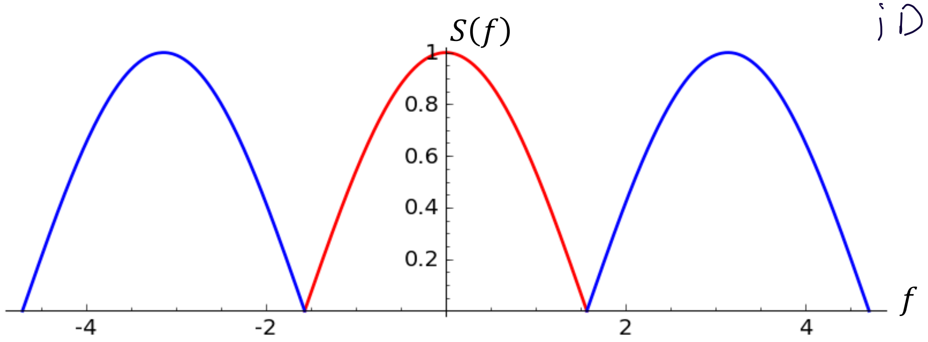
If we limit the signal in the example at frequency $f_M = 1.5$, we can disregard all the replicas (which have higher frequency):



Hence, if $s(t)$ is strictly limited in band (i.e. it has a spectrum null at frequences above f_M), we can choose f_c to ensure that the replicas of the spectrum of $s(t)$ do not overlap.

iDEALE

=> resta difficile avere segnali limitati nella banda



iDEAL: but hard have signal limited in band

⇓

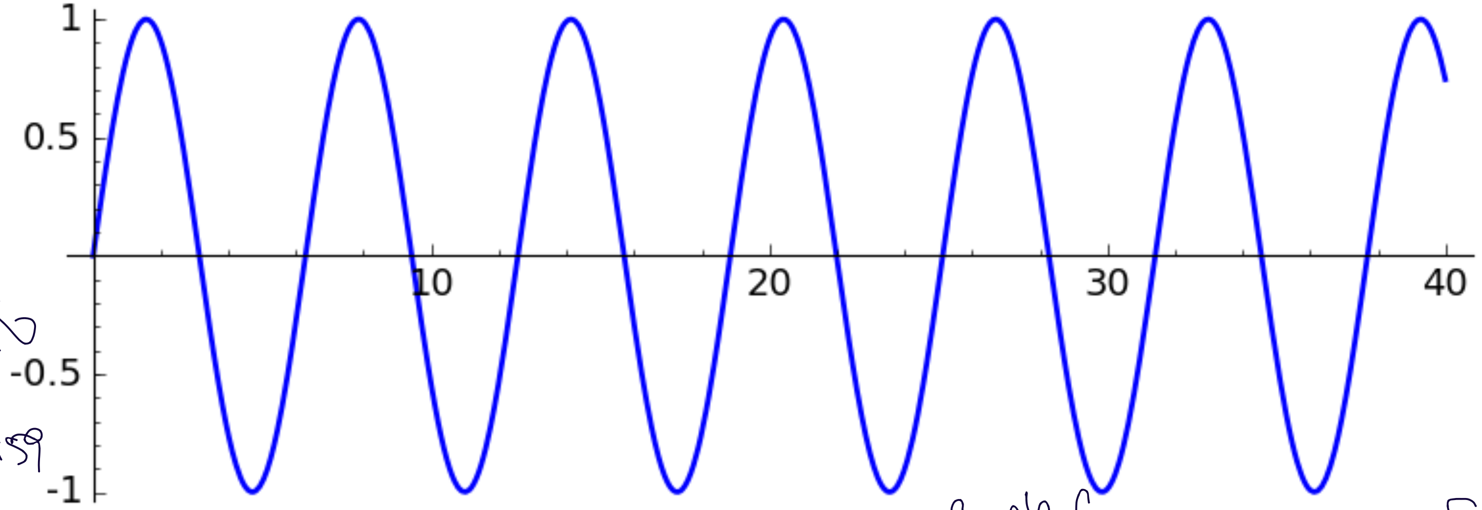
want very limited in

signal

Example

- Sampling frequency ci da un aiuto per capire la frequenza di campionamento
- Se campiono $s(t)$ $f = \frac{1}{2\pi} = 0{,}159$ ad una frequenza $f_c = 0{,}324 > 2f$

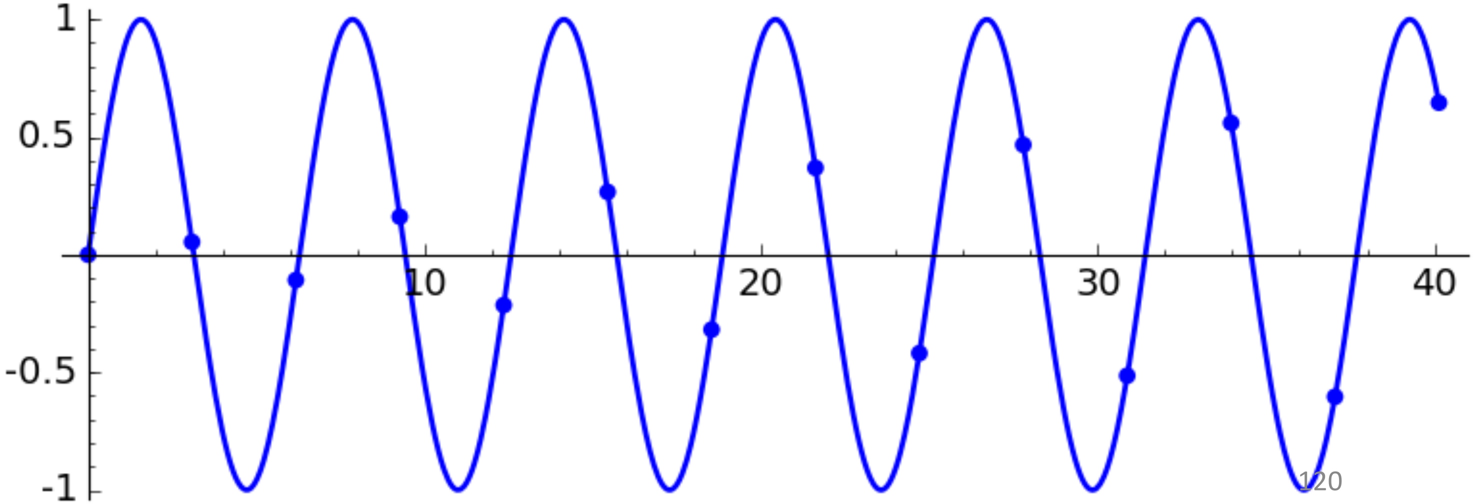
$s(t) = \sin(t)$, with frequency $f = 1/2\pi \sim 0.159$



