2.(a) let's use a indicator var Y_i to denote that the ith ball gets into bin 1. so $X_1 = \sum Y_i$ then $Var[X_1] = E[X_1^2] - E[X_1]^2 =$

$$\sum_i E[Y_i^2] + \sum_{\substack{\text{for all } i \\ i \neq j}} E[Y_i \cdot Y_j] - n^2 = \sum E[Y_i] + \sum_{i \neq j} E[Y_i]E[Y_j] - n^2$$

$$= n + 2\binom{2n}{2}\frac{1}{2}^2 - n^2 = n/2$$

by Chebyshev's inequality, $Pr[|X_1 - E[X_1]| \geq c\sqrt{n}] \leq \frac{var[X_1]}{c^2 n} = \frac{1}{2c^2}$

then $C = \frac{1}{\sqrt{2\varepsilon}}$ and we will have $Pr[|X_1 - E[X_1]| \geq c\sqrt{n}) \leq \varepsilon$.

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2.b $Pr[X_1 \geq n+c\sqrt{n}] = Pr[e^{rX_1} \geq e^{r(n+c\sqrt{n})}] \leq \frac{E[e^{rX_1}]}{e^{r(n+c\sqrt{n})}}$ from problem

$\frac{E[e^{rX_1}]}{e^{r(n+c\sqrt{n})}} \leq \frac{e^{rE[X_1]+r^2E[X_1]}}{e^{r(n+c\sqrt{n})}} = \frac{e^{rn+r^2n}}{e^{rn+cr\sqrt{n}}} = e^{r^2n-cr\sqrt{n}}$

set $r=\frac{\ln 2}{\sqrt{n}}$ so $r \leq \ln 2$ and r is positive, then $e^{r^2n-cr\sqrt{n}} = e^{\ln^2 2 - c\ln 2}$

similarly for $Pr[X_1 \leq n-c\sqrt{n}] = Pr[e^{-rX_1} \geq e^{rc\sqrt{n}-rn}] \leq \frac{E[e^{-rX_1}]}{e^{rc\sqrt{n}-rn}}$ then

$\leq \frac{e^{-rE[X_1]+r^2E[X_1]}}{e^{rc\sqrt{n}-rn}} = e^{r^2n-rc\sqrt{n}}$, set $r=\frac{\ln 2}{\sqrt{n}}$ so r is positive $r \leq \ln 2$

and $e^{r^2n-rc\sqrt{n}} = e^{\ln^2 2 - c\ln 2}$

then $Pr[|X_1-n| \geq c\sqrt{n}] = Pr[X_1 \geq n+c\sqrt{n}] + Pr[X_1 \leq n-c\sqrt{n}] \leq 2\cdot e^{\ln^2 2 - c\ln 2}$

if $c = \frac{\ln^2 2 - \ln\frac{\varepsilon}{2}}{\ln 2}$, then $2\cdot e^{\ln^2 2 - c\ln 2} = \varepsilon$, so that's the dependency.

(c) $Pr[X_1 - X_2 \geq c\sqrt{n}] = Pr[X_1 - (2n-X_1) \geq c\sqrt{n}] = Pr[2X_1 \geq 2n + c\sqrt{n}] = Pr[X_1 \geq n + \frac{c}{2}\sqrt{n}]$

$\leq \frac{E[e^{rX_1}]}{e^{rn+\frac{cr}{2}\sqrt{n}}} \leq \frac{e^{rE[X_1]+r^2E[X_1]}}{e^{rn+\frac{cr}{2}\sqrt{n}}} = e^{r^2n-\frac{cr}{2}\sqrt{n}}$

set $r=\frac{\ln 2}{\sqrt{n}}$ so r is positive and $r \leq \ln 2 \Rightarrow e^{r^2n-\frac{cr}{2}\sqrt{n}} = e^{\ln^2 2 - \frac{c\ln 2}{2}}$

if $c = \frac{2(\ln^2 2 - \ln\varepsilon)}{\ln 2}$

then we have $Pr[X_1 - X_2 \geq c\sqrt{n}] \leq e^{\ln^2 2 - \frac{c\ln 2}{2}} = \varepsilon$

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