1. Orders of ODE

   Definition 0.1. *An ODE is said to be of order n if the highest order derivative of the unknown function that appears in the ODE is of order n.*

2. Degree of ODE

   Definition 0.2. *An ODE is said to be of degree n if the highest power of the highest order derivative of the unknown function that appears in the ODE is n.*

   Example 0.1. *The ODE $y'' + y = 0$ is of order 2 and degree 1.*

3. Linear ODE

   Definition 0.3. *An ODE is said to be linear if it can be written in the form*

   $$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \tag{1}$$

   where $a_i(x)$ are functions of x and $f(x)$ is a function of x.

4. Homogeneous ODE

   Definition 0.4. *An ODE is said to be homogeneous if the term that does not involve the unknown function is zero.*

   Example 0.2. *The ODE $y'' + y = 0$ is homogeneous.*

   The ODE $y'' + y = x$ is not homogeneous.

5. The superposition principle

   Theorem 0.1. *If $y_1(x)$ and $y_2(x)$ are solutions of a homogeneous linear ODE, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of the ODE, where c_1 and c_2 are constants.*

6. Series solution of ODE

   Theorem 0.2. *It is possible to construct a series solution for most ODEs.*

   For ODE:

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2}$$

   If we are interested in finding a solution around $x = x_0$, we can assume the solution to be of the form:

   $$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{3}$$

   Then expanding P, Q, and R in Taylor series around $x = x_0$, we can substitute the series solution into the ODE, and solve for the coefficients a_n.

   Example 0.3. *Find the series solution of the ODE*

   $$(1 + x^2)y'' + xy' - y = 0 \tag{4}$$

   about $x = 0$.

We assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

Substituting the series solution into the ODE, we have:

$$(1+x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{6}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 0 \tag{7}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 0 \tag{8}$$

$$\sum_{n=0}^{\infty} \left(\left(n(n-1)+n-1\right) a_n + (n+2)(n+1)a_{n+2} \right) x^n = 0 \tag{9}$$

Since the series must be zero for all x, the coefficients of x^n must be zero for all n. Therefore, we have:

$$\begin{aligned} \left((n+1)(n-1)\right) a_n + (n+2)(n+1)a_{n+2} &= 0, \quad n \geq 0 && (10) \\ a_{n+2} &= -\frac{(n+1)(n-1)}{(n+2)(n+1)} a_n && (11) \\ a_{n+2} &= -\frac{n-1}{n+2} a_n && (12) \end{aligned}$$

And it is clear that the general solution depends on a_0 and a_1, and has a degeneracy of 2.

7. Ordinary points and singular points

Definition 0.5. *Given an ODE of the form:*

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{13}$$

A point $x = x_0$ is said to be an ordinary point if $Q(x)/P(x)$, and $R(x)/P(x)$ are analytic at $x = x_0$.

Otherwise, $x = x_0$ is said to be a singular point.

Trying to find a series solution around an ordinary point will result in a solution that has a degeneracy of 2.

8. Regular singular points

Definition 0.6. *Given an ODE of the form:*

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{14}$$

A singular point $x = x_0$ is said to be a regular singular point if $(x-x_0)Q(x)/P(x)$, and $(x-x_0)^2R(x)/P(x)$ are analytic at $x = x_0$. Otherwise, it is an irregular singular point.

Finding a solution around a regular singular point will need special treatment.

9. Euler Equation

Definition 0.7. *An ODE of the form:*

$$x^2y'' + axy' + by = 0 \tag{15}$$

is called an Euler equation.
If the quadraric eqation:

$$m^2 + (a-1)m + b = 0 \tag{16}$$

has two distinct roots m_1 and m_2, then the general solution of the Euler equation is:

$$y(x) = c_1x^{m_1} + c_2x^{m_2} \tag{17}$$

If the quadratic equation has a single root m, then the general solution of the Euler equation is:

$$y(x) = c_1x^m + c_2x^m \ln x \tag{18}$$

10. Frobenius method

Definition 0.8. *The Frobenius method is a method to find the series solution of ODEs around regular singular points.*
Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{19}$$

Assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+m} \tag{20}$$

Where m is the root of the indicial equation:

$$m^2 + (\lim_{x\to x_0} \frac{xQ}{P} - 1)m + \lim_{x\to x_0} \frac{x^2R}{P} = 0 \tag{21}$$

Substituting the series solution into the ODE, we can solve for the coefficients a_n.
If the indical equation has two same roots, then the general solution will be in the form

$$y(x) = \log(x)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+m} + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+m} \tag{22}$$

If given $m_1 > m_2$ and $m_1 - m_2 \in \mathbb{Z}$, then the general solution will be in the form

$$y(x) = \log(x)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+m_1} + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+m_2} \tag{23}$$

The Frobenius method is used when we try to find the series solution around a regular singular point.

