Assignment 3 - Demonstrating Understanding of SAT/SMT Solvers

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Question 1: Translating Facts into Propositional Phrases

Suppose we have a number of teachers and a number of courses and we want to assign the courses to the teachers. Let $T = \{T_1, \ldots, T_n\}$ be a set of teachers and let $C = \{C_1, \ldots, C_m\}$ be a set of courses. Each course $c \in C$ is taught by a set $T(c)$ of the teachers ($T(c) \subset T$).

Let k (such that $k \leq n$) be a given natural number. We need to know if we can hire at most k teachers to cover all the m courses. We define the following boolean variables:

- Teachers t_i for $i = 1, \ldots, n$, where t_i is true if and only if the teacher is hired.
- Courses c_j for $j = 1, \ldots, m$ where c_j is true if and only if there is a teacher that teaches c_j.

Consider the following points in answering Questions 1.1 to 1.4:

- You can only use the variables t_i for $i = 1, \ldots, n$ and c_j for $j = 1, \ldots, m$ in your answers to questions Questions 1.1 to 1.4.
- The answer to Questions 1.1 to 1.4 are supposed to be written in terms of SAT and SMT formulas. No English words are required to explain the answers. This part of the assignment is marked based on the correctness of the SAT/SMT formulas you have written and English explanations will be ignored.
- You can use the following function in your answer to Questions 1.1 to 1.4. Let $l \leq n$ be a given natural number. We define:

$$subsets(T, l) = \{T' \subseteq T \mid |T'| = l\}$$

  where $|T|$ denotes the cardinality of a set. So the function $subsets(T, l)$ denotes all subsets of T that include l elements.

Screenshot from Assignment Description

Question 1.1

A constraint (formula) that is true if and only if all the courses in C are taught by at least one teacher:

$$(c_1 \land c_2 \land \ldots \land c_m) = \bigwedge_{j=1}^{m} c_j$$

Question 1.2

A constraint (formula) that is true if and only if at most k teachers are hired:

$$\bigwedge_{S \in \text{subsets}(T, k+1)} \left(\bigvee_{i \in S} \neg t_i \right)$$

Question 1.3

A constraint (formula) that is true if a course is covered only if at least one teacher who teaches it is hired:

$$\bigwedge_{j=1}^{m} \left(\neg c_j \lor \bigvee_{i \in T(c_j)} t_i \right)$$

Question 1.4

A constraint (formula) that is true for when we want at least k teachers to be hired is:

$$\bigwedge_{S \in \text{subsets}(T, n-k+1)} \left(\bigvee_{i \in S} t_i \right)$$

For exactly k teachers the constraint would be:

$$\left(\bigwedge_{S \in \text{subsets}(T, k+1)} \left(\bigvee_{i \in S} \neg t_i \right) \right) \wedge \left(\bigwedge_{S \in \text{subsets}(T, n-k+1)} \left(\bigvee_{i \in S} t_i \right) \right)$$

Question 2: Tseitin's Transformation

Given $p \vee (q \wedge ((\neg p \wedge \neg r) \vee (p \wedge r)))$ we declare the auxiliary variables and derive the CNF from the *iff* clauses:
Recall that $A \iff B \equiv (\neg A \vee B) \wedge (\neg B \vee A)$