$s(nT_c)$ with $T_c = 3.09$; hence $f_c \sim 0.324 > 2f$

→ sample freq

→ freq of the signal $s(t)$

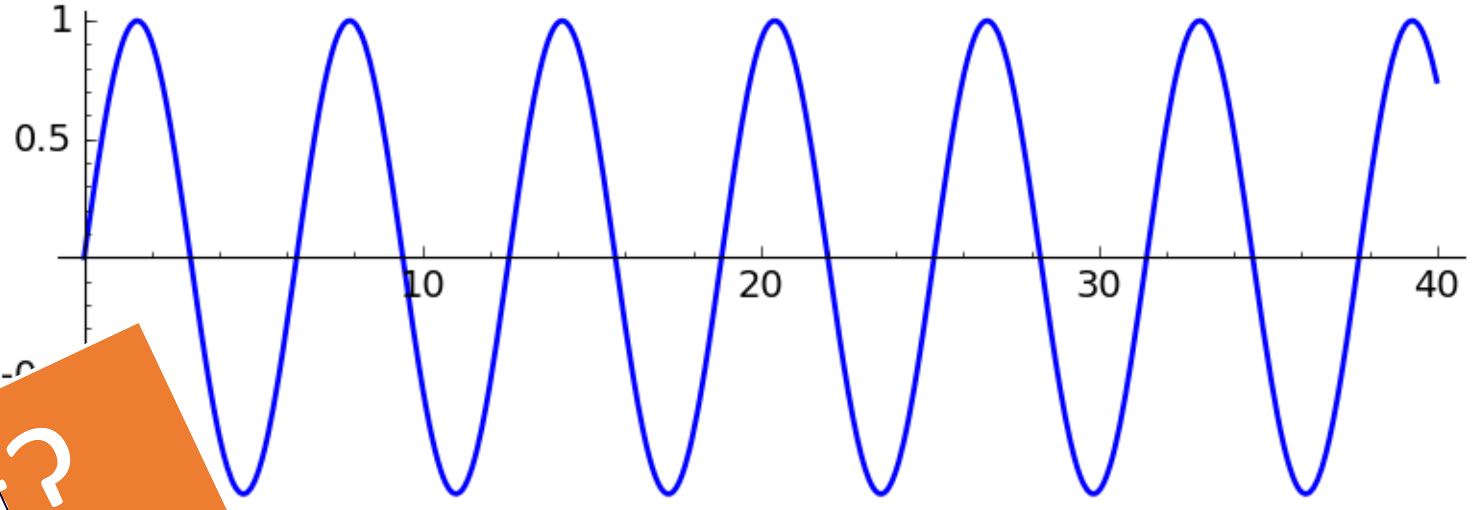


• se ricostruisco il segnale considerando un finito numero di samples (-15 to 15)

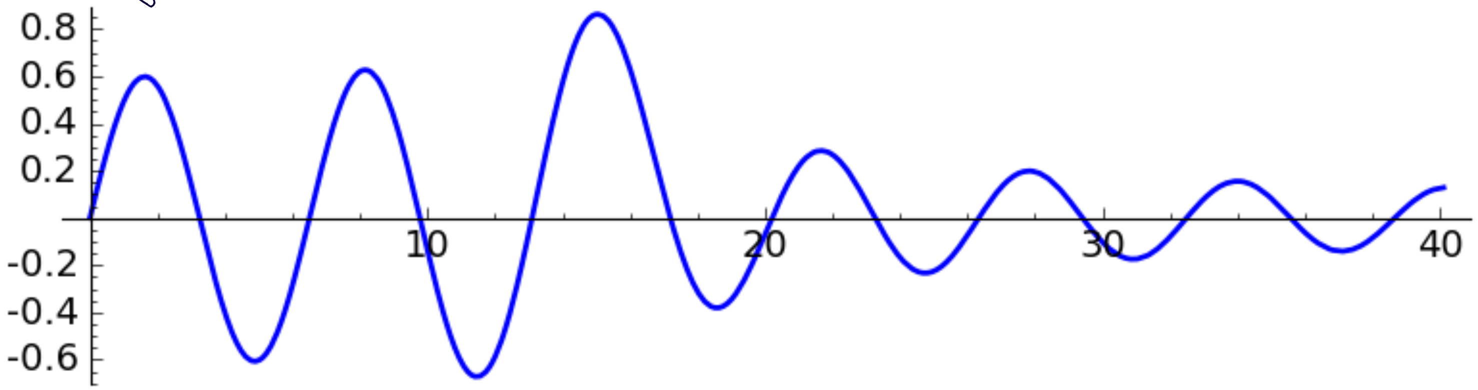
Example

OTTENGO

$s(t)$



$s(t)$ from interpolation (from -15 to +15…)



What's wrong?

