An introduction to the theory of signals

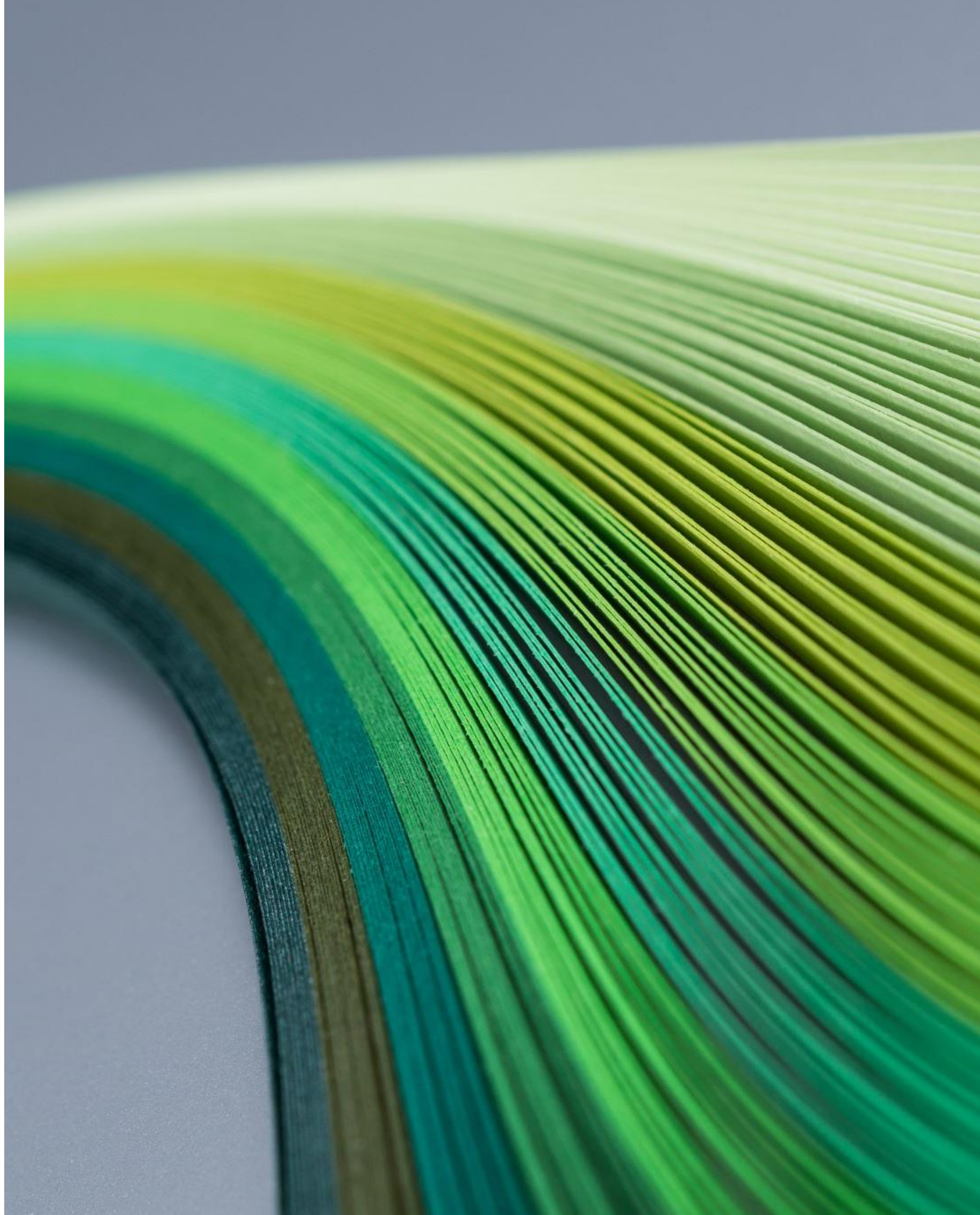
Mobile and Cyber Physical Systems

Learning objectives

- Classification of signals
- Fourier series
- Fourier transform
- Sampling and quantization

TOOLS

decomposing signal into infinite series



References

[1] chapters 1-4 of Alessandro Falaschi, «Trasmissione dei Segnali e Sistemi di telecomunicazione», in italian, free on internet:
https://teoriadeisegnali.it/libro/html/html/index.html

[2] Tim Wescott, «Sampling: What Nyquist Didn't Say, and What to Do About It», free on internet:

http://www.wescottdesign.com/articles/Sampling/sampling.pdf

- you can also refer to any other book that presents the fourier transforms. Usually in books on signal processing, theory of systems etc.

1.1 Trasmissione dell'informazione

Per inviare un messaggio informativo da un luogo ad un altro sono coinvolti un discreto numero di apparati differenti e cooperanti, spesso organizzati in rete. La trasmissione può riguardare un messaggio generato al tempo stesso della sua trasmissione, oppure esistente a priori. Il supporto fisico del messaggio, chiamato segnale, identifica due categorie molto generali: quella dei segnali analogici, e quella dei segnali numerici. Nel primo caso rientra ad esempio la voce umana, mentre esempi di segnali numerici sono i documenti conservati su di un computer.

Sorgente, destinatario e canale

L'origine del segnale da trasmettere è indicata (vedi Fig. 1.1 ↓) come sorgente, di tipo analogico o numerico per i due tipi di segnale. Ciò che giace tra sorgente e destinatario viene descritto da una entità astratta denominata canale di comunicazione, le cui caratteristiche condizionano i messaggi trasmessi.

Distorsioni e disturbi

Il canale può ad esempio imporre una limitazione alla banda di frequenze del segnale in transito[3]; cause fisiche ineliminabili producono inoltre, al lato

ricevente, l'insorgere di un segnale di disturbo additivo, comunemente indicato con il termine di rumore, che causa la ricezione di un segnale diverso da quello stesso presente all'uscita del canale. Pertanto, ci si preoccupa di caratterizzare il canale in modo da scegliere i metodi di trasmissione più idonei a rendere minima l'alterazione del messaggio trasmesso.

Rapporto segnale-rumore

L'entità delle alterazioni subite dal messaggio viene spesso quantificata nei termini del rapporto segnale rumore (SNR ↓ o Signal-to-Noise Ratio), che rappresenta un indice di qualità del collegamento stesso, e che per ora definiamo genericamente come il rapporto tra l'entità del segnale utile ricevuto e quello del rumore ad esso sovrapposto, indicato con n nella figura 1.1 ↓ .

Trasmissione

La Fig. 1.1 ↓ a) evidenzia come il canale, nella realtà fisica, è costituito da un mezzo trasmissivo su cui si propaga un segnale di natura elettromagnetica, che viene convertito in tale forma da appositi trasduttori di trasmissione e ricezione[4]. Considerando per il momento i trasduttori come facenti parte del canale stesso, proseguiamo l'analisi concentrandoci sugli ulteriori aspetti del processo di comunicazione.

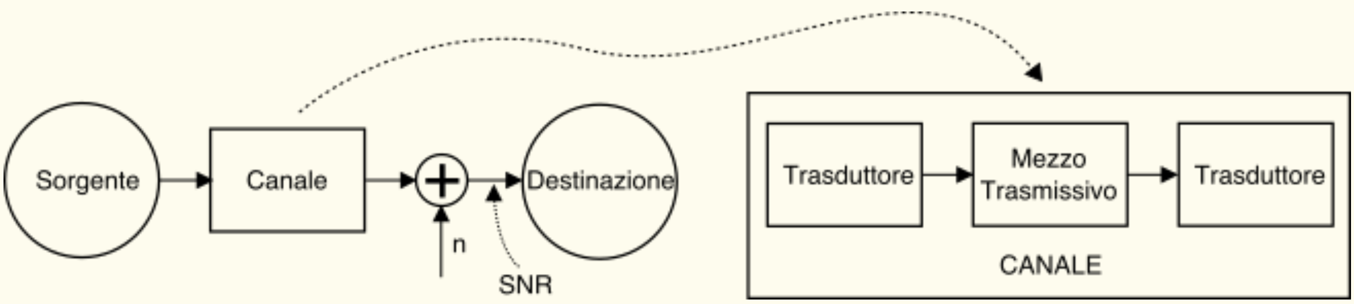


Fig. 1.1 a

Adattatori

La figura 1.1 ↑ b) evidenzia l'esistenza (in trasmissione, in ricezione od a entrambe le estremità) di dispositivi adattatori, che hanno lo scopo di ridurre od eliminare le cause di deterioramento del messaggio introdotte dalla trasmissione: si può ad esempio ricorrere ad equalizzatori per correggere la risposta in frequenza di un canale, ad amplificatori per contrastare l'attenuazione subita dal segnale, ovvero a codificatori di linea per rendere le caratteristiche del segnale idonee ad essere trasmesse sul canale a disposizione.

Rete

La trasmissione lungo un canale in uso esclusivo alla coppia sorgente-destinazione
figure Network.png
è piuttosto raro; di solito i collegamenti sono condivisi tra più comunicazioni, ognuna con differente origine e destinatario. Il problema della condivisione delle risorse trasmissive, ed il coordinamento di queste attività, produce la necessità di analizzare in modo esplicito le reti di telecomunicazione, che entrano a far parte integrante dei sistemi di trasmissione dell'informazione.
Gli aspetti delle telecomunicazioni brevemente accennati sono immediatamente applicabili a segnali di natura analogica, in cui il segnale è definito per tutti gli istanti di tempo, ed assume valori qualsiasi. Nel caso invece in cui il segnale è definito solo per istanti di tempo discreti e valori discreti, si entra nell'ambito delle trasmissioni numeriche. (o digitali)

Trasmissioni numeriche

Sorgente e destinazione numerica il messaggio informativo in questo caso è di natura discreta, ossia ad intervalli di tempo regolari sono prodotti simboli appartenenti ad un insieme finito, come ad es. le lettere dell'alfabeto;
Modem : i simboli che compongono un segnale numerico devono essere trasformati in un segnale analogico mediante l'utilizzo di dispositivi chiamati modem[5] (o codificatore di linea), come rappresentato dalla figura seguente, in cui è evidenziato come per una trasmissione unidirezionale[6] occorra solo metà delle funzioni del modem per entrambi i lati del collegamento, mentre nel caso di collegamento full duplex (in cui entrambi gli estremi possono essere contemporaneamente sorgente e destinazione) il modem opera allo stesso tempo nelle due direzioni.

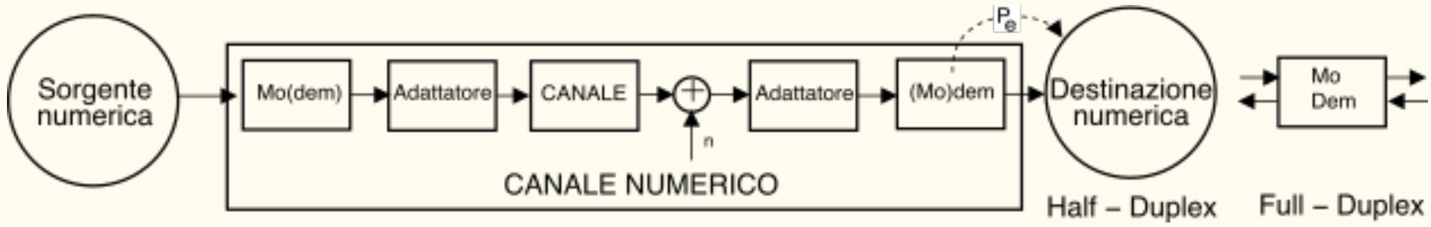


Fig 1.1 b

Introduction

Limits in precision, range of sampling

dev → gateway → Data

Sensors : devices read physical quantity, samples a signal,

the quantity: continuous in time instances and time to represent the quantity into digital, discrete signals

- sample real signals (e.g. temperature)
- transform them into discrete, digital signals
- process and transmit the discrete, digital signals

Wireless transmissions: Problem: msg (data packet as digital msg)

- TRANSMIT transform a data packet into an analog signal (often radio waves)
- RECEIVE receive analog radio signals and transform them into a discrete, digital signals

transmit in TRANS TRANSMISSION MEDIUM

transform analog signal (electromag. wave) are shaped to transport the MESSAGE TO DESTINATION

TRANSFORMATION

steps to do the transform in a good way!

