Sequences and Time Series

Time Series Modeling

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Strings, sequences, time series

| A *string* or *sequence*, $S = (c_1, c_2, \ldots, c_N)$, *is a finite sequence of symbols.*

abcbbbaabbaabcbbbaaabbc

| A *time series*, $T = (d_1, d_2, \ldots, d_N)$, *is a finite sequence of data values.*

Interest in the term “big data”

https://trends.google.com/trends/explore?date=2008-12-19%202018-01-19&q=big%20data

Comparing time series

Forecasting time series

Motifs

"Big Data"

"Machine Learning"

"Deep Learning"

Motifs

"Big Data"
"Machine Learning"
"Deep Learning"
"Time Series"
100
75
50
25
Note
Feb 24, 2013
Nov 23, 2014
Aug 21, 2016

Classification

Markers on human body [1]

Multi-variate time series tracking 7 markers on subject's left leg

1. https://en.wikipedia.org/wiki/Motion_capture

Time Series Models

- Question: Can we discover a closed form formula (or a model) that describes the time series?

- Stationary vs. non-stationary series
- Autoregressive and moving average series
- Trends, seasonal, or cyclic patterns

Stationary Series

- Stationary series
  - statistical properties, such as mean and variance, are constant over time.

Non-Stationary Series

- Non-stationary series
  - statistical properties change over time

Non-Stationary Series

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Cyclic with constant variance

Cyclic with time-dependent variance

Non-Stationary Series

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  - statistical properties change over time

Cyclic with time-dependent speed

Cyclic with constant speed

Stationary Series

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- Non-stationary series
  - statistical properties change over time

- Why important?
  - most statistical forecasting methods assume that the series can be rendered (approximately) stationary through mathematical transformations

Random Time Series Model

- Current observations only reflect current (random) error:

$$X_t = E_t \qquad \boxed{E_t \sim N(0, \sigma^2)}$$

- Intuitively, the "random error" represents a stochastic process that does not depend on the past

Autoregressive Time Series Model

- Current observations only reflect current (random) error and has no dependence on the past observations:

$$X_t = E_t$$

$$\boxed{E_t \sim N(0, \sigma^2)}$$

- AR(1): the current observation depends only the previous time instance and the current error

$$X_t = \alpha X_{t-1} + E_t + \lambda$$

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- AR(1): the current observation depends only the previous time instance and the current error

$$X_t = \alpha X_{t-1} + E_t + \lambda$$

- AR(2): the current observation depends only the previous 2 time instances and the current error

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-1} + E_t + \lambda$$

Moving Average Time Series Model

- MA(1): The current observation depends only to the error in the previous time instance and the current error

$$X_t = \beta E_{t-1} + E_t + \lambda$$

- MA(2): The current observation depends only to the error in the previous 2 time instances and the current error

$$X_t = \beta_1 E_{t-1} + \beta_2 E_{t-2} + E_t + \lambda$$

ARMA model

- ARMA(a,m):
  - AR(a): The model has a auto-regressive terms
  - MA(m): The model has m moving-average terms

$$X_t = \left(\alpha_1 X_{t-1} + .. + \alpha_a X_{t-a}\right) + \left(\beta_1 E_{t-1} + .. + \beta_m E_{t-m}\right) + E_t + \lambda$$

Models with Differencing

- "differencing" enables models to also consider speed of change and degree of acceleration in determining the current value:
  - Order-1 differencing:

$$X_t^{(1)} = X_t - X_{t-1}$$

speed of change

  - Order-2 differencing:

$$X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)}$$

degree of acceleration

  - …
  - Order-d differencing:

$$X_t^{(d)} = X_t^{(d-1)} - X_{t-1}^{(d-1)}$$

ARIMA model

- ARIMA(a,d,m)
  - AR(a): The model has a auto-regressive terms
  - MA(m): The model has m moving-average terms
  - Integrated with order-d differencing

$$X_t = \left(\alpha_1 X_{t-1} + .. + \alpha_a X_{t-a}\right) + \left(\beta_1 E_{t-1} + .. + \beta_m E_{t-m}\right) + E_t + \lambda + \left(\theta_1 X_{t-1}^{(d)} + .. + \theta_m X_{t-m}^{(d)}\right)$$

Models with Seasonal Differencing

- Seasonal "differencing" enables models to also consider speed of changes of values across a gap:
  - s seasonal differencing:

$$S_t^{(s)} = X_t - X_{t-s}$$

Models with Seasonal Differencing

- Seasonal "differencing" enables models to also consider speed of changes of values across a lag:
  - lag "s" seasonal differencing:

$$S_t^{(s)} = X_t - X_{t-s}$$

- Seasonal ARIMA models also incorporate seasonal terms
  - additive seasonal models

$$X_t = \mathrm{ARIMA}(a,d,m) + \mathrm{SEASONAL}_s(A,D,M)$$

  - Multiplicative seasonal models

$$X_t = \mathrm{ARIMA}(a,d,m) \times \mathrm{SEASONAL}_s(A,D,M)$$

  - A= # seasonal autoregressive terms, D= # seasonal differences, M = # seasonal moving average terms

