Optimal Behaviour of Regulated Firms in Solar Renewable Energy Certificate (SREC) Markets*

Arvind Shrivats and Sebastian Jaimungal†

Abstract. SREC markets are a relatively novel market-based system to incentivize the production of energy from solar means. A regulator imposes a floor on the amount of energy each regulated firm must generate from solar power in a given period, providing them with certificates for each generated MWh. Firms offset these certificates against the floor, paying a penalty for any lacking certificates. Certificates are tradable assets, allowing firms to purchase / sell them freely. In this work, we formulate a stochastic control problem for generating and trading in SREC markets for a regulated firm and discuss potential takeaways for both a regulated firm under this system and regulatory bodies in charge of designing them.

Key words. Commodity Markets, Stochastic Control, SREC, Cap and Trade, Market Design

AMS subject classifications. 37H10, 49L20, 39A14, 91G80

1. Introduction.

As the impacts of climate change continue to be felt worldwide, policies to reduce greenhouse gas emissions and promote renewable energy generation will increase in importance. One policy approach which encapsulates many policies is market-based solutions. The most well-known of the policies which fall under this umbrella are carbon cap-and-trade (C&T) markets.

In carbon C&T markets, regulators impose a limit on the amount of carbon dioxide (CO_2) that regulated firms can emit during a certain time period (referred to as a compliance period). They also distribute allowances (credits) to individual firms in the amount of this limit, each allowing for a unit of CO_2 emission, usually one tonne. Firms must offset each of their units of emissions with an allowance, or face a monetary penalty for each allowance they are lacking. These allowances are tradable assets, allowing firms who require more credits than what they were allocated to buy them, and firms who require less to sell them. In this way, C&T markets aim to find an efficient way of allocating the costs of CO_2 abatement across the regulated firms.

In practice, these systems regulate multiple consecutive and disjoint compliance periods, which are linked together through mechanisms such as *banking*, where unused allowances in period n can be carried over to period $n+1$. Other linking mechanisms include *borrowing* from future periods (where a firm may reduce its allotment of allowances in period $n+1$ in order to use them in period n) and *withdrawal*, where non-compliance in period n reduces period $n+1$ allowances by the amount of non-compliance (in addition to the monetary penalty mentioned above).

A closely related alternative to these cap-and-trade markets are *renewable energy certificate* markets (REC markets). A regulator sets a floor on the amount of energy generated from

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†Department of Statistical Sciences, University of Toronto, Toronto, ON (shrivats@utstat.toronto.edu,sebastian.jaimungal@utoronto.ca).

renewable sources for each firm (based on a percentage of their total energy generation), and provides certificates for each MWh of energy produced via these means[1]. To ensure compliance, they each firm must surrender certificates totaling the floor at the end of each compliance period, with a monetary penalty paid for each lacking certificate. The certificates are traded assets, allowing regulated LSEs to make a choice about whether to produce electricity from renewable means themselves, or purchase the certificates on the market (or a mix of both). A natural and interesting question is to determine the optimal behaviour for regulated firms.

REC markets can be used to encourage growth of a particular type of renewable energy. The most notable of these systems are Solar REC markets (SREC markets), which have been implemented in many areas of the northeastern United States[2], and are the focus of this work.

The similarities between SREC markets and carbon cap-and-trade markets are clear. However, there are also some notable differences. One key difference between the SREC market and traditional carbon cap-and-trade markets is the uncertainty in the former market is the supply of certificates (driven by some generation process), while in the latter, the uncertainty is in the demand for allowances (driven by an emissions process). In SREC markets, banking is implemented, but borrowing and withdrawal are not. Broadly speaking, SREC markets can be considered the inverse of a cap-and-trade system.

Prior works on SREC markets have focused on certificate price formation. [9] present a stochastic model for SREC generation, calibrate it to the New Jersey SREC market, and ultimately solve for the certificate price as a function of economy-wide generation capacity and banked SRECs, as well as investigating the role and impact of regulatory parameters on these markets. [13] study an alternate design scheme for SREC markets and show that it stabilizes SREC prices. The volatility of REC prices has been noted in other works, such as [3] and [12]. The latter focuses on the Swedish-Norwegian electricity certificate market and develops a stochastic model to analyze price dynamics and policy.

While literature in SREC markets is fairly limited, there is a great deal of work that has been done in carbon cap-and-trade markets, particularly in developing stochastic equilibrium models for emissions markets. [10] present a general stochastic framework for firm behaviour leading to the expression of allowance price as a strip of European binary options written on economy-wide emissions. Agents' optimal strategies and properties of allowance prices are also studied by [7] and [17] within a single compliance period setup, with the former also making significant contributions through detailed analyses of potential shortcomings of these markets and their alternatives. [6] also proposes a stochastic equilibrium model to explain allowance price formation and develop a model where abatement (switching from less green to more green fuel sources) costs are stochastic. There is also significant work on structural models for financial instruments in emissions markets, such as [11] and [5].

As noted earlier, a natural question that arises in these systems is how regulated LSEs should behave. Here, we use stochastic control techniques to characterize firm specific optimal behaviour through generation and trading and discuss potential takeaways from a market design perspective. We believe these results are of interest to both regulators, the designers

[1]Not all generators of renewable energy who participate in REC markets are regulated Load Serving Entities (LSEs), though in this work, we largely focus on the decisions faced by them

[2]The largest and most mature SREC market in North America is the New Jersey SREC Market

of REC markets, and the firms regulated by them.

Specifically, we explore a cost minimization problem of a single regulated firm in a single-period SREC market with the goal of understanding their optimal behaviour as a function of their current level of compliance and the market price of SRECs. To this end, we pose the problem as a continuous time stochastic control problem. We provide the optimality conditions, and analyze the form of the optimal controls in feedback form to illuminate features of the solution. In addition, we numerically solve for the optimal controls of the regulated firm as generation and trading costs vary, include a detailed analysis of various scenarios, sample paths, and comparisons to benchmark strategies. We also explore the sensitivity of the optimal controls to the various parameters in the model.

The differences between our work and the extant literature are several. Firstly, we focus on the SREC market, which is a new and burgeoning market and there are only a few studies (in comparison to carbon C&T markets). Secondly, we focus on the optimal behaviour of firms, rather than what optimal behaviour implies about the price of SRECs. Prior works concern themselves with a stochastic control problem in order to learn about the behaviour of the allowance prices, while we begin with assumed dynamics for the price process of an SREC (which regulated agents exert some control over) and are interested in how the agent should behave given this. The control that agents have over the price process of an SREC is assumed to be similar to the permanent price impacts seen in the literature of optimal execution (see [2], [8])

The remainder of this paper is organized as follows. Section 2 discusses our model and poses the general optimal behaviour problem in continuous time. Section 3 presents optimality results in a continuous time setting. Section 4 provides a discrete time formulation and numerically solves the dynamic programming equation to characterize the optimal behaviour of a regulated firm. Finally, in Section 5, we present the results including sensitivity analysis and the implications for market design.

2. Model.

2.1. SREC Market Rules. We assume the following set of rules for the SREC market. These rules are exogenous and fixed prior to the market beginning. In an n-period framework, a firm is required to submit $(R_1, ..., R_n)$ SRECs corresponding to the compliance periods $[0, T_1], ..., [T_{n-1}, T_n]$. For the period $[T_{i-1}, T_i]$, firms pay P_i for each SREC below R_i at T_i. Firms receive an SREC for each MWh of electricity they produce through solar energy. We assume firms may bank leftover SRECs not needed for compliance into the next period, with no expiry on SRECs. This is a simplifying assumption we make – many SREC markets have limitations on how long an SREC can be banked for (in New Jersey's SREC market, the largest and most mature in North America, an SREC can be banked for a maximum of four years). This assumption reduces the dimensionality of the state space. After T_n, all SRECs are forfeited.

In a single-period framework, we consider a regulated firm in an n-period framework who is only concerned with the first compliance period. That is, the regulated firm is required to submit R SRECs at time T, representing their required production for the compliance period $[0, T]$. A penalty P is imposed for each missing SREC at time T. The firm considers any costs/profits arising from the SREC system after T to be immaterial.

2.2. Firm Behaviours. We first restrict ourselves to a single firm who is optimizing their behaviour in a single compliance period SREC system. A regulated firm can control their generation rate (SRECs/year) at any given time $(g_t)_{t\in[0,T]}$ and their trading rate (SRECs/year) at any given time $(\Gamma_t)_{t\in[0,T]}$. The processes g and Γ constitute the firm's controls.

The trading rate may be positive or negative, reflecting that firms can either buy or sell SRECs at the prevailing market rate for SRECs. Firms also incur a trading penalty of $\frac{1}{2}\gamma\Gamma_t^2$, $\gamma > 0$, per unit time. This induces a constraint on their trading speed. In general, this could be any function convex in Γ_t.

We assume that a firm has a baseline deterministic generation level, represented by $h(t)$ (SRECs/year), at which there is no cost of generation. Methods similar to [9] may be used to estimate $h(t)$. Deviations from their baseline production incurs a cost, represented by $C(g,h) := \frac{1}{2}\zeta(g-h)^2$ per unit time, which is similar to [1]. This is both differentiable and convex (any choice of C with these properties would be appropriate).

All objects are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{F}$ is the natural filtration generated by the SREC price. The set of admissible controls equals the set of all progressively measurable (with respect to $\mathbb{F}$) processes g_t, Γ_t such that $\mathbb{E}[\int_0^T g_t^2\,dt] < \infty$ and $\mathbb{E}[\int_0^T \Gamma_t^2\,dt] < \infty$. We denote this set by $\mathcal{A}$. At time t the firm holds $b_t^{g,\Gamma}$ SRECs and the SRECs price process is denoted $S_t^{g,\Gamma}$.

The various processes satisfy the stochastic differential euqations (SDEs)

$$S_t^{g,\Gamma} = S_0 + \int_0^t \mu(u)\,du + \int_0^t (\eta\,\Gamma_u - \psi\,g_u)\,du + \int_0^t \sigma(u)\,dB_u, \qquad \text{and} \tag{2.1a}$$

$$b_t^{g,\Gamma} = b_0 + \int_0^t (g_u + \Gamma_u)\,du. \tag{2.1b}$$

$B = (B_t)_{t\geq 0}$ is a Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and μ, σ are deterministic functions. We assume $\int_0^T \sigma(u)^2 du < \infty$. Trading and generation impact the price in opposite directions, and we assume their impacts are linear. This is similar to the price impact commonly studied in optimal execution problems. Such impacts are due to excess buying (selling) of SRECs pushing the price up (down) and due to generation pushing the price downwards. In this way, a firm's behaviour impacts the rest of the market.

As such, a regulated firm's performance criterion for the single-period problem is as follows:

$$J(t,b,S,g,\Gamma) = \mathbb{E}_{t,b,S,g,\Gamma}\left[-\int_t^T C(g_u, h(u))\,du - \int_t^T \Gamma_u\,S_u^{g,\Gamma}\,du - \tfrac{\gamma}{2}\int_t^T \Gamma_u^2\,du - P(R - b_T^{g,\Gamma})_+\right] \tag{2.2}$$

where $\mathbb{E}_{t,b,S,g,\Gamma}[\cdot]$ represents taking expectation conditioned on $b_t = b$, $S_t = S$, $g_t = g$, and $\Gamma_t = \Gamma$. The firm's cost minimization is the strategy which attains the sup (if it exists) below and the value of the optimal strategy is

$$V(t,b,S) = \sup_{(g_s,\Gamma_s)_{s\in[t,T]}\in\mathcal{A}} J(t,b,S,g,\Gamma). \tag{2.3}$$

In the next section, we characterize the optimal trading strategy and the relationship to SREC price using the stochastic maximum principle as well as the dynamic programming equation approach.

3. Continuous time approach.

3.1. Stochastic Maximum Approach. One approach to solving problem (2.3) is through the Stochastic (Pontryagin) Maximum Principle (see, the seminal works of [14] and [15]). Here, we apply the stochastic maximum principle to our problem along the lines of [10]. In doing so, we characterize the optimal controls as a system of coupled forward backward stochastic differential equations (FBSDEs). The key result is contained in the following proposition.

Proposition 3.1 (Optimality Conditions). *The processes $(g,\Gamma) = (g_t,\Gamma_t)_{t\in[0,T]}$ satisfying the FBSDE*

$$d\Gamma_t = \tfrac{1}{\gamma}(dM_t - \mu(t)\,dt + \psi\, g_t dt), \qquad \Gamma_T = \tfrac{1}{\gamma}(P\,\mathbb{I}_{b_T<R} - S_T), \tag{3.1}$$

$$dg_t = \tfrac{1}{\zeta}(dZ_t + \big(\zeta\, h'(t) - \psi\,\Gamma_t\big)\,dt), \qquad g_T = \tfrac{1}{\zeta}(P\,\mathbb{I}_{b_T<R} + \zeta\, h(T)), \tag{3.2}$$

for all $t \in [0,T]$, where the processes $(M,Z) = (M_t,Z_t)_{t\in[0,T]}$ are martingales, are the optimal controls for problem (2.3).