Variable	Definition	CNF Derivation
e_1	$\neg p$	$e_1 \iff \neg p$ $\equiv (e_1 \implies \neg p) \wedge (\neg p \implies e_1)$ $\equiv (\neg e_1 \vee \neg p) \wedge (p \vee e_1)$
e_2	$\neg r$	$e_2 \iff \neg r$ $\equiv (\neg e_2 \vee \neg r) \wedge (r \vee e_2)$
e_3	$e_1 \wedge e_2$	$e_3 \iff e_1 \wedge e_2$ $\equiv (e_3 \implies (e_1 \wedge e_2)) \wedge ((e_1 \wedge e_2) \implies e_3)$ $\equiv (\neg e_3 \vee e_1) \wedge (\neg e_3 \vee e_2) \wedge (\neg(e_1 \wedge e_2) \vee e_3)$ $\equiv (\neg e_3 \vee e_1) \wedge (\neg e_3 \vee e_2) \wedge (\neg e_1 \vee \neg e_2 \vee e_3)$
e_4	$p \wedge r$	$e_4 \iff p \wedge r$ $\equiv (\neg e_4 \vee p) \wedge (\neg e_4 \vee r) \wedge (\neg p \vee \neg r \vee e_4)$
e_5	$e_3 \vee e_4$	$e_5 \iff e_3 \vee e_4$ $\equiv (e_5 \implies (e_3 \vee e_4)) \wedge ((e_3 \vee e_4) \implies e_5)$ $\equiv (\neg e_5 \vee e_3 \vee e_4) \wedge ((\neg e_3 \wedge \neg e_4) \vee e_5)$ $\equiv (\neg e_5 \vee e_3 \vee e_4) \wedge (\neg e_3 \vee e_5) \wedge (\neg e_4 \vee e_5)$
e_6	$q \wedge e_5$	$e_6 \iff q \wedge e_5$ $\equiv (\neg e_6 \vee q) \wedge (\neg e_6 \vee e_5) \wedge (\neg q \vee \neg e_5 \vee e_6)$
e_7 (root)	$p \vee e_6$	$e_7 \iff p \vee e_6$ $\equiv (\neg e_7 \vee p \vee e_6) \wedge (\neg p \vee e_7) \wedge (\neg e_6 \vee e_7)$

To complete the transformation, we conjoin all definition clauses:

$$e_7 \wedge \bigwedge_{i=1}^{7} \text{CNF}(e_i \iff \text{def}_i)$$

Question 3

This question is to determine whether given Boolean clauses are SAT (satisfiable) or UNSAT.

Statement: $p \vee (q \wedge r)$

In a disjunctive statement, we only have to satisfy either clause.

Case 1: when p = True No matter what boolean outcome $q \wedge r$ evaluates, the entire statement is satisfied (SAT).

Case 2: When q = True $q \wedge r$ evaluates True. No matter what value the literal p is, the statement is True (SAT).

Case 3: p = False Conversely, if p itself is False, as long as either q or r is True, it becomes the consideration that we encountered in case 2, which tells us that it is SAT.

Conclusion This statement, $p \vee (q \wedge r)$, is satisfiable. $\square$

Statement: $\neg((p \wedge (p \rightarrow r)) \rightarrow r)$

The outer negation implies $(p \wedge (p \rightarrow r)) \rightarrow r$ to be False, in order to evaluate the entire statement to be True, and hence, satisfied. We must recall the truth table for implications, so we know how $(p \wedge (p \rightarrow r) \rightarrow r)$ evaluates True.

The Truth Table

Given 2 literals, each of which forms a clause itself, these are all the possible outcomes:

P	$\rightarrow$	Q	Outcome
True		True	True
True		False	False
False		True	True
False		False	True

Applying the 2nd truth scenario Only the second scenario gives a Falsy outcome. If we apply the 2nd truth scenario into our sub-statement $(p \wedge (p \rightarrow r)) \rightarrow r)$, it gives us the following information:

- $p \wedge (p \rightarrow r))$ is True,
- r is False.

SAT Possibility In order to make $p \wedge (p \rightarrow r))$ True, we need both p and $(p \rightarrow r)$ to be True. In that case, p must be True such that the clause with p itself is True in the statement. But based on the truth table, if r is known to be False, $(p \rightarrow r)$ gives False. So, the entire statement $p \wedge (p \rightarrow r)$ will never be True.

