1. Orders of ODE

   Definition 0.1. *An ODE is said to be of order n if the highest order derivative of the unknown function that appears in the ODE is of order n.*

2. Degree of ODE

   Definition 0.2. *An ODE is said to be of degree n if the highest power of the highest order derivative of the unknown function that appears in the ODE is n.*

   Example 0.1. *The ODE $y'' + y = 0$ is of order 2 and degree 1.*

3. Linear ODE

   Definition 0.3. *An ODE is said to be linear if it can be written in the form*

   $$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \tag{1}$$

   where $a_i(x)$ are functions of x and $f(x)$ is a function of x.

4. Homogeneous ODE

   Definition 0.4. *An ODE is said to be homogeneous if the term that does not involve the unknown function is zero.*

   Example 0.2. *The ODE $y'' + y = 0$ is homogeneous.*

   The ODE $y'' + y = x$ is not homogeneous.

5. The superposition principle

   Theorem 0.1. *If $y_1(x)$ and $y_2(x)$ are solutions of a homogeneous linear ODE, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of the ODE, where c_1 and c_2 are constants.*

6. Series solution of ODE

   Theorem 0.2. *It is possible to construct a series solution for most ODEs.*

   For ODE:

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2}$$

   If we are interested in finding a solution around $x = x_0$, we can assume the solution to be of the form:

   $$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{3}$$

   Then expanding P, Q, and R in Taylor series around $x = x_0$, we can substitute the series solution into the ODE, and solve for the coefficients a_n.

   Example 0.3. *Find the series solution of the ODE*

   $$(1 + x^2)y'' + xy' - y = 0 \tag{4}$$

   about $x = 0$.

We assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

Substituting the series solution into the ODE, we have:

$$(1+x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{6}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 0 \tag{7}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 0 \tag{8}$$

$$\sum_{n=0}^{\infty} \left(\left(n(n-1) + n - 1\right) a_n + (n+2)(n+1)a_{n+2} \right) x^n = 0 \tag{9}$$

Since the series must be zero for all x, the coefficients of x^n must be zero for all n. Therefore, we have:

$$\begin{aligned}
\left((n+1)(n-1)\right) a_n + (n+2)(n+1)a_{n+2} &= 0, \quad n \geq 0 && (10)\\
a_{n+2} &= -\frac{(n+1)(n-1)}{(n+2)(n+1)} a_n && (11)\\
a_{n+2} &= -\frac{n-1}{n+2} a_n && (12)
\end{aligned}$$

And it is clear that the general solution depends on a_0 and a_1, and has a degeneracy of 2.

7. Ordinary points and singular points

   Definition 0.5. *Given an ODE of the form:*

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{13}$$

   A point $x = x_0$ is said to be an ordinary point if $Q(x)/P(x)$, and $R(x)/P(x)$ are analytic at $x = x_0$.

   Otherwise, $x = x_0$ is said to be a singular point.

   Trying to find a series solution around an ordinary point will result in a solution that has a degeneracy of 2.

8. Regular singular points

   Definition 0.6. *Given an ODE of the form:*

   $$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{14}$$

   A singular point $x = x_0$ is said to be a regular singular point if $(x - x_0)Q(x)/P(x)$, and $(x - x_0)^2 R(x)/P(x)$ are analytic at $x = x_0$. Otherwise, it is an irregular singular point.

   Finding a solution around a regular singular point will need special treatment.

9. Euler Equation

Definition 0.7. *An ODE of the form:*

$$x^2y'' + axy' + by = 0 \tag{15}$$

is called an Euler equation.

If the quadraric eqation:

$$m^2 + (a-1)m + b = 0 \tag{16}$$

has two distinct roots m_1 and m_2, then the general solution of the Euler equation is:

$$y(x) = c_1x^{m_1} + c_2x^{m_2} \tag{17}$$

If the quadratic equation has a single root m, then the general solution of the Euler equation is:

$$y(x) = c_1x^m + c_2x^m \ln x \tag{18}$$

10. Frobenius method

Definition 0.8. *The Frobenius method is a method to find the series solution of ODEs around regular singular points.*

Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{19}$$

Assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+m} \tag{20}$$

Where m is the root of the indicial equation:

$$m^2 + (\lim_{x\to x_0} \frac{xQ}{P} - 1)m + \lim_{x\to x_0} \frac{x^2R}{P} = 0 \tag{21}$$

Substituting the series solution into the ODE, we can solve for the coefficients a_n.

If the indical equation has two same roots, then the general solution will be in the form

$$y(x) = \log(x)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+m} + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+m} \tag{22}$$

The Frobenius method is used when we try to find the series solution around a regular singular point.

Example 0.4. *Find a series solution of the ODE:*

$$2xy'' + (3-2x)y' + y = 0 \tag{23}$$

It is easy to see that $x = 0$ is a regular singular point.

Solving the indicial equation:

$$m^2 + (\frac{3}{2} - 1)m = 0 \tag{24}$$

we have $m = 0$ *or* $m = 1$.

Therefore, assume the general solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \tag{25}$$

Substituting the series solution into the ODE, we have:

$$\begin{aligned}
2x \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-2} & & \quad (26)\\
+(3-2x) \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m} &= 0 & \quad (27)\\
2 \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-1} & & \quad (28)\\
+3 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} & & \quad (29)\\
-2 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} &= 0 & \quad (30)\\
2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1} & & \quad (31)\\
+2 \sum_{n=0}^{\infty} \left(\left((n+1+m)(2(n+m)+3) \right) a_{n+1} + \left(1 - 2(n+m)\right) a_n \right) x^{n+m} &= 0 & \quad (32)
\end{aligned}$$

$2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1}$ *is automatically satisfied, as it is the same with the indicial equation.*

Therefore, we have:

$$\left((n+1+m)(2(n+m)+3) \right) a_{n+1} + \left(1 - 2(n+m)\right) a_n = 0 \tag{33}$$

As m *has two values, the general solution will depend on* a_0 *and* m*, and has a degeneracy of* 2.