Proof. We consider the Hamiltonian

$$\begin{aligned}\mathcal{H}(t,b,S,g,\Gamma,\boldsymbol{y},\boldsymbol{z}) = &-\tfrac{\zeta}{2}(g-h(t))^2 - S\Gamma - \tfrac{\gamma}{2}\Gamma^2\\ &+ y_b\,(g+\Gamma) + y_S(\mu(t)+\eta\Gamma-\psi g) + \sigma(t) z_S\end{aligned} \tag{3.3}$$

where $\boldsymbol{y} = (y_b, y_S), \boldsymbol{z} = (z_b, z_S)$.

This is concave in the controls g,Γ and state variables b,S. Moreover, the adjoint processes $(y_b,y_S) = (y_{b,t}, y_{S,t})_{t\in[0,T]}$ satisfy the BSDEs

$$dy_{b,t} = z_{b,t}\,dB_t, \qquad y_{b,T} = P\,\mathbb{I}_{b_T<R} \tag{3.4}$$

$$dy_{S,t} = \Gamma_t\,dt + z_{S,t}\,dB_t, \qquad y_{S,T} = 0. \tag{3.5}$$

The stochastic maximum principle tells us that if there exists a solution $(\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$ to the BSDEs above, a strategy (g,Γ) that maximizes $\mathcal{H}(t,b,S,g,\Gamma,\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$ is the optimal control.

As both BSDEs have linear drivers, their solution is straightforward (see [16], Chapter 6) and given by

$$y_{b,t} = P\,\mathbb{P}_t(b_T<R)\,, \qquad \text{and} \qquad y_{S,t} = -\mathbb{E}_t\left[\int_t^T \Gamma_u du\right]. \tag{3.6}$$

Differentiating the Hamiltonian with respect to the controls, we obtain the first order conditions

$$\frac{\partial\mathcal{H}}{\partial\Gamma}: \quad y_b + \eta\, y_S - S - \gamma\,\Gamma = 0, \qquad \text{and} \tag{3.7a}$$

$$\frac{\partial\mathcal{H}}{\partial g}: \quad y_b - \psi\, y_S - \zeta\,(g-h) = 0, \tag{3.7b}$$

and substituting the solutions to the adjoint processes (3.6), we obtain the optimality conditions

$$P\,\mathbb{P}_t(b_T<R) - \eta\,\mathbb{E}_t\left[\int_t^T \Gamma_u du\right] - S_t - \Gamma_t\,\gamma = 0, \qquad \text{and} \tag{3.8a}$$

$$P\,\mathbb{P}_t(b_T<R) + \psi\,\mathbb{E}_t\left[\int_t^T \Gamma_u du\right] - \zeta\,(g_t - h(t)) = 0. \tag{3.8b}$$

The Stochastic Maximum Principle implies that solutions to (3.8a) and (3.8b) are the optimal controls we seek. We next rewrite the optimality conditions as the solution to FBSDEs. From (3.8a), we have

$$Y_t + \eta \int_0^t \Gamma_u du - S_t = \Gamma_t \gamma, \tag{3.9}$$

where $Y = (Y_t)_{t\in[0,T]}$ is the Doob-martingale defined by

$$Y_t = P\,\mathbb{P}_t(b_T < R) - \eta\,\mathbb{E}_t\left[\int_0^T \Gamma_u du\right]. \tag{3.10}$$

Rearranging (3.9), we arrive at (3.1) where the terminal condition follows immediately from (3.8a), and $M = (M_t)_{t\in[0,T]}$ is the martingale defined by

$$M_t = Y_t - \int_0^t \sigma(u) dB_u\,. \tag{3.11}$$

From (3.8b), we apply a similar argument to above, leading to (3.2) where $Z = (Z_t)_{t\in[0,T]}$ is the Doob-martingale defined by

$$Z_t = P\,\mathbb{P}_t(b_T < R) + \psi\,\mathbb{E}_t\left[\int_0^T \Gamma_u du\right]. \tag{3.12}$$

■

We end this subsection with a few comments regarding the results of this proposition and interpretations of the optimality conditions. The addition of the trading speed penalty and the impact of trading and generation on SREC prices modify the optimality conditions in [10], where the authors develop optimality conditions in a carbon C&T system. When $\eta = \psi = 0$, (3.8b) reduces to $P\,\mathbb{P}_t(b_T < R) = \zeta(g_t - h(t))$. This is similar to the result that the marginal cost of generation is equal to the product of the penalty and probability of non-compliance found in [10]. Moreover, when $\eta = \psi = 0$, (3.8a) reduces to $P\,\mathbb{P}_t(b_T < R) - \gamma\Gamma_t = S_t$. Hence, in this case, the SREC price equals the penalty scaled by the probability of non-compliance but modified by the optimal trading of the firm.

Similar behavior persists in the general case when $\eta > 0, \psi > 0$. From (3.8a), the SREC price equals the penalty scaled by the probability of non-compliance, but modified by the time-t marginal cost of the firm's trading and our expectations of their future trading. That is, low prices are associated with high rate of trading and high expected future rate of trading.

From (3.8b), the penalty scaled by the probability of non-compliance equals the difference between the marginal cost of generation and re-scaled (by ψ) expected future trading.

The FBSDEs (3.1)-(3.2) can be solved numerically using Least Square Monte Carlo techniques, however, we will consider a dynamic programming approach to solving the original problem (2.3).

3.2. HJB Approach. Dynamic programming principle provides additional insight into the solution of the problem. Here, for simplicity, we assume $\mu(t), \sigma(t)$ are constants represented by μ, σ. Using standard techniques (see e.g., [16]), the Dynamic Programming Principle applied to (2.3) implies that the value function is the unique viscosity solution to the dynamic programming equation (DPE) or Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t V(t,b,S) + \sup_{g,\Gamma} \left\{ \mathcal{L}^{b,S} V(t,b,S) + F(t,b,S,g,\Gamma) \right\} = 0\,, \tag{3.13}$$

$$V(T,b,S) = G(b)\,, \tag{3.14}$$

where $G(b) = P(R-b)_+$ and $F(t,b,S,g,\Gamma) = -\frac{1}{2}\,\zeta\,(g-h(t))^2 - S\,\Gamma - \frac{\gamma}{2}\,\Gamma^2$ and the operator $\mathcal{L}^{b,S}$ acts on functions as follows

$$\mathcal{L}^{b,S} V = (\mu + \eta\Gamma - \psi\, g)\,\partial_S V + (g+\Gamma)\,\partial_b V + \tfrac{1}{2}\sigma^2\,\partial_{SS} V\,. \tag{3.15}$$

The first order conditions provides the optimal controls in feedback form

$$g^*(t,b,S) = h(t) + \tfrac{1}{\zeta}\left(\partial_b V(t,b,S) - \psi\,\partial_S V(t,b,S)\right)\,, \qquad \text{and} \tag{3.16a}$$

$$\Gamma^*(t,b,S) = \tfrac{1}{\gamma}\left(\partial_b V(t,b,S) + \eta\,\partial_S V(t,b,S) - S\right)\,. \tag{3.16b}$$

From the above, the optimal level of trading has a negative linear relationship with respect to the SREC market price S. Hence, as S increases, the optimal level of trading decreases. That is, the firm buys less (or equivalently, sell more) as SREC prices increase.

Similarly, the optimal generation amount can be interpreted as the baseline amount of SRECs (h) plus the marginal value gained by the generation of an SREC (as generating an SREC increases b and negatively impacts S), modulated by the cost of generation parameter ζ.

Generation and trading have opposite dependence in their sensitivity to asset price; that is, the coefficients of the $\partial_S V$ terms in (3.16) have opposite signs. If an incremental change in SREC price increases (decreases) the value function, then it increases (reduces) trading and simultaneously reduces (increases) generation. One reason is that as SRECs are purchased (sold), trading impacts prices and pushes them upwards (downwards).

Substituting the feedback form into the HJB equation leads to the semi-linear parabolic PDE

$$\partial_t V + \mathcal{L}^S V - \tfrac{1}{2}\zeta\,h^2 + \tfrac{1}{2\zeta}\left(\partial_b V - \psi\partial_s V + \zeta h\right)^2 + \tfrac{1}{2\gamma}\left(\partial_b V + \eta\partial_S V - S\right)^2 = 0, \tag{3.17a}$$

$$V(T,b,S) = G(b), \tag{3.17b}$$

where $\mathcal{L}^S = \mu\,\partial_s + \frac{1}{2}\sigma^2\,\partial_{ss}$ is the generator of the no-impact SREC price. This PDE is difficult to solve analytically. One can solve it numerically using finite differences methods and then apply (3.16) to obtain the optimal controls. However, due to the lack of a convexity term in b, numerical instabilities occur and require large number of grid point methods, or more sophisticated finite-difference schemes. Instead, we formulate a discrete time version of the problem directly and solve it numerically.

4. Discrete time version of problem. Thus far, we formulated the cost minimization problem of a single regulated firm using continuous time optimal control techniques to characterize the solution and tease out some essential features of the optimal strategy. To obtain numerical solutions, however, we solve a discrete time version of the problem which we find has better numerical stability. Indeed, a discrete time formulation more closely approximates practice, as regulated firms typically take actions only at discrete time points within a compliance period.

To this end, let n be the number of decision points within a single compliance period (we are still in the single firm, single-period case), which occur at $0 = t_1 < t_2 < ... < t_n < T = t_{n+1}$. For simplicity, we assume these are equally spaced such that $t_k = k\Delta t$.

The processes g_t, Γ_t are now piecewise constant within $[t_i, t_{i+1})$, and the firm controls $\{g_{t_i}, \Gamma_{t_i}\}_{i\in\mathfrak{N}}$ where $\mathfrak{N} := \{0, \dots, n\}$. Intuitively, at each time point, the regulated firm chooses their trading and generating behaviour over the next interval of length Δt. In this section, g, Γ represent vectors whose elements are these controls.

Under the same assumptions as earlier, the performance criterion is

$$J(t, b, S, g, \Gamma) = \mathbb{E}\left[\tfrac{1}{2}\zeta\sum_{i=1}^{n}(g_{t_i} - h(t_i))^2\Delta t + \sum_{i=1}^{n}\Gamma_{t_i}S_{t_i}^{g,\Gamma}\Delta t + \tfrac{\gamma}{2}\sum_{i=0}^{n}\Gamma_{t_i}^2\Delta t + P(R - b_T^{g,\Gamma})_+\right]. \tag{4.1}$$

In the above, the dynamics of the state variables (b, S) are modified for discrete time to

$$S_{t_i}^{g,\Gamma} = \min(\max(S_{t_{i-1}}^{g,\Gamma} + \mu\Delta t + \eta\Gamma_{t_{i-1}}\Delta t - \psi g_{t_{i-1}}\Delta t + \sigma\sqrt{\Delta t}Z_{t_i}, 0), P) \tag{4.2a}$$

$$b_{t_i}^{g,\Gamma} = b_{t_{i-1}}^{g,\Gamma} + (g_{t_{i-1}} + \Gamma_{t_{i-1}})\Delta t \tag{4.2b}$$

where $Z_{t_i} \sim N(0, 1)$, iid, for all $i \in \mathfrak{N}$.

Note that (4.2a) is the discrete time analogue of (2.1a), which we further cap at P and floor at 0. The cap and floor is to ensure that SREC price never falls below 0 and never rises above P. Prices outside these ranges cannot occur in a real market.

We aim to optimize (4.1) with respect to g, Γ and determine the value of the position the regulated firm, as well as their optimal behaviour. Hence, we seek

$$V(t, b, S) = \inf_{g,\Gamma} J(t, b, S, g, \Gamma), \tag{4.3}$$

and the strategy that attains the sup, if it exists. Applying the Bellman Principle to (4.3) implies

$$\begin{aligned} V(t_i, b, S) = \inf_{g_{t_i},\Gamma_{t_i}} \Big\{ &\tfrac{1}{2}\zeta(g_{t_i} - h(t_i))^2\Delta t + \Gamma_{t_i}S_{t_i}^{g,\Gamma}\Delta t + \tfrac{\gamma}{2}\Gamma_{t_i}^2\Delta t \\ &+ \mathbb{E}_{t_i}\left[V(t_{i+1}, b_{t_{i+1}}^{g,\Gamma}, S_{t_{i+1}}^{g,\Gamma})\right]\Big\}, \end{aligned} \tag{4.4a}$$

$$V(T, b, S) = P(R - b)_+ \,. \tag{4.4b}$$

where we have

$$b_{t_{i+1}}^{g,\Gamma} = b + (g_{t_i} + \Gamma_{t_i})\Delta t \tag{4.5a}$$

$$S_{t_{i+1}}^{g,\Gamma} = \min(\max(S + \mu\Delta t + \eta\Gamma_{t_i}\Delta t - \psi g_{t_i}\Delta t + \sigma\sqrt{\Delta t}Z_{t_{i+1}}, 0), P) \tag{4.5b}$$

In the next section we describe how we numerically solve this optimization problem through backwards induction.