Introduction

To understand these steps, it is necessary to understand the nature of signals

MATH TOOLS to

- How they can be decomposed a signal
- How they can be reconstructed " "
- How they can be sampled " "

All this passes through the understanding of Fourier series

Classification of signals

Classification of signals

RANDOM SIGNAL = use stochastics

$\neq$ → is a $f(x)$

- We focus on ***deterministic*** signals
  - are known before they are produced
  - the other class **of *random* signals** are analyzed by using probabilistic methods and statistics
- A deterministic signal can be represented by a function $f(t)$ of a real value (generally the time)

↳ function of time ($\mathbb{R}$ ~~...~~)

. deterministic signal: $f(t): \mathfrak{D} \longrightarrow \mathfrak{C}$

↳ codomain (es temperature Time

value

The domain and the codomain can be either the set of real numbers $\mathbb{R}$ or a discrete set (e.g., the integers $\mathbb{Z}$), or even the set of complex numbers $\mathbb{C}$

- Using $\mathbb{C}$ as codomain allows to represent two independent signals combined together

Classification of signals

A signal is a real function of time

According to D & C are structured:

1) • Time-continuous and amplitude-continuous

2) • Time-continuos and quantized (only fixed set of values)
↳ discrete

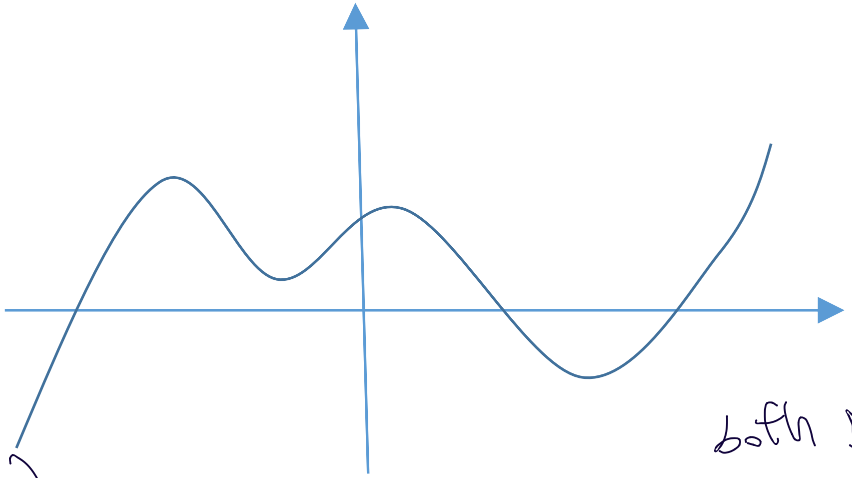
↳ value ~~of~~

3) • Time-discrete and amplitude-continuous

4) • Time-discrete and quantized

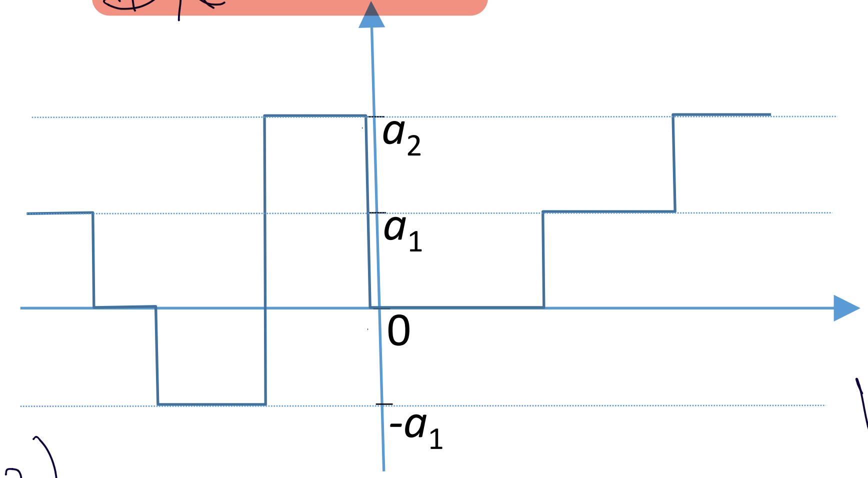
↳ discrete values on ~~amplitude~~ amplitude

Continuous-time signals



1) Continuous time and amplitude s

$\mathbb{D}, \mathbb{C} \longrightarrow \mathbb{R}$



2) Continuous time and discrete amplitude

$\mathbb{D} \to \mathbb{R}, \ \mathbb{C} \to \mathbb{Z}$

$\mathfrak{C}$ C

The domain $\mathfrak{D}$ is the set of real numbers $\mathbb{R}$

- also known as ***analog*** signals

both D & C are $\mathbb{R}$ $\Longrightarrow$

If the codomain $\mathfrak{C}$ is the set of real numbers $\mathbb{R}$: ***continuous amplitude*** signals

If the codomain $\mathfrak{C}$ is a discrete set (e.g. $\mathbb{Z}$): ***discrete amplitude*** signals

- AKA quantized signals

have a value $\forall$ instance of time, set codomain no arbitrary value set just concrete values

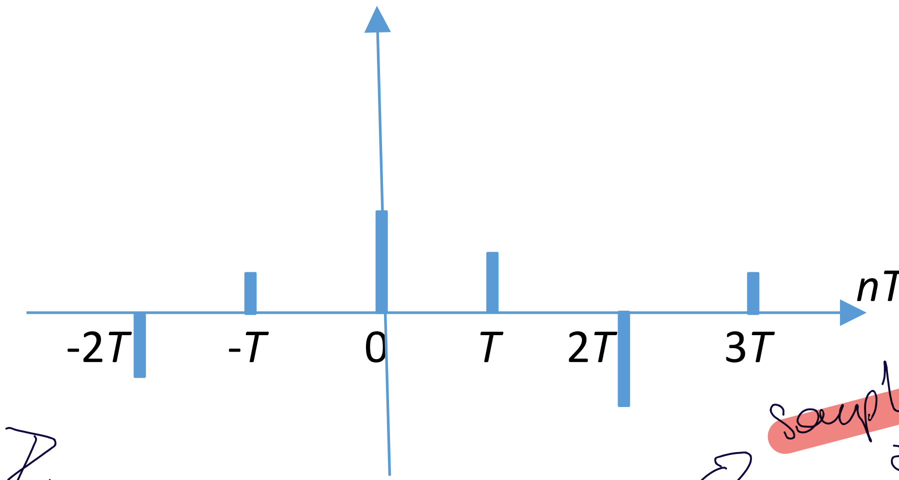
Discrete-time signals: have only limited discrete set for $\mathfrak{D}$

The domain $\mathfrak{D}$ is the set of integers $\mathbb{Z}(T)$

- where $\mathbb{Z}(T) = \{nT,\ \forall n \in \mathbb{Z}\ and\ T \in \mathbb{R}\}$
- they are also called *discrete* signals
- for example, $\mathbb{Z}(2) = \{\dots, -4, -2, 0, 2, 4, \dots\}$

$\mathbb{D} \to \mathbb{Z}$

$T \in \mathbb{R}$



Discrete time and continuous amplitude

sampling at a given instance of time

set values can be any

A discrete signal is called ***digital*** when the codomain is a finite set of symbols

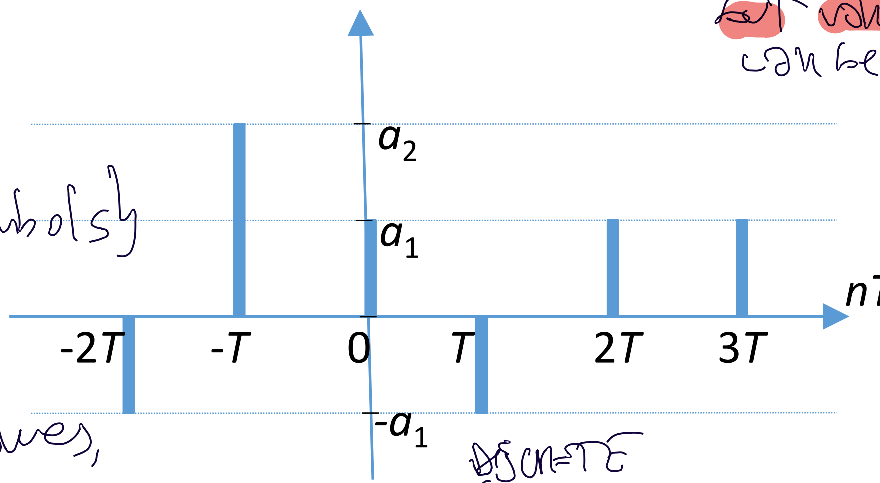
- a digital signal is also called a symbolic sequence
- a text is an example of symbolic signal

$\mathbb{D} \to \mathbb{Z}$

$\mathbb{C} \to \{\text{symbols}\}$

OK when a signal will be processed in a computer

$f(x)$ can assume just finite state of values, values just in a discrete set of time



Discrete time and DISCRETE amplitude

Digital signals - example

combine them to provide an info

Consider the alphabet of the 26 symbols A={a,b,c,…,x,y,z}.

A symbolic signal is any sequence of such symbols, e.g.:

- ababfxje…

2^0 2^1 2^2 2^3 2^4

1 2 4 8 16

Consider now the alphabet of two symbols B={0,1}:

- We can represent any symbol in A with a sequence in B
- For example: a ≡ 00000, b ≡ 00001, etc.

5 BITS

2^5 2^6 2^7

32 64 128

2^8

256

Hence, the digital signal «ababfxje» would become:

0000000001000000000100101101110100100100

Digital signals - examples

at a frequency

Assume a source able to transmit f symbols/second (symbol frequency), we have:

- 8 symbols with alphabet A $\Rightarrow$ the transmission lasts $^{8}/_{f}$ secs. (duration of transmission, # of bits, inverse of freq. f)
- 40 symbols with alphabet B $\Rightarrow$ the transmission lasts $^{40}/_{f}$ secs.
  - the symbol frequency is also called binary frequency if a binary alphabet is used
  - throughput usually refers to the binary frequency

bin freq in BIT/Sec from source to dest => binary freq (like temp)

Assume a source that samples an analog signal with f_c samples/second

Each sample is represented (quantized) with M bit/sample => need M bit to represent (the temp)

The source has a **throughput of $f_c \cdot M$ bit/second**

- Assume: - SOURCE is emitting symbols at given frequency (f)
  - Frequency: $\frac{f\text{-symbols}}{\text{seconds}}$

- transmit 8 symbols (signal) in Alphabet A, the transmission will last:

  \# of bits $\cdot$ inverse of frequency $\Rightarrow \boxed{8/f} \Rightarrow$ DURATION OF TRANSMISSION

- using binary alphabet to transmit same info (8 symbols $\in A$)

  $|A| = 26 \quad |B| = 2$

  $2^x = 26$ $\qquad$ 1 2 4 8 16

  $\log_2 2^x = \lceil \log_2 26 \rceil$ $\qquad$ $32 = 2^5$

  $\Rightarrow$ 8 symbols $\cdot$ 5 bits $= 40$

  $x = \lceil \log_2 26 \rceil = \lceil 4,7 \rceil = 5$

  $\Rightarrow$ DURATION OF TRANSMISSION $= \boxed{40/f}$ $\qquad \Rightarrow \boxed{\#\text{symbols}/f}$

- THROUGHPUT = frequency (in bit/sec) of information transmitted by the source is expressed in bit/sec

  $\Rightarrow$ binary frequency

  $\Rightarrow \boxed{f_c \cdot M\ \text{bit/sec}}$

  f_c: samples per sec

  M: \# bits to encode the message

period

$T = \frac{1}{f}$

$T \cdot f = \frac{1}{\cancel{f}} \cdot \cancel{f} \cdot \frac{}{f}$

Question

Considering a source that emits a digital signal encoded in an alphabet of 8 symbols, with a symbol frequency equal to 10 symbols per second. What is the throughput of the source?

