1. Orders of ODE

   Definition 0.1. *An ODE is said to be of order n if the highest order derivative of the unknown function that appears in the ODE is of order n.*

2. Degree of ODE

   Definition 0.2. *An ODE is said to be of degree n if the highest power of the highest order derivative of the unknown function that appears in the ODE is n.*

   Example 0.1. *The ODE $y'' + y = 0$ is of order 2 and degree 1.*

3. Linear ODE

   Definition 0.3. *An ODE is said to be linear if it can be written in the form*

   $$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \tag{1}$$

   where $a_i(x)$ are functions of x and $f(x)$ is a function of x.

4. Homogeneous ODE

   Definition 0.4. *An ODE is said to be homogeneous if the term that does not involve the unknown function is zero.*

   Example 0.2. *The ODE $y'' + y = 0$ is homogeneous.*

   The ODE $y'' + y = x$ is not homogeneous.

5. The superposition principle

   Theorem 0.1. *If $y_1(x)$ and $y_2(x)$ are solutions of a homogeneous linear ODE, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of the ODE, where c_1 and c_2 are constants.*

6. Series solution of ODE

   Theorem 0.2. *It is possible to construct a series solution for most ODEs.*

   For ODE:

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2}$$

   If we are interested in finding a solution around $x = x_0$, we can assume the solution to be of the form:

   $$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{3}$$

   Then expanding P, Q, and R in Taylor series around $x = x_0$, we can substitute the series solution into the ODE, and solve for the coefficients a_n.

   Example 0.3. *Find the series solution of the ODE*

   $$(1 + x^2)y'' + xy' - y = 0 \tag{4}$$

   about $x = 0$.

We assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

Substituting the series solution into the ODE, we have:

$$(1+x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{6}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 0 \tag{7}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 0 \tag{8}$$

$$\sum_{n=0}^{\infty} \left((n(n-1)+n-1)\, a_n + (n+2)(n+1)a_{n+2} \right) x^n = 0 \tag{9}$$

Since the series must be zero for all x, the coefficients of x^n must be zero for all n. Therefore, we have:

$$\begin{aligned}
((n+1)(n-1))\, a_n + (n+2)(n+1)a_{n+2} &= 0, \quad n \geq 0 && (10)\\
a_{n+2} &= -\frac{(n+1)(n-1)}{(n+2)(n+1)} a_n && (11)\\
a_{n+2} &= -\frac{n-1}{n+2} a_n && (12)
\end{aligned}$$

And it is clear that the general solution depends on a_0 and a_1, and has a degeneracy of 2.

7. Ordinary points and singular points

   Definition 0.5. *Given an ODE of the form:*

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{13}$$

   A point $x = x_0$ is said to be an ordinary point if $Q(x)/P(x)$, and $R(x)/P(x)$ are analytic at $x = x_0$.

   Otherwise, $x = x_0$ is said to be a singular point.

   Trying to find a series solution around an ordinary point will result in a solution that has a degeneracy of 2.

8. Regular singular points

   Definition 0.6. *Given an ODE of the form:*

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{14}$$

   A singular point $x = x_0$ is said to be a regular singular point if $(x-x_0)Q(x)/P(x)$, and $(x-x_0)^2 R(x)/P(x)$ are analytic at $x = x_0$. Otherwise, it is an irregular singular point.

   Finding a solution around a regular singular point will need special treatment.

9. Euler Equation

Definition 0.7. *An ODE of the form:*

$$x^2y'' + axy' + by = 0 \tag{15}$$

is called an Euler equation.
If the quadraric eqation:

$$m^2 + (a-1)m + b = 0 \tag{16}$$

has two distinct roots m_1 and m_2, then the general solution of the Euler equation is:

$$y(x) = c_1x^{m_1} + c_2x^{m_2} \tag{17}$$

If the quadratic equation has a single root m, then the general solution of the Euler equation is:

$$y(x) = c_1x^m + c_2x^m \ln x \tag{18}$$

10. Frobenius method

Definition 0.8. *The Frobenius method is a method to find the series solution of ODEs around regular singular points.*
Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{19}$$

Assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+m} \tag{20}$$

Where m is the root of the indicial equation:

$$m^2 + (\lim_{x\to x_0} \frac{xQ}{P} - 1)m + \lim_{x\to x_0} \frac{x^2R}{P} = 0 \tag{21}$$

Substituting the series solution into the ODE, we can solve for the coefficients a_n.
If the indical equation has two same roots, then the general solution will be in the form

$$y(x) = \log(x)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+m} + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+m} \tag{22}$$

If given $m_1 > m_2$ and $m_1 - m_2 \in \mathbb{Z}$, then the general solution will be in the form

$$y(x) = \log(x)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+m_1} + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+m_2} \tag{23}$$

The Frobenius method is used when we try to find the series solution around a regular singular point.

Example 0.4. *Find a series solution of the ODE:*

$$2xy'' + (3-2x)y' + y = 0 \tag{24}$$

It is easy to see that $x = 0$ is a regular singular point.
Solving the indicial equation:

$$m^2 + (\frac{3}{2} - 1)m = 0 \tag{25}$$

we have $m = 0$ or $m = 1$.
Therefore, assume the general solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \tag{26}$$

Substituting the series solution into the ODE, we have:

$$\begin{aligned}
& 2x \sum_{n=0}^{\infty} a_n (n+m)(n+m-1)x^{n+m-2} && (27)\\
+(3-2x)\sum_{n=0}^{\infty} a_n(n+m)x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m} &= 0 && (28)\\
& 2\sum_{n=0}^{\infty} a_n (n+m)(n+m-1)x^{n+m-1} && (29)\\
& +3\sum_{n=0}^{\infty} a_n(n+m)x^{n+m-1} && (30)\\
-2\sum_{n=0}^{\infty} a_n(n+m)x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} &= 0 && (31)\\
& 2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1} && (32)\\
+2\sum_{n=0}^{\infty} \left(\left((n+1+m)(2(n+m)+3) \right) a_{n+1} + (1-2(n+m))\, a_n \right) x^{n+m} &= 0 && (33)
\end{aligned}$$

$2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1}$ is automatically satisfied, as it is the same with the indicial equation.
Therefore, we have:

$$\left((n+1+m)(2(n+m)+3) \right) a_{n+1} + (1-2(n+m))\, a_n = 0 \tag{34}$$

As m has two values, the general solution will depend on a_0 and m, and has a degeneracy of 2.

11. Legendre Equation

Definition 0.9. *The Legendre equation is given by: and is given by:*

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{35}$$

As the Legendre equation has an ordinary point at $x = 0$, we can solve this equation for two linearly independent series solutions: $y_{0,\alpha}(x)$, and $y_{1,\alpha}(x)$.

If α is not a integer, then both $y_{0,\alpha}(x)$, and $y_{1,\alpha}(x)$ is an infinite series with a radius of convergence 1.

12. Legendre Polynomial

Definition 0.10. *If α in Equation 35 is an integer, then the series solution to Equation 35 is a polynomial, which is often known as the Legendre polynomial.*

Theorem 0.3 (Orthogonality of Legendre Polynomials)**.** *The Legendre polynomials are orthogonal on the interval $[-1, 1]$, that is:*

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2\delta_{nm}}{2n+1} \tag{36}$$

Theorem 0.4 (Negative index Legendre Polynomial)**.** *As the equation 35 is invariant under $\alpha \to -\alpha - 1$,*

$$P_n(x) = P_{-n-1}(x) \tag{37}$$

13. Bessel Equation

Definition 0.11. *The Bessel equation is given by:*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{38}$$

As the Bessel equation has a regular singular point at $x = 0$, we can solve this equation for two linearly independent series solutions.

14. Fourier Series

Definition 0.12. *The Fourier series of a function $f(x)$ is given by:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{39}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \tag{40}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{41}$$

15. Trigonometric Integration Formula

Definition 0.13. *The trigonometric integration formula is given by:*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn} \tag{42}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{43}$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0 \tag{44}$$

16. Fourier Convergence Theorem

Theorem 0.5. *If $f(x)$ is piecewise continuous on $[-L, L]$, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous, and converges to the average of the left and right limits of $f(x)$ at points where $f(x)$ is discontinuous.*

If the edge of the discontinuity is at $x = a$, then there will be the Gibbs phenomenon at $x = a$.

