

1. Orders of ODE

Definition 0.1. An ODE is said to be of order n if the highest order derivative of the unknown function that appears in the ODE is of order n .

2. Degree of ODE

Definition 0.2. An ODE is said to be of degree n if the highest power of the highest order derivative of the unknown function that appears in the ODE is n .

Example 0.1. The ODE $y'' + y = 0$ is of order 2 and degree 1.

3. Linear ODE

Definition 0.3. An ODE is said to be linear if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \quad (1)$$

where $a_i(x)$ are functions of x and $f(x)$ is a function of x .

4. Homogeneous ODE

Definition 0.4. An ODE is said to be homogeneous if the term that does not involve the unknown function is zero.

Example 0.2. The ODE $y'' + y = 0$ is homogeneous.

The ODE $y'' + y = x$ is not homogeneous.

5. The superposition principle

Theorem 0.1. If $y_1(x)$ and $y_2(x)$ are solutions of a homogeneous linear ODE, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of the ODE, where c_1 and c_2 are constants.

6. Series solution of ODE

Theorem 0.2. It is possible to construct a series solution for most ODEs.

For ODE:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (2)$$

If we are interested in finding a solution around $x = x_0$, we can assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad (3)$$

Then expanding P , Q , and R in Taylor series around $x = x_0$, we can substitute the series solution into the ODE, and solve for the coefficients a_n .

Example 0.3. Find the series solution of the ODE

$$(1 + x^2)y'' + xy' - y = 0 \quad (4)$$

about $x = 0$.

We assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (5)$$

Substituting the series solution into the ODE, we have:

$$(1+x^2) \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \quad (6)$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 0 \quad (7)$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 0 \quad (8)$$

$$\sum_{n=0}^{\infty} ((n(n-1) + n - 1) a_n + (n+2)(n+1)a_{n+2}) x^n = 0 \quad (9)$$

Since the series must be zero for all x , the coefficients of x^n must be zero for all n . Therefore, we have:

$$((n+1)(n-1)) a_n + (n+2)(n+1)a_{n+2} = 0, \quad n \geq 0 \quad (10)$$

$$a_{n+2} = -\frac{(n+1)(n-1)}{(n+2)(n+1)} a_n \quad (11)$$

$$a_{n+2} = -\frac{n-1}{n+2} a_n \quad (12)$$

And it is clear that the general solution depends on a_0 and a_1 , and has a degeneracy of 2.

7. Ordinary points and singular points

Definition 0.5. Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (13)$$

A point $x = x_0$ is said to be an ordinary point if $Q(x)/P(x)$, and $R(x)/P(x)$ are analytic at $x = x_0$.

Otherwise, $x = x_0$ is said to be a singular point.

Trying to find a series solution around an ordinary point will result in a solution that has a degeneracy of 2.

8. Regular singular points

Definition 0.6. Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (14)$$

A singular point $x = x_0$ is said to be a regular singular point if $(x - x_0)Q(x)/P(x)$, and $(x - x_0)^2 R(x)/P(x)$ are analytic at $x = x_0$. Otherwise, it is an irregular singular point.

Finding a solution around a regular singular point will need special treatment.

9. Euler Equation

Definition 0.7. An ODE of the form:

$$x^2 y'' + ax y' + by = 0 \quad (15)$$

is called an Euler equation.

If the quadratic equation:

$$m^2 + (a-1)m + b = 0 \quad (16)$$

has two distinct roots m_1 and m_2 , then the general solution of the Euler equation is:

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2} \quad (17)$$

If the quadratic equation has a single root m , then the general solution of the Euler equation is:

$$y(x) = c_1 x^m + c_2 x^m \ln x \quad (18)$$

10. Frobenius method

Definition 0.8. The Frobenius method is a method to find the series solution of ODEs around regular singular points.

Given an ODE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (19)$$

Assume the solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m} \quad (20)$$

Where m is the root of the indicial equation:

$$m^2 + \left(\lim_{x \rightarrow x_0} \frac{xQ}{P} - 1 \right) m + \lim_{x \rightarrow x_0} \frac{x^2 R}{P} = 0 \quad (21)$$

Substituting the series solution into the ODE, we can solve for the coefficients a_n .

If the indicial equation has two same roots, then the general solution will be in the form

$$y(x) = \log(x) \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m} + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+m} \quad (22)$$

If given $m_1 > m_2$ and $m_1 - m_2 \in \mathbb{Z}$, then the general solution will be in the form

$$y(x) = \log(x) \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m_1} + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+m_2} \quad (23)$$

The Frobenius method is used when we try to find the series solution around a regular singular point.

Example 0.4. Find a series solution of the ODE:

$$2xy'' + (3 - 2x)y' + y = 0 \quad (24)$$

It is easy to see that $x = 0$ is a regular singular point.

Solving the indicial equation:

$$m^2 + \left(\frac{3}{2} - 1\right)m = 0 \quad (25)$$

we have $m = 0$ or $m = 1$.

Therefore, assume the general solution to be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \quad (26)$$

Substituting the series solution into the ODE, we have:

$$2x \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-2} \quad (27)$$

$$+ (3 - 2x) \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \quad (28)$$

$$2 \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-1} \quad (29)$$

$$+ 3 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} \quad (30)$$

$$- 2 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \quad (31)$$

$$2a_0 m(m-1) x^{m-1} + 3a_0 m x^{m-1} \quad (32)$$

$$+ 2 \sum_{n=0}^{\infty} (((n+1+m)(2(n+m)+3)) a_{n+1} + (1-2(n+m)) a_n) x^{n+m} = 0 \quad (33)$$

$2a_0 m(m-1)x^{m-1} + 3a_0 m x^{m-1}$ is automatically satisfied, as it is the same with the indicial equation.

Therefore, we have:

$$((n+1+m)(2(n+m)+3)) a_{n+1} + (1-2(n+m)) a_n = 0 \quad (34)$$

As m has two values, the general solution will depend on a_0 and m , and has a degeneracy of 2.