$f_c = 10/sec$

→ binary frequence of the source

$|A| = 8$

$|B| = 2$

THROUGHPUT = $f_c \cdot M$ bit/sec =

$= \frac{10}{sec} \cdot 3 \frac{bit}{sec} = \boxed{30 \text{ bit/sec}}$ ✓

$x = \lceil \log_2 8 \rceil = 3$

Periodic continuous signals

CONTINUOUS IN TIME
& AMPLITUDE

A continuous signal $s(t)$: $\mathbb{R} \longrightarrow \mathbb{R}$ is ***periodic*** with period T if

$$s(t) = s(t+T) \; \forall t \in \mathbb{R}$$

period

- Example: $\sin nt$ and $\cos nt$ are periodic with period $T = \frac{2\pi}{n}$ $(\forall\, n \in \mathbb{Z})$

- non-periodic signals are called ***aperiodic***
- periodic signals can be studied in the period $[0, T]$ since their behavior remains the same in all its domain of existence

1

For a periodic signal also holds:

PROPERTIES

SHIFT

2

$$s(t) = s(t+T) = s(t+2T) = s(t+nT) \; \forall t \in \mathbb{R}, n \in \mathbb{Z}$$

if you SHIFT the function by a period T or multiples of T ⇒ you have the same value

Periodic continous signal

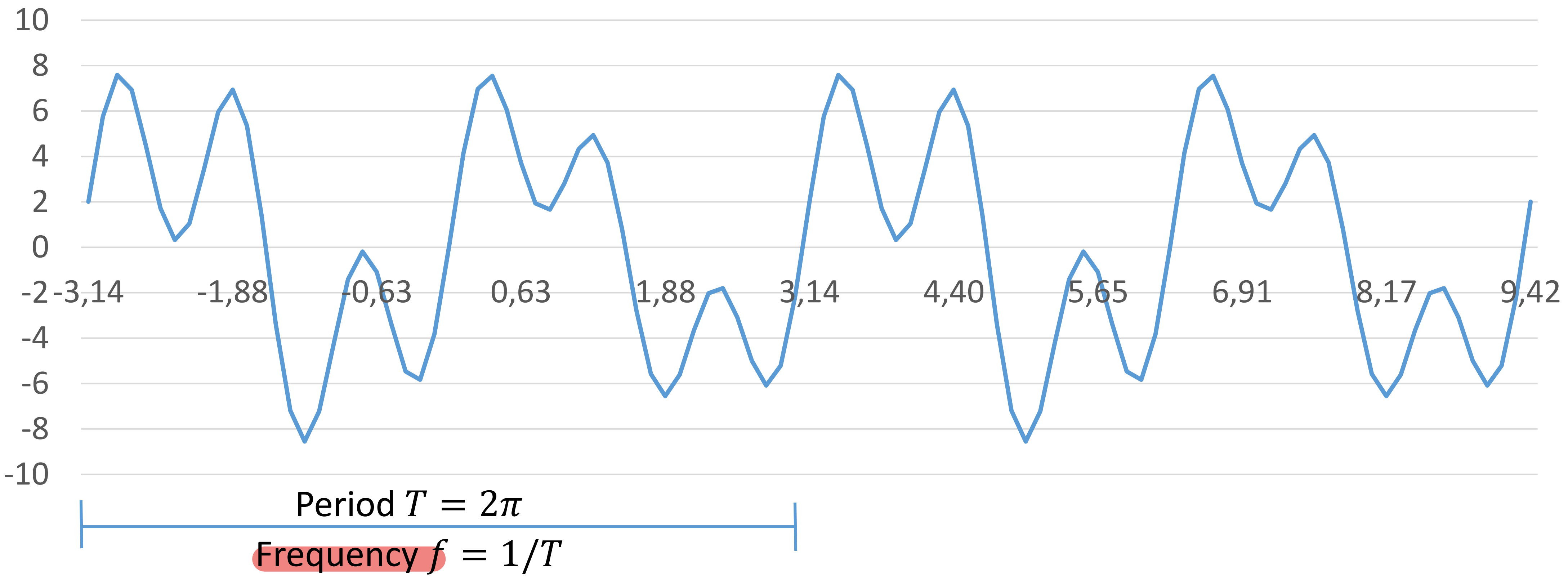
- if a signal $s(t)$ is periodic with Period T
⇒ $s(t) = s(t+T)$ $\forall t \in \mathbb{R}$ ⇒ the value of the function $s(t)$ is equal to itself + the period T

- Continuos Amp, time $s(t)$: $\mathbb{R} \to \mathbb{R}$
if $s(t) = s(t+T) =$
$= s(t+2T) = s(t+nT)$

Periodic continuous signals

• CAN combine periodic functions => still have a periodic function

$s(t) = \cos 2t + 5 \cdot \sin 2t - \cos 5t + 4 \cdot \sin 6t$ has period 2π



CAN THREAT A NOT-periodic signal as a PERIODIC SIGNAL

Periodic extension of aperiodic signals

- given a non-periodic signal defined in $[a,b)$, $\emptyset$ elsewhere (remaining domain)
⇒ SIGNAL LIMITED IN TIME ⇒ signal with limited support!

- Consider an aperiodic signal s with *support limited* to the interval $[a, b)$
  - i.e. $s(t) = 0 \; \forall t \notin [a, b)$ ⇒ null in the remaining domain

CAN DEFINE $\boxed{\text{PERIODIC EXTENSION}}$ = s^*

- The *periodic extension* s^* of s is defined as: infinite sum of s shifted by a period and multiple of period

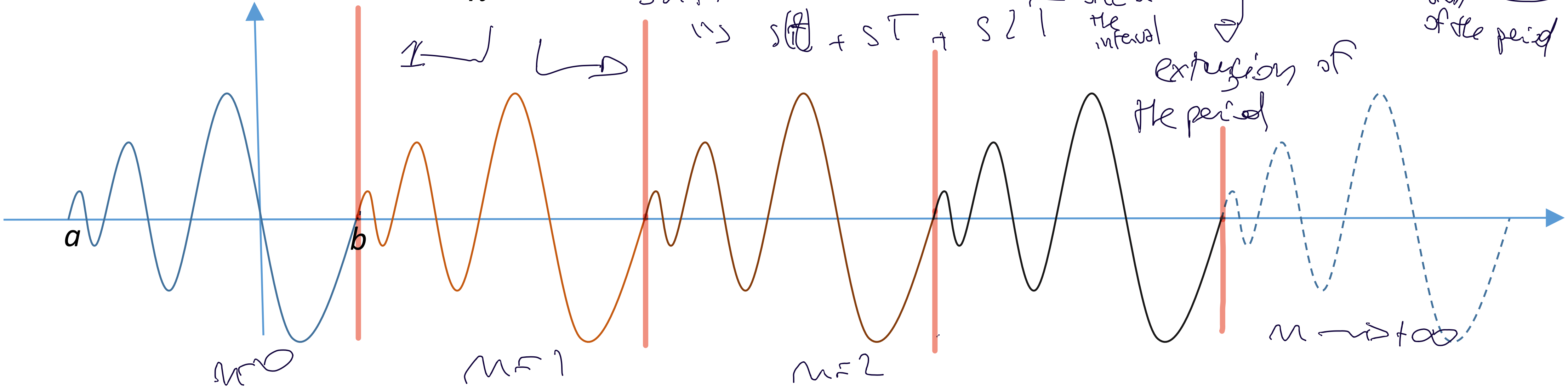
$$s^*(t) = \sum_{n=-\infty}^{\infty} s(t - nT) \qquad \text{where } T = b - a$$

shifted, ⇒ $s(t) + sT + s2T$

size of the interval

extension of the period

extension of the period



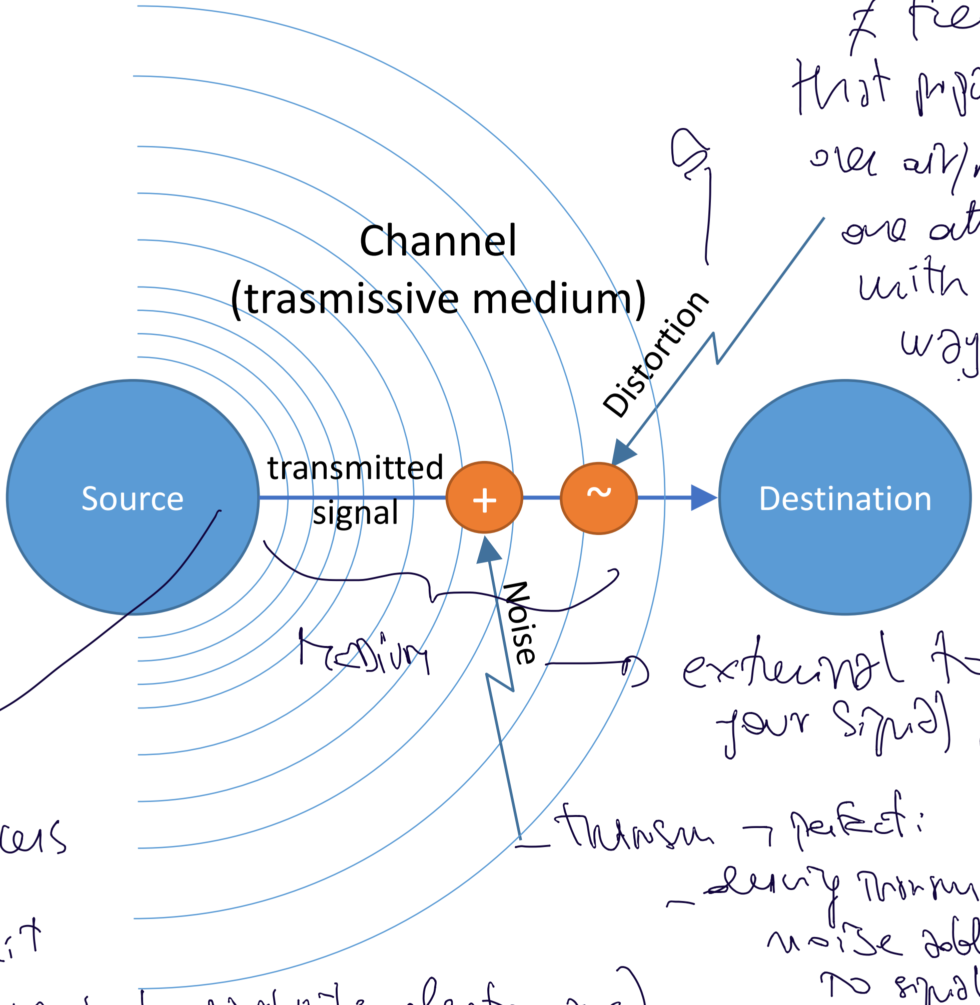
Transmission

Transmission of information

- Signals may have different nature, depending on the channel, e.g.:
  - e.g. electromagnetic, sound, optical waves etc.

  → AIR ⊗ → AIR → GUIDED MEDIUM ⊗
- At the source a transducer converts a message into a signal
- At the destination another transducer converts a signal into a message
  - for example, the antenna is a transducer for electromagnetic signals
- Signal to noise ratio (SNR): $\frac{\text{level of signal}}{\text{level of noise}}$
  - amount of changes suffered by the messages transmitted through a channel
  - property of a channel (not of a message)
  - expresses an index of quality of a channel

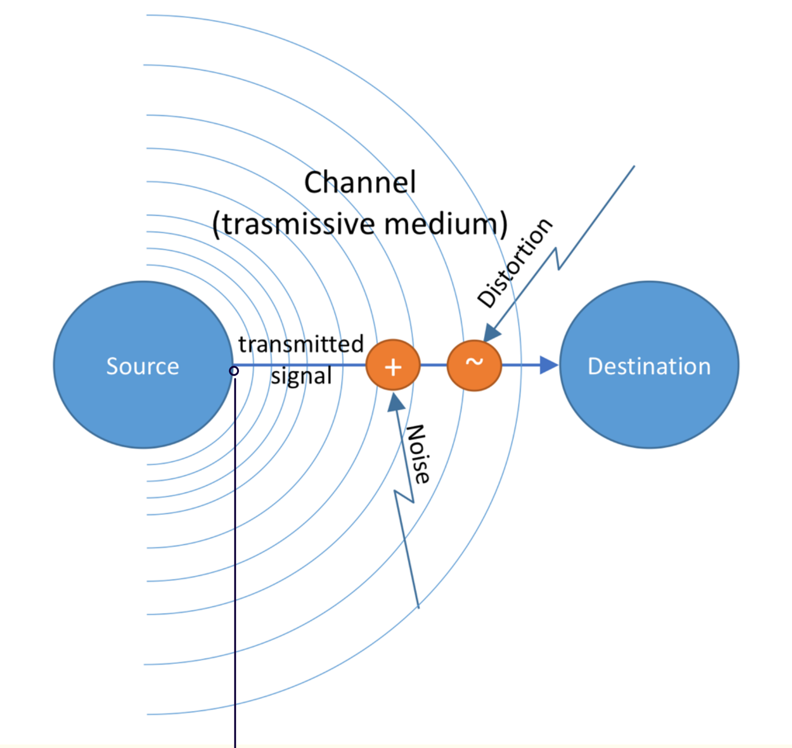
Signal composed of signal of ≠ freq that propagate over air/medium are attenuated with ≠ way



→ external to your signal (≠)

Transducers to transmit (antenna electro signal into electro wave)

Transm → perfect:
→ during transm noise added to signal or DISTORTION



- Different channels for transmission: air, optical...