• NUMERO DI CAMPIONI

⇒ perché il sampling theorem è una serie ∞

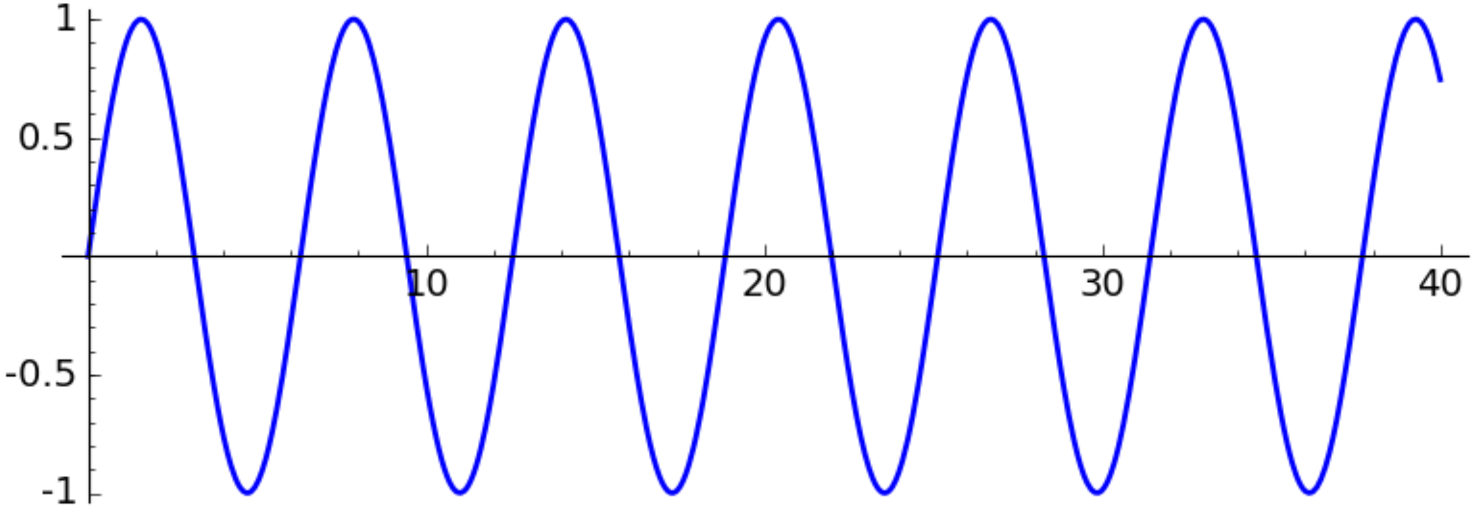
Example reloaded

Let's start again increasing the sampling rate…

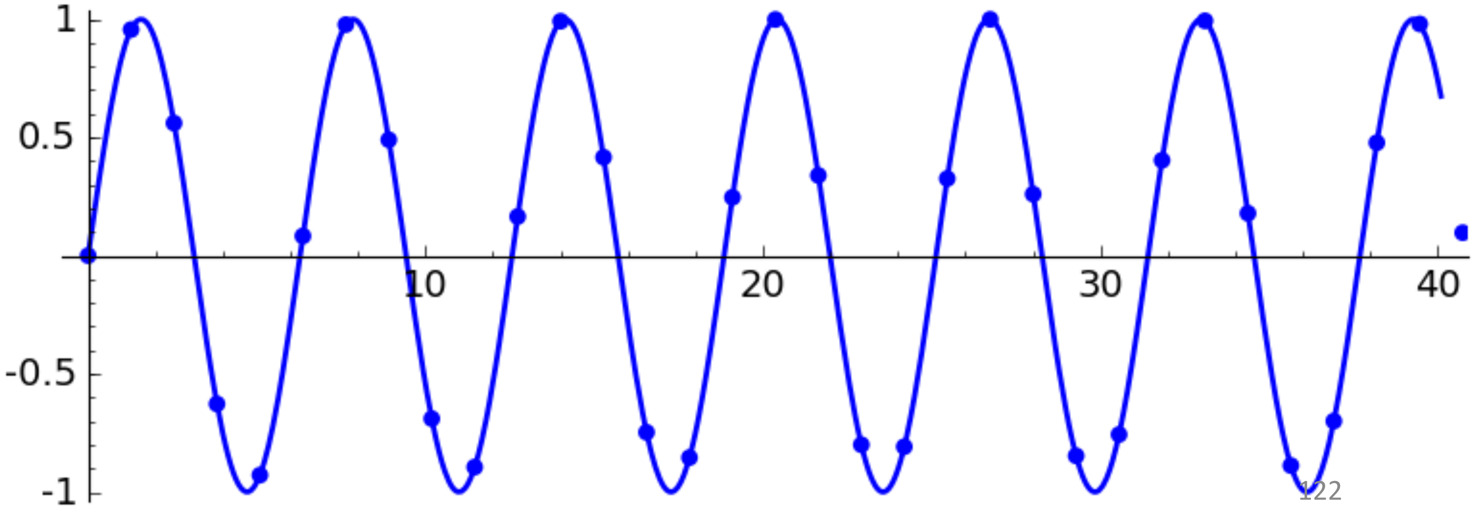
Now:

$$f_c \sim 0.787$$
$$f_c >> 2f$$

$s(t) = \sin(t)$

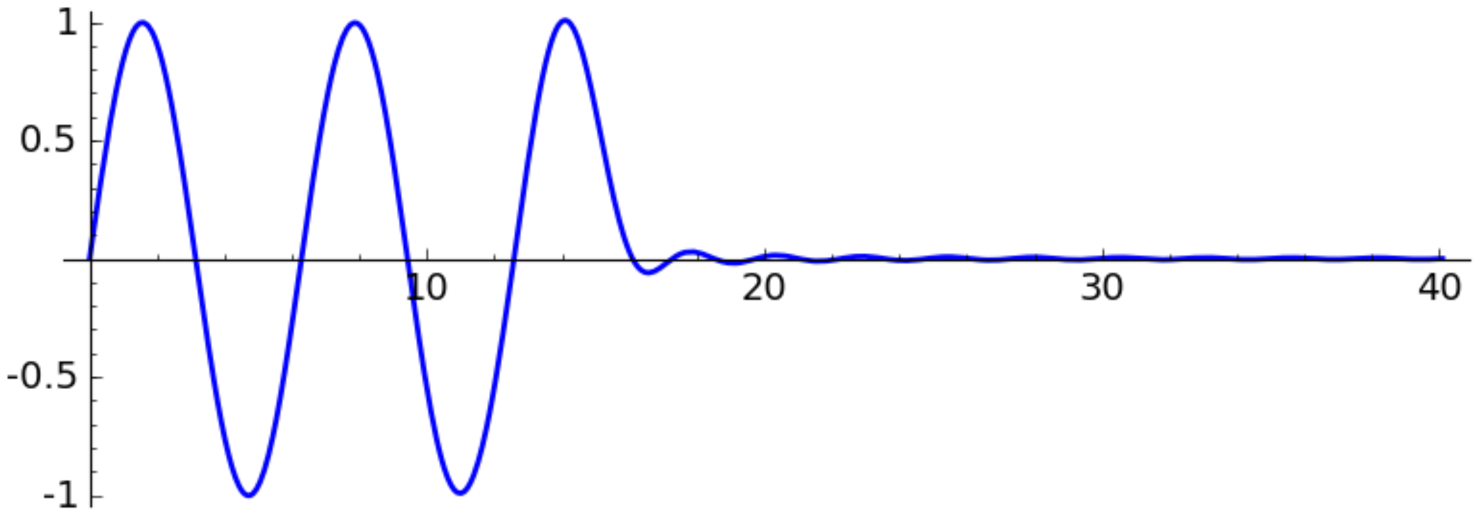


$s(nT_c) \quad (T_c = 1.27, n \in [0,40])$

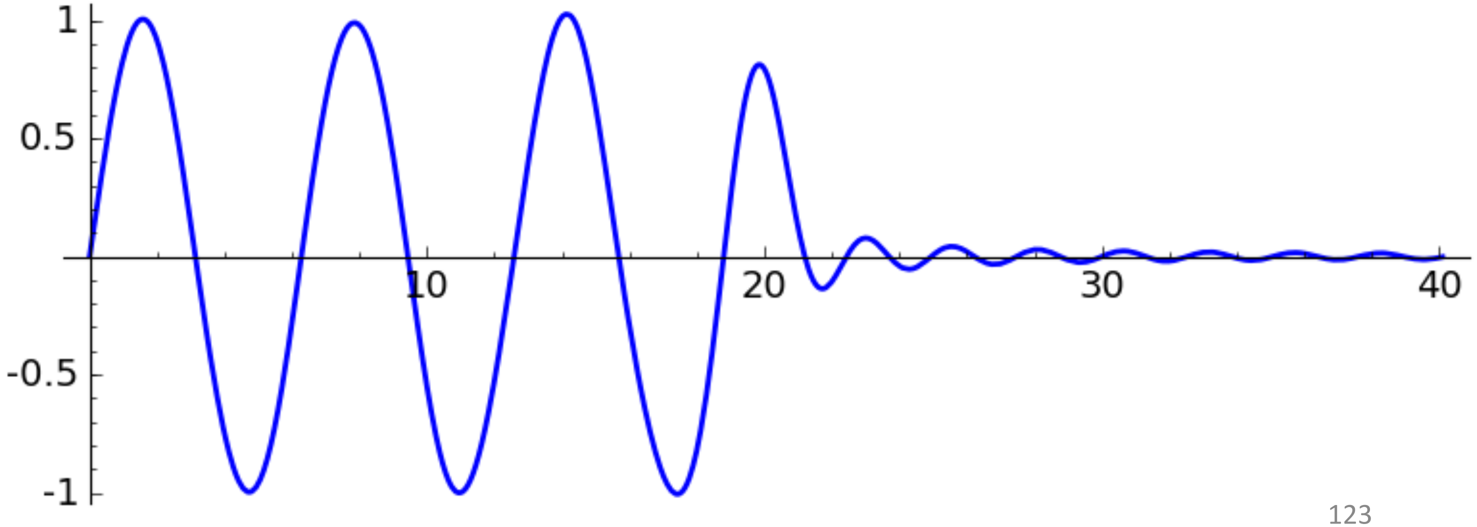


Example

now, interpolating from -15 to +15:



… interpolating from -20 to +20:



Now it's getting better…

But what's wrong with Nyquist?

Se ho un segnale con durata finita avrò uno spettro che copre tutte le frequenze

What Nyquist did say

Given a signal $s(t)$ with spectrum null at frequences above f

(LIMITED IN BAND)

However, no real-world signal is perfectly limited in bandwidth

- such a signal should not have energy beyond a frequency band
- but for this it must extend infinitely in time

However, even if the signal is bounded in frequency there is no mean to sample a signal from the real-world and to rebuild the signal perfectly…

… unless we take an infinite number of samples

- **The bound of Nyquist is on the frequency of sampling, not on the number of samples required…** (theory works with a ∞ # of samples

In any case, if we don't have an infinite time, we can sample a signal well enough to reasonably reconstruct it…

• possiamo applicare il teorema a segnali QUASI LIMITATI nella banda, così da focalizzarci SU UNA PARTE DELLO SPETTRO dove è concentrata la maggior parte dell'energia

Sampling

- In reality, no signal is strictly limited in band
  - this would mean that the signal is unlimited in time
  - and this fact has no physical meaningfullness
- In practice, many signals are **almost** limited in a band
  - … meaning that their spectrum beyond their band is small
  - i.e. their energy beyond the band is small
- In any sampling you can expect a little distorsion due to this fact
  - but with a «good» sampling this distortion is negligible
  - furthermore, it is also possible to use a **«cutoff» filter** that cuts out frequencies above a threshold from a signal in order so to meet the Nyquist requirements

To cut out components of higher freq.

=> cut-off filter => elimino le componenti del segnale ad alte frequenze e posso campionare il segnale

What Nyquist did not say

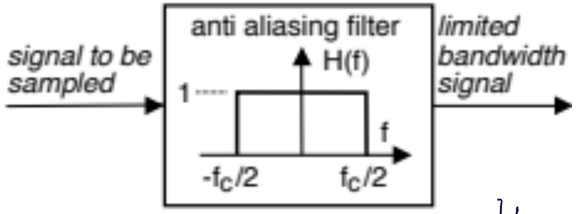
«I am going to sample at 8kHz, so I need to use a filter with a 4kHz cutoff»

- Nyquist didn't say that if you are sampling at rate N samples per second you can use an anti-aliasing filter with a cutoff at frequency $f = N/2$.
- No (analog) filter is perfect, and there is a significant amount of frequencies above 4KHz that pass, although attenuated.
- The result is that the aliased signal is not negligible and seriously disturbs your sampling

… and so what shall we do? apply:

- either you use more complex **anti-alias filtering**…

OR



- … or you simply **oversample**. aumentare la frequenza di sampling
- In this situation the (old, analog) telephone companies used a stronger cutoff (say at 3Khz), and they oversampled at 8 KHz

Low-pass filter (0, 3 KHz)

e campionando ad una frequenza di 8 KHz

aumenta la distanza dello spettro delle copie (campioni)

• APPLICANDO IL sampling THEOREM, è difficile dare regole generali per tutti i casi