5. Solution Algorithm and Results.

5.1. Parameter Choice and Optimal Behaviour. The solution algorithm to solve (4.4) with state variable dynamics in (4.2) is as follows:

1. Choose a grid of b and S values which we denote by $\mathfrak{G}$. We applied a uniform grid of 101 points in b from 0 to $2R$ (so that R is on the grid), and a uniform grid of S with $\Delta S = \sqrt{3\Delta t}\sigma$, with lower and upper bounds of 0 and P respectively (thus the number of grid points in S is dependent on the parameters σ and Δt). As with any numerical solution, the trade-off made in the choice of grid is between grid size (accuracy of the dynamic program solution) and run-time. The grid we employed provides an acceptable trade-off between accuracy and run-time.
2. Begin at time t_n and minimize (4.4a) with respect to g_{t_n}, Γ_{t_n}, for every point in $\mathfrak{G}$. This is a convex optimization problem at t_n.
3. Step backwards in time, minimizing (4.4a) with respect to g_{t_i}, Γ_{t_i} at time t_i for all points in $\mathfrak{G}$. To minimize, we require $\mathbb{E}_{t_i}\left[V(t_{i+1}, b^{g_{t_i},\Gamma_{t_i}}_{t_{i+1}}, S^{g_{t_i},\Gamma_{t_i}}_{t_{i+1}})\right]$ for candidate values of g_{t_i}, Γ_{t_i} and at all starting points in $\mathfrak{G}$. This can be achieved by simulating $b_{t_{i+1}}$ and $S_{t_{i+1}}$[3].
   - The former is a deterministic function of g_{t_i}, Γ_{t_i} (see (4.2b))).
   - For the latter, for given value of g_{t_i}, Γ_{t_i}, we simulate 100 scenarios of $S_{t_{i+1}}$ through the update equation (4.2a), resulting in 100 pairs of $(b_{t_{i+1}}, S_{t_{i+1}})$. For each of these 100 pairs, we evaluate the value function $V(t_{i+1}, b_{t_{i+1}}, S_{t_{i+1}})$, which has already been calculated, as we have iterated backwards in time. If necessary, we linearly interpolate between grid points.

   We set $\mathbb{E}[V(t_{i+1}, b^{g_{t_i},\Gamma_{t_i}}_{t_{i+1}}, S^{g_{t_i},\Gamma_{t_i}}_{t_{i+1}})]$ to be the mean of these 100 values. This allows us to calculate the RHS of (4.4a) at time t_i as a function of (g_{t_i}, Γ_{t_i}), and therefore minimize it with respect to those variables. In all cases, minimization was done through Matlab's 'fminsearch' function and across the grid we use the same set of random numbers to generate the one-step ahead values of $S_{t_{i+1}}$.

This procedure provides an estimate of the value function at all grid points $\mathfrak{G}$ and at all times $\mathfrak{T} := \{t_i\}_{i\in\mathfrak{N}}$, as well as the optimal generation and trading rates on $\mathfrak{G} \times \mathfrak{T}$.

For the first set of numerical experiments, we use the parameters reported in Tables 1 and 2.

n	T	P ($/ lacking SRECs)	R (SRECs)	$h(t)$ (SREC/y)
50	1	300	500	500

Table 1
Compliance parameters.

These parameters are chosen for illustrative purposes, as calibration to a particular firm is itself a non-trivial problem and requires proprietary knowledge of a firm's cost function and baseline production (which also varies significantly from firm to firm). Instead, we provide broad-level explanations and intuitions regarding the optimal behaviour of a firm in a single-

[3]For convenience, we drop the superscripts in this discussion of our algorithm

μ	σ	ψ	η	ζ	γ
0	10	0	0	0.6	0.6

Table 2
Model Parameters.

period SREC market with reasonable parameters. The penalty of $P = \$300$ is motivated by the New Jersey SREC market, where the non-compliance penalty in compliance period ending May 2018 is \$308. The choice $h(t) = \frac{1}{T}R$ implies the regulated firm we are considering has the capability to produce the required SRECs by generating at their baseline. We set $\mu = \psi = \eta = 0$ so that, in this baseline case, S is a martingale. This choice also implies that the firm's generation and trading has no price impact on prices.

The values of ζ and γ are motivated by the upper bounds they imply for g_t, Γ_t. Specifically, consider the case of a firm that cannot generate enough solar energy to meet the requirements, and hence will fail to comply. The benefit of generating SRECs is to reduce their non-compliance obligation, and with each generated SREC their obligation is reduced by P. Therefore, the costs and benefits of generation over a time-step are (independent of trading activity)

$$K_1(g_t) = \tfrac{1}{2}\zeta(g_t - h(t))^2\Delta t \qquad \text{and} \qquad B_1(g_t) = P g_t \Delta t\,. \tag{5.1}$$

The firm generates in order to minimize $N_1(g_t) := K_1(g_t) - B_1(g_t)$ which occurs at $g_t^* = \frac{P}{\zeta} + h(t)$. For the chosen parameters, $g^* = 1000$ which is exactly twice the baseline rate $h(t)$. In other words, this choice of ζ ensures that the firm's maximum generation rate is bounded by twice their baseline.

We conduct a similar exercise for Γ_t. Once again, consider a firm that will assuredly fail to comply. It is clear that in this scenario, a rational firm will purchase SRECs. As before, the benefit of a firm purchasing SRECs is to reduce their non-compliance obligation, with each generated SREC reducing the obligation by P. As such, the costs and benefits to purchase over the next time-step are (independent of generation activity):

$$K_2(\Gamma_t) = \tfrac{1}{2}\gamma\Gamma_t^2\Delta t + S_t\Gamma_t\Delta t \qquad \text{and} \qquad B_2(\Gamma_t) = P g_t \Delta t. \tag{5.2}$$

The firm purchases in order to minimize $N_2(g_t) := K_2(\Gamma_t) - B_2(\Gamma_t)$ which occurs at $\Gamma_t^* = \frac{P-S}{\gamma}$. For the chosen parameters, this is maximized when $S = 0$ and results in $\Gamma^* = 500$. The significance of this is that we have chosen parameters that result in a reasonable upper bound on the amount of trading a firm will partake in.

Repeating the same exercise for a firm that is guaranteed to comply (and thus is motivated to sell), we obtain $g_t^* = h(t) = 500$ and $\Gamma_t = -\frac{S}{\gamma}$ which is maximized (in absolute value) at -500 for the chosen parameters.

The result of this simple analysis is that, for the parameters chosen in Tables 1 and 2, generation and trading rates are restricted to the range $g_t \in [500, 1000]$ and $\Gamma_t \in [-500, 500]$, which is a reasonable range of possible values given our choices of $h(t)$ and R.

In Section 5.4, we consider other parameters. In particular, we explore how various levels of ζ, γ impact firm behaviour, as well as the effect of price impact ($\psi \neq 0, \eta \neq 0$). We restrict to $h(t)$ being constant but study alternative baselines in Section 5.4. As total generated SRECs is important from the perspective of a market designer, we investigate how the various parameters affect it.

A regulated firm's optimal behaviour is one of the key outputs from solving the Bellman equation. Figure 1 shows the dependence of the optimal trading and generation rate on banked SRECS for three SREC prices at six points in time.

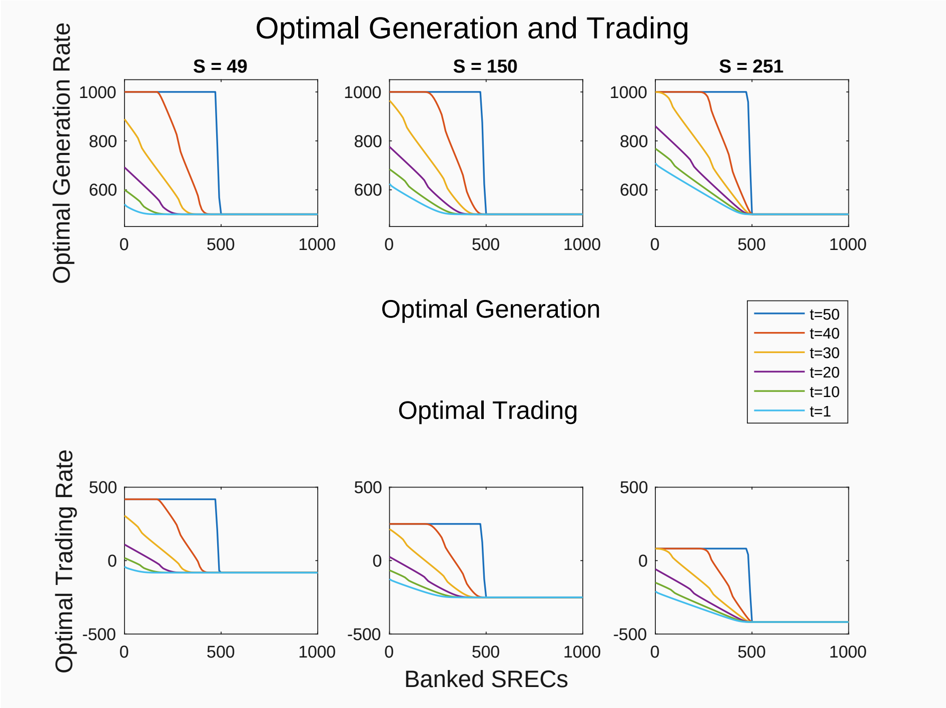

Figure 1. *Optimal firm behaviour as a function of banked SRECs across various time-steps and SREC market prices*

The most notable feature is the distinct regimes of generation/trading. For low levels of banked SRECs and nearly terminal date, the firm generates until the marginal cost of producing another SREC exceeds P, and purchases until the marginal cost of purchasing another SREC exceeds P, as the firm is almost assured to fail to comply. This follows the classic microeconomic adage of conducting an activity until the marginal benefit from the activity equals the marginal cost. In this regime, the marginal benefit of an additional SREC to the firm is P, as each additional SREC lowers their non-compliance obligation by P.

As b increases, the firm reaches a point where the marginal benefit from an additional SREC decreases from P. This occurs as the probability of compliance becomes non-negligible, as additional SRECs in excess of R provide smaller marginal benefit than P. This is a result of the sale price of an SREC being bounded above by P. This leads to a decrease in optimal generation and optimal trading, as the firm adjusts its behaviour so that its marginal costs are in line with this marginal benefit. This eventually leads to the firm selling as opposed to purchasing SRECs, as the net proceeds from the sale exceed the marginal value of retaining those certificates. This decrease continues until the firm no longer benefits from additional SRECs at all. That is, at a certain level of b, the marginal benefit of an additional SREC is zero. More concretely, having an additional SREC does not increase the firm's likelihood of compliance, nor can they sell the additional SREC to make a profit. This results in optimal generation plateauing at the baseline amount, $h(t)$, as the firm can produce $h(t)$ with zero marginal cost. Similarly, optimal trading plateaus at the level where the marginal revenue from trading equals the marginal cost.

We note that if we hold b, S constant, generation and trading are increasing in t. This is natural, as with less time until the end of a compliance period, the firm needs to accumulate more SRECs in order to comply.

It is also immediately clear that trading is heavily influenced by a change in SREC price, which is in accordance with our intuition and coincides with the theoretical results from Section 3. As SREC prices increase, the regulated firm chooses to purchase less, across all levels of b (banked SRECs). Meanwhile, we see that higher SREC prices also generally imply higher generation, as the firm chooses to generate their own SRECs, either to avoid paying a large price for them in the market, or to sell in the market and capitalize on the high prices (which of these two factors is the larger contributor depends on their value of b).

5.2. Sample Paths. We also model quantities of interest for an optimally behaving firm throughout the course of the compliance period, as shown in Figure 2. We set initial values of $S_0 = 150, b_0 = 0$. We simulate a path for S and at each time-step, the firm behaves optimally (per our solution) given their values of banked SRECs and the SREC price.

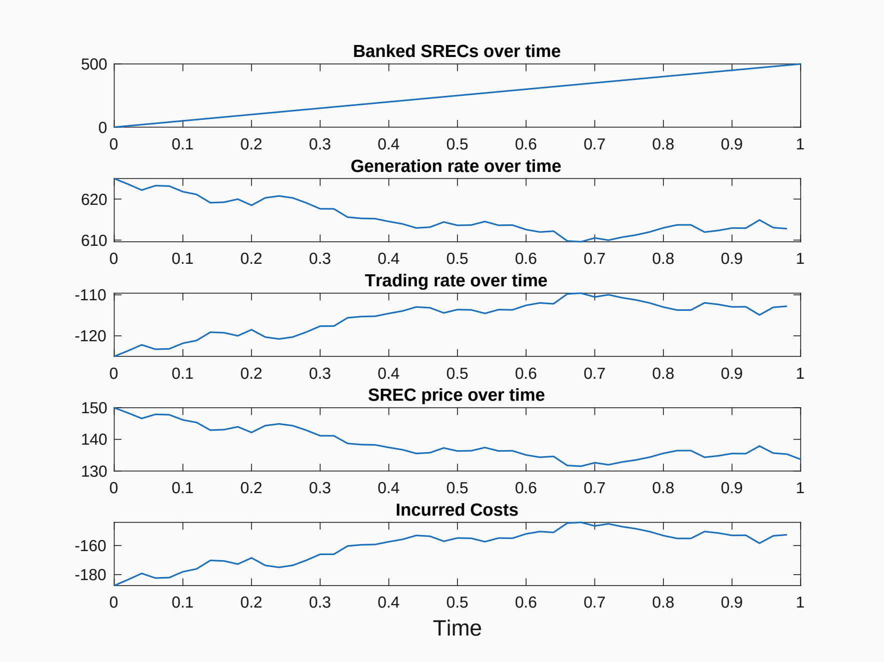

Figure 2. *Optimal firm behaviour with* $S_0 = 150, b_0 = 0$

From Figure 2, we see that the regulated firm acquires SRECs at a steady rate. However, the generation and trading processes exhibit notable variation over time. In particular, the inverse relationship between trading rate and SREC price is notable, as is the positive relationship between generation rate and SREC price[4]. Note that incurred costs being negative means that the firm is making profits (from the sales of SRECs) at each time-step.

