(a) $E[e^{rX}]$ by definition $X=\sum_i X_i \Rightarrow E[e^{rX}] = E[e^{r(X_1+X_2 \cdots +X_n)}]$

and since the variables are all independent $E[e^{r(X_1+X_2+\cdots+X_n)}] =$

$\prod_i E[e^{rX_i}]$, from inequility $e^x \leq 1+x+x^2 \Rightarrow E[e^{rX_i}] \leq E[1+rX_i+r^2X_i^2]$

$= 1+rE[X_i]+r^2E[X_i^2]$ and since $X_i \in [0,1]$ then $E[X_i^2] \leq E[X_i]$

so $1+rE[X_i]+r^2E[X_i^2] \leq 1+rE[X_i]+r^2E[X_i]$, ~~let~~ $\cancel{y = rE[X_i]+r^2E[X_i]}$

think

then $1+rE[X_i]+r^2E[X_i] \leq e^{rE[X_i]+r^2E[X_i]}$

then $\prod_i E[e^{rX_i}] \leq \prod_i e^{rE[X_i]+r^2E[X_i]} = e^{\sum_i (rE[X_i]+r^2E[X_i])}$

$= e^{rE[X]+r^2E[X]}$

so we proved $E[e^{rX}] \leq e^{rE[X]+r^2E[X]}$

(b) we used the independence of the X_i we say $E[e^{r(X_1+X_2+X_3 \cdots X_n)}]$

$= \prod_i E[e^{rX_i}]$, since $E[e^{r(X_1+X_2+\cdots X_n)}] = E[e^{rX_1} \cdot e^{rX_2} \cdot e^{rX_3} \cdots e^{rX_n}]$

only when $X_1, X_2 \cdots X_n$ are independent we can have $E[e^{rX_1} \cdot e^{rX_2} \cdot e^{rX_3} \cdots e^{rX_n}]$

$= \prod_i E[e^{rX_i}]$

(c) i. $Pr[X \geq (1+\varepsilon)E[X]] = Pr[rX \geq r(1+\varepsilon)E[X]]$

$= Pr[e^{rX} \geq e^{r(1+\varepsilon)E[X]}]$

by markov's inequality $Pr[e^{rX} \geq e^{r(1+\varepsilon)E[X]}] \leq \frac{E[e^{rX}]}{e^{r(1+\varepsilon)E[X]}}$

then $\frac{E[e^{rX}]}{e^{r(1+\varepsilon)E[X]}} \le \frac{e^{rE[X]+r^2E[X]}}{e^{r(1+\varepsilon)E[X]}} = e^{(r^2-r\varepsilon)E[X]}$ when $r=\varepsilon/2$ since

$0\le\varepsilon\le\ln 4$, $0\le r\le \ln 2$, $r\in[-\infty,\ln 2]$, then we have $Pr[X\ge(1+\varepsilon)E[X]]\le e^{-\varepsilon^2 E[X]}$

(ii) Similarly we have $Pr[X\ge(1+\varepsilon)E[X]] = Pr[e^{rX}\ge e^{r(1+\varepsilon)E[X]}]$

$Pr[e^{rX}\ge e^{r(1+\varepsilon)E[X]}] \le \frac{E[e^{rX}]}{e^{r(1+\varepsilon)E[X]}} \le e^{(r^2-r\varepsilon)E[X]}$ suppose $r=k\ln 2$

$e^{(r^2-r\varepsilon)E[X]} = 2^{k(k\ln 2-\varepsilon)E[X]}$, when $k^2\ln 2 - k\varepsilon = -\varepsilon/2 \Rightarrow 2k^2\ln 2 - 2k\varepsilon+\varepsilon = 0$

$k=\frac{2\varepsilon\pm\sqrt{4\varepsilon^2-8\ln 2\,\varepsilon}}{4\ln 2}$, we take $k=\frac{2\varepsilon-\sqrt{4\varepsilon^2-8\ln 2\,\varepsilon}}{4\ln 2}$, first since $\varepsilon\ge\ln 4$

$4\varepsilon^2-8\ln 2\,\varepsilon\ge 0$, so k is real, then since $\sqrt{4\varepsilon^2}=2\varepsilon>\sqrt{4\varepsilon^2-8\ln 2\,\varepsilon}$

k is positive. then $k=\frac{\varepsilon-\sqrt{\varepsilon^2-2\ln 2\,\varepsilon}}{2\ln 2}$, since $\sqrt{\varepsilon-2\ln 2}<\sqrt{\varepsilon}$ then

$\varepsilon-2\ln 2<\sqrt{\varepsilon^2-2\ln 2\,\varepsilon}$ so $k<1$ and so $r=k\ln 2<\ln 2\in[-\infty,\ln 2]$

then we know if $r=\frac{\varepsilon-\sqrt{\varepsilon^2-2\ln 2\cdot\varepsilon}}{2}$, $Pr[X\ge(1+\varepsilon)E[X]]\le 2^{-\varepsilon E[X]/2}$

(iii) $Pr[X\le(1-\varepsilon)E[X]] = Pr[-rX\ge -r(1-\varepsilon)E[X]] = Pr[e^{-rX}\ge e^{-r(1-\varepsilon)E[X]}]$

$\le\frac{E[e^{-rX}]}{e^{-r(1-\varepsilon)E[X]}} \le \frac{e^{-rE[X]+r^2E[X]}}{e^{-r(1-\varepsilon)E[X]}} = e^{(r^2-r\varepsilon)E[X]}$ when $r=\varepsilon/2$, since $0\le\varepsilon\le 1$

r is positive and $-r\in[-\infty,\ln 2]$, then $Pr[X\le(1-\varepsilon)E[X]]\le e^{-\varepsilon^2E[X]/4}$

$\cancel{E[x] + \varepsilon n/4 \leq 0}$

(iv)

For $Pr[|x-E[x]| \geq \varepsilon n] = Pr[x^2 - 2xE[x] + E[x^2] \geq \varepsilon^2 n^2]$

$$\leq \frac{e^{(r+r^2)E[x^2 - 2xE[x] + E[x^2]]}}{e^{r\varepsilon^2 n^2}} \quad \text{if } r \in [-\infty, \ln 2]$$

$$= \frac{e^{(r+r^2)\,E[x^2] - E[x]^2}}{e^{r\varepsilon^2 n^2}} \leq \frac{e^{(r+r^2)(E[x] - E[x]^2)}}{e^{r\varepsilon^2 n^2}} \quad \text{if } r \geq 0$$

since $E[x^2] < E[x]$

set $r = \frac{1}{n}$ $\cancel{r = \frac{1}{E[x]}}$ (positive $r \in (-\infty, \ln 2]$) $e^{(r+r^2)(E[x]-E[x]^2)} - r\varepsilon^2 n^2 \leq 2e^{-\varepsilon^2 n/4}$

since $-\frac{3}{4}\varepsilon^2 n + (\frac{n+1}{n^2})(E[x]-E[x]^2)^{-\ln 2} \leq 0$ (which is obvious when $\varepsilon \geq \ln 4$ and n is big)

So we know $P_r[|x-E[x]| \geq \varepsilon n] \leq 2e^{-\varepsilon^2 n/4}$ when $\varepsilon \geq \ln 4$ and from

(i) and (iii), when $0 \leq \varepsilon \leq \ln 4$ and $0 \leq \varepsilon \leq 1$, $Pr[|x-E[x]| \geq \varepsilon n] \leq 2e^{-\varepsilon^2 n/4}$

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So for $\varepsilon \geq 0$, it's alway correct.

by just change ε to $\frac{\varepsilon n}{E(X)}$