3.13 (i) If H is a proper subgroup of a finite group G, then G is not the union of all the conjugates of H.

(ii) If G is a finite group with conjugacy classes $C_1, \ldots, C_m$ and if $g_i \in C_i$, then $G = \langle g_1, \ldots, g_m \rangle$

Proof. (i) We consider the stabilizer of H regarding the conjugate action of G on the whole set of subgroups of G.

$$N(H) = \{g \in G \mid gHg^{-1} = H\} \tag{1}$$

Clearly, as $H \subset N(H)$, $|H| \leq |N(H)|$.
And clearly $[G : N(H)] = \frac{|G|}{|N(H)|}$ is the number of conjugates of H in G.
Consider the number of elements in the union of all the conjugates of H.

$$\left| \bigcup_{g \in G} gHg^{-1} \right| \leq 1 + (|H| - 1)\, [G : N(H)] \tag{2}$$

The first term 1 on the right-hand side is the identity element of G, as the conjugate of the identity element is the identity element itself. The other part of the right-hand side is from counting the conjugate of all the other elements of H in G, which two conjugates may give the same element in G.

$$\begin{aligned}
\left| \bigcup_{g \in G} gHg^{-1} \right| &\leq 1 + (|H| - 1)\, [G : N(H)] && (3)\\
&= 1 + (|H| - 1)\frac{|G|}{|N(H)|} && (4)\\
&\leq 1 + (|H| - 1)\frac{|G|}{|H|} && (5)\\
&= 1 + |G| - \frac{|G|}{|H|} && (6)\\
&< |G| && (7)
\end{aligned}$$

The final strict inequality is because H is a proper subgroup of G, and indicates that G is not the union of all the conjugates of H.

(ii) Let $H = \langle g_1, \ldots, g_m \rangle$.
If $H = \{1\}$, then G has only one conjugacy class, and the conclusion is trivial.
If $H = G$, then we prove the conclusion.
Otherwise, H is a proper subgroup of G.
We assert that $\bigcup_{g \in G} gHg^{-1} = G$, as for any $g \in G$, there is a k, and $h \in G$, such that $g = hg_kh^{-1}$, and $h \in H$.
However, this contradicts the conclusion of part i.

□