Conclusion Conversely, if we apply other truth scenarios to $(p \wedge (p \rightarrow r)) \rightarrow r$, it will give True. However, given the negation in the original statement, this makes $\neg((p \wedge (p \rightarrow r)) \rightarrow r)$ always Falsy, and unsatisfiable (UNSAT). $\square$

Statement: $\neg(p \wedge ((p \rightarrow q) \rightarrow q))$

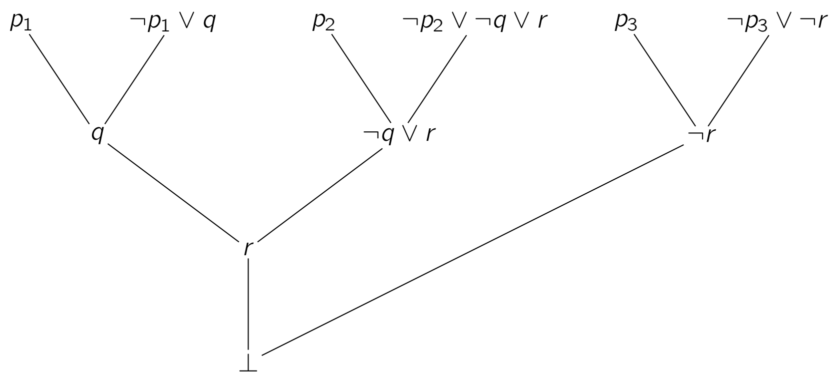
Similarly, it is overall negated, so we aim to make the sub-statement $p \wedge ((p \rightarrow q) \rightarrow q)$ False, such that after negation it becomes True. In order to make this conjunctive statement False, we need either p or $((p \rightarrow q) \rightarrow q)$ to be False.

Conclusion As we already found a case where p = False, satisfying the overall negated statement to be True, it is SAT. $\square$

Question 4

Construct a resolution refutation graph for the following unsatisfiable formula:

$$p_1 \wedge p_2 \wedge p_3 \wedge (\neg p_1 \vee q) \wedge (\neg p_2 \vee \neg q \vee r) \wedge (\neg p_3 \vee \neg r)$$



Conclusion Since the graph ends up with an empty clause, by eliminating absolutely impossible clauses like $a \wedge \neg a$, it is proven unsatisfiable (UNSAT). $\square$

Question 5

Apply the congruence closure algorithm to decide the satisfiability of the following formula:

$$f(g(x) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$$

Listing all Subterms

From the complex formula, we have:

- x
- y
- $f(x)$
- $g(x)$
- $f(y)$
- $f(g(x)$
- $g(f(x))$
- $g(f(y))$
- $f(g(f(y)))$

Applying the Given Equalities

From the formula, we know the following separate facts:

- $f(g(x) = g(f(x))$
- $f(g(f(y))) = x$
- $f(y) = x$

Applying Congruence Closure

Let C_j be a clause determined from the above equalities. By merging those equalities, we should have a list of different clauses which is formed by the union of terms from known equality facts:

- $C_1 = f(g(x), g(f(x))$
- $C_2 = f(g(f(y))), x$
- $C_3 = f(y), x$

Merging Clauses From clauses C_2 and C_3, we also got a new clause $C_4 = f(g(f(y))), f(y), x$ through x.

Congruence Propagation

Generally, given any equalities with functions, if arguments are equal, their function results must be equal. Since x is a simple literal, let's propagate the resulting congruences with substitution:

- $C_5 = g(x), g(f(y))$
- $C_6 = g(f(x)), g(f(f(y))), f(g(x), f(g(f(y))$

Connecting Terms with the simple literal Finally, connecting C_2 C_4 with C_6, we have: $g(f(x)), g(f(f(y))), f(g(x), f(g(f(y))),$

Applying the Given Inequalities

Conversely, we also know the following inequalities from the given formula:

- $g(f(x)) \neq x$

Conclusion

From the equalities, we derived $g(x) = x$ to be true. However, the inequalities states the opposite. Therefore, we can conclude that, this statement, $f(g(x) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$, is unsatisfiable (UNSAT). $\square$