If we simulate many paths similar to the above, we also obtain summary statistics and learn about the distribution of various interesting quantities for the firm, such as their final SREC total (b_T), total generated amount ($\int_0^T g_u du$), total traded amount ($\int_0^T \Gamma_u du$), and total costs incurred.

For the same parameter choice as above, and with $S_0 = 150$, we present summary statistics for these quantities, based on 1000 simulated paths of S. Note that we consider total profit below, which is simply the negative of total incurred cost.

Statistic	Mean	SD	1st Quartile	3rd Quartile	Skewness	Kurtosis
b_T	500.00	7.60×10^{-7}	500.00	500.00	3.75	21.98
$\int_0^T g_u du$	624.89	4.58	621.96	628.01	0.02	3.06
$\int_0^T \Gamma_u du$	-124.89	4.58	-128.01	-121.96	-0.02	3.06
Profit	9,380.00	687.25	8,9300.00	9.840.00	0.17	3.15

Table 3

Summary statistics of various quantities for 1000 sample paths of S with firm behaving optimally

In this one-period setup, the firm's optimal behaviour results in $b_T = 500$ SRECs in almost every path; there is minimal variation in the firm's final SREC total. This is logical, as there is no advantage to additional SRECs above the requirement in a single-period framework, and any strategy where $b_T < 500$ exposes the firm to the non-compliance penalty.

We turn our attention to the other quantities, noting that they exhibit little skewness or excess kurtosis. This is also apparent when looking at the histograms of each of these quantities across these simulations, as shown in Figure 3.

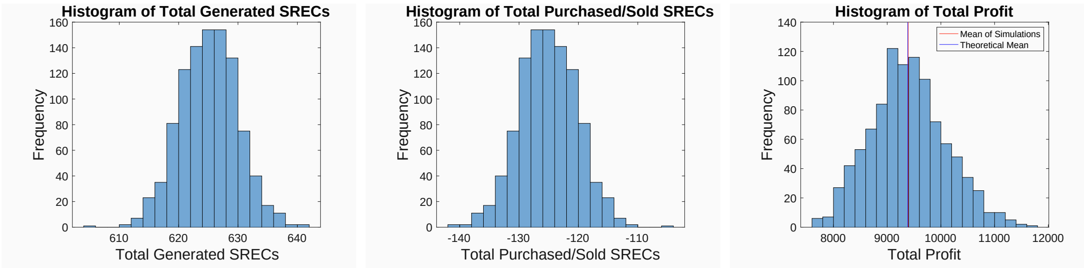

Figure 3. *Histogram of total generation, total trading, and profit over 1000 simulations*

We note that the mean profit of our simulations lies on top of the theoretical expected profit, represented by the value function evaluated at $t = 1, b = b_0, S = S_0$, which suggests our calculations are internally consistent. As the nature of the distribution of b_T would imply,

[4]As a corollary to this, we note that if the SREC price was constant, the optimal behaviours would be as well

we can also see that the histograms of $\int_0^T g_u du$ and $\int_0^T \Gamma_u du$ (if flipped horizontally) would have the exact same shape. That is, the instances of increased (decreased) total generation correspond exactly to increased (decreased) total SRECs sold, as the firm always ends up with 500 SRECs.

Between their symmetrical histograms, lack of skewness, and lack of excess kurtosis, there is some support for these quantities being normal. Examining their QQ-plots (in Figure 4), we observe that some deviation from normality exists in the tails of these quantities.

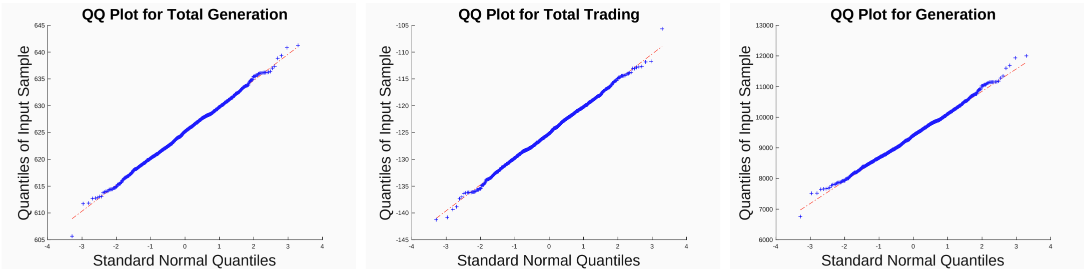

Figure 4. *QQ Plots of total generation, total trading, and profit over 1000 simulations*

5.3. Comparison With Other Strategies. A natural question to arise is how the optimal strategy suggested by the numerical solution to the control problem compares with other intuitive strategies. In particular, we compare the optimal strategy (for the parameter choice in Section 5.1) with the following strategies:

1. A constant generation strategy where $g_t = 500, \Gamma_t = 0$ at all time-steps t (recall $R = 500, h(t) = 500$). We denote this as the 'No Trade' strategy.
2. A constant strategy where the firm follows the mean behaviour elucidated in Section 5.2. That is, they produce constantly such that $\int_0^T g_u du = 625$ and $\int_0^T \Gamma_u du = -125$. We denote this as the 'Mean Behaviour' strategy.

It can be shown that the Mean Behaviour strategy proposed is the optimal strategy for this parameter set if we restrict the firm to constant behaviours for their controls (i.e. $g_t = g, \Gamma_t = \Gamma$ for every $t \in [0, T]$) and require that they exactly comply with the requirement $(g + \Gamma = R)$[5]. See Appendix A for details.

Over 1000 simulated paths of S with $S_0 = 150$, we calculate the profit from each strategy, as well as other summary statistics, detailed in the table below (we note that these are a separate set of simulations to those carried out in the previous section). The strategy suggested as the output to the dynamic program is referred to as the 'Output' strategy.

The No Trade strategy is trivial. As there is no randomness associated with generation, the profit from the No Trade strategy is deterministic. In particular, the profit is 0 in this case due to the cost function and the parameters we chose (in particular, that $h(t) = 500, R = 500$). This means the firm can generate SRECs 'cost-free' (in line with their baseline production)

[5]It is clear that any strategy with $g + \Gamma \neq R$ is not optimal for these parameters. In a one period model with $h(t) = R$, a strategy with $g + \Gamma > R$ implies the firm spends money to generate additional SRECs, some of which expire valueless. If $g + \Gamma < R$, the firm incurs non-compliance penalties that cannot be made up for by sales, as $S < P$

Strategy	Mean Profit	Std. Dev. of Profit	Q1 Profit	Q3 Profit
No Trade	0	0	0	0
Output	9380.00	723.56	8920.00	9850.00
Mean Behaviour	9360.00	724.05	8910.00	9830.00

Table 4
Summary of various strategies

and produce enough to comply with the requirement. However, it is clear that this strategy is not optimal for this parameter set. It is worth noting that there is a parameter set for which this strategy is optimal. Specifically, this will be the case if γ was taken towards ∞.

The Output and Mean Behaviour strategies have more similar results. In order to understand the performance of these strategies further, we explore these two strategies in Figure 5 by plotting the histogram of the differences in profit between each strategy across sample paths.

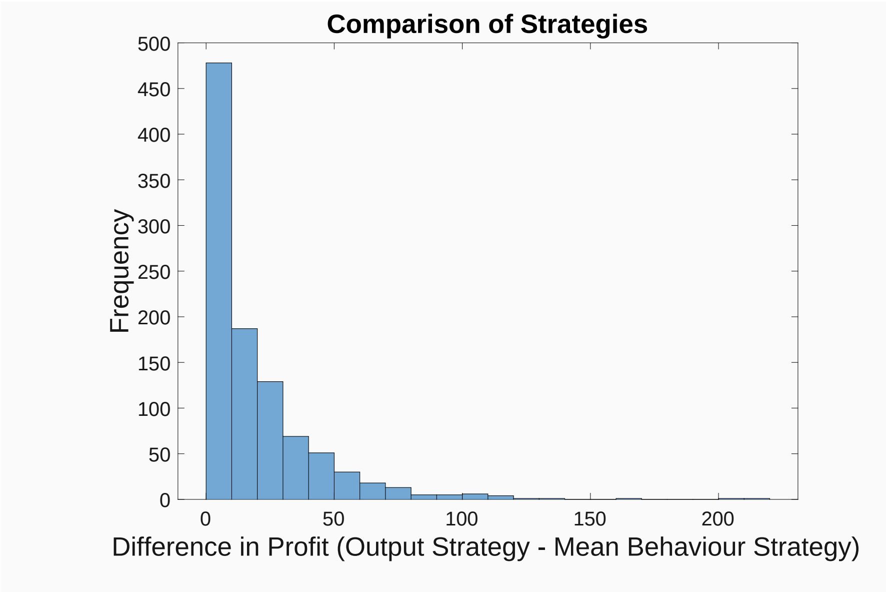

Figure 5. *Comparison between strategies*

From Figure 5, we observe that the Output strategy outperformed the Mean Behaviour strategy in every sample path of S. Intuitively, this means that the firm adjusting their behaviour depending on the evolution of S provides value over a static optimal control regardless of the path of S.

5.4. Parameter Sensitivity. As in any numerical model, parameter choices are of great importance. As such, it is critical to understand how perturbing the parameters modifies the output of the model, and the intuition behind the resulting changes.

5.4.1. Activating Price Impacts. One parameter choice made in previous subsections was that $\eta = \psi = 0$. This was the assumption that the regulated firm whose behaviour we optimize has no impact on the overall SREC price (S). We analyze the firm's behaviour when these are non-zero in order to understand how this unique aspect of our model influences their controls. In particular, we choose $\eta = 0.01, \psi = 0.005$ as reasonable parameters that demonstrate the impact of our model choice. We justify these choices in a similar manner to in Section 5.1. Recall that previously, we discussed that under the parameter choices of Section 5.1, $g_t \leq 1000, |\Gamma_t| \leq 500$. Under these choices of generation and trading, we would see price impacts of $500 \times 0.01 \times \frac{1}{50} = 0.1$ per time-step for trading and $1000 \times\ 0.005 \times \frac{1}{50} = 0.1$ per time-step for generation. This ensures that the firm's price impacts will not be so large as to warp the model entirely.

As before, we plot the optimal behaviours of the regulated firm as a function of banked SRECs, across three different prices of S and at six points in time.

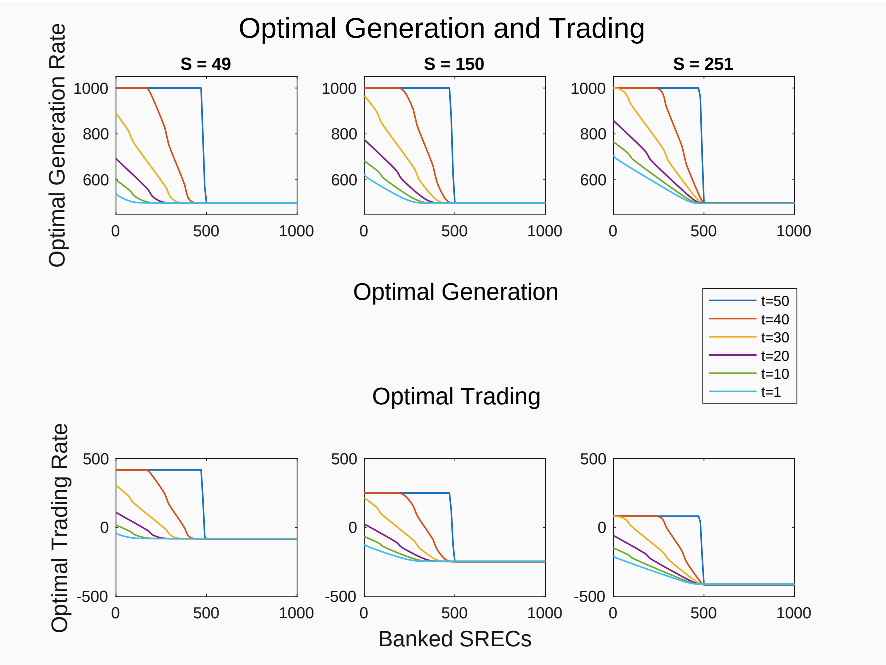

Figure 6. *Optimal firm behaviour as a function of banked SRECs across various time-steps and SREC market prices*

Comparing Figure 6 with Figure 1, we see that the optimal controls have the same shape, but with additional interesting properties. The optimal controls at each of the 'plateaus' now vary slightly with time-step. This was not the case when price impacts of controls were not activated, as in Figure 1. Specifically, at high values of b, firms generate less and purchase more (equivalently, sell less) at earlier time-steps than they do at later time-steps. This is to lower the impact of their behaviour on S, and in particular, to try and keep S as high as possible in order to capitalize on future sales. The inverse behaviour happens for low values of b. Firms generate more and purchase more in order to push the price down and make compliance more attainable. These effects are proportional to the magnitude of η and ψ.

