1. Stochastic process

   Definition 0.1. *A stochastic process is a collection of random variables* $\{X_t\}_{t\in T}$*, where* T *is an index set.*

2. Markov Property

   Definition 0.2. *A stochastic process* $\{X_t\}_{t\in T}$ *has the Markov property if*

$$P(X_{t_{n+1}} = x_{n+1}|X_{t_1} = x_1, \ldots, X_{t_n} = x_n) = P(X_{t_{n+1}} = x_{n+1}|X_{t_n} = x_n) \tag{1}$$

3. Markov Chain

   Definition 0.3. *A Markov chain is a stochastic process with the Markov property.*

4. Time-homogeneous Markov Chain

   Definition 0.4. *A Markov chain is time-homogeneous if the transition probabilities are independent of time.*

5. Transition Probability

   Definition 0.5. *The transition probability* P_{ij} *is the probability of transitioning from state* i *to state* j*.*

6. Transition Probability Matrix

   Definition 0.6. *The transition probability matrix* P *is a matrix where* P_{ij} *is the probability of transitioning from state* i *to state* j*.*

7. n-step Transition Probability Matrix

   Definition 0.7. *The n-step transition probability matrix* $P^{(n)}$ *is a matrix where* $P_{ij}^{(n)}$ *is the probability of transitioning from state* i *to state* j *in* n *steps.*

$$P_{ij}^{(n)} = P(X_{t_n} = j|X_{t_0} = i) \tag{2}$$

   where $t_n = t_0 + n$*.*

8. Chapman-Kolmogorov Equation

   Theorem 0.1. *The Chapman-Kolmogorov equation is given by*

$$P_{ij}^{(n+m)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)} \tag{3}$$

9. Hitting Probability

   Definition 0.8. *The hitting probability* f_{ij} *is the probability of reaching state* j *in the future starting from state* i*.*

$$\begin{aligned} P_i(hit\ j) = f_{ij} &= P(reach\ state\ j|start\ at\ state\ i) && (4)\\ & \sum_{n=1}^{\infty} P(first\ reach\ state\ j\ in\ n\ steps|start\ at\ state\ i) && (5)\end{aligned}$$

10. First Visit Probability

    Definition 0.9. *The first visit probability* $f_{ij}^{(n)}$ *is the probability of reaching state* j *in exactly* n *steps starting from state* i*.*

    $$f_{ij}^{(n)} = P(\textit{reach state } j \textit{ in exactly } n \textit{ steps} | \textit{start at state } i) \tag{6}$$

11. Hitting Time

    Definition 0.10. *Given a stochastic path* $\{X_t\}_{t \in T}$*, the hitting time* T_i^j *is the first time the path reaches state* j *starting from state* i*.*

    We use T^j *to denote the hitting time of state* j *starting from the current state.*

12. Communicate

    Definition 0.11. *Two states* i *is said to Communicate with* j *and is written as* $i \to j$ *if there exists a natural number* n *such that* $P_{ij}^{(n)} > 0$*.*

13. Inter Communicate

    Definition 0.12. *Two states* i *and* j *are said to Inter Communicate and are written as* $i \leftrightarrow j$ *if* $i \to j$ *and* $j \to i$*.*

    Example 0.1. $i \leftrightarrow i$ *as* $P_{ii}^{(0)} = 1$*.*

14. Irreducible Class of States

    Definition 0.13. *A class of states* $C = \{i_1, i_2, \ldots, i_n\}$ *is said to be Irreducible if for all* $i, j \in C$*,* $i \leftrightarrow j$*.*

15. Closed Class of States

    Definition 0.14. *A class of states* $C = \{i_1, i_2, \ldots, i_n\}$ *is said to be Closed if for all* $i \in C$*, there is no* $j \notin C$ *such that* $i \to j$*.*

16. Absorbing State

    Definition 0.15. *A state* i *is said to be Absorbing if* $P_{ii} = 1$*.*

17. Recurrent (Persistent) State

    Definition 0.16. *A state* i *is said to be Recurrent or Persistent if the hitting probability* $f_{ii} = 1$*.*

18. Transient State

    Definition 0.17. *A state* i *is said to be Transient if it is not Recurrent.*

19. Recurrent Markov Chain

    Definition 0.18. *A Markov chain is said to be Recurrent if all states are Recurrent.*

20. Transient Markov Chain

Definition 0.19. *A Markov chain is said to be Transient if all states are Transient.*

21. Identification of Recurrent States

    Theorem 0.2. *A state i is Recurrent if and only if the series*

    $$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \tag{7}$$

    Theorem 0.3. *If a state i is Transient, then*

    $$\lim_{n\to\infty} P_{ji}^{(n)} = 0, \forall j \tag{8}$$

    Note, that the converse of this theorem is not true.