Example 0.4. *Find a series solution of the ODE:*

$$2xy'' + (3 - 2x)y' + y = 0 \tag{24}$$

It is easy to see that $x = 0$ is a regular singular point.
Solving the indicial equation:

$$m^2 + (\frac{3}{2} - 1)m = 0 \tag{25}$$

we have $m = 0$ or $m = 1$.
Therefore, assume the general solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \tag{26}$$

Substituting the series solution into the ODE, we have:

$$
\begin{aligned}
2x \sum_{n=0}^{\infty} a_n (n+m)(n+m-1)x^{n+m-2} && (27)\\
+(3-2x) \sum_{n=0}^{\infty} a_n (n+m)x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m} &= 0 & (28)\\
2 \sum_{n=0}^{\infty} a_n (n+m)(n+m-1)x^{n+m-1} && (29)\\
+3 \sum_{n=0}^{\infty} a_n (n+m)x^{n+m-1} && (30)\\
-2 \sum_{n=0}^{\infty} a_n (n+m)x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} &= 0 & (31)\\
2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1} && (32)\\
+2 \sum_{n=0}^{\infty} \left(\left((n+1+m)(2(n+m)+3) \right) a_{n+1} + \left(1 - 2(n+m)\right) a_n \right) x^{n+m} &= 0 & (33)
\end{aligned}
$$

$2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1}$ is automatically satisfied, as it is the same with the indicial equation.
Therefore, we have:

$$\left((n+1+m)(2(n+m)+3) \right) a_{n+1} + \left(1 - 2(n+m)\right) a_n = 0 \tag{34}$$

As m has two values, the general solution will depend on a_0 and m, and has a degeneracy of 2.

11. Legendre Equation

Definition 0.9. *The Legendre equation is given by: and is given by:*

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \tag{35}$$

As the Legendre equation has an ordinary point at $x = 0$, we can solve this equation for two linearly independent series solutions: $y_{0,\alpha}(x)$, and $y_{1,\alpha}(x)$.

If α is not a integer, then both $y_{0,\alpha}(x)$, and $y_{1,\alpha}(x)$ is an infinite series with a radius of convergence 1.

12. Legendre Polynomial

Definition 0.10. *If α in Equation 35 is an integer, then the series solution to Equation 35 is a polynomial, which is often known as the Legendre polynomial.*

Theorem 0.3 (Orthogonality of Legendre Polynomials)**.** *The Legendre polynomials are orthogonal on the interval $[-1, 1]$, that is:*

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2\delta_{nm}}{2n+1} \tag{36}$$

Theorem 0.4 (Negative index Legendre Polynomial)**.** *As the equation 35 is invariant under $\alpha \to -\alpha - 1$,*

$$P_n(x) = P_{-n-1}(x) \tag{37}$$

13. Bessel Equation

Definition 0.11. *The Bessel equation is given by:*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{38}$$

As the Bessel equation has a regular singular point at $x = 0$, we can solve this equation for two linearly independent series solutions.

14. Fourier Series

Definition 0.12. *The Fourier series of a function $f(x)$ is given by:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{39}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \tag{40}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{41}$$

15. Trigonometric Integration Formula

Definition 0.13. *The trigonometric integration formula is given by:*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn} \tag{42}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{43}$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0 \tag{44}$$

16. Fourier Convergence Theorem

    Theorem 0.5. *If $f(x)$ is piecewise continuous on $[-L, L]$, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous, and converges to the average of the left and right limits of $f(x)$ at points where $f(x)$ is discontinuous.*

    If the edge of the discontinuity is at $x = a$, then there will be the Gibbs phenomenon at $x = a$.

17. Gibbs Phenomenon

    Definition 0.14. *The Gibbs phenomenon is the overshoot of the Fourier series at the edge of a discontinuity.*

18. Use Fourier Series to Solve ODE with Periodic Forcing Term

    Example 0.5. *Solve the ODE:*

$$y'' + \omega^2 y = F(x) \tag{45}$$

Where F is a 2π periodic function with definition $F(x)$ on $[-\pi, \pi]$:

$$F(x) = \begin{cases} \pi + t, & -\pi \leq x < 0 \\ \pi - t, & 0 \leq x < \pi \end{cases} \tag{46}$$

We first derive the Fourier series of $F(x)$:

$$\begin{aligned}
a_n &= \int_{-\pi}^{\pi} F(x)\cos(nx)\,dx && (47)\\
&= 2\int_0^{\pi} (\pi - t)\cos(nx)\,dx && (48)\\
&= 2\left(\frac{1}{n}(\pi - t)\sin(nx)\,|_0^{\pi} + \int_0^{\pi} \frac{1}{n}\sin(nx)\right) && (49)\\
&= 2\left(+\frac{1}{n^2}(-\cos(nx))\,|_0^{\pi}\right) && (50)\\
&= -\frac{2}{n^2}\left(\cos(n\pi) - 1\right) && (51)\\
&= -\frac{2}{n^2}\left((-1)^n - 1\right) && (52)\\
b_n &= \int_{-\pi}^{\pi} F(x)\sin(nx)\,dx && (53)\\
&= 0 && (54)\\
a_0 &= \int_{-\pi}^{\pi} F(x)\cos(nx)\,dx && (55)\\
&= 2\int_0^{\pi} (\pi - t)\,dx && (56)\\
&= \pi^2 && (57)
\end{aligned}$$

For the particular solution, we assume the solution to be of the form:

$$y_p(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n\cos(nx) + B_n\sin(nx)\right) \tag{58}$$

Then we have:

$$\begin{cases} \omega^2 \frac{A_0}{2} = \frac{a_0}{2} \\ -A_n n^2 + \omega^2 A_n = a_n \\ -B_n n^2 + \omega^2 B_n = b_n = 0 \end{cases} \tag{59}$$

$$\begin{cases} A_0 = \frac{a_0}{\omega^2} \\ A_n = \frac{a_n}{n^2 - \omega^2} \\ B_n = 0 \end{cases} \tag{60}$$

The general solution is then:

$$y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x) + y_p(x) \tag{61}$$