- TRANSDUCERS (antennas): transform message into a signal (electro signal → electro wave)

- TRASMISSION THRU CHANNEL:
  - Noise: can added to the signal: external source for the noise (thermal noise, like moving of electrons in da medium)
  - Distorsion: if signal is composed by signals of ≠ frequencies ⇒ different waves propagates all air/medium → or signals ≠ waways for delay = 1 wave hits obstacol

⇒ Signal received is not the same as the one transmitted from the source

- SNR = how much medium is good

$$= \frac{\text{level of signal}}{\text{level of noise}} \Rightarrow 20 \log(SNR) \text{ in dB}$$

Transmission of information

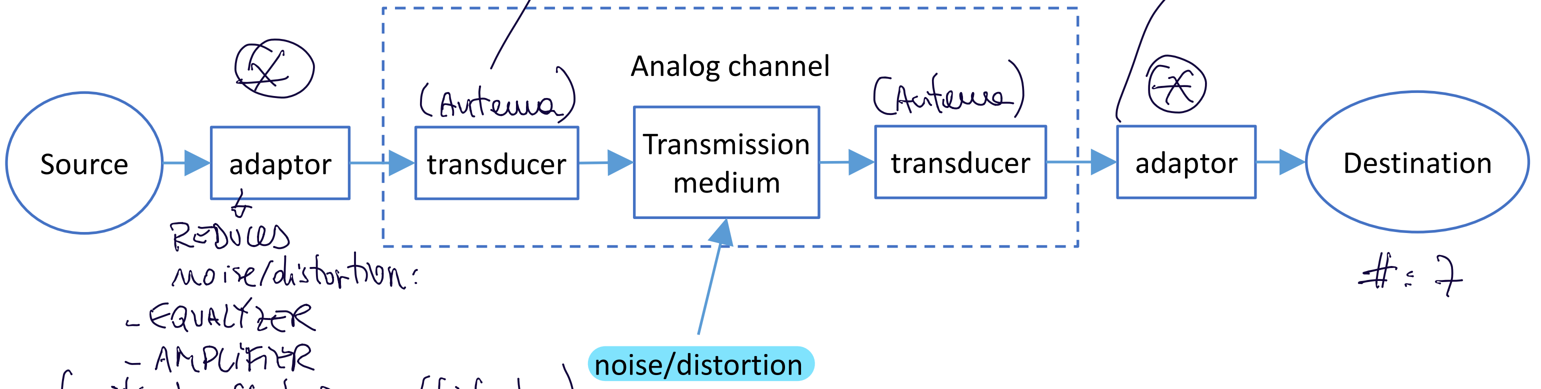
→ Source:
From: Electronic signal
to: Electronic waves

→ Destination
From: Electronic waves
To: Electronic signal

Analog transmission

use of adaptors to improve the quality of a signal

- reduce the effect of noise and/or distortions)
- e.g. equalizers, amplifiers, etc.



REDUCES noise/distortion:
- EQUALYZER
- AMPLIFIER
(mitigate effect of noise/distortion)

#: 7

Digital transmission

exchange of discrete messages

- discrete (over time) sequence of symbols

even over an analog channel:

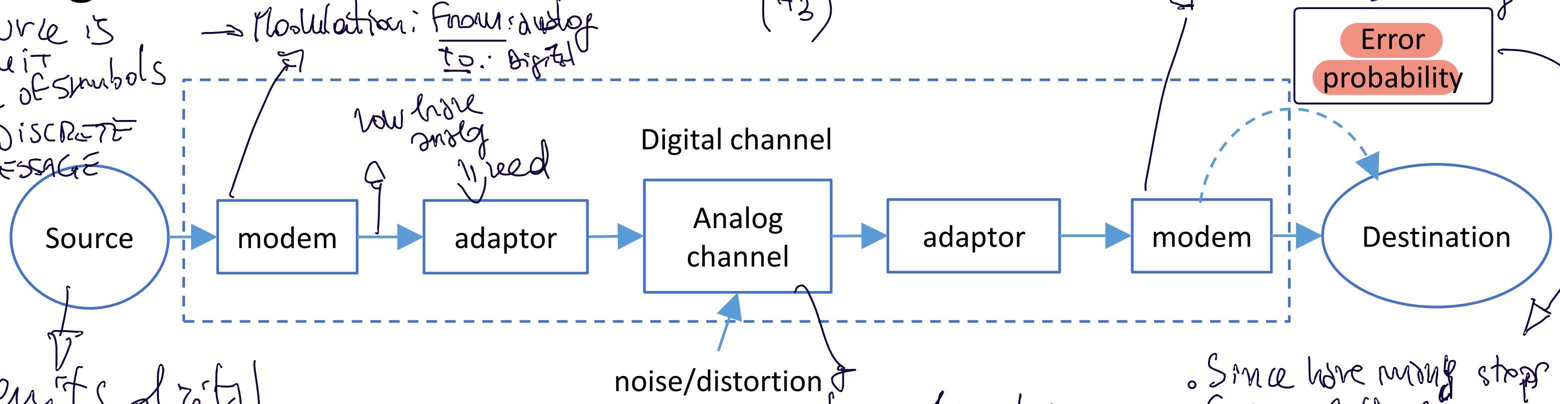
- … by means of modems that convert digital signals into analog signals (and viceversa)

Source is emit seq. of symbols => DISCRETE MESSAGE

→ Modulation: From: analog to: Digital

#: 7 (+3)

→ Demodulation: From: digital To: analog



now have analog, need

emits digital signal => discrete seq of symbols (bits)

antenna, medium, antenna

· Since have many steps from S/D, have errors in reconstructing the signal
=> try to minimize Error Probability with:

(Source)

CHANNEL ENCODING: add some redundancy bit to information transmitted, to allow receiver to make checks

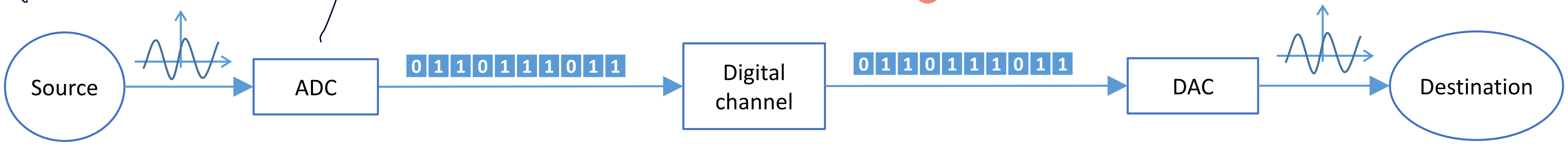
Digital transmission: sampling & quantization

FOCUS ON SOURCE

- At the transmitter:
  - sample the analog signal at discrete intervals («discrete intervals» imply a quantization), by means of an analog to digital converter (ADC)
  - the analog signal is now represented as a sequence of symbols
- At receiver:
  - the symbols must be converted again into an analog signal
  - by means of a digital to analog converter (DAC)

↳ process that constrains an input (continuos or large set of values) to a discrete set (like $\mathbb{Z}$)

produces digital msg from an analog signal



Digital transmission: sampling & quantization

(how we can sample)

However, sampling and quantization introduce a further distortion of the analog signal (*quantization noise*), due to:

- frequency of sampling
- resolution of the digital symbols
  - how many different symbols used to represent a single analog value

to have very low errors but I want the trade off => we'd have to transmit a big amount of bits

High resolution and high sampling rate ⇒ small quantization error

- but also a larger number of symbols to transmit

problem in reconstructing the original signal
=> reconstruct an approx of source's signal

Fourier series

• How can SAMPLE an Analog signal? => Analize signal in It's frequency components

Frequency Domain Analysis

=> How a signal can be composed by signals at ≠ frequencies

- Frequency domain analysis and Fourier transforms are a cornerstone of signal and system analysis.
  (Base)

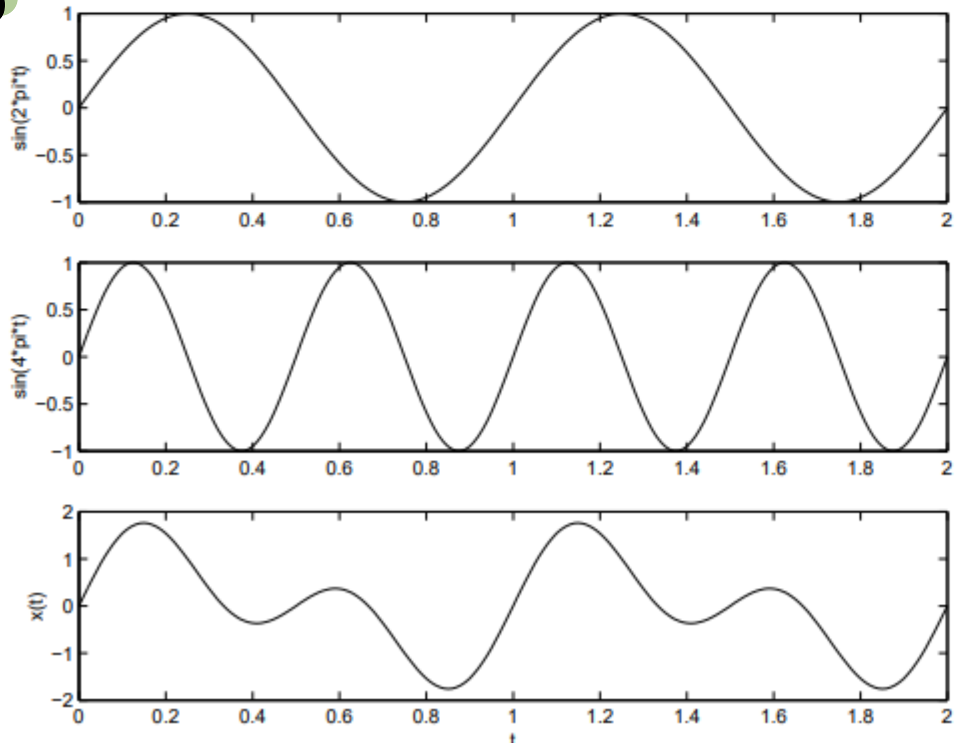
○ ω = omega
○ φ = phi

- General expression for a sinusoid at frequency ω (or frequency f in Hertz)

$$x(t) = a\sin(\underset{\text{Freq } \omega}{\omega} t + \varphi) = a\sin(\underbrace{2\pi f}_{\text{frequency in Hz}} t + \varphi)$$

- We can combine two sinusoids

$$x(t) = \sin(2\pi t) + \sin(4\pi t)$$



• Understand how a signal can be composed of base signals at different frequencies