Comparing the optimal behaviour with price impacts active to the optimal behaviour with price impacts inactive is of importance. To do so, we replicate Figure 2 in Figure 7, for price impacts being both active and inactive (with the same random seed for the path of S, with $S_0 = 150, b_0 = 0$).

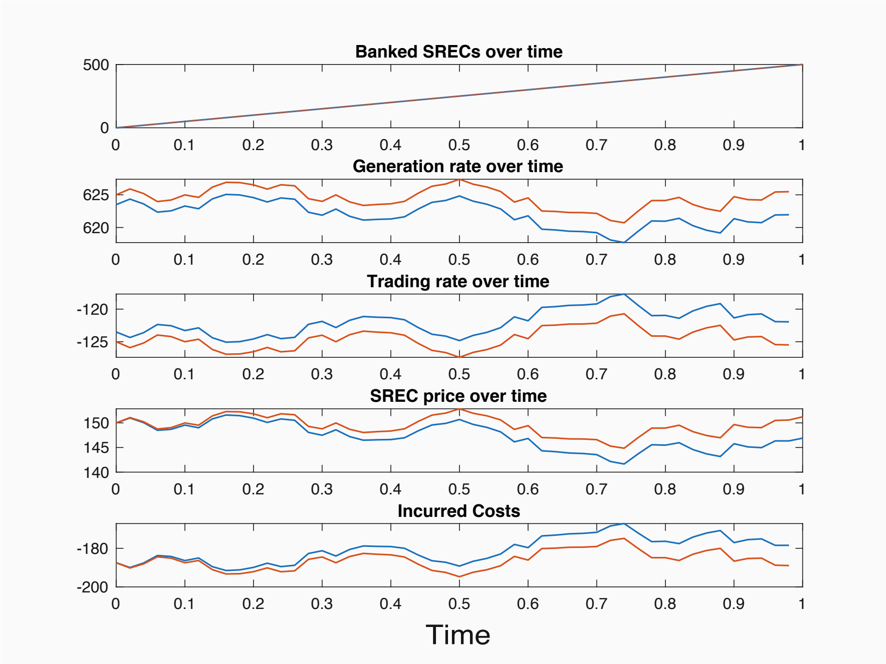

Figure 7. *Optimal firm behaviour with $S_0 = 150, b_0 = 0$ with price impacts active (blue) and inactive (red)*

The shape of the optimal controls are similar whether price impacts are active or inactive, reflecting the features discussed regarding Figure 2. In particular, in both cases, the firm accumulates banked SRECs at identical rates. However, we see that the firm under price impacts will generate and sell less in order to mitigate their price impact. Nonetheless, the deviation between the paths of S is also clear, with the price impacts resulting in a lower SREC price due to the firm behaviour of generation and selling. Finally, we see that the firm generates less revenue when price impacts are active, as the market will always move against their behaviour. We also observe that as time progresses, the difference in generated revenue between the two parameter sets increases.

We also produce summary statistics after simulating many paths in the same manner with price impacts activated.

There are clear similarities to the quantities in Table 3, as the main properties discussed previously continue to hold. As in the case without price impacts, banked SRECs are universally equal to the requirement R. In comparison to the figures in Table 3, we see that the firm generates and sells less, which is also reflected in Figure 7. Once again, the total generation, trading, and profit exhibit little skewness or excess kurtosis.

As before, we observe that the histograms of $\int_0^T g_u du$ and $\int_0^T \Gamma_u du$ (if flipped horizontally)

Statistic	Mean	SD	1st Quartile	3rd Quartile	Skewness	Kurtosis
b_T	500.00	9.20e-07	500.00	500.00	3.09	14.52
$\int_0^T g_u du$	622.47	4.73	619.13	625.66	0.05	2.94
$\int_0^T \Gamma_u du$	-122.47	4.73	-125.66	-119.13	-0.05	2.94
Profit	9130.00	700.85	8630.00	9590.00	0.18	2.98

Table 5

Summary statistics for 1000 simulations of sample path of S, with price impacts activated

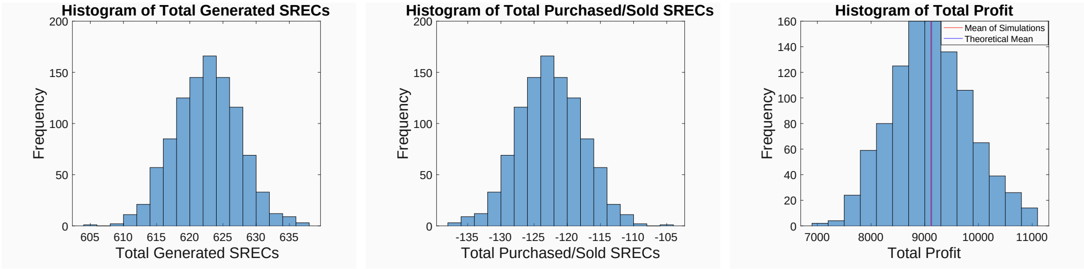

Figure 8. *Histogram of total generation, total trading, and profit over 1000 simulations with price impacts activated*

would have the same shape. This means that total generation and total trading exist on the line $500 - \int_0^T g_u du - \int_0^T \Gamma_u du = 0$, and the correlation between $\int_0^T g_u du$ and $\int_0^T \Gamma_u du$ is -1.

We display the QQ plots of each of these distributions, shown in Figure 9. As in the no price impact case, these quantities exhibit some evidence of normality, though we see some deviation away from normality in the tails of the distribution.

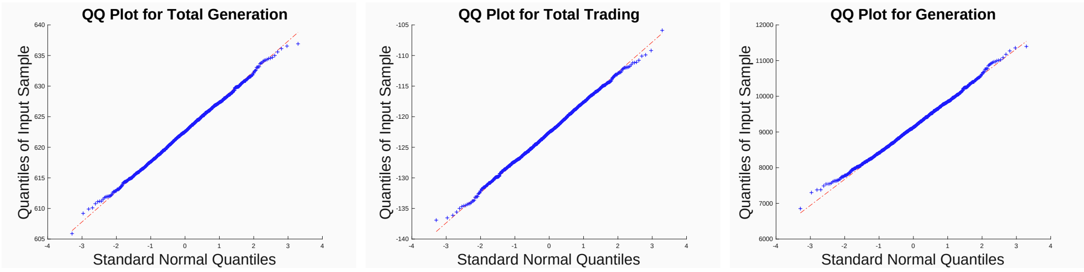

Figure 9. *Histogram of total generation, total trading, and profit over 1000 simulations with price impacts activated*

5.4.2. Sensitivity to $h(t)$. We have predominantly explored the numerical solution to this optimal control problem from the perspective of a firm with baseline SREC generation of $h(t) = R$. However, in reality, a wide array of LSEs are regulated by SREC systems, with varying levels of investment into solar and thus, capability to generate SRECs. As such, it is important to consider the optimal behaviours of firms with various levels of production capability.

In particular, this has important ramifications from the perspective of a regulator in charge of market design. Ultimately, the goal of SREC systems is to promote investment into solar energy generation. Consequently, the amount of SRECs generated by a regulated firm is a quantity that would be monitored carefully by regulators and market designers. To briefly study the impact of $h(t)$ on total SRECs generated by a firm, we simulate 1000 paths of S starting from S_0, with a regulated firm behaving optimally at each time-step. We examine how the distribution of $\int_0^T g_u du$ changes across varying levels of $h(t)$ for the firm, with all other parameters reprising their value from Section 5.1. Specifically, we look at (constant) values of $h(t)$ of $0.75R$, R, and $1.25R$ (corresponding to values of $375, 500$, and 625, respectively). We also consider various initialization points of the path of S: $S_0 = 49, 150, 251$ (which represent low, medium, and high prices of S respectively).

$h(t)$	$S_0 = 49$		$S_0 = 150$		$S_0 = 251$	
	$\mathbb{E}[\int_0^T g_u du]$	$\mathrm{SD}(\int_0^T g_u du)$	$\mathbb{E}[\int_0^T g_u du]$	$\mathrm{SD}(\int_0^T g_u du)$	$\mathbb{E}[\int_0^T g_u du]$	$\mathrm{SD}(\int_0^T g_u du)$
$0.75R$	478.45	4.78	562.45	4.59	646.48	4.95
R	541.01	4.79	624.95	4.59	708.98	4.95
$1.25R$	625.00	0.00	687.45	4.59	771.48	4.95

Table 6

Total generation across varying levels of $h(t)$ and S_0

From Table 6, we confirm the relatively obvious fact that a firm with a higher baseline will produce more SRECs on average given the same level of S_0 than a firm with a lower baseline. It is also apparent that firms produce more as S_0 increases, which is consistent with previous results in this work.

Note also that in every scenario, firms produce above their baseline production. The only scenario in which a firm produces exactly its baseline occurs with a high baseline firm with low SREC prices. In this case, the firm produces its baseline and sells as much as possible given the SREC price and cost parameters, as it is all but guaranteed compliance.

We also note that the low-production firm is incentivized to increase production over the course of the period through the requirement R to a more substantial degree than the other firms, regardless of S_0. This is evidenced by the fact that their optimal generation exceeds their baseline to a greater degree than the mid-production and high-production firms across all values of S_0. That is, they respond to an 'ambitious' requirement by investing into generation, which is the ultimate goal of SREC markets. This provides some evidence that setting R to be significantly above economy-wide baseline generation can incite higher degrees of investment into SREC generation. However, this is a very simplistic setting and analysis that does not consider the impact to the firm's profit, political and lobbying pressures against high requirements, and other important factors. This remains an interesting area for future work which we hope to continue to build towards, with the model presented in this work being a significant first step.

5.4.3. Sensitivity to ζ, γ. Finally, we explore sensitivity to two particularly important parameters: ζ and γ, which represent the generation cost parameter and trading speed penalty parameter respectively. In the figure below, we show how optimal behaviours change for various values of ζ and γ, at a given time-step ($t = 50$) and SREC price level ($S_n = 150$).

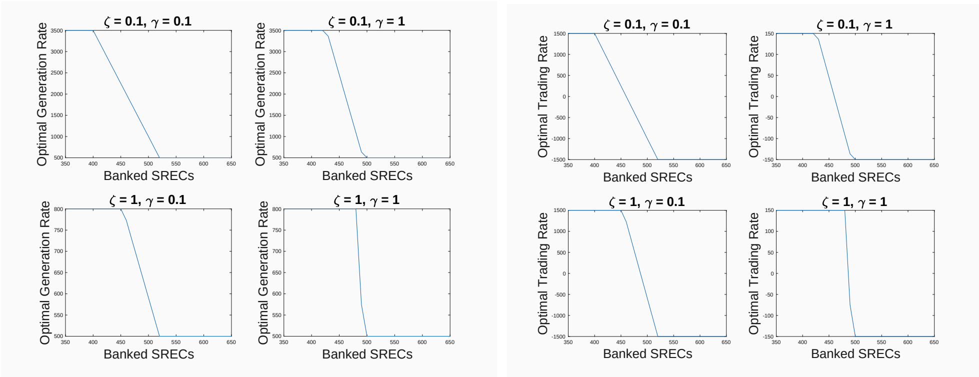

Figure 10. *Optimal generation and trading for differing levels of* ζ, γ ***MAKE THE AXES NUMBERS BIGGER***

Both optimal generation and trading respond in very similar ways. First, we notice that a change in γ impacts the level of optimal trading significantly, while a change in ζ impacts the level of optimal generation. This is in accordance with the fact that the former represents the trading speed penalty and the latter represents the generation penalty.

Additionally, it is clear that lowering both ζ and γ widens the range over which the firm's optimal behaviour has non-zero derivative in b. Recall from Section 5.1 that the regimes observed in the optimal behaviours correspond to the marginal benefit of the firm holding an additional SREC. In particular, the optimal behaviour having non-zero derivative corresponds to the values of b, S where the marginal benefit of an additional SREC is between 0 and P. The expansion of this range is the result of the increased generation and trading capacity that the firm experiences due to lower costs.

5.5. Multi-period models. Thus far, we have considered a single period compliance framework, as described in Section 2. In practice, SREC markets consist of multiple periods. In this section, we present the results from numerically solving for the optimal behaviour of a regulated firm in a multi-period SREC market. Much of the behaviour and intuition discussed in the earlier parts of this section also applies to the multi-period model.