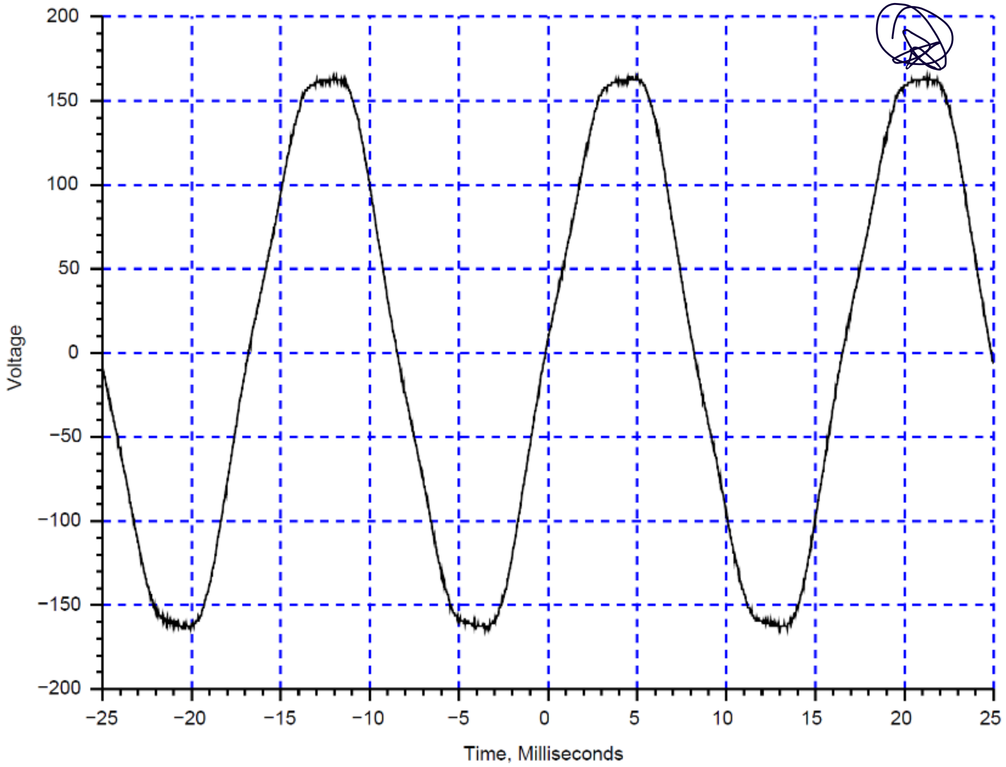
=> ogni segnale va STUDIATO

What Nyquist did not say

«I need to monitor the 60Hz power line, so I need to sample at 120Hz» NON VA

«We have a signal at 1kHz we need to detect, so I need to sample at 2kHz»

- Nyquist didn't say that a signal that repeats N times a second has a bandwidth of N Hertz. Ho dei componenti con frequenze PIÙ ALTE
- In fact the powerline (SEGNALE) for example is nominally at 60 Hz, but in fact you can see a lot of distortion, meaning that there are many other frequencies involved…



120 Hz non è buono

=> aumentare

s(t) freq. = 60 Hz

=> segnale con limitato nella banda (ossia che max freq. sia 60 Hz)

What Nyquist did not say

«I need to monitor the 60Hz power line, so I need to sample at 120Hz»

«We have a signal at 1kHz we need to detect, so I need to sample at 2kHz»

- Nyquist didn't say that a signal that repeats N times a second has a bandwidth of N Hertz.
- In fact the powerline for example is nominally at 60 Hz, but in fact you can see a lot of distortion, meaning that there many other frequencies involved…

- If you don't use anti-aliasing filters you need to figure out the maximum frequency of the signal,
- If it comes out that, as in this case, up to the fifth harmonic (300 Hz) is meaningful, you should sample at least twice that frequency (600 Hz)
- … a bit more actually because you don't know the phase of the signal…
- …and if you wish an accurate sampling in a short time you should better oversample seriously

What Nyquist did not say

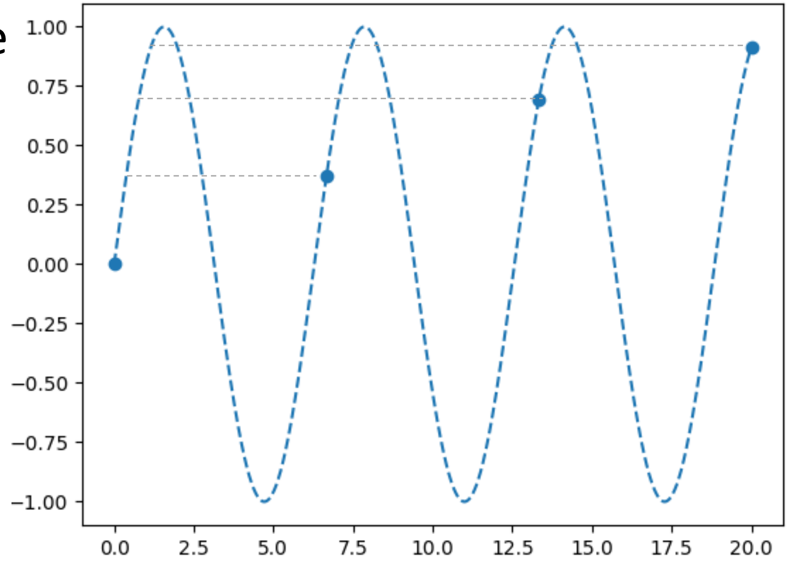
So the quest should be rephrased: «I need to monitor the 60Hz power line, so I guess need to sample at well over 120Hz» …

no variations in time

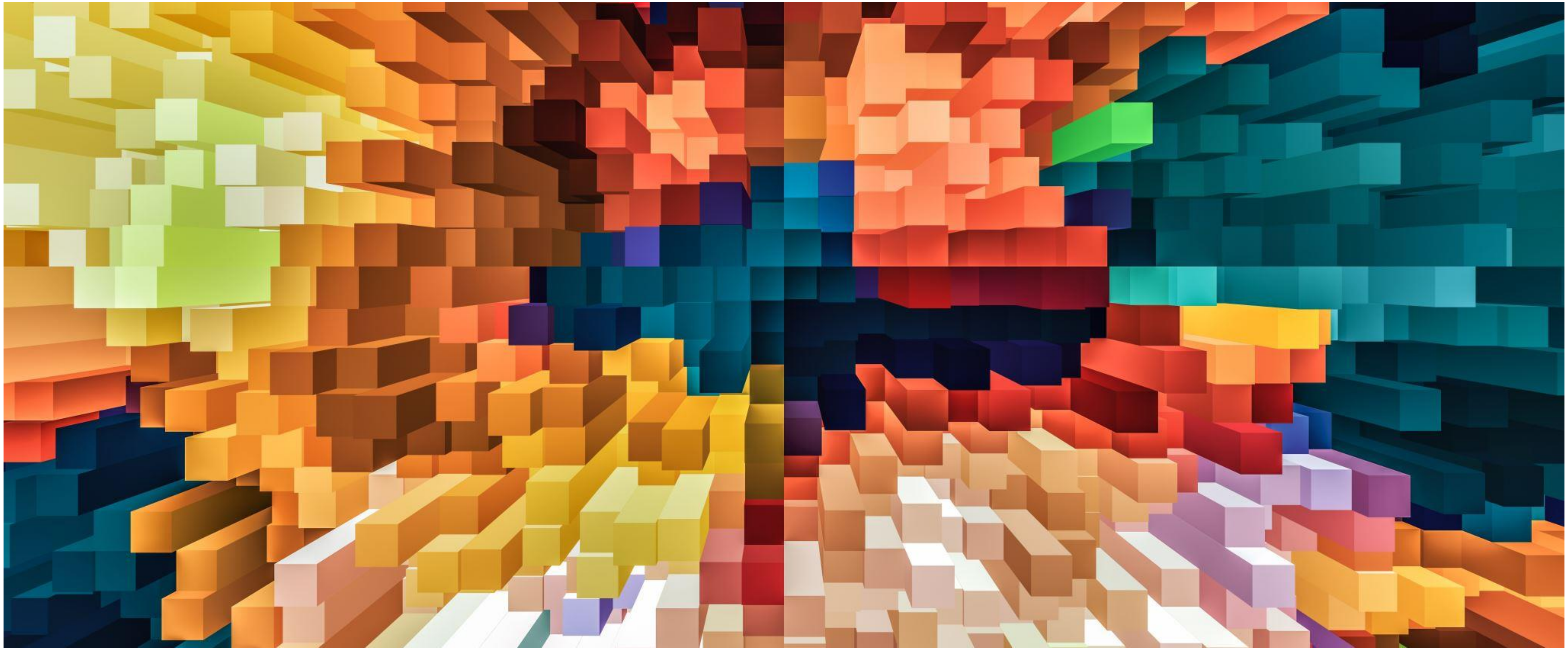
- Well… no. In some cases we could even downsample (if we have the patience to wait)
- In this specific case, **because the signal repeats continuously, it is steady and with a known and steady frequency**, if we downsample (say at a frequency slightly lower than 20Hz) we may be able to reconstruct the signal well enough