22. There is at least one Recurrent State for finite state space

    Theorem 0.4. *For a finite class of closed state space, there is at least one Recurrent State.*

    Proof. Let I be a finite class of closed state space. If all states are Transient, then

    $$\lim_{n\to\infty} P_{ii}^{(n)} = 0, \forall i \in I \tag{9}$$

    .

    Given some $i \in I$, we have

    $$P_{ii}^{(n)} = \sum_{j\in I} P_{ij} P_{ji}^{(n-1)} \tag{10}$$

    For all $j \in I$ such that $P_{ij} > 0$, we have

    $$\lim_{n\to\infty} P_{ji}^{(n-1)} = 0 \tag{11}$$

    if not, equation 10 will not hold.
    Thus, for all $i \in I$, and all $j \in I$ such that $P_{ij} > 0$, we have

    $$\lim_{n\to\infty} P_{ii}^{(n)} = 0 \tag{12}$$

    Then, for all $n > 1$ we have

    $$\begin{aligned} 1 &= \sum_{j\in I} P_{ij}^{(n)} & (13)\\ \lim_{n\to\infty} 1 &= \lim_{n\to\infty} \sum_{j\in I} P_{ij}^{(n)} & (14)\\ 1 &= \sum_{j\in I} \lim_{n\to\infty} P_{ij}^{(n)} & (15)\\ 1 &= \sum_{j\in I} 0 & (16)\\ 1 &= 0 & (17) \end{aligned}$$

    This is a contradiction. Thus, there is at least one Recurrent State. □

23. Inter Communicate Classes have the same Recurrence

    Theorem 0.5. *If two states i and j Inter Communicate, then i is Recurrent if and only if j is Recurrent.*

24. Number of visits to a state

    Definition 0.20. *Given a stochastic path $\{X_t\}_{t \in T}$, the number of visits to state j is given by*

$$N_j = \sum_{n=1}^{\infty} 1_{\{X_n = j\}} \tag{18}$$

25. Number of visits to a state at infinity

    Theorem 0.6.

$$P(N_i = \infty) = 1, \forall i \text{ Recurrent} \tag{19}$$

$$P(N_i = \infty) = 0, \forall i \text{ Transient} \tag{20}$$

26. Decomposition of State Space

    Theorem 0.7. *The state space can be decomposed into a set of Irreducible Closed Recurrent classes and a set of Transient classes.*

    Proof. Given state space I, as the InterCommunicate relation is an equivalence relation, we have a partition of I into Inter Communicate classes. Let the classes be $I_\alpha, \alpha \in A$.

    As all states in an InnerCommunicate class have the same Recurrence, let

$$B = \{\alpha \in A | I_\alpha \text{ is Transient}\} \tag{21}$$

    Let $C = A - B$. Then, for all $\alpha \in C$, I_α is Recurrent.

    We then prove that I_α is Closed for all $\alpha \in C$., which finished the proof of this theorem.

    If I_α is not closed, then there exists $i \in I_\alpha$ and $j \notin I_\alpha$, and $n \in \mathbb{N}$ such that $P_{ij}^{(n)} > 0$.

    Also, as $i \in I_\alpha$, i is Recurrent, and thus

$$f_{ii} = 1 \tag{22}$$

    Also, in this case, j can not Communicate with i, as if it does, j will be in I_α, which means

$$f_{ji} = 0 \tag{23}$$

Then we have

$$\begin{aligned}
1 &= f_{ii} \\
&= \sum_{k \in I} P_{ik}^{(n)} f_{ki} \\
&= P_{ij}^{(n)} * f_{ji} + \sum_{k \in I, k \neq j} P_{ik}^{(n)} f_{ki} \\
&= \sum_{k \in I, k \neq j} P_{ik}^{(n)} f_{ki} \\
&\leq \sum_{k \in I, k \neq j} P_{ik}^{(n)} \\
&= 1 - P_{ij}^{(n)} \\
&< 1
\end{aligned}$$