5.5.1. Multi-period models without price impacts. We retain the assumptions specified in Section 2 regarding a multi-period compliance framework. Specifically, we assume that P and R are constant across each of the n periods. We denote the end points of the i-th period by T_i. We also assume that firms may bank unused certificates in one period to future periods with no expiry. In SREC markets, certificates generally cannot be banked indefinitely, but for a finite amount of periods (in New Jersey, unused certificates can be banked for four periods). Assuming firms can bank unused certificates indefinitely reduces the dimensionality of the problem significantly, making it more computationally tractable. We adjust our solution algorithm described in Section 5.1 to account for the assumptions stated above, using the same parameters, and choosing $n = 5$. We denote the current period by m.

In Figure 11, we plot the dependence of the optimal generation and trading rate of the firm in the first period of the 5−period model ($m = 1$) against banked SRECs, for three SREC prices, at six points in time.

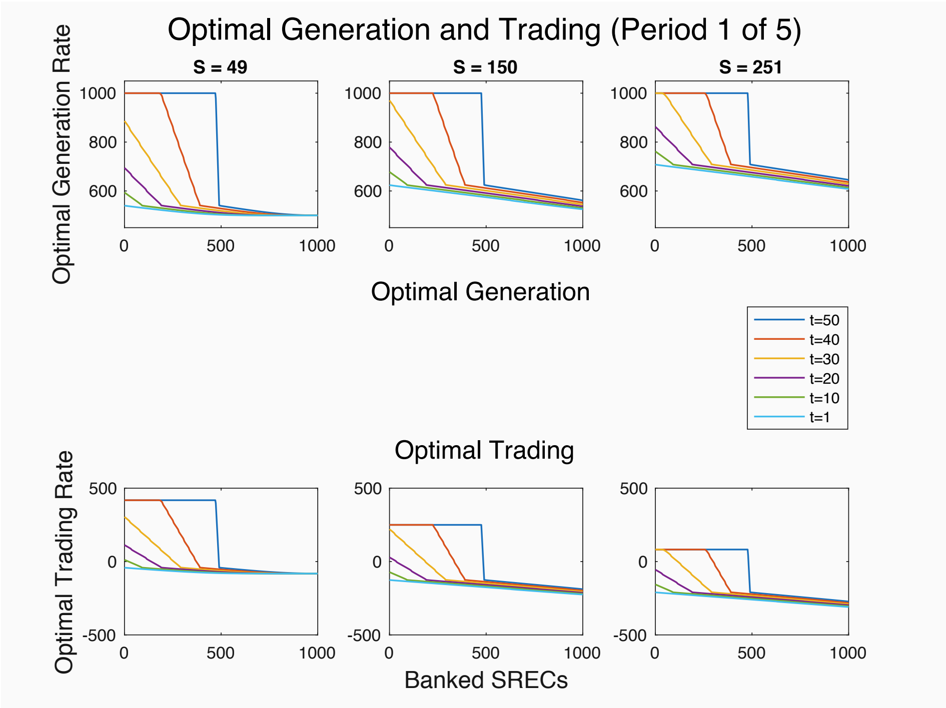

Figure 11. *Optimal firm behaviour as a function of banked SRECs across various time-steps and SREC market prices. This represents the firm's optimal behaviour in the first period of a 5-period model*

We begin by discussing the generic shape shared by each subplot. Much of the intuition we presented in explaining Figure 1 applies here, though there are obvious differences between Figures 1 and 11. As before, for low levels of banked SRECs, across all values of S, and nearly terminal date, the firm generates until the marginal cost of producing another SREC exceeds P, and purchases until the marginal cost of purchasing another SREC exceeds P, as the firm is almost assured to fail to comply. In this regime, the marginal benefit of an additional SREC to the firm is P, as each additional SREC lowers their non-compliance obligation by P.

As b increases, the firm reaches a point where the marginal benefit from an additional SREC decreases from P. This occurs as the probability of compliance becomes non-negligible, as additional SRECs in excess of R provide smaller marginal benefit than P. This leads to a decrease in optimal generation and optimal trading, as the firm adjusts its behaviour so that its marginal costs are in line with this marginal benefit. Thus far, this is the same interpretation as the single-period setting. As b continues to increase, the firm holds sufficient banked SRECs such that they will be able to acquire surplus certificates above R. These surplus SRECs have no value in the current period to the firm. However, they may be banked for the future, putting the firm in a better position for future compliance periods. In the single-period case, these SRECs lack this utility, as there is no future period to position oneself for. As a result, we see an abrupt change in the slope of the optimal controls, and a slower decay in generation and purchasing rate when compared to Figure 1.

This decrease continues until the firm no longer benefits from additional SRECs. That is, at a certain level of b, the marginal benefit of an additional SREC is zero. Specifically, having an additional SREC does not increase the firm's likelihood of compliance in current or future periods, nor can the firm sell the additional SREC for a profit (taking into account their trading costs and S). As in Figure 1, this results in optimal generation plateauing at the baseline amount, $h(t)$ and optimal trading plateauing at the level where the marginal revenue from trading equals the marginal cost. We observe that this plateau is not visible in every subplot in the figure above, due to the axes limits chosen, and the fact that $m = 1$. As m increases, this plateau occurs for lower values of b, as there are fewer future periods to position oneself for. See Appendix B for the figures when $m = 2, 3, 4, 5$.

We now turn our attention to the differences between subplots. It is immediately clear that purchasing is negatively correlated to changes in SREC price and that generation is positively correlated to changes in SREC price. As before, we note that if we hold b, S constant, generation and purchasing are increasing in t. As S increases, the value of b above which optimal generation and trading is invariant also increases. This is more apparent for $m = 2, 3, 4$.

The optimal controls for all non-terminal periods behave similarly[6]. As m increases, the firm has fewer future periods to position themselves for. Consequently, the firm's optimal generation and purchasing behaviour decays more quickly for larger m, once they are in the regime where they may acquire more than R SRECs. See Appendix B for more figures.

In addition to the plot of the optimal controls as a function of b and S, we model the paths of various quantities of interest for three optimally behaving firms (with the same cost functions) throughout the course of the 5-period SREC market, with each period lasting 1 year, as shown in Figure 12. Firm 1 has $b_0 = 0$, Firm 2 has $b_0 = 250$ and Firm 3 has $b_0 = 500$. We set $S_0 = 150$ and simulate a path for S. At each time-step, each firm behaves optimally (per our solution) given their values of banked SRECs and the SREC price.

There are clear similarities between Figure 12 and Figure 2. Specifically, we see the same positive relationship between SREC price and generation rate, and the same inverse relationship between SREC price and purchasing. In this sample path, the SREC price generally decreases over the compliance period, and accordingly, we observe each firm generating less and selling less as this occurs. We also see the banked SRECs for all three firms converge at $R(500)$ as $t \longrightarrow 5$. This is expected - as before, there is no advantage to banking SRECs in the terminal period, so an optimally behaving firm will aim to end up with exactly R SRECs in the terminal period, if compliance is possible. This also implies that Firm 3 accumulates SRECs at a slower rate than Firm 2, who accumulates SRECs at a slower rate than Firm 1. The rate at which each firm accumulates SRECs is also consistent over time, and between periods. This results in the 'converging sawtooth' pattern we see in the first subplot of Figure 12.

The optimal behaviours of each firm are nearly parallel to one another, suggesting that they react similarly to changes in S. The primary difference in their behaviours are the magnitude of trading and generation, which are influenced by their initial amount of banked

[6]In terminal periods, the optimal behaviours are equivalent to the behaviours of the firm regulated in a single-period compliance system, hence it is not addressed in detail in this section

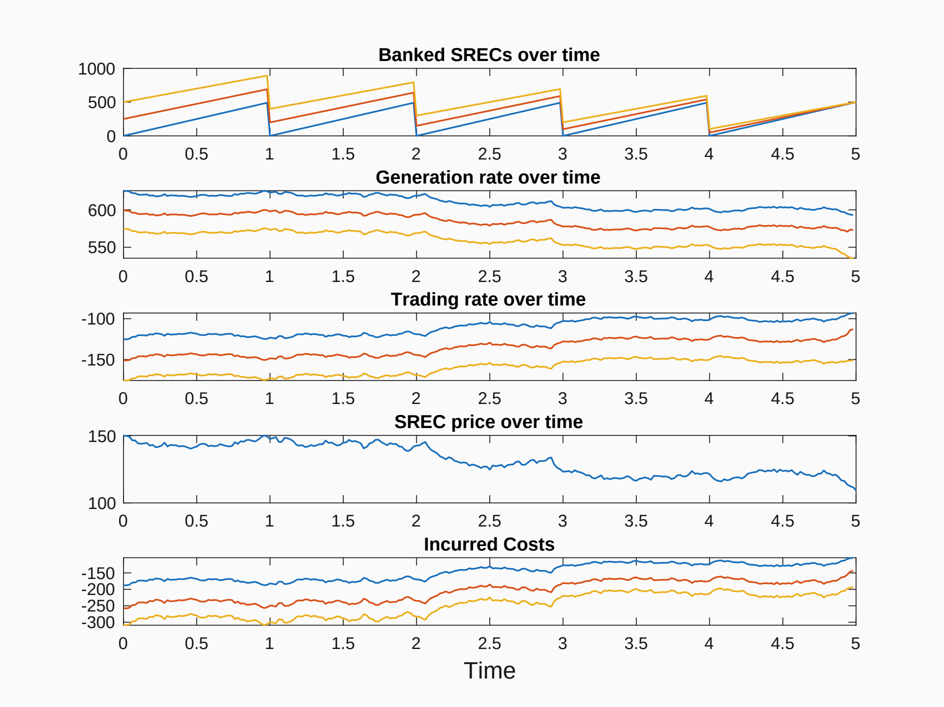

Figure 12. *Paths of three optimally behaving firms in a 5-period compliance system with* $S_0 = 150, b_0 = 0$ *(blue),* $b_0 = 250$ *(red),* $b_0 = 500$ *(yellow)*

SRECs, $b0$. Firm 1 has no spare SRECs at $t = 0$, and thus they generate the most and sell the least. Firm 3 has 500 spare SRECs at $t = 0$ - enough for an entire period of compliance. As such, they produce the least and sell the most. Firm 2 operates between Firm 1 and Firm 3. Naturally, Firm 3 profits the most from this system, due to their initial position.

As in Section 5.2, we simulate many paths similar to the method above in order to obtain summary statistics and learn about the distribution of various quantities for each firm. In Figures 13 and 14, we plot the histograms of total generated SRECs and total traded SRECs for each firm in each period, based on 1000 sample paths of S, with $S_0 = 150$.

In Figures 13 and 14, the (i, j) histogram corresponds to the histogram of the aggregate behaviour of Firm i (of 3) in period j (of 5). Each histogram exhibits symmetry. The most notable aspect of these plots is that for each row, the mean of total generated SRECs and total traded SRECs does not change significantly as the period changes. Consequently, given the initial value of the SREC price S_0, we would expect the firm to have similar aggregate behaviour across all periods. This is a result of the martingale property of S_t, for our choice of parameters, as specified earlier in this subsection and Section 5.1[7].

However, the variance of each firm's aggregate behaviour increases as the periods progress. This is the result of simulating forward paths of S_t conditioning on $\mathcal{F}_0$, as $\mathrm{Var}(S_t|S_0)$ is

[7]While not covered in detail here, this property does not hold if price impacts are activated, as S_t would no longer be a martingale

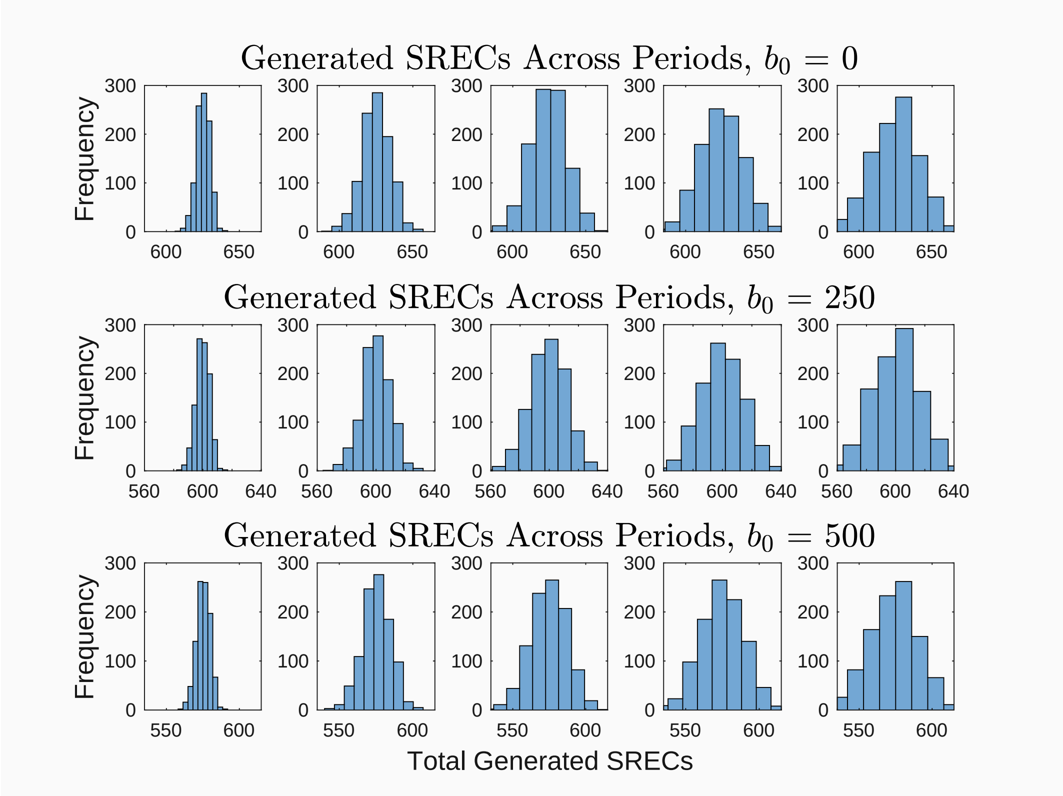

Figure 13. *Histogram of firm generation across each period, with* $S_0 = 150$

increasing in t. The increased variance in the paths of S as time passes corresponds to increased variance in total generation and trading.