Box-Jenkins procedure

- Remove any seasonal patterns and deterministic trends that may hide valuable information and patterns through differencing

- When the mean trend is stochastic, differencing the series may yield a stationary stochastic process and, thus, may help convert a non-stationary series to a stationary one

Original Series

Differenced Series

Box-Jenkins procedure

- Remove any seasonal patterns and deterministic trends that may hide valuable information and patterns through differencing

- When the mean trend is stochastic, differencing the series may yield a stationary stochastic process and, thus, may help convert a non-stationary series to a stationary one

- Series that can be rendered stationary by D time differencing are called "integrated processes of order D"

Model based Time Series Analysis

- Find the parameters of the model
  - Model fitting
    - the model should be as simple as possible (contain as few terms as possible)
    - the fit to historic data should be as good as possible

Model based Time Series Analysis

- Find the parameters of the model
  - Model fitting
  - Plot analysis
    - Autocorrelation Function (ACF) helps observe linear relationships between lagged values of a time series

$$\mathrm{ACF}(X, lag) = \frac{E\left[\left(X_t - \mu\right)\left(X_{t+lag} - \mu\right)\right]}{\sigma^2} = \frac{\mathrm{Covar}(X_t, X_{t+lag})}{\sigma^2}$$

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    - Partial Autocorrelation Function (PACF) discount the impact of intermediate values

$$\mathrm{PACF}(X, lag) = \frac{\mathrm{Covar}(X_t, X_{t+lag} \mid X_{t+1}, \ldots, X_{t+lag-1})}{\sigma^2}$$

Key Properties of ACF and PACF

- For an autoregressive, AR(a), series, the partial autocorrelation function (PACF) gives 0 at lag >= a + 1

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- For a moving average, MA(m), series,
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  - the autocorrelation function (ACF) has non-zero autocorrelation only at the lags of the model.

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- For seasonal series autocorrelation function (ACF) at the seasonal lag will be large and positive.
- For trend series autocorrelation function (ACF) slowly decays.

Autoregressive model

Series A

$$X_t = X_{t-1} + E_t$$

An AR(1) model

- Slowly decaying ACF indicates a trend
- If the series has positive autocorrelations for high number of lags, then it may need differencing to remove the trend or seasonal patterns

Autoregressive model

Series A

$$X_t = X_{t-1} + E_t$$

An AR(1) model

- Partial autocorrelation of an AR(a) process is zero at lag >= a + 1

Moving average model

Series B

$$X_t = 0.5E_{t-1} + E_t$$

An MA(1) model

- Autocorrelation of an MA(*m*) process has non-zero autocorrelation only at the lags of the model

Moving average model

Series B

$$X_t = 0.5E_{t-1} + E_t$$

An MA(1) model

- Partial autocorrelation of an MA(*m*) process does not "shut off" at a fixed lag, but moves toward 0

Moving average model

Series B

$$X_t = 0.5E_{t-1} + 0.3E_{t-3} + E_t$$

An MA(3) model

- Autocorrelation of an MA(*m*) process has non-zero autocorrelation only at the lags of the model

Moving average model

Series B

$$X_t = 0.5E_{t-1} + 0.3E_{t-3} + E_t$$

An MA(3) model

- Partial autocorrelation of an MA(*m*) process does not "shut off" at a fixed lag, but moves toward 0

Non-stationary (cyclic) model

Series B

- No decay to zero – not stationary
- If the series has positive autocorrelations for high number of lags, then it may need differencing to remove the trend or seasonal patterns
- High values at fixed interval - seasonal autoregressive term
- (seasonal pattern with period of 24)

Non-stationary (cyclic) model

Series B

- The overall autocorrelation pattern indicates a cyclic pattern with period of 24

Non-stationary (cyclic) model

Series B

- The partial autocorrelation pattern may indicate a cyclic pattern with 12 unit half lag

Complex model

Complex model

- Slowly decaying ACF indicates a trend
- If the series has positive autocorrelations for high number of lags, then it may need differencing to remove the trend or seasonal patterns

Complex model

- Autocorrelation of an MA(*m*) process has non-zero autocorrelation only at the lags of the model
- Indicates MA(5) and MA(10) components.

Complex model

- Partial autocorrelation of an MA(*m*) process does not "shut off" at a fixed lag, but moves toward 0

Complex model

- The differenced series appears to be a moving average series with MA(5) and MA(10) components.

$$D_t = X_t - X_{t-1} = \alpha E_{t-5} + \beta E_{t-10} + E_t$$

$$X_t = X_{t-1} + \alpha E_{t-5} + \beta E_{t-10} + E_t$$

$$\boxed{X_t = X_{t-1} + 0.5E_{t-5} + 0.5E_{t-10} + E_t}$$

Complex model

- But isn't this series cyclic?

$$X_t = X_{t-1} + 0.5E_{t-5} + 0.5E_{t-10} + E_t$$