This is a contradiction. Thus, I_α is closed. □

27. Mean Recurrence Time

Definition 0.21. *The mean recurrence time μ_i is the expected number of steps to return to state i starting from state i.*

$$\mu_i = \begin{cases} \sum_{n=1}^{\infty} n f_{ii}^{(n)} & \text{if } i \text{ is Recurrent} \\ \infty & \text{if } i \text{ is Transient} \end{cases} \tag{24}$$

28. Generating Function of Mean Recurrence Time

Definition 0.22. *Define*

$$F_{ij}(s) = \sum_{n=1}^{\infty} f_{ij}^{(n)} s^n \tag{25}$$

Then we have

$$\mu_i = F'_{ii}(1) \tag{26}$$

29. Positive Recurrent and Null Recurrent

Definition 0.23. *A state i is said to be Positive Recurrent if*

$$\mu_i < \infty \tag{27}$$

A state i is said to be Null Recurrent if

$$\mu_i = \infty \tag{28}$$

30. Identification of Positive Recurrent States

Theorem 0.8. *A state i is Null Recurrent if and only if*

$$\lim_{n \infty} P_{ii}^{(n)} = 0 \tag{29}$$

31. Item in the same Irreducible Class have the same Recurrence

    Theorem 0.9. *If two states i and j are in the same Irreducible class, then i is Positive Recurrent if and only if j is Positive Recurrent.*

    i is Null Recurrent if and only if j is Null Recurrent.

32. Finite State Class has at least one Positive Recurrent State

    Theorem 0.10. *For a finite class of closed state space, there is at least one Positive Recurrent State.*

33. The Expected First Hit Time

    Definition 0.24. *The expected first hit time β_{ij} is the expected number of steps to reach state j starting from state i.*

$$\beta_{ij} = \begin{cases} \sum_{n=1}^{\infty} n f_{ij}^{(n)} & \text{if } i \neq j \\ \mu_i & \text{if } i = j \end{cases} \tag{30}$$

34. Periodicity

    Definition 0.25. *A state i is said to be periodic if the greatest common divisor of the set*

$$\{n \in \mathbb{N} | P_{ii}^{(n)} > 0\} \tag{31}$$

    is greater than 1.

    If the greatest common divisor is 1, then the state is said to be aperiodic.

35. State in Irreducible State Class Have Same Periodicity

    Theorem 0.11. *Given two states $i \leftrightarrow j$, then i and j have the same period.*

36. Equilibrium (Stationary) Distribution of a Markov Chain

    Definition 0.26. *A distribution π is said to be an equilibrium distribution if*

$$\sum_{i \in S} \pi_i = 1 \tag{32}$$

$$\pi = \pi P \tag{33}$$

37. Relationship Between Stationary State and Mean Recurrence Time

    Theorem 0.12. *An Irreducible Markov Chain is Positive Recurrent if and only if there exists a unique stationary distribution π.*

    In this case, the stationary distribution is given by

$$\pi_i = \frac{1}{\mu_i} \tag{34}$$

38. Ergodic Markov Chain Limiting Distribution

Definition 0.27. *A Markov Chain is said to be Ergodic if it is Irreducible, Positive Recurrent, and Aperiodic.*

Theorem 0.13. *For an Ergodic Markov Chain, the limiting distribution exists and is given by*

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j, \forall i, j \tag{35}$$

39. Passage Time to a Closed State Space

Definition 0.28. *Given a closed state space A, and given a path $\{X_t\}_{t\in T}$, where $X_0 = i$ the first passage time to A is given by*

$$T_i^A = \inf\{t > 0 | X_t \in A\} \tag{36}$$

40. Absorption Probability to a Closed State Space

Definition 0.29. *Given a closed state space A, and given a path $\{X_t\}_{t\in T}$, where $X_0 = i$ the absorption probability to A is given by*

$$\alpha_i^A = P(T_i^A < \infty) \tag{37}$$

41. Mean Passage Time to a Closed State Space

Definition 0.30. *Given a closed state space A, and given a path $\{X_t\}_{t\in T}$, where $X_0 = i$ the mean passage time to A is given by*

$$\beta_i^A = E[T_i^A] \tag{38}$$