As in the single-period case, the correlation coefficient between total generation and total trading in each period is -1, indicating that the two quantities exist on a line in $\mathbb{R}^2$. In particular, they satisfy the linear relation $\int_0^{T_i} g_u du + \int_0^{T_i} \Gamma_u du = \frac{nR-b_0}{n}$ for all $i \in [n]$. This further implies that for all $i \in [n]$, the distribution of b_{T_i} is a point mass.

5.5.2. Multi-period models with price impacts. Thus far, we have discussed multi-period SREC frameworks where $\psi = \eta = 0$. As in Section 5.4.1, we now consider the firm's optimal behaviour when this is not the case. As before, we choose $\eta = 0.01, \psi = 0.005$ as the price impacts of trading and generation respectively.

Similar to Figure 6, we plot the optimal behaviours of the regulated firm as a function of banked SRECs, across three different prices of S and at six points in time.

In comparing Figure 15 to Figure 11, we observe the same characteristics as in Section 5.4.1. Specifically, the optimal controls at each of the 'plateaus' now vary slightly with time-step. Furthermore, with price impacts activated firms generate more (less) and purchase less (more) when far from compliance (near compliance) relative to their optimal behaviour without price impacts activated. This behaviour ensures that they influence the price of SRECs to their advantage (relative to their no-price-impact behaviour). That is, when far

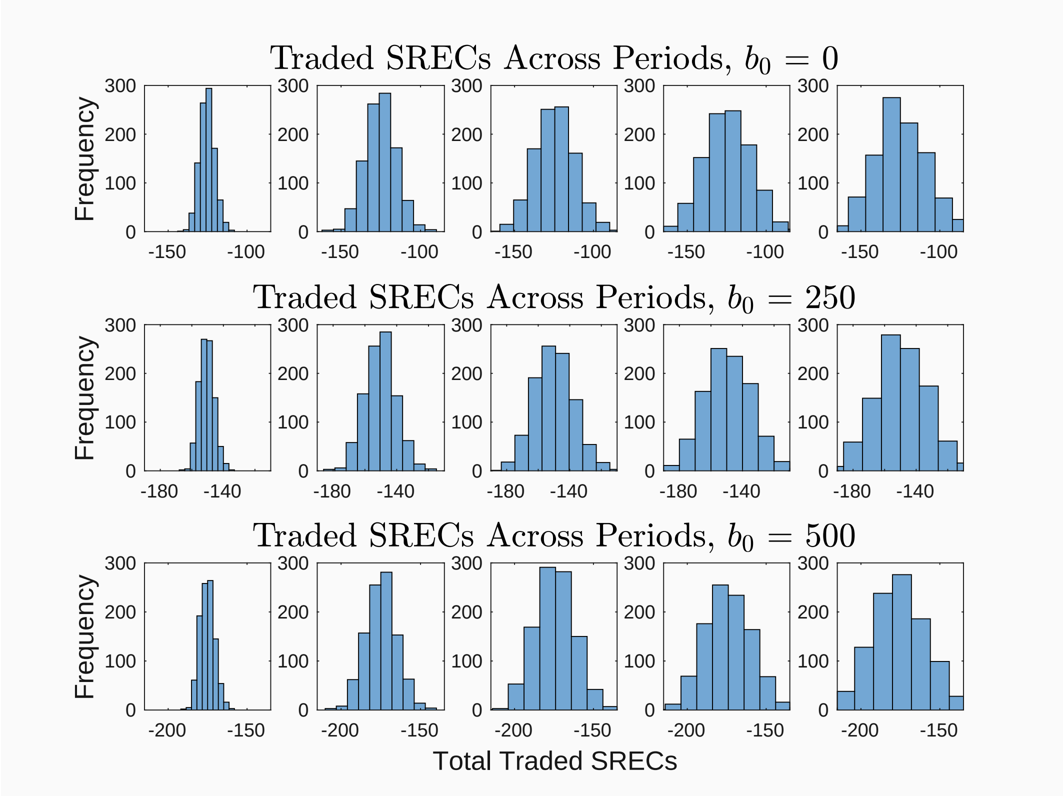

Figure 14. *Histogram of firm trading across each period, with* $S_0 = 150$

from compliance, their behaviours will keep S relatively lower than if they reprised their no-price-impact behaviour, and when on track to comply, their behaviours will keep S relatively higher.

We now explore how the behaviour of a regulated firm in a 5-period SREC framework changes with the activation of price impacts. To do so, we replicate Figure 7 in the multi-period setting. Here, we choose $S_0 = 150, b_0 = 0$. The results are shown in Figure 16.

Regardless of the activation of price impacts, the optimally behaving firm accumulates SRECs at nearly identical rates, which are consistent across periods, as in Figure 12, resulting in the same saw-tooth pattern in the first subplot of Figure 16. As expected, the regulated firm generates less and purchases less when price impacts are active, in order to mitigate their price impact. Nonetheless, we see that the price paths diverge over time, with the path of S when price impacts are active being dominated by the path of S when price impacts are inactive. Lastly, we note that activating price impacts results in lower profit for the firm. Each of these properties were exhibited in the single-period setting, in Figure 7.

6. Conclusion. In this work, we characterize the optimal behaviour of a single regulated LSE in a single-period SREC market. In particular, we characterize their optimal generation and trading behaviour as the solution to a continuous time stochastic control problem. In doing so, we characterize the solution and tease out some essential features of the optimal

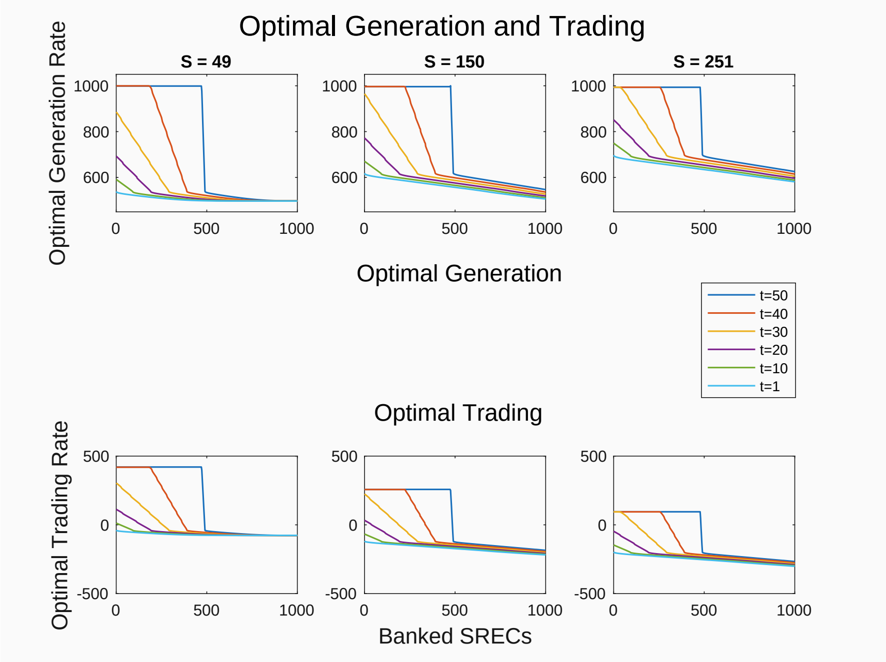

Figure 15. *Optimal firm behaviour as a function of banked SRECs across various time-steps and SREC market prices, with price impacts activated. This represents the firm's optimal behaviour in the first period of a 5-period model*

strategy. We also numerically solve for the system in a discrete time setting for both single and multi-period SREC frameworks. In doing so, we also provide intuition and reasoning for the resulting optimal behaviour, including detailed analysis of various sample paths, summary statistics, strategies, and parameter groups.

Many further extensions are possible. Interactions between agents are a critical component of real SREC markets that are largely ignored in this single-firm setup. In particular incorporating partial information of firms would be a very challenging but mathematically interesting problem that would more closely mimic the realities of SREC markets. This could potentially necessitate the use of a mean field games approach. Improved calibration to real world parameters would also increase the applicability of this work for use by regulated firms and regulators.

However, even our simple model reveals salient facts about the nature of these systems and how firms should behave when regulated by them. Our single-period model reveals that the optimal generation and trading of regulated firms broadly exists in three regimes, depending on the marginal benefit received from holding an additional SREC. We observe that a firm's trading behaviour is more sensitive to changes in S than its generation behaviour, and that higher SREC prices imply greater generation and lower purchasing (more selling). We also show that the optimal behaviour outperforms the optimal constant behaviour strategy for our parameter choice, and study the effects of including price impacts in our model, as well as sensitivity to other parameters.

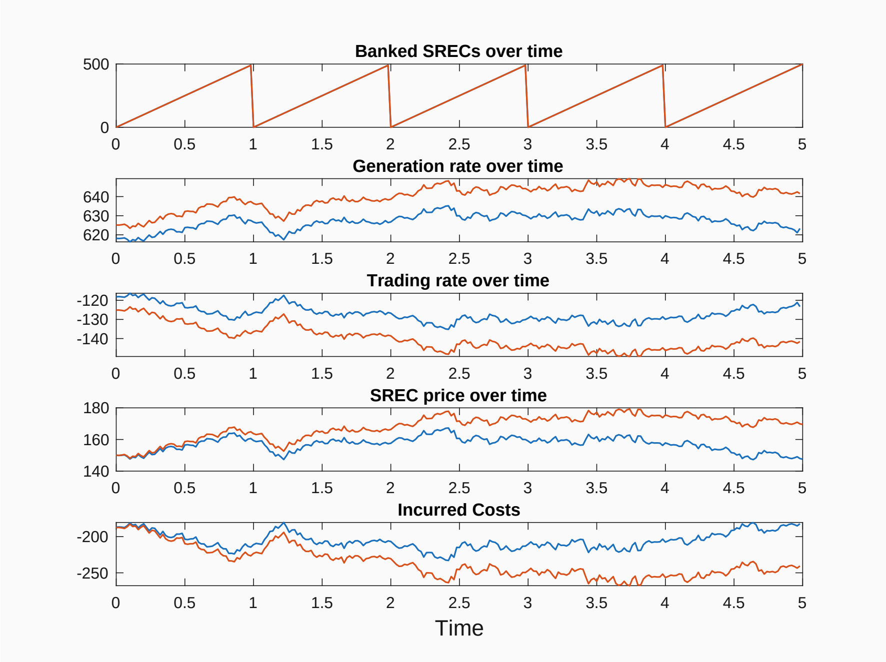

Figure 16. *Optimal firm behaviour with* $S_0 = 150, b_0 = 0$ *with price impacts active (blue) and inactive (red)*

When extending to the multiple-period framework, we observe many similarities, but also the key difference that a fourth regime exists in the optimal generation and trading of regulated firms; that is, the regime where a marginal SREC does not provide value in the current period, but may be banked to provide value in the future. Additionally, we compare and contrast the optimal behaviours of firms throughout the multiple-period framework based on different initialization points. Lastly, we discover that conditional on S_0 and b_0, the mean of a firm's total generation and total trading does not change across periods, but the variance of said quantities increase.

In providing these results, we have produced a framework and numerical solution that would be of use for both regulated firms and regulatory bodies who both have immense interest in understanding the optimal behaviour of regulated LSEs in these systems.

Appendix A. Optimal Constant Behaviour Strategy.

A sub-problem to the one considered in prior subsections is how a firm should behave if it is restricted to behaving in a constant manner over the course of the compliance period. This provides insight into how their controls will change with respect to parameter changes. As such, consider a regulated firm that is optimizing their behaviour at $t = 0$, with $b_0 = 0$ and $T = 1$. For simplicity, further assume that $h(t), \mu(t), \sigma(t)$ are constants. That is, $h(t) = h, \mu(t) = \mu, \sigma(t) = \sigma$. In particular, we consider the case where we constrain $g + \Gamma = R$

Proposition A.1 (Optimal Constant Behaviours).