$x(t) = a \sin(\omega t + \varphi) = a \sin(2\pi f t + \varphi)$

Frequency

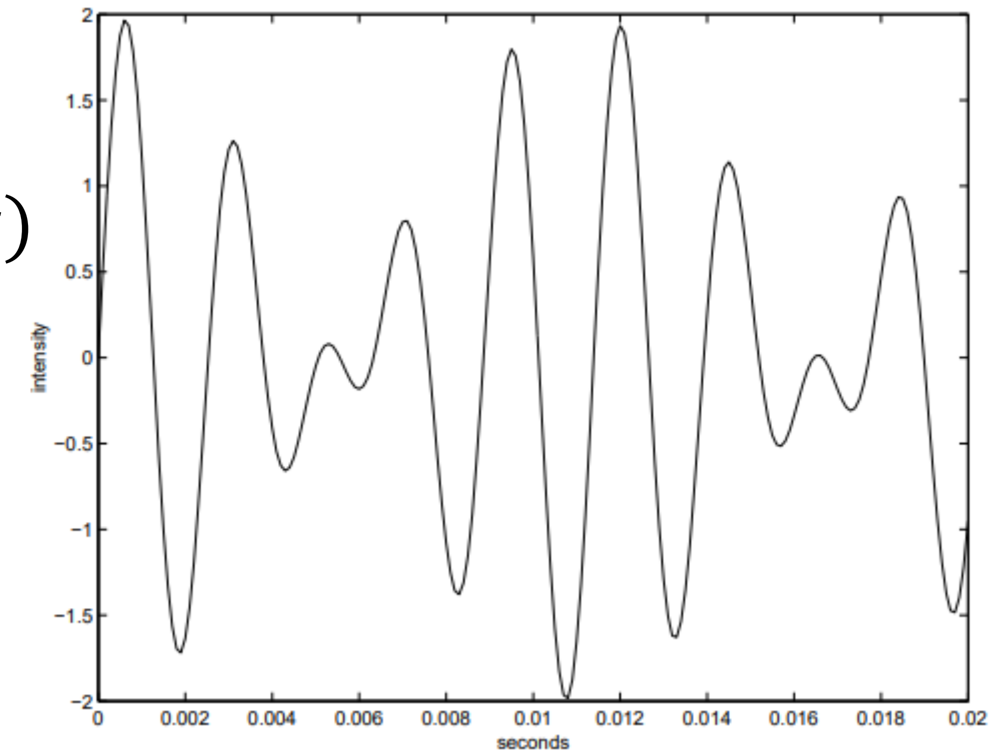
frequency f in Hertz

Frequency Domain Analysis

- Analogy with audio signals
- e.g. a dial tone of US telephony line

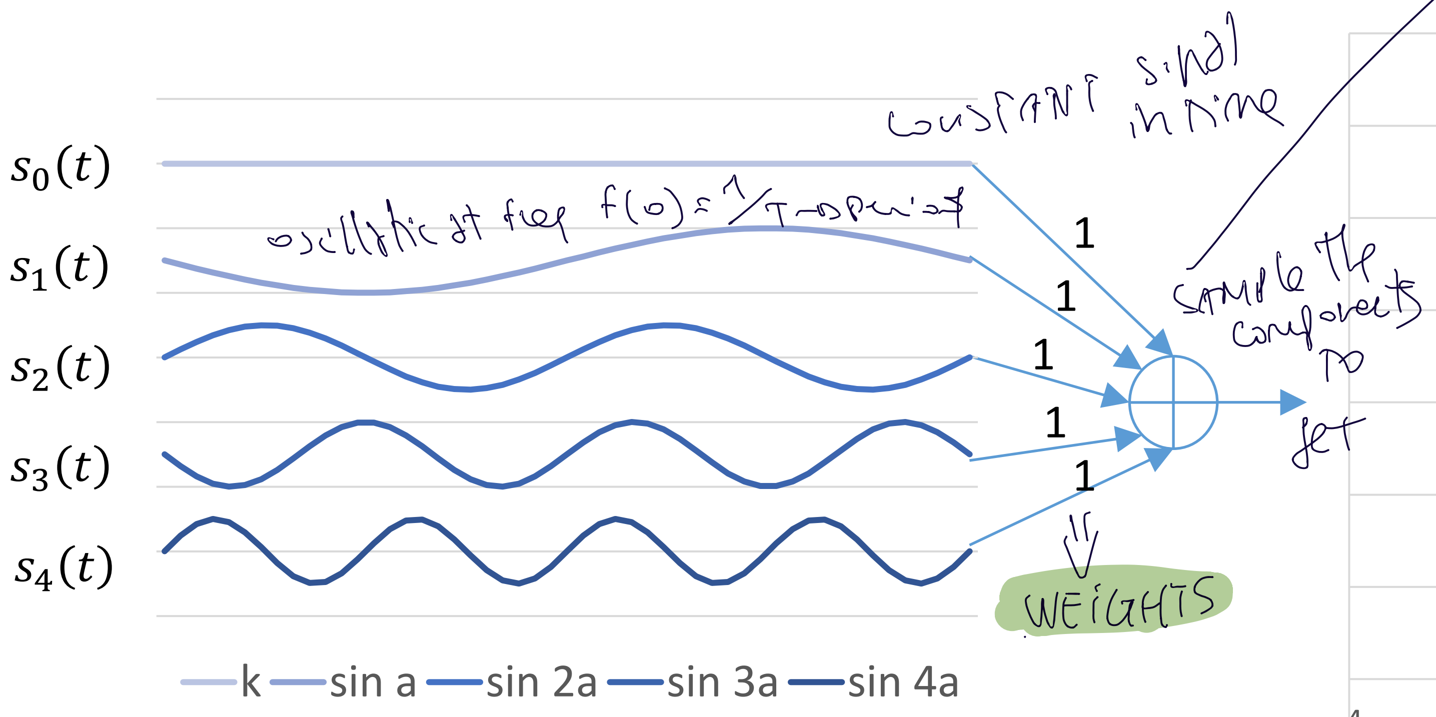
$$x(t) = \sin(2\pi * \underset{Hz}{\boxed{350}} * t) + \sin(2\pi * \underset{Hz}{\boxed{440}} * t)$$

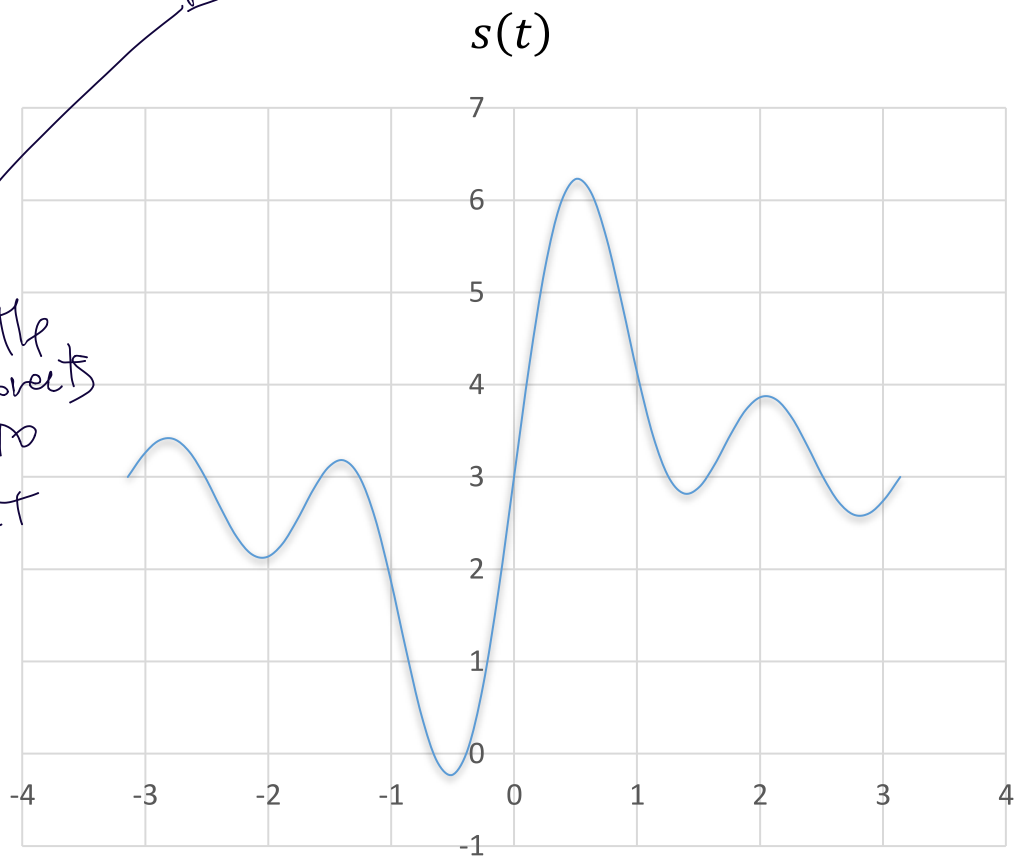
○ Sounds are oscillating at 2 ≠ frequency



Combine those signals to get a signal: CONTINUOS in TIME & AMPLITUDE:

Frequencies and signals





$s_0(t)$: constant component (frequency 0)

OSCILLATING AT

$s_1(t)$: fundamental (frequency $f_0 = 1/T$) FUNDAMENTAL FREQ.

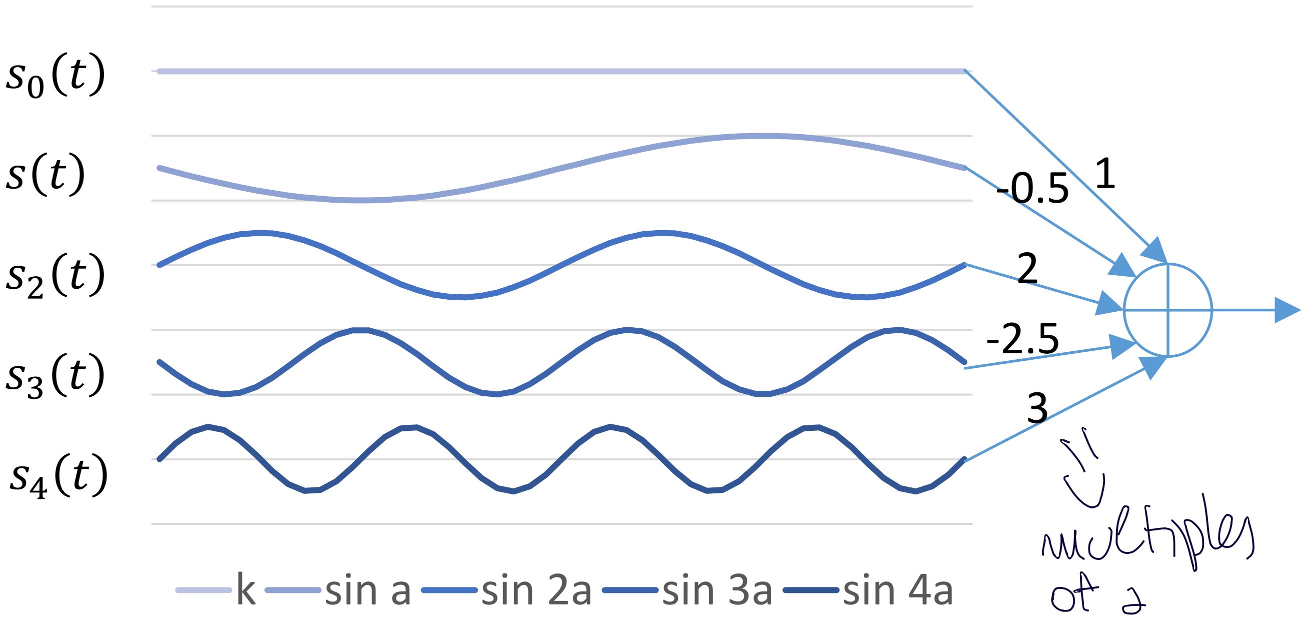
$s_2(t)$: second harmonic (frequency $f_1 = 2/T = 2f_0$) oscillating at multiple f_0

$s_3(t)$: third harmonic (frequency $f_2 = 3/T = 3f_0$)

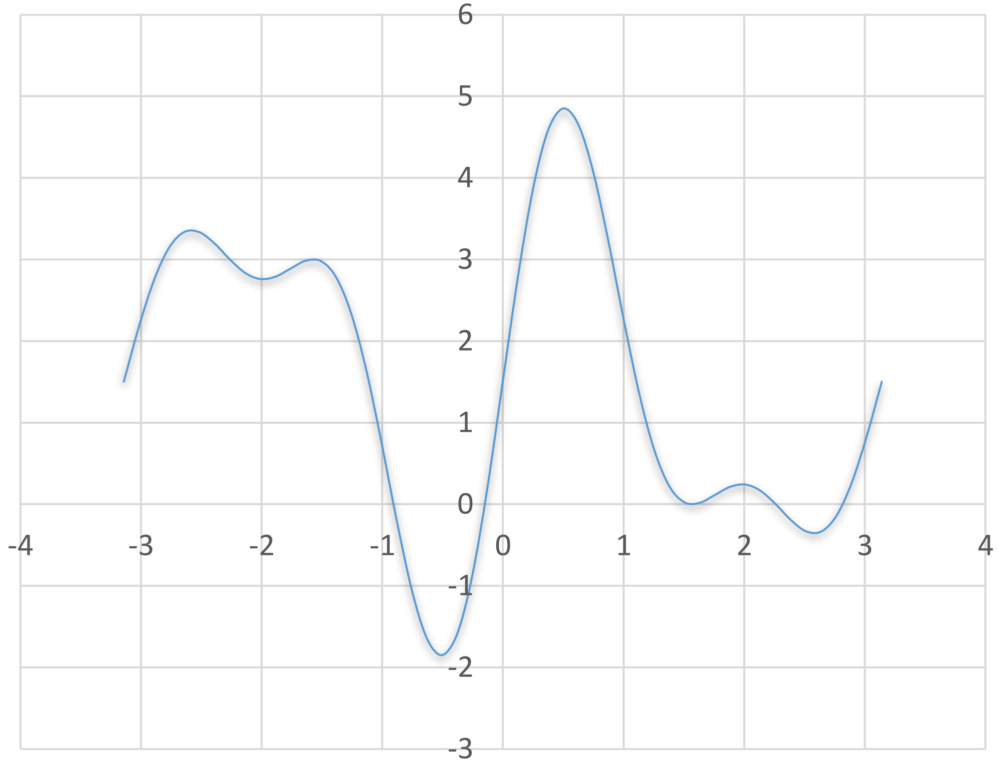
$s_4(t)$: fourth harmonic (frequency $f_3 = 4/T = 4f_0$) ARMONICS

