

1. Orders of ODE

**Definition 0.1.** An ODE is said to be of order  $n$  if the highest order derivative of the unknown function that appears in the ODE is of order  $n$ .

2. Degree of ODE

**Definition 0.2.** An ODE is said to be of degree  $n$  if the highest power of the highest order derivative of the unknown function that appears in the ODE is  $n$ .

**Example 0.1.** The ODE  $y'' + y = 0$  is of order 2 and degree 1.

3. Linear ODE

**Definition 0.3.** An ODE is said to be linear if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \quad (1)$$

where  $a_i(x)$  are functions of  $x$  and  $f(x)$  is a function of  $x$ .

4. Homogeneous ODE

**Definition 0.4.** An ODE is said to be homogeneous if the term that does not involve the unknown function is zero.

**Example 0.2.** The ODE  $y'' + y = 0$  is homogeneous.

The ODE  $y'' + y = x$  is not homogeneous.

5. The superposition principle

**Theorem 0.1.** If  $y_1(x)$  and  $y_2(x)$  are solutions of a homogeneous linear ODE, then  $c_1y_1(x) + c_2y_2(x)$  is also a solution of the ODE, where  $c_1$  and  $c_2$  are constants.

6. Series solution of ODE

**Theorem 0.2.** It is possible to construct a series solution for most ODEs.

For ODE:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (2)$$

If we are interested in finding a solution around  $x = x_0$ , we can assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad (3)$$

Then expanding  $P$ ,  $Q$ , and  $R$  in Taylor series around  $x = x_0$ , we can substitute the series solution into the ODE, and solve for the coefficients  $a_n$ .

**Example 0.3.** Find the series solution of the ODE

$$(1 + x^2)y'' + xy' - y = 0 \quad (4)$$

about  $x = 0$ .

We assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (5)$$

Substituting the series solution into the ODE, we have:

$$(1+x^2) \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \quad (6)$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 0 \quad (7)$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 0 \quad (8)$$

$$\sum_{n=0}^{\infty} ((n(n-1) + n - 1) a_n + (n+2)(n+1)a_{n+2}) x^n = 0 \quad (9)$$

Since the series must be zero for all  $x$ , the coefficients of  $x^n$  must be zero for all  $n$ . Therefore, we have:

$$((n+1)(n-1)) a_n + (n+2)(n+1)a_{n+2} = 0, \quad n \geq 0 \quad (10)$$

$$a_{n+2} = -\frac{(n+1)(n-1)}{(n+2)(n+1)} a_n \quad (11)$$

$$a_{n+2} = -\frac{n-1}{n+2} a_n \quad (12)$$

And it is clear that the general solution depends on  $a_0$  and  $a_1$ , and has a degeneracy of 2.

## 7. Ordinary points and singular points

**Definition 0.5.** Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (13)$$

A point  $x = x_0$  is said to be an ordinary point if  $Q(x)/P(x)$ , and  $R(x)/P(x)$  are analytic at  $x = x_0$ .

Otherwise,  $x = x_0$  is said to be a singular point.

Trying to find a series solution around an ordinary point will result in a solution that has a degeneracy of 2.

## 8. Regular singular points

**Definition 0.6.** Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (14)$$

A singular point  $x = x_0$  is said to be a regular singular point if  $(x - x_0)Q(x)/P(x)$ , and  $(x - x_0)^2 R(x)/P(x)$  are analytic at  $x = x_0$ . Otherwise, it is an irregular singular point.

Finding a solution around a regular singular point will need special treatment.

## 9. Euler Equation

**Definition 0.7.** An ODE of the form:

$$x^2 y'' + axy' + by = 0 \quad (15)$$

is called an Euler equation.

If the quadratic equation:

$$m^2 + (a-1)m + b = 0 \quad (16)$$

has two distinct roots  $m_1$  and  $m_2$ , then the general solution of the Euler equation is:

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2} \quad (17)$$

If the quadratic equation has a single root  $m$ , then the general solution of the Euler equation is:

$$y(x) = c_1 x^m + c_2 x^m \ln x \quad (18)$$

## 10. Frobenius method

**Definition 0.8.** The Frobenius method is a method to find the series solution of ODEs around regular singular points.

Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (19)$$

Assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m} \quad (20)$$

Where  $m$  is the root of the indicial equation:

$$m^2 + \left( \lim_{x \rightarrow x_0} \frac{xQ}{P} - 1 \right) m + \lim_{x \rightarrow x_0} \frac{x^2 R}{P} = 0 \quad (21)$$

Substituting the series solution into the ODE, we can solve for the coefficients  $a_n$ .

If the indicial equation has two same roots, then the general solution will be in the form

$$y(x) = \log(x) \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m} + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+m} \quad (22)$$

If given  $m_1 > m_2$  and  $m_1 - m_2 \in \mathbb{Z}$ , then the general solution will be in the form

$$y(x) = \log(x) \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m_1} + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+m_2} \quad (23)$$

The Frobenius method is used when we try to find the series solution around a regular singular point.