Consider a single firm that is regulated in a single-period SREC market, with the following additional assumptions:

- $h(t) = h, \mu(t) = \mu, \sigma(t) = \sigma$
- $b_0 = 0, T = 1$
- *The controls* g_t, Γ_t *must be constant across the period (i.e.* $g_t = g, \Gamma_t = \Gamma$ *for all* $t \in [0, T]$
- $g + \Gamma = R$

Therefore, the firm aims to maximize the following:

$$J(g, \Gamma) = \mathbb{E}\left[-\int_0^T \frac{\zeta}{2}(g-h)^2 du - \int_0^T \Gamma S_u^{g,\Gamma} du - \int_0^T \frac{\gamma}{2}\Gamma^2 du\right] \tag{A.1}$$

The optimal control is given by $(g^*, \Gamma^*) = \left(\frac{2S_0+\mu+R(\psi+2\eta+2\gamma)+2\zeta h}{2(\eta+\gamma+\psi+\zeta)}, -\frac{2S_0+\mu-R(\psi+2\zeta)+2\zeta h}{2(\eta+\gamma+\psi+\zeta)}\right)$

Proof. The RHS of (A.1) is identical to the RHS of (2.2) updated for the additional assumptions made (note that we can remove the non-compliance penalty as the constraint $g + \Gamma = R$ ensures compliance). This is a constrained optimization problem, which we solve through the use of Lagrange multipliers.

$$\begin{aligned}
J(g, \Gamma) &= \mathbb{E}\left[-\int_0^T \frac{\zeta}{2}(g-h)^2 du - \int_0^T \Gamma S_u^{g,\Gamma} du - \int_0^T \frac{\gamma}{2}\Gamma^2 du\right] \\
&= -\frac{\zeta}{2}(g-h)^2 - \Gamma\mathbb{E}\left[\int_0^T S_u^{g,\Gamma} du\right] - \frac{\gamma}{2}\Gamma^2 \\
&= -\frac{\zeta}{2}(g-h)^2 - \Gamma\int_0^T \mathbb{E}[S_u] du - \frac{\gamma}{2}\Gamma^2 && \text{(Fubini's Theorem)} \\
&= -\frac{\zeta}{2}(g-h)^2 - \Gamma\int_0^T (S_0 + (\mu + \eta\Gamma - \psi g)u) du - \frac{\gamma}{2}\Gamma^2 \\
&= -\frac{\zeta}{2}(g-h)^2 - \Gamma(S_0 + \frac{\mu}{2}) - \frac{\Gamma^2}{2}(\eta+\gamma) + \frac{\psi}{2}g\Gamma && \text{(A.2)}
\end{aligned}$$

We introduce λ as an auxiliary variable, and define:

$$\mathcal{L}(g, \Gamma, \lambda) := -\frac{\zeta}{2}(g-h)^2 - \Gamma(S_0 + \frac{\mu}{2}) - \frac{\Gamma^2}{2}(\eta+\gamma) + \frac{\psi}{2}g\Gamma - \lambda(g + \Gamma - R) \tag{A.3}$$

Set $\nabla\mathcal{L} = 0$ to obtain the following system of equations, which is a necessary condition for a candidate optimizer to satisfy:

$$-\zeta(g-h) + \frac{\psi}{2}\Gamma - \lambda = 0 \tag{A.4}$$

$$-S_0 - \frac{\mu}{2} - (\eta+\gamma)\Gamma + \frac{\psi}{2}g - \lambda = 0 \tag{A.5}$$

$$-g - \Gamma + R = 0 \tag{A.6}$$

This is a system of three equations and three unknowns. The solution to this system is given by (with λ omitted):

$$g = \frac{2S_0 + \mu + R(\psi + 2\eta + 2\gamma) + 2\zeta h}{2(\eta + \gamma + \psi + \zeta)} \qquad \Gamma = -\frac{2S_0 + \mu - R(\psi + 2\zeta) + 2\zeta h}{2(\eta + \gamma + \psi + \zeta)} \tag{A.7}$$

Since J is concave and the equality constraint is linear, a point satisfying the necessary conditions for an optimizer is in fact the global optimizer (see [4]). Therefore, g^*, Γ^* take the forms seen in (A.7) ■

In Section 5.3, we consider the Mean Behaviour strategy given by $g = 625, \Gamma = -125$. By substituting the applicable parameters from Section 5.1 into (A.7), we see that these values coincide with the optimal behaviours given by A.1. For these parameters, it can also be shown that the Mean Behaviour strategy is optimal among all constant behaviour strategies.

Appendix B. Additional Figures.

Included below are plots of the regulated firm's optimal behaviour in the context of Section 5.5, for periods 2-5 of a 5-period model.

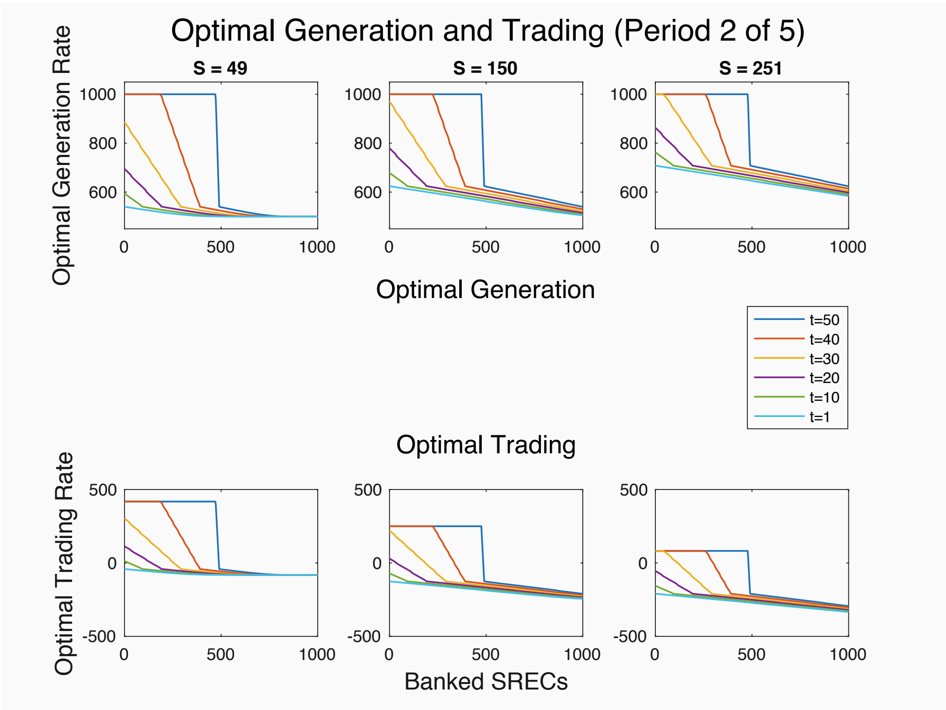

Figure 17.

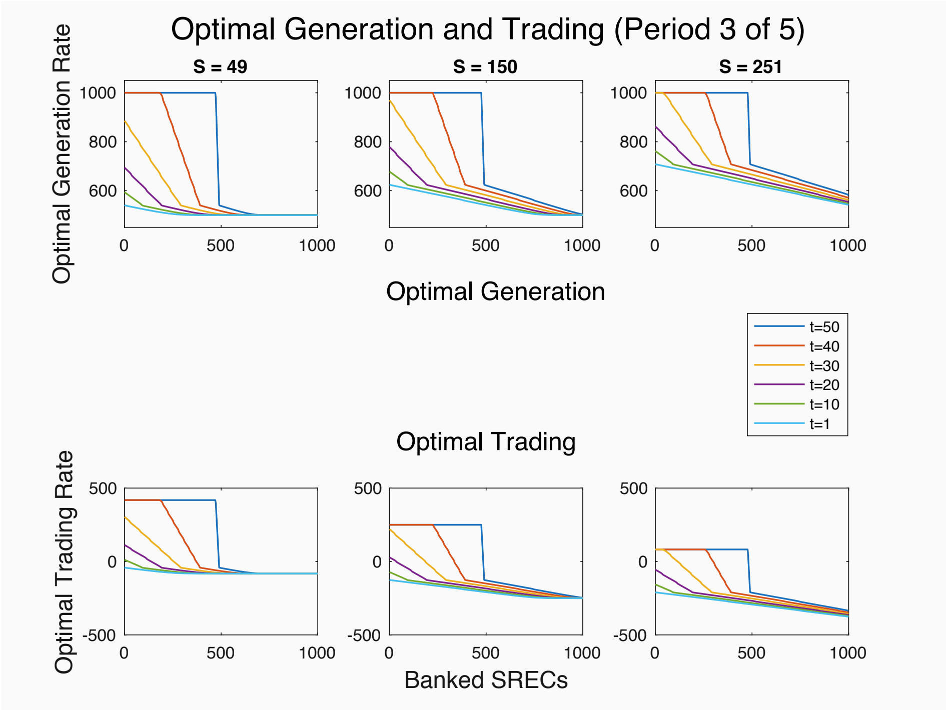

Figure 18.

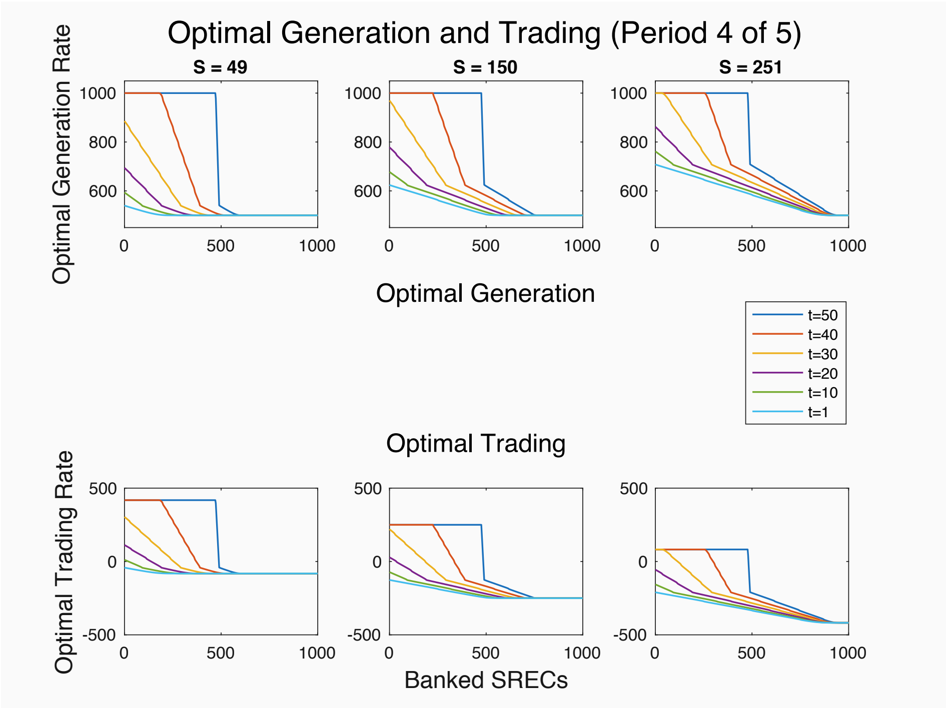

Figure 19.

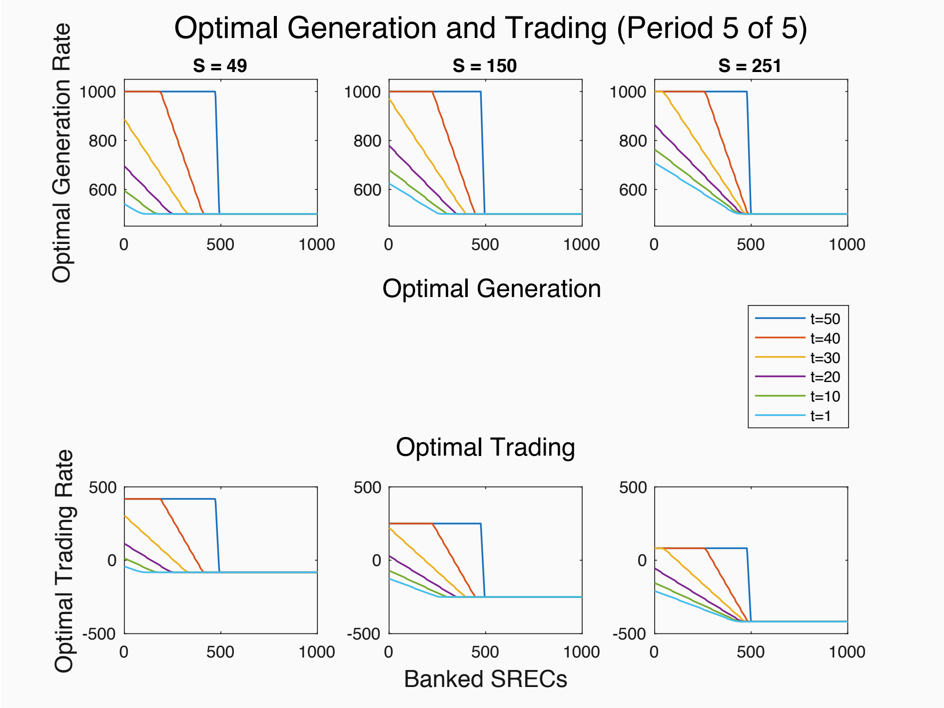

Figure 20.

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