You can see how in [2]. A hint is that the signal phase within a cycle will advance a little bit with each new sample. If we sample long enough, we can have enough information to reconstruct the original signal

…but we need to know that it is a steady, repetitive signal with a steady frequency!



Quantization



SAMPLING: abbiamo un set di campioni finito da un segnale continuo analogico, dobbiamo rappresentarlo con un insieme finito di bit

Quantization

⇓

represent campioni with finite # of values

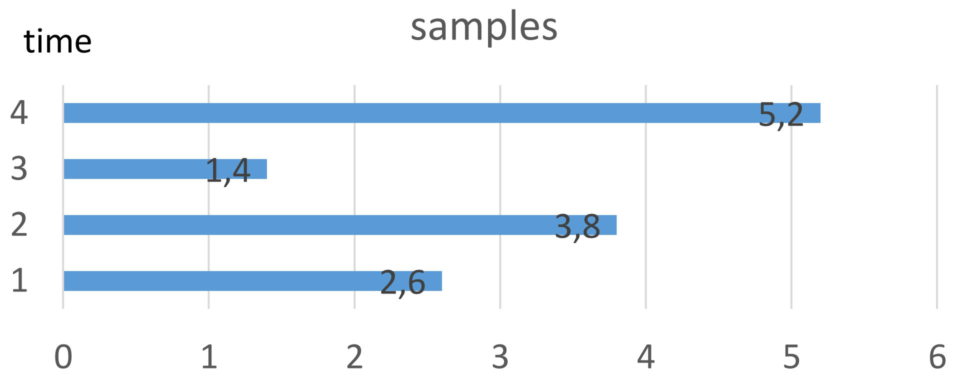
In general, the values obtained with sampling are Real numbers

However, in some applications even a representation in floating point (32/64 bits) is considered prohibitive (e.g. in IoT)

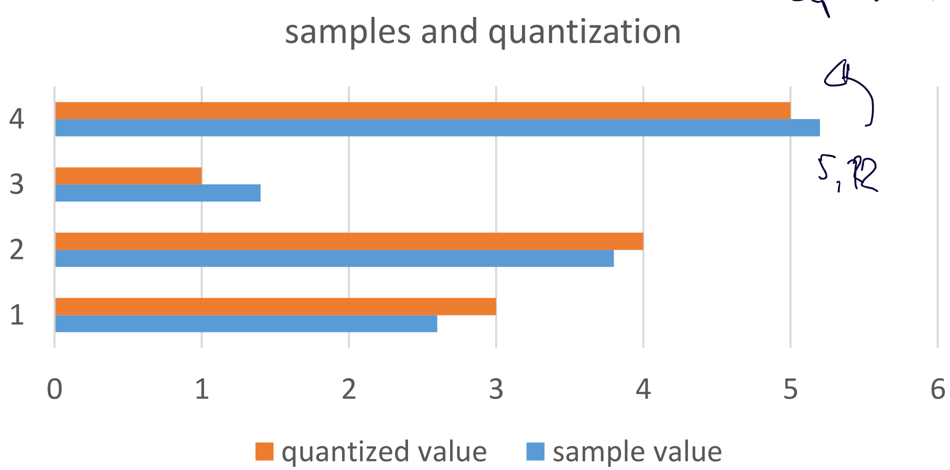
Usually the samples are represented in fixed points

- for example, in Arduino uno uses 10 bits per sample

(output of ADC)



represent integer closest = 5



5

5,2

Quantization means to approximate the value of a sample so that it can be represented with an integer in range $[0, 2^R - 1]$

1023

Posso rappresentare l'intero più vicino

Quantization

Let us consider:

- a signal $s(t)$ sampled at frequency f_c
  - hence samples are taken at intervals $T_c = 1/f_c$
- let $x_k \in \mathbb{R}$ be the sample read at the time interval I_k
- the quantization encodes x_k in an integer value $y_k \in [0, 2^R - 1]$
  - we thus use a R-bit scalar quantizer
  - … it is called «scalar» because the samples are quantized individually
- this requires R bits per sample (quantization rate)
- the bit rate for a signal sampled at frequency f_c is:

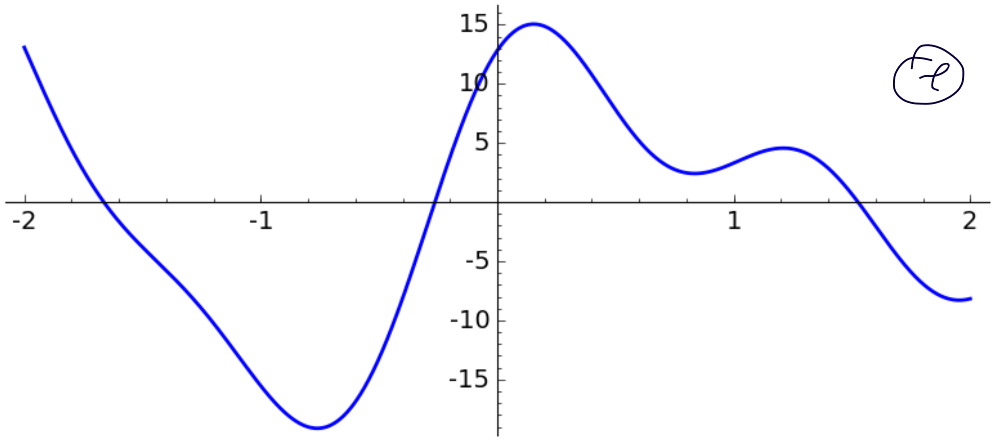
$$R_b = R \cdot f_c \;\; bit/sec$$

bits to encode the symbol

frequenza con cui sto campionando

Quantization

• QUANTIZZANDO PRODUCO UN ERRORE (approssimo all'intero più vicino)