Frequencies and signals

can change weights => modelling a different signal $\forall\, s(t)$

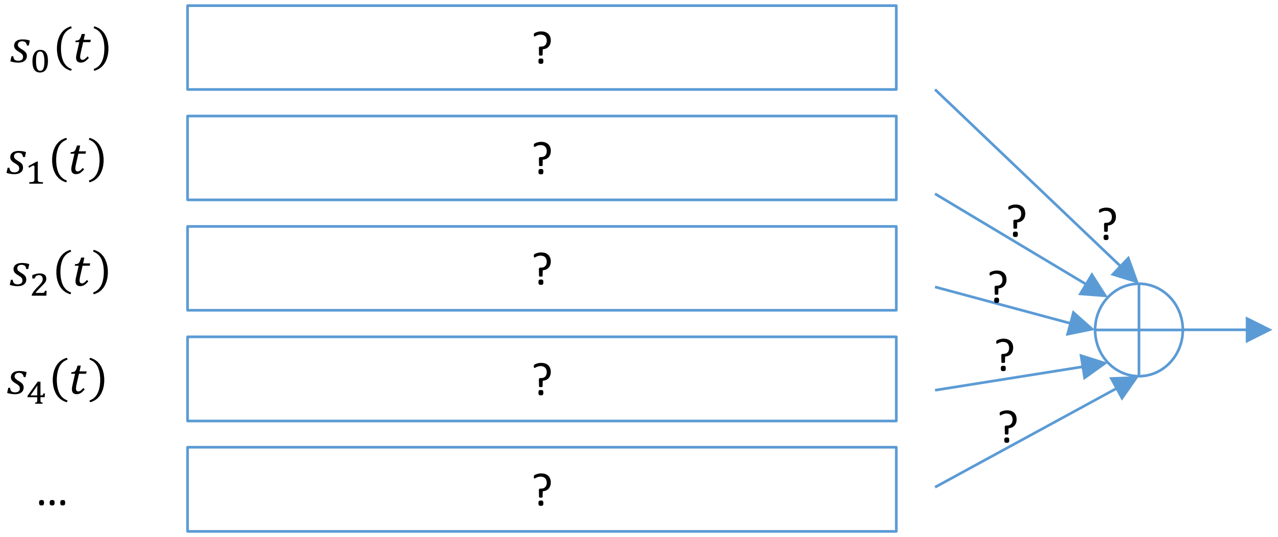


$\Downarrow$ multiples of a given frequency

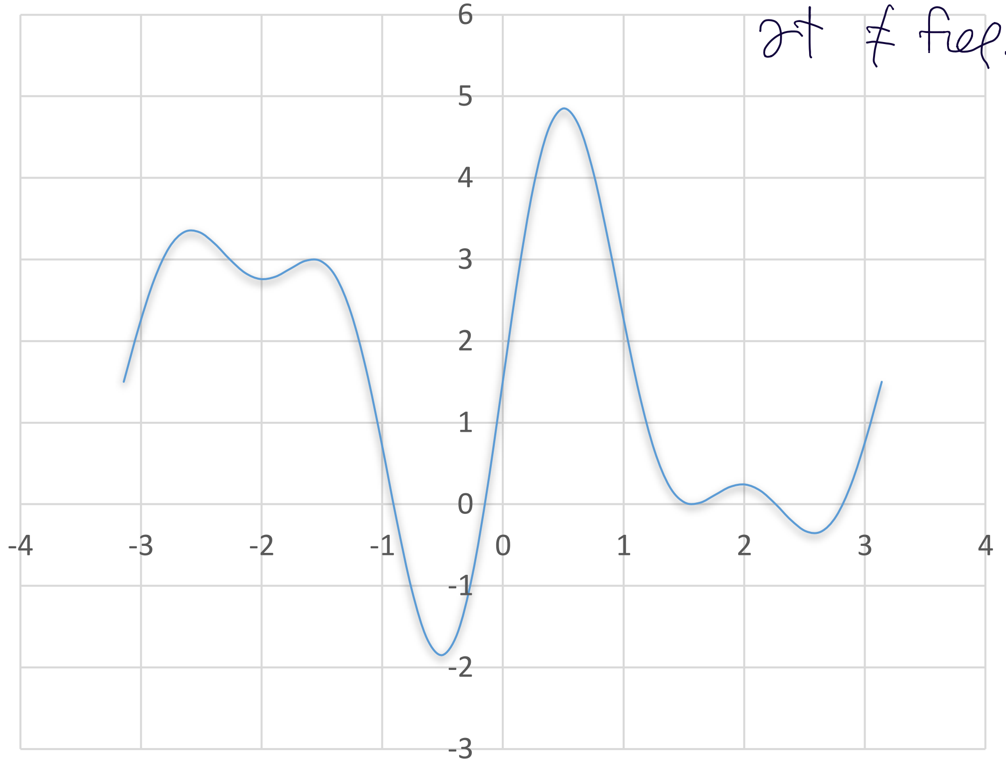


Is it possible to invert the process?

from sign continuous and periodic to get periodic f(x)s oscillating at ≠ freq.



$s(t)$



how compute weigths to obtain the signal?

↳ Fourier serie

=> COMPUTE WEIGHTS = compute the values which represent the contribution of each frequency to the resulting signal

Fourier series

GOAL!

- Decomposes a function (signal) as the **sum of an infinite number of continuous functions**, oscillating at **different frequencies**
  - Represents a change of coordinates: from time domain to frequency domain
- This set of continuous functions defines the base of decomposition
  - the Fourier series has a base represented by a set of functions $\varphi_n(t), n \in \mathbb{Z}$
  - this set of functions must be orthogonal (as in the case of decomposition of vectors in a vector space)

Fourier series

GOAL:

Decompose a signal as sum of ∞ number of continuous functions, oscillating at ≠ frequencies

• Domain = time • SHIFTING from time domain to frequency domain

Due to Joseph Fourier (around 1800)

Assumptions

Given a ***continuous*** signal $s(t)\colon \mathbb{R} \longrightarrow \mathbb{R}$, ***periodic*** in the interval $[-\pi, \pi]$ its Fourier series is defined as:

(from Real to Real numbers)

=> $\boxed{\text{period is } 2\pi}$

$$s(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Signal (function in time)

& CONVERGES IN $\mathbb{R}$

Can compute FT if signal is:

=> PERIODIC & PIECEWISE CONTINUOUS

↳ continuous with FINITE # of discontinuity

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\,dt;\;\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\cos nt\,dt;\;\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\sin nt\,dt$$

$S(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

where: $\frac{1}{2} a_0$: constant component

a_n, b_n: amplitude of armonics

$\cos nt$; $\sin nt$: armonics

Fourier series

: WAY to decompose a f(x) (a signal in time) as an ∞ sum of sinusoid and cosinusoid oscillating at some freq. => Fondamental = Free inverse of period of the signal armonic → others are multiple

Due to Joseph Fourier (around 1800)

Given a ***continuous*** signal $s(t)$: $\mathbb{R} \longrightarrow \mathbb{R}$, ***periodic*** in the interval $[-\pi, \pi]$ its Fourier series is defined as:

(does not oscillate)

$$s(t) = \boxed{\frac{1}{2} a_0} + \sum_{n=1}^{\infty} (\boxed{a_n}\,\boxed{\cos nt} + \boxed{b_n}\,\boxed{\sin nt})$$

∀ signal have to compute the weights (a_n, b_n)

Constant component

(not oscillating)

AVG value of this signal

Amplitude of the harmonics

Harmonics

oscillating at multiple freq.

where

For each signal, compute

a_0, a_n, b_n ⇓ values of the weights

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t)\, dt; \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \cos nt\, dt; \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \sin nt\, dt$$

$$a_0 = \frac{\int_{-\pi}^{\pi} s(t)\,dt}{\pi}$$

$$a_n = \frac{\int_{-\pi}^{\pi} s(t)\cos nt\,dt}{\pi}$$

$$b_n = \frac{\int_{-\pi}^{\pi} s(t)\sin nt\,dt}{\pi}$$

Fourier series

is continuos with finite numbers of discontinuity

if $s(t)$ is periodic and piecewise continuos

=> can compute Fourier series of $s(t)$
and converges in $\mathbb{R}$

It is rather complicate to assess the conditions under which an arbitrary $s(t)$ can be developed in a Fourier series

- In particular necessary conditions are not known
- However, there are sufficient conditions (Dirichlet theorem):

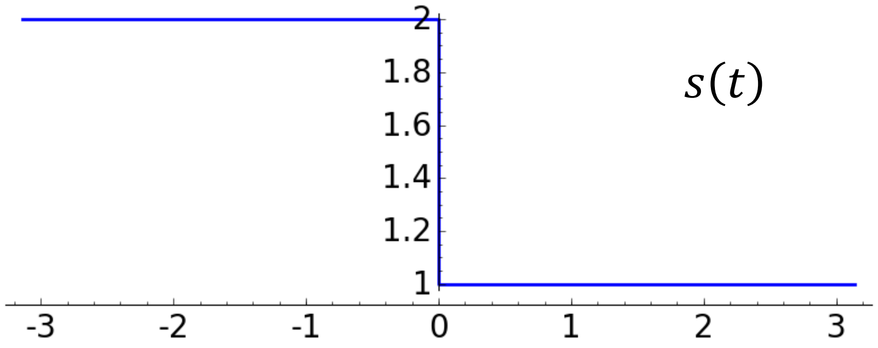
if $s(t)$ is periodic

and $s(t)$ is piecewise continuous

continous and have frequent ~~#~~ of discontinuity in time

Can compute $\Rightarrow$ the Fourier series of $s(t)$ exists and converges in $\mathbb{R}$

Example



Consider $s(t) = \begin{cases} 2 & if \ -\pi < t < 0 \\ 1 & if \ 0 \leq t \leq \pi \end{cases}$ (periodic, with period 2π)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\sin nt\,dt$$

- Given a CONTINOUS signal and PERIODIC in $[-\pi, \pi]$
  $\to s(t): \mathbb{R} \to \mathbb{R}$

- the Fourier series is:

$$s(t) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

where a_0: constant component $\Rightarrow$ not oscillating
a_n, b_n: amplitude of armonics
$\cos(nt), \sin(nt)$: armonics

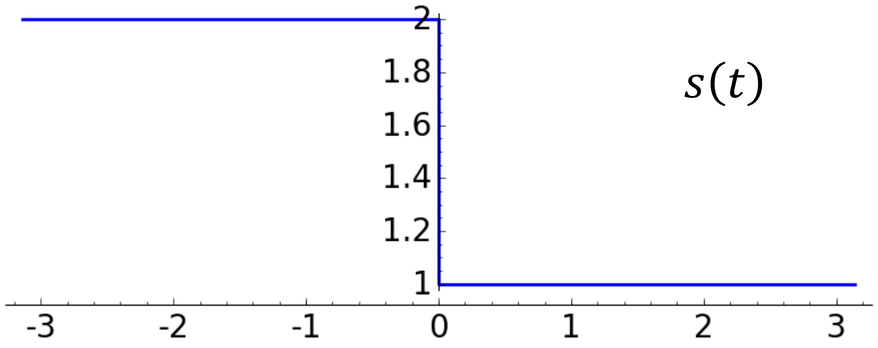
- and

$$a_0 = \int_{-\pi}^{\pi}\frac{s(t)\,dt}{\pi} \quad a_n = \int_{-\pi}^{\pi}\frac{s(t)\cos(nt)\,dt}{\pi}$$

$$b_n = \int_{-\pi}^{\pi}\frac{s(t)\sin(nt)\,dt}{\pi}$$

✓

Example



Consider $s(t) = \begin{cases} 2 & if\ -\pi < t < 0 \\ 1 & if\ 0 \leq t \leq \pi \end{cases}$ (periodic, with period 2π)

The coefficients of the Fourier series are:

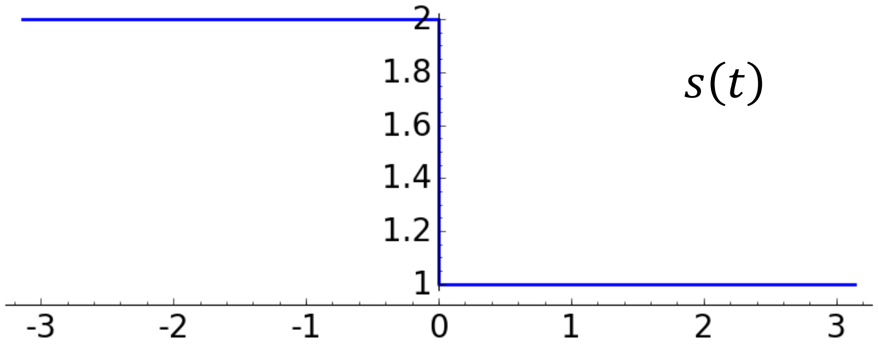
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\,dt + \int_{0}^{\pi} 1\,dt\right) = 3$$

$-(-\pi\cdot 2)$

$$\frac{1}{\pi}\left(\left[2x\right]_{-\pi}^{0} + \left[x\right]_{0}^{\pi}\right) = \frac{1}{\pi}\left(0 + 2\pi + \pi - 0\right) =$$

$$= \frac{1}{\cancel{\pi}}\cdot 3\cancel{\pi} = 3$$

Example



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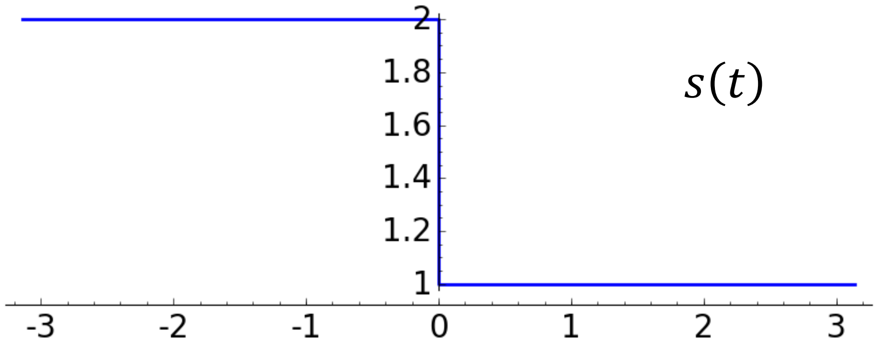
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \cos nt\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2 \cos nt\,dt + \int_{0}^{\pi} 1 \cos nt\,dt \right) = 0$$

Example



Consider $s(t) = \begin{cases} 2 & if\ -\pi < t < 0 \\ 1 & if\ 0 \le t \le \pi \end{cases}$ (periodic, with period 2π)

The coefficients of the Fourier series are:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\,dt + \int_{0}^{\pi} 1\,dt\right) = 3$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\cos nt\,dt + \int_{0}^{\pi} 1\cos nt\,dt\right) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\sin nt\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\sin nt\,dt + \int_{0}^{\pi} 1\sin nt\,dt\right) = \begin{cases} 0 & if\ n\ is\ even \\ -2/n\pi & if\ n\ is\ odd \end{cases}$$