**Example 0.4.** Find a series solution of the ODE:

$$2xy'' + (3 - 2x)y' + y = 0 \quad (24)$$

It is easy to see that  $x = 0$  is a regular singular point.

Solving the indicial equation:

$$m^2 + \left(\frac{3}{2} - 1\right)m = 0 \quad (25)$$

we have  $m = 0$  or  $m = 1$ .

Therefore, assume the general solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \quad (26)$$

Substituting the series solution into the ODE, we have:

$$2x \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-2} \quad (27)$$

$$+ (3 - 2x) \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \quad (28)$$

$$2 \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-1} \quad (29)$$

$$+ 3 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} \quad (30)$$

$$- 2 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \quad (31)$$

$$2a_0 m(m-1) x^{m-1} + 3a_0 m x^{m-1} \quad (32)$$

$$+ 2 \sum_{n=0}^{\infty} (((n+1+m)(2(n+m)+3)) a_{n+1} + (1-2(n+m)) a_n) x^{n+m} = 0 \quad (33)$$

$2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1}$  is automatically satisfied, as it is the same with the indicial equation.

Therefore, we have:

$$((n+1+m)(2(n+m)+3)) a_{n+1} + (1-2(n+m)) a_n = 0 \quad (34)$$

As  $m$  has two values, the general solution will depend on  $a_0$  and  $m$ , and has a degeneracy of 2.

## 11. Legendre Equation

**Definition 0.9.** The Legendre equation is given by: and is given by:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \quad (35)$$

As the Legendre equation has an ordinary point at  $x = 0$ , we can solve this equation for two linearly independent series solutions:  $y_{0,\alpha}(x)$ , and  $y_{1,\alpha}(x)$ .

If  $\alpha$  is not a integer, then both  $y_{0,\alpha}(x)$ , and  $y_{1,\alpha}(x)$  is an infinite series with a radius of convergence 1.

## 12. Legendre Polynomial

**Definition 0.10.** If  $\alpha$  in Equation 35 is an integer, then the series solution to Equation 35 is a polynomial, which is often known as the Legendre polynomial.

**Theorem 0.3** (Orthogonality of Legendre Polynomials). The Legendre polynomials are orthogonal on the interval  $[-1, 1]$ , that is:

$$\int_{-1}^1 P_n(x)P_m(x) dx = \frac{2\delta_{nm}}{2n+1} \quad (36)$$

**Theorem 0.4** (Negative index Legendre Polynomial). As the equation 35 is invariant under  $\alpha \rightarrow -\alpha - 1$ ,

$$P_n(x) = P_{-n-1}(x) \quad (37)$$

## 13. Bessel Equation

**Definition 0.11.** The Bessel equation is given by:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad (38)$$

As the Bessel equation has a regular singular point at  $x = 0$ , we can solve this equation for two linearly independent series solutions.

## 14. Fourier Series

**Definition 0.12.** The Fourier series of a function  $f(x)$  is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad (39)$$

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad (40)$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (41)$$

## 15. Trigonometric Integration Formula

**Definition 0.13.** The trigonometric integration formula is given by:

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \pi \delta_{mn} \quad (42)$$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \pi \delta_{mn} \quad (43)$$

$$\int_{-\pi}^{\pi} \cos(mx) \sin(nx) dx = 0 \quad (44)$$

16. Fourier Convergence Theorem

**Theorem 0.5.** *If  $f(x)$  is piecewise continuous on  $[-L, L]$ , then the Fourier series of  $f(x)$  converges to  $f(x)$  at all points where  $f(x)$  is continuous, and converges to the average of the left and right limits of  $f(x)$  at points where  $f(x)$  is discontinuous.*

*If the edge of the discontinuity is at  $x = a$ , then there will be the Gibbs phenomenon at  $x = a$ .*

17. Gibbs Phenomenon

**Definition 0.14.** *The Gibbs phenomenon is the overshoot of the Fourier series at the edge of a discontinuity.*

18. Use Fourier Series to Solve ODE with Periodic Forcing Term

**Example 0.5.** *Solve the ODE:*

$$y'' + \omega^2 y = F(x) \quad (45)$$

Where  $F$  is a  $2\pi$  periodic function with definition  $F(x)$  on  $[-\pi, \pi]$ :

$$F(x) = \begin{cases} \pi + t, & -\pi \leq x < 0 \\ \pi - t, & 0 \leq x < \pi \end{cases} \quad (46)$$