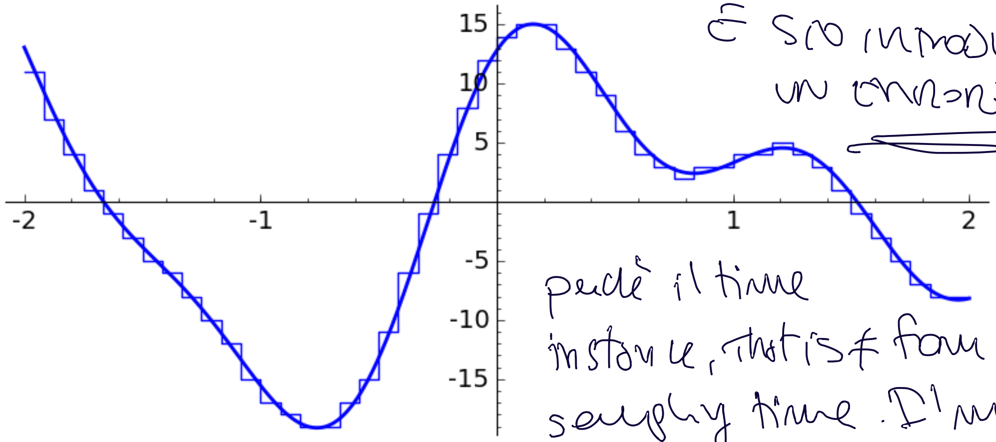


mio segnale

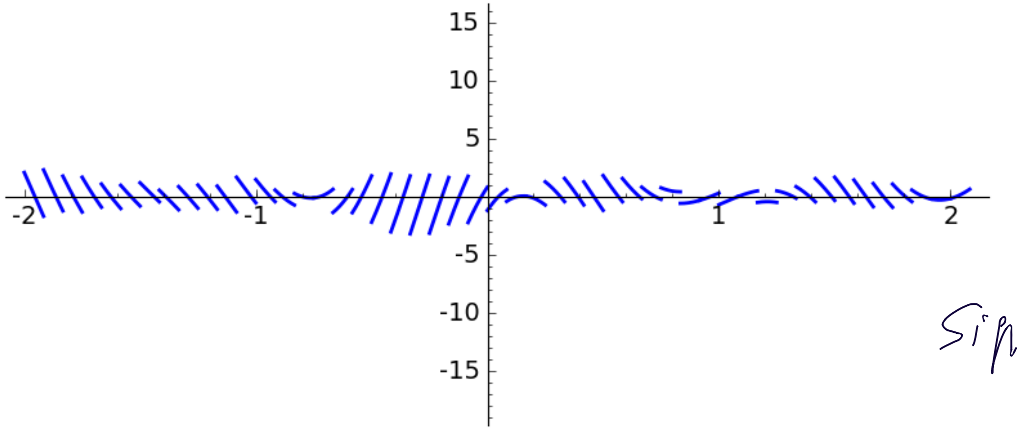
signal $s(t)$

② Campionando rappresento il segnale quantizzato (∅-OLD interpolation) e sto introducendo un errore

Quantized signal



perché il time instance, that is ≠ from the sampling time. I'm repeating the quantized value con il segnale



Quantization error

$s(t) - y_k$

signal

quantized value (valore che posso rappresentare)

Succede quando faccio una quantizzazione uniforme

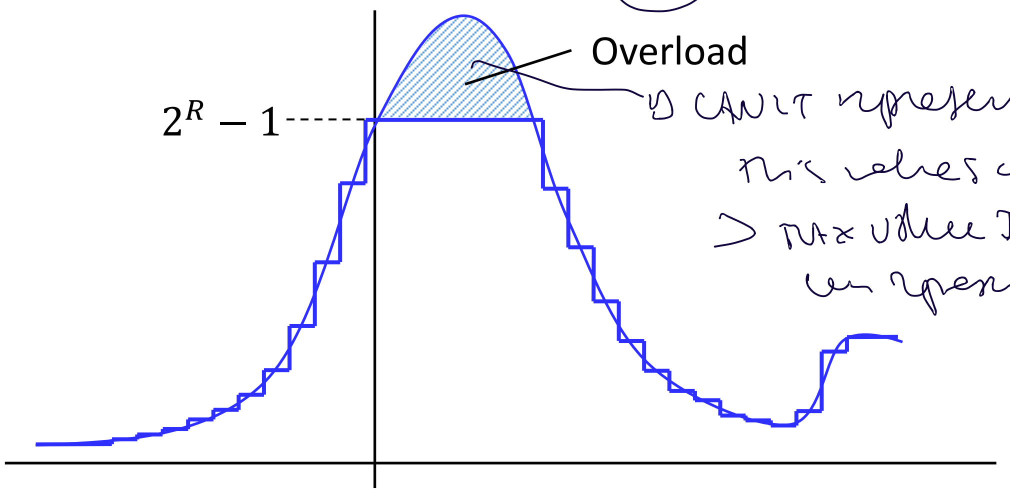
altri effetti

Quantization noise

① **Overload**: occurs when $s(t) > 2^R - 1$ for some t

R bits

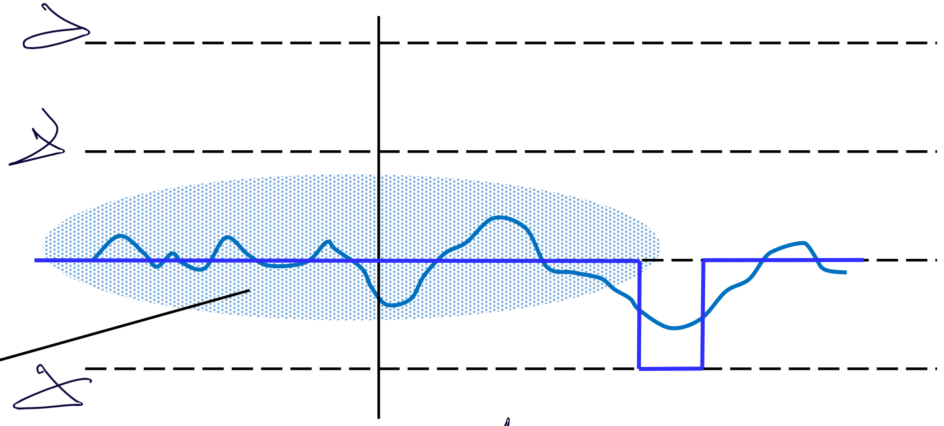
⇓ non uniform quantization



① can't represent this values cuz > max value I can represent

② **Granular noise**: occurs because all values in an interval I_k are represented with a unique y_k

quando tutti i valori di un intervallo vengono rappresentati ~~con~~ un unico valore



all these values are represented with the same y_k

can represent x

⇒ can't put those values

Example

- The human ear can hear between 20 and 20,000 Hz (20 kHz)
- With an acceptable degradation in quality, it can be cut at around 4kHz
  - as we have seen before even less, at 3KHz…
  - this is the level usually accepted at the phone
- Hence:
  - sampling rate: $f_c = 8KHz$
  - each sample quantized on 8 bits ($R = 8$)
  - the bit rate is then $R_b = 64Kbps$