Example

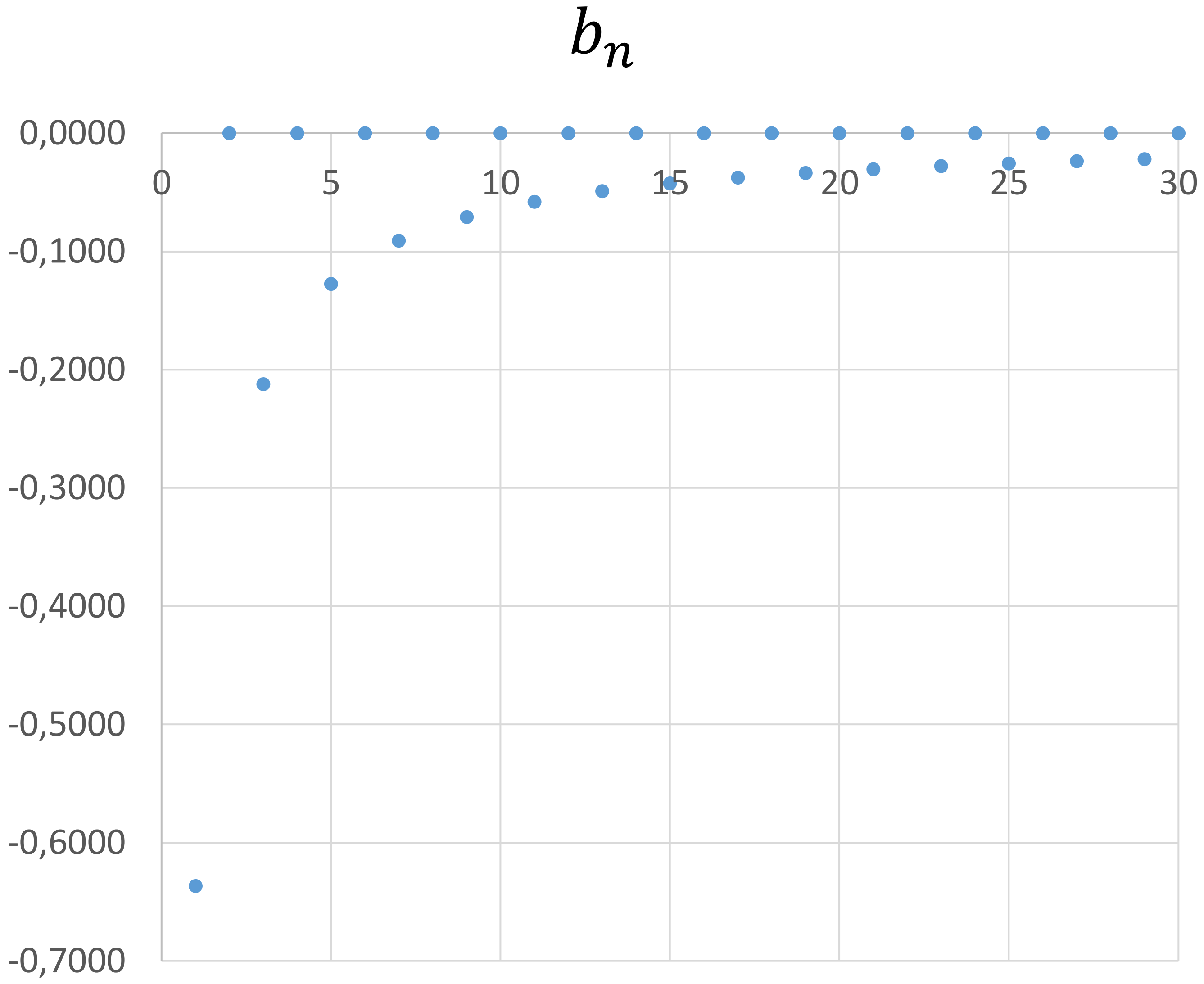
$$s(t) = \begin{cases} 2 & if\ -\pi < t < 0 \\ 1 & if\ 0 \le t \le \pi \end{cases}$$

The coefficients are:

$a_0 = 3;$

$a_n = 0;$

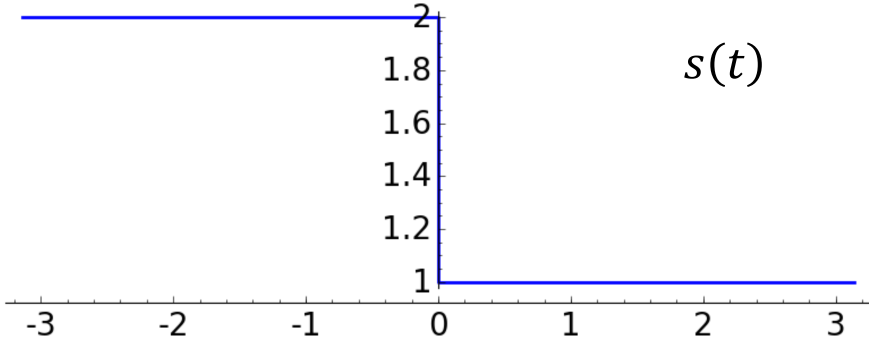
$$b_n = \begin{cases} 0 & if\ n\ is\ even \\ -\frac{2}{n\pi} & if\ n\ is\ odd \end{cases}$$



Theorem

Decompose the signal

$$s(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$



Example

Hence, letting $n = 2k - 1$ for all $k > 0$:

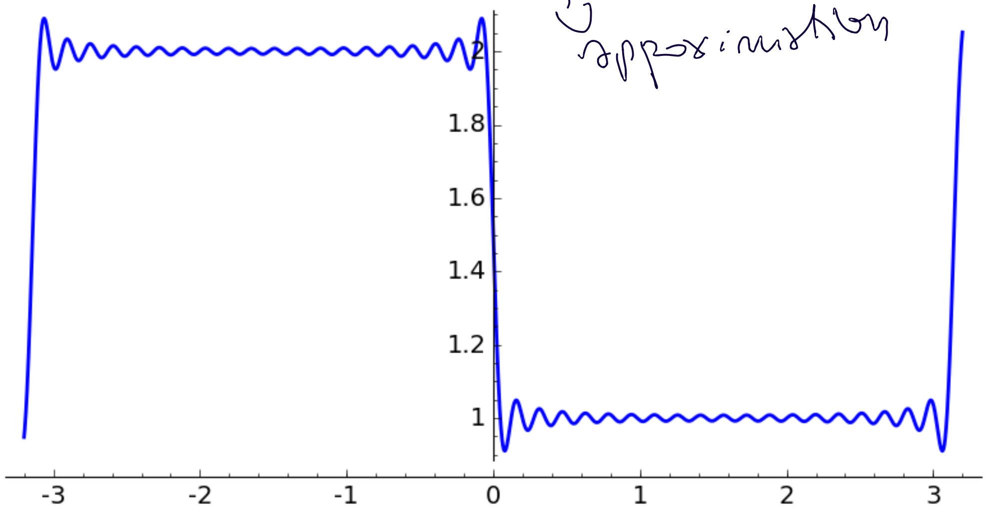
If we sum just the sum of a threshold

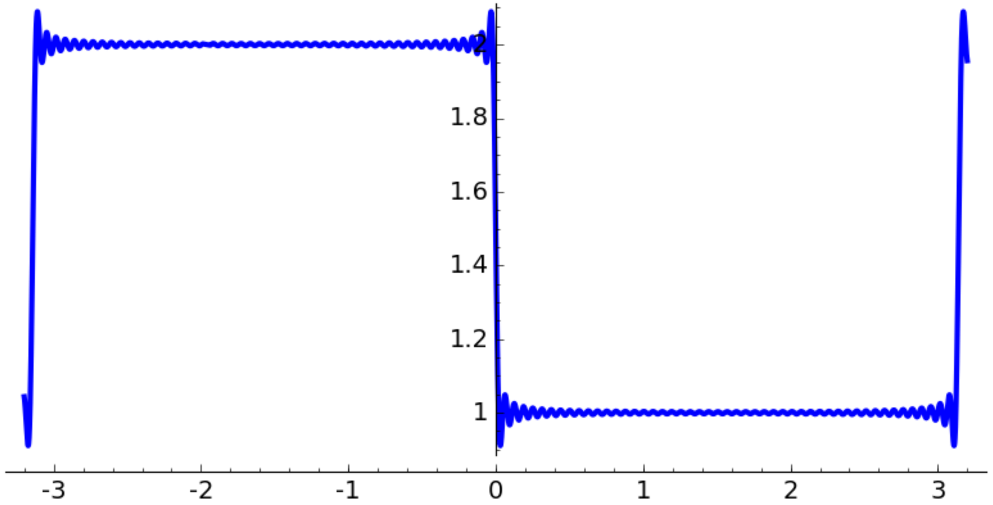
$$s(t) = \frac{3}{2} - \sum_{k=1}^{\infty}\left(\frac{2}{(2k-1)\pi}\sin((2k-1)t)\right)$$

Plot of the first 20 harmonics

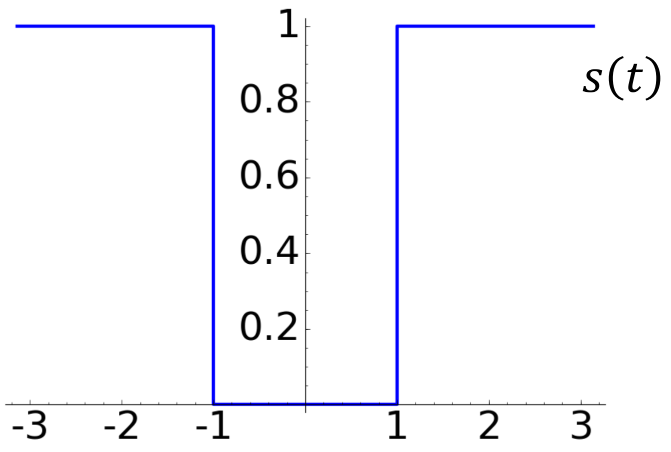
approximation

Plot of the first 50 harmonics (components)





Exercise



Consider $s(t) = \begin{cases} 1 & if \ -\pi < t < -1 \\ 0 & if \ -1 \le t < 1 \\ 1 & if \ 1 \le t \le \pi \end{cases}$

periodic, with period 2π

Compute the Fourier series of $s(t)$

Remember:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t)\, dt;$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \cos nt\, dt;$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \sin nt\, dt$$

Hint:

$$\int \sin(t)\, dt = -\cos(t)$$

$$\int \cos(t)\, dt = \sin(t)$$

$$\int \cos(nt)\, dt = \frac{\sin(nt)}{n}$$

$$s(t) = \boxed{\frac{1}{2}a_0} + \sum_{n=1}^{\infty}\left(\boxed{a_n}\boxed{\cos nt} + \boxed{b_n}\boxed{\sin nt}\right)$$

Constant component (does not oscillate)

Amplitude of the harmonics

Harmonics

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\,dt;\ \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\cos nt\,dt;\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\sin nt\,dt$$

or

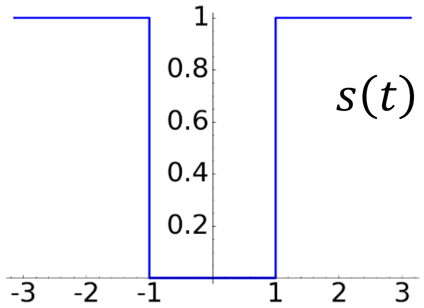
- $a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{-1} 1\,dt + \int_{-1}^{1} 0\,dt + \int_{1}^{\pi} 1\right)$ ✓

- $a_n = \frac{1}{\pi}\left(\int_{-\pi}^{-1} 1\cos nt\,dt + \int_{-1}^{1} 0\cos nt\,dt + \int_{1}^{\pi} 1\cos nt\,dt\right)$ ✓

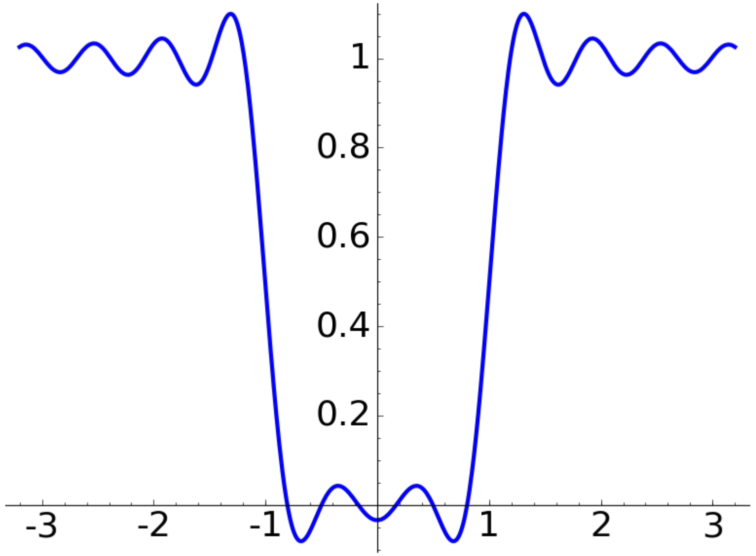
- $b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-1} 1\sin nt + \int_{-1}^{1} 0\sin nt + \int_{1}^{\pi} 1\sin nt\right]$ ✓