We first derive the Fourier series of  $F(x)$ :

$$a_n = \int_{-\pi}^{\pi} F(x) \cos(nx) dx \quad (47)$$

$$= 2 \int_0^{\pi} (\pi - t) \cos(nx) dx \quad (48)$$

$$= 2 \left( \frac{1}{n} (\pi - t) \sin(nx) \Big|_0^{\pi} + \int_0^{\pi} \frac{1}{n} \sin(nx) \right) \quad (49)$$

$$= 2 \left( + \frac{1}{n^2} (-\cos(nx)) \Big|_0^{\pi} \right) \quad (50)$$

$$= -\frac{2}{n^2} (\cos(n\pi) - 1) \quad (51)$$

$$= -\frac{2}{n^2} ((-1)^n - 1) \quad (52)$$

$$b_n = \int_{-\pi}^{\pi} F(x) \sin(nx) dx \quad (53)$$

$$= 0 \quad (54)$$

$$a_0 = \int_{-\pi}^{\pi} F(x) \cos(nx) dx \quad (55)$$

$$= 2 \int_0^{\pi} (\pi - t) dx \quad (56)$$

$$= \pi^2 \quad (57)$$

For the particular solution, we assume the solution to be of the form:

$$y_p(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx)) \quad (58)$$

Then we have:

$$\begin{cases} \omega^2 \frac{A_0}{2} = \frac{a_0}{2} \\ -A_n n^2 + \omega^2 A_n = a_n \\ -B_n n^2 + \omega^2 B_n = b_n = 0 \end{cases} \quad (59)$$

$$\begin{cases} A_0 = \frac{a_0}{\omega^2} \\ A_n = \frac{a_n}{n^2 - \omega^2} \\ B_n = 0 \end{cases} \quad (60)$$

The general solution is then:

$$y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x) + y_p(x) \quad (61)$$

#### 19. Laplace Equation

**Definition 0.15.** The Laplace equation is given by:

$$\nabla^2 \phi = 0 \quad (62)$$

#### 20. Wave Equation

**Definition 0.16.** The wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (63)$$

#### 21. Heat Equation

**Definition 0.17.** The heat equation is given by:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (64)$$

#### 22. Using Change of Variable to Solve Wave Equation

*Proof.* Give the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (65)$$

We can use the change of variable:

$$2\xi = x - ct \quad (66)$$

$$2\eta = x + ct \quad (67)$$

Then we have:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta} \quad (68)$$

$$\frac{\partial}{\partial t} = -c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta} \quad (69)$$

Substituting the change of variable into the wave equation, we have:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (70)$$

$$\left(-c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}\right) \left(-c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}\right) u = c^2 \left(\frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}\right) \left(\frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}\right) u \quad (71)$$

$$\left(c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2}\right) = c^2 \left(\frac{\partial^2 u}{\partial \xi^2} + 2c \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2}\right) \quad (72)$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad (73)$$

Integrating the equation, we have:

$$u = f(\xi) + g(\eta) \quad (74)$$

which is the general solution of the wave equation.  $\square$

### 23. Solving Spherical Laplace Equation

*Proof.* We are interested in solving the Laplace equation in spherical coordinates, with the assumption that the solution is independent of  $\phi$ .

The general solution is then:

$$\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) + \sum_{l=-1}^{-\infty} B_l r^l P_l(\cos \theta) \quad (75)$$

$\square$

### 24. Fourier Transform

**Definition 0.18.** The Fourier transform of a function  $f(x)$  is given by:

$$\mathcal{F}(f) = F(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) dx \quad (76)$$

The inverse Fourier transform of a function  $F(k)$  is given by:

$$\mathcal{F}^{-1}(F) = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} F(k) dk \quad (77)$$

### 25. Properties of Fourier Transform

**Theorem 0.6.** The Fourier transform is linear, that is:

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g) \quad (78)$$

The Fourier transform of the derivative of a function is given by:

$$\mathcal{F}\left(\frac{df}{dx}\right) = -ik\mathcal{F}(f) \quad (79)$$



The Fourier transform of the convolution of two functions is given by:

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) \quad (80)$$

Double Fourier transform is the original function with a negative sign and a constant:

$$\mathcal{F}(\mathcal{F}(f(x))) = 2\pi f(-x) \quad (81)$$

## 26. Laplace Transform

**Definition 0.19.** The Laplace transform of a function  $f(t)$  is given by:

$$\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (82)$$

The inverse Laplace transform needs knowledge of complex analysis, which is not covered in this note.

## 27. Properties of Laplace Transform

**Definition 0.20.** The Laplace transform is linear, that is:

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g) \quad (83)$$

Laplace transform of indefinite integral is given by:

$$\mathcal{L}\left(\int_0^t f(x)dx\right) = \int_0^{\infty} e^{-st} \int_0^t f(x)dx dt \quad (84)$$

$$= \left[ -\frac{1}{s} e^{-st} \int_0^t f(x)dx \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} f(t) dt \quad (85)$$

$$= \frac{1}{s} \mathcal{L}(f) \quad (86)$$