17. Gibbs Phenomenon

Definition 0.14. *The Gibbs phenomenon is the overshoot of the Fourier series at the edge of a discontinuity.*

18. Use Fourier Series to Solve ODE with Periodic Forcing Term

Example 0.5. *Solve the ODE:*

$$y'' + \omega^2 y = F(x) \tag{45}$$

Where F is a 2π periodic function with definition $F(x)$ on $[-\pi, \pi]$:

$$F(x) = \begin{cases} \pi + t, & -\pi \leq x < 0 \\ \pi - t, & 0 \leq x < \pi \end{cases} \tag{46}$$

We first derive the Fourier series of $F(x)$:

$$\begin{aligned}
a_n &= \int_{-\pi}^{\pi} F(x)\cos(nx)\,dx && (47)\\
&= 2\int_{0}^{\pi} (\pi - t)\cos(nx)\,dx && (48)\\
&= 2\left(\frac{1}{n}(\pi - t)\sin(nx)\mid_0^\pi + \int_0^\pi \frac{1}{n}\sin(nx)\right) && (49)\\
&= 2\left(+\frac{1}{n^2}(-\cos(nx))\mid_0^\pi\right) && (50)\\
&= -\frac{2}{n^2}\left(\cos(n\pi) - 1\right) && (51)\\
&= -\frac{2}{n^2}\left((-1)^n - 1\right) && (52)\\
b_n &= \int_{-\pi}^{\pi} F(x)\sin(nx)\,dx && (53)\\
&= 0 && (54)\\
a_0 &= \int_{-\pi}^{\pi} F(x)\cos(nx)\,dx && (55)\\
&= 2\int_{0}^{\pi} (\pi - t)\,dx && (56)\\
&= \pi^2 && (57)
\end{aligned}$$

For the particular solution, we assume the solution to be of the form:

$$y_p(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos(nx) + B_n\sin(nx)\right) \tag{58}$$

Then we have:

$$\begin{cases} \omega^2 \frac{A_0}{2} = \frac{a_0}{2} \\ -A_n n^2 + \omega^2 A_n = a_n \\ -B_n n^2 + \omega^2 B_n = b_n = 0 \end{cases} \tag{59}$$

$$\begin{cases} A_0 = \frac{a_0}{\omega^2} \\ A_n = \frac{a_n}{n^2 - \omega^2} \\ B_n = 0 \end{cases} \tag{60}$$

The general solution is then:

$$y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x) + y_p(x) \tag{61}$$

19. Laplace Equation

Definition 0.15. *The Laplace equation is given by:*

$$\nabla^2 \phi = 0 \tag{62}$$

20. Wave Equation

Definition 0.16. *The wave equation is given by:*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{63}$$

21. Heat Equation

Definition 0.17. *The heat equation is given by:*

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{64}$$

22. Using Change of Variable to Solve Wave Equation

Proof. Give the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{65}$$

We can use the change of variable:

$$2\xi = x - ct \tag{66}$$

$$2\eta = x + ct \tag{67}$$

Then we have:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \tag{68}$$

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \tag{69}$$

Substituting the change of variable into the wave equation, we have:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} && (70)\\
\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right) u &= c^2 \left(\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right) u && (71)\\
\left(c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}\right) &= c^2 \left(\frac{\partial^2 u}{\partial \xi^2} + 2c\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}\right) && (72)\\
\frac{\partial^2 u}{\partial \xi \partial \eta} &= 0 && (73)
\end{aligned}$$

Integrating the equation, we have:

$$u = f(\xi) + g(\eta) \tag{74}$$

which is the general solution of the wave equation. □

23. Solving Spherical Laplace Equation

Proof. We are interested in solving the Laplace equation in spherical coordinates, with the assumption that the solution is independent of ϕ.

The general solution is then:

$$\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \sum_{l=-1}^{-\infty} B_l r^l P_l(\cos\theta) \tag{75}$$

□