Solution

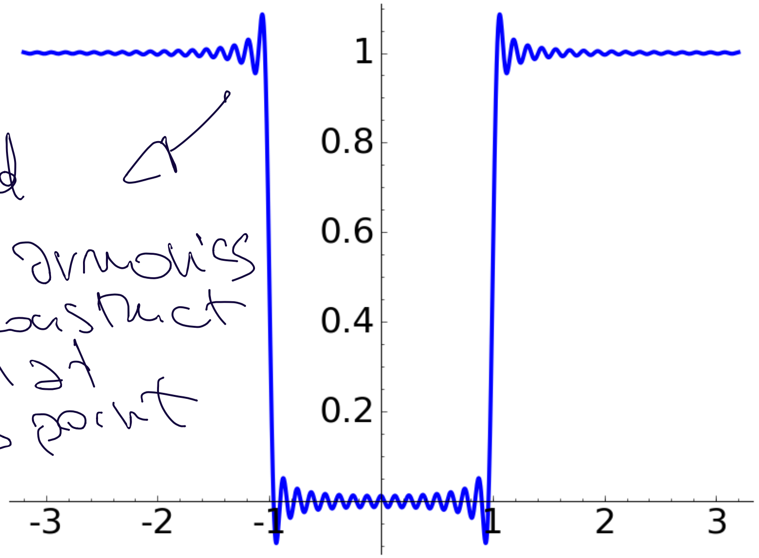
$$s(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$



Plot of the first 10 harmonics



Plot of the first 50 harmonics



need more armonics to re-construct the signal at this point

$2 + \frac{T}{2} = \frac{4+T}{2} = \frac{\cancel{4}^{2}T}{\cancel{2}_{1}} = 2T$

Fourier series with arbitrary period

continuos signal

⇓

can apply also Fourier to periodic signals with arbitrary period

The Fourier series is defined also for signals with arbitrary period:

- Given a continuous signal $s(t)\colon \mathbb{R} \longrightarrow \mathbb{R}$, periodic in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$
- Using the substitution $y = \frac{2\pi t}{T}$ $\Rightarrow$ $\frac{T}{2\pi} \cdot y = \frac{\cancel{2\pi} t}{\cancel{T}}$ $\quad \frac{\cancel{T}}{\cancel{2\pi}}$

$f(y) = s\left(\frac{T}{2\pi} y\right)$ is periodic in the interval $[-\pi, \pi]$ $\quad t = \frac{T}{2\pi} \cdot y$

$$\boxed{f(y) = s\left(\frac{T}{2\pi} y\right) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny)}$$

Returning to the initial variable (i.e., $y = \frac{2\pi}{T} t$)

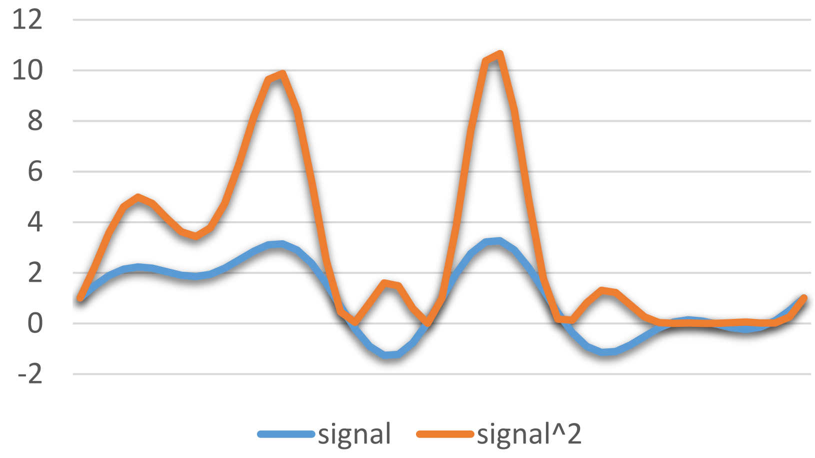
$$s(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right)$$

with

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} s(t)\, dt; \quad a_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t) \cos \frac{2\pi n t}{T} dt; \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t) \sin \frac{2\pi n t}{T} dt$$

period

Energy of a continuous signal



Given a signal $s(t)$ defined in the interval $[-\frac{T}{2}, \frac{T}{2}]$
$[-T/2, T/2]$, the ***energy*** of the signal is defined as:

Integral of the square of the signal

$$E_s(T) \triangleq \int_{-\frac{T}{2}}^{\frac{T}{2}} |s(t)|^2 dt$$

energy of the signal

this also has a meaning in physics: if $s(t)$ represents the voltage applied to a 1Ω resistor, $E_s(T)$ is the energy dissipated in the period T

A signal is with finite energy (energy signal) if the limit: energy signal

when computing the limit of the integral of the square of the signal

$$E_s = \lim_{T\to\infty} E_s(T) = \int_{-\infty}^{\infty} |s(t)|^2 dt > 0$$

$$\&\& \quad E_s < +\infty$$

⇒ End if is a finite # ~~∞~~

SIGNAL WITH FINITE ENERGY

- this includes signals with either finite or infinite duration
- if the signal has infinite duration, then $E_s \to 0$ as fast as $1/t$ (or faster)
- in the physical world, all signals have finite energy (no signal lasts forever…)

Real world:

Signal with fined Duration has FINITE ENERGY

(limited duration, cont)

• Power over a period T => if $\lim_{T\to\infty} \frac{\int_{-T/2}^{T/2} (s(t))^2}{T}$ so

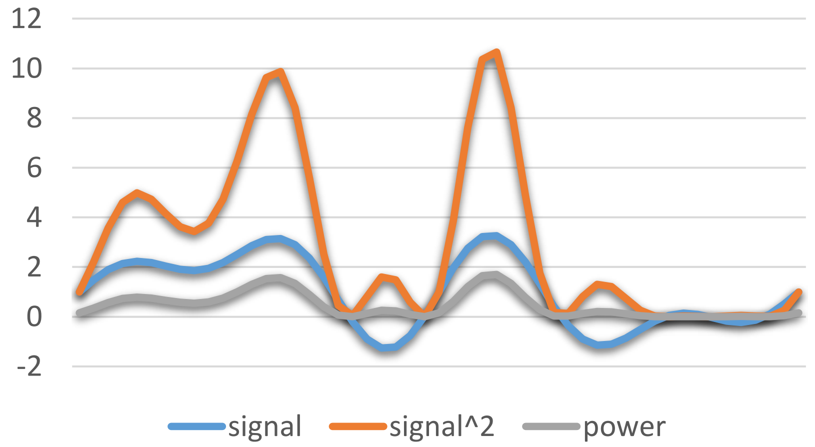
&& FINITE

Power of a signal

Given a signal $s(t)$ defined in the interval $[-T/2, T/2]$, the average power of the signal is defined as: the AVG of the energy over the period T

$$P_f(T) \triangleq \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |s(t)|^2 dt = \boxed{\frac{E_s(T)}{T}}$$

AVG



A signal is with finite power (*power signal*) if the limit:

$$P_s = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |s(t)|^2 dt > 0$$

$$\&\& \quad P_s < +\infty$$

PERIODIC SIGNALS

Periodic signals are an important class of signals with finite power: and ∞ energy

- they have infinite energy, BUT CONSTANT POWER
- their average power equals the average power computed in a period

=> for a periodic signal, can compute the power (same computing $T \to \infty$ || T = period of signal) and is finito

=> AVG Power: AVG power computed in a period

Energy and power

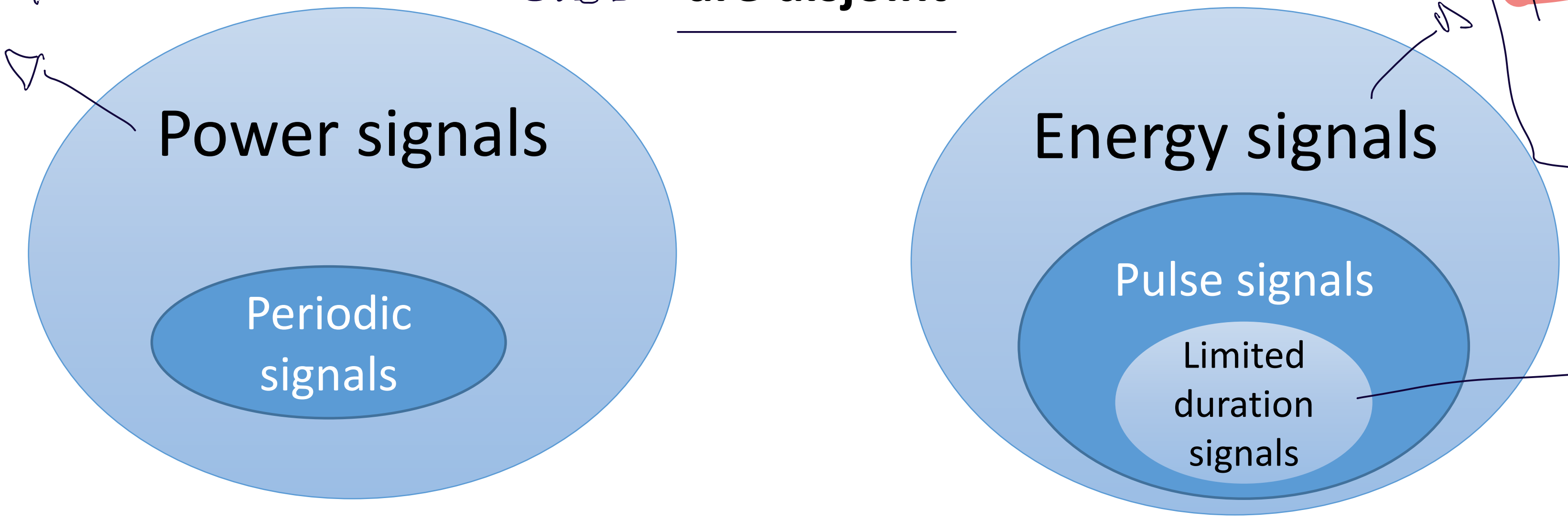
If a signal has finite energy its average power is zero, hence the classes of signals:

- with finite energy (energy signals)
- with finite average power (power signals) > 0

Ener is FINITE >0

Classes **are disjoint**

power => 0 but finite energy



real world signals

0 < Power < +∞

>0 and finite

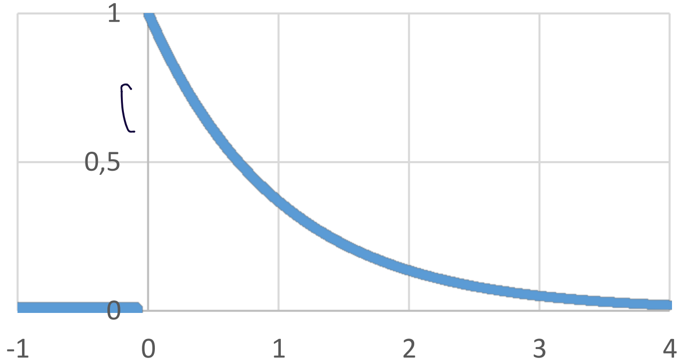
Unlike Power=0 but have finite energy

Examples

Exp signal

The exponential signal:

$$s(t) = \begin{cases} 0 & \forall t < 0 \\ ae^{-bt} & \forall t \geq 0 \end{cases}$$



Has finite energy:

$$E_s = \int_0^\infty |s(t)|^2 dt = \frac{a^2}{2b} < \infty$$

Has null average power:

$$P_s = \lim_{T\to\infty} \frac{1}{T} \int_0^{\frac{T}{2}} |s(t)|^2 dt = \lim_{T\to\infty} \frac{a^2}{2bT} = 0$$

⇒ finite energy

The periodic signal: $s(t) = \cos(t) \quad \forall t$

periodic in period 2π

Has infinite energy and finite average power:

$$P_s = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t) dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} dt + \underbrace{\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{2} \cos(2t)\, dt}_{\text{null}} = \frac{1}{2}$$

Question

ENERGY => FINITE DURATION SIGNAL

POWER SIGNAL => periodic signal

Classify the following signals as Energy and Power signals and explain why

Power

Periodic signal

- $s(t) = \cos(t)$ Power signal cuz is periodic => power is periodic
- $s(t) = \begin{cases} 3 & -2 < t < -1 \\ 0 & otherwise \end{cases}$ => energy signal => final duration
- $s(t) = \begin{cases} \frac{3}{t} & 1 < t < 10 \\ 0 & otherwise \end{cases}$ => energy => final duration

  energy is a final value