$\rightarrow 2 \cdot 4kHz < f_c$

cut off freq

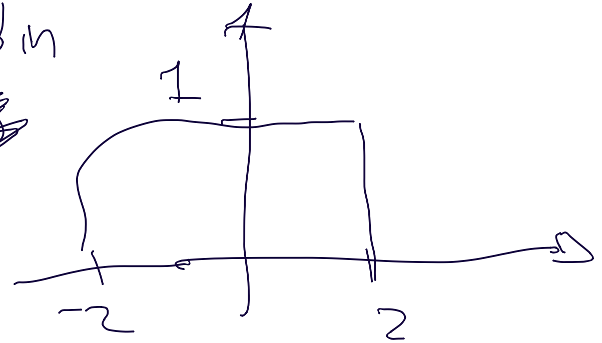
Exercise

What is the Nyquist sampling frequency of the following signals:

- 1) $s(t) = \sin(3 \cdot t)$
- 2) $s(t) = \begin{cases} 1 & -2 < t < 2 \\ 0 & otherwise \end{cases}$ ⇒ $1 \cdot 2 = 2$
- 3) $s(t) = \sin(2 \cdot t + 5) + \sin(8 \cdot t - 1)$ ⇒ not limited in BAND signal
- 4) $s(t) = 7 + \sin(2 \cdot t - 1)$ not limited in band

$2 \cdot \frac{18}{2\pi} = \frac{18}{\pi}$

$2 \cdot \frac{2}{2\pi}$



Exercise

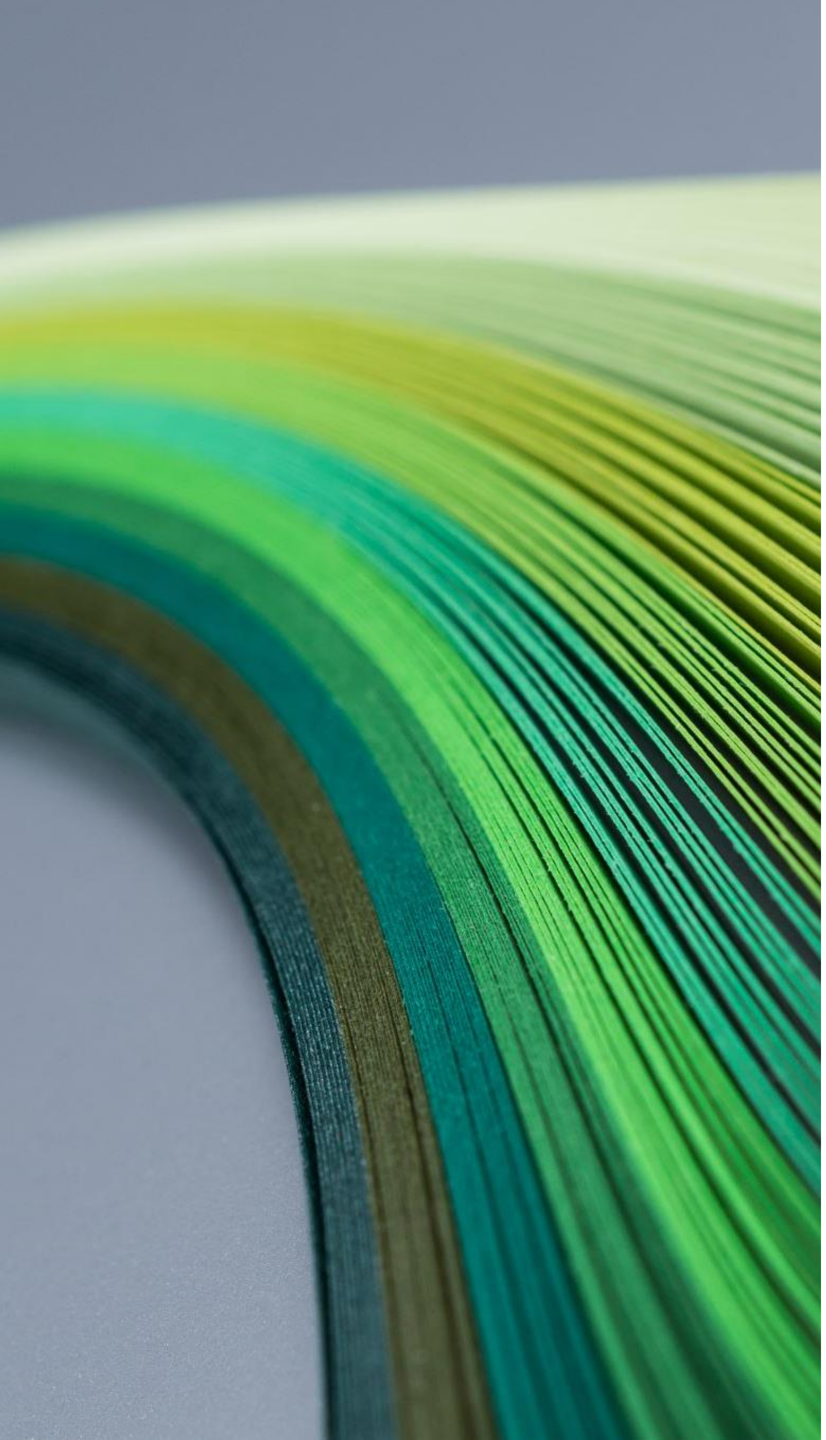
half of resolution

Considering ADCs that perform Q-bits quantization of signals with values ranging in $[0, M]$, compute the resolution of the quantization and the maximum absolute quantization error.

$[0, 5]$

$\frac{M}{2^Q}$

Bits of quantization Q	Max.value of the input signal M	Quantization resolution	Maximum quantization error
10	5	Q bits = $\frac{M}{1024}$	1024 - 5 = 1019
12	5	Q bits = $\frac{M}{4096}$	4096 - 5 = 4091
8	3,5	1 bit = $\frac{M}{256}$	256 - 3,5 = 252
10	3,5	1024 \ M	1024 - 3,5



Summary

- Analog, discrete and digital signals
- Decomposition of signals into harmonics by means of the Fourier series
- Switching from the domain of time to the domain of frequences with the Fourier Transform
- Band of a signal
- From Analog to digital signals: sampling and quantization and the Nyquist theorem